Modular forms and two new integer sequences at level 7

A Thesis presented in partial fulfilment of the requirements for the degree of

Master of Science
in
Mathematics

at Massey University, Albany, New Zealand

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2016

> No mathematician can be a complete mathematician unless he is also something of a poet. - K.Weierstrass [55]

In the magical world of modular forms
Double series of theta functions transforms
To Fourier expansions of functions of q.
Some gaps in the theory left plenty to do.

So in level seven I fossicked around
Two infinite integer sequences found
In functions with coefficients polynomial
But, I could not find a form binomial.

Recurrence relations were produced
From this the theorems were deduced.
The proofs I have will be revealed
By reading this thesis in which they're concealed.

Lynette O'Brien (6/7/16)

ABSTRACT

Integer sequences resulting from recurrence relations with polynomial coefficients are rare. Two new integer sequences have been discovered and are the main result in this thesis. They consist of a three-term quadratic recurrence

$$(n+1)^2 c_7(n+1) = (26n^2+13n+2)c_7(n) + 3(3n-1)(3n-2)c_7(n-1)$$

with initial conditions $c_7(-1)=0$ and $c_7(0)=1$, and a five-term quartic recurrence

$$(n+1)^4 u_7(n+1) = -Pu_7(n) - Qu_7(n-1) - Ru_7(n-2) - Su_7(n-3)$$

where

$$\begin{aligned} P &= 26n^4+52n^3+58n^2+32n+7, \\ Q &= 267n^4+268n^2+18, \\ R &= 1274n^4-2548n^3+2842n^2-1568n+343, \\ S &= 2401(n-1)^4 \end{aligned}$$

with initial conditions $u_7(0)=1$ and $u_7(-1)=u_7(-2)=u_7(-3)=0$. The experimental procedure used in their discovery utilizes the mathematical software Maple. Proofs are given that rely on the theory of modular forms for level 7, Ramanujan's Eisenstein series, theta functions and Euler products. Differential equations associated with theta functions are solved to reveal these recurrence relations. Interesting properties are investigated including an analogue of Clausen's identity, asymptotic behaviour of the sequences and finally two conjectures for congruence properties are given.

Acknowledgements

It has been a privilege to have Associate Professor Shaun Cooper as my supervisor. His excellent teaching, patience and attention to detail has guided me through this thesis. I also want to thank my husband Graeme O'Brien who has encouraged me and for his proof-reading of my thesis.

PREFACE

This thesis is the original work of the author Lynette A. O'Brien. It consists of ten chapters. The introductory chapter gives a brief outline of the main results and the motivation. In Chapter 2 we give a brief historical overview of modular forms and some background theory. In Chapters 3–5 we give definitions, derivatives, differential equations and proofs. Our main results are revealed in Chapter 6. Then in Chapters 7–9 we look at consequences of our findings. First we discuss an analogue to Clausen's identity for the square of the hypergeometric series, then look at asymptotics and congruences. We finish with conclusions and suggestions for further work.

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