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SCALAR-TENSOR THEORIES OF GRAVITATION

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Chapter One

INTRODUCTION

The problem the writer wishes to consider here, is essentially one related to the classical field description of Nature.

The framework of General Relativity provides a theory for the geometry of the four dimensional space-time manifold and at the same time gives a description of the gravitational field in terms of the metric tensor, while the electromagnetic field can be interpreted in terms of a particular second rank, skew-symmetric tensor — the covariant curl of a vector field defined on the manifold. However the scalar field, the simplest geometric object that could be defined on the manifold, does not seem to be experimentally evident when it is interpreted as a third, classical long range field. In spite of this lack of experimental evidence and as there appears to be no theoretical objection to the existence of such a long range field, the problem is to introduce the scalar field into the classical scheme of things and to construct a viable theory containing all three long range fields.

It is interesting to compare the physical descriptions involved with these fields. Both the gravitational and the electromagnetic fields have gauge-like degrees of freedom and before a situation could be physically relevant these degrees of freedom must be fixed — for the gravitational field by imposing coordinate conditions while for the electromagnetic field, after coordinate conditions have been imposed the gauge of the field potential must be chosen. As a consequence of these gauge freedoms, in order that the fields couple consistently with matter sources, the energy momentum tensor of the source must be covariantly conserved and the electro-

magnetic current density of the source must be conserved. The scalar field on the other hand has no gauge-like degree of freedom and consequently has no conserved "charge" as a source. Thus for example, in contrast to the other two fields, no constraints exist by which the scalar field could be separated into a source "bound" part and a free "wave" part.

In recent years the problem of incorporating the scalar field into the description of gravitation has led to the investigation of a special class of gravitation theories — the scalar-tensor gravitation theories.

With the previously mentioned problem in mind, the goals of this thesis are

- (i) to review work that has been done on these theories and
- (ii) to discuss them in a way that compares them to the theory of gravitation given in General Relativity.

Chapter Two basically gives an historical background and introduces more specific motives for considering the scalar field as a fundamental physical field.

Chapter Three considers the important class of scalar-tensor gravitation theories based on a Riemann space-time and Chapter Four continues this theme by looking at the "most developed" and perhaps simplest member - the Brans-Dicke theory.

For completeness the "massive Brans-Dicke" theories and some special scalar-tensor theories are looked at briefly in Chapter Five.

Chapter Six returns to the scalar-tensor model of gravitation developed in Chapter Three and looks at the implications for the model in more general space-times.

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Chapter Two

BACKGROUND

The electromagnetic and gravitational fields were described in the introduction as classical long range fields. These fields are responsible for forces that fall off inversely proportional to the square of the distance apart of the interacting bodies (sources); in contrast to short range forces which show an exponential behaviour. Einstein (1916) (1), attributed to the space-time manifold a Riemann structure and gave "meaning" to the gravitational field in terms of curvature through his gravitational field equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \quad , \quad 2.1$$

where G is Newton's gravitational constant.

The notation established here is used in most sections. Units of length and time are chosen such that $c = 1$, although with this understanding some formulae may still contain c . Greek indices range over the values $\{0, 1, 2, 3\}$, the coordinates x^0 and x^i , ($i = 1, 2, 3$), are assumed time-like and space-like respectively and the signature of the space-time metric, $g_{\mu\nu}$ is $- + + +$.

The Riemann and Ricci tensors have the respective forms

$$R^{\lambda}{}_{\mu\nu\rho} = \left\{ \begin{matrix} \lambda \\ \mu\rho \end{matrix} \right\}_{,\nu} - \left\{ \begin{matrix} \lambda \\ \nu\mu \end{matrix} \right\}_{,\rho} + \left\{ \begin{matrix} \eta \\ \mu\rho \end{matrix} \right\} \left\{ \begin{matrix} \lambda \\ \nu\eta \end{matrix} \right\} - \left\{ \begin{matrix} \eta \\ \mu\nu \end{matrix} \right\} \left\{ \begin{matrix} \lambda \\ \rho\eta \end{matrix} \right\}$$

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}_{,\lambda} - \left\{ \begin{matrix} \lambda \\ \mu\lambda \end{matrix} \right\}_{,\nu} + \left\{ \begin{matrix} \eta \\ \mu\nu \end{matrix} \right\} \left\{ \begin{matrix} \lambda \\ \lambda\eta \end{matrix} \right\} - \left\{ \begin{matrix} \eta \\ \mu\lambda \end{matrix} \right\} \left\{ \begin{matrix} \lambda \\ \nu\eta \end{matrix} \right\}$$

where $\left\{ \begin{matrix} \sigma \\ \mu\nu \end{matrix} \right\} = \frac{1}{2g^{\lambda\sigma}} (\epsilon_{\mu\lambda,\nu} + \epsilon_{\nu\lambda,\mu} - \epsilon_{\mu\nu,\lambda})$ and partial differentiation is denoted by a comma.

The close relation between the Riemann curvature tensor and gravitational effects is further illustrated, for example, in the equations of geodesic deviation, (2)

$$\frac{D^2}{D\tau^2} \delta x^{\mu} = R^{\alpha}{}_{\mu\nu\rho} \delta x^{\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\rho}}{d\tau} \quad , \quad 2.2$$

where $x^\alpha(\tau)$, $x^\alpha(\tau) + \delta x^\alpha(\tau)$ define the paths of a pair of neighbouring, freely falling particles and $\frac{D}{D\tau}$ denotes the absolute derivative along the curve $x^\alpha(\tau)$. A freely falling particle is at rest in a coordinate frame falling with it, whereas a pair of neighbouring freely falling particles will show a relative acceleration given by eq. 2.2. To an observer travelling with the frame this motion will indicate the presence of a gravitational field.

The electromagnetic field on the other hand, appears in this picture as a field "embedded" in space-time, the geometry of which is determined by gravitation. A resolution of this difference in the roles of the two long range fields was proposed by Weyl (1918), (3, 4). However, along with other attempts at unification it was generally considered to be physically unsatisfactory, and so except for some special references, the electromagnetic field is included in the source side (i.e. the right hand side of eq. 2.1) of Einstein's field equations or of these equations in any subsequently modified form.

Basic to Weyl's approach was a generalisation of Riemann space-time - the Weyl space-time, for short. This space-time has been revived quite recently by some authors (e.g. Ross, Lord, and Omote , (5, 6, 7)) as a framework for scalar-tensor gravitation theories and for this reason it deserves a few comments about its historical origins, in addition to the treatment given in Chapter Six.

Curvature in Riemann space-time can be related to the idea of the parallel displacement of a vector - the transport of a vector by parallel displacement around a closed curve resulting in the final direction of the vector being different from its initial direction. Weyl supposed that the transported vector has a different length as well as a different direction and so for Weyl

space-time, unless two points are infinitesimally close together lengths at these points can only be compared with respect to a path joining them. Because a determination of length at one point leads to only a first order approximation to a determination of length at neighbouring points one must set up, arbitrarily, a standard of length at each point and with lengths referred to this local standard a definite number can be given for the length of a vector at a point. If a vector which has length, l , at a point with coordinates x^α is parallelly displaced to the point with coordinates $x^\alpha + \delta x^\alpha$ then its change of length Weyl gave to be

$$\delta l^2 = - l^2 \phi_{\mu} \delta x^{\mu}, \quad 2.3$$

where ϕ_{μ} are the components of a vector field.

For parallel displacement around a small closed curve the total change of length of the transported vector turns out to be

$$\Delta l^2 = - l^2 f_{\mu\lambda} \delta a^{\mu\lambda}, \quad 2.4$$

where $\delta a^{\mu\lambda}$ describes the element of area enclosed by the curve, and

$$f_{\mu\lambda} = \phi_{[\mu, \lambda]} - \phi_{[\lambda, \mu]}. \quad 2.5$$

Weyl set ϕ_{μ} proportional to the electromagnetic potential A_{μ} and so $f_{\mu\lambda}$ is made proportional to the electromagnetic field tensor, $F_{\mu\lambda}$. Thus the electromagnetic potential determines by eq. 2.3 the behaviour of length on parallel displacement and the electric and magnetic fields find expression in the derived tensor, $f_{\mu\lambda}$. This tensor can be shown to be independent of the initial choice of length standard, which is a necessary condition if it is to be physically meaningful.

A difficulty of the theory was an apparent conflict between eq. 2.3 and the interpretation given above, with the idea that atomic standards of length and

time appear absolute and independent of space-time position.* If the coefficient of proportionality between ϕ_μ and A_μ is assumed real and put equal to $\frac{2C}{e}$ (e is the **charge** of an electron and C is dimensionless) then a recent experiment, (8), places an upper bound on C of 10^{-47} . Such a figure, however, does not exclude the geometric interpretation of the electromagnetic field in terms of the Weyl space-time if, for example, Weyl's original idea of equivalent initial length standards is modified to give special status to atomic standards.

Aside from introducing the Weyl space-time these comments emphasize a feature of length standards in Riemann space-time, where once they are defined in terms of atomic standards at a point, parallel displacement allows the comparison of lengths taken at separated points. Without getting involved in problems of measurement we shall just assume that on this basis, the physical descriptions of atomic systems are independent of space-time position and that by using these systems the space-time interval measured between neighbouring events is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad 2.6$$

where $g_{\mu\nu}$ is identified with the gravitational field variable appearing in eq. 2.1.

With this brief and rather indirect look at some of the ideas involved in the Riemann space-time we return to look at Einstein's field equations and his description of gravitation in order to provide a background before introducing the scalar field.

Because the dynamical variable of the gravitational field in General Relativity is the metric tensor, it plays an important geometry-determining role in space-time and the field equations can be understood to couple the geometry of space-time to matter. Thus the only physical constraint imposed by these equations

* so under parallel displacement a vector maintains its length with respect to atomic standards.

on the nature of matter is that its energy-momentum tensor has zero divergence.

Implicit in all of the discussion so far has been the division between the gravitational field and matter. In his discussion of the action principle formulation of his field equations, Einstein (9), stated an assumption to ensure that this distinction carried over to the action principle — that is, the Lagrangian density could be divided into two parts, one of which refers to the gravitational field and contains only the metric tensor and its derivatives. The appropriate density for this part is the Riemann scalar density and apart from a cosmological term the resulting action for the free gravitational field is unique in giving field equations which are linear in the second derivatives of the metric and which in the weak-field limit give the Newtonian case.

In spite of this success in giving empty space-time field equations that are unique modulo a cosmological term, the action principle without further assumptions, does not offer much insight into the nature of the energy-momentum tensor. So the action principle remains an important method for constructing field equations.

In order to make progress later on, much use is made of the Principle of Minimal Coupling, (e.g. 10), which Anderson notes is not an essential part of General Relativity. If a material system is considered in Special Relativity as a set, χ of matter fields then its Euler-Lagrange equations of motion, in some inertial frame, will follow from an action principle

$$\delta \int L_M d^4x = 0 \quad ,$$

for suitable variations of the variables χ . The action ($\delta \int L_M d^4x$) will depend on the Lorentz metric, $\eta_{\mu\nu}$, and with $\eta_{\mu\nu}$ replaced by $g_{\mu\nu}$ this action when added to

the free gravitation field action gives the required action¹ for the gravitational and matter field equations. If \mathcal{L}_{NG} denotes the matter or nongravitational part of the full Lagrangian density (i.e. with the principle assumed, \mathcal{L}_{NG} is the minimally coupled L_M) then the energy-momentum tensor of the material system is defined in terms of the system's "response" to the metric field by

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \left[\frac{\partial \mathcal{L}_{\text{NG}}}{\partial g_{\mu\nu}} - \frac{\partial}{\partial x^\alpha} \left(\frac{\partial \mathcal{L}_{\text{NG}}}{\partial x^\alpha} \right) \right] + \dots \quad 2.8$$

This definition still holds without appeal to the Principle of Minimal Coupling but in this case the connection described between \mathcal{L}_{NG} and L_M could not be supposed to hold.

An aspect of the field equations noted here then, is that the nature of the energy-momentum tensor is determined by criteria outside of General Relativity. To work within the framework of General Relativity leads to an extreme position such as suggested by McCrea (11), that the Einstein tensor is to be interpreted or identified as an energy-momentum tensor and the central question is then which geometric constraints imposed by the field equations are physically meaningful. This rather formal approach has been developed a little by Harrison, (12), in a way, to suggest that scalar-tensor gravitation theories are in fact derivative from Einstein's theory by suitable interpretations of the energy-momentum tensor. However this view is a bit unorthodox and we shall return to it later.

It was mentioned in the introduction that the gravitational field, as described by Einstein, has a gauge-like degree of freedom. This of course corresponds to

¹ this minimal coupling prescription is sometimes, when needed, supplemented with the rule that :-

partial derivatives \rightarrow covariant derivatives.

the coordinate transformations of the metric tensor and one can always introduce at a point a local coordinate system, sometimes characterised as a locally freely falling system, in which for a sufficiently small neighbourhood of the point the metric is the Lorentz metric. By describing physics in this neighbourhood in terms of such a coordinate system the effects of the gravitational field are transformed away. Essentially it is this feature of General Relativity - that gravitational or cosmological effects can be made to vanish in the small, which has been questioned and led to the scalar-tensor gravitation theories.

In 1937 Dirac, (13), (and 1938 (14)), suggested that an expanding model of the Universe not only provided a cosmic time scale but also allowed the possibility that the gravitational constant may vary with this time. By taking the ratio of the age of the Universe to a unit of time fixed by atomic constants (e.g. $\frac{e^2}{mc^3}$ or $\frac{h}{mc^2}$) one obtains a number, t , of the order 10^{40} , and by taking the ratio of the gravitational force to the electric force between typically charged particles one obtains a dimensionless expression, $\frac{Gm^2}{e^2}$, of the order 10^{-40} . So for this epoch

$$\frac{Gm^2}{e^2} \sim t^{-1}, \quad 2.9$$

and with m and e supposed constant this relation becomes

$$G \sim t^{-1}, \quad 2.10$$

which Dirac's hypothesis implied, held for all epochs. In more general terms Dirac's hypothesis, (14), stated that "any two of the very large dimensionless numbers occurring in Nature are connected by a simple mathematical relation in which the coefficients are of the order of magnitude unity" and as a consequence if a number varies with epoch then other dimensionless numbers may be required to vary with epoch in order to keep the relations between them.

A feature of the cosmology which Dirac was led to, is that fundamental significance need not be given to these numbers. For example, from eq. 2.9, the ratio of gravitational to electric forces is small because the Universe is old. Although Dirac's cosmology could almost be ruled out by present observations, the idea remains that fundamental constants and in particular, the gravitational constant, may not in fact be constant. Some observable effects of a variable gravitational constant have been discussed by Jordan, (15), and by Dicke, (16), but because these effects are geophysical or cosmological, the systems involved are complex and the numerical data available is insufficient as evidence for variation of the gravitational constant. Recent results by Shapiro, (17), using planetary radar systems, and atomic clocks put an experimental limit on the fractional time variation of the gravitational constant as 4×10^{-10} /year and so the idea of a variable gravitational constant is still a conjecture which has not been established by direct observation.

Einstein's equations appear at present to describe local gravitational effects quite adequately and one could expect these equations to hold for the Universe as a whole. But, since the equations require the gravitational constant, when measured in units defined by atomic standards, to be constant they need to be modified if Dirac's hypothesis is assumed to be valid. A simple way to introduce a variable gravitational constant into the field equations is to make the gravitational constant a new local scalar field variable depending on position in space-time.

Historically, Jordan (1948) was the first to use this approach to incorporate a variable gravitational constant in a field theory of gravitation. He originally used the five-dimensional representation of General Relativity developed by Kaluza (1921) and Klein (1926) and later (1955), (18), he and others developed the theory as a four-dimensional

scalar-tensor formalism. These earlier references to Jordan's theory are given more completely by Pauli, (19), and a comprehensive review of Jordan's theory is given in an article by Brill, (20). The most widely known theory of gravitation which includes a variable gravitational constant is the Brans-Dicke theory (1961) which is formally, very closely related to Jordan's theory. From 1961 onwards, the existence of such a long-range scalar field seemed feasible (but perhaps experimentally doubtful) and in the writer's opinion the most interesting developments to come from the Brans-Dicke theory relate to the problem of constructing dynamical laws involving the gravitational field variable, the scalar field variable and matter field variables. Finally, one notes that, to introduce the scalar field as a long range cosmological field for the purpose of obtaining a variable gravitational constant is by no means the only way of giving expression to Dirac's hypothesis.

In a pattern similar to that described above, other authors have postulated a scalar field and introduced scalar field terms into Einstein's field equations in order to deduce from these equations preferable models of the Universe. Hoyle's equations (1948) (21) implied that matter was not conserved and gave a steady-state model of the Universe. Here the scalar field was related to the creation of matter, in contrast to the scalar field postulated by Rosen (1969) (22) which had no interaction with matter. Rosen's equations gave an oscillating model of the Universe.

The Machian idea of a connection between local physical laws and properties of the Universe as a whole has already been partly met, with Dirac's hypothesis. In an effort to explain inertia, the Brans-Dicke theory was based more on Mach's Principle than on Dirac's hypothesis. Some further references to these ideas are given in Chapter Four while passing mention is made here to Caloi

and Firmani's (1970) (23) modified Brans-Dicke theory in which radiation is given a more Machian property in determining along with matter, the inertia of a body. However their theory is restrictive and applies only to a homogeneous, isotropic space-time in which the matter content can be represented as a perfect fluid. Gursev's (24) theory is Machian motivated in a different kind of way and this theory is discussed in Chapter Five.

It is apparent that the scalar field has been introduced into the General Relativistic framework to incorporate many quite different physical features which have been thought desirable and found not to follow from the usual interpretations of Einstein's field equations. The field equations of the scalar-tensor gravitation theories that have been devised, possess cosmological solutions describing a variety of models of the Universe. So with these rather general comments summarizing (and substituting for) what could have been a lengthy look at the individual theories, the relation between the scalar-tensor and Einstein's descriptions of gravitation is taken up with reference to Harrison's papers (12, 25).

Harrison, (12), states that the scalar-tensor field equations "constitute in fact a limited and particular class of equations that derive from General Relativity and are of lesser generality". He arrives at this view after showing that the forms of the action principles of different scalar-tensor theories can be transformed into each other and into the form of the action principle for General Relativity, by recalibrations (i.e. conformal or scaling transformations) of the field variables. Thus, together with the observation noted earlier that the physical nature of the energy-momentum tensor lies outside of the scope of the theory of General Relativity, a scalar-tensor gravitation theory seems to be, (25), "a specialised application of the theory of General Relativity". In this way the scalar-tensor and

Einstein's theories of gravitation do not have the same status as gravitation theories. General Relativity becomes in a sense a generic theory where one works in a Riemann space-time and postulates field equations based on assumptions about the content of the energy-momentum tensor in Einstein's field equations. A classic example of this procedure is given by McCrea (1951), (26), who found that Hoyle's results (1948) could be derived from Einstein's field equations if negative stress was allowed in the energy-momentum tensor of the Universe. Another example is implied by remarks of Dirac (1938) that, assuming the gravitational constant was variable with respect to atomic standards of measurement, Einstein's equations should hold for units which vary appropriately with respect to the atomic standards.

Perhaps this view emphasizes the geometrization of gravitation achieved by General Relativity and the special importance placed on the interpretation of the Einstein tensor.

In contrast, the assumption of the following chapters is that one wants the scalar field to be an integral part of the description of gravitation — for the purpose of giving position dependence to the gravitational constant, inertial mass or just to offer new models of the Universe — and therefore the scalar-tensor and Einstein's theories of gravitation are to be on equal footing as gravitation theories.

Chapter Three

THE SCALAR-TENSOR MODEL OF GRAVITATION

Having introduced some of the motivation that has been given for including a scalar field in the description of gravitation, the natural procedure is to obtain an appropriate scalar-tensor gravitation theory as a modification of General Relativity. The basic equations of this "new" theory would generally follow from an action principle and so the relevant class of gravitation theories to which the "new" theory would belong, would be the class of Lagrangian - based scalar-tensor gravitation theories.

Before looking at this class of theories in more detail it is convenient to first establish some definitions and concepts by looking at some general properties of a broader class of physical theories and then specialising to the class of gravitation theories. For this purpose some of the definitions and concepts given by Anderson (1), Trautman (2) and developed by Thorne, Lee and Lightman (3) are summarised under the headings

- 3.1 Space-time theories
- 3.2 Gravitation theories

3.1 Space-time theories

The basic element of these theories is the four dimensional space-time manifold which assumes that physical events can in some way, be associated with a continuum of points and that the points "fit together" sufficiently smoothly to form a four dimensional differentiable manifold, M .

The manifold mapping group (MMG) (3) is the group of diffeomorphisms of the space-time manifold onto itself. For a diffeomorphism, λ , an initial coordinate system

$x^\alpha(P)$ transforms to a new coordinate system given by

$$x^{\alpha'}(P) = x^\alpha(\lambda^{-1}P), \quad P \in M \quad 3.1$$

A geometric object (field) (2) is a correspondence

$$y: (P, \{x^\alpha\}) \rightarrow (y_1, y_2, \dots, y_N) \in \mathbb{R}^N \quad 3.2$$

which associates with every point $P \in M$ and every system of local coordinates $\{x^\alpha\}$ around P a set of N real numbers (the components of the geometric object) together with a rule which determines $(y_1', y_2', \dots, y_N')$ given by

$$y: (P, \{x^{\alpha'}\}) \rightarrow (y_1', y_2', \dots, y_N') \quad 3.3$$

in terms of (y_1, y_2, \dots, y_N) and the values at P of the functions and their partial derivatives which relate the coordinate systems $\{x^\alpha\}$ and $\{x^{\alpha'}\}$.

A space-time theory then, (3), is a theory that possesses a mathematical representation constructed from a four dimensional space-time manifold and from geometric objects defined on that manifold. The geometric objects of a particular representation are called its variables and the equations which the variables must satisfy are called the physical laws of the representation.

A kinematically possible trajectory (kpt) (1, 3) of a particular representation of a space-time theory is any set of values for the components of all the variables in any coordinate system.

A dynamically possible trajectory (dpt) (3) is any kpt that satisfies all the physical laws of the representation.

A covariance group of a representation (3) is a group G which (i) maps kpt of the representation into kpt
(ii) maps dpt into dpt

and for which distinct group elements produce distinct mappings of the kpt.

An internal covariance group is a covariance group that involves no diffeomorphisms of the space-time onto itself, in contrast to an external covariance group which is a covariance group and also a subgroup of MMG. The complete covariance group is the largest covariance group of the representation and referring to (3) the effect of a particular group element G is characterised as follows :

Suppose G consists of a diffeomorphism h and an internal transformation H and write

$$G = (h, H) \quad , \quad 3.4$$

where it is understood that if G is an external transformation, H is the identity transformation, and if G is an internal transformation then h is the identity mapping.

If y is a geometric object (eq. 3.2, 3.3) and a component is written in functional notation as

$$y_A(P, \{x^\alpha\}) \quad , \quad 3.5$$

then the set of functions

$$y_A(P, \{x^\alpha\}), P \text{ varying and } \{x^\alpha\} \text{ fixed} \quad , \quad 3.6$$

defines a kpt. Under h this kpt maps into

$$y_A(P, \{x^{\alpha'}\})$$

where $\{x^{\alpha'}\}$ is defined by eq. 3.1 and under H, y_A transforms into a new geometric object

$$y' \equiv Hy \quad 3.7$$

The net effect of G on the kpt (eq. 3.6) is

$$G: y_A(P, \{x^\alpha\}) \rightarrow y'_A(P, \{x^{\alpha'}\}) \quad . \quad 3.8$$

The "changes in y" which characterise G are defined by

$$\begin{aligned} \bar{\delta}y_A(P, \{x^{\alpha'}\}) &= y'_A(P, \{x^{\alpha'}\}) - y_A(h^{-1}P, \{x^{\alpha'}\}) \quad 3.9 \\ &= y'_A|_{\text{evaluated at } x^{\alpha'}}(P) \\ &\quad - y_A|_{\text{evaluated at } x^\alpha = x^{\alpha'}}(P) \quad . \end{aligned}$$

If $\bar{\delta}_{y_A}(P, \{x^{\alpha'}\}) = 0$ at all P and for all coordinate systems $\{x^{\alpha'}\}$, then G is called a symmetry transformation of the geometric object, y , and the set of all such elements G form the symmetry group of this object. 3.10

If a space-time theory has a representation for which MMG is a covariance group then the variables of this representation can be classified according to three types - confined, absolute and dynamical.

The confined variables (3) are those that do not constitute the basis of a faithful realization of MMG e.g. universal constants.

An unconfined variable, B , is termed an absolute or a dynamical variable by the following test (3) :

- (i) Choose an arbitrary dpt and let $B_A(x^\alpha)$ be the functions which describe the components of B for this dpt.
- (ii) Define the equivalence class of a dpt as the set of dpt which map into the given dpt for some element of the complete covariance group of the representation.
- (iii) Check to see if the same functions $B_A(x^\alpha)$ appear in each equivalence class. If they do for every equivalence class and for every choice of the arbitrary initial dpt then B is an absolute variable. If they do not for some particular choice of the initial dpt and for some particular equivalence class then B is a dynamical variable.

Also for this representation a set of variables is called irrelevant, (3), if (i) its variables are not

coupled by the physical laws to the remaining variables of the representation and (ii) its variables can be eliminated from the representation without altering the structure of the equivalence classes of dpt and without destroying the covariance of the representation under MMG.

Finally, for a representation of a given space-time theory the physical laws can be classified into four sets (3)

- (i) boundary conditions - these laws which involve only confined variables
- (ii) prior geometric constraints - those laws which involve absolute (and perhaps confined) variables, but not dynamical variables
- (iii) decomposition equations - those which express a dynamical variable algebraically in terms of other variables
- and (iv) dynamical laws - all other physical laws.

As examples :-

If the electromagnetic gauge group is ignored, the complete covariance group of General Relativity is MMG and all the unconfined variables are dynamical and contain no absolute parts. On the other hand, for Jordan's theory the complete covariance group is the direct product of MMG with the conformal group and because of this internal covariance group the dynamical variables, metric and scalar contain irrelevant parts.

3.2 Gravitation Theories

Scalar-tensor gravitation theories belong to the general class of space-time theories and, as in General Relativity, they seek to combine gravitational and non-gravitational laws by means of an equivalence principle. Thus gravitation is described by a set of gravitational fields including the metric tensor, $g_{\mu\nu}$, and it is required that in the local Lorentz frame of $g_{\mu\nu}$ all non-gravitational laws go over to their standard special relativistic forms.

Some of the ideas involved here have been made more precise in a foundation analysis of gravitation theories given by Thorne, Lee and Lightman, (3). They distinguish between gravitational and non-gravitational phenomena by regarding gravitational phenomena as either prior geometric effects or effects generated by mass-energy and they further clarify this by introducing the concept of a local test-experiment. For the purpose of formulating the equivalence principles, the essential features of such an experiment are that

- (i) it is performed anywhere in space-time
 - (ii) it is performed with freely-falling apparatus
- and (iii) it is performed over a region of space-time, sufficiently small for the inhomogeneities in all external fields to be irrelevant.

A special kind of local test experiment is the local, non-gravitational, test experiment (3) which is performed in a region of space-time with a nearly uniform gravitational potential throughout it, as calculated using Newton's theory, and which if repeated with successively smaller mass-energies in the region, that leave the characteristics of the various parts - e.g. charge, angular momenta etc. - unchanged, gives an experimental result which does not change.

Dicke's Weak Equivalence Principle (WEP) (3, 4) states that if a test particle is placed at an initial event in space-time and is given an initial velocity there, then its subsequent world line will be independent of its internal structure and composition. Following (3) a test particle is taken to be an uncharged body with sufficiently small "self-gravitational energy" as calculated by Newton's theory and with sufficiently small size to guarantee that any test of WEP is a local, non-gravitational test experiment. Given that WEP is valid, the world lines of test particles are a preferred family of curves in space-time - with a unique curve in a given direction through each given event.

Einstein Equivalence Principle (EEP), (3) states that

- (i) WEP is valid
- and (ii) the outcome of any local, non-gravitational test experiment is independent of where and when in the universe it is performed and independent of the velocity of the freely-falling apparatus.

Dicke's Strong Equivalence Principle (SEP) (3, 4) states that

- (i) WEP is valid
- and (ii) the outcome of any local, gravitational or non-gravitational, test experiment is independent of where and when in the universe it is performed and independent of the velocity of the freely-falling apparatus.

The scalar-tensor gravitation theories violate SEP because the relevant scalar fields introduce preferred location effects, while they satisfy* EEP and thus possess metric representations (3), in which a metric is defined

* for experimental limitation see sect. 3.4.

on the space-time and the world lines of test particles are the geodesics of that metric. Also, for a metric representation, the metric involved in EEP is called the physical metric, while the other gravitational fields, that is (3) unconfined, relevant variables of the representation which in the absence of gravity (e.g. as in the analysis of a local, non-gravitational experiment) reduce to constant, or absolute or irrelevant variables, are called auxiliary gravitational fields.

Two further general properties of scalar-tensor gravitation theories are that

- (i) MMG is a covariance group
- and (ii) for a particular representation the dynamical laws are assumed to follow from an action integral which is made stationary with respect to variations of all dynamical variables. With the action principle written as

$$\delta \int \mathcal{L} d^4x = 0 \quad , \quad 3.11$$

the Lagrangian density, (a scalar density of weight + 1*) can be split into two parts

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_{NG} \quad , \quad 3.12$$

where \mathcal{L}_G , the gravitational part, (3), is the largest part which contains only gravitational fields, and \mathcal{L}_{NG} is the non-gravitational part which will be often referred to as representing the matter fields or matter "content" in a space-time.

This representation is said to be universally coupled (3) if

- (i) \mathcal{L}_{NG} contains a second rank symmetric tensor, $\psi_{\eta\nu}$, of the same signature as the Lorentz metric, as the only gravitational field

* This restriction on \mathcal{L} is sufficient (e.g. (1)) to guarantee that any transform (by an element of MMG) of a kpt satisfying eq. 3.11, also satisfies eq. 3.11.

- (ii) in the limit as gravity is "made absent" $\psi_{\mu\nu}$ becomes a Riemann - flat second rank symmetric tensor, $\eta_{\mu\nu}$ and whenever such a $\eta_{\mu\nu}$ replaces $\psi_{\mu\nu}$, \mathcal{L}_{NG} becomes the total special relativistic Lagrangian
- and (iii) the prediction for the outcome of any local non-gravitational experiment remains unchanged when $\psi_{\mu\nu}$ in the region of the experiment is replaced by a Riemann-flat second rank symmetric tensor.

This concludes, more or less, a glossary of terms begun in Section 3.1. It consists of definitions and concepts that have been borrowed largely as formulated in a very recent paper by Thorne, Lee and Lightman and adapted to provide what the writer thinks is a relevant context for the scalar-tensor theories as gravitation theories.

3.3

The Scalar-Tensor Model of Gravitation

In this model gravitation is described in terms of two fields defined on the space-time manifold - a metric tensor field, $g_{\mu\nu}$ and an always positive scalar field, ϕ . The most general \mathcal{L}_G , (to within a divergence term) that gives dynamical laws of no higher than second differential order and with the second derivatives appearing linearly has the form (5)

$$\mathcal{L}_G = (f_1(\phi)R + f_2(\phi)g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + f_3(\phi))\sqrt{-g}, \quad 3.13$$

where f_1 and f_2 are arbitrary functions of the scalar field, but may be reduced to constants by suitable recalibrations of the variables $g_{\mu\nu}$ and ϕ respectively. f_3 is an arbitrary function of the scalar field. R is the Riemann curvature scalar.

On writing the non-gravitational part of the Lagrangian density (\mathcal{L}) as $\mathcal{L}_{NG} = \mathcal{L}_{NG}(\chi, \varepsilon_{\mu\nu}, \phi)$, where

the variable χ collectively stands for all the non-gravitational fields and where for simplicity it is assumed that no higher than first derivatives of the dynamical variables appear, the action principle for the model becomes

$$\delta \int [(f_1(\phi)R + f_2(\phi)g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + f_3(\phi))\sqrt{-g} + \mathcal{L}_{NG}] d^4x = 0. \quad 3.14$$

For the model to obey EEP restrictions need to be put on the functional dependence of \mathcal{L}_{NG} on the gravitational fields. As this is so far a purely formal element of the model these restrictions will involve a choice of units so that WEP holds and an interpretation of \mathcal{L}_{NG} so that EEP holds. In particular, as the non-gravitational laws follow from eq. 3.14 in the absence of gravity, for variations of the non-gravitational fields, \mathcal{L}_{NG} in the absence of gravity must be the total special relativistic Lagrangian which is denoted by $\mathcal{L}_{NG}(\chi, \eta_{\mu\nu})$.

The natural choice of units is based on atomic standards - e.g. as Ohanian, (6) suggests, one takes a neutral, massive spin-zero meson and defines the unit of length (time) as the Compton wavelength of the meson and the unit of mass as the meson mass. Using an argument due to Fierz, (7), he shows that for this choice of units the free meson falls along the geodesics of $g_{\mu\nu}$. This would also apply to any localised system which could be treated as a test particle.

Bergmann, (5) using a quite general argument has shown that, unless \mathcal{L}_{NG} has no scalar field dependence the motion of a test particle is indeterminate. In a simplified form due to Ohanian, (6) one first derives the four differential identities for the matter field. Rather than using the field equations as in (6), these follow more easily from the fact that infinitesimal coordinate transformations are symmetry transformations of the non-gravitational part of the action.

$$\text{i.e. } \delta \int \mathcal{L}_{NG} d^4x = 0, \text{ because } \int \mathcal{L}_{NG} d^4x \text{ is a scalar.}$$

For variations of the variables and their first derivatives which vanish on the boundary of the region of integration eq. 3.15 can be written

$$\int \left[\frac{\delta \mathcal{L}_{\text{NG}}}{\delta \chi} \bar{\delta} \chi + \frac{\delta \mathcal{L}_{\text{NG}}}{\delta g_{\mu\nu}} \bar{\delta} g_{\mu\nu} + \frac{\delta \mathcal{L}_{\text{NG}}}{\delta \phi} \bar{\delta} \phi \right] d^4 x = 0 \quad , \quad 3.16$$

where the Hamiltonian derivatives are defined, as e.g.

$$\left. \begin{aligned} \frac{\delta \mathcal{L}_{\text{NG}}}{\delta g_{\mu\nu}} &\equiv \frac{\partial \mathcal{L}_{\text{NG}}}{\partial g_{\mu\nu}} - \frac{\partial}{\partial x^\alpha} \left(\frac{\partial \mathcal{L}_{\text{NG}}}{\partial g_{\mu\nu, \alpha}} \right) \\ \frac{\delta \mathcal{L}_{\text{NG}}}{\delta \phi} &\equiv \frac{\partial \mathcal{L}_{\text{NG}}}{\partial \phi} - \frac{\partial}{\partial x^\alpha} \left(\frac{\partial \mathcal{L}_{\text{NG}}}{\partial \phi_{, \alpha}} \right) \end{aligned} \right\} \quad 3.17$$

The Euler-Lagrange equations of motion for χ give $\frac{\delta \mathcal{L}_{\text{NG}}}{\delta \chi} = 0$ and so

$$\int \left[\frac{\delta \mathcal{L}_{\text{NG}}}{\delta g_{\mu\nu}} \bar{\delta} g_{\mu\nu} + \frac{\delta \mathcal{L}_{\text{NG}}}{\delta \phi} \bar{\delta} \phi \right] d^4 x = 0 \quad . \quad 3.18$$

Writing the coordinate transformation as

$$x^{\alpha'} = x^\alpha + \xi^\alpha, \quad |\xi^\alpha| \ll 1 \quad , \quad 3.19$$

$$\text{eq. 3.10 gives } \bar{\delta} g_{\mu\nu} = -\xi_{\mu; \nu} - \xi_{\nu; \mu} \quad 3.20$$

$$\bar{\delta} \phi = -\phi_{, \alpha} \xi^\alpha \quad , \quad 3.21$$

where the covariant derivative " ; " is with respect to the Christoffel symbols .

Eqs. 3.20, 3.21 in 3.18 give on integration by parts and for arbitrary ξ^α

$$\bar{\delta} g_{\mu\nu} \left(\frac{\delta \mathcal{L}_{\text{NG}}}{\delta g_{\mu\rho}} \right)_{; \rho} - \frac{\delta \mathcal{L}_{\text{NG}}}{\delta \phi} \phi_{, \nu} = 0 \quad , \quad \nu = 0, 1, 2, 3 \quad 3.22$$

$$\text{or equivalently } T^{\mu\nu}_{; \mu} = \phi_{, \nu} \frac{\delta \mathcal{L}_{\text{NG}}}{\delta \phi} \quad 3.23$$

where

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_{\text{NG}}}{\delta g_{\mu\nu}} \quad . \quad 3.24$$

In the external fields $g_{\mu\nu}$ and ϕ , a locally freely falling frame can be introduced in which eq. 3.23 becomes for a sufficiently small system and with its self-gravitational field assumed negligible,

$$T^{\mu\nu}_{,\mu} = \phi^{,\nu} \frac{\delta \mathcal{L}_{NG}}{\delta \phi} \quad , \quad 3.25$$

where \mathcal{L}_{NG} refers to the system and $T^{\mu\nu}$ now denotes the special relativistic energy-momentum tensor of the system.

Integrating over the volume of the system and with $p^{\nu} = \int T^{0\nu} d^3x$, eq. 3.25

$$\text{defn.} \\ \text{gives } \frac{d}{dx} p^{\nu} = \int \phi^{,\nu} \frac{\delta \mathcal{L}_{NG}}{\delta \phi} d^3x \quad . \quad 3.26$$

Thus the system will experience an acceleration relative to the local freely falling frame unless $\int \phi^{,\nu} \frac{\delta \mathcal{L}_{NG}}{\delta \phi} d^3x = 0$ 3.27 or because of the approximation involved

$$\int \frac{\delta \mathcal{L}_{NG}}{\delta \phi} d^3x = 0 \quad . \quad 3.28$$

This restriction holds if the gravitational fields enter into \mathcal{L}_{NG} according to $\mathcal{L}_{NG} = \mathcal{L}_{NG}(\chi, f_{l_t}(\phi)g_{\mu\nu})$, where f_{l_t} is an arbitrary function of the scalar field, as for example postulated for the scalar-tensor model considered by Wagoner, (8). A metric recalibration can reduce f_{l_t} to a constant and in this representation eq. 3.27 or 3.28 is satisfied.

For WEP to hold then, one can take \mathcal{L}_{NG} to have no scalar-field dependence, and for EEP to hold one can take \mathcal{L}_{NG} to be $\mathcal{L}_{NG}(\chi, g_{\mu\nu})$, — the minimally coupled total special relativistic Lagrangian density for the system.

$$\text{i.e. } \mathcal{L}_{NG}(\chi, \eta_{\mu\nu}) \rightarrow \mathcal{L}_{NG}(\chi, \varepsilon_{\mu\nu}) \text{ as } \eta_{\mu\nu} \rightarrow \varepsilon_{\mu\nu} \quad .$$

In this way a metric and at the same time universally coupled representation can be arrived at with the action principle given by

$$\delta \int [(f_1(\phi)R + f_2(\phi)g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + f_3(\phi)\sqrt{-g} + \mathcal{L}_{NG}(\chi, \varepsilon_{\mu\nu})] d^4x = 0 \quad 3.29$$

Because ϕ can be recalibrated, independently of $\varepsilon_{\mu\nu}$, to reduce f_2 to a constant this representation has effectively two arbitrary functions and an arbitrary (dimensionless) constant and so describes a family of scalar-tensor theories.

The main problem for the scalar-tensor model is to select a satisfactory theory by eliminating some of these arbitrary features in an acceptable way.

e.g. the Brans-Dicke theory is obtained from the metric representation by putting $f_1 = \phi$, $f_2 = -\omega\phi^{-1}$ and $f_3 = 0$ which gives a simple wave equation as the field equation for ϕ .

To conclude this section, mention is made of a paper by Hart, (9), in which a scalar-tensor model is given by the action principle, eq. 3.14, with $\mathcal{L}_{NG} = f_5(\phi)\mathcal{L}_{NG}(\chi, g_{\mu\nu})$, where f_5 is an arbitrary function of the scalar field. From the differential identities for \mathcal{L}_G (i.e. eq. 3.22 with \mathcal{L}_{NG} replaced by \mathcal{L}_G) the scalar-tensor conservation laws are derived in a way analogous to General Relativity. If f_5 is a constant (as one needs for a metric or universally coupled representation) and/or a dimensionless function it is found that the total scalar-tensor conserved quantities have appropriate units of energy or momentum.

Hence, as ϕ generally has dimensions of some power of G , in order to have conserved quantities that are meaningful unit-wise, at least, it is sufficient to take f_5 as a dimensionless constant, in which case the scalar-tensor model considered is given by eq. 3.29.

Using Noether's theorem (e.g. Trautmann, (2),) which associates a differential conservation law with an infinitesimal coordinate transformation, one can obtain differential scalar-tensor conservation laws, (9), that depend on the choice of ξ^α in the coordinate transformation eq. 3.19. Because of the infinite order of MMG there is a corresponding infinity of conserved single index quantities (such as Komar's vector) and if the physically important quantities are generated by infinitesimal coordinate transformations which are symmetry transformations of the gravitational fields then from eq. 3.20 and eq. 3.21

$$\left. \begin{aligned} \Theta &= \bar{\delta}g_{\mu\nu} = -\xi_{\mu;\nu} - \xi_{\nu;\mu} \\ \text{and} \quad 0 &= \bar{\delta}\phi = -\phi_{,\alpha}\xi^\alpha \end{aligned} \right\} 3.30$$

Killing vectors with components which satisfy the latter equation, Hart calls restricted Killing vectors and an outstanding problem he suggests, is whether or not the Killing vectors are actually restricted for all gravitational fields. Also in this connection he suggests that the relationship between the restricted Killing vectors and the scalar-tensor conservation laws needs further study.

3.4 Experimental Tests and the Scalar-Tensor Model of Gravitation

The metric representation (eq. 3.29) of the scalar tensor model can by a recalibration of the metric and the scalar field variables be transformed into the representation with an action principle

$$\delta \int [(R - n g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + f_6(\phi)) \sqrt{-g} + \mathcal{L}_{NG}(\chi, f_7(\phi) g_{\mu\nu})] d^4x = 0 \quad 3.34$$

where f_6 and f_7 are arbitrary functions of the scalar field and $n = \pm 1$. For this representation Wagoner, (8), has considered the linearized weak-field limit in relation to the solar-system experiments and arrived at two possible restrictions :

- (i) the locally measured gravitational constant is the same as Newton's gravitational constant, and f_6 gives rise to a massive short range scalar field
- or (ii) the locally measured gravitational constant depends on the scalar field and f_6 corresponds to a "cosmological term".

The first case leads to the "massive Brans-Dicke"

theories (see Chapter Five) which give the same predictions for the solar-system experiments as General Relativity.

The second case leads to predictions for the light deflection and perihelion shift observations, which depend on the first and possibly second-order terms in the expansion for f_7 . However, depending on the sign of n , the model does not seem to be inconsistent with present observations and certainly no severe restrictions are placed on the form of the action principle for the model.

A prediction of the scalar-tensor model does lead to a violation of WEP. Ohanian, (6), has shown that for a massive system the inertial and gravitational masses differ by a term which is of the order of the self-gravitational energy. This result holds also, and in particular, for the Brans-Dicke theory, (10). (10a), but because of the size of the violation there is no conflict with the Eotvös - Dicke experiments. Thus the violation of WEP seems to be a matter of principle and as such adds against the scalar-tensor model of gravitation.

3.5 Conformal Invariance and the Scalar-tensor Model

Previously, in section 3.3, it was shown that the scalar-tensor model possesses a metric (and universally coupled) representation which describes a family of scalar-tensor theories based on an action principle given by eq. 3.29. After recalibration of the scalar-field variable this action principle can be put into a form characterised by two arbitrary functions f_1 and f_3 of the scalar field and an arbitrary constant ζ ,

$$\text{i.e. } \delta \int \{ [f_1(\phi)(R - \zeta \phi^{-2} \phi_{,\mu} \phi_{,\nu} g^{\mu\nu}) + f_3(\phi) \sqrt{-g} + \mathcal{L}_{\text{MG}}(\chi, \varepsilon_{\mu\nu})] \} d^4x = 0 \quad 3.32$$

and since observation does not rule out the existence of the cosmological term, $f_3(\phi)$ or the presence of $f_1(\phi)$ the problem of removing at least some of the arbitrariness from the dynamical laws remains.

For this purpose then, it is interesting to look at the restrictions placed on the functions $f_1(\phi)$ and $f_3(\phi)$ when the conformal group defined by

$$(P, \{x^\alpha\}) \rightarrow (P, \{x'^\alpha\}) \quad \left. \vphantom{(P, \{x^\alpha\})} \right\} \quad 3.33$$

and $\varepsilon_{\mu\nu} \rightarrow \bar{\varepsilon}_{\mu\nu} = \lambda \varepsilon_{\mu\nu}$, for λ an arbitrary positive function of position,

is postulated to be an internal covariance group.

If the group is first restricted to be a symmetry group of the gravitational part of the action principle then

$$\begin{aligned} & \int \{ [f_1(\phi)(R - \zeta \phi^{-2} \phi_{,\mu} \phi_{,\nu} g^{\mu\nu}) + f_3(\phi) \sqrt{-g}] \} d^4x \\ & = \int \{ [f_1(\bar{\phi})(\bar{R} - \zeta \bar{\phi}^{-2} \bar{\phi}_{,\mu} \bar{\phi}_{,\nu} \bar{g}^{\mu\nu}) + f_3(\bar{\phi}) \sqrt{-\bar{g}}] \} d^4x \end{aligned} \quad 3.34$$

where barred variables represent transformed variables, requires that, (11)

$$f_1(\phi) \propto \phi^\eta \quad 3.35$$

$$f_3(\phi) \propto \phi^{2\eta} \quad 3.36$$

$$\eta^2 = -\frac{2}{3} \zeta \quad 3.37$$

and the transformation of ϕ to be

$$\phi \rightarrow \bar{\phi} = \lambda \frac{1}{\eta} \phi . \quad 3.38$$

Thus the Lagrangian density for the free gravitational fields becomes

$$\mathcal{L}_G = [\phi^\eta (R - \zeta \phi^{-2} \phi_{,\mu} \phi_{,\nu} g^{\mu\nu}) + \lambda_0 \phi^{2\eta}] \sqrt{-g} \quad 3.39$$

where $\eta^2 = -\frac{2}{3} \zeta$ and λ_0 is an arbitrary constant.

From the form of \mathcal{L}_G one can recognise the gravitational part of Jordan's Lagrangian density (see Chapter Five), if λ_0 is put equal to zero and ζ is considered to be arbitrary, and in particular if η is put equal to unity \mathcal{L}_G is the gravitational part of the Brans-Dicke Lagrangian density.

(The idea of using conformal transformations to formally identify the Brans-Dicke and Jordan theories is also looked at in Chapter Five).

Returning to the conformally invariant Lagrangian density \mathcal{L}_G , (eq. 3.39), one notes that it is conformally equivalent to

$$\mathcal{L}_{\bar{G}} = (\bar{R} + \lambda_0) \sqrt{-\bar{g}} \quad , \quad \text{where } \bar{g}_{\mu\nu} = \phi^\eta g_{\mu\nu} ,$$

and so with $\mathcal{L}_{\bar{NG}} = \mathcal{L}_{NG}(\chi, \bar{g}_{\mu\nu})$, one has the Lagrangian density for Einstein's field equations with a cosmological constant. This choice of $\mathcal{L}_{\bar{NG}}$ implies that in the original representation one takes the matter Lagrangian density \mathcal{L}_{NG} to be minimally coupled to the "metric" $\phi^\eta g_{\mu\nu}$. For a different theory one can take \mathcal{L}_{NG} to be minimally coupled to $g_{\mu\nu}$ and with $\eta = 1$, ($\zeta = -\frac{3}{2}$) for simplicity, the action principle for the metric (and universally coupled) representation is

$$\delta \int [(\phi R + \frac{3}{2} \phi^{-1} \phi_{,\mu} \phi_{,\nu} g^{\mu\nu} + \lambda_0 \phi^2) \sqrt{-g} + \mathcal{L}_{NG}(\chi, g_{\mu\nu})] d^4x = 0 \quad 3.40$$

The field equations are, for arbitrary variations of the variables $g_{\mu\nu}$ and ϕ , and their first derivatives which vanish on the boundary of the region of integration,

[e.g. 11]

$$G_{\mu\nu} + \frac{3}{2} \phi^{-2} (\phi_{;\mu} \phi_{;\nu} - \frac{1}{2} \epsilon_{\mu\nu} \phi_{;\alpha} \phi^{;\alpha}) - \phi^{-1} (\phi_{;\mu;\nu} - \epsilon_{\mu\nu} \square \phi) - \frac{1}{2} \epsilon_{\mu\nu} \lambda_0 \phi = \frac{\phi^{-1}}{2} T_{\mu\nu} \quad 3.41$$

$$R + \frac{3}{2} \phi^{-2} \phi_{;\alpha} \phi^{;\alpha} - 3\phi^{-1} \square \phi + 2\lambda_0 \phi = 0 \quad 3.42$$

where $\square \phi = g^{\mu\nu} \phi_{;\mu;\nu}$, by defn.

Contracting eq. 3.41 and comparing with eq. 3.42 implies for consistency that,

$$(11) \quad T^\alpha_{\alpha} = 0. \quad 3.43$$

Hence in the metric representation, for the gravitational fields, $g_{\mu\nu}$ and ϕ , which enter into a conformally invariant \mathcal{L}_G to couple consistently with matter, the trace of the energy-momentum tensor must vanish. This condition is in fact the condition for \mathcal{L}_{NG} to be conformally invariant (see eq. eq 3.47). Thus in particular, as shown explicitly by Anderson, (12), the Brans -Dicke theory is conformally invariant if $\omega = -\frac{3}{2}$ and $T^\alpha_{\alpha} = 0$.

To allow the gravitational fields to couple with massive systems ($T^\alpha_{\alpha} \neq 0$), one approach is to extend the complete covariance group of the scalar-tensor model by including the conformal group and let a scalar field interaction appear in \mathcal{L}_{NG} . Thus the action principle is written

$$\delta \int [(\phi R + \frac{3}{2} \phi^{-1} \phi_{;\mu} \phi_{;\nu} g^{\mu\nu} + \lambda_0 \phi^2) \sqrt{-g} + \mathcal{L}_{NG}(\chi, g_{\mu\nu}, \phi)] d^4x = 0 \quad 3.44$$

and because the additional covariance group elements are specified in terms of a single position dependent function, the field equations satisfy an additional Bianchi-type identity.

This follows for example from

$$\bar{\delta} \int \mathcal{L}_{NG} d^4x = 0, \quad 3.45$$

for the infinitesimal conformal transformation

$$\left. \begin{aligned} \bar{\delta} g_{\mu\nu} &= g_{\mu\nu} \delta \lambda \\ \bar{\delta} \phi &= -\phi \delta \lambda \end{aligned} \right\} \quad 3.46$$

which gives [e.g. 11] $T^\alpha_{\alpha} = \frac{\partial \phi}{\sqrt{-g}} \frac{\delta \mathcal{L}_{NG}}{\delta \phi}$ 3.47

where as usual

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_{NG}}{\delta g_{\mu\nu}}.$$

O'Hanlon and Tupper in their paper, (11), show that the covariant Lagrangian densities for massive fields of non-zero spin could be put into a conformally invariant form if one postulates a scalar-field interaction with mass as

$$m \rightarrow m \phi^{\frac{1}{2}}, \quad 3.48$$

which is in keeping with a conclusion of (13) that without this mass transformation there is difficulty in making non-gravitational laws and equations of motion conformally invariant.

Effectively (for simple cases anyway) the same result occurs if one takes following Anderson (12)

$$\mathcal{L}_{NG} = \mathcal{L}_{NG}(\lambda, \phi, g_{\mu\nu}) \quad , \quad 3.49$$

where the mass transformation has been accounted for in the coefficient of $g_{\mu\nu}$. A metric representation is obtained by choosing $\lambda = \phi^{-1}$ in the transformation eq. 3.33 and the action principle (eq. 3.44), in the barred form is then essentially no different from the one for General Relativity (with a cosmological constant).

Although postulating the conformal group as a covariance group seems to be too strong a condition on the scalar-tensor model to give an interesting scalar-tensor theory, it is perhaps still appropriate to add a few general remarks. Under the postulate, the complete covariance group of the scalar-tensor model is the direct product of \overline{MNC} with the conformal group. The conformal group introduces irrelevant parts into the dynamical variables $g_{\mu\nu}$ and ϕ and a simple way to covariantly eliminate these parts is to carry out the transformation eq. 3.33 for some predetermined λ . In the barred form the dynamical variables become relevant variables and in particular, for the choice $\lambda = \phi^{-1}$ the scalar field variable is made completely irrelevant. This enforces the result shown above that conformal invariance can lead to a purely tensor theory of gravitation.

Interestingly, a rather philosophical objection that could be raised against the scalar-tensor model (eq. 3.32) of gravitation, remains for the model with the

extended complete covariance group. Pauli, (13), has stated a formal principle that "only irreducible quantities should be used in field theories". Under the extended complete covariance group the metric and scalar fields still transform independently of each other, and thus together, under the group, they constitute a reducible geometric object. Also in the philosophical vein, it has been felt that physical laws should be invariant under the widest group of transformations possible. In this respect the idea that the conformal group be a subgroup of the complete covariance group of space-time theories, has appeal and perhaps a step in this direction is the use of more general space-times (such as the Weyl or Lyra space-time - Chapter Six) where, in the absence of matter it is required that field equations be conformally invariant.

Dicke (14) and Hoyle and Narlikar (15) have been recent advocates for conformal invariance. Dicke for example states, (14), "that the equations of motion of matter must be invariant under a coordinate dependent transformation of units". The connection with conformal transformations is simple. If the interval between two infinitesimally close events is measured with respect to a system of units as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad , \quad 3.50$$

then with respect to a transformed set of units, the interval can be written as

$$\overline{ds}^2 = \lambda g_{\mu\nu} dx^\mu dx^\nu \quad , \quad 3.51$$

where λ is a space-time position dependent function. So the metrics established in the two sets of units are conformally related.

It seems at some stage that a conformally or scale (units) invariant theory needs some convention about the comparison of a particular unit of length (time) at separated points. For without this, one has a problem noted by Anderson (12) "that two observers separated by

a space-like interval would have no way of informing each other of the particular value of λ which they were going to use in carrying out the transformation". (eq. 3.33). Although the usual way out of this situation is to assert that atomic standards are position independent and so make the properties (mass, charge etc.) of atomic systems independent of space-time position, Brans and Dicke (16) consider this an arbitrary choice.

In a similar fashion, Hoyle and Narlikar (15) say that "for a physical theory not to be conformally invariant it is necessary that there be some form of propagation from a point, A say, to another point, B say, whereby information concerning the length unit at A is carried to B. Then the length units at A and B can be compared "

With these few asides we return to the main discussion of the action principle (eq. 3.40). This principle was obtained from the original metric representation of the scalar-tensor model by postulating that the gravitational fields enter into \mathcal{L}_G in a conformally invariant way. It was found that such a restriction on the gravitational fields meant that they could couple only to matter whose energy-momentum tensor had a vanishing trace.

There is a second approach to allow massive systems to couple with these fields. Following Deser, (17), one can introduce a conformal invariance breaking term into the action principle

$$\text{e.g. } \mathcal{L}_{CB} = l(\phi)\sqrt{-g}. \quad 3.52$$

where $l(\phi)$ is not proportional to ϕ^2 as this would make \mathcal{L}_{CB} conformally invariant.

\mathcal{L}_{CB} adds to the left hand side of eq. 3.41

$$\text{a term } -\frac{1}{2}l\phi^{-1}g_{\mu\nu} \quad -\frac{1}{4}l'\phi^{-1}g_{\mu\nu}$$

and to the right hand side of eq. 3.42

$$\text{a term } -l^2,$$

where $l' = \frac{dl}{d\phi}$.

The new field equations give

$$T_{\alpha}^{\alpha} = 2 l' \phi - 4l \quad .$$

Deser considers the simplest symmetry breaking term

$$\mathcal{L}_{CB} = \frac{1}{2} \mu^2 \phi \sqrt{-g} \quad , \mu \text{ constant},$$

which gives $T_{\alpha}^{\alpha} = -\mu^2 \phi$, 3.53

and with $\lambda_0 = 0$ he arrives at a scalar-tensor theory of gravitation (Sec. 3.6) which corresponds to the singular choice $\omega = -\frac{3}{2}$ in the Brans-Dicke theory.

3.6 Deser's Scalar-Tensor Theory

In contrast to the Brans-Dicke or Jordan theories, Deser's theory, (17), is quite unusual by having an algebraic relation between the scalar field variable and the trace of the energy-momentum tensor

$$\text{i.e. (c.f.* eq. 3.53)} \quad \mu^2 \phi^2 = -T_{\alpha}^{\alpha} \quad .$$

He arrives at his field equations as a modification of Einstein's field equations by first requiring that a physical system be scale invariant in order that it couple consistently to gravitation. This means that no dimensional parameters appear in the description of the system and in particular that the flat space-time energy-momentum tensor, $T_{\mu\nu}(\eta_{\mu\nu})$ has a vanishing trace.

For the scalar field ($\eta^{\mu\nu} \phi_{,\mu\nu} = 0$) this can be achieved by adding a divergenceless term to its energy-momentum tensor to obtain a new, traceless energy-momentum tensor,

$$T_{\mu\nu}(\phi, \eta_{\mu\nu}) = \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} \eta_{\mu\nu} \phi_{,\rho} \phi^{,\rho} + \frac{1}{6} [\eta_{\mu\nu} (\phi^2)^{,\rho}{}_{,\rho} - (\phi^2)_{,\mu\nu}] \quad .$$

In the presence of gravitation the additional term can be introduced into the action principle by coupling the

* the notation of Deser's paper is used here. ϕ in the previous sections has been replaced by ϕ^2 .

scalar field in a non-minimal way to the gravitational field, $g_{\mu\nu}$

$$\text{viz } \delta \int \left[\frac{1}{6} \phi^2 R + g^{\mu\nu} \phi_{, \mu} \phi_{, \nu} \right] \sqrt{-g} d^4x = 0 \quad ,$$

which gives

$$G_{\mu\nu} = -6\phi^{-2} T_{\mu\nu}(\phi) \quad 3.54$$

where now $T_{\mu\nu}(\phi) = T_{\mu\nu}(\phi, g_{\mu\nu})$

$$= \phi_{, \mu} \phi_{, \nu} - \frac{1}{2} g_{\mu\nu} \phi_{, \rho} \phi_{, \rho} + \frac{1}{6} [g_{\mu\nu} \square \phi^2 - (\phi^2)_{; \mu} ; \nu] \quad 3.55$$

$$\text{and } (\square - \frac{1}{6} R) \phi = 0 \quad . \quad 3.56$$

Other matter $(T^M_{\mu\nu})$ is incorporated by including $T^M_{\mu\nu}$ as follows

$$G_{\mu\nu} = 6\phi^{-2} [-T_{\mu\nu}(\phi) + T^M_{\mu\nu}] \quad . \quad 3.57$$

At this point one notes that eqs. 3.56 and 3.57 are conformally invariant and have the property that

$$T^{M\alpha}_{\alpha} = 0 \quad .$$

Because the Lagrangian density in the action principle cannot be split to give a gravitational part, $R\sqrt{-g}$, but rather to give

$$\mathcal{L}_G = \frac{1}{2} \left[\frac{1}{6} \phi^2 R + g^{\mu\nu} \phi_{, \mu} \phi_{, \nu} \right] \sqrt{-g} \quad ,$$

one has a scalar-tensor theory of gravitation.

As indicated at the end of Section 3.5 Deser then introduced a scale invariance breaking term to allow massive systems to couple with the gravitational fields. The final field equations are

$$G_{\mu\nu} = 6\phi^{-2} [-T_{\mu\nu}(\phi) + T^M_{\mu\nu}] + 3\mu^2 g_{\mu\nu} \quad 3.58$$

$$\square \phi - \left(\frac{1}{6} R + \mu^2 \right) \phi = 0 \quad . \quad 3.59$$

Deser interprets the invariance breaking as due to the finite range, μ^{-1} , of the scalar field which he requires to be of the order of the radius, R_0 , of the observable Universe. For this level of approximation the average value of T^α_α is

$\langle T^\alpha_\alpha \rangle \sim MR_0^{-3}$ where M is the mass of the Universe, and using the approximate relation that the radius of the Universe is of the order of its Schwarzschild radius,

$GM \sim R_0$ where G is the gravitational constant, he obtains

$$\phi^{-2} \sim G.$$

Having used all the available information from the scalar field equation (eq. 3.59) eq. 3.58 becomes

$$G_{\mu\nu} - 3\mu^2 g_{\mu\nu} = G^\mu_{\mu\nu}.$$

i.e. Einstein's field equations with a cosmological constant.

Aragone and Restuccia (18, 19) have given a spherically symmetric, static solution and a class of Friedmannian perfect fluid solutions for eqs. 3.58 and 3.59. They maintain that the matter energy-momentum tensor is not conserved and some of their solutions bear this out. (19). However the writer feels that all is not well because direct calculation from the field equations shows that the matter energy-momentum tensor is divergenceless.

(That matter is conserved follows perhaps more simply from the matter L_a Lagrangian density being invariant under infinitesimal coordinate transformations - cf. section 3.3)

Chapter Four

THE BRANS-DICKE (BD) SCALAR-TENSOR THEORY

In a previous section (sec 3.3) the BD theory was mentioned as being a member of the class of Lagrangian-based, scalar-tensor gravitation theories. To illustrate more specific assumptions in the BD theory, the field equations and their relation to the experimental tests are looked at briefly. Following this, recent applications of the BD field equations are discussed under the headings :

- 4.2 Vacuum Solutions
- 4.3 Standard BD cosmology
- 4.4 Units transformations and the BD theory
- 4.5 Standard BD cosmology (ctd.)
- 4.6 Non-standard BD cosmology
- 4.7 Scalar-Matter field gauge freedom
- 4.8 Nordstrom-Reissner type solutions
- 4.9 ADM formulation in the BD theory
- 4.10 BD theory and Mach's Principle.

4.1 The BD Scalar-Tensor Theory

As a modification of General Relativity the BD theory was motivated (e.g. Brans and Dicke, (1),) by a form of Mach's Principle which states that the inertial mass of a material particle, rather than being a constant intrinsic to the particle, is due to the particle's interaction with a cosmic field. For this purpose a long range scalar field was chosen and coupled to the matter of the Universe as a source. Dicke (2) has suggested that in such a field the mass of an interacting particle is required to be a function of the scalar field. Briefly, in the particular case of a static field, a particle accelerates because it experiences a force proportional to the gradient of the scalar field while

the time rate of change of its energy (the zero'th component of the momentum four vector) is zero. Thus to maintain constant energy some transfer of energy, as rest mass energy to kinetic energy must be provided and it proves sufficient to take the rest mass of the particle as a function of the scalar field.

Because the scale of particle masses is provided by measurements involving the product GM (e.g. as in the gravitational acceleration $\frac{GM}{r^2}$ of a particle) a conclusion equivalent to that of allowing mass to vary in terms of the scalar field, is that the gravitational constant, G , should be determined by the average value of the scalar field. With this conclusion, units of length (time) and mass defined in terms of atomic standards are appropriate and the simplest covariant second order differential equation for the scalar field, ϕ , would be

$$\square \phi = 4 \pi \nu T^{\mu\nu} \quad 4.1$$

where ν is a coupling constant. $T^{\mu\nu}$ is the energy-momentum tensor of the matter field (excluding the metric and scalar fields) of the Universe and it is defined in exactly the same way as in General Relativity. From the last definition it follows by eq. 4.1 that the scalar field does not couple with or have any direct interaction with electromagnetic or radiation fields.

In keeping with the Machian motivation, the scalar field is to be interpreted, (1), as an "advanced wave" integral over all matter.

A rough estimate (3) of the mean value of ϕ , $\langle \phi \rangle$, based on the "central potential" solution of eq. 4.1, of a gas sphere with density of the order of the cosmic mass density and with a radius equal to the radius of the observable Universe, suggests that if ν is taken to be a dimensionless number of order unity then

$$\langle \phi \rangle \approx G^{-1} .$$

Introducing¹ the scalar field in a way analogous to the inverse gravitational constant into the Einstein

¹ This derivation is given in detail by Weinberg (3)

field equations gives the BD form

$$G^{\mu\nu} = 8\pi\phi^{-1} (T^{\mu\nu} + T_{\phi}^{\mu\nu}), \quad 4.2$$

where $T_{\phi}^{\mu\nu}$ is the energy-momentum tensor for the scalar field and is included in the source of the Einstein tensor. $T_{\phi}^{\mu\nu}$ is taken to be the most general symmetric tensor involving the derivatives of one or two scalar fields.

$$\text{viz: } T_{\phi}^{\mu\nu} = A(\phi)\phi_{,\rho}\phi_{,\nu} + B(\phi)\delta_{\nu}^{\mu}\phi_{,\rho}\phi_{,\rho} \\ + C(\phi)\phi_{,\nu} + D(\phi)\delta_{\nu}^{\mu}\square\phi, \quad 4.3$$

and by requiring that the matter energy-momentum tensor satisfy

$$T^{\mu}{}_{\nu;\mu} = 0, \quad 4.4$$

eqs. 4.1 and 4.2 become the usual BD field equations

$$G_{\mu\nu} = 8\pi\phi^{-1}T_{\mu\nu} + \frac{\omega}{\phi^2} (\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\rho}\phi_{,\rho}) + \frac{1}{\phi} (\phi_{,\mu;\nu} - g_{\mu\nu}\square\phi) \quad 4.5$$

$$\square\phi = \frac{8\pi}{3+2\omega} T, \quad T = T^{\alpha}{}_{\alpha}, \quad 4.6$$

where ω is a dimensionless constant given by $\nu = \frac{2}{3+2\omega}$, $\omega \neq -\frac{3}{2}$.

ν was estimated to be of order unity and so it could be expected that ω would also be of order unity. For ω very large, eq. 4.6 becomes

$$\square\phi = 0 \left(\frac{1}{\omega}\right)$$

$$\text{with solution } \phi = \langle\phi\rangle + O\left(\frac{1}{\omega}\right) \\ = G^{-1} + O\left(\frac{1}{\omega}\right),$$

and eq. 4.5 becomes

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + O\left(\frac{1}{\omega}\right).$$

In the limit as $\omega \rightarrow \infty$ the BD field equations reduce to the Einstein field equations.

Although ϕ^{-1} appears formally in the field equations as the gravitational constant it is necessary (e.g. Brans (4)) to determine the relation between ϕ^{-1} and what is defined as the locally measured gravitational constant G^1 . Involved here is a comparison with the observed

¹ e.g. as measured by a Cavendish experiment using test particles

accelerations of test particles near a gravitating mass, as predicted by Newton's theory. So, for this purpose it is sufficient to note that from the post-Newtonian approximation (3) form of the BD equations the analogue of Poisson's equation is

$$\nabla^2 \overset{2}{g}_{00} = 8\pi \phi_0^{-1} \left(\frac{2\omega + 4}{2\omega + 3} \right) \overset{0}{T}_{00} \quad 4.7$$

where ∇^2 is the flat 3-space Laplacian and the variables $g_{\mu\nu}$, $T_{\mu\nu}$ are expanded in powers of a parameter $\left(\frac{GM}{r}\right)$.

$\overset{2}{g}_{00}$ is the 2nd order term in the expansion of g_{00}
 $\overset{0}{T}_{00}$, the negative of the rest mass density, is the zero'th order term in the expansion for T_{00}

The expansion for ϕ is written in terms of ψ where

$$\overset{\text{defn}}{\phi} = \phi_0 (1 + \psi), \quad \phi_0 \text{ is a constant and } \psi \text{ satisfies}$$

$$\square \psi = \frac{8\pi}{3 + 2\omega} \phi_0^{-1} T_{00} T^\mu{}_\mu \text{ with } \psi \rightarrow 0 \text{ as } r \rightarrow \infty \quad 4.8$$

For the post-Newtonian approximation eq. 4.8 gives

$$\nabla^2 \overset{2}{\psi} = - \frac{8\pi}{3 + 2\omega} \phi_0^{-1} \overset{0}{T}_{00} \quad 4.9$$

where $\overset{2}{\psi}$ is the 2nd order term in the expansion of ψ .

Poisson's equation is achieved from eq. 4.7 by the identification

$$\overset{2}{g}_{00} = -2\phi_{\text{NEW}} \quad 4.10$$

where ϕ_{NEW} denotes the Newtonian potential

$$\text{and } G = \left(\frac{2\omega + 4}{2\omega + 3} \right) \phi_0^{-1} \quad 4.11$$

$$\text{which give } \nabla^2 \phi_{\text{NEW}} = -4\pi G \overset{0}{T}_{00} \quad 4.12$$

Thus the gravitational constant as measured in terms of slowly moving particles in a weak gravitational field is

$$\left(\frac{2\omega + 4}{2\omega + 3}\right) \phi_0^{-1} \quad .$$

Using the post-Newtonian approximation, the field equations can be approximately solved in the region outside of a static spherically symmetric mass and when the metric is put into the general Eddington-Robertson form the Robertson parameters are (3)

$$\alpha = 1, \beta = 1, \gamma = \frac{\omega + 1}{\omega + 2} = \frac{1 + 1/\omega}{1 + 2/\omega}, \quad 4.13$$

compared to the values for General Relativity

$$\alpha = 1, \beta = 1, \gamma = 1 \quad , \quad 4.14$$

Provided $\omega \geq 6$, 4.13 and 4.14 are indistinguishable for present solar-system experiments (42) and depending on how measurements are refined the form of 4.13 suggests that the (fractional time) variation of the gravitational constant could be made small enough, not to violate observations.

The specific violation of the strong equivalence principle can be seen (3) from eqs. 4.9 and 4.12 when at any point a locally freely falling coordinate system is introduced. By definition $g_{\mu\nu} = \eta_{\mu\nu}$ and $\left\{ \begin{smallmatrix} \lambda \\ \mu\nu \end{smallmatrix} \right\} = 0$ at this point but the scalar field does not vanish or necessarily become constant as

$$\psi \simeq \tilde{\psi} = -(\omega + 2)^{-1} \phi_{\text{NEW}} \quad .$$

Thus the gravitational field of a very small mass placed at the point could be calculated from Einstein's field equations (because terms involving derivatives of the scalar field are of a higher order approximation in the BD equations), with the effective gravitational constant G_E (the coefficient of $8\pi T_{\mu\nu}$) given by (3)

$$G_E = G(1 + \psi) \simeq G(1 + (\omega + 2)^{-1} \phi_{\text{NEW}}) \quad .$$

Summarizing then, the Brans-Dicke modification of General Relativity is to introduce through the field equations only, a scalar field into the description of the gravitational field. The new field equations (BD) determine the relative coupling of the metric and scalar fields to matter via the parameter ω and it is this division of coupling strength which gives rise to the differences from General Relativity in the predictions for the solar system experiments.

Because the geometrical interpretation of the gravitational field (and for example the relation between curvature and the matter distribution) that exists in General Relativity is partly lost, it is convenient to describe the BD gravitational field as having a scalar field component and a metric field component. The physical meaning of the scalar field component, ϕ , is that $\phi^{-1} \left(\frac{2\omega + 4}{2\omega + 3} \right)$ is the locally measured gravitational constant. The metric field component, as in General Relativity, describes (locally) the geometry of a space-time and with the geodesic equations describes the inertial properties of particles. Anticipating a conclusion suggested by solutions that have been found recently, the extra "degree of freedom" relating to the scalar field in the description of the BD gravitational field, allows the BD gravitational field to have quite different properties (e.g. symmetry) to the associated space-time geometry.

An interesting relation due to Quale (5), exists between the linearized weak field limits of the BD and Einstein gravitation theories. The linearized BD theory can be put into the form characteristic of a class of Lorentz covariant, linear tensor theories of gravitation together with an additional constraint on the class. In such a form the BD theory (in a restricted sense) and the linearized Einstein theory are examples of the only two, physically distinct theories that can exist in this class.

Briefly, to give a better context to these ideas the method used in (5), is indicated as follows :

The class of gravitation theories referred to have the properties that

- (a) the gravitational field is described by a symmetric second rank tensor, $h_{\mu\nu}$
 and (b) the field equation for $h_{\mu\nu}$ is linear, of the second differential order and Lorentz covariant with a source term proportional to the energy-momentum tensor, $T_{\mu\nu}$, of matter.

The further property that

$$T^{\lambda}_{\nu,\mu} = 0$$

restricts the class of theories to those with field equations of the form shown by Weyl (6) to be

$$D_{\mu\nu}(h) = \overset{\circ}{D}_{\mu\nu}(h) + \beta P_{\mu\nu}(h) = T_{\mu\nu}, \quad 4.15$$

where for convenience units are chosen to make the coefficient of $T_{\mu\nu}$ unity and the operator notation is defined by

$$\begin{aligned} \overset{\circ}{D}_{\mu\nu}(h) &= h_{\mu\nu}^{\lambda\lambda} - 2h^{\lambda}{}_{(\mu,\nu)\lambda} + h^{\rho\lambda}{}_{,\rho\lambda} \eta_{\mu\nu} \\ P_{\mu\nu}(h) &= h^{\lambda}{}_{\lambda,\mu\nu} - h^{\lambda}{}_{\lambda,\rho}{}^{\rho} \eta_{\mu\nu}. \end{aligned}$$

For the left hand side indices are dropped ($h_{\mu\nu} \rightarrow h$) for simplicity.

β is an arbitrary parameter for which two cases can be distinguished, dividing the class of physical theories into two kinds.

- (i) $\beta \neq \frac{1}{2}$ By a non-singular mapping²

$$h_{\mu\nu} \rightarrow h''_{\mu\nu} = h_{\mu\nu} + \frac{1}{2}(\beta - 1) h^{\lambda}{}_{\lambda} \eta_{\mu\nu},$$

eq. 4.15 can be put into the form, dropping dashes

$$\overset{E}{D}_{\mu\nu}(h) = \overset{\circ}{D}_{\mu\nu}(h) + P_{\mu\nu}(h) = T_{\mu\nu}, \quad 4.16$$

which is the linearized Einstein field equation for a first order perturbation of the Lorentz metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1.$$

$T^{\mu\nu}$ is independent of $h_{\mu\nu}$

2 for $\beta = \frac{1}{2}$ the mapping is singular. The new tensor $h''_{\mu\nu}$ is traceless and because it has fewer independent components than $h_{\mu\nu}$, the field descriptions by $h_{\mu\nu}$, $h''_{\mu\nu}$ cannot be physically equivalent.

Because the mapping only rearranges the field components any theory of the class with $\beta \neq \frac{1}{2}$ may be considered physically equivalent to the linearized Einstein theory.

The gauge group of eq. 4.16, or the group of substitutions under which the equation is forminvariant, is the group of infinitesimal coordinate transformations applied to the metric tensor $g_{\mu\nu}$ (as $\eta_{\mu\nu}$ is invariant),

$$\left. \begin{aligned} h_{\mu\nu} &\rightarrow h'_{\mu\nu} = h_{\mu\nu} + \delta h_{\mu\nu} \\ \delta h_{\mu\nu} &= 2\xi_{(\mu, \nu)} \end{aligned} \right\} 4.17$$

(ii) $\beta = \frac{1}{2}$ defines the Weyl theory which Quale develops and shows, together with an imposed constraint to give the linearized BD theory.

Eq. 4.15 takes the form

$$D_{\mu\nu}^W(h) = D_{\mu\nu}^E(h) + \frac{1}{2}P_{\mu\nu}(h) = T_{\mu\nu}, \quad 4.18$$

and the gauge group of this equation is given as

$$\left. \begin{aligned} h_{\mu\nu} &\rightarrow h'_{\mu\nu} = h_{\mu\nu} + \delta h_{\mu\nu} \\ \delta h_{\mu\nu} &= 2\xi_{(\mu, \nu)} + \mu \eta_{\mu\nu} \end{aligned} \right\} 4.19$$

where μ is an arbitrary scalar and ξ_{μ} is an arbitrary vector subject to $\xi^{\mu}_{;\mu} = \dot{x}$: the scalar \dot{x} , the general solution of $P_{\mu\nu}(\dot{x}) = 0$, has the form

$$\dot{x} = c_{\lambda} x^{\lambda} + c \quad \text{with } c_{\lambda} \text{ for } \lambda = 0 - 3, \text{ and } c \text{ arbitrary constants.}$$

In contrast to the Einstein group δ^E , defined by eq. 4.17, the Weyl group δ^W , defined by eq. 4.19 has a greater range of freedom by allowing the variations of δh and $\delta h^{\wedge}_{\mu\nu}$, defined as variations of the trace and traceless parts of $h_{\mu\nu}$, to be made independently of each other.

For completeness, a result proved by Quale, relating to the gauge groups δ^E , δ^W is stated — " the necessary and sufficient condition that the gauge of $h_{\mu\nu}$ can be completely fixed within each of the linearized Einstein and Weyl theories using the appropriate gauge groups to give formally identical field equations is that

$T \equiv 0$ ". With this understanding the two kinds of theories are physically distinct for $T \neq 0$.

The BD equations for

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1$$

$$\phi = 1 + \psi, \quad |\psi| \ll 1,$$

where the metric $g_{\mu\nu}$ differs infinitesimally from the Lorentz metric and the scalar field, ϕ differs infinitesimally from a constant equal to unity with the previous choice of units, give in the weak field limit the linearized BD equations

$$\overset{D}{D}_{\mu\nu}(h, \psi) \equiv \overset{E}{D}_{\mu\nu}(h) + 2\overset{E}{P}_{\mu\nu}(\psi) = T_{\mu\nu} \quad . \quad 4.20$$

$$\psi^{\lambda}_{;\lambda} = \frac{1}{3 + 2\omega} T \quad . \quad 4.21$$

By a choice of gauge

$$h^{\lambda}_{\lambda} = -4\psi + h^0 \quad \text{where } F_{\mu\nu}(h^0) = 0, \quad 4.22$$

eq. 4.20 reduces to the form

$$\overset{W}{D}_{\mu\nu}(h) = T_{\mu\nu} \quad . \quad 4.23$$

Taking the trace gives

$$2\hat{h}^{\rho\lambda}_{;\rho\lambda} = T \quad \text{where } \hat{h}_{\mu\nu} \equiv h_{\mu\nu} - 1/4 h^{\lambda}_{\lambda} \eta_{\mu\nu} \quad 4.24$$

and for consistency with eqs. 4.21 and 4.22 an additional gauge condition needs to be imposed on $h_{\mu\nu}$

$$\text{i.e. } h^{\rho\lambda}_{;\rho\lambda} = \frac{1 + 2\omega}{2(3 + 2\omega)} T \quad . \quad 4.25$$

Thus, if ψ is defined by eq. 4.21 and the gauge of $h_{\mu\nu}$ is fixed in terms of T by

$$h^{\lambda}_{\lambda}{}^{;\rho}{}_{\rho} = -\frac{4}{3 + 2\omega} T \quad \text{(using eqs. 4.21 and 4.22)} \quad , \quad 4.26$$

then the linearized BD theory could be interpreted as the Weyl theory with an externally imposed constraint - eq. 4.26 (one notes that eq. 4.26 with eq. 4.24 implies eq. 4.25). Although in this form the gravitational field is described by one component, the tensor field $\overset{W}{h}_{\mu\nu}$, the theory is still physically equivalent to the original

*rather than being a metric deviation from the Lorentz metric, (5)

$$\overset{W}{h}_{\mu\nu} = h_{\mu\nu} + \alpha\psi\eta_{\mu\nu}, \quad \text{for constant } \alpha \neq 1 \quad .$$

formulation of the linearized BD theory in terms of the two fields $h_{\mu\nu}$ and ψ . The trace of $h_{\mu\nu}$ has been partly fixed by eq. 4.26 (or alternatively by $P_{\mu\nu}(h^\lambda{}_\lambda + 4\psi) = 0$ using eqs. 4.21 and 4.22) and as a consequence the symmetry group of the theory is not the full Weyl group $\overset{W}{S}$, but the "reduced" Weyl group given by $\overset{RW}{\delta}$

$$\overset{RW}{\delta} h_{\mu\nu} = 2\tilde{\xi}_{(\mu, \nu)} + \beta \eta_{\mu\nu}, \quad 4.27$$

where $\tilde{\xi}_\mu$ is an arbitrary vector subject to $\tilde{\xi}_{,\mu} = 0$, $P_{\mu\nu}(\tilde{x}) = 0$ and β is a scalar such that $P_{\mu\nu}(\beta) = 0$.

It is perhaps worthwhile mentioning that Quale's result quoted earlier also holds for the linearized Einstein theory with gauge group $\overset{E}{\delta}$, and the linearized BD theory in the Weyl form with gauge group $\overset{RW}{\delta}$. The proof remains the same and so for $T = 0$ the linearized Einstein and BD theories are physically equivalent.

To conclude this section a brief look is given to the linearized BD theory in relation to the generally covariant, non-linear BD theory expressed by eqs. 4.5 and 4.6. The linearized BD theory has been put into the form of a Lorentz covariant, linear tensor theory of gravitation :

$$\overset{W}{D}_{\mu\nu}(h) = T_{\mu\nu}, \quad 4.28$$

with an auxillary condition

$$P_{\mu\nu}(h^\lambda{}_\lambda + 4\psi) = 0 \text{ where } \psi \text{ is defined by eq. 4.21, 4.29}$$

and as with the linearized Einstein theory, eq. 4.28 is inconsistent when the gravitational field is interacting with matter - for energy and momentum can be interchanged between the two and the source term must then contain a contribution from the gravitational field and become dependent on the tensor field and its derivatives. So by analogy with General Relativity it seems that the generally covariant, non-linear BD theory or a related non-linear theory could be derived from the Weyl form, eqs. 4.28 and

4.29, in terms of a self-interaction of the tensor field, provided consistency is taken into account between the gauge group of the field equations and the equations of motion for material particles.¹

Such a procedure is not possible it appears from a general argument given by Quale, who was concerned with the question as to whether a generally covariant, non-linear Weyl theory exists with the linearized weak-field limit given by the Weyl theory, in the same sense that the generally covariant, non-linear Einstein theory is related to its linearized weak field approximation.

If the non-linear Weyl theory is supposed to have the general form (5) :

$$\Delta_{\mu\nu}(H, G) = 0 \quad , \quad 4.30$$

where $\Delta_{\mu\nu}$ represents a non-linear, tensorial, differential operator acting on a symmetric tensor density, H and G denotes other geometric objects which may or may not be dynamical variables, then Quale showed that the symmetry group of the weak-field limit form of eq. 4.30 could not be the Weyl or reduced Weyl groups. So one concludes that a generally covariant non-linear theory, constructed as suggested above from eq. 4.28, does not exist.

Instead of the interpretation of the linearized BD theory given on page 47 there is another physically equivalent interpretation (5). The linearized BD theory (eqs. 4.20 and 4.21) involved two dynamical variables, ψ and $h_{\mu\nu}$, with independent gauge groups given by δ^S and δ^E respectively, where

$$\delta^S \psi = \psi \text{ for } P_{\mu\nu}(\psi) = 0 \quad . \quad 4.31$$

By using the Einstein group δ^E , $h_{\mu\nu}$ was adjusted to the gauge choice

$$P_{\mu\nu}(h^\lambda_\lambda + 4\psi) = 0 \quad 4.32$$

which meant for eq. 4.20 that the scalar ψ and its gauge

¹ This problem occurs in General Relativity (e.g. Sexl (2)) where although the field equations are gauge invariant the equations of motion are not. Specialising the gauge transformations to make the latter equations gauge invariant leads to MNC

degree of freedom (eq. 4.31) had been absorbed into the trace of $h_{\mu\nu}$ transforming eq. 4.20 into the form of eq. 4.23. From eqs. 4.23 and 4.21 $h^{\rho\lambda},_{\rho\lambda}$ was fixed gauge-invariantly in terms of T (eq. 4.25) and the remaining gauge degrees of freedom for $h_{\mu\nu}$ are given by the "reduced" Einstein group δ^{RE} , (5)

$$\delta^{RE} h_{\mu\nu} = 2 \xi_{(\mu, \nu)} \quad 4.33$$

This group is obtained from the full group (eq. 4.17) by imposing the constraint eq. 4.32.

So the linearized BD theory as looked at in terms of the two separate fields, $h_{\mu\nu}$ and ψ , has the reduced Einstein group for its symmetry group. The dynamical equation for $h_{\mu\nu}$ is

$$D_{\mu\nu}(h) = T_{\mu\nu}, \quad 4.34$$

subject to $P_{\mu\nu} (h^\lambda{}_\lambda + 4\psi) = 0$,

and it remains to make eq. 4.21 compatible with eq. 4.34 either by taking eq. 4.25 as a dynamical equation or perhaps from eqs. 4.25 and 4.21

$$h^\lambda{}_\lambda{}^{,\rho}{}_\rho = -4\psi{}^{,\lambda}{}_\lambda$$

as a dynamical equation.

Again the problem of constructing a non-linear theory by a self-interaction of the tensor field suggests itself. Unfortunately, the key paper for further details, Grunberg (8) is written in German, and the writer on obtaining a copy late, has been unable to translate it sufficiently to justify writing about it - except to note that the work appears independent of Quale's study and also connects the linearized BD theory with the Weyl form e.g. eq. 4.34.

4.2 Vacuum Solutions

The BD field equations (eqs. 4.5 and 4.6) in the absence of matter ($T_{\mu\nu} = 0$) give the vacuum BD equations

$$G_{\mu\nu} = \omega\phi^{-2} (\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\rho}\phi^{;\rho}) + \phi^{-2}\phi_{;\mu;\nu} \quad 4.28$$

$$\square\phi = 0, \quad 4.29$$

the solutions of which describe the BD gravitational fields that could exist in empty space-times.

For reference the Brans (4) static, spherically symmetric solution in isotropic coordinates is stated. With the line element written as

$$ds^2 = - e^{\alpha(r)} dt^2 + e^{\beta(r)} (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\psi^2),$$

the four types of solution corresponding to different relationships between the constants ω and C as indicated, are

TYPE I

$$e^{\frac{\alpha}{2}} = e^{\frac{\alpha_0}{2}} \left[\frac{1 - B/r}{1 + B/r} \right]$$

$$e^{\frac{\beta}{2}} = e^{\frac{\beta_0}{2}} [1 + B/r]^2 \left[\frac{1 - B/r}{1 + B/r} \right]^{-(\lambda - C - 1)/\lambda}$$

$$\phi = \phi_0 \left[\frac{1 - B/r}{1 + B/r} \right]^{C/\lambda}$$

$$\lambda^2 = (C + 1)^2 - C(1 - \omega^{C/2}) > 0$$

TYPE II

$$\frac{\alpha}{2} = \frac{\alpha_0}{2} + \frac{2}{\Lambda} \tan^{-1}(r/B)$$

$$\frac{\beta}{2} = \frac{\beta_0}{2} - 2 \frac{(C+1)}{\Lambda} \tan^{-1}(r/B) - \ln[r^2/r^2 + B^2]$$

$$\phi = \phi_0 e^{2C/\Lambda \tan^{-1}(r/B)}$$

$$\Lambda^2 = C(1 - \omega^{C/2}) - (C+1)^2 > 0$$

TYPE III

$$\frac{\alpha}{2} = \frac{\alpha_0}{2} - r/B$$

$$\frac{\beta}{2} = \frac{\beta_0}{2} - 2 \ln r/B + (C+1)r/B$$

$$\phi = \phi_0 e^{-Cr/B}$$

$$C = \frac{-1 \pm (-2\omega - 3)^{\frac{1}{2}}}{\omega + 2}$$

TYPE IV

$$\frac{\alpha}{2} = \frac{\alpha_0}{2} - 1/Br$$

$$\frac{\beta}{2} = \frac{\beta_0}{2} + (C+1)/Br$$

$$\phi = \phi_0 e^{-C/Br}$$

$$C = \frac{-1 \pm (-2\omega - 3)^{\frac{1}{2}}}{\omega + 2}$$

where α_0 , β_0 , ϕ_0 and B are constants.

Essentially there are only three types of solution as Buchdahl has pointed out (9) that TYPE III solution goes over to TYPE IV solution when the coordinate r is replaced by $1/r$.

If Lorentz conditions are required at spatial infinity

$$\text{i.e. } \alpha \rightarrow 0$$

$$\beta \rightarrow 0$$

$$\phi \rightarrow \frac{\kappa^2}{G}, \quad \kappa^2 = \left(\frac{2\omega + 4}{2\omega + 3} \right),$$

all as $r \rightarrow \infty$,

then the constants α_0 , β_0 and ϕ_0 can be chosen appropriately but B remains undetermined.

TYPE I solution is the only solution which remains physically possible for all positive ω . In the limit as $\omega \rightarrow \infty$ this solution goes over to the exterior Schwarzschild solution in General Relativity provided one puts

$$B = \frac{GM}{2c^2}, \quad G = \frac{\kappa^2}{\phi_0}.$$

M is the mass of the point source of the exterior Schwarzschild solution. For finite ω the TYPE I solution could be interpreted as the solution for a point mass, and if such a mass is small enough B could be determined (as in General Relativity) by comparison with the weak field limit solution.

O'Hanlon and Tupper (10) have found solutions of the vacuum BD equations, with metrics of the Robertson-Walker type and with the associated scalar fields allowed to be functions of position or time or both. So there is the possibility of curved, uniform, empty space-times in which are defined non-uniform BD gravitational fields; the non-uniformity arising from the scalar field component.

From their paper, the metric-scalar field solutions based on the line element

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \quad 4.36$$

$$0 \leq r < 1$$

where $k = +1, -1, 0$ for closed, open, flat 3-spaces respectively, can be divided into two classes.

CLASS A $\frac{\partial \phi}{\partial r} = 0$ i.e. $\phi = \phi(t)$

<p>(1) $\omega = \frac{3}{2}$ $k=0$</p>	$ds^2 = -dt^2 + a_0^2 \left(\frac{t}{t_0} \right)^{2q} [dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\psi^2],$ $\phi = \phi_0 \left(\frac{t}{t_0} \right)^r \quad 2r?$ <p>where a_0, ϕ_0, t_0 are constants.</p> $\frac{1}{r} = -\frac{1}{2} \left[1 \pm \sqrt{3(2\omega + 3)} \right]$ $q = \frac{1}{3} (1 - r)$ <p>For $\omega = -\frac{4}{3}$ take + sign in expression for r ; the solution for the -ve sign is given below</p>
<p>(2) $\omega = \frac{4}{3}$ $k=0$</p>	$ds^2 = -dt^2 \cdot \exp\left(\frac{2t}{R}\right) [dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\psi^2]$ $\phi = \phi_0 \exp\left[-\frac{3t}{R}\right]$ <p>for ϕ_0 constants. <i>Dopie R</i></p>
<p>(3) $\omega=0$ $k=0, \pm 1$</p>	$ds^2 = -dt^2 + (D - kt^2) \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\psi^2 \right]$ $\phi = t(D - dt^2)^{-\frac{1}{2}} \quad k?$ <p>for D constant.</p>
<p>(4) $\omega = -\frac{3}{2}$ $k=0, -1$</p>	$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\psi^2 \right]$ $\phi = \phi(t)$ <p>Where a, ϕ satisfy</p> $\frac{\dot{a}}{a} = -\frac{\dot{\phi}}{2\phi} \pm \frac{1}{a} \sqrt{-k} \quad \dot{\phi} = \frac{d\phi}{dt}$ $\dot{a} = \frac{da}{dt}$

CLASS B $\frac{\partial \phi}{\partial r} \neq 0, \frac{\partial \phi}{\partial t} \neq 0$ i.e. $\phi = \phi(r, t)$

<p>(5) $\omega = -\frac{4}{3}$</p>	<p>$k=+1$ $ds^2 = -dt^2 + R^2 \cosh^2 \frac{t}{R} \left[\frac{dr^2}{1-r^2} + r^2 d\theta^2 + \sin^2 \theta d\psi^2 \right]$</p> <p>$\phi = \left[-\frac{2}{3} RA \left(\sinh \frac{t}{R} \pm \sqrt{1-r^2} \cosh \frac{t}{R} \right) \right]^{-3}$</p> <p>where R, A are constants.</p>
	<p>$k=0$ $ds^2 = -dt^2 + \exp \left[\frac{2t}{R} \right] \left[dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\psi^2 \right]$</p> <p>$\phi = \left[-\frac{1}{3} A \left(r^2 \exp \left(\frac{t}{R} \right) - R^2 \exp \left(-\frac{t}{R} \right) \right) \right]^{-3}$</p>
	<p>$k=-1$ $ds^2 = -dt^2 + R^2 \sinh^2 \left(\frac{t}{R} \right) \left[\frac{dr^2}{1+r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\psi^2 \right]$</p> <p>$\phi = \left[-\frac{2}{3} RA \left(\cosh \frac{t}{R} \pm \sqrt{1+r^2} \sinh \frac{t}{R} \right) \right]^{-3}$</p>
<p>(6) $\omega = -\frac{3}{2}$</p>	<p>$k=\pm 1$ $ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\psi^2 \right]$</p> <p>$\phi = [A(b - ak \sqrt{1-kr^2})]^{-2}$</p> <p>where $a(t), b(t)$ are arbitrary subject to the conditions</p> <p>$(\dot{a}b - a\dot{b})^2 = -4ab.$</p>

A third possible class not mentioned, is given for $\phi = \phi(r)$ in which case

$$a = a_0 \text{ a constant}$$

$$\text{and } \phi = \left[\left(\frac{2Ar^2}{1 + \sqrt{1-kr^2}} + B \right) \omega + 1 \right] \frac{1}{\omega + 1}$$

where A and B are constants.

Remarks on Vacuum Solutions

Solution 1 is similar to the BD cosmological solution (Section 4.3) and admits an expanding empty space-time with an increasing ϕ if $\omega > 0$ and the negative sign ^{is dependent on} is taken.

Solution 3 for $\omega = 0$ and $D > 0$ gives two cases.

- (i) For $k < 0$ the 3-space expands indefinitely from an initially non-singular state .
- (ii) For $k > 0$ an initially finite space-time contracts to a singular state in finite time .

A further discussion of this solution is given in Section 4.7.

Solutions 4 and 6 for $\omega = -\frac{3}{2}$ should be the conformal maps of vacuum solutions of Einstein's field equations where the conformal factor ϕ satisfies $\square\phi = 0$.

Solutions 2, 5 and 6 for $\omega = -\frac{4}{3}$ give the three forms of the de-Sitter space-time (e.g. (11)). This space-time is static and solution 6 for example can be put into a form with an explicitly static line element (10)

$$ds^2 = - \left(1 - \frac{\rho^2}{R^2}\right) dr^2 + \frac{d\rho^2}{1 - \frac{\rho^2}{R^2}} + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\psi^2 \quad 4.31$$

while ϕ remains time dependent (10)

$$\phi = \left[\frac{1}{3} AR^2 \exp\left(-\frac{r}{R}\right) \sqrt{1 - \frac{\rho^2}{R^2}} \right] \quad 4.32$$

Thus the vacuum BD equations allow static, empty, non-singular space-times with a time dependent scalar field and a Birkhoff theorem for the BD gravitational field could not exist.

Class B Solutions O'Hanlon and Tupper note, have a "curious feature" in that the uniform space-times have corresponding scalar field values which, because they are functions of both r and t , cannot be uniform. Although for the de-Sitter space-time written in the static form, the origin is a preferred point (as test particles are subjected to radial accelerations away from the origin - see e.g. (12)) the coordinate system of solution 6 has no preferred point but the scalar field has a minimum value at $r = 0$ for constant t . The other solutions of Class B have stationary values at for constant t .

This existence of a preferred point and so preferred coordinate systems they suggest may arise from the scalar field being an energetic system and the General Relativistic idea of the vacuum not being well-defined in the BD theory. Taken further, this suggests that the BD vacuum equations can be written in a form in which the scalar field terms collect together to form the source term for the Einstein tensor and the proper vacuum situation would then be obtained by putting this energy-momentum tensor equal to zero. In this case the only vacuum solutions would be the vacuum solutions of General Relativity, each with an everywhere constant scalar field.

However, if the energy-momentum tensor constructed above is considered as that of a matter source, minimally coupled to the metric field, such a source may not always have positive definite flat-space energy and so rather than continuing this interpretation it will be interesting to turn to the idea of the BD gravitational field being described by metric and scalar field components. An advantage of this description is that it suggests that the BD gravitational field need not have the same direct relation to the structure of space-time as does the Einstein gravitational field. In fact the vacuum solutions show that the knowledge of a space-time geometry is not always sufficient to determine the BD gravitational field that could exist in the space-time.

It appears for example that the symmetry group of an empty space-time need not be the symmetry group of an associated BD gravitational field and thus preferred coordinate systems determined by the symmetry of the BD field could exist. More specifically, for the $k = 0$ de-Sitter solution 6 in cartesian coordinates (x^0, x^1, x^2, x^3)

$$ds^2 = -dx^0{}^2 + a^2(x^0) (dx^1{}^2 + dx^2{}^2 + dx^3{}^2) \quad 4.33$$

$$\text{and } \phi = \left[-\frac{1}{3}A \left(r^2 \exp\left(\frac{x^0}{R}\right) - R^2 \exp\left(-\frac{x^0}{R}\right) \right) \right], \quad 4.34$$

where $\alpha = \exp\left(\frac{x^0}{R}\right)$ and $r^2 = \sum_{i=1}^3 x_i^2$.

The symmetry group of the space-time is the de-Sitter group. The ten independent Killing vectors have components (e.g. Hart (13))

$$\left. \begin{aligned} \xi^0 &= E + B_\alpha x^\alpha \\ \xi^\alpha &= C^\alpha_\beta x^\beta + B^\alpha \int \frac{dx^0}{a^2(x^0)} + D^\alpha - \frac{\dot{a}}{a} x^\alpha (E + B_\alpha x^\alpha) \end{aligned} \right\} 4.35$$

where $\dot{a} = \frac{da}{dx^0}$, $-C^\alpha_\beta = C^\beta_\alpha$, $B^\alpha = B_\alpha$ and the ten parameters E , B_α , C^α_β and D^α correspond to temporal translations, generalised Lorentz transformations, spatial rotations and spatial translations respectively.

The symmetry group of the BD gravitational field is the de-Sitter group restricted by the condition that the symmetry transformations leave the scalar field invariant.

i.e. the Killing vector components are given by

$$\xi^{\alpha;\beta} + \xi^{\beta;\alpha} = 0,$$

subject to $\phi_{,\alpha} \xi^\alpha = 0$ (cf. eq. 3.30),

which implies that E , B^α and D^α all vanish. The Killing vectors for the BD gravitational field have components

$$\xi^\alpha = C^\alpha_\beta x^\beta, \quad 4.36$$

and the symmetry transformations are the group of spatial rotations. This group is intransitive, implying that the BD gravitational field is (spatially) inhomogeneous - a feature which was observed in the solution.

Effectively then, the influence of the scalar field component has been to introduce preferred location effects.

4.3 Standard BD Cosmology

By asserting that the Universe is spatially homogeneous and isotropic the cosmological Principle distinguishes as privileged, a set of observers for which it could hold time. For these observers it can be shown that a universal cosmic time exists (e.g. Bondi, (14)) and the Cosmological Principle can then be re-expressed to assert the existence of coordinate systems (with the first coordinate label identified with the cosmic time coordinate) which are equivalent for the purpose of describing the gross structure and the history of the Universe.

The coordinate transformation between a pair of such equivalent coordinate systems

$$\text{c.s. } \{ x^\alpha \} \rightarrow \{ x^{\alpha'} \}$$

where $x^0 \rightarrow x^{0'} = x^0$ denotes the cosmic time, t ,

$$x^i \rightarrow x^{i'} (x^1, x^2, x^3) , \quad i = 1 - 3 ,$$

must then be an isometry for the space-time model of the Universe. So for example, at any point P and for any element of the subgroup of MIG that gives coordinate transformations of the kind above,

$$\bar{\delta}_{\mu\nu} = 0 ,$$

which implies (e.g. Weinberg, (3)) that the metric takes the Robertson Walker form

$$ds^2 = - dt^2 + a^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\psi^2 \right] \quad 4.37$$

$$k = 0, \pm 1 .$$

The question arises for the BD theory as to whether or not the scalar and matter fields are to respect these isometries. (See Sect. 4.6). This is in contrast to the General Relativity case where the field equations guarantee that the space-time isometries described above also apply to the mixed energy-momentum tensor of the matter field.

For the Standard BD cosmology (big-bang cosmology described by using the Cosmological Principle and the BD field equations) it is assumed that the cosmic (e.g. scalar and matter) fields are form-invariant under coordinate transformations between equivalent coordinate systems. This requires, e.g. (3), the scalar field to be a function of the cosmic time only and the energy-momentum tensor to have the form of a perfect-fluid energy-momentum tensor with the density and pressure components also functions of cosmic time only

$$\text{i.e. } T^{\mu\nu} = \text{diag}(-\rho(t), p(t), p(t), p(t)), \quad \phi = \phi(t). \quad 4.38$$

Using 4.37 and 4.38 in the field equations (4.5, 4.6) the fundamental equations for the BD Friedmann models can be taken as the set, (3),

$$\frac{d}{dt} (\dot{\phi} a^3) = \frac{8\pi}{3+2\omega} (\rho - 3p) a^3 \quad 4.39$$

$$\ddot{\phi} = -3\frac{\dot{\phi}}{a} (\rho + p), \quad 4.40$$

$$\frac{\ddot{a}}{a} + \frac{k}{a^2} = \frac{8\pi\rho}{3\dot{\phi}} - \frac{\dot{\phi}^2}{9a} + \frac{\omega\dot{\phi}^2}{6\phi} \quad 4.41$$

where $\dot{\phi} = \frac{d\phi}{dt}$, $\dot{\rho} = \frac{d\rho}{dt}$, $\dot{a} = \frac{da}{dt}$.

Thus for a given equation of state $p = p(\rho)$, the three variables $a(t)$, $\phi(t)$ and $p(t)$ are determined by one 2nd order and two 1st order differential equations. A unique solution then exists for fixed ω and k if the values of the four variables a , \dot{a} , ρ , and ϕ can be specified at some time t . Solutions in general, have a singularity at $a = 0$ at some finite time; so on defining $t = 0$ for this time, equation 4.39 gives

$$\dot{\phi} a^3(t) = \frac{8\pi}{3+2\omega} \int_0^t [\rho(t') - 3p(t')] a^3(t') dt' - C, \quad 4.42$$

where C is the integration constant.

For $C = 0$ a three parameter family of solutions is obtained which satisfy the constraint Brans and Dicke, (1), used to eliminate the extra degree of freedom implied by the initial conditions.

$$\text{i.e. } \dot{\phi} a^3(t) \rightarrow 0 \text{ for } a \rightarrow 0 .$$

For $C \neq 0$ an extra parameter is needed to fix C and so a four parameter family of solutions is obtained .

For the pressureless, zero curvature case

$$p = 0 , k = 0 ,$$

there are three kinds of solutions which for late epochs have similar behaviour while for very early epochs, differ quite considerably.

I. $C = 0$ Brans and Dicke (1)

$$\phi \propto t^{2/(4+3\omega)} \quad 4.4.1$$

$$a \propto t^{2(\omega+1)/(4+3\omega)} \quad 4.4.2$$

$$\frac{p t^3}{\phi} = \frac{2\omega+3}{\pi(3\omega+4)} . \quad 4.4.3$$

These solutions go over smoothly to the Einstein-Friedmann model solutions

$$\phi = \text{constant} \quad 4.4.4$$

$$a \propto t^{2/3}$$

$$p \propto t^{-2}$$

for large ω .

Although explicit solutions for $C \neq 0$ could be found their behaviour for very early and late epochs tends to be more informative

II. $C > 0$, for $t \rightarrow 0$, Dicke (15)

$$\phi \propto t [1 - 3(1 + 2/3\omega)^{1/2}] / (4 + 3\omega) \quad 4.4.7$$

$$a \propto t [1 + \omega + (1 + 2/3\omega)^{1/2}] / (4 + 3\omega) \quad 4.4.8$$

$$p \propto a^{-2} \quad 4.4.9$$

III. C < 0, for $t \rightarrow 0$ Dicke (15)

$$\phi \propto t^{[1 + 3(1 + 2/3\omega)^{1/2}]/(4 + 3\omega)} \quad 4.50$$

$$a \propto t^{[1 + \omega - (1 + 2/3\omega)^{1/2}]/(4 + 3\omega)} \quad 4.51$$

$$\rho \propto a^{-2} \quad 4.52$$

For large t the behaviour of solutions type II and III approach that of type I.

A peculiarity* of solutions of type II and III is that for finite ω , as $t \rightarrow 0$.

$$\text{TYPE II : } \phi \rightarrow \infty$$

$$\text{TYPE III : } \phi \rightarrow 0 \quad .$$

Solutions for $k \neq 0, p \neq 0$ Weinberg (3) suggests would have limiting behaviour ($t \rightarrow 0, \infty$) similar in classification to the three types described above.

Since appropriate observations of early epochs are not available, C may be negligible. In this case for large ω the relations between curvature, mass density (ρ_0), epoch (t_0), Hubble's constant (H_0) and the deceleration parameter (q_0) would be expected to be very similar to those of the Einstein Friedmann model. For example, for the zero pressure, zero curvature model in the limit of large t eqs. 4.41 - 4.43 give (3)

$$H_0 t_0 = \left(\frac{2 + 2\omega}{4 + 3\omega} \right)$$

$$q_0 = \left(\frac{\omega + 2}{2\omega + 2} \right)$$

$$\frac{4\pi G \rho_0}{H_0^2} = \frac{(4 + 3\omega)(4 + 2\omega)}{(2 + 2\omega)^2} \quad .$$

With $\omega \geq 6$ these quantities differ from those of the Einstein Friedmann model by less than five per cent.

* The same peculiarities occur in the anisotropic models considered by Matzner, Ryan and Toon, (34).

However, if the integration constant is not negligible it is an additional parameter with which to fit observations. The latitude for achieving a fit is greater and consequently the bound implied on the fractional time variation of the gravitational constant is not accurate enough (16), at present to decide between the BD and Einstein-Friedmann models.

4.4 Units transformations and the BD theory

By a suitable position dependent scaling of length, time and inverse mass, Dicke, (17), has shown that the original form of the BD theory in which the gravitational constant $G \sim \phi^{-1}$ varies, can be put into a barred form in which \bar{G} is constant and particle masses, \bar{M} , vary with space-time position. In the original form numerical quantities are measured relative to constant atomic units (particle masses, M_1 , constant; G varying) and in the barred form they are measured relative to constant gravitational units. (particle masses, \bar{M} , varying; \bar{G} constant).

The barred form, or representation, is interesting because the scalar field appears as a matter field minimally coupled to the barred metric field. In the linearized weak field limit the two fields are independent and both give rise to a universal inverse square law attractive force. If it is supposed that units are based on atomic standards then the scalar field interacts with material particles through the mass of a particle being a function of the scalar field.

$$\text{i.e. } \bar{M} \propto \lambda^{-\frac{1}{2}},$$

where the scalar field, λ , appears in the transformed BD equations, (17),

$$\bar{G}_{\mu\nu} = 8\pi\bar{G}(\bar{T}_{\mu\nu} + \Lambda_{\mu\nu}) \quad 4.53$$

$$\bar{\square}(1/\lambda) = \frac{8\pi\bar{G}}{2\omega + 3} \bar{T} \quad 4.54$$

with $\Lambda_{\mu\nu} = \frac{(2\omega + 3)}{16\pi\bar{G}\lambda} [\lambda_{,\mu}\lambda_{,\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\lambda_{,\alpha}\lambda^{,\alpha}]$, $\bar{T} = \bar{T}^{\alpha}_{\alpha}$, and barred operations are performed with respect to barred symbols o.g. $\bar{g}_{\mu\nu}$.

*except for electromagnetic fields which they ignore.

The mass variation with space-time position is universal and in principle could be observed by determining the ratio of the mass of some elementary particle to the characteristic gravitational mass, $(\hbar c/G)^{\frac{1}{2}}$. Using the de Broglie relation for an elementary particle

$$\begin{aligned}
 p &= \hbar k, & p &= \text{3 - momentum} \\
 & & k &= \text{wave no. vector} \\
 & & \hbar &= \text{Planck's constant} / 2\pi
 \end{aligned}$$

the variation in mass determines an identical variation for inverse length and in a similar way for time rates. As with the mass variation Dicke (17), suggests that the length and time variations with space-time position could be observed by comparison with the characteristic length and time invariants $(GM/c^3)^{\frac{1}{2}}$ and $(GM/c^5)^{\frac{1}{2}}$.

This representation is clearly non-metric as the equations of motion of a test particle derive from $0 = \delta \int m ds$, where $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, and are geodesics of the metric $g_{\mu\nu}$ rather than the metric $\bar{g}_{\mu\nu}$.

A generalisation of Dicke's units transformation by Morganstern, (18), which likewise leaves Planck's constant, \hbar and c invariant, allows for the constancy of products of G and M , such as for example the Schwarzschild radius $2GM/c^2$. This is achieved by taking

$\phi = \phi_0 \lambda$ in the original BD equations, where ϕ_0 is a constant and λ is a dimensionless scalar field.

The units transformation for length (time), L , and mass M is defined by

$$L \rightarrow \bar{L} = \lambda^{\frac{1}{2}(1-\alpha)} L \quad 4.56$$

$$M \rightarrow \bar{M} = \lambda^{-\frac{1}{2}(1-\alpha)} M \quad 4.57$$

under which

$$g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} = \lambda^{(1-\alpha)} g_{\mu\nu} \quad 4.58$$

$$\phi \rightarrow \bar{\phi} = \bar{\phi}_0 \lambda = \phi_0 \lambda^\alpha, \quad 4.59$$

where α is an arbitrary parameter characterising the transformed units. The BD equations can be then written as, (18),

$$\bar{G}_{\mu\nu} = \text{Scp}_0^{-1} \lambda^{-2} \bar{T}_{\mu\nu} + \frac{1}{2} (2\omega + 3 - \alpha^2) \Lambda_{,;\mu}^{\Lambda}{}_{,;\nu} - (2\omega + 3 + \alpha^2) \bar{G}_{\mu\nu} \Lambda_{,;\alpha}^{\Lambda}{}_{,;\bar{\alpha}} + \alpha (\Lambda_{,;\mu}{}_{,;\nu} \bar{G}_{\mu\nu} \square \Lambda) \quad 4.60$$

$$\square \Lambda + \alpha \Lambda_{,;\alpha}^{\Lambda}{}_{,;\bar{\alpha}} = \text{Scp}_0^{-1} \frac{T}{(2\omega + 3)}, \quad \Lambda = \text{In} \lambda. \quad 4.61$$

The transformed field equations 4.60, 4.61, show that, irrespective of the value of the parameter α , matter makes an always positive contribution to the scalar field Λ , (rather than to the scalar field $\bar{\Phi}$), (18). Thus the Machian requirement that matter give a positive contribution to the inverse gravitational constant is independent of the choice of units when interpreted in terms of the dimensionless scaling field Λ , (18).

Also independent of the value of α , are the predictions, (18), for the three tests (and for Schiff's gyroscope test) which could have been expected as angular measurements are generally considered to be dimensionless.

Hence the BD theory can be given various representations, each generated from the original by a units transformation and clearly for a given representation the input data (e.g. mass-energy density etc.) must be expressed in units appropriate to the representation. It should perhaps be noted that this interpretation of a representation as a units transformed BD theory is regarded as superfluous by some authors. e.g. O'Hanlon (39), who states that ~~such~~ an interpretation "is illegal since units are legally (and reasonably so) defined in terms of atomic standards". Thus a representation is a mathematical convenience.

However, on considering how atomic standards of length and time are defined (e.g. Sec. 3.1), one feels that this microcosmos bias could be countered in the spirit of remarks made by Eddington (12) which although taken out of context suggest that an elementary particle adjusts its size to the spatial curvature of the Universe.

The BD theory was designed to apply on the cosmological scale. With this view it is useful, although as mentioned not necessary, to interpret the conformally related representations (eq. 4.60, 4.61) as units transformed representations and look at the appropriateness of a system of units and its associated representation. Thus for local physics, because of the weakness of gravitational interaction, the relative variation between atomic and for example gravitational units that exists in the BD theory should be unimportant while for expanding universes it may be important. On this latter scale, the problem arises as Morganstern, (19), points out, that the numerical value of a given quantity obtained from observation must be further analysed to determine which set of units is constant in the measurements.

4.5 Standard BD cosmology (Contd.)

In a series of papers (18, 19, 20, 21) Morganstern has developed and given solutions to the Friedman-type models based on the transformed field equations (eqs. 4.60 and 4.61) and with the energy-momentum tensor in standard form

$$\bar{T}^{\mu}_{\nu} = \text{diag}(-\bar{\rho}, \bar{p}, \bar{p}, \bar{p}), \quad 4.62$$

where $\bar{\rho} = \bar{\rho}(t)$, $\bar{p} = \bar{p}(t)$ and the equation of state is given by

$$\bar{p} = \epsilon \bar{\rho}, \quad 0 \leq \epsilon \leq \frac{1}{3}. \quad 4.63$$

Although not giving full justice to the calculations, the purpose here will be to give a brief summary to extend and complement the previous section.

A privileged observer views the Universe as homogeneous and isotropic and then for his particular choice (α) of units takes the line element in comoving coordinates as

$$d\bar{s}^2 = -d\bar{t}^2 + \bar{a}^2(\bar{t}) \left[\frac{d\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2 \theta d\psi^2 \right] \quad 4.64$$

where $k = 0, \pm 1$ and the barred notation as usual denotes the choice of units. The cosmic time, \bar{t} , will depend on this choice.

In the original units of the BD theory the line element takes the form

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta dy^2 \right] \quad 4.65$$

and because of the dimensionality given to the coordinate t , and the function $a(t)$, the units transformation between the forms 4.64 and 4.65 needs to be extended.*

$$\text{viz: } \bar{a} = \lambda^{\frac{1}{2}(1-\alpha)} a \quad 4.66$$

$$d\bar{t} = \lambda^{\frac{1}{2}(1-\alpha)} dt \quad 4.67$$

Hence the solutions $\bar{a}(\bar{t})$, $\lambda(\bar{t})$ and $\bar{\rho}(\bar{t})$ could be expected in general to be different functions of \bar{t} for each set of units.

For early epochs, as $t \rightarrow 0$, it is found that (21)

- (a) for the radiation filled Friedmann models, the solutions a , λ and ρ show the same time dependence for all spatial curvatures ,
- (b) for matter filled Friedmann models the solutions a and λ have the same time and unit dependence. The density however, increases less rapidly than for the radiation case (which is less rapid than the t^{-2} behaviour of the Einstein-Friedmann model). In contrast, the trend is for the assignment of numerical values to physical parameters (e.g. epoch,

* the transformation on the time differential can be avoided (19) by defining .

$$\frac{d\bar{t}}{dt} = \frac{\bar{a}(\eta)}{a(\eta)} \frac{d\eta}{dt}$$

where η is a dimensionless parameter. Under a units transformation these equations imply the transformation eq. 4.67.

matter density etc.) to depend very much on the units employed.

- (c) If ρ_λ denotes the scalar field energy density (which for $\alpha = 0$ is proportional to Λ^2) then $\frac{\rho}{\rho_\lambda} \rightarrow 0$ independently of spatial curvature and units. So in the limit the scalar field energy density dominates the matter energy density.

For late epochs, as $t \rightarrow \infty$

- (d) $\frac{\rho}{\rho_\lambda} \rightarrow \text{constant} \sim 1$ for $\epsilon \neq 1/3$, suggesting that both the matter and scalar fields are important, while for the radiation case ($\epsilon = 1/3$) $\frac{\rho}{\rho_\lambda} \rightarrow \infty$ and $\lambda \rightarrow \text{constant}$.

The general conclusion based on present observational constraints (22) is that the scalar field, if it exists, although of negligible importance for the present epoch, seems important near the initial singularity. However, one notes from the Hamiltonian formulation, e.g. (33) and (34), for similar but anisotropic cosmologies that the presence of the increasing scalar field energy density for early epochs is not sufficient to prevent the existence of the initial singularity.

4.6 "Non-Standard" BD Cosmology

So far only the Friedmann type cosmologies have been looked at. For these models the assumptions of spatial isotropy and homogeneity apply also to the scalar and matter fields and as a consequence these fields are functions of cosmic time only.

McIntosh (23) and Halford (24) have shown that perfect fluid-type Friedmann solutions exist for which the scalar and matter fields are position and time dependent.

e.g. McIntosh (23) for the flat - space
Robertson-Walker metric

$$ds^2 = - dt^2 + a^2(t)[dx^2 + dy^2 + dz^2]$$

obtains

$$(i) \quad \underline{\omega = 0}, \quad a(t) = At^{\frac{1}{2}}$$

$$\phi(t, x, y, z) = B^{\frac{1}{2}}[K(x^2 + y^2 + z^2) + Lx + My + Nz + P]$$

$$\delta\pi_p(t, x, y, z) = \frac{3\phi}{2t^2} - \frac{6KB}{A^2 t^{\frac{3}{2}}}$$

$$\delta\pi_p(t) = \frac{4KB_1}{A^2 t^{\frac{3}{2}}}$$

$$(ii) \quad \underline{\omega = -\frac{1}{2}}, \quad a(t) = At^{\frac{2}{5}}$$

$$\phi(t, x, y, z) = Bt^{\frac{4}{5}}(Lx + My + Nz)^2$$

$$\delta\pi_p(t, x, y, z) = \frac{8\phi}{5t^2} - \frac{(L^2 + M^2 + N^2)B}{A^2}$$

$$\delta\pi_p(t) = \frac{(L^2 + M^2 + N^2)B}{A^2}$$

where A, B, K, L, M, N, and P are constants.

Although initial or boundary conditions are

not given to determine these constants, it is not clear how these can be given, even in a frame dependent way and make physical sense.

A further difficulty arises because the \mathcal{J} -space is the open Euclidean \mathcal{J} -space. For the two solutions, the scalar field can be zero at some points (because it has a factor containing independent polynomial expressions in x , y , and z). The scalar field is unphysical at such points and the matter density is negative. Alternatively, by using symmetry transformations of the space-time, for example spatial rotations and translations, it appears that the matter density at a given point may be made negative.

This suggests that in looking for position-dependent solutions of the perfect fluid-type Friedmann the choice of functional position dependence is important. For depending on the choice, if the symmetry group of the BD gravitational field or matter field is not the same as the symmetry group of the associated space-time then the situation may arise where coordinate transformations leave the metric form-invariant while giving an unphysical BD gravitational field or matter distribution.

4.7 Scalar-Matter field gauge freedom

In Section 4.2 it was shown that in the absence of matter, the geometry of a space-time does not in general uniquely determine the BD gravitational field that could exist in it. This suggests for the full BD theory that the matter content of a space-time is likewise, not uniquely determined by the geometry of the space-time - in contrast to General Relativity where geometry uniquely determines matter content almost trivially via the Einstein tensor.

O'Hanlon (25) has made this idea more explicit by showing that geometry determines the matter content only up to a space-time position dependent gauge-transformation. The BD equations

$$G_{\mu\nu} = 8\pi\phi^{-1}T_{\mu\nu} + \omega\phi^{-2}[\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}\varepsilon_{\mu\nu}\phi_{;\rho}\phi^{;\rho}] + \phi^{-1}[\phi_{;\mu;\nu} - \varepsilon_{\mu\nu}\square\phi] \quad 4.68$$

$$\square\phi = \frac{8\pi T}{3 + 2\omega} \quad , \quad 4.69$$

are form-invariant under the group of geometry preserving transformations (25)

$$\phi \rightarrow \phi' = \lambda\phi \text{ where } \lambda > 0 \text{ to maintain attractive } (\phi > 0) \text{ gravitation } 4.70$$

$$T_{\mu\nu} \rightarrow T'_{\mu\nu} = (8\pi)^{-1}\phi[\omega\phi^{-1}(\lambda_{;\mu}\lambda_{;\nu} - \frac{1}{2}\varepsilon_{\mu\nu}\lambda_{;\rho}\lambda^{;\rho}) + (\lambda_{;\mu;\nu} - \varepsilon_{\mu\nu}\square\lambda) + 2(\omega + 1)\phi^{-1}\lambda_{;\mu}\lambda_{;\nu} - (\omega + 2)\phi^{-1}\varepsilon_{\mu\nu}\lambda_{;\rho}\lambda^{;\rho}] \quad 4.71$$

$$\text{only if } \omega(\square\lambda + \lambda_{;\rho}\lambda^{;\rho}) = 0 \text{ where } \lambda = \ln\phi, \mu = \lambda^{\frac{1}{2}}. \quad 4.72$$

In this case the BD equations (eqs. 4.68 and 4.69) become

$$G_{\mu\nu} = 8\pi\phi^{-1}T'_{\mu\nu} + \omega\phi'^{-2}[\phi'_{;\mu}\phi'_{;\nu} - \frac{1}{2}\varepsilon_{\mu\nu}\phi'_{;\rho}\phi'^{;\rho}] + \phi'^{-1}[\phi'_{;\mu;\nu} - \varepsilon_{\mu\nu}\square\phi'] \quad 4.73$$

$$\square\phi' = \frac{8\pi T'}{3 + 2\omega} \quad , \quad 4.74$$

which would follow from the action principle

$$\delta \int (\phi'R - \omega\phi'^{-1}\phi'_{;\rho}\phi'^{;\rho} + L') \sqrt{-g}d^4x = 0 \quad 4.75$$

where L' , for the new matter field is independent of ϕ' .

Thus the scalar field dependence of the new energy-momentum

tensor in eq. 4.71 appears only by construction.

From then a solution $\{g_{\mu\nu}, \phi, T_{\mu\nu}\}$, if for some $\lambda > 0$ eq. 4.72 is satisfied, another solution $\{g'_{\mu\nu}, \phi', T'_{\mu\nu}\}$ can be generated by the transformation given by eqs. 4.70 and 4.71. However, without some gauge condition imposed, O'Hanlon has shown that some physically acceptable solutions can generate solutions which are unphysical in the sense that a new energy-momentum tensor may have a negative energy density or inappropriate boundary or limit properties.

Equivalent solutions are defined (25) as those solutions which for fixed ω give the same space-time geometry. The set of equivalent solutions defined by eqs. 4.70 and 4.71 subject to the constraint eq. 4.72 is an equivalence class and some such equivalence classes can be generated from solutions of Einstein's field equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad \text{provided } T = 0 \quad . \quad 4.76$$

This can be seen when

- (a) $\omega = 0$ in which case from eq. 4.72 λ is an arbitrary space-time position dependent function and the BD equations are fully covariant under this transformation group. In particular the choice $\lambda = \phi^{-1}$ reduces eqs. 4.73 and 4.74 to the form of eq. 4.76.
- or (b) $\omega \neq 0$ and $R = 0$ in which case the choice $\lambda = \phi^{-1}$ is again acceptable as eq. 4.72 becomes a field equation derivable from eqs. 4.68 and 4.69 (by contraction).

Two examples (25) based on space-times with Robertson-Walker metrics

$$\text{i.e. } ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\psi^2 \right]$$

$$k = 0, \pm 1,$$

illustrate some of the possible matter - BD gravitational fields which satisfy the field equations for a given space-time.

- (i) Referring to the vacuum solution (3)
(Sec. 4.2)

$$a(t) = (D-kt^2)^{\frac{1}{2}}, \quad D > 0, \quad k = \pm 1,$$

where for $k = +1$ the 3-space contracts to a singularity in finite time and for $k = -1$ the 3-space expands indefinitely from an initially non-singular state.

Also $\phi(t) = ta^{-1}$

$$T_{\mu\nu} = 0, \quad \text{all } \mu, \nu,$$

and $\omega = 0$, implying by eq. 4.72 that λ is arbitrary. Eqs. 4.70 and 4.71 give a family of perfect-fluid type cosmological models with

$$\rho' = -3k(\partial\pi)^{-1}a^{-3}t^{2\lambda}, \quad \lambda = \frac{\partial\lambda}{\partial t}$$

$$p' = -(\partial\pi)^{-1}(at)^{-2}t^{2\lambda}$$

and $\phi' = \lambda ta^{-1}$.

Particular choices of λ (25)

- (a) $\lambda = t^{-1}$ gives a dust model

$$\rho' = 3k(\partial\pi)^{-1}a^{-3}$$

$$p' = 0,$$

with a varying gravitational constant implied by $\phi' = a^{-1}$.

- (b) $\lambda = (Dt)^{-1}a$ gives a radiation model

$$\rho' = 3p'$$

$$= 3k(\partial\pi)^{-1}a^{-4},$$

with $\phi' = \text{constant}$.

Thus for the given space-time the BD equations do not distinguish between the case with no matter present and with the scalar field defined by

$$\phi(t) = ta^{-1},$$

and the case where a particular radiation field is present while the scalar field is constant.

For either model, (a) or (b), the energy density for the matter distribution is positive or negative according to whether $k = +1$ or -1 respectively.

(ii) Referring to the BD flat space dust model (See Sec. 4.3)

$$a = a_0 t^q, \quad k = 0$$

$$\rho = \rho_0 t^{-3q}$$

$$p = 0$$

$$\text{and } \phi = \phi_0 t^r, \quad r = 2/(4 + 3\omega), \quad q = r(1 + \omega).$$

Eq. 4.72 gives (25)

$$\lambda = \kappa^2 t^{-2} (1 + k_1 t)^2, \quad \kappa, k_1 \text{ constants}$$

and the new density and pressure are, (25),

$$\rho' = \kappa^2 \rho_0 t^{r-4} [k_1 t^2 - 3(\omega + 1)^2 (3 + 2\omega)^{-1}]$$

$$p' = -\kappa^2 \rho_0 t^{r-4} [(4 + 3\omega)(\omega + 1)(3 + 2\omega)^{-1}]$$

$$\text{and } \phi' = \kappa^2 \phi_0 t^{r-2} (1 + k_1 t)^2.$$

The energy density is positive for

$$t^2 > 3k_1^{-2} (\omega + 1)^2 (3 + 2\omega)^{-1}.$$

The pressure is negative* for $\omega > -3/2$.

* It is interesting to note that some authors regard negative cosmological pressure as acceptable suggesting that this pressure need not be given the usual kinetic interpretation
 ———— McCrea (43), Pachner (44)

Although this model is valid for all positive ω the limit as $\omega \rightarrow \infty$ does not exist for some parameters and consequently the model does not have a counterpart in General Relativity.

Finally one notes that if the metric is written in terms of cartesian coordinates then a more general solution of eq. 4.72 is

$$\lambda = \lambda_0 t(Lx + My + Nz) \quad L, M, N \text{ constants,}$$

for which $T'_{\mu\nu}$, $\mu \neq \nu$, would have, in general, non-zero components.

REMARKS

Because there are few exact solutions of eqs. 4.68 and 4.69 to work with, the gauge freedom between the scalar and matter fields poses two problems. The main one has to do with interpretation, while the second is related to a remark (25) that "it is possible that some as yet unknown gauge condition will select out of each equivalence class the solution describing the "real" physical situation". In contrast one could ask :

given a solution that describes a "real" physical situation, which other equivalent solutions give equally "real" physical situations.

For the case $\omega = 0$, λ is arbitrary and the space-time geometry does not even in a qualitative manner determine the matter content as demonstrated by examples (a) and (b) in which both dust and radiation models of matter are consistent with the geometry.

So far it has been convenient to interpret the BD gravitational field equations in terms of two general ideas — the BD gravitational field with the metric field component determining the space-time geometry while the scalar field component determines the locally measured gravitational constant and the matter content of a space-time as a local source of the BD field. The important gravitational aspect of the matter content is the energy-momentum tensor, $T_{\mu\nu}$, that can be assigned to it. $T_{\mu\nu}$ is locally defined in terms of the dynamical interaction of the matter field with the metric field and for simple matter fields (e.g. fluid , or electromagnetic) $T_{\mu\nu}$ is continuous in finite (or infinite) regions. For these cases a space-time could be divided into separable regions characterised by vanishing or non-vanishing matter content. In the former regions the vacuum BD field equations hold

and in the latter regions the full BD equations hold. Continuing the interpretation, the transformation given by eqs. 4.70, 4.71, and 4.72 relates some possible matter and BD gravitational fields that could co-exist in regions of a given space-time and suggests that unless some restriction is put on the form, or boundary or limit properties of the energy-momentum tensor, different matter fields could be consistent with the one space-time region.

A similar, but in other respects different, problem arose in the cosmology sections where once the Robertson-Walker line element was given for a space-time model of the Universe, further assumptions had to be made about the functional dependence of the energy-momentum tensor of the Universe and of the scalar field.

Thus because the interpretation of a solution as describing a physical situation depends on which covariantly conserved second rank tensors could be specified as being the energy-momentum tensors of physical matter fields, a general criterion beyond interpretation for determining the physical solutions of an equivalence class seems doubtful. Moreover, for an empty space-time general features such as the sign of the energy density of matter fields compatible with the space-time, seem awkward to find. In short one is not sure what can be put into a space-time without disturbing the pre-existing geometric relations.

The scalar-matter field gauge freedom expressed by eqs. 4.70, 4.71 and 4.72 does not seem to apply in the linearized weak field limit. For in this limit, using the result of Quale, ((5), Sec. 4.), one can choose a gauge consistent with the symmetry group of the theory so that the scalar field variable is effectively "absorbed" into the trace of the metric field variable. The gravitational field is then described by a tensor field with the symmetry group defined by the reduced Weyl group. This conclusion suggests that to interpret eqs. 4.70, 71, 72 as (25), " a scale change in the unit of mass which preserves the conservation laws " would involve a rather arbitrary classification

of matter sources; for example strong and weak, before such a scaling could be applied.

Despite this doubt about interpretation the writer feels it is appropriate to refer to the group of transformations as the mass-gauge group (26) with the understanding that the group is not a covariance group (Sec. 3.1) because under the group dpt are mapped into dpt but the map of kpt is undefined.

The origin of the group seems to be related to the ideas that the representation of the BD theory used is metric (and universally coupled) with EMG as the complete covariance group and that the scalar field is a long range field. The first idea means that $\Gamma_{\mu\nu}^{\lambda}$ is independent of the scalar field and satisfies the only Bianchi type identities

$$T^{\mu\nu}_{;\mu} = 0$$

while the second idea excludes from the action principle for the representation a cosmological term which would have forbidden transformations of the kind eqs. 4.70 - 4.72. Such a term would give the scalar field a non-zero rest mass and so a finite range.

In view of these comments a mass-gauge group should exist for the metric representation of any scalar-tensor theory involving a long range scalar field.

TABLE I. The four types of solutions for a charged mass point.

	Parameters	ϕ	$e^{\alpha/2}$	$e^{\beta/2}$
I	$B^2 > 0, C > 0,$ $\frac{A}{\lambda} = \frac{g_0}{2CB}, \lambda^2 = (1+A)^2 - A(1 - \frac{1}{2}\omega A),$ $K^2 = \frac{4\pi\epsilon_0^2}{c^4 g_0^2} \frac{\phi_0 A^2}{(1 + \frac{1}{2}A)^2}$	$\phi_0 \left(\frac{\rho - B}{\rho + B} \right)^{A/\lambda}$	$\frac{e^{\alpha_0/2} \left(\frac{\rho - B}{\rho + B} \right)^{1/\lambda}}{1 - \frac{K^2 e^{\alpha_0}}{4} \left(\frac{\rho - B}{\rho + B} \right)^{2(1+A/2)/\lambda}}$	$e^{\beta_0/2} \left(1 + \frac{B}{\rho} \right)^2 \left(\frac{\rho - B}{\rho + B} \right)^{(\lambda - A - 1)/\lambda}$ $\times \left[1 - \frac{K^2 e^{\alpha_0}}{4} \left(\frac{\rho - B}{\rho + B} \right)^{2(1+A/2)/\lambda} \right]$
II	$B^2 < 0, b^2 = -B^2, C > 0,$ $\frac{A}{\lambda} = \frac{g_0}{2Cb}, \lambda^2 = (1 - \frac{1}{2}\omega A) - (1+A)^2,$ $K^2 = \frac{4\pi\epsilon_0^2}{c^4 g_0^2} \frac{\phi_0 A^2}{(1 + \frac{1}{2}A)^2}$	$\phi_0 \exp \left[\frac{2A}{\lambda} \tan^{-1}(\rho/b) \right]$	$\frac{\exp \left[\frac{\alpha_0}{2} + \frac{2}{\lambda} \tan^{-1}(\rho/b) \right]}{1 - \frac{1}{4} K^2 \exp \left[\alpha_0 + \frac{4}{\lambda} (1 + \frac{1}{2}A) \tan^{-1}(\rho/b) \right]}$	$\left(\frac{\rho^2 + b^2}{\rho^2} \right)^2 \exp \left[\frac{\beta_0}{2} - \frac{2}{\lambda} (1+A) \tan^{-1}(\rho/b) \right]$ $\times \left\{ 1 - \frac{1}{4} K^2 \exp \left[\alpha_0 + \frac{4}{\lambda} (1 + \frac{1}{2}A) \tan^{-1}(\rho/b) \right] \right\}$
III	$B^2 = 0, C > 0,$ $\frac{a}{b} = \frac{g_0}{C}, a = \frac{-1 \pm (-3 - 2\omega)^{1/2}}{\omega + 2},$ $K^2 = \frac{4\pi\epsilon_0^2}{c^4 g_0^2} \frac{\phi_0 a^2}{(1 + \frac{1}{2}a)^2}$	$\phi_0 e^{-a/b\rho}$	$\frac{\frac{\alpha_0}{e^2} - \frac{1}{b\rho}}{1 - \frac{1}{4} K^2 \exp \left[\alpha_0 - \frac{2}{b\rho} (1 + \frac{1}{2}a) \right]}$	$\frac{\beta_0}{e^2} + \frac{(1+a)}{b\rho} \left\{ 1 - \frac{1}{4} K^2 \exp \left[\alpha_0 - \frac{2}{b\rho} (1 + \frac{1}{2}a) \right] \right\}$
IV	$B^2 = +\infty, C = 0, CB^2 = \text{const},$ $\frac{a}{b} = \frac{g_0}{CB^2}, a = \frac{-1 \pm (-3 - 2\omega)^{1/2}}{\omega + 2},$ $K^2 = \frac{4\pi\epsilon_0^2}{c^4 g_0^2} \frac{\phi_0 a^2}{(1 + \frac{1}{2}a)^2}$	$\phi_0 e^{-a\rho/b}$	$\frac{\frac{\alpha_0}{e^2} - \frac{\rho}{b}}{1 - \frac{1}{4} K^2 \exp \left[\alpha_0 - \frac{2}{b} (1 + \frac{1}{2}a) \rho \right]}$	$\frac{b^2}{\rho^2} \exp \left[\frac{\beta_0}{2} + \left(\frac{1+a}{b} \right) \rho \right] \left\{ 1 - \frac{1}{4} K^2 \exp \left[\alpha_0 - \frac{2}{b} (1 + \frac{1}{2}a) \rho \right] \right\}$

4.8 Nordstrom-Reissner Type Solutions

The static solution quoted here is analogous to the Nordstrom-Reissner solution in General Relativity. It is derived from the BD field equations with the energy-momentum tensor given by the Maxwell stress tensor

$$T^{\mu\nu} = F^{\mu}_{\gamma} F^{\nu\gamma} - \frac{1}{4} g^{\mu\nu} F_{\gamma\delta} F^{\gamma\delta} \quad 4.77$$

where $(\sqrt{-g} F^{\mu\nu})_{,\nu} = 0$, 4.78

and with an appropriate identification of the integration constants the solution can be interpreted to describe the BD gravitational field of a charged, mass point source.

With the line element in isotropic coordinates

$$ds^2 = -e^{\alpha(\rho)} c^2 dt^2 + e^{\beta(\rho)} [d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\psi^2]$$

the solution of eq. 4.78 in matrix form is

$$F^{\mu\nu} = \frac{\epsilon_0}{\rho^2} e^{-(\alpha + 3\beta)/2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \epsilon_0 \text{ a constant,}$$

while the metric and scalar fields have been solved for 4.79
by Luke and Szamosi (29) who give four types of solution. These are stated in Table I and the classification is according to the relationships of the parameters B and C

Eqs. 4.78 and 4.79 imply that the radial electric force component is (27)

$$E_{\rho} \equiv e^{(\alpha + \beta)/2} \frac{\epsilon_0}{\rho^2} \quad . \quad 4.80$$

Thus ϵ_0 can be identified with the electric charge of a point source at the origin ($\rho = 0$); for if Lorentz conditions are imposed at spatial infinity

i.e. $\alpha \rightarrow 0$
and $\beta \rightarrow 0$
as $\rho \rightarrow \infty$,

then eq. 4.80 goes over to the classical form for a point charge.

With $\epsilon_0 = 0$ solution I becomes (apart from a trivial redefinition of constants) the Brans Type I vacuum solution (Sect. 4.2).

Similarly solution II becomes the Brans Type II vacuum solution. Curiously, in the same manner, if the numerators for $e^{\alpha/2}$ of solutions III and IV are replaced as

$$e^{\alpha_0/2} - \frac{1}{b\rho} \longrightarrow \exp\left[\frac{\alpha_0}{2} - \frac{1}{b\rho}\right] ,$$

$$e^{\alpha_0/2} - \frac{\rho}{b} \longrightarrow \exp\left[\frac{\alpha_0}{2} - \frac{\rho}{b}\right] , \text{ respectively}$$

and also for solution III, $e^{\beta/2}$ is replaced as

$$e^{\beta_0/2} + \frac{1+a}{b\rho} \left\{ 1 - \frac{1}{4} K^2 \exp\left[\alpha_0 - \frac{2}{b\rho} (1 + \frac{1}{2}a)\right] \right\} \longrightarrow$$

$$\exp\left[\frac{\beta_0}{2} + \left(\frac{1+a}{b\rho}\right)\right] \left\{ 1 - \frac{1}{4} K^2 \exp\left[\alpha_0 - \frac{2}{b\rho} (1 + \frac{1}{2}a)\right] \right\} ,$$

then solution III becomes the Brans Type IV vacuum solution and solution IV becomes the Brans Type III vacuum solution. ξ_0 from their derivation (28) is an integration constant, which if zero gives a constant scalar field. In this case solution I appears to be the only physically reasonable solution and they show that it is the Nordstrom-Reissner solution (in isotropic coordinates) of General Relativity.

Further, if $\epsilon_0 = 0$ then the Schwarzschild solution (in isotropic coordinates) is obtained with the identification

$$B = \frac{GM}{2c^2} , G_{11} = \frac{\kappa^2}{\phi_0} , \kappa^2 = \frac{2\omega + 4}{2\omega + 3} ,$$

Where G is the locally measured gravitational constant and M is the mass of the point source.

However, in general for solution I, if Lorentz conditions are imposed at spatial infinity

$$\text{i.e. } \phi \rightarrow \frac{\kappa^2}{G_0}$$

$$\alpha \rightarrow 0$$

$$\text{and } \beta \rightarrow 0, \text{ all as } \rho \rightarrow \infty,$$

then the constants B and G_0 remain undetermined and in this respect the solution is incomplete.

Finally, Buchdahl (9) has independently considered the same problem. By first simplifying the action integral for the problem, he obtains a relatively simple set of equations for which he gives exact solutions. The integration constants are related to the source parameters - electric charge, active and passive masses, by comparing the asymptotic form of the solutions, at spatial infinity, with the linearized weak-field limit solutions. Such a comparison would presumably hold if the linearized weak-field limit solution is valid also near the source.

4.9

ADM Formulation in the BD Theory

Briefly, this method has so far been used in the BD theory for the investigation of two areas.

- (a) the solutions of "point like" sources
- and (b) the dynamics of expanding homogeneous models of the Universe.

(a) The approach here, following ADM (29), is to construct a physical model of a point source by considering a spherical-shell distribution of pressureless dust and allowing the coordinate radius to tend to zero. Such a source is point-like to the extent that the coordinate

radius vanishes and the proper circumference vanishes, while depending on the fields present the proper radius may be finite.

The resulting initial-value problem has been discussed by Toton (30, 30a, 30b) and in a slightly different context by Deser and Higbie (31) and Higbie (32). Rather than making full use of their papers, their conclusions relating only to the Nordstrom-Reissner and Schwarzschild type solutions will be stated.

Denoting the source parameters by

- m_0 : the mass of the source in the absence of gravitation,
- ϵ_0 : the charge of the source,
- ϵ : the source's coordinate extension,

it is found (31) that for an uncharged source, a unique, nonsingular static solution exists for every set of source parameters (m_0, ϵ) and coupling constant ω , and in the limit as $\omega \rightarrow \infty$ the General Relativistic ~~case~~ results. Also, as $\epsilon \rightarrow 0$ the total mass of the source vanishes (as in General Relativity) and the invariant or proper radius of the source vanishes (unlike in General Relativity)

For a charged source a restriction (32)

$$|\epsilon_0| < \frac{1}{2} G^{\frac{1}{2}} m_0$$

is needed for the existence of static solutions and so in this case there is not a unique solution for every set source parameters ($m_0, \epsilon_0, \epsilon$).

In the limit as $\omega \rightarrow \infty$ the solutions go over to the Nordstrom-Reissner solution of General Relativity. However these "limit" solutions still satisfy the above restriction in contrast to the situation in General Relativity.

(b) Likewise for this area, little more than references will be given. The anisotropic generalisations of the BD Friedmann models for $k = 0$ and $k = +1$,

that is the anisotropic models of Bianchi type I and IX have been independently discussed by Nairai (33) and Matzner, Ryan, and Toton (34). However, because similar problems have been different treatments, a detailed comparison of their methods would be necessary for a suitable discussion.

4.10 BD Theory and Mach's Principle

A purpose of this Chapter has been to bring together references for a range of topics and in so doing detail and depth have sometimes been neglected while discussion has sometimes been minimal. In particular the field theoretic construction of the BD theory needed further discussion if only because this method may have proved interesting when it was related to the scalar-tensor model looked at in Chapter Three. The Hamiltonian approach to BD cosmology also deserves more study. Some applications of the BD theory to different cosmological and astrophysical problems have been discussed by Nairai (35, 36) but not mentioned here. The special topic, the integral formulation of the BD theory, Nairai (37, 38), has also been excluded.

The topics chosen were not really involved applications of the gravitational field equations and almost all of them have counterparts in General Relativity. In this way the BD theory was presented mainly as a departure from General Relativity.

In section 4.1 the role of the scalar field was observed to be restricted to the gravitational field equations and so once the metric field has been solved for, the gravitational effects on physical systems follow in the same way as they do in General Relativity - via the equivalence principle or by means of universal coupling. One can take a problem in General Relativity, work out an analogous one in the BD theory and for many cases the BD

solution is well-defined for all positive ω and in the limit for large ω this solution goes over to the corresponding solution in General Relativity. However, a problem characteristic to the BD theory arose when one is given the functional form of the metric tensor and one solves for the scalar field and/or the components of the energy-momentum tensor.

For the vacuum situation it was possible to have a curved, non-singular space-time with a related non-uniform scalar field. Thus a gravitational field would be observed on the basis of the equations for geodesic deviation and it is natural to assume that the source of this field is the scalar field. Another peculiarity of the scalar field is that its symmetry properties may help to define preferred coordinate systems. In some respects then, the scalar field can show the behaviour of a matter field (one notes that it is only in the representation with gravitational units, \bar{G} constant, that the scalar field can in general be fully treated as a matter field)- for the special space-time (example (b) page 73 Section 4.5) the scalar field was indistinguishable from a radiation field in so far as gravitational effects are concerned. Because vacuum solutions exist, the scalar field can exist in the absence of matter and the interpretation given in section 4.1 that the scalar field is in a sense generated by the matter field of the Universe needs slight modification.

For the cosmological situation the introduction of the scalar field was accompanied by additional assumptions. It was noted that the symmetries of the space-time were not sufficient to determine the functional form of either the scalar field or the components of the energy-momentum tensor of the Universe*. There was no substantial difference between the BD and Einstein-Friedmann models and for reasonable values of ω these differences would be practically undetectable.

* the usual form may be arrived at by kinetic considerations but the concern here is with what the field equations can give.

That the BD theory is basically a departure from General Relativity has been stressed in the interpretation of the BD gravitational field as having a scalar field component. The scalar field through the field equations has two consequences for matter fields. A perhaps un-Machian division has been made between matter fields because electromagnetic or radiation fields cannot be direct sources for the scalar field. The second consequence is that an ambiguity arises when one tries to determine an energy-momentum tensor as a source term for a given Einstein tensor. The energy-momentum tensor is not always unique and this suggests that there may be problems if a matter gravitational source is to be deduced from the motions of for example, a planet.

The main feature of the BD theory is a variable gravitational constant which the structure of the theory makes dependent on the arbitrary parameter ω . It appears that experiment can do little more than to put a lower bound on ω and upper bound on the variation of the gravitational constant.

Brans and Dicke (1) claim that their theory is more compatible than General Relativity with Mach's Principle.

For both the BD theory and General Relativity an inertial frame may be defined as a locally freely falling coordinate system - so an inertial frame is determined by the local metric field which is itself determined by the matter field of the Universe. In such a frame the laws of special Relativity hold (at least to first order) as required. If local space-time inhomogeneities can be neglected the cosmological background geometry determines the class of inertial frames - and inertial effects now appear as a result of accelerations relative to this cosmological background. Mach's Principle goes beyond these ideas and it is sufficient for the needs here to use the statements of Dicke (39, 40)

- (1) that "the inertial frame ought to be uniquely determined by the matter distribution of the Universe"
- and (2) that "the inertial reaction may be considered to be of gravitational origin".

By the previous comments the first idea receives the same support in the BD theory as it does in General Relativity. However, the idea may be slightly more evident in the BD theory than in General Relativity. Brill (41) has shown that the rotation of a local inertial frame brought about by a rotating mass shell, is in the weak field approximation the same as in General Relativity (the Lens-Thirring effect) except for a factor depending on ω . But the conditions for the limit of perfect dragging are the same for both theories.

The second idea requires a little elaboration. A body accelerating relative to the cosmological background experiences inertial forces and the suggestion is that these forces are actually gravitational. In the local inertial frame of the body the cosmological background appears **anisotropic** and inhomogeneous and the implied redistribution of the background matter field exerts on the body, gravitational forces which by (2) are identified with the inertial forces. This equivalence of inertial and gravitational forces leads to a relation (attributed to Sciama (42) e.g. in Brans and Dicke (1)),

$$\frac{GM}{Rc^2} \sim 1 \quad , \quad 4.81$$

where M is the mass and R is of radius of the observable Universe.

In a closed expanding Universe this relation can admit a variety of interpretations, notable among them being Dirac's hypothesis (in a strict form) with

$$G \sim t, \quad M \sim t^{2*}, \quad R \sim t,$$

and the two representations of the BD theory in which (a) G is a constant and the mass distribution $\frac{M}{R}$ is determined

* the number of particles in the Universe, a measure of its total mass, is a dimensionless number of the order of 10^{80} and by the comments in Chapter Two $M \sim t^2$, where t = age of Universe/atomic unit of time

by the field equations and/or boundary conditions,

or (b) G is variable and a function of $\frac{M}{R}$

For the representation of the BD theory characterised by constant gravitational units, the mass of a particle is a function of the scalar field and hence mass becomes dependent on distant matter. One could hope that through the scalar field the inertia of a body would be completely due to the matter of the Universe. A measure of the inertia of a body is its self-energy and for General Relativity the self-energy of a neutral point particle is zero while the self-energy of a point charge is equal to the magnitude of its charge (29). For the BD representation **Toton** (30, 30a) has shown that the situation is unchanged and more importantly that if the boundary value of the scalar field is changed by introducing matter at infinity then the situation still remains the same.

The general observation is that Mach's Principle is no more compatible with the BD theory than it is with General Relativity.

Finally in the absence of decisive experimental evidence against the BD theory it seems important to ask if in any sense the BD theory is an advance on Einstein's theory of gravitation. In comparison to Einstein's theory the BD theory involves no new physical principles (except to replace SEP by EEP.). However, the BD theory violates WEP for massive physical systems. In these respects there is not much justification for the BD theory as a departure from Einstein's theory.

Chapter Five

SPECIAL SCALAR-TENSOR THEORIES

5.1 Gursey's theory

Although Gursey's theory (1) is formally similar to the BD theory for $\omega = -\frac{3}{2}$ (or Deser's theory without the scale-invariance breaking term) his theory involves a scalar density field rather than a scalar field and consequently it is not strictly a scalar-tensor gravitation theory. However the theory deserves to be mentioned because it is based on interpretations of Mach's Principle similar to those used by Brans and Dicke, with the difference that the theory is constructed within the framework of General Relativity.

The discussion of Mach's Principle in Section 4.10 proceeded by separating the space-time geometry into two parts - one due to the cosmological matter field (i.e. distant matter sources) and the other due to local matter distributions. Gursey chose the cosmological background to have a de-Sitter geometry and an inertial frame is then determined to within the de-Sitter group. Transformations of this group acting on an inertial frame always give inertial frames. However inertial forces appear in a non-inertial frame that is derived from an inertial frame by coordinate transformations not belonging to the de-Sitter group. Mach's Principle is expressed as before, to say that these forces are gravitational - in fact due to the redistribution of the matter of the Universe implied by the change of frame.

The de-Sitter metric can be put into a conformally flat form (1)

$$ds^2 = \Phi^2(r) \eta_{\mu\nu} dx^\mu dx^\nu , \quad 5.1$$

and for an actual space-time with metric $g_{\mu\nu}$ the boundary

conditions at spatial infinity for Machian solutions (in General Relativity) are stated in an inertial frame to be (1),

$$\varepsilon_{\mu\nu} \rightarrow \Phi^2(\tau)\eta_{\mu\nu} \quad . \quad 5.2$$

Gursey rewrites the metric tensor as

$$\varepsilon_{\mu\nu} = \phi_G^2 \gamma_{\mu\nu} \quad , \quad 5.3$$

where ϕ_G is a scalar density of weight $\frac{1}{4}$ and $\gamma_{\mu\nu}$ is a tensor density of weight $-\frac{1}{2}$. The boundary conditions can be restated then as

$$\left. \begin{array}{l} \phi_G \rightarrow \Phi \\ \text{and } \gamma_{\mu\nu} \rightarrow \eta_{\mu\nu} \end{array} \right\} \quad 5.4$$

In an inertial frame Einstein's field equations, when reformulated in terms of the variables ϕ_G and $\gamma_{\mu\nu}$ are clearly going to have a form identical to the conformally invariant field equations looked at in Chapter Three. The geodesic equations can be similarly reformulated.

It turns out that the effective gravitational constant is $G\phi_G^{-1}$ rather than G and Gursey shows that the effective gravitational constant can be related to the radius and mass of the closed, de-Sitter model of the Universe (as in the Machian relation eq. 4.81). ϕ_G appears as an inertial coefficient in the equations of motion of a test particle moving against the de-Sitter background. Hence the inertial mass of the particle depends on the cosmological background in a way which is more satisfactory than in the analogous situation for the BD theory where mass depends on the BD scalar field. There, a coordinate transformation taking an inertial frame to a non-inertial frame has no effect on the inertial mass of the particle despite the redistribution of matter seen by the particle. For Gursey's theory such a transformation alters the inertia because ϕ_G is a scalar density and makes the inertial mass of the particle more dependent on the cosmological background.

$$* \quad | \det \gamma_{\mu\nu} | = 1$$

Interestingly (1), for a particle in an otherwise empty Universe the boundary conditions (eq. 5.4) are sufficient to make its inertial mass vanish.

Gursey's reformulation of General Relativity then, has important Machian properties and it suggests further developments within General Relativity such as using more general cosmological backgrounds against which to consider inertial reactions and gravitational forces. However in the context of this thesis Gursey's theory may be considered to represent a turning point.

Gravitation in his theory is described by two dynamical variables, a scalar density and a tensor density. He found that the scalar density field leads to repulsive forces for the cosmological background (and so to expanding models of the Universe) while departures of the tensor field from the Lorentz metric lead to attractive gravitational forces. Thus for local gravitational interactions the important dynamical variable is the tensor density.

With this tensor density defining a metric on the space-time, the interval between two neighbouring events will not be invariant under the de-Sitter group and so the resulting geometry will not be Riemannian. In fact, except for the restriction to the inertial frames, this geometry is very similar to a formulation of Weyl's geometry given by Bergmann (2). Bergmann made the physical metric, $\xi_{\mu\nu}$ a tensor density by imposing a "normalising" condition

$$\sqrt{-g} = 1 \quad ,$$

which was to be invariant under the full group of coordinate transformations. For the construction of covariants in Weyl space-time this formalism is equivalent to the usual formulation given in Chapter Six.

5.2 Massive Brans-Dicke Theories

A feature of the BD theory is the existence of the mass gauge group. In section 4.7 it was suggested that this is a general property of a metric (and uni-versally coupled) representation of the scalar-tensor model provided there is no scalar field self interaction term in the action principle (i.e. $f_3 = 0$ in eq. 3.29). The action of the group is to generate from a solution $\{\epsilon_{\mu\nu}, \phi, T_{\mu\nu}\}$ of the field equations an equivalence class of solutions, each member of which has the same metric field, and it is quite possible that an equivalence class may contain more than one physically acceptable solution (e.g. for the BD theory examples are given in Section 4.7). This situation it was suggested is at least partly due to the complete covariance group being MMG. For if the complete covariance group is extended (e.g. by taking the direct product with the conformal group) the additional Bianchi-type identities may lead to each equivalence class having only one element. Another contributing factor is the scalar field being a long range, massless field. The purpose of this section is to introduce the massive scalar field and in particular to look very briefly at the family of massive BD theories.

The scalar field, ϕ with finite mass m , satisfies (in contrast to eq. 4.1) the field equation

$$\square \phi - m^2 F(\phi) = 4\pi\lambda T \quad , \quad 5.5$$

where F is a scalar function of the scalar field and λ is a coupling constant. Following the method of Weinberg, (3), (eq. ? cf. eqs. 4.1 \rightarrow 4.6) Acharya and Hogan, (4), show that the desired field equations are

$$\begin{aligned} G_{\mu\nu} = & 8\pi\phi^{-1} T_{\mu\nu} + \omega\phi^{-2} (\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}\epsilon_{\mu\nu}\phi_{;\rho}\phi^{;\rho}) \\ & + \phi^{-1} (\phi_{;\mu;\nu} - \epsilon_{\mu\nu}\square\phi + 3\pi\epsilon_{\mu\nu}E(\phi)) \end{aligned} \quad 5.6$$

$$\square\phi - m^2 F(\phi) = \frac{8\pi}{2 + \frac{2\omega}{2\omega}} T \quad , \quad 5.7$$

where $\omega = \frac{1}{\lambda} - \frac{3}{2}$, and E is a scalar function of the scalar field satisfying

$$E' - 2\phi^{-1}E = \frac{-m^2}{16\pi} (3 + 2\omega)\phi^{-1}F(\phi) \quad , \quad E' \equiv \frac{dE}{d\phi} \quad . \quad 5.8$$

For these equations to reduce to the BD equations when $m = 0$ one requires $E(\phi_0) = 0$, where $\phi_0 \equiv G^{-1}$, 5.9

and in addition for these equations to reduce to Einstein's field equations one requires $F(\phi_0) = 0$ 5.10

Eqs. 5.6, 5.7, 5.8, subject to 5.9 and 5.10, describe a family of massive BD theories. Their predictions (4) for the classical tests are identical to the predictions of General Relativity and so no lower bound can be set on ω .

O'Hanlon, (5), has considered the special case where

$$\omega = 0 \quad 5.11$$

$$\text{and } F(\phi) = \phi - \phi_0 \quad . \quad 5.12$$

In the weak field limit the modified Newtonian gravitational potential turns out to be

$$V(r) = -(Gm/r) \left(1 + \frac{1}{3} e^{-mr} \right) \quad 5.13$$

and it is the presence of the exponential term that gives a short range component to the Newtonian gravitational force.

Non-Newtonian behaviour of this kind was first proposed by Fujii, (6, 7) as evidence for a dilaton with mass m . Such a particle with zero mass is associated with scale or dilation invariance in particle physics, (8), and by allowing non-zero mass this symmetry is broken. Based on present observations Fujii, (6), puts restrictions on m^{-1} as

$$\left. \begin{aligned} 10m \lesssim m^{-1} \lesssim 1km, \\ \text{or } m^{-1} \lesssim 1cm. \end{aligned} \right\} \quad 5.14$$

Although the massive BD theory with $\omega = 0$ may not be an appropriate modification of General Relativity if the massive dilaton is observed, O'Hanlon, (5), concludes with some general comments about this theory.

Without the scalar field mass term, the field equations are form invariant under arbitrary mass-scalar field transformations (eqs. 4.70, 71, 72). With the mass term included this invariance is broken with an associated massive scalar field and this situation has a formal counterpart in particle physics with scale invariance breaking being related to the massive dilaton. The suggestion is that scale invariance is related to the mass gauge group rather than with the conformal group.

One notes that conformal invariance breaking in the scalar-tensor model led to the Deser scalar-tensor theory in which the range of the scalar field was cosmological and did not satisfy the restrictions, eq. 5.14.

5.3 Other Scalar-Tensor theories

It is beyond the scope of this thesis to make a comparative study of the individual scalar-tensor theories but what can be done is to briefly take up a reference made in Chapter Two to Harrison's paper, (9).

One can write a simplified version of the action principle for the scalar-tensor model as

$$\delta \int \left[\phi^A R - \zeta \phi^A - 2 \phi_{, \alpha} \phi^{, \alpha} + 16\pi \phi^B L_M \right] \sqrt{-g} d^4x = 0, \quad 5.15$$

where ζ is a coupling constant. Values for the constants A and B generate a family of scalar-tensor theories which includes the more important theories that have been suggested. Some of the action principles which have

been used can be summarized following (9), by a Table (see below), and depending on the magnitude of the constants A and B and the form of L_M one may have representations characterised by a combination of features such as

1. test particles moving on geodesics
2. world-lines intersecting a hypersurface not being conserved - i.e. mass or particle creation,
3. the gravitational constant being variable,

or their alternatives .

Author :		A	B
Scherffer	(10)	2	= 0
Deser and Pirani, Deser, Anderson	(11,12,13)	2	0
Jordan	(14)	1	2
Hoyle, Yilmaz	(15,16)	1	1
Brans & Dicke	(17)	1	0
Dicke	(18)	0	0
Jordan	(19)	-1	0

Harrison showed that an action principle for one theory could be obtained from the action principle for another theory by a suitably defined conformal transformation and in this way the various scalar-tensor theories could be formally identified with each other and with General Relativity. However this does not mean that the scalar-tensor theories (in the Table) are equivalent because different interpretations are given to the matter Lagrangians - thus non-conservation of the energy-momentum tensor may not for some theories involve the creation of matter but rather a rescaling of the

matter variables. This is illustrated by looking at a connection established by Morganstern, (20), between representations of the BD theory and the class of Jordan theories.

If B is put equal to unity, eq. 5.15 becomes with $\xi = \phi/\phi_0$, $\eta \equiv \Lambda$

$$\delta \int \left\{ \phi_0^{\eta} \xi^{\eta} \left[R - \zeta \xi^{-2} \xi_{,\alpha} \xi^{,\alpha} + 16\pi \phi_0^{-1} \xi L_M^J \right] \right\} \sqrt{-g} d^4x = 0, \quad 5.16$$

which with a slight change of notation is identical to Jordan's action principle (e.g. Brill, (21)). The action principle is formally invariant under a Pauli conformal transformation (21)

$$\left. \begin{aligned} \eta' &= \eta - \gamma & \xi'_{\mu\nu} &= \xi^{\gamma} \xi_{\mu\nu} \\ \zeta' &= \zeta - 3\eta\gamma - 3/2\gamma^2 & L_M^J &= \xi^{-\gamma} L_M^J \end{aligned} \right\} 5.17$$

and so effectively the class of Jordan theories is described by one parameter (η).

The action principle for the BD theory is

$$\delta \int \left\{ \phi_0 \lambda R - \omega \phi_0 \lambda^{-1} \lambda_{,\alpha} \lambda^{,\alpha} + 16\pi L_M^{BD} \right\} \sqrt{-g} d^4x = 0,$$

where $\phi = \phi_0 \lambda$, ϕ_0 constant, and L_M^{BD} is the matter Lagrangian. Under a units transformation one has (cf. Section 4.4)

$$\left. \begin{aligned} \xi_{\mu\nu} &\rightarrow \bar{\xi}_{\mu\nu} = \lambda^{(1-\alpha)} \xi_{\mu\nu} \\ m &\rightarrow \bar{m} = \lambda^{-(1-\alpha)/2} m, \quad \alpha \text{ arbitrary} \end{aligned} \right\} 5.18$$

and the action principle becomes (20)

$$\delta \int \left\{ \phi_0 \lambda^{\alpha} \left[\bar{R} - \frac{1}{2}(2\omega + 3(1-\alpha^2)) \lambda^{-2} \lambda_{,\alpha} \lambda^{,\alpha} + 16\pi \phi_0^{-1} \bar{L}_M^{BD} \right] \right\} \sqrt{-g} d^4x = 0, \quad 5.19$$

where barred operations are performed with $\bar{\xi}_{\mu\nu}$ and the bar as in \bar{R} and \bar{L}_M^{BD} indicates their functional dependence on $\bar{\xi}_{\mu\nu}$ while the subscript \bar{M} , means that M is replaced by \bar{M} . Eq. 5.19 describes a family of representation of the BD theory parametrized by α . $\bar{\xi}_{\mu\nu}$ is the new metric field to be varied and in this respect it has no λ dependence.

and its explicit λ dependence is

$$\bar{L}_M^{BD} = \bar{L}_M^{BD} \lambda^{-(1-\alpha)/2} \quad . \quad 5.20$$

In order to identify the metric fields in a BD representation and a Jordan theory eq. 5.16 can be rewritten in a barred form

$$\delta \int \left\{ \phi_0 + \eta \bar{\xi}^\eta \left[\bar{R} - \zeta \bar{\xi}^{-2} \bar{\xi}_{,\alpha} \bar{\xi}^{\alpha} + 16\pi \phi_0^{-1} \bar{\xi} \bar{L}_M^J \right] \right\} \sqrt{-g} d^4x = 0 \quad , \quad 5.21$$

where \bar{L}_M^J has its mass subscript unbarred because in the Jordan theory mass is taken to be independent of the scalar field λ . Identification of the last terms in eqs. 5.19 and 5.21 gives

$$\bar{\xi}^\eta + \bar{L}_M^J = \bar{L}_M^{BD} = \bar{L}_M^{BD} \lambda^{-(1-\alpha)/2} \quad (\text{using eq. 5.20}) \quad ,$$

and further identification of the field variables and Lagrangians of the BD representation and the Jordan theory becomes somewhat arbitrary.

Morganstern introduces a parameter γ and makes the choice

$$\left. \begin{aligned} \lambda^{-\gamma} &= \bar{\xi}^{\eta+1} \\ \bar{L}_M^J &= \bar{L}_M^{BD} \lambda^\gamma = \bar{L}_M^{BD} \lambda^{\gamma-(1-\alpha)/2} \quad , \end{aligned} \right\} \quad 5.22$$

which satisfies eq. 5.20 and which implies for eqs. 5.19 and 5.21 that

$$\left. \begin{aligned} \eta &= -\alpha/(\alpha + \gamma) \\ \zeta &= [2\omega + 3(1 - \alpha^2)]/[2(\gamma + 2)^2] \\ \bar{\xi} &= \lambda^{-(\alpha + \gamma)} \quad . \end{aligned} \right\} \quad 5.23$$

Hence by means of eqs. 5.22 and 5.23 the BD representation (eq. 5.19) can be interconverted with the Jordan theory (eq. 5.21) and it remains now to look more closely at the matter Lagrangians.

For the BD representation \bar{L}_M^{BD} is a units transformed L_M^{BD} and so depends on both λ and α . The mass \bar{M} appearing in this matter Lagrangian depends on λ and the resulting non-conservation of the

associated energy-momentum tensor $\bar{T}_{\mu\nu}^{\text{BD}}$ does not imply the creation or annihilation of matter but just a rescaling of the matter variables.

For the Jordan theory the matter Lagrangian is a purely formal element, independent of η and it requires interpretation. In the BD form the implied non-conservation of the energy-momentum tensor cannot be related to, say, the rescaling of particle masses and so it involves creation or annihilation of matter - or more simply a change in the number of particles.

Chapter Six

THE SCALAR-TENSOR MODEL OF GRAVITATION (CTD)

Back in Chapter Three the scalar field was brought into the description of gravitation by using the framework of General Relativity. Thus the two dynamical variables for gravitation in this model were a scalar and a tensor field - the tensor field being a metric tensor defined on the space-time manifold. By means of a covariant derivative defined on the manifold, the gravitational laws were set up as second order differential equations in these variables but it was found that simple laws could not be arrived at in a natural manner because of some arbitrariness in the form and strength of the coupling of the scalar field with the tensor field. In particular, for the metric and universally coupled representation of the model (e.g. eq. 3.29) there was present in the action principle two undetermined functions of the scalar field and an arbitrary constant. One of the scalar field functions could have been discarded because it was probably important only for problems in cosmology.

The field equations for this representation would have implied that the energy-momentum tensor of matter had a vanishing divergence and so it would have followed, as in General Relativity that the equations of motion for test particles were identical to the geodesics of the metric. From WEP the paths of freely falling test particles are independent of the nature of the particles and so with these paths being the same as geodesics of a suitably curved space-time, gravitation had been geometrized. However, gravitation had become only partly geometrical because the scalar field did not appear explicitly in the components of the curvature tensor or in the geodesic equations. The scalar field in contrast to the metric field was confined to the gravitational field equations and had no fundamental geometric role.

In this Chapter a different approach is taken towards the scalar-tensor model with the result that many ideas such as length standards, conformal transformations, and the variability of dimensionless numbers are united in quite a satisfactory way.

The physical space-time manifold is considered separately as a Weyl and a Lyra manifold and in analogy with General Relativity the gravitational field equations are to be comprised in an action principle consisting basically of two parts - one giving the contribution of the gravitational fields via the "curvature function" of the manifold and the other giving the interaction with all other forms of matter and energy.

Generally, the gravitational part turns out to contain three dynamical variables - a scalar, vector and a (metric) tensor field.

The geometry of the space-time given by either manifold is unchanged for transformations belonging to MMG or to the conformal group. Gravitation theories are therefore required to be conformally invariant and the complete covariance group of these theories would be the direct product of MMG with the conformal group. Thus by generalising the geometric structure of space-time the complete covariance group of space-time theories has been enlarged in a rather natural way.

It was noted in Section 3.5 that the conformally invariant scalar-tensor model developed there could be put into a barred form which is a special case of a reduced form where the dynamical variables contain no irrelevant parts*. In the barred form the gravitational field equations appeared to be no different from those of General Relativity and at this point the identification of the two theories was implied to be complete. For

* which describe the gauge freedom due to the conformal group.

in this form the equations giving the path of a freely-falling particle would have been the geodesic equations of a metric and this metric would have been identical to the tensor variable occurring in the barred form of the field equations. It remained just to assert that the metric defined an interval between neighbouring events which is the one measured by atomic systems.

The simplest gravitation theories that can be set up on the Weyl or Lyra manifold are similar to the conformally invariant scalar-tensor model in the respect that there is a reduced form of their field equations which is the same as Einstein's field equations (with or without a cosmological constant).

Again, as with Riemann space-time the non-gravitational properties (e.g. rest mass, spectral frequencies) of atomic systems are assumed to be independent of position and past history in the space-time manifold and one can therefore measure an absolute interval between neighbouring events. It follows however that the related "absolute" metric is not determined by a gravitation theory because the physical laws are conformally invariant and thus their metric variable is specified only up to a position dependent factor. One concludes then that it is an arbitrary choice to identify a reduced metric variable of the field equations or of the equations of motion for physical systems, with the metric determined by atomic systems.

Depending on the choice, one can arrive at theories similar to Hoyle and Narlikar's theory (1) with creation of matter, or to the Brans-Dicke theory with a variable gravitational constant or as above, to Einstein's theory with matter conserved and the gravitational constant a true constant.

A global formulation of the Weyl manifold has been given by Folland, (2), and for the Lyra manifold by

Sen and Vanstone, (3); the latter two authors pointing out that both manifolds are instances of differentiable manifolds with more general linear connections. From a linear connection one can define a curvature tensor, a covariant derivative and a family of distinguished curves on a manifold. Hence Sen and Vanstone's approach will be useful to briefly give non-rigorous and comparative descriptions of each manifold as a space-time manifold.

6.1 Linear connections - definitions.

Let M denote the four dimensional space-time manifold

$T_P(M)$ the tangent vector space to M at P

$\mathfrak{F}(M)$ the ring of C^∞ functions on M

$\chi(M)$ the Lie algebra of C^∞ vector functions on M

$\Lambda^1(M)$ the $\mathfrak{F}(M)$ module of C^∞ 1 - forms on M ,

and let g be a second order symmetric covariant tensor field, the metric, defined on M , where the induced form ξ_P on $T_P(M) \times T_P(M)$ is non-singular and has signature $- + + +$.

A linear connection ∇ on M is defined as the mapping

$$\nabla : \chi(M) \times \chi(M) \rightarrow \chi(M)$$

where $(X, Y) \rightarrow \nabla_X Y$ such that

$$(a) \nabla_{f \cdot X + g \cdot Y}(Z) = f \cdot \nabla_X Z + g \cdot \nabla_Y Z$$

$$(b) \nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$$

$$(c) \nabla_X(f \cdot Y) = X(f) \cdot Y + f \nabla_X Y$$

for all $f, g \in \mathfrak{F}(M)$ and $X, Y, Z \in \chi(M)$.

The torsion of ∇ is defined to be the mapping

$$\text{Tor}_{\nabla}: \chi(M) \times \chi(M) \rightarrow \chi(M)$$

where $(X, Y) \rightarrow \text{Tor}_{\nabla}(X, Y) \equiv \nabla_X Y - \nabla_Y X - [X, Y]$

for all $X, Y \in \chi(M)$.

Following Sen and Vanstone, a connection ∇ on M is uniquely determined by

$$\begin{aligned} \nabla_Z \xi(X, Y) \\ \text{Tor}_{\nabla}(X, Y) \end{aligned} \quad ,$$

only if these two covariant tensor fields on M are respectively symmetric and skew-symmetric in (X, Y) for all $X, Y, Z \in \chi(M)$.

A Weyl connection on M , (3), can be thus defined by

$$\begin{aligned} \nabla_Z \xi(X, Y) &= -\phi(Z) \cdot \xi(X, Y) \\ \text{Tor}_{\nabla}(X, Y) &= 0 \end{aligned} \quad ,$$

for a 1-form $\phi \in \Lambda^1(M)$ and for all $X, Y, Z \in \chi(M)$.

A Lyra connection on M , (3), is defined by

$$\begin{aligned} \nabla_Z \xi(X, Y) &= 0 \\ \text{Tor}_{\nabla}(X, Y) &= \frac{1}{2}[\phi(Y) \odot X - \phi(X) \odot Y] \end{aligned} \quad ,$$

for a 1-form $\phi \in \Lambda^1(M)$ and for all $X, Y, Z \in \chi(M)$, in comparison to a Riemann connection defined by

$$\nabla_Z \xi(X, Y) = 0 = \text{Tor}_{\nabla}(X, Y) \quad ,$$

for all $X, Y, Z \in \chi(M)$.

For reference :

1. $\nabla_Z G(X, Y) = Z(G(X, Y)) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)$,
for $X, Y, Z \in \chi(M)$.
2. ∇ is said to be metric preserving (or have integrable ^{length} path transfer)
if $\nabla_Z G(X, Y) = 0$ for all $X, Y, Z \in \chi(M)$.
3. The curvature tensor of ∇ is a linear transformation valued tensor R that assigns to each pair of vectors $X, Y \in \chi(M)$ linear transformation $R(X, Y)$ of $T_p(M)$ into itself, where

$$R(X, Y)Z = (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z)_P$$

and $[X, Y] = \frac{1}{2}[XY - YX]$.

4. If C is a differentiable curve in M , with tangent vector field X , then C is a geodesic of ∇ if

$$\nabla_X X = 0 \quad \text{on } C .$$

6.2 Weyl Space-time

In this case the space-time manifold is characterised by a metric g and 1-form $\phi [\in \Lambda^1]$ which define as in Section 6.1 a Weyl connection, ∇ on the manifold.

*There is a change of **convention** implied here e.g. the Riemann curvature tensor components are the **negative** of those defined in Chapter Two.

Some properties

In local coordinates $\{x^\alpha\}$ and with $\{e_\alpha = \frac{\partial}{\partial x^\alpha}\}$ denoting the set of coordinate vector fields let $\varepsilon_{\alpha\beta} = \varepsilon(e_\alpha, e_\beta)$, $\phi_\alpha = \phi(e_\alpha)$ and $\phi^\alpha = \varepsilon^{\alpha\gamma} \phi_\gamma$.

W.1 The components $\Gamma_{\alpha\beta}^Y$ of the Weyl connection defined by

$$\nabla_{e_\alpha} e_\beta = \Gamma_{\alpha\beta}^Y e_Y,$$

are $\Gamma_{\alpha\beta}^Y = \left\{ \begin{matrix} Y \\ \alpha\beta \end{matrix} \right\} + \frac{1}{2} \left[\delta_\alpha^Y \phi_\beta + \delta_\beta^Y \phi_\alpha - \varepsilon_{\alpha\beta} \phi^Y \right]$.

W.2 If the curvature tensor of the Weyl connection is denoted by K (the curvature tensor of the Riemann connection is denoted by R) then the components $K_{\alpha\beta}^Y$ of K defined by

$$K(e_\alpha, e_\beta)e_\gamma = K_{\alpha\beta}^Y e_Y,$$

are $K_{\alpha\beta}^Y = \Gamma_{\delta\beta, \alpha}^Y - \Gamma_{\delta\alpha, \beta}^Y + \Gamma_{\eta\alpha}^Y \Gamma_{\delta\beta}^\eta - \Gamma_{\eta\beta}^Y \Gamma_{\delta\alpha}^\eta$.

The generalised Ricci tensor has components

$$\begin{aligned} K_{\alpha\beta} &= K_{\alpha\beta}^Y \\ &= R_{\alpha\beta} + \frac{1}{2}(\phi_{\alpha; \beta} + \phi_{\beta; \alpha}) - \frac{1}{2}\varepsilon_{\alpha\beta} \phi^Y_{; Y} + \frac{1}{2}(\phi_\alpha \phi_\beta - \varepsilon_{\alpha\beta} \phi^Y \phi_Y) \end{aligned}$$

and the curvature scalar is

$$\begin{aligned} K &= K^\alpha_\alpha \\ &= R + 3\phi^Y_{; Y} + \frac{3}{2}\phi^Y \phi_Y. \end{aligned}$$

W.3 In M , if C is a differentiable curve parameterized by t and X is its associated tangent vector field then for $Y \in \chi(M)$ the value of Y at one point on C determines the value of Y at another point on C by parallel displacement if

$$\nabla_X Y = 0 \text{ on } C,$$

* although the curvature tensor and curvature scalar are denoted by the same symbol the context of the symbol indicates its meaning.

and in local coordinates

$$\frac{d}{dt} Y^\gamma + \Gamma_{\alpha\beta}^\gamma \frac{dx^\alpha}{dt} Y^\beta = 0 .$$

Similarly, for the change in length of the tangent vector Y on parallel displacement along C

$$X_\alpha (Y, Y) = -\phi(X) \cdot g(Y, Y)$$

(using the definition of the Weyl connection and reference 1 in Section 6.1), or in local coordinates

$$\frac{dl^2}{dt} = -\phi_\alpha l^2 \frac{dx^\alpha}{dt} \quad \text{where } l^2 = g(Y, Y) .$$

W. 4 The curve C defined in W.3 is a geodesic of the Weyl connection

$$\text{if } \frac{d^2 Y^\gamma}{dt^2} + \Gamma_{\alpha\beta}^\gamma \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0$$

W. 5 As well as defining a Weyl connection on M , the choice of g and ϕ defines a Weyl structure on M . Folland has formulated this idea as follows :

$$\text{Let } G = \{g^\lambda \mid g^\lambda = e^\lambda g \text{ for } \lambda \in \mathcal{F}(M)\} .$$

Then G is the equivalence class of metrics strictly conformal to g and a Weyl structure on M is the mapping

$$F: G \rightarrow \Lambda^1(M)$$

given by $F(e^\lambda g) = \phi - d\lambda$.

λ determines a gauge on M and under the gauge transformation*

$$g \rightarrow g^\lambda$$

one has

$$\phi \rightarrow \phi^\lambda = \phi - d\lambda$$

or in local coordinates

$$\phi_\alpha \rightarrow \phi_\alpha^\lambda = \phi_\alpha - \lambda_{,\alpha} .$$

* in keeping with historical usage this transformation is referred to from now on, as a gauge transformation rather than a conformal transformation.

A special case of the Weyl space-time is given when ϕ is exact

$$\text{i.e. } d\phi = 0 \text{ or } \phi = df \text{ for some } f \in \mathcal{F}(M),$$

in which case the space-time can be reduced by a suitable choice of gauge to a Riemann space-time. In this sense the Weyl space-time could be considered equivalent to the Riemann space-time.

6.3 Gravitation in Weyl Space-time

From the discussion of the Weyl space-time given in Chapter Two, a gauge is the same as an arbitrary length standard at each point of the space-time and a gauge transformation is just a transformation of this standard.

Under the gauge transformation

$$ds \rightarrow ds' = \lambda ds \quad (\text{W.5 in local coordinates})$$

where $ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$

$$g_{\mu\nu} \rightarrow g'_{\mu\nu} = \lambda^2 g_{\mu\nu}$$

because dx^μ is taken to be unaffected by the gauge transformation. If a physical quantity denoted by X , transforms as

$$X \rightarrow X' = \lambda^n X$$

then n is called the dimensional number (Lord, (4)) of X . So the dimensional number of $g_{\mu\nu}$ is $+2$.

Further, if X corresponds to the tensor density of weight W then its covariant derivative as implied by the Weyl connection can be defined by, (4)

$$X_{\nu \dots \rho}^{\mu \dots}{}_{; \rho} = X_{\nu \dots, \rho}^{\mu \dots} + \Gamma_{\lambda \rho}^{\mu} X_{\nu \dots}^{\lambda \dots} + -\Gamma_{\nu \rho}^{\lambda} X_{\lambda}^{\mu \dots} + \frac{N}{2} \phi_{\rho} X_{\nu \dots}^{\mu \dots} - W \Gamma_{\rho}^{\gamma} X_{\nu \dots}^{\mu \dots} \quad 6.1$$

which is covariant with respect to both coordinate and gauge transformations.

The Riemann covariant derivative is as usual defined by

$$X_{\nu}^{\mu}{}_{;\rho} = X_{\nu}^{\mu}{}_{,\rho} + \Gamma_{\lambda\rho}^{\mu} X_{\nu}^{\lambda}{}_{;\rho} - \Gamma_{\nu\rho}^{\lambda} X_{\lambda}^{\mu}{}_{;\rho} - W \Gamma_{\lambda\rho}^{\lambda} X_{\nu}^{\mu}{}_{;\rho} \quad 6.12$$

One has $G_{\mu\nu}{}_{;\rho} \equiv 0 \equiv G_{\mu\nu}{}_{|\rho}$ and so both covariant derivatives commute with the raising and lowering of indices.

The basic hypothesis of a gravitation theory in Weyl space-time is that physical laws can be written in a form that is manifestly covariant under the group of coordinate transformations and the group of gauge transformations. The field equations for the theory are to follow from an action principle whose action has the form

$$I = \int \sqrt{-G} d^4x. \quad 6.3$$

I must have zero weight and zero dimensional number. So W is a scalar with dimensional number -4 . The curvature scalar, K , of the Weyl manifold has dimensional number -2 and to avoid involving complicated expressions such as K^2 in I , it is sufficient to postulate a scalar field, σ , with dimensional number -1 .

An appropriate action for the vacuum field equations would be

$$I_G = \int \left[a\sigma^2 K + \frac{1}{2} \sigma_{|\alpha} \sigma^{|\alpha} + a^2 f_{\mu\nu} f^{\mu\nu} + b\sigma^4 \right] \sqrt{-G} d^4x \quad 6.4$$

where $f_{\mu\nu} = \phi_{\mu|\nu} - \phi_{\nu|\mu}$ by definition, and contributions to the action from the scalar and vector fields have been included. Eq. 6.4 can be rearranged to give, after neglecting an integral of a divergence term and the "cosmological" term,

$$\int b\sigma^4 \sqrt{-G} d^4x,$$

$$I_G = \int \left[a\sigma^2 R + \frac{1}{2} \sigma_{,\alpha} \sigma^{,\alpha} - (6a + \frac{1}{2}) \sigma \sigma_{,\alpha} \sigma^{,\alpha} + \left(\frac{3}{2}a + \frac{1}{8} \right) \phi_{\alpha} \phi^{\alpha} \sigma^2 + a f_{\mu\nu} f^{\mu\nu} \right]$$

$$\sqrt{-G} d^4x \quad 6.5$$

As in Chapter Three a term, I_{NG} , representing the interaction of the metric, vector and scalar fields with matter is added to I_G to give the full action for the theory,

$$I = I_G + I_{NG} \quad 6.6$$

The field equations,

$$\delta I = 0 \quad 6.7$$

are not all independent because of five Bianchi-type identities.

With ^{*}
$$\delta I_{NG} = - \int \left[\frac{T^{\mu\nu}}{2} \delta g_{\mu\nu} - \frac{1}{2} j^\alpha \delta \phi_\alpha + S \delta \sigma \right] \sqrt{-g} d^4x; \quad 6.8$$

variations, that describe infinitesimal coordinate and gauge transformations and that vanish on the boundary of the region of integration, give the five identities between the matter "sources", $T^{\mu\nu}$, j^α and S .

$$\text{viz: } T^{\mu\nu}{}_{;\nu} = - \frac{1}{2} j^{\mu\nu}{}_{;\nu} + \sigma^{\mu\nu} S \quad 6.9$$

$$T^{\mu\mu}{}_{;\mu} = j^{\mu\mu}{}_{;\mu} + \sigma S \quad 6.10$$

Eqs. 6.7 - 6.10 formally describe in Weyl space-time, a model for interactions between the three simplest long range fields and for their interactions with matter. Models of this type have been considered, independently by Lord (4) and Dirac (5). Two other authors, Ross (6) and Omote (7) have looked at relations between Weyl space-time and scalar-tensor theories but their theories are not physically relevant because they do not include matter.

Gauge transformations have been interpreted by some authors (Anderson (8), Omote, Lord) as space-time position dependent scale (or units) transformations. Once a system of units for length (time) and mass has been established the group of such transformations is usually restricted to those under which h and c are invariant. Inverse mass, length and time scale the same way and physical quantities can then be expressed in terms of a power of length —

* $T_{\mu\nu}$ is exactly the same as in previous Chapters. The change of convention for the curvature tensor has been compensated in the action principle by a change of sign for the matter Lagrangian. I_{NG} is the negative of previous matter actions.

the dimensional number defined earlier. The coupling of matter with the three fields has been partly limited by this scale invariance requirement and it seems that the full generality of Weyl space-time has not been utilised.

Rather than interpret the group of gauge transformations as a special group of units transformations, the original idea of gauge transformations as just length standard transformations is used. Except for two examples the construction of gauge invariant matter actions is left indeterminate.

Example 1 : The coordinate invariant action for the electromagnetic field

$$I_{EM} = \int F_{\mu\nu} F^{\mu\nu} \sqrt{-g} d^4x \quad 6.11$$

is gauge invariant.

Example 2 : The action for a neutral* test particle of mass m (as measured by atomic standards) can be taken as

$$I_P = \int m ds. \quad 6.12$$

With the coordinates x^α of the particle, functions of the proper time t measured along the particle's world line, appropriate curve variations give the equations of motion of the particle in a vacuum as

$$\frac{d}{dt} \left(\sigma \frac{dx^\alpha}{dt} \right) + \sigma \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt} = 0 \quad 6.13$$

These equations are the geodesics of the "metric" $g_{\mu\nu}$ and because they are independent of the vector field they cannot in general be the geodesics of the Weyl connection. This perhaps could have been anticipated because the distinguishing feature of Weyl space-time is that with respect to a particular gauge the length of a vector does not remain constant on parallel displacement. In flat-Riemann space-time the Law of Inertia requires the four-momentum of a

* Neutral signifies that the vector field has no influence on the state of motion of the particle

particle to keep constant magnitude and direction, suggesting for Weyl space-time that this law could not be immediately generalised.

Lord takes an action, I, in which the scalar field term enters with a negative sign relative to K, as compared to the positive sign in eq. 6.4. He chooses a gauge for which the scalar field is constant everywhere and with the restriction that

$$r_{\mu\nu} = 0 \quad 6.14$$

he arrives at a reduced theory which is almost identical to Hoyle and Narlikar's theory. The specified gauge is identified with atomic standards of length.

Below in contrast, a Brans-Dicke type theory is derived from eq. 6.7 with the restriction, eq. 6.14, and a condition

$$a \neq -\frac{1}{12}, \quad 6.15$$

which is also used by Lord.

Eq. 6.14 implies that $\phi_{,\mu}$ is the gradient of a scalar field. This field can be transformed away by a suitable choice of gauge in the gauge transformation, (W.5), and on dropping dashes and putting $a\sigma^2 = \phi_{,\omega} = \frac{1}{\sigma^2}$ the field equations (eq. 6.7) become

$$G_{\mu\nu} = -\frac{\phi^{-1}}{2} T_{\mu\nu} - \omega\phi^{-2} (\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}) - \phi^{-1}(\phi_{;\mu;\nu} - \frac{1}{2}g_{\mu\nu}\square\phi) \quad 6.16$$

$$\square\phi = \frac{1}{3+2\omega} [T^{\mu}_{\mu} - \sigma S], \quad 6.17$$

with four identities

$$T^{\mu\nu}_{;\nu} = \sigma^{;\mu} S \quad 6.18$$

The identity, eq. 6.10 is used to eliminate j_{α} .

The restriction, eq. 6.14 on the vector field means that the geometry of the Weyl space-time is Riemannian. The Weyl and Riemann connections are identical but the equations of motion of a test particle are geodesics of neither connection. On parallel displacement vectors maintain their length with respect to the gauge choice, which we shall identify with atomic standards of length. Thus there are two distinct intervals between neighbouring events.

$$(a) \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

which is measured by atomic systems - e.g. using the radar method, and

(b) the interval determined by the equations of motion of a test particle. For a vacuum this interval is

$$dt^2 = \sigma_{\mu\nu}^a dx^\mu dx^\nu$$

In the gauge chosen, the action for a test particle (eq. 6.12) gives, using eq. 6.8,

$$T^{\mu\nu} = \frac{m}{\sqrt{-g}} \int \frac{dx^\mu dx^\nu}{dt dt} \delta^4(x^\alpha - x^\alpha(t)) dt \quad 6.19$$

$$j^\alpha = 0$$

$$\text{and} \quad S = \frac{m}{\sqrt{-g}} \int \delta^4(x^\alpha - x^\alpha(t)) ds \quad , \quad 6.21$$

where the scalar density, $\delta^4(x^\alpha - x^\alpha(t))$ is the four dimensional Dirac function. If matter is considered as a system of pressure-less dust particles then the three sources, $T_{\mu\nu}^m$, j_m^α , and $S_{\mu\nu}$ are just the sums over the particles, of eqs. 6.19, 6.20, and 6.21 respectively.

The field equations for this model of matter imply a variable gravitational constant $\sim \sigma^{-1}$.

The theory derived here has two unsatisfactory aspects in common with Lord's theory. The first is the restriction (eq. 6.14) which is an artificial constraint *in the gauge chosen.

on the Weyl space-time. Two scalar field variables now appear in the description of gravitation.

The other aspect that $a \neq -\frac{1}{12}$ means that the action principle (eq. 6.7) is not the simplest which could be set up in the space-time .

A more fundamental objection to the theory is that it determines a **unique** and preferred gauge. This suggests that despite the theory's gauge invariance there is in fact no need for this invariance.

6.4 Dirac's Theory

Dirac's paper was published very recently and it has unexpectedly turned out to be the culmination point for the discussion of the scalar-tensor model begun back in Chapter Three.

In the more developed scalar-tensor theories such as the Brans-Dicke or Jordan theories, the scalar field gave position dependence to the locally measured gravitational constant and in this way Dirac's hypothesis was at least partly brought within the framework of a gravitation theory.

Dirac proposed that the appropriate modification of General Relativity needed to give a variable gravitational constant is contained in the Weyl space-time.

By putting $a = -\frac{1}{12}$ the action (eq. 6.5) for

the vacuum field equations simplifies to

$$I_G = \int [\sigma^2 R - 6\sigma_{;\alpha}\sigma^{;\alpha} + f_{\mu\nu}f^{\mu\nu}] \sqrt{-g} d^4x \quad 6.22$$

where now the vector field variable does not appear explicitly. In the presence of matter the full field equations can be written.

$$\delta I = 0 \text{ where } I = I_G + I_{NG}$$

between which the same identities as eqs. 6.9 and 6.10 hold. These equations are essentially the field equations which Dirac gives. The vacuum field equations involve three dynamical variables - a scalar, vector and a metric tensor field. The vector field Dirac identifies with the electromagnetic field potential because both variables have the same transformation properties.

In the absence of electromagnetic fields the field equations correspond to those of the Brans-Dicke theory provided $\omega = -\frac{3}{2}$. However the predictions for the solar system experiments are identical to the predictions of General Relativity because the field equations and particle equations of motion in the particular gauge $\sigma = 1$, go over to Einstein's equations.

As the theory is gauge invariant there is no preferred gauge and so the gauge choice $\sigma = 1$, is just as valid as any other gauge choice. Some such choice is necessary before any relation between the theory and experiment could be set up.

Dirac lists three special gauges :

- (a) The natural gauge . For $f_{\mu\nu} = 0$ one takes $\phi_a = 0$ while for $f_{\mu\nu} \neq 0$ this gauge is not well defined.
- (b) The Einstein gauge . This is the gauge choice given above $\sigma = 1$.
- (c) The atomic gauge. This is the gauge in which the metric defines the interval that is measured by atomic systems.

The vacuum field equations in the natural gauge are independent of the vector field and so for this case the

metric may be transformed by a gauge transformation without an associated transformation of the vector field. The natural gauge can therefore be transformed into the Einstein' or atomic gauges.

In contrast to the Brans-Dicke type theory developed earlier, no gauge in general (i.e. for non-vanishing vector field) exists which could be naturally identified with atomic standards. Thus the gauge of the metric is not directly observable.

Dirac supposes that the Einstein and atomic gauges are distinct and clearly it is this difference which leads to the gravitational constant being variable when it is expressed in terms of atomic standards. For one can take a simple dust or fluid model of matter in the same way as for the Brans-Dicke type theory and one finds from the field equations that the gravitational constant is variable (as σ^{-1}) because in the atomic gauge σ by hypothesis is not constant.

Because the three long range fields are considered in the Weyl space-time, this formal framework has an interesting consequence. The interaction between the gravitational and electromagnetic fields may lead to the breaking of two symmetries - symmetry under charge conjugation and symmetry under time reversal.

Referring to Dirac's paper, one can take a simple case of the parallel displacement law (w.3)

$$\delta \ell^2 = -\ell^2 \phi_0 \delta x^0$$

which gives the change in length of a length ℓ at some point P when ℓ is parallelly displaced into the future by a distance δx^0 . The electromagnetic potential is taken to be due to a charged particle whose world line is close to the point P and the coordinate system has been chosen so that the particle appears at rest. Thus ϕ_0 is primarily the Coulomb potential of the particle.

If ℓ increases when it is displaced into the future, for a given charge on the particle, then ℓ will decrease if instead there is a charge of opposite sign on the particle.

Hence there is no symmetry between positive and negative charge.

Also if λ increases when it is displaced into the future then it will decrease when it is displaced into the past. Thus there is no symmetry between future and past. Symmetry can be restored if the sign of the charge on the particle is changed. With the interchange of positive and negative charges denoted by the operator C and the reversal of time denoted by the operator T then one has both the C and T symmetries broken but the CT symmetry conserved.

However the symmetry breaking does not apply to the atomic gauge because in this gauge length is preserved on parallel displacement. Consequently the symmetry breaking can't be directly measured in the geometry of space-time. Rather, as Dirac points out, symmetry breaking will show up only in the equations of motion. Furthermore it will not occur for simple, charged particles* but for particles whose action has for example a term of the form (Dirac),

$$I = \int \sigma^{-3} \sigma^{\lambda\alpha} \sigma_{\lambda\alpha} ds$$

or

$$I = \int [\sigma^{-2} \sigma^{\alpha} \sigma_{,\alpha} - \sigma^{-2} \sigma_{,\alpha} \phi^{\alpha} + \sigma^{-1} \phi_{,\alpha} \phi^{\alpha}] ds$$

For the middle expression :

Under C, ϕ_{α} changes sign \rightarrow symmetry breaking

Under T, σ_{α} changes sign \rightarrow symmetry breaking

while under CT there is no overall sign change.

*The breaking of the C and T symmetries has been observed so far only for the K meson, (5).

6.5 Lyra Space-Time

With the notation of Sections 6.1 and 6.2 a Lyra space-time manifold is characterised by a metric g and a 1-form ϕ which define a Lyra connection, ∇ .

One could proceed to develop properties of this space-time in the same way as was done for the Weyl space-time. However, Lyra (3) has introduced the idea of a reference system. Following Sen and Vanstone, (3), if $\{x^\alpha\}$ denotes a set of local coordinates on M and ϕ is a real, non-zero function on R^4 , called the gauge function, then a local reference system is specified by $\{\bar{x}^\alpha, x^\alpha\}$.

The induced basis in $T_p(M)$ is $\{\bar{e}_\alpha(P) = [(\bar{x})^{-1} \frac{\partial}{\partial x^\alpha}]_P\}$ and so the set of reference vector fields is $\{\bar{e}_\alpha = (\bar{x})^{-1} \frac{\partial}{\partial x^\alpha}\}$. The natural basis dual to $\{\bar{e}_\alpha(P)\}$ is

$$\{\bar{e}^\alpha(P) = [x dx^\alpha]_P\} \quad .$$

In a local reference system $\{\bar{x}, x^\alpha\}$ the components $G_{\mu\nu}$ of the metric tensor g are defined by

$$G_{\mu\nu} = G(\bar{e}_\mu, \bar{e}_\nu)$$

and so the interval between two infinitesimally close points with coordinates x^α and $x^\alpha + dx^\alpha$ is

$$ds^2 = G_{\mu\nu}(\bar{x})^2 dx^\mu dx^\nu \quad .$$

The properties analogous to those given for the Weyl space-time in section 6.2 are as follows.

L. 1 The components $\bar{\Gamma}_{\alpha\beta}^{\gamma}$ of the Lyra connection defined by

$$\bar{\nabla}_{\bar{c}_{\beta}} \bar{c}_{\alpha} = \bar{\Gamma}_{\alpha\beta}^{\gamma} \bar{c}_{\gamma}$$

$$\text{are } \bar{\Gamma}_{\alpha\beta}^{\gamma} = (\overset{\circ}{x})^{-1} \left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\} + \frac{1}{2} \left[\delta_{\beta}^{\gamma} \phi_{\alpha} - \varepsilon_{\alpha\beta} \phi^{\gamma} \right]$$

where $\phi_{\alpha} = \phi(\bar{c}_{\alpha}) + \phi_{\alpha}^{\circ}$ and $\phi_{\alpha}^{\circ} = \left(-2(\overset{\circ}{x})^{-1} \right)_{,\alpha}$.
In contrast to the symmetric components of the Weyl connection, $\bar{\Gamma}_{\alpha\beta}^{\gamma}$ are non-symmetric.

L. 2 If the curvature tensor of the Lyra connection is denoted by \bar{K} then the components $\bar{K}_{\delta\alpha\beta}^{\gamma}$ of \bar{K} defined by

$$\bar{K}(\bar{c}_{\alpha}, \bar{c}_{\beta}) \bar{c}_{\delta} = \bar{K}_{\delta\alpha\beta}^{\gamma} \bar{c}_{\gamma}$$

are

$$\bar{K}_{\delta\alpha\beta}^{\gamma} = \bar{\Gamma}_{\delta\beta,\alpha}^{\gamma} - \bar{\Gamma}_{\delta\alpha,\beta}^{\gamma} + \bar{\Gamma}_{\eta\alpha}^{\gamma} \bar{\Gamma}_{\delta\beta}^{\eta} - \bar{\Gamma}_{\eta\beta}^{\gamma} \bar{\Gamma}_{\delta\alpha}^{\eta}$$

the generalised Ricci tensor has components

$$\begin{aligned} \bar{K}_{\alpha\beta} &= \bar{K}_{\alpha\beta}^{\gamma} \\ &= \bar{K}(\alpha\beta) + \bar{K}[\alpha\beta] \end{aligned}$$

where the symmetric part

$$\begin{aligned} \bar{K}(\alpha\beta) &= (\overset{\circ}{x})^2 R_{\alpha\beta} + \frac{1}{2} (\overset{\circ}{x})^{-1} (\phi_{\alpha;\beta} + \phi_{\beta;\alpha}) + \frac{1}{2} \varepsilon_{\alpha\beta} \phi^{\gamma}_{;\gamma} - \frac{1}{2} \phi_{\alpha} \phi_{\beta} \\ &\quad + \frac{1}{2} \varepsilon_{\alpha\beta} \phi^{\gamma} + \frac{1}{4} \varepsilon_{\alpha\beta} \phi^{\gamma} + \frac{1}{4} (\phi_{\alpha} \phi_{\beta} + \phi_{\beta} \phi_{\alpha}) \end{aligned}$$

and the skew symmetric part

$$\bar{K}[\alpha\beta] = \frac{1}{2} (\overset{\circ}{x})^{-1} (\phi_{\alpha;\beta} - \phi_{\beta;\alpha}) - \frac{1}{4} (\phi_{\alpha} \phi_{\beta} - \phi_{\beta} \phi_{\alpha})$$

The curvature scalar of the Lyra manifold is

$$\begin{aligned} \bar{K} &= \bar{K}^{\alpha}_{\alpha} \\ &= (\overset{\circ}{x})^{-2} R + 3(\overset{\circ}{x})^{-1} \phi^{\alpha}_{;\alpha} + \frac{3}{2} \phi_{\alpha} \phi^{\alpha} + \frac{3}{2} \phi_{\alpha}^{\circ} \phi^{\alpha} \end{aligned}$$

L. 3 Using the definition for parallel displacement given in W.3 (Section 6.2) the equations for the parallel displacement of a vector with components Y^{α} along a curve parameterized by t are

$$\frac{d}{dt} Y^{\alpha} + \Gamma^{\alpha}_{\beta\gamma} \frac{dx^{\gamma}}{dt} Y^{\beta} = 0$$

In contrast to the Weyl space-time, length is preserved on parallel transfer.

L. 4 The ^{autoparallels} geodesics of the Lyra connection are

$$\frac{d^2 x^{\alpha}}{dt^2} + \overset{0}{\Gamma}^{\alpha}_{\beta\gamma} \frac{dx^{\gamma}}{dt} \frac{dx^{\beta}}{dt} = \frac{0}{2} (\phi_{\alpha} - \phi_{\alpha}) \frac{dx^{\alpha}}{dt} \frac{dx^{\gamma}}{dt}$$

and because length is integrable there is a second class of distinguished curves in the manifold. - viz: those of stationary length $\delta \int ds = 0$,

or

$$\frac{d^2 x^{\alpha}}{dt^2} + \left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\} \frac{dx^{\alpha}}{dt} \frac{dx^{\beta}}{dt} + \frac{0}{2} \left(\delta_{\alpha}^{\gamma} \phi_{\beta} + \delta_{\beta}^{\gamma} \phi_{\alpha} - g_{\alpha\beta}^{\gamma} \right) \frac{dx^{\alpha}}{dt} \frac{dx^{\beta}}{dt} = 0$$

L. 5 For the Weyl space-time it was found that the choice of g and ϕ in the Weyl connection defined a Weyl structure and it seems for the Lyra space-time that the role of reference systems is to express the same idea.

Thus a gauge in Weyl space-time corresponds to a specification on the gauge function or first coordinate in all the reference systems defined in the Lyra space-time - and vice versa. A gauge transformation in Weyl space-time corresponds in the Lyra space-time to a transformation which takes a reference system into another by changing the gauge function but keeps the local coordinates

the same - and vice versa.

In general, under a transformation of local reference systems

$$\{\bar{x}, x^\alpha\} \rightarrow \{\bar{x}', x'^\alpha\}$$

$$\bar{e}_{\alpha'} = \lambda^{-1} \Lambda_{\alpha'}^{\alpha} \bar{e}_{\alpha} \quad \text{where } \lambda = \frac{\partial \bar{x}'}{\partial \bar{x}}, \quad \Lambda_{\alpha'}^{\alpha} = \frac{\partial x^\alpha}{\partial x'^{\alpha'}}$$

and the components X^{α} of a vector field $X = X^{\alpha} \bar{e}_{\alpha}$ transform as $X'^{\alpha'} = \Lambda_{\alpha'}^{\alpha} X^{\alpha}$.

$$\text{Also } G_{\alpha'\beta'} = \lambda^{-2} \Lambda_{\alpha'}^{\alpha} \Lambda_{\beta'}^{\beta} G_{\alpha\beta}$$

$$\text{and } \phi_{\alpha'} = \lambda^{-1} \Lambda_{\alpha'}^{\alpha} (\phi_{\alpha} + (\bar{x})^{-1} (\ln \lambda^2)_{,\alpha})$$

So ϕ_{α} transforms differently under a gauge transformation than ordinary vector components. If ϕ is exact then there exists a local reference system in which $\phi_{\alpha} = 0$.

6.6 Gravitation in Lyra space-time

Sen and Dunn, (9) have formulated a scalar-tensor theory in the Lyra space-time. For their field equations Halford, (10), has given a static, spherically symmetric solution and he, (11), has also looked at a class of Friedmann-type solutions. However he (12) has pointed out that one set of field equations is not gauge invariant* and a subsequent examination of Sen and Dunn's derivation shows more inadequacies. A further unsatisfactory feature of their theory is that the gauge function appears as a dynamical variable but one which has not been varied in the original action principle. Their theory is therefore

* i.e. invariant under a change of reference system

short of one equation.

Accordingly a similar model of gravitation is set up as follows :

Following Sen and Dunn the action for the vacuum field equations is taken as

$$\bar{I}_G = \int \bar{K} \sqrt{-g} d^4x ,$$

where \bar{K} is the curvature scalar of the Lyra space-time and \bar{I}_G by construction is invariant under a change of reference systems. As for the scalar-tensor model in Weyl space-time, a term \bar{I}_{NG} is added to \bar{I}_G to give the full action for the gravitational field and matter equations, and on writing

$$\delta \bar{I}_{NG} = - \int \left[\frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + 3T^\alpha \delta \phi_\alpha \right] \sqrt{-g} d^4x , \quad 6.23$$

the field equations follow from arbitrary variations of the metric and vector fields and their derivatives that vanish on the boundary of the region of integration

$$G_{\mu\nu} = - \frac{Q^2}{2} T_{\mu\nu} - \frac{3}{2} Q^2 \phi_{;\mu} \phi_{;\nu} + \frac{3}{4} Q^2 \delta_{\mu\nu} \phi_{;\alpha} \phi^{;\alpha} + \frac{3}{2} Q^2 \phi_{(\mu} \phi_{\nu)} - \frac{3}{2} Q^2 G_{\mu\nu} \phi_{;\alpha} \phi^{;\alpha} - Q^{-2} [(Q^2)_{;\mu} \nu - G_{\mu\nu} \square (Q^2)] , \quad 6.24$$

$$\phi_\alpha = \overset{0}{\phi}_\alpha + T_\alpha . \quad 6.25$$

Also for variations, $\bar{\delta}$, arising from a change of reference system

$$\bar{\delta} \bar{I}_{NG} = 0 \quad 6.26$$

or

$$T^{\mu\nu}{}_{;\mu} + 2T^{\mu\nu} \frac{Q_{;\mu}}{Q} - 3T^{\mu\nu} T_{\mu\nu} + 3T^{\mu\nu}{}_{;\mu} \phi_\nu + 6 \frac{Q_{;\mu}}{Q} T^{\mu\nu} \phi_\nu = 0 \quad 6.27$$

$$T^\alpha_{;\alpha} + 3T^\alpha \phi_{;\alpha} + 6T^\alpha{}_{;\alpha} Q^{-1} + 9T^\alpha \phi_\alpha = 0 \quad 6.28$$

where

$$\bar{\delta} \phi_\alpha = \phi_{;\beta} - \phi_{\beta;\alpha} . \quad 6.29$$

Using eq. 6.25 to eliminate T^α from eq. 6.28, gives

$$0 = T^\alpha + 3\phi_\alpha \phi^\alpha + 6\phi_\alpha \phi^\alpha - 9\phi_\alpha \phi^\alpha + 6x^{-1}\phi^\alpha_{;\alpha} - 6x^{-1}\phi^\alpha_{;\alpha} \quad 6.30$$

and taking the trace of eq. 6.24 and comparing with eq. 6.30 gives

$$\phi^\alpha_{;\alpha} + x\phi_\alpha \phi^\alpha + \frac{1}{5}x^{-1}R = 0, \quad 6.31$$

This equation is not in general an identity. In fact it is the same as

$$\frac{\delta \bar{I}_G}{\delta x} = 0 \quad 6.32$$

Thus for consistency between the field equations (eqs. 6.24 and 6.25) and the identities (eqs. 6.27 & 6.28) it is sufficient to take as the field equations,

$$G_{\mu\nu} = \frac{(x^2)^{-2}}{2} T_{\mu\nu} - \frac{3}{2} x^2 \phi_\mu \phi_\nu + \frac{3}{4} x^2 g_{\mu\nu} \phi_\alpha \phi^\alpha + \frac{3}{2} x^2 \phi_{(\mu} \phi_{\nu)} - \frac{3}{2} x^2 g_{\mu\nu} \phi_\alpha \phi^\alpha - \frac{x^{-2}}{2} [(x^2)_{;\mu;\nu} - g_{\mu\nu} \square (x^2)] \quad 6.34$$

$$T^\alpha_\alpha = -3\phi_\alpha \phi^\alpha - 6\phi_\alpha \phi^\alpha + 9\phi_\alpha \phi^\alpha - 6x^{-1}\phi^\alpha_{;\alpha} + 6x^{-1}\phi^\alpha_{;\alpha} \quad 6.35$$

which identically satisfy eq. 6.31.

In analogy with section 6.3 two examples of gauge invariant matter action can be given.

Example 1 For the electromagnetic field²

$$\bar{I}_{EM} = \int \bar{F}_{\mu\nu} \bar{F}^{\mu\nu} \sqrt{-G} d^4x \quad .$$

Example 2 For a neutral test particle of mass m (with respect to atomic standards)

$$\bar{I}_P = \int m ds, \quad .$$

where

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad .$$

It follows then that the equations of motion of the test particle are geodesics of the metric $x^2 g_{\mu\nu}$ (e.g. of

1. The second equation is effectively the field equation Sen and Durn missed out.

2. $\bar{F}_{\mu\nu}$ is the simplest 2nd order skew-symmetric tensor on the manifold.

the form of e.g. 6.13 with σ replaced by $\overset{\circ}{x}$).

Suppose ϕ_α is the electromagnetic field potential. Then in the absence of electromagnetic fields eqs 6.34 and 6.35 become

$$G_{\mu\nu} = -\frac{(\overset{\circ}{x})^{-2}}{2} T_{\mu\nu} - \overset{\circ}{x}^{-2} [(\overset{\circ}{x}^2) j_\mu j_\nu - g_{\mu\nu} \square(\overset{\circ}{x}^2)] \quad 6.36$$

$$\square(\overset{\circ}{x}^2) = \frac{(\overset{\circ}{x})^4}{6} T, \quad 6.37$$

which are similar in form to the Brans-Dicke field equations for $\omega = 0$.

By using the action for a test particle given in Example 2, the action for a pressureless dust model of matter could be taken as a summation over the dust particles of terms

$$\bar{I}_m = \int m ds \delta^4(x^\alpha - x^\alpha(t)) \overset{\circ}{x}^3 d^3x, \quad 6.38$$

where the Dirac function is defined in the Lyra manifold as

$$\int f(x) \delta^4(x-y) \overset{\circ}{x}^4 d^4x = f(y) \quad 6.39$$

For this model of matter it is apparent from the definition of $T_{\mu\nu}$ (see eq. 6.23) and the field equations that the gravitational constant varies as $\overset{\circ}{x}^{-2}$. Consequently the field equations are identical ($\phi \leftrightarrow \overset{\circ}{x}^2$) to the Brans-Dicke field equations for $\omega = 0$.

If a reference system is chosen such that $\overset{\circ}{x} = 1$, then the field equations and the particle equations of motion are the same as those in General Relativity. As a result the predictions for the solar system experiments are the same as the predictions of General Relativity.

Because ϕ_α has been identified with the electromagnetic field potential an extra term giving the action for the electromagnetic field should strictly be added to the full action principle if non-vanishing electromagnetic fields are to be considered. The simplest second order skew-symmetric tensor

in the Lyra manifold seems to be, (3)

$$\bar{F}_{\mu\nu} = \overset{\circ}{x}^{-1} \bar{F}_{\mu\nu} - \frac{1}{2} (\overset{\circ}{\phi}_{\mu} \phi_{\nu} - \overset{\circ}{\phi}_{\nu} \phi_{\mu}) \quad 6.40$$

and the additional action required is

$$\bar{I}_{EM} = \int \bar{F}_{\mu\nu} \bar{F}^{\mu\nu} \sqrt{-g} \overset{\circ}{x}^4 d^4x. \quad 6.41$$

The resulting action principle Sen and Vanstone give as an example of a unified field theory in the Lyra space-time. However, the vacuum field equations even in the $\overset{\circ}{x} = 1$ gauge contain the electromagnetic field potential explicitly and this could be considered a complicating feature.

If one forgoes the tradition of General Relativity in using the curvature scalar of the space-time manifold as the Lagrangian for the free "gravitational" fields then the simplest action principle involving the three long range fields and matter, that could be set up in the Lyra space-time is

$$\delta \bar{I} = 0, \text{ for } \bar{I} = \bar{I}_G + \bar{I}_{NG} \quad 6.42$$

where

$$\bar{I}_G = \int [\overset{\circ}{x}^2 R - \frac{\overset{\circ}{x}^4}{2} \phi_{\alpha} \phi^{\alpha} - \overset{\circ}{x}^4 \bar{F}_{\mu\nu} \bar{F}^{\mu\nu}] \sqrt{-g} d^4x \quad 6.43$$

The variables ϕ_{α} and $g_{\mu\nu}$ are varied and as before, an extra field equation is obtained as a consequence of the Bianchi type identities. One finds that the field equations in the $\overset{\circ}{x} = 1$ gauge are identical to Dirac's field equations in the $\sigma = 1$ gauge and also that the particle equations of motion of each theory in their appropriate gauges are identical. One concludes then that the two theories are physically equivalent for in their respective gauges the dynamical variables of each theory contain no irrelevant parts and the physical laws of one theory are identical to those of the other.

Although Dirac arrives at the prediction of the C and T symmetry breaking by considering the non-integrability of length in Weyl space-time he notes that such a breaking is

only observable through the equations of motion and therefore the same symmetry breaking is expected to appear in the Lyra space-time. Finally because length is integrable in Lyra space-time the interval, $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ between two neighbouring events can be identified with that measured by atomic systems and in contrast to Dirac's theory there is no need for arbitrary length standards. However the gauge of the metric tensor is not directly observable and so following Dirac one supposes that the metric appearing in the $\dot{x} = 1$ gauge is not that determined by atomic systems. This completes the statement of equivalence between the two theories.

Chapter Seven

CONCLUSION

The theories set up in the Weyl or Lyra space-times provide an answer for the problem posed in the Introduction. These theories incorporate a scalar field into the description of gravitation in a manner which is less artificial than as for example in the Brans-Dicke theory. In particular, for Dirac's theory in the Weyl space-time the scalar field was necessary in order to construct a simple field theory while for the formulation of Dirac's theory in the Lyra space-time the scalar field has a more geometric role.

The modification involved here of the General Relativistic view of the space-time manifold is quite substantial and the observation of a variable gravitational constant would **certainly** not justify it. However the modification appears as a natural generalisation because it is essentially based on enlarging the complete covariance group of space-time theories to include the conformal group.

An immediate consequence of postulating conformal or gauge invariance is that the metric tensor has an extra gauge-like degree of freedom. Thus in local coordinates the interval between neighbouring events is no longer a physically measurable quantity.

From the point of view of construction and structure the most interesting scalar-tensor theory that has been discussed is Dirac's theory. The main result of Chapter **Six** is that this theory can be formulated with no loss of physics, in the Lyra space-time rather than in the Weyl space-time. Although the use of one space-time for Dirac's theory rather than the other **space-time** must be a matter of convenience it seems that the Weyl formulation could be simpler.

The basic **hypothesis** of Dirac's theory is the existence of two metrics -one referred to in the measurements given by atomic systems and the other is the metric that appears in

the equations of motion. This hypothesis ties in with Chapter Three when it is thought of as a third approach by which massive physical systems are allowed to couple with conformally invariant gravitational fields.

Dirac's theory has two advantages in contrast to the Brans-Dicke or related scalar-tensor theories which derive from the **scalar-tensor** model discussed in Chapter Three.

1. The mass-gauge group is trivial (i.e. it contains only the identity element), and
 2. WEP is not violated by massive physical systems.
- Both of these properties arise from Dirac's theory being identical to Einstein's theory when the $\dot{\alpha} = 1$ or $\sigma = 1$ gauge is chosen and when no electromagnetic fields are present.

An intention of this thesis has been to look at the scalar-tensor formalism and an important historical motive for considering these theories has been Dirac's hypothesis. Thus the conclusion arrived at is that if this hypothesis is to be taken seriously then the applications of conformal invariance in conjunction with the hypothesis mentioned earlier need to be studied further.

APPENDIX

A note on reference systems

These appear necessary if the equivalent of a Weyl structure is to be defined on the Lyra manifold. For if one begins with a set of local coordinates on the manifold, under the transformation

$$g \rightarrow g' = \lambda g$$

the coordinate vector fields are unchanged and from the definition (L.1) for the components of the Lyra connection, $\bar{\Gamma}_{\alpha\beta}^{\gamma}$ are also unchanged. Using the expression (L.1) for these components one finds that a transformation law for ϕ_{α} is not well-defined unless λ is a constant. Thus the complete covariance group of a gravitation theory in the Lyra space-time is MMG.

An action principle for a simple field theory is given by

$$\delta \int [\bar{K} + \bar{L}] \sqrt{-g} d^4x = 0.$$

However the resulting theory is of little interest here because it could be formulated equally well in the Riemann space-time.

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