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A DESCRIPTOR APPROACH TO SINGULAR LQG CONTROL PROBLEMS USING WIENER-HOPF METHODS

A thesis presented in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Production Technology at Massey University.

Ian Harvey Noell

1994

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Errata

Page 53 Line 6:

... the function f(t) must be absolutely integrable over $(-\infty,\infty)$.

Page 116 Section 5.3.1, 2nd line:

... the state-space representations (4.22) and (4.23).

Page 135 Equation (5.76):

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left(\mathrm{Tr}\left\{ \Delta T \Gamma \Gamma^* T^* \Delta^* - M_{\oplus} \Gamma^* T^* \Delta^* - \Delta T \Gamma M_{\oplus}^* - T M_{\vartheta}^* + M_{\vartheta} T^* \right\} + \mathrm{Tr}\left\{ Q P_d \Phi_d P_d^* \right\} \right) ds \qquad (5.76)$$

Page 157 Example 6.1 the state-space representation of P_d is:

$$P_d = (sI-A)^{-1}E = \left[\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right]$$

Abstract

Wiener-Hopf methods are used in this thesis to solve the output feedback Linear Quadratic Gaussian (LQG) control problem for continuous, linear time-invariant systems where the weighting on the control inputs and the measurement noise intensity may be singular. Some outstanding issues regarding the closed loop stability of Wiener-Hopf solutions and its connection with partial fraction expansion are resolved.

The main tools in this study are state-space representations and Linear Matrix Inequalities. The relationship between Linear Matrix Inequalities and Wiener-Hopf solutions is studied; the role of the Linear Matrix Inequality in determining spectral factors, the partial fraction expansion step, the form of the controller, and the value of the performance index is demonstrated.

One of the main contributions of this thesis is the derivation of some new descriptor forms for singular LQG controllers which depend on the solution to the Linear Matrix Inequalities. These forms are used to establish the separation theorem for singular LQG control problems and to investigate the order of singular LQG controllers.

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He would have been at a loss to explain what was so arresting about this notion; he simply felt stricken to the heart and stood there in a terror that was almost mystical. A moment passed and everything before him seemed to expand; instead of horror - light and gladness, ecstasy; he began to struggle for breath and . . . but the moment passed. Thank God, it wasn't that! He took a deep breath and looked about him.

Fyodor Dostoevsky
The Idiot.

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CHAPTER 1

INTRODUCTION

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1.1. BACKGROUND

Following the development of the LQG solution to the optimal state feedback control problem by Kalman (1960), LQG theory dominated the subject of control theory. This thesis provides a further contribution to this subject, specifically tackling the LQG output feedback problem: Wiener-Hopf techniques are used to investigate the stability of the resulting closed loop system, and to derive a new descriptor form for singular LQG controllers.

The Wiener-Hopf solution to the LQG problem by Newton *et al.* (1957) actually predates Kalman's solution. This solution is limited to stable, Single Input Single Output (SISO) systems whereas Kalman's solution is also valid for unstable, multivariable systems. Wiener-Hopf solutions differ from the Kalman approach in that they are frequency domain based. Transfer function descriptions are used instead of linear, time-invariant systems of differential equations, and spectral factors replace Riccati equations. However, Wiener-Hopf techniques require two extra steps: partial fraction expansion, and simplification of the resulting controller by cancelling the common factors in the controller's numerator and denominator. The partial fraction expansion step arises from the requirement that the closed loop system is stable. A major advantage of Wiener-Hopf techniques is that they are able to solve singular problems (those with singular weightings on the control input or singular measurement noise intensities). This property of Wiener-Hopf techniques is used in this thesis to study singular LQG control problems

Wiener-Hopf techniques were studied further in the 1970's by a number of researchers (Youla *et al.* 1976a,b; Shaked 1976a,b; Austin 1979; Kucera 1979; Grimble 1978). These researchers independently extended the Wiener-Hopf technique to handle unstable, multivariable plants. While conceptually quite different in their approaches all these methods resolved the problem of ensuring that the resulting closed loop system is stable.

The early approach to ensuring closed loop stability was to assume that the closed loop system is stable and to show the necessary form of the controller under this assumption. This approach was used by Newton *et al.* (1957), Weston and Bongiorno (1972), Shaked (1976a,b) and Austin (1979). These methods will be collectively referred to as classical Wiener-Hopf methods. In addition to showing the necessary form that the controller should have if the closed loop system is stable, Youla *et al.* (1976b) and Kucera (1979) showed that their respective controller solutions actually guaranteed the stability of the closed loop system. That is, the controller form resulting from this solution is necessary and sufficient for closed loop stability. The method of Youla *et al.* (1976b) was termed a modern Wiener-Hopf method, and the method of Kucera a

polynomial solution. One aim of this thesis is to establish that the solution obtained by the classical method of Austin (1979) results in a stable closed loop system.

This thesis builds on the method of Austin (1979) which is a fully frequency domain development of the method of Shaked (1976b). The central features of this method are the use of generalised spectral factors and partial fraction expansion. In his derivation Austin made assumptions restricting the positions of the open loop poles (Austin 1979, Chapter 4). While these assumptions coincide with assumptions that Shaked (1976a) made with regard to the uniqueness of generalised spectral factors they are made for different reasons. Furthermore, these assumptions do not appear in any of the other LQG methods, frequency or time domain. The removal of these assumptions from is one of the aims of this thesis.

Grimble and Johnson (1986) criticised classical Wiener-Hopf methods, stating that they may lead to an unstable closed loop system. The example they provided (Grimble and Johnson 1986, 1:111-114) is only unstable when a nonminimal form of the controller is used. However, if the minimal form of the controller is used, the resulting closed loop system is stable. The issue of using nonminimal representations of controllers is far wider than when applied to classical Wiener-Hopf methods. Grimble and Johnson's criticism of this method can be taken as a warning that the resulting controller needs to be simplified by removing the common factors. The issue of producing a minimal controller is also addressed in this thesis.

One of the distinguishing features of the different Wiener-Hopf methods is that different forms are used to represent systems. The different system representations arise from the different ways in which the requirement of closed loop system stability is built into the solution. These forms include rational transfer functions, polynomials, fractional, state-space, and descriptor forms. All of these forms are used in this thesis. Rational transfer functions and polynomial forms were initially used in Wiener-Hopf techniques as they are easy to use for SISO systems; multivariable techniques have since been developed for these representations (Kucera 1979). One of the strengths of state-space forms is that multivariable systems are handled just as simply as SISO systems. Another advantage of state-space forms is that the internal as well as the input/output characteristics can be studied. For these reasons state-space representations are used extensively in this thesis.

Nonsingular LQG controllers are defined in state-space representations by the solution to two Riccati equations which determine the LQ state feedback controller and the Kalman filter. Macfarlane (1970) showed that there is a relationship between the Riccati equations and the spectral factors. The partial fraction expansion step required by Wiener-Hopf methods is redundant as this step can be related to the Riccati equations

(Shaked 1976b). The state-space form of nonsingular LQG controllers is generally minimal and so the simplification of the controller, required by Wiener-Hopf methods, is not required. For these reasons state-space methods using Riccati equations are generally preferred for nonsingular LQG problems. A state-space approach to singular LQG controller design which has the advantages of the Riccati equation methods is developed in this thesis.

As the Riccati equations require the invertibility of the weighting on control inputs and measurement noise intensities they are not defined for singular LQG problems. For singular LQG problems the role of the Riccati equation is taken by the Linear Matrix Inequality (LMI) (Willems 1971). The importance of the LMI to the LQG problem cannot be understated: as well as being valid for singular LQG problems, LMI contain Riccati equations as a special case (for nonsingular LQG problems) and as such provides a more complete description of the LQG problem. For the output feedback problem there are two LMI: one associated with the state and input weightings, and the other associated with the disturbance noise processes. The relationship between the spectral factors and the LMI was shown by Willems (1971). Willems (1971) derived the LMI from the dissipation inequality; a derivation starting from the definition of spectral factors is given in this thesis. While it did not prove possible to derive all the properties of the LMI, this approach provides extra insight into the link between state-space and transfer function interpretation of spectral factors. In spite of the importance of LMI in singular LQG problems they have not been used in Wiener-Hopf methods to date. This omission is rectified in this thesis.

Using these LMI, it is possible to extend the Riccati equation based methods (Shaked 1976b; Park and Youla 1992) to include singular LQG problems. The relationship between the partial fraction expansion step and closed loop stability is studied using the LMI. The partial fraction expansion for singular LQG problems is shown to be defined by the LMI in the same way as it is for nonsingular LQG problems. This approach allows the assumption associated with the method of Austin (1979) to be removed and establishes that the solution results in a stable closed loop system.

The value of the LQG performance index is sometimes used as a yardstick for measuring the comparative performance of controllers. For nonsingular LQG problems the performance index can be evaluated using the solution to the Riccati equations. In comparison the minimum value for the performance index in Wiener-Hopf methods is given by a contour integral. The LMI are used to derive expressions for the value of the performance index from the contour integral. These expressions are direct generalisations of the Riccati equation expressions for nonsingular problems which are valid for singular LQG problems.

Wiener-Hopf techniques determine a closed loop transfer function: the controller is not found directly, but can be determined from this transfer function and knowledge of the open loop system. While nonsingular LQG controllers are formed directly from the

solution to two Riccati equations the LMI are not currently used to form the controllers directly. Rather the LMI are used to form the closed loop system, from which a controller is determined. This approach has been used recently by Stoorvogel (1992a), Chen *et al.* (1993) and Stoorvogel *et al.* (1993) where the LMI are used to set up a disturbance decoupling problem which is solved to give the controller. This method is limited to forming proper controllers. For singular LQG problems, the resulting controller is not necessarily proper (that is, it can contain derivative action) and hence the method of Stoorvogel (1992a) is unsuitable for many problems. A new approach which forms the controller directly from the LMI, and which is not limited to proper controllers, is developed in this thesis.

Other studies of singular LQG problems have used inverse transfer functions to represent the improper nature of the controller (Soroka and Shaked 1988a). However such a representation obscures the role of the LMI in determining the structure of the controller. Another possibility for representing improper controllers, which is more consistent with the state-space approach of this thesis, is that of descriptor systems (also known as singular or generalised state-space systems). Descriptor forms were first used in control theory by Rosenbrook (1974) and Luenburger (1977). The subject has received further attention in the 1980's by Verghese *et al.* (1980), Lewis (1984), Cobb (1984), and Zhou *et al.* (1987). Most of this work has been concerned with the structural properties of descriptor systems and with some basic control problems (for example, state feedback and observer problems).

One of the main contributions of this thesis is the development of some explicit formulae for the controllers using a descriptor approach. These descriptor forms clearly show the role of the LMI in determining the improper nature of the controller and are used to investigate the problem of existence and order of the resulting controllers. Furthermore, the descriptor forms of the controller have a similar structure for both singular and nonsingular LQG controllers, and thus provide a more unified approach to the subject.

The central feature of nonsingular LQG problems is the separation structure which states that the LQG controller is composed of an LQ state feedback controller and a minimum variance estimator of the states (Kalman filter). The separation structure does not generally extend to singular LQG problems (Stoorvogel 1992). The problem is that singular state controllers and minimum variance estimators cannot always be connected. The descriptor forms derived in this thesis provide further insight into the separation structure of the controller. Specifically, the descriptor forms of the LQG controller allow the state feedback controller and the minimum variance estimator to be identified in the LQG controller. The difference between singular and nonsingular LQG

controllers is that a singular feedback loop replaces the constant gains of nonsingular LQG controllers.

Although the closed loop transfer function determined by Wiener-Hopf methods is well-defined, there may not be a controller which gives rise to this transfer function. This problem was discussed by Stoorvogel (1992a) who provided an example. However, a comprehensive classification of open loop systems for which there does not exist a well-defined solution has not yet been produced and a further classification of this problem is developed in this thesis.

The initial aim of this thesis was to provide a further study of the outstanding theoretical issues in the Austin method. In attempting to achieve this aim, the major tool has been state-space representations. These representations give extra insight into the structure of the solution which, in the author's opinion, is not clear from rational function representations of transfer functions. The central feature of the LQG problem in state-space terms is the LMI. A new innovation in LQG theory presented in this thesis is an explicit descriptor form which is completely determined from the LMI and the open loop system. These descriptor forms of the LQG controller arise naturally from the application of the Austin method. A feature of these descriptor forms is that they allow a more unified understanding of the structure of singular and nonsingular LQG controllers.

1.2. OUTLINE OF THESIS

The thesis begins with a review of methods for representing linear systems: state-space, polynomial, fractional, and descriptor forms are considered. Gaussian white noise processes and the response of linear systems to these processes are studied. The regulator controller configuration is defined and the stability of the resulting closed loop system is studied.

Chapter 3 contains an introduction to the LQG control problem together with a review of Wiener-Hopf techniques. The presentation is unified where possible with emphasis on the similarities of the approaches. Although the optimisation of the LQG performance index was originally done in different ways, the method of completing the square of Austin (1979) is used as a common starting point. The presentation then diverges to show the contrasting way in which the requirement that the resulting closed loop system is stable is built into the solution. The method of Newton *et al.* (1957) is presented first for historical reasons. The general transfer function method of Austin (1979) is then presented in detail. As a contrast in methodology the modern Wiener-Hopf method of Youla *et al.* (1976b) and the polynomial method of Kucera (1979) are also presented. Emphasis is placed on the role of partial fraction expansion in ensuring that the closed loop system is stable. The different methods are then contrasted from both conceptual and computational points of view.

In Chapter 4, state-space techniques for the study of Wiener-Hopf methods are presented. One of the main state-space tools in LQG theory is the Sylvester equation which has application in stability analysis, stochastic linear system theory, quadratic expressions and partial fraction expansion. Other topics which are studied using state-space techniques are the order of a rational function, contour integration, and descriptor forms.

Linear Matrix Inequalities (LMI), which are the key to a state-space approach to Wiener-Hopf methods, are studied in Chapter 5. The LMI are derived from consideration of the spectral factor equations. The partial fraction expansion step is shown to be related to the LMI and to result in a stable closed loop system. The LMI are then used to evaluate the performance index. While the solution method developed requires that the spectral factors are invertible a solution can still exist if the spectral factors are not invertible. The nature of the such solutions is investigated and the state feedback problem is studied as it is an important example of such an LQG problem. Wiener-Hopf methods can also be used to solve minimum variance estimation problems; the state-space techniques developed in this chapter are used to study this problem.

In Chapter 5 only the closed loop transfer function from the outputs to the control inputs was derived. In Chapter 6 the form of the controller is considered; some new descriptor forms for singular LQG controllers are derived. Singular state feedback control problems and singular state estimation problems are then studied as a prelude to studying the separation structure of singular LQG controllers. A feedback interpretation is given for the descriptor forms which allows block diagram representations for singular LQG controllers to be given. These block diagrams have a similar structure to the structure of nonsingular LQG controllers with the difference being that the constant state feedback and Kalman filter gains are replaced by singular loops. The order of LQG controllers is studied followed by a discussion of some design issues that need to be considered when designing singular LQG controllers. Chapter 6 concludes with some comments about the existence and uniqueness of LQG controllers.

The thesis concludes with a summary and some ideas for future work. The appendix contains the paper "Linear Quadratic Gaussian Controllers for Perfect Measurements" which was presented at the 12th World Congress of the International Federation of Automatic Control in 1993.

1.3. NOTATION

In this section some common notation which is used throughout the thesis is introduced.

The open loop system to be controlled is defined by the following linear, continuous, time-invariant system:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) + E\mathbf{d}(t) \quad , \quad \mathbf{x}(0) = \mathbf{0}$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + \mathbf{n}(t)$$

where

$\mathbf{x}(t)$ is the vector of system states of dimension n,
$\mathbf{y}(t)$ is the vector of system measurements of dimension m,
$\mathbf{u}(t)$ is the vector of control inputs of dimension r,
$\mathbf{d}(t)$ is the vector of disturbance noise of dimension d.
$\mathbf{n}(t)$ is the vector of measurement noise of dimension m.

Attention is limited to systems for which m, r, and d are all less than n. A, B, C and E are constant matrices of compatible dimensions. The transfer functions associated with this system and their fractional representations are:

$$P(s) = (sI-A)^{-1}B = N_p(D_p)^{-1} = (\tilde{D}_p)^{-1}\tilde{N}_p$$

$$P_d(s) = (sI-A)^{-1}E = N_D(D_D)^{-1} = (\tilde{D}_D)^{-1}\tilde{N}_D$$

$$G(s) = C(sI-A)^{-1}B = N_g(D_g)^{-1} = (\tilde{D}_g)^{-1}\tilde{N}_g$$

$$G_d(s) = C(sI-A)^{-1}E = N_d(D_d)^{-1} = (\tilde{D}_g)^{-1}\tilde{N}_g$$

The regulator controller

$$\mathbf{u}(s) = -H(s)\mathbf{y}(s)$$

is used to produce the closed loop system

$$\mathbf{u}(s) = -T(s)\mathbf{y}(s)$$

where

$$T(s) = (I+H(s)G(s))^{-1}H(s)$$

The following symbols are used:

$:=$	is defined as,
$\sim$	is similar to
A^T	is the matrix transpose,
$G^*(s)$	is the adjoint transfer function defined by $G^T(-s)$,
A^R	is a right inverse of A ($AA^R = I$),
A^L	is a left inverse of A ($A^LA = I$),
$Tr\{\cdot\}$	is the matrix trace,
$\mathcal{E}[\cdot]$	is the expected value over time $t \in [0.\infty)$,
$\det(\cdot)$	is the matrix determinant.

The LQG problem is given by:

$$J = \mathcal{E}\left[\begin{bmatrix}\mathbf{x}^T(t) , \mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix} Q & S \\ S^T & R\end{bmatrix}\begin{bmatrix}\mathbf{x}(t) \\ \mathbf{u}(t)\end{bmatrix}\right]$$

The weightings on the states and inputs in an LQG problem are given by:

$$W = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} C_1^T \\ D_1^T \end{bmatrix}\begin{bmatrix} C_1 , D_1 \end{bmatrix}$$

The spectral factor Δ is defined by:

$$\Delta^*\Delta = B^T(-sI-A^T)^{-1}Q(sI-A)^{-1}B + B^T(-sI-A^T)^{-1}S + S^T(sI-A)^{-1}B + R$$
$$= \Delta_1^*\Delta_1$$

where $\Delta_1 = C_1(sI-A)^{-1}B+D_1$. The spectral factor Δ has the state-space form:

$$\Delta = K(sI-A)^{-1}B+D = \Lambda(D_p)^{-1}$$

where K and D are determined from the LMI:

$$LMI(X) := \begin{bmatrix} A^TX+XA+Q & XB+S \\ B^TX+S^T & R \end{bmatrix} = \begin{bmatrix} K^T \\ D^T \end{bmatrix}\begin{bmatrix} K , D \end{bmatrix}$$

The noise intensities of the disturbances acting on the open loop system are:

$$\Phi = \begin{bmatrix} E\Phi_dE^T & E\Phi_{dn} \\ \Phi_{dn}^TE^T & \Phi_n \end{bmatrix} = \begin{bmatrix} B_1 \\ V_1 \end{bmatrix}\begin{bmatrix} B_1^T , V_1^T \end{bmatrix}$$

The disturbance spectral factor Γ is defined by:

$$\Gamma\Gamma^* = C(sI-A)^{-1}E\Phi_dE^T(-sI-A^T)^{-1}C^T + \Phi_{dn}^TE^T(-sI-A^T)^{-1}C^T + C(sI-A)^{-1}E\Phi_{dn} + \Phi_n$$
$$= \Gamma_1\Gamma_1^*$$

where $\Gamma_1 = C(sI-A)^{-1}B_1+V_1$. The spectral factor Γ has the state-space form:

$$\Gamma = C(sI-A)^{-1}L+V = (\tilde{D}_p)^{-1}\Omega$$

where L and V are determined from the LMI:

$$LMI(Y) := \begin{bmatrix} YA^T+AY+E\Phi_dE^T & YC^T+E\Phi_{dn} \\ CY+\Phi_{dn}^TE^T & \Phi_n \end{bmatrix} = \begin{bmatrix} L , V \end{bmatrix}\begin{bmatrix} L^T \\ V^T \end{bmatrix}$$

The transfer function M(s) is defined as:

$$M(s) = (\Delta^*)^{-1}\left(P^*Q+S^T\right)(sI-A)^{-1}\left(E\Phi_dG_d^*+E\Phi_{dn}\right)(\Gamma^*)^{-1}$$

Associated with the descriptor forms are the matrices:

$E_1 = I - K^R(I-D)K$

$E_{F1} = I - K_F^R(I-D)K_F$ where $K_F = K-DF_1$

$E_2 = I - L(I-V)L^L$

$E_{F2} = I - L_F(I-V)L_F^L$ where $L_F = L-F_2V$

Occasionally a geometric approach will be used. Matrices are to be thought of as being maps from one space to another. Consider two linear spaces $\mathcal{X}$ and $\mathcal{Y}$. Let A map $\mathcal{X}$ to $\mathcal{Y}$.

The kernel and image of the mapping A are defined as follows:

$Ker\ A := \{\mathbf{x} \in \mathcal{X}$ where $A\mathbf{x} = \mathbf{0}\}$

$Im\ A := \{\mathbf{y} \in \mathcal{Y}$ where there exists $\mathbf{x} \in \mathcal{X}$, $\mathbf{y} = A\mathbf{x}\}$

CHAPTER 2

LINEAR SYSTEMS AND CONTROLLER CONFIGURATION

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2.1. INTRODUCTION

A basic description of linear system representations, Gaussian white noise processes and controller configurations is given in this chapter. These are important elements of the LQG control problem.

A number of different ways of representing time-invariant, continuous-time, finite-dimensional linear systems are presented; state-space, polynomial, fractional and descriptor representations are considered. The results in this chapter are intended to provide a common reference point, and to establish notational conventions. An indepth discussion of system representations can be found in one of the many texts on the linear systems (Kailath 1980; Vidyasagar 1985; Kwakernaak and Sivan 1972). The Gaussian white noise process and the response of linear systems to white noise are studied following the approach used by Kwakernaak and Sivan (1972). The general description of the system to be controlled with an LQG controller is introduced and the regulator controller configuration for this system is specified. Closed loop stability is then studied with particular emphasis on transfer function definitions of closed loop stability.

2.2. REPRESENTATION OF LINEAR SYSTEMS

The amenability of linear systems to analysis has been exploited in many areas, both within and outside the field of control theory. The solution of the LQG problem is, perhaps, one of the greatest successes of linear systems theory. Before studying the LQG problem in detail, methods for representing linear systems are introduced. Attention in this thesis is limited to continuous-time, time-invariant, finite-dimensional linear systems. Four methods of representing these linear systems are presented in this section.

Before proceeding to study each method in detail a brief history of linear systems representations in control theory is presented. Initially, frequency domain descriptions of linear systems were used (Bode 1945; Wiener 1949). These techniques were mainly limited to Single Input Single Output (SISO) systems. Following the work of Kalman and Bellman, the main tool in the study of linear systems has been state-space analysis. The main advantages of state-space techniques is the inherent ability to describe multivariable systems and that the internal structure of systems is more readily studied with these methods. In the 1970's, Rosenbrook and Macfarlane headed a renewed interest in frequency domain techniques in which many of the SISO techniques were extended to multivariable systems. From this study the polynomial and fractional

representations, due to Kucera and Desoer respectively, were developed. One of the main disadvantages of state-space representations is that they can only be used to represent proper transfer functions (numerator degree is less than or equal to the denominator degree). A generalisation of state-space forms which can be used to represent improper transfer functions is the descriptor form (also known as generalised state-space systems, semi-state systems, or singular systems). These representations were introduced to the control literature by Rosenbrook and Luenburger and have been the subject of much research in the last ten years.

The focus of this thesis is in using frequency domain descriptions of systems. The fundamental tool of frequency domain analysis is the Laplace transform, which is defined by:

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt \tag{2.1}$$

The complex variable s can be interpreted as a frequency $s=j\omega$ for some real ω for functions f(t) which are absolutely integratable on $(-\infty,\infty)$. The inverse Laplace transform is defined by:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\,ds \tag{2.2}$$

where σ is greater than the abscissa of convergence. When F(s) is a rational function in s, σ is chosen to be to the right of all the poles of F(s). A particular case which will be considered throughout this thesis is the use of stable transfer functions in which case σ may be taken to be zero. Operational properties of the Laplace Transform are readily available in most elementary mathematics textbooks (Kreyszig 1983).

The rest of the section is organised as follows: state-space representations are discussed in Section 2.2.1, polynomial representations in Section 2.2.2, fractional representations in Section 2.2.3 and descriptor representations in Section 2.2.4.

2.2.1. State-Space Representations

Consider a system described by the following linear, time-invariant system of differential equations:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \quad , \quad \mathbf{x}(0) = \mathbf{x}_0. \\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t) \end{aligned} \tag{2.3}$$

where:

x(t) is the state vector (of dimension n);
y(t) is the system output vector (of dimension m);
u(t) is the input vector (of dimension r).

and the matrices A, B, C, and D are matrices of compatible dimension. The system is stable if all the eigenvalues $\lambda_i(A)$ of A are in the left half of the complex plane.

The time domain solution of this system is:

$$\mathbf{x}(t) = e^{At}\mathbf{x_0} + \int_0^t e^{A(t-\tau)}\mathbf{u}(\tau)\, d\tau \tag{2.4}$$

The transfer function of (2.3) is:

$$\mathbf{y}(s) = G(s)\mathbf{u}(s) + C(sI-A)^{-1}\mathbf{x_0} \tag{2.5}$$

where

$$G(s) = C(sI-A)^{-1}B+D \tag{2.6}$$

G(s) is referred to as the transfer function between **u**(s) and **y**(s). A proper transfer function G(s) has a state-space representation defined by four constant matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$. Normally the indeterminate 's' in G(s) is omitted. Two alternative ways of writing G(s) are used as follows:

$$G(s) = \mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{B}+\mathcal{D} = [\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}] \tag{2.7}$$

Throughout this thesis the script letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ refer to the matrices in equation (2.7) of a state-space representation of a linear system (for example: the $\mathcal{A}$ matrix of the system [A , B , C , D] is A.).

Some standard state-space formulae for combinations of transfer functions are introduced below:

Definition 2.1 Standard State-Space Formulae (Francis 1987)

The conjugate transpose of a transfer function G(s) = [A , B , C , D] is defined by :

$$G^*(s) = G^T(-s) = [-A^T, -C^T, B^T, D^T] \tag{2.8}$$

For two transfer functions $G_1(s) = [A_1 , B_1 , C_1 , D_1]$ and $G_2(s) = [A_2 , B_2 , C_2 , D_2]$:

(i) The product of two transfer functions is given by:

$$G_1(s)G_2(s) = \left[\begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix}, [C_1 , D_1C_2] , D_1D_2\right] \tag{2.9}$$

Multiplication of two transfer functions corresponds to placing the two transfer functions in series such that the output of $G_2(s)$ becomes the input to $G_1(s)$.

(ii) The sum of two transfer functions is given by :

$$G_1(s) + G_2(s) = \left[\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, [C_1 , C_2] , D_1 + D_2\right] \tag{2.10}$$

Addition of transfer functions corresponds to placing the transfer functions in parallel ($\mathbf{u}_1(s) = \mathbf{u}_2(s)$) and adding the outputs ($\mathbf{y}(s) = \mathbf{y}_1(s) + \mathbf{y}_2(s)$).

(iii) Two state-space representations are said to be equivalent if there exists a nonsingular T such that:

$$A_1 = TA_2T^{-1} , B_1 = TB_2 , C_1 = C_2T^{-1} \text{ and } D_1 = D_2 . \tag{2.11}$$

The equivalence is an output equivalence ($G_1(s)=G_2(s)$). ♦

Controllability, Stabilisability and Observability, Detectability

The concepts of controllability and stabilisability are concerned with the manipulation of the response of the states $\mathbf{x}(t)$ with the external inputs $\mathbf{u}(t)$. Formally controllability and stabilisability are defined as follows:

Definition 2.2 Controllability/Stabilisability

A system is said to be completely controllable if, given an initial state $\mathbf{x}(0)\neq\mathbf{0}$, the states can be driven to zero by the inputs $\mathbf{u}(t)$ in a finite amount of time. The following conditions are equivalent:

(i) The pair (A,B) is completely controllable;

(ii) $P = \left[B , AB , A^2B , \ldots , A^{n-1}B\right]$ is full rank;

(iii) $\text{rank}\left(\left[\lambda I\text{-}A \;,\; B\right]\right) = n \quad \text{for all } \lambda.$

The system is stabilisable if a non-zero initial state can be driven asymptotically to zero. A system is stabilisable if $\text{rank}\left(\left[\lambda I\text{-}A \;,\; B\right]\right) = n$ holds for all λ such that $\text{Re}(\lambda)\geq 0$.

Any value of λ which causes $\left[\lambda I\text{-}A \;,\; B\right]$ to lose rank is necessarily an eigenvalue of A. Therefore these values of λ correspond to uncontrollable eigenvalues or modes.

The concepts of observability and detectability are concerned with the ability to detect the presence of non-zero states in the output **y**(t). Formally observability and detectability are defined as follows:

Definition 2.3 Observability/Detectability

A system is said to be observable if the output is non-zero for all non-zero initial conditions. The following conditions are equivalent:

(i) The pair (C,A) is completely observable;

(ii) $Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ \cdots \\ CA^{n-1} \end{bmatrix}$ is full rank;

(iii) $\text{rank}\left(\begin{bmatrix} C \\ \lambda I\text{-}A \end{bmatrix}\right) = n \quad \text{for all } \lambda.$

A system is detectable if the unobservable states tend asymptotically to zero. The system is detectable if $\text{rank}\left(\begin{bmatrix} C \\ \lambda I\text{-}A \end{bmatrix}\right) = n$ holds for all λ such that $\text{Re}(\lambda)\geq 0$.

It should be noted that observability is dual to controllability (that is, (C,A) is observable if and only if (A^T,C^T) is controllable. Detectability is dual to stabilisability.

Zeros of a Transfer Function

A zero of a SISO transfer function is any complex number for which the numerator of the transfer function equals zero. To extend the concept of zeros to multivariable systems is not straightforward; there are several types of multivariable zeros.

Definition 2.4 Multivariable Zeros of a Transfer Function (Macfarlane and Karcanias 1976)

A zero of a multivariable transfer function is defined as a frequency for which the output is zero for some non-zero state and/or input. Multivariable zeros are classified as follows:

(i) **Invariant Zeros** includes all zeros z for which the matrix $\begin{bmatrix} zI-A & -B \\ C & D \end{bmatrix}$ loses rank.

(ii) **Decoupling Zeros** correspond to uncontrollable and unobservable modes (Definitions 2.2. and 2.3).

(iii) **Transmission Zeros** includes all zeros z for which the matrix $\begin{bmatrix} zI-A & -B \\ C & D \end{bmatrix}$ loses rank, provided these are not decoupling zeros.

(iv) **System Zeros** includes all transmission and decoupling zeros. Note that for a square system the system and invariant zeros coincide.

For square systems, invariant zeros z of $G(s)=C(sI-A)^{-1}B+D$ satisfy the generalised eigenvalue problem:

$$\begin{bmatrix} zI-A & -B \\ C & D \end{bmatrix}\begin{bmatrix} X \\ U \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.12}$$

The generalised eigenvector is comprised of X, the zero direction, and U, the zero input direction. The generalised eigenvalues for this problem can be finite or infinite. The maximum number of finite zeros of a transfer function $G(s)= C(sI-A)^{-1}B$ is given in the following lemma:

Lemma 2.5 Maximum Number of Finite Zeros
(Kouvaritakis and Shaked 1976)

The maximum number of finite zeros for a square transfer function $G(s) = C(sI-A)^{-1}B$ is n-m-d where n is the dimension of A, m is the number of inputs and outputs, and rank(CB)=m-d. If d=0 (that is, CB is full rank) then the number of finite zeros is precisely n-m.

An alternative definition can be used if $G(s) = C(sI-A)^{-1}B$ is an mxm matrix transfer function. The zeros are given by the solution to the generalised eigenvalue problem:

$$\det(sNM-NAM) = 0 \qquad (2.13)$$

where NB=0 and CM=0.

Inverse of a Transfer Function

The inverse of a transfer function has many applications in system theory. Extensive use will be made of inverses of transfer functions in this thesis in connection with LQG problems. The following lemma establishes conditions under which the inverse of a transfer function exists:

Lemma 2.6 Inverse of a Transfer Function

The inverse of a square transfer function $G(s)=C(sI-A)^{-1}B+D$ will exist if and only if:

$$\det(C(sI-A)^{-1}B+D) = \det\left(\begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix}\right) \not\equiv 0$$

A necessary condition for the inverse to exist is that $\begin{bmatrix} B \\ D \end{bmatrix}$ has full column rank and $\begin{bmatrix} C , D \end{bmatrix}$ has full row rank. The poles of the inverse correspond to the invariant zeros of the original transfer function. If D is nonsingular, the inverse has the state-space form:

$$G^{-1}(s) = [A-BD^{-1}C , BD^{-1} , -D^{-1}C , D^{-1}] \qquad (2.14)$$

Minimal State-Space Representations

An important issue in forming a state-space representation is the question of forming a minimal representation (a state-space representation with the smallest number of states). A state-space representation is said to be minimal in an input-output sense if the output from two systems is the same for a given set of inputs. The conditions under which a state-space representation is minimal in an input-output sense are defined in the following lemma:

Lemma 2.7 Minimal State-Space Representations.

A state-space representation with a zero initial state is said to be minimal in an input/output sense if and only if the system is controllable and observable. If the initial state is nonzero, the representation is minimal if and only if the system is observable.

2.2.2. Polynomial Descriptions

As an alternative to the transfer function descriptions in (2.6), the transfer function can be represented as the ratio of two polynomials in s. To account for matrix multiplication being non-commutative, there are left and right factors for each of the transfer functions. Consider a rational function with polynomial representation:

$$G(s) = N(s)D^{-1}(s) = \tilde{D}^{-1}(s)\tilde{N}(s) \qquad (2.15)$$

The rational function G(s), by contrast with the requirement for state-space representations that G(s) is proper, does not necessarily have to be proper. A polynomial factorisation is not unique, as one could arbitrarily introduce any unimodular polynomial (a polynomial matrix is unimodular if its inverse is also a polynomial) into the numerator and denominator of (2.15). The numerators and denominators in (2.15) have no common factors if they satisfy the Bezout identities below; in this case the factors are said to be coprime.

Definition 2.8 Coprime Factors and Bezout Identities

Consider the factors N and D of a right polynomial representation $G=ND^{-1}$. D and N are right coprime if there exist $\tilde{X}$, $\tilde{Y}$ such that:

$$\tilde{X}D + \tilde{Y}N = I \tag{2.16}$$

Similarly the factors of a left polynomial representation $G=\tilde{D}^{-1}\tilde{N}$. $\tilde{D}$ and $\tilde{N}$ are left coprime if there exist X and Y such that:

$$\tilde{D}X + \tilde{N}Y = I \tag{2.17}$$

Equations (2.16) and (2.17) are known as Bezout Identities ♦

A discussion of common factors for two polynomial matrices can be found in Kailath (1980, Section 6.3).

2.2.3. Fractional Representations

Instead of polynomials, the fractional representations of Desoer *et al.* (1980) can be used. A fractional representation has factors which are stable, proper transfer functions. There are a number of conceptual advantages in using fractional representations (see Vidyasagar 1985). If the original transfer function is proper, there are simple state-space forms for fractional representations which are based on state feedback concepts. These state-space forms for fractional representations are presented in the following lemma:

Lemma 2.9 State-Space forms for Fractional Representations (Vidyasagar 1985, 82-85)

Consider a transfer function G(s) = [A , B , C , D] with (A,B) stabilisable and (A ,C) detectable and with left and right fractional representations:

$$G(s) = N_g(D_g)^{-1} = (\tilde{D}_g)^{-1}\tilde{N}_g \tag{2.18}$$

To form left and right fractional representations of G(s), choose F_1 and F_2 such that $A-F_1C$ and $A-BF_2$ are stable. The various factors are then given by:

$$\tilde{D}_g = I - C(sI-A+F_1C)^{-1}F_1 \tag{2.19}$$

$$\tilde{N}_g = D + C(sI-A+F_1C)^{-1}(B-F_1D) \tag{2.20}$$

$$D_g = I - F_2(sI-A+BF_2)^{-1}B \tag{2.21}$$

$$N_g = D + (C-DF_2)(sI-A+BF_2)^{-1}B \tag{2.22}$$

It is also possible to form the Bezout Identity for these fractional representations such that:

$$\begin{bmatrix} \tilde{X} & \tilde{Y} \\ -\tilde{N}_g & \tilde{D}_g \end{bmatrix}\begin{bmatrix} D_g & -Y \\ N_g & X \end{bmatrix} = I \tag{2.23}$$

The state-space forms of X, Y, $\tilde{X}$ and $\tilde{Y}$ are:

$$\tilde{Y} = F_2(sI-A+F_1C)^{-1}F_1 \tag{2.24}$$

$$\tilde{X} = I + F_2(sI-A+F_1C)^{-1}(B-F_1D) \tag{2.25}$$

$$Y = F_2(sI-A+BF_2)^{-1}F_1 \tag{2.26}$$

$$X = I + (C-DF_2)(sI-A+BF_2)^{-1}F_1 \tag{2.27}$$

2.2.4. Descriptor Forms

State-space forms have the disadvantage that they can only be used to represent proper transfer functions. To represent improper transfer functions a more general system representation is necessary. The polynomial or fractional forms which were presented in the previous two sections can be used to represent these transfer functions. Another approach is to use generalised state-space or descriptor forms. In this section, some standard properties of descriptor systems are presented.

A descriptor system has the form:

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \qquad \mathbf{x}(0^-)=E\mathbf{x}_0$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t) \tag{2.28}$$

where E may be singular. The dimensions of the matrices E, A, C, B and D are nxn, nxn, mxn, nxr and mxr respectively. Following Luenburger (1977) the vector $\mathbf{x}(t)$ is called the vector of descriptor states to distinguish it from states in state-space forms. The individual matrices will be referred to in the same manner as the state-space form in (2.7) as:

$[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}]$

If det(sE-A)≠0, then the system is said to be regular and the transfer function is given by:

$$G(s) = C(sE-A)^{-1}B+D \tag{2.29}$$

The regularity of a descriptor system can be determined using the following lemma.

Lemma 2.10 Regularity of Descriptor Forms
(Özçaldiran and Lewis 1990)

For the descriptor system (2.28), the pair E and A satisfy the generalised eigenvalue problem:

$$\lambda E\mathbf{v} = A\mathbf{v} \tag{2.30}$$

Let V_f span the eigenspace corresponding to the finite eigenvalues and V_∞ span the eigenspace corresponding to the infinite eigenvalues. The following statements are equivalent:

(i) The pair E,A is regular (det(sE-A)≠0) ,

(ii) $\text{rank}(EV_f) = \text{rank}(V_f)$,

(iii) $\text{rank}(AV_\infty) = \text{rank}(V_\infty)$,

(iv) $\text{rank}\left\{\left[EV_f , AV_\infty\right]\right\} = n$.

As a descriptor system has finite and infinite eigenvalues, its response includes impulses as well as finite frequencies. The order and response of a descriptor system is described in the following definition:

Definition 2.11 Order of a Descriptor System
(Verghese *et al.* 1980)

The order and nature of the response of a regular descriptor system (2.28) are described by:

(i) The generalised order of the system is $f = \text{rank}(E) \leq n$,

(ii) The number of finite frequencies of the system is $g = \text{degree}(\det(sE-A)) \leq f$,

(iii) The remaining $f-g$ modes correspond to impulsive modes. ♦

The finite modes of a regular descriptor form can be determined using the following result:

Lemma 2.12 Finite Modes of a Descriptor Form
(Zhou *et al.* 1987)

For a regular descriptor form $(sE-A)^{-1}$, choose α such that $\det(\alpha E-A)\neq 0$. The g finite eigenvalues of sE-A are given by:

$$\lambda = \frac{1}{\alpha - s} \quad \text{or} \quad s = \alpha - \frac{1}{\lambda}$$

where λ is a nonzero eigenvalue of $(\alpha E-A)^{-1}E$.

The equivalence of two state-space representations was described by a nonsingular transformation of the states in Definition 2.1. The situation is more complicated with descriptor forms as the direct feedthrough matrix D can be incorporated as descriptor states. For this reason the definition of equivalence needs to be extended for descriptor representations. The following definition of equivalence is from Verghese *et al.* (1980):

Definition 2.13 Strong Equivalence
(Verghese *et al.* 1980)

Two descriptor systems $C_1(sE_1-A_1)^{-1}B_1+D_1$ and $C_2(sE_2-A_2)^{-1}B_2+D_2$ are said to strongly equivalent if:

$$\begin{bmatrix} M & 0 \\ Q & I \end{bmatrix}\begin{bmatrix} sE_1-A_1 & -B_1 \\ C_1 & D_1 \end{bmatrix}\begin{bmatrix} N & R \\ 0 & I \end{bmatrix}=\begin{bmatrix} sE_2-A_2 & -B_2 \\ C_2 & D_2 \end{bmatrix} \tag{2.31}$$

where M and N are nonsingular and $QE_1 = 0$ and $RE_1 = 0$. An additional modification that can be made to a descriptor form is trivial augmentation:

$$\begin{bmatrix} C_1 , 0 \end{bmatrix}\begin{bmatrix} sE_1-A_1 & 0 \\ 0 & I \end{bmatrix}^{-1}\begin{bmatrix} B_1 \\ 0 \end{bmatrix} + D_1 = C_1(sE_1-A_1)^{-1}B_1+D_1 \tag{2.32}$$

The reverse of this process is known as trivial deflation.

Expanding (2.31) gives:

$$\begin{bmatrix} M(sE_1-A_1)N & MA_1R-MB_1 \\ C_1N-QA_1N & D_1-QA_1R-QB_1+C_1R \end{bmatrix}=\begin{bmatrix} sE_2-A_2 & -B_2 \\ C_2 & D_2 \end{bmatrix} \tag{2.33}$$

♦

As a descriptor system can have modes at infinity, the definition of controllability and observability (Definitions 2.2 and 2.3) has to be modified to include these modes. A discussion of the extension of the concepts of controllability can be found in Özçaldiran and Lewis (1990) and Cobb (1984).

Definition 2.14 Controllability, Observability
(Özçaldiran and Lewis 1990; Cobb 1984)

A regular descriptor system is controllable if and only if:

$$\text{rank}\left(\begin{bmatrix} E\lambda-A\mu , B \end{bmatrix}\right) = n \quad \text{for all } (\lambda,\mu) \neq (0,0) \tag{2.34}$$

Similarly a regular descriptor system is observable if and only if

$$\text{rank}\left(\begin{bmatrix} C \\ E\lambda-A\mu \end{bmatrix}\right) = n \quad \text{for all } (\lambda,\mu) \neq (0,0) \tag{2.35}$$

The system is controllable at infinity if (2.34) is satisfied for $\mu=0$. Similarly the system is observable at infinity if (2.35) is satisfied for $\mu=0$.

With state-space representations, the system will have the same input-output behaviour if any unobservable and uncontrollable modes are deleted from the model (Lemma 2.7).

This is also necessary with descriptor representations. However an additional requirement for descriptor forms to be minimal is that there is an absence of nondynamic modes in $(sE-A)^{-1}$. This requirement is necessary as these nondynamic modes can be deleted from the descriptor states and added to the direct feedthrough term D. This leads to the following definition of a minimal descriptor system:

Lemma 2.15 Minimal Descriptor Forms
(Grimm 1988)

A descriptor representation is minimal if and only if it is controllable and observable as defined in Definition 2.14 and

$$A\ Ker\ E \subseteq Im\ E \tag{2.36}$$

The condition in (2.36) is equivalent to the absence of nondynamic modes in $(sE-A)^{-1}$. From a practical point of view, controllers with impulsive (or derivative) action are undesirable as they are sensitive to noise in the system. A simple test for the properness of a descriptor form can be derived from the condition in (2.36):

Lemma 2.16 Properness of a Descriptor Form
(Kautsky *et al.* 1989)

Let S_∞ and T_∞ be full rank matrices whose columns span the null spaces *Ker* E and *Ker* E^T, respectively. Then for a regular descriptor form, the following are equivalent:

(i) $(sE-A)^{-1}$ is proper

(ii) $\text{rank}([\ E\ ,\ AS_\infty\ ]) = n$

(iii) $\text{rank}\left(\begin{bmatrix} E \\ T_\infty^T A \end{bmatrix}\right) = n$

This lemma can be related to Lemma 2.15 by noting that, geometrically, Condition (ii) Lemma 2.16 can be written as:

$$Im\ E + A\ Ker\ E = \boldsymbol{R}^n$$

The condition in Lemma 2.15, (2.36) can be rewritten as:

$$Im\,E + A\,Ker\,E = Im E$$

This condition is equivalent to the absence of nondynamic modes, while Condition (ii) Lemma 2.16 means that the only non-finite modes are nondynamic.

2.3. STOCHASTIC PROCESSES

In a typical control system there are external disturbances acting on the system which the control engineer has no control over. These disturbances may be deterministic, stochastic, or a combination of both. The disturbances acting on the system to be controlled by an LQG controller are assumed to be Gaussian white noise. This stochastic process is described in Section 2.3.1. The response of linear systems to this process is then studied in Section 2.3.2. The notation $\mathcal{E}[\cdot]$ is used to denote the expected value over positive time $[0,\infty)$.

2.3.1. Gaussian White Noise

A Gaussian white noise process is defined as follows:

Definition 2.17 Gaussian Stochastic Process
(Kwakernaak and Sivan 1972, 89-90)

A Gaussian stochastic process $\mathbf{v}(t)$ is a stochastic process where for each set of instants of time $t_1, t_2, \ldots, t_m \geq t_0$ the n-dimensional vector stochastic processes: $\mathbf{v}(t_1), \mathbf{v}(t_2), \ldots, \mathbf{v}(t_m)$ have a Gaussian joint probability distribution:

$$P\left[\mathbf{v}(t_1),\mathbf{v}(t_2),\ldots,\mathbf{v}(t_m)\right] = \frac{1}{((2\pi)^{mn}\det(R))^{1/2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}[\mathbf{v}(t_i)-\mathbf{m}(t_i)]^T\Lambda_{ij}[\mathbf{v}(t_j)-\mathbf{m}(t_j)]\right\}$$

where R is the covariance matrix of $\mathbf{v}(t)$ with elements $R_{ij} = \mathcal{E}\left[\mathbf{v}(t_i)\mathbf{v}^T(t_j)\right]$ and $\Lambda=R^{-1}$ has nxn partitioned matrices Λ_{ij}. If the mean $\mathbf{m}(t)$ is constant with respect to time and the covariances $R_{ij} = \mathcal{E}\left[\mathbf{v}(t_i)\mathbf{v}^T(t_j)\right]$ depend only on t_i-t_j then the Gaussian process is stationary.

It should be noted that a Gaussian process is completely characterised by its mean and covariance. One particular Gaussian process which is used extensively in this thesis is the Gaussian white noise process. This process is derived from a Brownian motion process:

Definition 2.18 Gaussian White Noise
(Kwakernaak and Sivan 1972, 97)

A white noise process $\mathbf{w}(t)$ has zero mean and a covariance function:

$$\mathcal{E}\left[\mathbf{w}(t+\tau)\mathbf{w}^T(t)\right] = \Phi_w\delta(\tau) \tag{2.37}$$

where Φ_w, the intensity matrix, is a constant, positive semidefinite matrix and $\delta(\tau)$ is the Dirac delta function. The spectral density function, which is the Laplace transform of the covariance function, of a white noise process is:

$$\mathcal{L}\left\{\mathcal{E}\left[\mathbf{w}(t+\tau)\mathbf{w}^T(t)\right]\right\} = \Phi_w \tag{2.38}$$

which is a constant matrix.

The covariance of white noise indicates that the process $\mathbf{w}(t+\tau)$ is uncorrelated to $\mathbf{w}(t)$ except when $\tau=0$. In practice, white noise does not exist, but is an idealised form of a process which is uncorrelated when $|\tau|>\varepsilon$ for some "small" ε. White noise is not well-defined mathematically, as it can be shown to be the "derivative" of a process with no derivative (Kwakernaak and Sivan 1972, 99). To rigorously define Gaussian noise processes it is necessary to model a process with uncorrelated increments. This approach will not be pursued in this thesis. Rather, the *ad hoc* approach of Kwakernaak and Sivan (1972, Section 1.11) is adopted.

2.3.2. Response of Linear Systems to White Noise

In this section, the response of linear systems to a white noise input signal is studied. Consider the following stable linear systems:

$$\begin{aligned}\dot{\mathbf{x}}_1(t) &= A_1\mathbf{x}_1(t) + B_1\mathbf{w}(t)\,, \quad \mathbf{x}_1(0) = \mathbf{0}\\ \mathbf{y}_1(t) &= C_1\mathbf{x}_1(t)\end{aligned} \tag{2.39a}$$

and

$$\dot{\mathbf{x}}_2(t) = A_2\mathbf{x}_2(t) + B_2\mathbf{w}(t)\,, \quad \mathbf{x}_2(0) = \mathbf{0}$$
$$\mathbf{y}_2(t) = C_2\mathbf{x}_2(t) \tag{2.39b}$$

where $\mathbf{w}(t)$ is white noise with spectral density Φ_w. These are two linear systems driven by the same white noise signal $\mathbf{w}(t)$.

In the study of stochastic systems, emphasis is placed on studying the expected response rather than a particular response of a linear system. The most general expected response that will be studied is the covariance of $\mathbf{y}_1(t+\tau)$ and $\mathbf{y}_2(t)$ which is denoted:

$$\text{cov}\left[\mathbf{y}_1(t+\tau)\,,\,\mathbf{y}_2(t)\right] = \mathcal{E}\left[\mathbf{y}_1(t+\tau)\mathbf{y}_2^T(t)\right] \tag{2.40}$$

Special cases of the covariance $\mathcal{E}\left[\mathbf{y}_1(t+\tau)\mathbf{y}_2^T(t)\right]$ that are considered are: $\mathcal{E}\left[\mathbf{y}_1(t)\mathbf{y}_2^T(t)\right]$, and variances $\mathcal{E}\left[\mathbf{y}(t)\mathbf{y}^T(t)\right]$. The following theorem gives time domain results for the determination of covariances (this is the time-invariant case of Theorem 1.52 in Kwakernaak and Sivan (1972, 101)).

Theorem 2.19 Covariance of the Response of Stochastic Linear Systems

For the linear systems in (2.39) the steady-state covariance of $\mathbf{x}_1(t+\tau)$ and $\mathbf{x}_2(t)$ is given by:

$$\mathcal{E}\left[\mathbf{x}_1(t+\tau)\mathbf{x}_2^T(t)\right] = \int_0^\infty e^{A_1(t+\tau)}B_1\Phi_w B_2^T e^{A_2^T t}\,dt \tag{2.41}$$

The covariance is also given by $e^{A_1\tau}Q$ where Q is the solution to the Lyapunov equation:

$$A_1Q + QA_2^T + B_1\Phi_w B_2^T = 0 \tag{2.42}$$

The covariance of the outputs $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ is:

$$\mathcal{E}\left[\mathbf{y}_1(t+\tau)\mathbf{y}_2^T(t)\right] = C_1e^{A_1\tau}QC_2^T \tag{2.43}$$

The Lyapunov equation (2.42) plays a central role in time domain studies of stochastic linear systems (Kwakernaak and Sivan 1972). In frequency domain techniques, the

contour integrals which are described in the following theorem are used instead of (2.42):

Theorem 2.20 Contour Integrals

The covariance of the response of (2.39), given by the solution to the Lyapunov equation (2.42), can be evaluated by the contour integral:

$$\mathcal{E}\left[\mathbf{y}_1(t+\tau)\mathbf{y}_2^T(t)\right] = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} G_1(s)\Phi_w G_2^*(s)e^{s\tau}\,ds \tag{2.44}$$

The covariance is given by:

$$\mathcal{E}\left[\mathbf{y}_1(t)\mathbf{y}_2^T(t)\right] = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} G_1(s)\Phi_w G_2^*(s)\,ds \tag{2.45}$$

where $G_1(s) = C_1(sI-A_1)^{-1}B_1$, $G_2(s) = C_2(sI-A_2)^{-1}B_2$ and $G^*(s) = G^T(-s)$. The spectral density function of the two processes $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ is given by:

$$\Phi_{y_1y_2}(s) = \mathcal{L}\left\{\mathcal{E}\left[\mathbf{y}_1(t)\mathbf{y}_2^T(t)\right]\right\} = G_1(s)\Phi_w G_2^*(s) \tag{2.46}$$

Proof

From (2.41) the covariance is given by:

$$\begin{aligned}\mathcal{E}\left[\mathbf{x}_1(t+\tau)\mathbf{x}_2^T(t)\right] &= \int_0^\infty e^{A_1(t+\tau)}B_1\Phi_w B_2^T e^{A_2^T t}\,dt \\ &= \int_0^\infty \left\{\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}(sI-A_1)^{-1}e^{s(t+\tau)}\,ds\right\} B_1\Phi_w B_2^T e^{A_2^T t}\,dt\end{aligned}$$

The internal integral is an inverse Laplace transform. The order of integration can be changed due to Fubini's results for double integrals (Goldberg 1976, 350):

$$\begin{aligned}\mathcal{E}\left[\mathbf{x}_1(t+\tau)\mathbf{x}_2^T(t)\right] &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}(sI-A_1)^{-1}B_1\Phi_w B_2^T\left\{\int_0^\infty e^{A_2^T t}e^{s(t+\tau)}\,dt\right\}ds \\ &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}(sI-A_1)^{-1}B_1\Phi_w B_2^T(-sI-A_2^T)^{-1}e^{s\tau}\,ds\end{aligned} \tag{2.47}$$

Equation (2.44) then follows using (2.43). The expression for the covariance (2.45) follows by setting $\tau=0$. ♦

The Laplace transform of a white noise $\mathbf{w}(t)$ does not exist as $\mathbf{w}(t)$ has an infinite number of discontinuities. This means that a transfer function description of the systems in (2.39) cannot be used. However, the notation:

$$\mathbf{y}(s) = G(s)\mathbf{w}(s) \tag{2.48}$$

is retained because of the use of transfer functions determining covariances in Theorem 2.20.

2.4. FEEDBACK CONFIGURATION

In Section 2.2, some methods for representing linear systems were presented. Often the response of these systems to typical disturbances is unacceptable. One way of modifying the response of the system is to use feedback control in which the output from the system is used to determine some of the inputs to the system. To do this it is necessary to distinguish between the external inputs which a user can manipulate and those which cannot be manipulated. There are a number of possible feedback configurations: regulator, tracking, feedforward, and two or three degree of freedom. The feedback configuration considered in this thesis is a regulator configuration.

The general system to be controlled is presented in Section 2.4.1. The regulator controller and the closed loop response under this configuration is introduced in Section 2.4.2. In Section 2.4.3 the stability of the closed loop system is studied. The emphasis in this section is on transfer function criteria for stability.

2.4.1. Open Loop Description

Consider an open loop system described by the linear, time-invariant system of differential equations:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) + E\mathbf{d}(t) \qquad , \quad \mathbf{x}(0) = \mathbf{0}. \\ \mathbf{y}(t) &= C\mathbf{x}(t) + \mathbf{n}(t) \end{aligned} \tag{2.49}$$

where

$\mathbf{y}(t)$ is the system output vector,

$\mathbf{u}(t)$ is the input vector,

$\mathbf{d}(t)$ is the disturbance input vector,

$\mathbf{n}(t)$ is the measurement noise vector.

The following assumptions are made about the system:

Assumption 2.21 Stabilisability and Detectability of System

The pair (A,B) is assumed to be stabilisable and the pair (A,C) is assumed to be detectable. ♦

These assumptions are reasonable for a system which is to be controlled. It is necessary that the unstable modes can be manipulated (stabilisability) and that the unstable modes can be detected in the output (detectability) otherwise the controller cannot be expected to control the system.

The Laplace transform of the system described in (2.49) is:

$$\mathbf{x}(s) = P(s)\mathbf{u}(s) + P_d(s)\mathbf{d}(s) \tag{2.50}$$

$$\mathbf{y}(s) = C\mathbf{x}(s) + \mathbf{n}(s) = G(s)\mathbf{u}(s) + G_d(s)\mathbf{d}(s) + \mathbf{n}(s) \tag{2.51}$$

where

$$P(s) = (sI-A)^{-1}B \quad , \; G(s) = CP(s) = C(sI-A)^{-1}B \tag{2.52}$$

and

$$P_d(s) = (sI-A)^{-1}E \quad , \; G_d(s) = CP_d(s) = C(sI-A)^{-1}E. \tag{2.53}$$

Laplace transforms such as P(s), G(s), $P_d(s)$ and $G_d(s)$ are known as transfer functions.

2.4.2. Closed Loop Description

The feedback description used for this thesis is the regulator configuration shown in Figure 2.1.

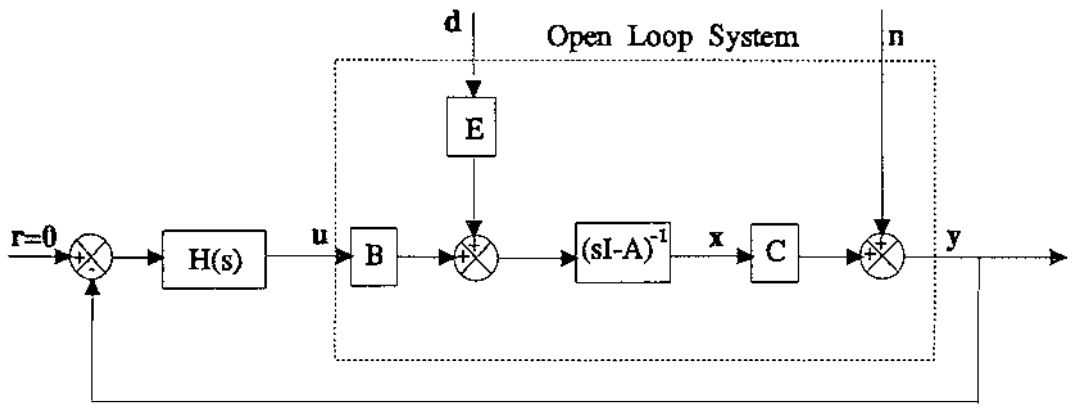

Figure 2.1
Regulator Controller Configuration

The following very well known matrix identities are used extensively in manipulating transfer function descriptions of closed loop systems:

Lemma 2.22 Useful Matrix Identities

The following two matrix identities:

$$(I+L)^{-1} = I - L(I+L)^{-1} = I - (I+L)^{-1}L \tag{2.54}$$

and

$$A(I+BA)^{-1} = (I+AB)^{-1}A \tag{2.55}$$

hold, provided the appropriate inverses exist.

Proof

The first identity follows from:

$$(I+L)(I+L)^{-1} = I$$

and the second follows from:

$$A(I+BA)(I+BA)^{-1} = A$$

It should be noted that the existence of one of the inverses in (2.55) is implied by the existence of the other, as det(I+AB)=det(I+BA). ♦

The controller H(s) is defined such that :

$$\mathbf{u}(s) = -H(s)\mathbf{y}(s). \tag{2.56}$$

Under this feedback configuration, the following expressions for $\mathbf{y}(s)$ and $\mathbf{u}(s)$ can be derived using (2.50) to (2.53) and the matrix identities in Lemma 2.22:

$$\begin{aligned}\mathbf{x}(s) &= [I+P(s)H(s)C]^{-1}\left[P_d(s)\mathbf{d}(s) - P(s)H(s)\mathbf{n}(s)\right] \\ &= \left[I - P(s)T(s)C\right]P_d(s)\mathbf{d}(s) - P(s)T(s)\mathbf{n}(s) \\ &= \left[P_d(s) - P(s)T(s)G_d(s)\right]\mathbf{d}(s) - P(s)T(s)\mathbf{n}(s)\end{aligned} \tag{2.57}$$

$$\begin{aligned}\mathbf{u}(s) &= [I+H(s)G(s)]^{-1}\left[-H(s)G_d(s)\mathbf{d}(s) - H(s)\mathbf{n}(s)\right] \\ &= -T(s)G_d(s)\mathbf{d}(s) - T(s)\mathbf{n}(s)\end{aligned} \tag{2.58}$$

where

$$T(s) = (I+H(s)G(s))^{-1}H(s) \tag{2.59}$$

Writing equations (2.57) and (2.58) in augmented form leads to:

$$\begin{bmatrix}\mathbf{x}(s)\\ \mathbf{u}(s)\end{bmatrix} = \begin{bmatrix} P_d-PTG_d & -PT \\ -TG_d & -T \end{bmatrix}\begin{bmatrix}\mathbf{d}(s)\\ \mathbf{n}(s)\end{bmatrix} \tag{2.60}$$

This augmented matrix plays an important role in establishing the closed loop stability of the system. This subject is discussed in the next section.

2.4.3. Closed Loop Stability

The feedback controlled closed loop system is said to be stable if all four transfer functions in (2.60) are stable.

Definition 2.23 Closed Loop Stability

The closed loop system is stable if the transfer functions **d→x**,**n→x**,**d→u**, and **n→u**, namely (P_d-PTG_d) , PT , TG_d and T are all stable transfer functions (that is, all the poles are in the left half of the complex plane). ♦

It should be noted that although it is necessary that all four transfer functions in Definition 2.23 are stable, a T satisfying this definition does not guarantee that the closed loop system is well defined. This is illustrated in the following example:

Consider an open loop plant G(s) which is minimum phase and invertible. Although $T = G^{-1}(s)$ satisfies Definition 2.23 no controller H(s) satisfying (2.59) and $T = G^{-1}(s)$ can be found, since from (2.59):

$$\begin{aligned} [I-TG]\,H &= T \\ [I-G^{-1}G]\,H &= G^{-1} \\ [\,0\,]\,H &= G^{-1} \end{aligned} \qquad (2.61)$$

Equation (2.61) is clearly inconsistent and therefore no controller exists such that $T = G^{-1}(s)$. This closed loop system is said to be ill-defined.

It is possible to produce a single condition which satisfies the requirements of Definition 2.23. This condition is described in the following lemma:

Theorem 2.24 Test for Closed Loop Stability

The closed loop system is asymptotically stable if and only if Ψ^{-1} is a stable transfer function where:

$$\Psi = \tilde{D}_1 D_2 + \tilde{N}_1 N_2 \qquad (2.62)$$

and $\tilde{N}_1$, $\tilde{D}_1$, N_2 and D_2 are defined in Figure 2.2. In terms of the configuration in Figure 2.1 and the fractional forms in Lemma 2.9:

$$\Psi = \tilde{D}_g D_c + \tilde{N}_g N_c \qquad (2.63)$$

where $H(s) = N_c D_c^{-1}$.

Proof

The standard feedback regulator configuration can be rewritten in fractional form as in Figure 2.2.

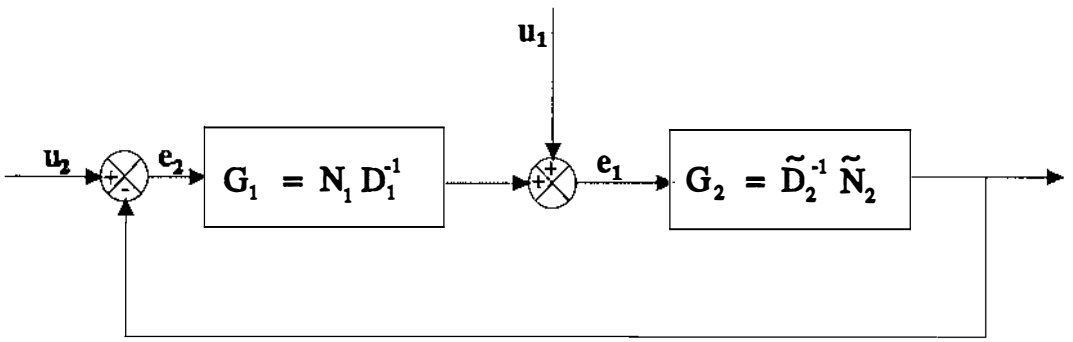

Figure 2.2
Block Diagram for Theorem 2.24

The closed loop description from the inputs $\mathbf{u}_1$ and $\mathbf{u}_2$ to the outputs $\mathbf{e}_1$ and $\mathbf{e}_2$ is given by:

$$\begin{bmatrix} e_1(s) \\ e_2(s) \end{bmatrix} = \begin{bmatrix} I-G_1(I+G_2G_1)^{-1}G_2 & -G_1(I+G_2G_1)^{-1} \\ -(I+G_2G_1)^{-1}G_2 & (I+G_2G_1)^{-1} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} \tag{2.64}$$

The closed loop system is stable if $\begin{bmatrix} I-G_1(I+G_2G_1)^{-1}G_2 & -G_1(I+G_2G_1)^{-1} \\ -(I+G_2G_1)^{-1}G_2 & (I+G_2G_1)^{-1} \end{bmatrix}$ is stable.

Substituting the fractional representations $G_1 = N_1D_1^{-1}$ and $G_2 = \tilde{D}_2^{-1}\tilde{N}_2$ into (2.64) leads to:

$$\begin{bmatrix} e_1(s) \\ e_2(s) \end{bmatrix} = \begin{bmatrix} I-N_1\Psi^{-1}\tilde{N}_2 & -N_1\Psi^{-1}\tilde{D}_2 \\ -D_1\Psi^{-1}\tilde{N}_2 & D_1\Psi^{-1}\tilde{D}_2 \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} \tag{2.65}$$

Suppose Ψ^{-1} is stable. Then it is immediate from (2.65) that the closed loop system is also stable.

Suppose the closed loop system is stable. Then the transfer function in (2.65) and:

$$\begin{bmatrix} N_1\Psi^{-1}\tilde{N}_2 & N_1\Psi^{-1}\tilde{D}_2 \\ D_1\Psi^{-1}\tilde{N}_2 & D_1\Psi^{-1}\tilde{D}_2 \end{bmatrix}$$

are stable. As the fractional forms $\tilde{N}_1$, $\tilde{D}_1$ are left coprime and N_2 and D_2 are right coprime respectively, there exist matrices $\tilde{X}_1$, $\tilde{Y}_1$ and X_2 , Y_2 such that:

$$\begin{aligned} \tilde{X}_1D_1 + \tilde{Y}_1N_1 &= I \\ \tilde{D}_2X_2 + \tilde{N}_2Y_2 &= I \end{aligned} \qquad (2.66)$$

Thus:

$$\begin{bmatrix} \tilde{Y}_1 , \tilde{X}_1 \end{bmatrix} \begin{bmatrix} N_1\Psi^{-1}\tilde{N}_2 & N_1\Psi^{-1}\tilde{D}_2 \\ D_1\Psi^{-1}\tilde{N}_2 & D_1\Psi^{-1}\tilde{D}_2 \end{bmatrix} \begin{bmatrix} Y_2 \\ X_2 \end{bmatrix} = \Psi^{-1} \qquad (2.67)$$

As the left hand side is a product of stable transfer functions, it follows that Ψ^{-1} is stable.

The feedback configuration in Figure 2.2 is the same as in Figure 2.1 although the transfer functions are slightly different. These transfer functions can be reconciled as follows:

$$\begin{aligned} G(s) &= \tilde{D}_g^{-1}\tilde{N}_g \\ &= G_2B \\ &= \tilde{D}_2^{-1}\tilde{N}_2B \end{aligned}$$

Therefore $\tilde{D}_2 = \tilde{D}_g$ and $\tilde{N}_g = \tilde{N}_2B$. Similarly with G_1 :

$$\begin{aligned} G_1(s) &= D_1^{-1}N_1 \\ &= HB \\ &= D_c^{-1}N_cB \end{aligned}$$

Therefore $D_1 = D_c$ and $N_1 = N_cB$, which leads to the form of Ψ in (2.63). This form of Ψ is very useful in establishing the closed loop stability of a given controller. ♦

This theorem is slightly different from that of Vidyasagar *et al.* (1982) as the transfer functions are not restricted to being proper rational functions. This theorem is used to derive the Youla parameterisation of controllers in Chapter 3.

2.5. SUMMARY

This chapter has served as an introductory chapter for the subjects of system representations, noise processes and feedback configurations.

State-space, polynomial, fractional and descriptor representations of linear systems were discussed in Section 2.2. Some useful properties of these representations were presented in this section for reference in future chapters.

The (Gaussian) white noise process and the response of a linear system to a white noise signal was introduced in Section 2.3. The presentation followed that of Kwakernaak and Sivan. The emphasis in this section was the frequency domain based expressions in Theorem 2.20.

The regulator feedback configuration was introduced in Section 2.4 for a general linear system described by (2.49). The closed loop system transfer function for a regulator controller was derived and the stability of the this system was studied. The closed loop configuration introduced in this section will be used to obtain the solution to the LQG problem in the next chapter.

CHAPTER 3

WIENER-HOPF METHODS FOR THE LQG CONTROL PROBLEM

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3.1. INTRODUCTION

In the previous chapter the regulator controller configuration was introduced and the closed loop response and the stability of the system were studied. The discussion in those sections was limited to a regulator configuration. The actual design of the controller was not mentioned. In this chapter the well-known Linear Quadratic Gaussian (LQG) controller design method is introduced and solved using Wiener-Hopf methods.

The LQG controller design problem is specified, and usually solved, in the time domain. However, the LQG problem can also be solved using frequency domain techniques. This is the approach taken in this thesis. Following the time domain specification of the LQG problem, an equivalent frequency domain form of the performance index is derived. The resulting performance index is a quadratic expression in an unknown closed loop transfer function. The concept of spectral factors is introduced and used to manipulate the performance index into a perfect square from which the LQG problem can be solved (Austin 1979); this is used as a common starting point for the presentation of frequency domain solution methods.

A requirement for any controller design method is that the resulting closed loop system must be stable. Newton *et al.* (1957) showed the form that the LQG controller should have by using frequency domain techniques to solve a Wiener-Hopf integral. The essence of this method is spectral factorisation and partial fraction expansion. This methodology is however limited to stable open loop systems. The extension of this method to unstable open loop systems was not completed until the 1970's when it was solved independently by three groups of researchers: Austin (1979), Youla *et al.* (1976b), and Kucera (1980). All of these methods are presented with the emphasis on the relationship between partial fraction expansion and closed loop stability. The method of Austin (1979) which provided the basis for the work of this thesis is presented in detail. The methods of Youla *et al.* (1976b) and Kucera (1980) are presented as a contrast in methodology.

3.2. LQG CONTROLLER DESIGN SPECIFICATION

In this section the LQG controller design method is introduced. The LQG problem considered here is general, permitting cross-weightings between states and inputs, and correlation between the disturbance and measurement noise. The reason for considering cross-weightings and correlations is not just as an algebraic exercise, but to provide a

more 'complete' picture of the LQG problem. The sense in which these weightings provide this more 'complete' picture of the LQG problem will be discussed in Chapter 5.

The LQG control problem is defined below:

Definition 3.1 LQG Control Problem

The LQG controller minimises the following performance index:

$$J = \mathcal{E}\left[\mathbf{x}^T(t)Q\mathbf{x}(t) + \mathbf{u}^T(t)R\mathbf{u}(t) + \mathbf{x}^T(t)S\mathbf{u}(t) + \mathbf{u}^T(t)S^T\mathbf{x}(t)\right]$$

$$= \mathcal{E}\left[\begin{bmatrix}\mathbf{x}^T(t) , \mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix} Q & S \\ S^T & R\end{bmatrix}\begin{bmatrix}\mathbf{x}(t) \\ \mathbf{u}(t)\end{bmatrix}\right] \tag{3.1}$$

where $\mathbf{x}(t)$ satisfies:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) + E\mathbf{d}(t) \quad , \quad \mathbf{x}(0) = \mathbf{0} \tag{3.2a}$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + \mathbf{n}(t) \tag{3.2b}$$

The dimensions of the vectors $\mathbf{x}(t)$, $\mathbf{u}(t)$, $\mathbf{d}(t)$ and $\mathbf{y}(t)$ are n, r, d and m respectively. It is also assumed that both r and m are less than or equal to n. The pair (A,B) are assumed to be stabilisable and (C,A) to be detectable. The control inputs are specified by the feedback regulator controller:

$$\mathbf{u}(s) = -H(s)\mathbf{y}(s) \tag{3.3}$$

The weighting matrix $\begin{bmatrix} Q & S \\ S^T & R\end{bmatrix}$ is required to be positive semidefinite. The noise processes $\mathbf{d}(t)$ and $\mathbf{n}(t)$ are assumed to be white noise processes with joint spectral densities:

$$\mathcal{L}\left\{\mathcal{E}\left[\begin{bmatrix}\mathbf{d}(t) \\ \mathbf{n}(t)\end{bmatrix}\begin{bmatrix}\mathbf{d}^T(t) , \mathbf{n}^T(t)\end{bmatrix}\right]\right\} = \begin{bmatrix} \Phi_d & \Phi_{dn} \\ \Phi_{dn}^T & \Phi_n\end{bmatrix} \tag{3.4}$$

♦

The weighting matrix and the noise intensities can in general be time varying. Only the case where these quantities are constant is considered in this thesis. If the noise process is the response of a stable linear system to a white noise input the states of this process can be augmented to the states of the open loop system $\mathbf{x}(t)$ to form an LQG problem

with constant noise intensities as in Definition 3.1. Dynamic weightings which can be represented by stable linear systems can be handled similarly. The open loop system (3.2) with a feedback regulator controller (3.3) was studied in Section 2.4.

The performance index (3.1) consists of three terms: state variances $\mathcal{E}\left[\mathbf{x}^T(t)Q\mathbf{x}(t)\right]$, input variances $\mathcal{E}\left[\mathbf{u}^T(t)R\mathbf{u}(t)\right]$, and power terms $\mathcal{E}\left[\mathbf{x}^T(t)S^T\mathbf{u}(t)+\mathbf{u}^T(t)S\mathbf{x}(t)\right]$. With a regulator controller, a control engineer aims to minimise the state or output deviations. However there is generally a limit on the size of control input $\mathbf{u}(t)$ that can be implemented. Occasionally the available 'power' is also limited. The cross-weighting terms between the states and inputs are called 'power' terms by analogy with an electrical circuit with a variable input current I and the voltage V being measured. A controller could be designed to minimise the total power V*I (Anderson and Moore 1990). Generally LQG controller design involves a trade-off between these three competing factors. The LQG control problem specifies the deviations of states, inputs and power as variances from the steady state value. The weights Q, S and R allow more 'weight' to be put on one factor compared with the others. There are also techniques which use the weights and noise intensities as design parameters to determine the closed loop response. These methods will be discussed further in Chapter 5.

Although the LQG formulation in Definition 3.1 is generally used some results are more naturally expressed in terms of the so-called H_2 problem which is defined as follows:

Definition 3.2 H_2 Formulation of the LQG Problem

The performance index for the H_2 problem is given by:

$$J = \mathcal{E}\left[\mathbf{z}^T(t)\mathbf{z}(t)\right] \tag{3.5}$$

where:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) + B_1\mathbf{w}(t) \quad , \quad \mathbf{x}(0) = \mathbf{0} \tag{3.6a}$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + V_1\mathbf{w}(t) \tag{3.6b}$$

$$\mathbf{z}(t) = C_1\mathbf{x}(t) + D_1\mathbf{u}(t) \tag{3.6c}$$

The noise process $\mathbf{w}(t)$ has spectral density function Φ_w=I.

The two formulations can be reconciled by noting that the weightings on the states and the inputs are represented by:

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} C_1^T \\ D_1^T \end{bmatrix} \begin{bmatrix} C_1 , D_1 \end{bmatrix} \tag{3.7}$$

The noise process **w**(t) includes both disturbance **d**(t) and measurement noise **n**(t):

$$\begin{bmatrix} E\Phi_d E^T & E\Phi_{dn} \\ \Phi_{dn}^T E^T & \Phi_n \end{bmatrix} = \begin{bmatrix} B_1 \\ V_1 \end{bmatrix} \begin{bmatrix} B_1^T , V_1^T \end{bmatrix} \tag{3.8}$$

There are a number of methods of solving the LQG problem. The best known solutions are in the time domain and are based on Pontryagin's maximum principle (Athans and Falb 1966) or Hamiltonian methods (Anderson and Moore 1990). Frequency domain techniques for solving this problem are presented in the rest of this chapter. Firstly it is necessary to transform the time domain criterion of equation (3.1) into the frequency domain and to introduce some techniques for manipulating the resulting expression. This task is performed in the next section.

3.3. A FREQUENCY DOMAIN EXPRESSION FOR THE LQG PERFORMANCE INDEX

The LQG performance index (3.1) is specified in the time domain. The results of Section 2.3 are now used to transform the performance index into the frequency domain. The resulting expression is a quadratic in the unknown closed loop transfer function T specified in (2.59). The key concept in solving the LQG problem in the frequency domain is that of the spectral factor. Spectral factors are introduced in Section 3.3.1. In Section 3.3.2 the performance index is manipulated into a perfect square from which the LQG problem can be solved. The manipulation on the performance index follows that of Austin (1979). However the resulting expression can be used to solve the LQG problem using any of the frequency domain methods.

To proceed the following assumption is made:

Assumption 3.3 Finite Performance Index

The closed loop system resulting from the application of the LQG controller is asymptotically stable and leads to a finite, non-negative performance index, (3.1). ♦

The solution obtained will be tested to check that this assumption is valid.

The performance index (3.1) is composed of quadratic terms of the form: $\mathcal{E}\left[\mathbf{y}_2^T(t)Q\mathbf{y}_1(t)\right]$. These terms can be manipulated using the following result:

Lemma 3.4 Quadratic Expressions

Consider the linear systems in (2.36). Then:

$$\mathcal{E}\left[\mathbf{y}_2^T(t)Q\mathbf{y}_1(t)\right] = \mathrm{Tr}\left\{Q\mathcal{E}\left[\mathbf{y}_1(t)\mathbf{y}_2^T(t)\right]\right\} \tag{3.9}$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{Tr}\left\{QG_1(s)\Phi_w G_2^*(s)\right\} ds \tag{3.10}$$

where $\mathrm{Tr}\{\cdot\}$ is the matrix trace.

Proof

The relationship in (3.9) is shown in Kwakernaak and Sivan (1972, 95). The significance of this relationship is that it allows quadratic responses to be studied in terms of covariances which were studied in Section 2.3. Applying Theorem 2.20 leads to (3.10). ♦

Using Lemma 3.4 allows the performance index in equation (3.1) to be expressed in the frequency domain as:

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left(\mathrm{Tr}\left\{Q\Phi_x(s)\right\} + \mathrm{Tr}\left\{R\Phi_u(s)\right\} + \mathrm{Tr}\left\{S\Phi_{xu}(s)\right\} + \mathrm{Tr}\left\{S^T\Phi_{ux}(s)\right\}\right) ds$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{Tr}\left\{\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}\begin{bmatrix} \Phi_x & \Phi_{xu} \\ \Phi_{xu}^* & \Phi_u \end{bmatrix}\right\} ds \tag{3.11}$$

From (2.60) and Theorem 2.20, the spectral densities of **x** and **u** are:

$$\begin{bmatrix} \Phi_x & \Phi_{xu} \\ \Phi_{xu}^* & \Phi_u \end{bmatrix} = \begin{bmatrix} P_d\text{-}PTG_d & -PT \\ -TG_d & -T \end{bmatrix}\begin{bmatrix} \Phi_d & \Phi_{dn} \\ \Phi_{dn}^T & \Phi_n \end{bmatrix}\begin{bmatrix} P_d^*\text{-}G_d^*T^*P^* & -G_d^*T^* \\ -T^*P^* & -T^* \end{bmatrix} \tag{3.12}$$

Expanding the terms in (3.12) leads to the following expressions for the individual spectral densities:

$$\begin{aligned}\Phi_x(s) &= (I - PTC)P_d\Phi_dP_d^*(I - C^TT^*P^*) - PT\Phi_{dn}^TP_d^*(I - C^TT^*P^*)\\ &\quad - (I - PTC)P_d\Phi_{dn}T^*P^* + PT\Phi_nT^*P^*\\ &= P_d\Phi_dP_d^* + PT(G_d\Phi_dG_d^* + G_d\Phi_{dn} + \Phi_{dn}^TG_d^* + \Phi_n)T^*P^*\\ &\quad - P_d(\Phi_dG_d^* + \Phi_{dn})T^*P^* - PT(G_d\Phi_d - \Phi_{dn}^T)P_d^* \qquad (3.13)\end{aligned}$$

$$\Phi_u(s) = T(G_d\Phi_dG_d^* + G_d\Phi_{dn} + \Phi_{dn}^TG_d^* + \Phi_n)T^* \qquad (3.14)$$

$$\begin{aligned}\Phi_{xu}(s) &= -(I - PTC)P_d\Phi_dG_d^*T^* - PT\Phi_{dn}^TG_d^* - (I - PTC)P_d\Phi_{dn}T^* + PT\Phi_nT^*\\ &= PT(G_d\Phi_dG_d^* + G_d\Phi_{dn} + \Phi_{dn}^TG_d^* + \Phi_n)T^* - P_d(\Phi_dG_d^* + \Phi_{dn})T^* \qquad (3.15)\end{aligned}$$

It should be noted that covariances Φ_x , Φ_u and Φ_{xu} are not necessarily strictly proper transfer functions and therefore the results of Section 2.3 may not apply. However Assumption 3.3 ensures that:

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}\begin{bmatrix} \Phi_x & \Phi_{xu} \\ \Phi_{xu}^* & \Phi_u \end{bmatrix}$$

is strictly proper. Therefore Lemma 3.4 can be applied to the performance index as a whole. Using the expressions in equations (3.13), (3.14) and (3.15), the performance index in equation (3.1) becomes:

$$\begin{aligned}J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\Big(&Tr\Big\{(P^*QP+S^TP+P^*S+R)T(G_d\Phi_dG_d^*+G_d\Phi_{dn}+\Phi_{dn}^*G_d^*+\Phi_n)T^*\\ &- (P^*Q+S^T)P_d(\Phi_dG_d^*+\Phi_{dn})T^* - T(G_d\Phi_d+\Phi_{dn}^*)P_d^*(S+QP)\Big\} + Tr\Big\{QP_d\Phi_dP_d^*\Big\}\Big)ds\end{aligned} \qquad (3.16)$$

The integrand of the expression for the performance index in equation (3.16) is a quadratic in T. As the other terms in this expression do not depend on the controller used, all that is required is to find the T which minimises the performance index. Then the controller H(s) can be found from T(s). One way to do this is to complete the square as in Austin (1979). There are several stages in calculating the optimal controller: the first, and most important stage, is that of spectral factorisation which is introduced in the next section.

3.3.1. Spectral Factors

The first stage in completing the square of the quadratic integrand in (3.16) is to transform the quadratic into a standard form:

$$YY^* + AY^* + YA^* + B$$

where Y is indeterminate and A and B are constant rational functions. The quadratic term in (3.16):

$$Tr\left\{(P^*QP+S^TP+P^*S+R)T(G_d\Phi_dG_d^*+G_d\Phi_{dn}+\Phi_{dn}^TG_d^*+\Phi_n)T^*\right\}$$

can be written in the form, YY^* by making the following factorisations:

$$\Delta^*\Delta = P^*QP + S^TP + P^*S + R \tag{3.17}$$

$$\Gamma\Gamma^* = G_d\Phi_dG_d^* + G_d\Phi_{dn} + \Phi_{dn}^TG_d^* + \Phi_n \tag{3.18}$$

Finding Δ and Γ is analogous to finding a square root of a rational matrix function. There are many possible solutions to this factorisation problem (Willems 1971). However, while this factorisation is initially introduced as a means of manipulating the performance index, the way these terms are factorised is crucial to the solution of the LQG problem. In fact, it will be shown that the solution of the LQG problem is completely and solely determined by Δ and Γ. A particular factorisation of equations (3.17) and (3.18), known as generalised spectral factorisation, was defined by Shaked (1976a). The following definition is adapted from there:

Definition 3.5 Generalised Spectral Factors

The generalised spectral factor Δ, satisfying equation (3.17) is defined as follows:

(i) a spectral factor is a real rational matrix function,

(ii) the poles of a spectral factor are exactly the poles of the system (that is, the poles of P(s)),

(iii) all the invariant zeros of a spectral factor are in the left half of the complex plane,

(iv) the spectral factor is invertible,

(v) $$\lim_{s\to\infty} \Delta(s) = \left[\lim_{s\to\infty} \Delta^*(s)\Delta(s)\right]^{1/2}. \qquad (3.19)$$

The spectral factor Γ is defined similarly. ♦

As a consequence of parts (iii) and (iv) of Definition 3.5 the inverse of a spectral factor exists and has all its poles in the left half of the complex plane. It is possible to relax condition (iv) to left invertibility for Δ and right invertibility for Γ. This relaxation of the conditions will be deferred until Chapter 5. Another difficulty can occur in LQG problems when Δ and/or Γ have zeros on the imaginary axis. There is no solution to such LQG problems. Consequently, in all subsequent sections it is assumed that the spectral factors Δ and Γ have no zeros on the imaginary axis.

The conditions for existence and uniqueness of the spectral factors in (3.17) and (3.18) are given by the following lemma, due to Shaked (1976a).

Lemma 3.6 Existence and Uniqueness of Generalised Spectral Factors (Shaked 1976a)

There always exists a generalised spectral factor, Δ, of (3.17). If P(s) and $P^*(s)$ have no common poles and if none of the zeros of (3.17) coincides with the right half plane poles of P(s), the generalised spectral factor is unique up to a product, from the left, by a constant unitary matrix. ♦

A similar result holds for Γ defined by (3.18).

The major difficulty with this lemma is the set of conditions which are required for uniqueness. If the open loop system is stable, then these conditions do not arise and the spectral factors are unique (up to a unitary product to the left). For unstable systems one way to avoid the difficulty associated with these conditions is to find the spectral factor of an associated problem which has a stable open loop system. This is done using Youla spectral factors.

Youla (1960) defined spectral factors as having poles only in the left half plane. There is no difference between these and the generalised spectral factors of Shaked (1976a) when the open loop system P(s) is stable. For unstable open loop systems, it is still

possible to use the Youla definition by using polynomial or fractional representations. The polynomial or fractional representation for $P = N_p D_p^{-1}$ leads to the following form for Δ:

$$\begin{aligned}\Delta^*\Delta &= (D_p^*)^{-1}N_p^*QN_p(D_p)^{-1} + (D_p^*)^{-1}N_p^*S^T + SN_p(D_p)^{-1} + R \\ &= (D_p^*)^{-1}\left[N_p^*QN_p + N_p^*S^TD_p + D_p^*SN_p + D_p^*RD_p\right](D_p)^{-1} \\ &= (D_p^*)^{-1}(\Lambda^*\Lambda)(D_p)^{-1} \end{aligned} \tag{3.20}$$

where Λ is a spectral factor which has poles only in the left half plane. This approach was used by Youla *et al.* (1976b) and Kucera (1979) in their development of frequency domain methods for solving the LQG problem. The two definitions are related by:

$$\Delta = \Lambda(D_p)^{-1} \tag{3.21}$$

A similar technique can be used for Γ, using the fractional representation $G_d = \tilde{D}_d^{-1}\tilde{N}_d$:

$$\Omega\Omega^* = \tilde{N}_d\Phi_d\tilde{N}_d^* + \tilde{D}_d\Phi_{dn}\tilde{N}_d^* + \tilde{N}_d\Phi_{dn}^T\tilde{D}_d^* + \tilde{D}_d\Phi_n\tilde{D}_d^* \tag{3.22}$$

$$\Gamma = \tilde{D}_d^{-1}\Omega \tag{3.23}$$

Using the state-space forms in Lemma 2.9 for the fractional representations, it follows that $\tilde{D}_d = \tilde{D}_g$ where $G = \tilde{D}_g^{-1}\tilde{N}_g$.

By defining the generalised spectral factors, Δ and Γ, by (3.21) and (3.23) in addition to the properties in Definition 3.5 it is possible to define a unique generalised spectral factor without the need for the conditions in Lemma 3.6.

Definition 3.7 Youla Definition of Spectral Factors

The generalised spectral factor, Δ, satisfies the properties in Definition 3.5 and is related to the Youla spectral factor in (3.20) by (3.21). This generalised spectral factor is unique up to a a product, from the left, by a constant unitary matrix. A similar modification can be made to the generalised spectral factor Γ using (3.23). ♦

Another method of representing the spectral factorisation problem is to use the H_2 formulation given in Definition 3.2:

Definition 3.8 H_2 Definition of Spectral Factors

For the H_2 formulation of the LQG problem in Definition 3.2, the spectral factors are defined by:

$$\Delta^*\Delta = \Delta_1^*\Delta_1 \tag{3.24}$$

where

$$\Delta_1 = C_1(sI-A)^{-1}B+D_1 \tag{3.25}$$

and

$$\Gamma\Gamma^* = \Gamma_1\Gamma_1^* \tag{3.26}$$

where

$$\Gamma_1 = C(sI-A)^{-1}B_1+V_1 \tag{3.27}$$

The factors Δ_1 and Γ_1 are not necessarily square. ♦

3.3.2. Completing the Square in the Frequency Domain

The integrand of the performance index (3.16) is a quadratic expression in an unknown, T. In this section, the spectral factors introduced in the previous section are used in the performance index (3.16) to transform the quadratic into a form for which it is relatively straightforward to see what the minimum value is. The technique used is that of completing the square which is a standard technique for solving quadratic equations. This technique has been used in Wiener-Hopf methods by Austin (1979), Kucera (1980) and Barrett (1976). The presentation given below follows Austin (1979).

Substituting the expressions for the spectral factors Δ and Γ (equations (3.17) and (3.18)) into the expression for the performance index in (3.16) leads to:

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Big(\mathrm{Tr}\Big\{\Delta^*\Delta T\Gamma\Gamma^*T^* - (P^*Q+S^T)P_d(\Phi_dG_d^*+\Phi_{dn})T^* - T(G_d\Phi_d+\Phi_{dn}^T)P_d^*(S+QP)\Big\} + \mathrm{Tr}\Big\{QP_d\Phi_dP_d^*\Big\}\Big)\,ds \tag{3.28}$$

This can be made into a normalised quadratic in $\Delta T\Gamma$ by introducing the factors $\Delta^{-1}\Delta$ and $\Gamma\Gamma^{-1}$ into the linear terms in T. The performance index is then given by:

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left(\mathrm{Tr}\left\{\Delta T\Gamma\Gamma^* T^* \Delta^* - M\Gamma^* T^* \Delta^* - \Delta T\Gamma M^*\right\} + \mathrm{Tr}\left\{QP_d\Phi_d P_d^*\right\}\right) ds \qquad (3.29)$$

where:

$$M = (\Delta^*)^{-1}(P^*Q+S^T)P_d(\Phi_d G_d^* + \Phi_{dn})(\Gamma^*)^{-1} \qquad (3.30)$$

The quadratic can now be factorised by completing the square:

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left(\mathrm{Tr}\left\{(\Delta T\Gamma - M)(\Delta T\Gamma - M)^* - MM^*\right\} + \mathrm{Tr}\left\{QP_d\Phi_d P_d^*\right\}\right) ds \qquad (3.31)$$

As the integrand in this expression for the performance index is Hermitian, the following properties hold:

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{Tr}\left\{(\Delta T\Gamma - M)(\Delta T\Gamma - M)^*\right\} ds \geq 0 \quad \text{for all } (\Delta T\Gamma - M)$$

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{Tr}\left\{(\Delta T\Gamma - M)(\Delta T\Gamma - M)^*\right\} ds = 0 \Leftrightarrow (\Delta T\Gamma - M) = 0 \qquad (3.32)$$

Hence the minimal expression for the performance index would require $\Delta T\Gamma = M$, However such a choice of T would generally result in an unstable system. The problem of choosing T such that the performance index (3.31) is minimised and the resulting closed loop system is stable is discussed in Section 3.4, after a description of the following output-weighted LQG problem.

3.3.3. Outputs-Weighted LQG Problems

The most general LQG problem allows the states $\mathbf{x}(t)$ to be weighted, as in (3.1). However, significant simplification in the analysis can be achieved if the outputs $\mathbf{z}(t) = C\mathbf{x}(t)$ are weighted instead of the states.

$$\begin{aligned} J &= \mathcal{E}\left[\mathbf{z}^T(t)Q_o(t)\mathbf{z}(t) + \mathbf{u}^T(t)R(t)\mathbf{u}(t) + \mathbf{z}^T(t)S_o(t)\mathbf{u}(t) + \mathbf{u}^T(t)S_o^T(t)\mathbf{z}(t)\right] \\ &= \mathcal{E}\left[\mathbf{x}^T(t)C^TQ_o(t)C\mathbf{x}(t) + \mathbf{u}^T(t)R(t)\mathbf{u}(t) + \mathbf{x}^T(t)C^TS_o(t)\mathbf{u}(t) + \mathbf{u}^T(t)S_o^T(t)C\mathbf{x}(t)\right] \end{aligned} \qquad (3.33)$$

This performance index can be manipulated to give an expression similar to (3.31). The algebra involved is almost identical and so the details are omitted. There are only two differences. The spectral factor, Δ, in (3.17) is modified to Δ_o satisfying:

$$\Delta_o^*\Delta_o = G^*Q_oG + G^*S_o + S_o^TG + R \tag{3.34}$$

The Youla form of this spectral factor is given by:

$$\Lambda_o^*\Lambda_o = \left(N_g^*Q_oN_g + N_g^*S_o^TD_g + D_g^*S_oN_g + D_g^*RD_g\right) \tag{3.35}$$

and

$$\Delta_o = \Lambda_oD_g^{-1} = \Lambda_oD_p^{-1} \tag{3.36}$$

The most significant change is in the definition of M (equation (3.30)), which becomes:

$$\begin{aligned} M &= (\Delta_o^*)^{-1}(G^*Q_o+S_o^T)G_d(\Phi_dG_d^* + \Phi_{dn})(\Gamma^*)^- \\ &= (\Delta_o^*)^{-1}(G^*Q_o+S_o^T)(\Gamma\Gamma^* - (\Phi_n + \Phi_{dn}^TG_d^*))(\Gamma^*)^{-1} \\ &= (\Delta_o^*)^{-1}(G^*Q_o+S_o^T)\Gamma - (\Delta_o^*)^{-1}(G^*Q_o+S_o^T)(\Phi_n + \Phi_{dn}^TG_d^*)(\Gamma^*)^{-1} \end{aligned} \tag{3.37}$$

This expression for M allows significant simplification to be made in the solution methods, which are given in the next section.

3.4. STABLE SOLUTIONS TO LQG PROBLEMS USING WIENER-HOPF TECHNIQUES

The requirement that the closed loop system is stable acts as a constraint on the minimisation of the performance index, (3.31). In the frequency domain this constraint is normally applied to the performance index by way of a partial fraction expansion of the transfer functions in the integrand. This connection follows from the solution of a Wiener-Hopf integral in the frequency domain (Newton *et al.* 1957). Wiener-Hopf methods were first applied to LQG problems for stable open loop systems by Newton *et al.* (1957). This methodology is presented in Section 3.4.1. The extension of this methodology to systems which are unstable was not completed until the 1970's following independent investigations by several groups of researchers: Shaked (1976a,b) and Austin (1979), Youla *et al.* (1976a,b), and Kucera (1979). These three methods are studied in Sections 3.4.2 to 3.4.4 and are compared in Section 3.4.5.

3.4.1. Causality and Wiener-Hopf Problems

The subject of least square criteria for controller design has its origin in the least squares filtering design of Wiener (1949). This method was applied to controller design

by Newton *et al.* (1957). The two key steps in the method of Newton *et al.* (1957) are spectral factorisation and causality. The role of spectral factors, which were introduced in Section 3.3.1 as a means of manipulating the performance index, is further explained in this section. It was noted in Section 3.3.2 that simply choosing $\Delta T\Gamma = M$ as a solution to the LQG problem can result in an unstable closed loop system. This problem is now resolved in this section by restricting the solution of the LQG problem to be a causal function. Causality is defined as follows:

Definition 3.9 Causality

A system $\mathbf{y}(t) = f(t)\mathbf{u}(t)$ is said to be causal if:

$$\mathbf{u}(t) \equiv 0 \text{ for all } t < 0 \;\Rightarrow\; \mathbf{y}(t) \equiv 0 \;\forall t < 0$$

A causal function is non-zero over positive time and will be denoted $f_+(t)$. A noncausal function is a function which is non-zero over negative time is denoted $f_-(t)$. A function which is both causal and noncausal can be written as: $f(t) = f_+(t) + f_-(t)$. A causal system is one whose inputs affect only current or future outputs. ♦

Some frequency domain expressions for the covariance of the response of a linear system to Gaussian white noise were derived in Section 2.3. These expressions involved adjoint transfer functions $G^*(s)=G^T(-s)$ which are noncausal as shown by the following lemma:

Lemma 3.10 Adjoints (Grimble and Johnson 1986, 91)

The adjoint of a causal transfer function is noncausal.

Proof

Consider a transfer function G(s) with $\mathbf{y}(s) = G^*(s)\mathbf{u}(s)$ with the inputs $\mathbf{u}(t) \equiv 0 \;\forall t < 0$. Then:

$$\begin{aligned}\mathbf{y}(t) &= \int_t^\infty g^T(\tau - t)\mathbf{u}(\tau)\, d\tau \\ &= \int_0^\infty g^T(\tau - t)\mathbf{u}(\tau)\, d\tau \quad \text{when } t < 0 \\ &\neq 0 \end{aligned} \tag{3.38}$$

That is, the process $\mathbf{y}(t)$ is noncausal. ♦

The Fourier Transform, as opposed to the Laplace Transform, is used in this section; this is defined by:

$$F(s) = \mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt \qquad \text{where } s = j\omega. \tag{3.39}$$

The main difference between the Laplace and Fourier transform is that for the Fourier transform the function f(t) must be absolutely convergent over $(-\infty,\infty)$. That is:

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty. \tag{3.40}$$

The Fourier Transform of the open loop system (3.2) is taken to be:

$$\mathbf{y}(s) = \mathcal{F}(h(t)\mathbf{y}(t)) = \int_{-\infty}^{\infty} h(t)\mathbf{y}(t)\, e^{-st}\, dt \quad \text{where } s = j\omega \tag{3.41}$$

where h(t) is the Heaviside function:

$$h(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

This restricts open loop systems to those which are stable.

For the regulator configuration to be a realisable system T, which is the closed loop transfer function between the measurement noise $\mathbf{n}(t)$ and the control inputs $\mathbf{u}(t)$, is required to be a causal transfer function.

A causal function which has a Fourier transform can only have poles in the left half of the complex plane. Consequently Δ, Γ, Δ^{-1} and Γ^{-1} are causal. Therefore, it is only necessary to find the optimal causal solution to $\Delta T \Gamma$ to find the optimal causal solution to T.

$$\{\mathcal{F}^{-1}(\Delta T \Gamma)\}_{+} = \mathcal{F}^{-1}(\Delta T \Gamma) \tag{3.42}$$

The transfer function M contains a mixture of causal and noncausal functions as it has adjoints of transfer functions as factors.

$$m(t) = \mathcal{F}^{-1}(M(s)) = m_{+}(t) + m_{-}(t) \tag{3.43}$$

The solution $\Delta T\Gamma=M$ would make $\Delta T\Gamma$ and therefore T noncausal. This closed loop transfer function T would not be realisable. Applying the causality condition to the performance index (3.31) leads to the optimal solution:

$$\{\mathcal{F}^{-1}(\Delta T\Gamma - M)\}_+ = 0 \tag{3.44}$$

This equation is the Wiener-Hopf integral equation. In the time domain a Wiener-Hopf integral has the form (Newton *et al.* 1957, Formula 5.4-1) :

$$\int_{-\infty}^{\infty} \psi(t)\Delta(\tau-t)\,dt - \Gamma(\tau) = 0 \quad \text{for } \tau \geq 0 \tag{3.45}$$

This integral will not be used explicitly in this thesis. The time domain solution to this equation is:

$$\mathcal{F}^{-1}(\Delta T\Gamma) = m_+(t) \tag{3.46}$$

or in the frequency domain:

$$\Delta T\Gamma = M_+(s) \tag{3.47}$$

where $M_+(s) = \mathcal{F}(m_+(t))$.

One method of evaluating M_+ is to transform M(s) into the time domain and use a Wiener-Hopf integral of the form in (3.45) to solve for the positive time part of m(t) (Grimble and Johnson 1986, 1:101,112). In practice this calculation is too complicated in all but the simplest examples.

The major advantage of using frequency domain methods is that there is a very simple way of evaluating the causal part of M(s) using partial fraction expansion. A causal function which has a Fourier transform can only have poles in the left half of the complex plane, otherwise the integral in (3.39) does not converge. Similarly a noncausal function only has poles in the right half plane. The causal part of M(s) can therefore be calculated by taking the terms in the partial fraction expansion of M(s) that have poles in the left half plane. It is straight forward to show that this solution satisfies the stability requirements of Definition 2.23. This result is summarised in the following theorem:

Theorem 3.11 Newton *et al.* (1957, Chapter 5) Solution.

The stabilising solution to an LQG problem where the open loop system (3.2) is stable is given by:

$$T = \Delta^{-1} M_+ \Gamma^{-1} \tag{3.48}$$

where Δ and Γ are the spectral factors satisfying (3.17) and (3.18) and M_+ is the terms in the partial fraction expansion of M (3.30) that have poles in the left half plane. ♦

As unstable transfer functions do not have a Fourier transform, this argument does not extend to unstable open loop systems. The extension of Wiener-Hopf methods to include systems with unstable open loop poles is considered in the next three sections.

3.4.2. A General Transfer Function Method

Shaked (1976b) developed an extension to the method of Newton *et al.* (1957) to include unstable, multivariable systems which does not directly use causality, but works from the partial fraction expansion step. In his derivation Shaked used the Riccati equations associated with the state feedback control and Kalman filter problems. This derivation is only valid for nonsingular LQG problems. Austin (1979) provided a completely frequency domain derivation of the method of Shaked (1976b) which is valid for singular and nonsingular LQG problems. The two major extensions in Shaked and Austin's method were: the use of the generalised spectral factors of Shaked (1976a), as discussed in Section 3.3.1, and the partial fraction expansion of M (3.30) which is defined as follows:

Definition 3.12 Partial Fraction Expansion

A transfer function M(s) can be expanded as:

$$\begin{aligned} M &= M_+ + M_- + M_\infty \\ &= M_+ + M_0 + M_\theta + M_\infty \\ &= M_\oplus + M_\theta + M_\infty \end{aligned} \tag{3.49}$$

where:

$[M(s)]_+$ represents terms in the partial fraction expansion of M(s) that have poles in the left half plane,

$[M(s)]_-$ represents terms in the partial fraction expansion of M(s) that have poles in the right half plane (including those on the imaginary axis),

$[M(s)]_0$ represents terms in the partial fraction expansion of M(s) that have poles at the right half plane system poles (that is poles of G(s)) of M(s) ,

$[M(s)]_\theta$ represents terms in the partial fraction expansion of M(s) that have poles at the right half plane non system poles,

$[M(s)]_\oplus$ represents terms in the partial fraction expansion of M(s) that have poles at system poles (whether in the right or left half plane),

$[M(s)]_\infty$ represents the improper part of M(s) including the constant gain. ♦

Note that the notation M_+ for the left plane terms ($Re(s) \leq 0$) is a carry over from the causality approach of Newton *et al.* (1957) which was outlined in the previous section. The '+' refers to positive time rather than the positive real part of the complex plane and has become standard notation in frequency domain LQG methods.

Consider the partial fraction expansion of $\Delta T\Gamma$-M as:

$$\Delta T\Gamma\text{-}M = [\Delta T\Gamma\text{-}M]_+ + [\Delta T\Gamma\text{-}M]_- \tag{3.50}$$

Assuming $\Delta T\Gamma$-M is analytic on the imaginary axis, and strictly proper, it follows that:

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} Tr\left\{[\Delta T\Gamma\text{-}M]_+[\Delta T\Gamma\text{-}M]_-^*\right\} ds = 0 \tag{3.51}$$

because $Tr\left\{[\Delta T\Gamma\text{-}M]_+[\Delta T\Gamma\text{-}M]_-^*\right\}$ is analytic in the right half plane. Similarly:

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} Tr\left\{[\Delta T\Gamma\text{-}M]_-[\Delta T\Gamma\text{-}M]_+^*\right\} ds = 0 \tag{3.52}$$

An alternative state-space justification of this step will be given in Section 4.5. Using (3.51) and (3.52) the performance index (3.31) can be expanded as:

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left\{ Tr\left\{(\Delta T\Gamma\text{-}M)_+(\Delta T\Gamma\text{-}M)_+^* + (\Delta T\Gamma\text{-}M)_-(\Delta T\Gamma\text{-}M)_-^* - MM^*\right\}\right.$$

$$+ \mathrm{Tr}\left\{QP_d\Phi_d P_d^*\right\}\Big] \, ds \tag{3.53}$$

Before applying the stability constraint, two preliminary results are necessary.

Lemma 3.13 $P\Delta^{-1}$ and $\Gamma^{-1}G_d$

The products $P\Delta^{-1}$ and $\Gamma^{-1}G_d$ are analytic in the right half plane.

Proof

This result was proved for nonsingular R and Φ_n by Shaked (1976b). It can be proved more generally from the Youla definition of Δ in (3.21) as follows:

$$P\Delta^{-1} = N_p\Lambda^{-1} \tag{3.54}$$

which is stable. A similar approach can be used to show that $\Gamma^{-1}G_d$ is analytic in the right half plane. ♦

Austin (1979) made the following assumption restricting the position of open loop poles.

Assumption 3.14 Restrictions on LQG Problems for Austin's Method

The spectral factors associated with the LQG problem satisfy the following property:

$(\Delta^*)^{-1}$ and $(\Gamma^*)^{-1}$ have no poles in common with the open loop system poles (that is, the poles of G(s))

This assumption was also one of the assumptions used by Shaked (1976a) for the uniqueness of generalised spectral factors (Lemma 3.6). This fact caused Austin (1979) not to be concerned about using this assumption in his derivation. However, the definition of generalised spectral factors was modified in Definition 3.7 to avoid the difficulties with nonuniqueness in Definition 3.5.

The following theorem provides the connection between stability and partial fraction expansion in forming the LQG solution. Austin (1979) showed the form the solution should have if the resulting closed loop system is stable.

Theorem 3.15 Austin Stability Theorem (1979)

If the closed loop system is stable, then subject to Assumption 3.14 (that $(\Delta^*)^{-1}$ and $(\Gamma^*)^{-1}$ have no poles in common with the open loop system poles):

$$[\Delta T\Gamma - M]_0 = 0 \quad \text{and} \quad [\Delta T\Gamma]_\theta = 0 \tag{3.55}$$

Proof

Suppose that the closed loop system is asymptotically stable. From Definition 2.23:

$$P_d - PTG_d \,,\; PT \,,\; TG_d \text{ and } T$$

have no poles in the right half plane. Now:

$$\begin{aligned}
\Delta T\Gamma &= (\Delta^*)^{-1}\Delta^*\Delta T\,\Gamma\Gamma^*(\Gamma^*)^{-1} \\
&= (\Delta^*)^{-1}(P^*QP + S^TP + P^*S + R)T(G_d\Phi_dG_d^* + G_d\Phi_{dn} + \Phi_{dn}^TG_d^* + \Phi_n)(\Gamma^*)^{-1} \\
&= (\Delta^*)^{-1}(P^*Q+S^T)PTG_d(\Phi_dG_d^*+\Phi_{dn})(\Gamma^*)^{-1} + (\Delta^*)^{-1}(P^*S+R)T(\Phi_{dn}^TG_d^*+\Phi_n)(\Gamma^*)^{-1} \\
&\quad + (\Delta^*)^{-1}(P^*S+R)TG_d(\Phi_dG_d^*+\Phi_{dn})(\Gamma^*)^{-1} + (\Delta^*)^{-1}(P^*Q+S^T)PT(\Phi_{dn}^TG_d^*+\Phi_n)(\Gamma^*)^{-1}
\end{aligned} \tag{3.56}$$

From Assumption 3.14, $(\Delta^*)^{-1}$ and $(\Gamma^*)^{-1}$ have no poles at the right half plane system poles. From this assumption and Lemma 3.13 the terms $(\Delta^*)^{-1}P^*$ and $G_d^*(\Delta^*)^{-1}$ also have no right half plane system poles. The last three terms of this expansion of $\Delta T\Gamma$ therefore have no poles at the right half plane system poles as T, TG_d and PT are all stable if the closed loop system is stable (Definition 2.23).

Therefore:

$$[\Delta T\Gamma]_0 = [\,(\Delta^*)^{-1}(P^*Q+S^T)PTG_d(\Phi_dG_d^*+\Phi_{dn})(\Gamma^*)^{-1}\,]_0 \tag{3.57}$$

but from this and (3.30):

$$\begin{aligned}
[\Delta T\Gamma - M]_0 &= [(\Delta^*)^{-1}(P^*Q+S^T)(PTG_d - P_d)(\Phi_dG_d^*+\Phi_{dn})(\Gamma^*)^{-1}]_0 \\
&= 0
\end{aligned} \tag{3.58}$$

The second equality follows because $PTG_d - P_d$ is stable if the closed loop system is stable (Definition 2.23).

Also $[\Delta T\Gamma]_\theta = 0$, because Δ and Γ only have right half plane poles corresponding with system poles and T has no poles in the right half plane because of the stability conditions (Definition 2.23). ♦

This result is used to form a stabilising solution for T. From Theorem 3.15 the right half plane expansion of $\Delta T\Gamma$ is:

$$[\Delta T\Gamma]_- = [\Delta T\Gamma]_0 + [\Delta T\Gamma]_\theta = M_0 + 0 \tag{3.59}$$

or

$$[\Delta T\Gamma - M]_- = M_\theta$$

From these restrictions it follows that $[\Delta T\Gamma - M]_-$ is analytic on the imaginary axis and therefore the expansion in (3.53) is valid. Applying this expansion to the cost function in equation (3.53) leads to:

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left(\mathrm{Tr}\left\{[\Delta T\Gamma - M]_+[\Delta T\Gamma - M]_+^* + M_\theta M_\theta^* - MM^*\right\} + \mathrm{Tr}\left\{QP_d\Phi_d P_d^*\right\}\right) ds \tag{3.60}$$

The optimal, stabilising solution is now given by:

$$[\Delta T\Gamma]_+ = M_+ \tag{3.61}$$

Combining equations (3.59) and (3.61) leads to:

$$\begin{aligned} \Delta T\Gamma &= [\Delta T\Gamma]_+ + [\Delta T\Gamma]_- \\ &= M_+ + M_0 = M_\oplus \end{aligned} \tag{3.62}$$

The optimal T is therefore given by:

$$T = \Delta^{-1} M_\oplus \Gamma^{-1} \tag{3.63}$$

The controller H can then be calculated from T using (2.59) by:

$$\begin{aligned} H(s) &= [I - TG]^{-1}T \\ &= T[I - GT]^{-1} \end{aligned} \tag{3.64}$$

provided $[I-TG]^{-1}$ exists. Some examples of systems for which these inverses do not exist will be presented in Chapter 6.

Austin (1979) used the frequency domain analysis for the LQG problem to obtain an expression for the performance index. The performance index was then used to compare the performance of various control strategies (feedback, feedforward and feedback/feedforward). From (3.60) and (3.61), the minimal value of the performance index is given by the contour integral:

$$J_{min} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left(\mathrm{Tr}\left\{M_\theta M_\theta^* - MM^*\right\} + \mathrm{Tr}\left\{QP_d\Phi_d P_d^*\right\}\right) ds$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left(\mathrm{Tr}\left\{QP_d\Phi_d P_d^*\right\} - \mathrm{Tr}\left\{M_\theta M_\oplus^* + M_\oplus M_\theta^* + M_\oplus M_\oplus^*\right\} \right) ds \qquad (3.65)$$

For this integral to converge it is necessary that the integrand has numerator degree at least two less than the denominator degree and is analytic on the imaginary axis. A sufficient condition for the integrand to have numerator degree at least two less than the denominator is that the transfer function M is strictly proper; this is proved in Chapter 4. There are potentially some problems in using this method for evaluating the performance index if the open loop system has poles on the imaginary axis. The reason for this is that P_d and M will have poles on the imaginary axis. In all examples that were tried these poles disappear when the whole of the integrand (3.65) is formed. It is not clear, using techniques so far presented, whether or not this happens generally. However, using techniques which are developed in Chapter 5, the integrand can be shown to be analytic on the imaginary axis. Provided the integral (3.65) converges it can be evaluated using standard contour integration techniques (Kreyzig 1983, Chapter 14):

$$J_{min} = \sum_{\text{LHP poles}} \mathrm{Res}\left(\mathrm{Tr}\left\{M_\theta M_\theta^* - MM^*\right\} + \mathrm{Tr}\left\{QP_d\Phi_d P_d^*\right\}\right) \qquad (3.66)$$

where Res(F(s)) represents the residues of F(s).

The general transfer function method of Austin (1979) is summarised in the following theorem:

Theorem 3.16 General Transfer Function Solution
(Austin 1979)

The optimal, stabilising solution to the Wiener-Hopf problem is given by:

$$T = \Delta^{-1} M_\oplus \Gamma^{-1} \qquad (3.63)$$

where $M_{\oplus}$ is the partial fraction expansion defined in Definition 3.12 provided Assumption 3.14 (that $(\Delta^*)^{-1}$ and $(\Gamma^*)^{-1}$ have no poles in common with the open loop poles of G(s)) is satisfied. The controller can be determined from T using:

$$H=[I-TG]^{-1}T \tag{3.64}$$

The minimal value of the performance index can be evaluated using the contour integral:

$$J_{min} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left(Tr\left\{M_\theta M_\theta^* - MM^*\right\} + Tr\left\{QP_d\Phi_d P_d^*\right\}\right) ds \tag{3.65}$$

There are two shortcomings with this method. Firstly, it only shows the form the solution should have if the closed loop system is stable but it does not show that the restrictions in equation (3.55) guarantee that the resulting closed loop system is stable. This is a feature of classical Wiener-Hopf methods (Newton *et al.* 1957; Weston and Bongiorno 1972; Shaked 1976b). The methods of Youla *et al.* (1976b) and Kucera (1979), which are considered in the next two sections, also show that their solutions guarantee that the resulting closed loop system is stable. One of the aims of this thesis is to address this shortcoming.

The second difficulty with this method is with Definition 3.12 when there are poles of M(s) which are both system, M_0, and non-system poles, M_θ. The reason for Assumption 3.14 is to avoid this ambiguity. As these restrictions are not required for other LQG solution methods it would be desirable to remove them in this method; this is done in Chapter 5. The following example shows some of the problems that occur when Assumption 3.14 is not satisfied.

Example 3.1

Consider the LQG problem with the open loop system described by:

$$G = G_d = \frac{s+2}{s(s-2)}$$

and with output weightings and covariances given by:

$$Q_o = R = \Phi_d = \Phi_n = 1 \ , \ S_o = \Phi_{dn} = 0$$

The spectral factors for this problem are:

$$\Delta_o = \Gamma = \frac{(s+2)(s+1)}{s(s-2)}$$

For this problem $(\Delta_o^*)^{-1}$ has a pole at s=2 which is a system pole, and therefore Assumption 3.14 is not satisfied. The transfer function M (3.37) is:

$$\begin{aligned} M &= \frac{(-s+2)}{(-s+2)(-s+1)} \frac{s+2}{s(s-2)} \frac{(-s+2)}{(-s+2)(-s+1)} \\ &= \frac{s+2}{s(s-2)(-s+1)^2} \end{aligned}$$

Using standard partial fraction expansion techniques, the appropriate expansion of M is:

$$\begin{aligned} M_\oplus &= \frac{-1}{s} + \frac{2}{s-2} \\ &= \frac{s+2}{s(s-2)} \end{aligned}$$

This solution does not satisfy the requirements for stability in Definition 2.23 as:

$$\begin{aligned} (I-GT)G_d &= (I-G\Delta_o^{-1}M_\oplus\Gamma^{-1})G_d \\ &= \left\{1 - \frac{1}{(s+1)^2}\right\} \frac{s+2}{s(s-2)} \\ &= \frac{(s+2)^2}{(s-2)(s+1)^2} \end{aligned}$$

is unstable. This example shows that it is necessary to include Assumption 3.14 in Theorem 3.15. The controller resulting from this solution is:

$$\begin{aligned} H(s) &= (I-\Delta_o^{-1}M_\oplus\Gamma^{-1}G)^{-1}\Delta_o^{-1}M_\oplus\Gamma^{-1} \\ &= \frac{s-2}{(s+2)^2} \end{aligned}$$

This is not a valid controller as it has a zero at s=2 which cancels the unstable open loop pole at s=2. ♦

The ambiguity in the partial fraction expansion (Definition 3.12), and the inability to ensure that closed loop system is stable, provided the initial motivation for this thesis. The ambiguity of Definition 3.12 is studied further in Chapter 4 using state-space techniques for partial fraction expansion. The methodology that is presented in Chapter 5 resolves both this ambiguity and shows that if the partial fraction expansion is defined appropriately, the restrictions in equation (3.55) do guarantee closed loop stability.

3.4.3. The Modern Wiener-Hopf Method

The modern Wiener-Hopf method for the design of LQG controllers was developed by Youla *et al.* (1976b) and is the first published fully frequency domain solution of the LQG problem which handles unstable open loop systems. This method retains the emphasis on partial fraction expansion which was used by Newton *et al.* (1957) but it incorporates some additional requirements to ensure that the resulting closed loop system is stable.

The key to this method is the parameterisation of all controllers which result in a stable closed loop system. This parameterisation is presented in Lemma 3.17. The Youla *et al.* (1976b) solution to the Wiener-Hopf problem is to optimise the performance index (3.16) over this class of controllers.

Lemma 3.17 Youla Parameterisation of Controllers

For an open loop system:

$$G(s) = \tilde{D}_g^{-1}\tilde{N}_g = N_g D_g^{-1}$$

the feedback controller

$$H(s) = N_c D_c^{-1}$$

is able to stabilise the system G(s) if and only if:

$$N_c = Y + D_g K \quad \text{and} \quad D_c = X - N_g K \tag{3.67}$$

where X and Y satisfy the Bezout identity:

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ \tilde{N}_g & \tilde{D}_g \end{bmatrix} \begin{bmatrix} D_g & Y \\ -N_g & X \end{bmatrix} = I \tag{3.68}$$

and K is an arbitrary stable rational function.

Proof

This proof is due to Middleton and Goodwin (1990).

Suppose K is stable, N_c and D_c satisfy (3.67), and X and Y satisfy (3.68). Then:

$$\begin{aligned}\Psi &= \tilde{D}_g D_c + \tilde{N}_g N_c \\ &= \tilde{D}_g(X - N_g K) + \tilde{N}_g(Y + D_g K) \\ &= \tilde{D}_g X + \tilde{N}_g Y + (\tilde{N}_g D_g - \tilde{D}_g N_g)K \\ &= I\end{aligned}$$

As $\Psi^{-1} = I$ is stable, it follows from Lemma 2.24 that the closed loop system is stable.

Suppose the controller, $H = N_c D_c^{-1}$ stabilises the system. Then, from Lemma 2.24:

$$\tilde{D}_g D_c + \tilde{N}_g N_c = \Psi \tag{3.69}$$

where Ψ^{-1} is stable. Also from the Bezout identity (3.68):

$$\tilde{D}_g X\Psi + \tilde{N}_g Y\Psi = \Psi \tag{3.70}$$

Subtracting these two equations (3.69) and (3.70) gives:

$$\tilde{D}_g(X\Psi - D_c) + \tilde{N}_g(Y\Psi - N_c) = 0 \tag{3.71}$$

As the factors of the Bezout identity (3.68) are inverses, the order of multiplication can be reversed to give the reverse Bezout identity:

$$\begin{bmatrix} D_g & Y \\ -N_g & X \end{bmatrix} \begin{bmatrix} \tilde{X} & -\tilde{Y} \\ \tilde{N}_g & \tilde{D}_g \end{bmatrix} = I \tag{3.72}$$

Postmultiplying this reverse Bezout identity by $\begin{bmatrix} (Y\Psi - N_c) \\ (X\Psi - D_c) \end{bmatrix}$ and using (3.71) gives:

$$\begin{bmatrix} D_g\left[\tilde{X}(Y\Psi - N_c) - \tilde{Y}(X\Psi - D_c)\right] \\ -N_g\left[\tilde{X}(Y\Psi - N_c) - \tilde{Y}(X\Psi - D_c)\right] \end{bmatrix} = \begin{bmatrix} (Y\Psi - N_c) \\ (X\Psi - D_c) \end{bmatrix}$$

from which:

$$\begin{bmatrix} N_c\Psi^{-1} \\ D_c\Psi^{-1} \end{bmatrix} = \begin{bmatrix} Y + D_g K \\ X - N_g K \end{bmatrix} \tag{3.73}$$

where

$$K = -\left[\tilde{X}(Y\Psi - N_c) - \tilde{Y}(X\Psi - D_c)\right]\Psi^{-1} \tag{3.74}$$

As K is constructed from the product and sum of stable transfer functions it is stable. The common right factors, Ψ^{-1}, on the left hand side of (3.73) can be cancelled to give the required result. ♦

There are two differences between the original paper of Youla *et al.* (1976b) and the following presentation. The first difference is in the performance index (3.1): weightings on the states instead of the outputs, and cross-weightings between inputs and states are permitted. The method of Youla *et al.* (1976b) has not been presented before using this extended performance index. The second difference is that the fractional forms of Section 2.2.3 are preferred to the polynomial forms which were used in the original paper (Youla *et al.* 1976b). There are several conceptual advantages in using fractional forms, which are discussed in Desoer *et al.* (1980) and Vidyasagar (1985).

The starting point for this method is the frequency domain expression for the performance index in (3.31). Before deriving the modern Wiener-Hopf solution a preliminary result that is needed in the derivation is presented. This result is presented separately as its length would otherwise obscure the derivation of the main result. This result is needed to ensure that the performance index is finite.

Lemma 3.18 M-$\Delta Y\Omega$ is Analytic on the Imaginary Axis

The transfer function M-$\Delta Y\Omega$ is analytic on the imaginary axis where M, Δ, Y and Ω are defined by (3.30), (3.17), (3.68) and (3.22) respectively.

Proof

Using fractional representations and Youla spectral factors (3.21) and (3.23) M-$\Delta Y\Omega$ can be written as:

$$\begin{aligned} M - \Delta Y\Omega &= (\Lambda^*)^{-1}\left[N_p^*Q+D_p^*S^T\right]P_d\left[\Phi_d N_d^*+\Phi_{dn}D_d^*\right](\Omega^*)^{-1} - \Lambda D_p^{-1}Y\Omega \\ &= (\Lambda^*)^{-1}\left(\left[N_p^*Q+D_p^*S^T\right]P_d\left[\Phi_d N_d^*+\Phi_{dn}D_d^*\right] - \Lambda^*\Lambda D_p^{-1}Y\Omega\Omega^*\right)(\Omega^*)^{-1} \end{aligned} \qquad (3.75)$$

As $(\Lambda^*)^{-1}$ and $(\Omega^*)^{-1}$ are analytic on the imaginary axis, (3.75) will also be if:

$$\left(\left[N_p^*Q+D_p^*S^T\right]P_d\left[\Phi_d N_d^*+\Phi_{dn}D_d^*\right] - \Lambda^*\Lambda D_p^{-1}Y\Omega\Omega^*\right) \qquad (3.76)$$

is. Attention can therefore be restricted to this transfer function. Using the definition of $\Lambda^*\Lambda$ in (3.20) to expand $\Lambda^*\Lambda D_p^{-1}Y\Omega\Omega^*$ leads to:

$$\Lambda^*\Lambda D_p^{-1}Y\Omega\Omega^* = \left(N_p^*QN_p + D_p^*S^TN_p + N_p^*SD_p + D_p^*RD_p\right)D_p^{-1}Y\Omega\Omega^*$$
$$= \left(N_p^*Q+D_p^*S^T\right)N_pD_p^{-1}Y\Omega\Omega^* + \left(N_p^*S+D_p^*R\right)Y\Omega\Omega^*$$

The second term is analytic on the imaginary axis and so can be ignored. Using this expansion in (3.76) gives:

$$\left(N_p^*Q+D_p^*S^T\right)\left(P_d\left\{\Phi_dN_d^*+\Phi_{dn}D_d^*\right\} - N_pD_p^{-1}Y\Omega\Omega^*\right) \tag{3.77}$$

Expanding the term $N_pD_p^{-1}Y\Omega\Omega^*$ using the reverse Bezout Identity (3.72), $D_p^{-1}Y = \tilde{Y}\tilde{D}_g^{-1}$, and the definition of $\Omega\Omega^*$ in (3.22) results in:

$$N_pD_p^{-1}Y\Omega\Omega^* = N_p\tilde{Y}\tilde{D}_g^{-1}\Omega\Omega^*$$
$$= N_p\tilde{Y}\tilde{D}_g^{-1}\left(N_d\Phi_dN_d^* + N_d\Phi_{dn}D_d^* + D_d\Phi_{dn}^TN_d^* + D_d\Phi_nD_d^*\right)$$
$$= N_p\tilde{Y}\tilde{D}_g^{-1}N_d\left(\Phi_dN_d^* + \Phi_{dn}D_d^*\right) + N_p\tilde{Y}\left(\Phi_{dn}^TN_d^* + \Phi_nD_d^*\right)$$

The second term is analytic on the imaginary axis and so can be ignored. Using this expansion in (3.77) gives:

$$\left(N_p^*Q+D_p^*S^T\right)\left(P_d - N_p\tilde{Y}G_d\right)\left(\Phi_dN_d^*+\Phi_{dn}D_d^*\right) \tag{3.78}$$

As $\left(N_p^*Q+D_p^*S^T\right)$ and $\left(\Phi_dN_d^*+\Phi_{dn}D_d^*\right)$ are analytic on the imaginary axis, (3.78) will also be if $P_d-N_p\tilde{Y}G_d$ is. Choosing a particular fractional representation for P(s) with $F_{1p} = F_{1g}C$ results in $\tilde{Y}C = \tilde{Y}_p$. Using this fact, and the reverse Bezout identity (3.72), $N_p\tilde{Y}_p + \tilde{X}_pD_p = I$, leads to:

$$P_d - N_p\tilde{Y}G_d = P_d - N_p\tilde{Y}_pP_d$$
$$= \left(I - (I-X_p\tilde{D}_p)\right)P_d$$
$$= X_p\tilde{D}_p\tilde{D}_D^{-1}\tilde{N}_D \qquad (\ P_d = \tilde{D}_D^{-1}\tilde{N}_D \text{ and } \tilde{D}_D = \tilde{D}_p\ )$$
$$= X_p\tilde{N}_D$$

This expression is the product of two fractional factors and is therefore analytic on the imaginary axis. The transfer function $M-\Delta Y\Omega$ is therefore analytic on the imaginary axis. ♦

The modern Wiener-Hopf solution of Youla *et al.* (1976b) is given in the following theorem.

Theorem 3.19 Modern Wiener-Hopf Solution

The modern Wiener-Hopf solution to the LQG problem (Definition 3.1) is given by:

$$T = \Delta^{-1}\left[M_+ + [\Lambda D_p^{-1} Y \Omega]_- \right]\Gamma^{-1} \tag{3.79}$$

where Y satisfies the Bezout identity (3.68), M is defined by (3.30) and the partial fraction expansion notation is defined in Definition 3.12. The controller is given by:

$$H = \Delta^{-1}\left[M_+ + [\Lambda D_p^{-1} Y \Omega]_- \right]\left(\Gamma - G\Delta^{-1}\left[M_+ + [\Lambda D_p^{-1} Y \Omega]_- \right]\right)^{-1} \tag{3.80}$$

Proof

Using the parameterisation of controllers in Lemma 3.17, the closed loop transfer function T, defined in (2.59), can be written as:

$$T = (Y + D_p K)\tilde{D}_g \tag{3.81}$$

Note that $D_g = D_p$; see Section 2.2.3. This expansion of T allows $\Delta T\Gamma - M$ to be expressed as:

$$\begin{aligned} \Delta T\Gamma - M &= \Delta(Y + D_p K)\tilde{D}_g\Gamma - M \\ &= \Lambda K\Omega - (M - \Delta Y\Omega) \\ &= \Lambda K\Omega - \left[[M - \Delta Y\Omega]_+ + [M - \Delta Y\Omega]_- + [M - \Delta Y\Omega]_\infty \right] \\ &= \left[\Lambda K\Omega - [M - \Delta Y\Omega]_+ + [\Delta Y\Omega]_\infty \right] - \left[M - \Delta Y\Omega \right]_- \end{aligned} \tag{3.82}$$

as M is assumed to be strictly proper (proved in Section 4.2). Since K is restricted in Lemma 3.17 to be a stable transfer function, it follows that $\left[\Lambda K\Omega - [M - \Delta Y\Omega]_+ + [\Delta Y\Omega]_\infty \right]$ is stable. This expansion has split $\Delta T\Gamma - M$ into stable and unstable parts.

Using the expression in (3.82) to expand the integral (3.31) leads to:

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Big(\mathrm{Tr}\Big\{\Big[\Lambda K\Omega - [M-\Delta Y\Omega]_+ + [\Delta Y\Omega]_\infty\Big]\Big[\Lambda K\Omega - [M-\Delta Y\Omega]_+ + [\Delta Y\Omega]_\infty\Big]^*$$
$$- \Big[\Lambda K\Omega - [M-\Delta Y\Omega]_+ + [\Delta Y\Omega]_\infty\Big]\Big[M - \Delta Y\Omega\Big]_-^*$$
$$- \Big[M - \Delta Y\Omega\Big]_-\Big[\Lambda K\Omega - [M-\Delta Y\Omega]_+ + [\Delta Y\Omega]_\infty\Big]^*$$
$$+ \Big[M - \Delta Y\Omega\Big]_-\Big[M - \Delta Y\Omega\Big]_-^* - MM^*\Big\} + \mathrm{Tr}\Big\{QP_d\Phi_dP_d^*\Big\}\Big) ds \qquad (3.83)$$

Assuming $\Big[\Lambda K\Omega - [M-\Delta Y\Omega]_+ - [\Delta Y\Omega]_\infty\Big]$ is strictly proper, and recalling from Lemma 3.18 that $\Big[M - \Delta Y\Omega\Big]_-$ is analytic on the imaginary axis, it follows that:

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Big(\mathrm{Tr}\Big\{\Big[\Lambda K\Omega - [M-\Delta Y\Omega]_+ - [\Delta Y\Omega]_\infty\Big]\Big[M - \Delta Y\Omega\Big]_-^*\Big\}\Big) ds = 0$$

The cost function then becomes:

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Big(\mathrm{Tr}\Big\{\Big[\Lambda K\Omega - [M-\Delta Y\Omega]_+ + [\Delta Y\Omega]_\infty\Big]\Big[\Lambda K\Omega - [M-\Delta Y\Omega]_+ + [\Delta Y\Omega]_\infty\Big]^*$$
$$+ \Big[M - \Delta Y\Omega\Big]_-\Big[M - \Delta Y\Omega\Big]_-^* - MM^*\Big\} + \mathrm{Tr}\Big\{QP_d\Phi_dP_d^*\Big\}\Big) ds \qquad (3.84)$$

As K is restricted to stable transfer functions, the minimal value of the performance index occurs when:

$$\Lambda K\Omega = \Big[[M - \Delta Y\Omega]_+ - [\Delta Y\Omega]_\infty\Big] \qquad (3.85)$$

This choice of $\Delta K\Omega$ satisfies the assumption that $\Big[\Lambda K\Omega - [M - \Delta Y\Omega]_+ + [\Delta Y\Omega]_\infty\Big]$ is strictly proper made above.

From the optimal K given by (3.85), and (3.81), the optimal T is:

$$\begin{aligned} T &= (Y+D_pK)\tilde{D}_g \\ &= Y\tilde{D}_g + D_p\Lambda^{-1}\Big[[M - \Delta Y\Omega]_+ - [\Delta Y\Omega]_\infty\Big]\Omega^{-1}\tilde{D}_g \\ &= Y\tilde{D}_g + D_p\Lambda^{-1}\Big[[M]_+ - (\Delta Y\Omega - [\Delta Y\Omega]_-)\Big]\Omega^{-1}\tilde{D}_g \qquad \text{using } W-W_- = W_+ + W_\infty \\ &= \Delta^{-1}\Big[M_+ + [\Lambda D_p^{-1}Y\Omega]_-\Big]\Gamma^{-1} \end{aligned} \qquad (3.86)$$

The controller (3.80) can be derived from this solution for T, (3.64) and the identities in Lemma 2.22. ♦

The minimal value for the performance index is given by:

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(\mathrm{Tr}\left\{\left[M - \Delta Y\Omega\right]\left[M - \Delta Y\Omega\right]^{*} - MM^{*}\right\} + \mathrm{Tr}\left\{QP_d\Phi_dP_d^{*}\right\}\right) ds \qquad (3.87)$$

In the modern Wiener-Hopf method, the transfer function $[M-\Delta Y\Omega]_-$ takes the place of M_θ in the general transfer function method of Section 3.4.2.

The main advantage of using the method of Youla *et al.* (1976b) is that the term $\Lambda D_p^{-1}Y\Omega$ only has right half plane poles at the system poles and so $[\Lambda D_p^{-1}Y\Omega]_-$ is equivalent to taking the partial fraction expansion of all the right half plane poles of $\Lambda D_p^{-1}Y\Omega$. This means that in this method there is no possibility of the ambiguity with the definition of the partial fraction expansion that occurred with the method of Austin (1979). A disadvantage of the method of Youla *et al.* (1976b) is that extra computational effort in calculating Y satisfying (3.68) is required. To conclude this section the modern Wiener-Hopf method is used to solve the LQG problem in Example 3.1.

Example 3.2

Consider the LQG problem in Example 3.1. The Bezout identity (3.68) for this problem is:

$$s(s-2)X + (s+2)Y = 1$$

which is satisfied by the pair $X = \frac{1}{8}$ and $Y = -\frac{1}{8}s + \frac{1}{2}$. From this identity and the spectral factors for this problem, given in Example 3.1:

$$\Delta Y\Omega = \frac{(s+2)(s+1)\left(-\frac{1}{8}s + \frac{1}{2}\right)(s+2)(s+1)}{s(s-2)}$$

The partial fraction expansion of the right half plane and imaginary poles of this term is:

$$[\Delta Y\Omega]_- = \frac{-1}{s} + \frac{18}{s-2}$$
$$= \frac{17s+2}{s(s-2)}$$

As there are no stable open loop poles in this problem $M_+=0$. The closed loop transfer function T(s) can then be calculated from (3.79):

$$T(s) = \Delta^{-1}[\Delta Y \Omega]_{-}\Gamma^{-1} = \frac{(17s+2)s(s-2)}{(s+1)^2(s+2)^2}$$

and the controller, from (3.80), is given by:

$$H(s) = (I-TG)^{-1}T = \frac{17s+2}{(s+6)(s+2)}$$

It can easily be verified the closed loop system under this controller is stable, as is guaranteed by Theorem 3.19. The modern Wiener-Hopf method of Youla *et al.* (1976b) can handle the classes of LQG problems which were specifically excluded from the method of Austin (1979) in Theorem 3.16. ♦

3.4.4. A Polynomial Approach

The third method for solving the LQG problem in the frequency domain is the method of Kucera (1980) which uses polynomial descriptions of the various transfer functions. The method of Kucera differs from the general transfer function method of Austin (1979) and the modern Wiener-Hopf method of Youla *et al.* (1976b) in that it does not use partial fraction expansion. Instead, the polynomial approach uses two diophantine equations. The relationship of these diophantine equations to the partial fraction expansion is shown in Theorem 3.20.

As in Section 3.3.3, attention is limited to weightings on the system outputs ($C\mathbf{x}(t)$). Some thought was given to extending this method to include weightings on states but the extra algebra and the complexity of the resulting formulae do not make this extension worth pursuing. An extension to the LQG problem considered from that considered by Kucera (1980) is that cross-weightings between the inputs and outputs as are permitted. The method of Kucera (1980) is presented in the following theorem:

Theorem 3.20 A Polynomial Solution

The LQG controller for the output weighting problem described in Section 3.3.3 is given by:

$$H = Y_1^{-1}X_1 \tag{3.88}$$

where

$$\left[\, Y\Omega_3^{-1} \;\; X\Omega_2^{-1} \,\right] = \Omega_f^{-1}\left[\, Y_1 \,,\, X_1 \,\right] \tag{3.89}$$

X and Y are given by the solution to the diophantine equations:

$$\Lambda_o^* X + Z D_2 = N_g^* Q_o \Omega_2 + D_g^* S_o^T \Omega_2 \tag{3.90}$$

$$\Lambda_o^* Y - Z N_3 = D_g^* R \Omega_3 + N_g^* S_o \Omega_3 \tag{3.91}$$

The spectral factors Λ_o and Ω are polynomial forms of (3.35) and (3.22) respectively. The additional terms in (3.90) and (3.91) are defined by the left and right factors of the following terms:

$$\Gamma = \tilde{D}_g^{-1}\Omega = \Omega_2 D_2^{-1} \tag{3.92}$$

$$\Gamma^{-1} G = \Omega^{-1}\tilde{N}_g = N_3 \Omega_3^{-1} \tag{3.93}$$

Proof

For the output LQG problem, the transfer function M in (3.30) is given by (3.37). In polynomial form, (3.37) is:

$$M = (\Lambda_o^*)^{-1}\left[N_g^* Q_o + D_g^* S_o^T\right]\tilde{D}_g^{-1}\Omega - (\Lambda_o^*)^{-1}\left[N_g^* Q_o + D_g^* S_o^T\right]\left[\Phi_n \tilde{D}_g^* + \Phi_{dn}^T \tilde{N}_d^*\right](\Omega^*)^{-1}$$

$$= M_1 + M_2 \tag{3.94}$$

Only the first term of (3.94) contains system poles. In Kucera's method two diophantine equations are used instead of partial fraction expansion. The first diophantine equation is equivalent to the partial fraction step. The first term of M in (3.94) can be expanded using (3.90) and (3.92) as:

$$\begin{aligned} M_1 &= (\Lambda_o^*)^{-1}\left[N_g^* Q_o + D_g^* S_o^T\right]\Omega_2 D_2^{-1} && \text{from (3.92)} \\ &= (\Lambda_o^*)^{-1}\left[N_g^* Q_o \Omega_2 + D_g^* S_o^T \Omega_2\right] D_2^{-1} \\ &= (\Lambda_o^*)^{-1}\left[\Lambda_o^* X + Z D_2\right] D_2^{-1} && \text{from (3.90)} \\ &= X D_2^{-1} + (\Lambda_o^*)^{-1} Z \end{aligned} \tag{3.95}$$

This is a partial fraction expansion of M_1 into poles of the system (D_2^{-1}) and noncausal poles $(\Lambda_o^*)^{-1}$. Applying this expansion of M_1 to the performance index (3.31) leads to:

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} Tr\Big\{\Big(\Delta_o T\Gamma-(XD_2^{-1}+(\Lambda_o^*)^{-1}Z+M_2)\Big)^*\Big(\Delta_o T\Gamma-(XD_2^{-1}+(\Lambda_o^*)^{-1}Z+M_2)\Big) - M^*M + Q_oG_d\Phi_dG_d^*\Big\}\,ds$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} Tr\Big\{\Big(\Delta_o T\Gamma-XD_2^{-1}\Big)^*\Big(\Delta_o T\Gamma-XD_2^{-1}\Big) + \Big((\Lambda_o^*)^{-1}Z+M_2\Big)^*\Big((\Lambda_o^*)^{-1}Z+M_2\Big) - \Big((\Lambda_o^*)^{-1}Z+M_2\Big)^*\Big(\Delta_o T\Gamma-XD_2^{-1}\Big) - \Big(\Delta_o T\Gamma-XD_2^{-1}\Big)^*\Big((\Lambda_o^*)^{-1}Z+M_2\Big) - M^*M + Q_oG_d\Phi_dG_d^*\Big\}\,ds$$

$((\Lambda_o^*)^{-1}Z+M_2)$ is analytic on the imaginary axis and strictly proper. Thus, assuming $(\Delta_o T\Gamma-XD_2^{-1})$ is also analytic on the imaginary axis and strictly proper, it follows that:

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} Tr\Big\{\Big(\Delta_o T\Gamma-XD_2^{-1}\Big)^*\Big((\Lambda_o^*)^{-1}Z+M_2\Big)\Big\}\,ds = 0$$

The minimal value of the performance index then occurs when:

$$\Delta_o T\Gamma = XD_2^{-1}$$

or

$$T = \Delta_o^{-1}XD_2^{-1}\Gamma^{-1}$$
$$= \Delta_o^{-1}X\Omega_2^{-1} \qquad (3.96)$$

The controller could then be calculated using (3.64) as:

$$H = [I-\Delta_o^{-1}X\Omega_2^{-1}G]^{-1}\Delta_o^{-1}X\Omega_2^{-1}$$
$$= [\Delta_o - X\Omega_2^{-1}G]^{-1}X\Omega_2^{-1} \qquad (3.97)$$

However, this step by itself would not ensure a stable closed loop system. It is now shown that the second diophantine equation (3.91) ensures that the resulting closed loop system is stable.

Postmultiplying (3.90) by $\Omega_2^{-1}N_g$ leads to:

$$A_o^*X\Omega_2^{-1}N_g + ZD_2\Omega_2^{-1}N_g = N_g^*Q_oN_g + D_g^*S_o^TN_g$$

and, using (3.92):

$$\Lambda_o^* X \Omega_2^{-1} N_g + Z \Omega^{-1} \tilde{D}_g N_g = N_g^* Q_o N_g + D_g^* S_o^T N_g \tag{3.98}$$

Similarly, postmultiplying the second diophantine equation (3.91) by $\Omega_3^{-1} D_g$ leads to:

$$\Lambda_o^* Y \Omega_3^{-1} D_g - Z N_3 \Omega_3^{-1} D_g = D_g^* R D_g + N_g^* S_o D_g$$

and, using (3.93):

$$\Lambda_o^* Y \Omega_3^{-1} D_g - Z \Omega^{-1} \tilde{N}_g D_g = D_g^* R D_g + N_g^* S_o D_g \tag{3.99}$$

Adding (3.98) and (3.99) leads to:

$$\begin{aligned} &\Lambda_o^* X \Omega_2^{-1} N_g + \Lambda_o^* Y \Omega_3^{-1} D_g + Z \Omega^{-1} (\tilde{D}_g N_g - \tilde{N}_g D_g) \\ &\quad = N_g^* Q_o N_g + N_g^* S_o D_g + D_g^* S_o^T N_g + D_g^* R D_g \\ &\quad = \Lambda_o^* \Lambda_o \end{aligned} \tag{3.100}$$

Now as $\tilde{D}_g N_g = \tilde{N}_g D_g$, (3.100) simplifies to:

$$X \Omega_2^{-1} N_g + Y \Omega_3^{-1} D_g = \Lambda_o \tag{3.101}$$

Substituting this equation into (3.97) leads to:

$$\begin{aligned} H &= [(X \Omega_2^{-1} N_g + Y \Omega_3^{-1} D_g) D_g^{-1} - X \Omega_2^{-1} G]^{-1} X \Omega_2^{-1} \\ &= [Y \Omega_3^{-1}]^{-1} X \Omega_2^{-1} \end{aligned} \tag{3.102}$$

This is a non-minimal form of the controller as the terms Ω_2^{-1} and Ω_3^{-1}, defined in (3.92) and (3.93) respectively, have common factors. To make the controller minimal (3.89) is used as follows:

$$\Omega_f^{-1} X_1 N_g + \Omega_f^{-1} Y_1 D_g = \Lambda_o$$

or

$$X_1 N_g + Y_1 D_g = \Omega_f \Lambda_o \tag{3.103}$$

As $(\Omega_f \Lambda_o)^{-1}$ is analytic in the right half plane (as all zeros of Ω and Λ_o are in the left half plane) the closed loop system is stable from Lemma 2.24. ♦

The main point to note from this method is the connection between partial fraction expansion and the diophantine equation (3.90) in (3.95). The second diophantine

equation (3.91) ensures the resulting closed loop system is stable. To conclude this section Kucera's method is illustrated with the example which was used in Examples 3.1 and 3.2.

Example 3.3

Consider the LQG problem in Example 3.1. This example is a SISO plant and so the left factors are the same as the right factors in (3.92) and (3.93). The diophantine equations (3.90) and (3.91) for this problem are:

$$(-s+2)(-s+1)X + Zs(s-2) = (-s+2)(s+2)(s+1)$$

$$(-s+2)(-s+1)Y + Z(s+2) = -s(-s-2)(s+2)(s+1)$$

The solution to these equations is: $X = 17s+2$, $Y = (s+2)(s+6)$, $Z = -18s+12$. The partial fraction expansion associated with the first diophantine equation is:

$$M_1 = \frac{-7s+2}{s(s-2)} = \frac{17s+2}{s(s-2)} + \frac{-18s+12}{(-s+1)(-s+2)}$$

This partial fraction expansion should be compared with the partial fraction expansion in Examples 3.1 and 3.2. The controller for this problem is:

$$H(s) = Y^{-1}X = \frac{17s+2}{(s+2)(s+6)}$$

which is the same as was calculated using the modern Wiener-Hopf method in Example 3.2. ♦

3.4.5. Comparison of Different Methods

In the previous sections four frequency domain methods of calculating the LQG controller have been presented. The essential difference between the four methods is the means by which the requirement of closed loop stability is built into the solution. With the exception of Kucera (1980) who used diophantine equations, the methods use partial fraction expansion to ensure that the closed loop system is stable. The relationship between the diophantine equations in the method of Kucera (1980) and the partial fraction expansion step in the other methods was shown in Section 3.4.4. Table 3.1 summarises the different methods from a point of view of system representations, calculation steps to form the controller, and limitations inherent in the methods.

Table 3.1 - Comparison of Methods

Method	System Representation	Calculation Steps	Limitations
Newton *et al.*	• rational transfer functions	• 2 spectral factors • partial fraction expansion • calculate controller from closed loop transfer function T and simplify	• stable open loop systems • stability not guaranteed
Austin General Transfer Function method	• rational transfer functions	• 2 generalised spectral factors • partial fraction expansion • calculate controller from closed loop transfer function T and simplify	• $(\Delta^*)^{-1}$ and $(\Gamma^*)^{-1}$ cannot have poles in common with the poles of G(s) (Assumption 3.14) • stability not guaranteed
Youla *et al.* Modern Wiener-Hopf method	• rational transfer functions and fractional representations *or* • rational transfer functions and polynomials	• 2 spectral factors • Bezout identity • partial fraction expansion • calculate controller from closed loop transfer function T and simplify	
Kucera Polynomial method	• polynomials	• 2 spectral factors • 2 diophantine equations • 3 left/right factors	• output weightings

There are a number of numerical routines available to calculate spectral factors. These can be divided into state-space methods and rational function methods. State-space methods for spectral factorisation are discussed in Chapter 5. Rational function methods

include the symmetric extraction method of Davis (1963), triangular factorisation method of Bauer (1955) and the Newton method of Wilson (1969). An efficient algorithm for the calculation of polynomial spectral factors, based on the Newton method, has been published by Jecek and Kucera (1985). Kucera (1979) has a chapter of his book *Discrete Linear Control: The Polynomial Equation Approach* devoted to numerical routines concerning polynomials.

Youla and Bongiorno (1985), and Park and Bongiorno (1989,1990) suggest that the modern Wiener-Hopf method and its extensions use state-space algorithms to calculate the controller: Riccati equations are used for spectral factorisation, and Lyapunov-like equations for the partial fraction expansion step. Park and Youla (1992) give details of these algorithms for their three-degree-of-freedom controller configuration. However this methodology is limited to nonsingular LQG problems and extension of these formulae to encompass singular LQG problems is not available in the literature. A contribution of this thesis is to provide a state-space methodology for the regulator configuration which is valid for singular as well as nonsingular LQG problems.

3.5. SUMMARY

In this chapter the LQG control problem has been introduced. This problem was then transformed into the frequency domain in Section 3.3 and solved in Section 3.4 using Wiener-Hopf methods. The key concepts with Wiener-Hopf methods are spectral factorisation and partial fraction expansion. Four different frequency domain solution methods due to Newton *et al.* (1957), Austin (1979), Youla *et al.* (1976b) and Kucera (1980) were presented. The presentation emphasises the relationship between the requirement that the closed loop system is stable and the partial fraction expansion.

The general transfer function method of Austin (1979) forms the focus of the rest of this thesis. Two difficulties with this method were noted: first, in the proof of Theorem 3.15, the solution was derived under the assumption that the resulting closed loop system is stable; it was not shown that this solution leads to a stable closed loop system. Secondly, there were two assumptions about the open loop system (Assumption 3.14) which were made in the proof of Theorem 3.15 which are not needed in any of the other methods. Example 3.1 showed the necessity of these assumptions. The partial fraction expansion step is studied further in Chapter 4 using state-space techniques. The methodology that is presented in Chapter 5 resolves both the definition of partial fraction expansion and the stability problems. It is shown that if the partial fraction expansion step is defined appropriately the restrictions in equation (3.55) do guarantee that the closed loop system is stable.

CHAPTER 4

STATE-SPACE TECHNIQUES FOR WIENER-HOPF METHODS

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4.1. INTRODUCTION

In the previous chapter a review of frequency domain methods for solving the LQG problem using transfer function or polynomial descriptions was presented. The main system representation used in the rest of this thesis is the state-space representation. In this chapter, some state-space techniques which are used to study Wiener-Hopf methods are introduced. In solving an LQG problem in the frequency domain, the first and most important step is that of spectral factorisation. A complete study of state-space methods for Wiener-Hopf problems should therefore begin with a study of state-space methods for spectral factorisation. This task is not attempted in this chapter. Rather, a collection of state-space methods which can be used to solve Wiener-Hopf problems is presented. The issue of state-space representations of spectral factors is studied as a subject in its own right in Chapter 5.

The determination of the order of a rational function is studied using the concept of row and column reduced transfer functions (Krishnarao and Chen 1984). These results are used to show that the contour integral for the LQG performance index (3.65) converges. One of the most fundamental equations in linear system theory is the Sylvester or Lyapunov equation. The existence and uniqueness of a solution to the Sylvester equation is discussed first. The Sylvester equation is then used to develop state-space methods for partial fraction expansion and contour integration.

It is well-known that singular LQG problems can lead to controllers which are improper. As state-space representations can only be used to represent proper transfer functions it is necessary to use descriptor forms to represent singular LQG controllers. A new descriptor form which can be used to represent singular LQG controllers is derived.

4.2. THE ORDER OF A RATIONAL FUNCTION

The order of a rational function is important in the study of continuous-time Wiener-Hopf problems as it is necessary to determine whether contour integrals of the type (3.65) converge. It is desirable that the order of these transfer functions can be determined *a priori*. This subject has been studied by Kailath (1980) using polynomial representations and Krishnarao and Chen (1984) using rational transfer functions. These results were used by Halevi and Palmor (1988) to investigate the order of closed loop transfer functions for singular LQG problems. In Section 3.4.2 it was noted that the

contour integral for the LQG performance index converges if the transfer function M(s) (3.30) is strictly proper. It is shown in this section that M(s) is always strictly proper.

4.2.1. Row and Column Reduced Forms

The term "order of a transfer function" is defined as the maximum difference between the transfer function numerator and denominator degrees. The specific question of interest is: "What is the order of $G(s) = A^{-1}(s)B(s) = B_1(s)A_1^{-1}(s)$ if A(s) and B(s), and $A_1(s)$ and $B_1(s)$ are rational transfer functions of known degree?" The row and column orders of the various factors of G(s) can be used to investigate the order of G(s) using the following results from Halevi and Palmor (1988). The concept of row reduced rational functions are presented first.

Lemma 4.1 Row Reduced Forms

Consider the following row expansion of a rational function, G(s):

$$G(s) = \Lambda(s)G_{hr} + \bar{G}(s) \tag{4.1}$$

where

$$\Lambda(s) = \text{diag}\{ s^{k_i} \}$$

G_{hr} is the matrix of leading coefficients, and the order of the ith row of $\bar{G}(s)$ is less than k_i . G(s) is proper if $k_i \leq 0$ for all i and is strictly proper if $k_i < 0$ for all i.

If G_{hr} has full row rank then G(s) is called row reduced.

For any G(s) with full row rank there exists a unimodular rational matrix U(s) such that $G_1(s) = U(s)G(s)$ is row reduced.

Given $G(s) = A^{-1}(s)B(s)$ where A(s) and B(s) are rational transfer functions, then the order of G(s) is given by:

$$k_i \geq \max_i(b_i - a_i) \tag{4.2}$$

where b_i and a_i are the orders of the ith rows of B(s) and A(s) respectively. If A(s) is row reduced then the inequality in (4.2) becomes an equality. ◆

The leading coefficient matrix G_{hr} can be determined from the state-space form $G(s) = C(sI-A)^{-1}B+D$ as follows (Halevi and Palmor 1988): let c_i and d_i denote the ith row of C and D respectively; the order of the ith row k_i is zero if $d_i \neq 0$ or else is $-k_i$ where:

$$c_i A^{j-1} B = 0 \quad 1 \leq j < k_i$$

$$c_i A^{k_i-1} B \neq 0$$

and the ith row of G_{hr} is given by:

$$(G_{hr})_i = \begin{cases} d_i & \text{if } d_i \neq 0 \\ c_i A^{k_i-1} B & \text{otherwise} \end{cases} \tag{4.3}$$

The row order of a rational function is preserved under the operation of spectral factorisation.

Lemma 4.2 Order of Transfer Functions under Spectral Factorisation (Halevi and Palmor 1988)

Let A(s) be an rxm rational matrix and C(s) is an rxq rational matrix satisfying:

$$C(s)C^T(-s) = A(s)A^T(-s)$$

then the row expansions of A(s) and C(s) given in (4.1) satisfy $C_{hr}C_{hr}^T = A_{hr}A_{hr}^T$. C(s) has the same row order as A(s) and if A(s) is row reduced, so is C(s). ♦

Row reduced forms were used in Lemma 4.1 to investigate the order of a rational function $G(s) = A^{-1}(s)B(s)$ and in Lemma 4.2 to investigate the left spectral factors of the form $C(s)C^T(-s)$. To investigate rational functions of the form $G(s) = B_1(s)A_1^{-1}(s)$ and right spectral factors $C^T(-s)C(s)$ column reduced forms, which are defined as follows, are used.

Definition 4.3 Column Reduced Forms

A rational matrix G(s) is column reduced if $G^T(s)$ is row reduced.

Using this definition dual forms of Lemmas 4.1 and 4.2 can be derived.

4.2.2. The Order of some Wiener-Hopf Transfer Functions

It was noted in Section 3.4.2 that a sufficient condition for the convergence of contour integral for the LQG performance index (3.65) is for M(s) (3.30) to be strictly proper. It was not shown that this is always the case. The following lemma shows that this is always the case.

Lemma 4.4 The Order of the Transfer Function M(s)

The transfer functions $(\Delta^*)^{-1}(P^*Q+S^T)$ and $E(\Phi_{dn}+\Phi_d G_d^*)(\Gamma^*)^{-1}$ are proper and therefore M(s) defined by (3.30) is strictly proper.

Proof

First note that the transfer functions $(\Delta^*)^{-1}(P^*Q+S^T)$ and $E(\Phi_{dn}+\Phi_d G_d^*)(\Gamma^*)^{-1}$ can be written as:

$$(\Delta^*)^{-1}(P^*Q+S^T) = \left[(C_1(sI-A)^{-1}B+D_1)\Delta^{-1}\right]^* C_1$$

and

$$E(\Phi_{dn}+\Phi_d G_d^*)(\Gamma^*)^{-1} = B_1\left[\Gamma^{-1}(C(sI-A)^{-1}B_1+V_1)\right]^*$$

where the transfer functions $C_1(sI-A)^{-1}B+D_1$ and $C(sI-A)^{-1}B_1+V_1$ were defined in the H_2 formulation of the LQG problem (Definition 3.2). To prove this lemma it will be shown that $(C_1(sI-A)^{-1}B+D_1)\Delta^{-1}$ and $\Gamma^{-1}(C(sI-A)^{-1}B_1+V_1)$ are proper. Since:

$$(C(sI-A)^{-1}B_1+V_1)(C(sI-A)^{-1}B_1+V_1)^* = \Gamma\Gamma^*$$

it follows from Lemma 4.2 that Γ and $C(sI-A)^{-1}B_1+V_1$ have the same row degree. If Γ is not row reduced, it can be made row reduced by premultiplying by a unimodular matrix U(s). Now $U\Gamma$ and $U(C(sI-A)^{-1}B_1+V_1)$ have the same row degree and $U\Gamma$ is row reduced. From Lemma 4.1 the order of:

$$\Gamma^{-1}(C(sI-A)^{-1}B_1+V_1) = (U\Gamma)^{-1}(U(C(sI-A)^{-1}B_1+V_1)) \qquad (4.4)$$

is zero, and therefore proper. Similarly, $(C_1(sI-A)^{-1}B+D_1)\Delta^{-1}$ can be shown to be proper.

The transfer function M(s) (3.30) is given by:

$$M = (\Delta^*)^{-1}\left[-B^T(sI+A^T)^{-1}Q+S^T\right](sI-A)^{-1}\left[E\Phi_d E^T(sI+A^T)^{-1}(-C^T)+E\Phi_{dn}\right](\Gamma^*)^{-1}$$
$$= (\Delta^*)^{-1}\left[C_1(sI-A)^{-1}B+D_1\right]^*C_1(sI-A)^{-1}B_1\left[C(sI-A)^{-1}B_1+V_1\right]^*(\Gamma^*)^{-1}$$

The transfer function M is composed of a product of a proper, a strictly proper and a proper transfer function. The resulting transfer function is therefore strictly proper. ♦

4.3. THE SYLVESTER EQUATION

The Sylvester equation is one of the most fundamental relationships in linear systems theory and arises in a number of contexts, for example: stability analysis, partial fraction expansion, quadratic integrals, stochastic differential equations, and contour integration. The use of the Sylvester equation in connection with the response of linear systems to white noise was considered in Theorem 2.19, (2.42). Before discussing some of the applications of the Sylvester equation to the Wiener-Hopf problem, some general results are presented. Most of the results presented appear in standard texts on matrix theory (Gantmacher 1959; Cullen 1972; Barnett 1990; Lewis 1991). The results on the existence of nonunique solutions, which are apparently not so well known, are presented in greater detail.

The Sylvester equation is given by:

$$XB-AX+C=0 \tag{4.5}$$

where $A \in \boldsymbol{R}^{nxn}$, $B \in \boldsymbol{R}^{mxm}$ and $C \in \boldsymbol{R}^{nxm}$.

A special Sylvester equation is the Lyapunov equation:

$$XA+A^TX+Q=0$$

where Q is symmetric.

First, it should be noted that the Sylvester equation (4.5) is a linear system of equations in X. This can be seen by using the Kronecker product expansion as in Barnett (1990, 109). The following lemma is a well known result for the existence of a unique solution:

Lemma 4.5 Uniqueness of the Solution to a Sylvester Equation

The Sylvester equation (4.5) has a unique solution if and only if $\lambda_i(A) \neq \lambda_j(B)$ for all i and j where $\lambda_i(A)$ is an eigenvalue of A.

Various proofs of this lemma are contained in the literature (Gantmacher 1959, 1:225; Frame 1964; Barnett 1990, 109,146-147). ♦

4.3.1. Nonunique Solutions to Sylvester Equations

When there is not a unique solution to the Sylvester equation (4.5), there is either no solution or nonunique solutions. The existence of a solution depends on the structure of the matrix C (Gantmacher 1959, 1:225). However, Gantmacher does not specify the structure of C which leads to a solution of (4.5). The existence of a solution is established in this section. The Sylvester equation (4.5) can be represented in an augmented matrix form as:

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \tag{4.6}$$

and since:

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = I \tag{4.7}$$

there exists a solution to (4.5) if and only if $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ is similar to $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Two matrices are similar if and only if they have the same Jordan block structure (Cullen 1972). The Jordan block structure can be determined using the following result:

Lemma 4.6 Determination of the Jordan Block Structure of a Matrix (Cullen 1972, 201-202)

The number of simple Jordan blocks, $\jmath_m(\lambda_i)$ with $m \geq k$ in a matrix A is:

$$\text{rank}\left((\lambda_i I - A)^{k-1} \right) - \text{rank}\left((\lambda_i I - A)^{k} \right) \text{ for } k = 1,2,\ldots \tag{4.8}$$

From this result, the Jordan block structure of $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, and hence the existence of a solution to (4.5) can be determined. This result is summarised in the following theorem:

Theorem 4.7 Existence of a Solution to the Sylvester Equation

The Sylvester Equation (4.5) will have a solution if and only if, for all eigenvalues λ of $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$:

$$\begin{aligned} &\operatorname{rank}\left(\left(\lambda I - \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right)^{k-1} \right) - \operatorname{rank}\left(\left(\lambda I - \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right)^{k} \right) \\ &\quad = \operatorname{rank}\left(\left(\lambda I - \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)^{k-1} \right) - \operatorname{rank}\left(\left(\lambda I - \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)^{k} \right) \\ &\quad = \operatorname{rank}((\lambda I - A)^{k-1}) - \operatorname{rank}((\lambda I - A)^{k}) + \\ &\qquad \operatorname{rank}((\lambda I - B)^{k-1}) - \operatorname{rank}((\lambda I - B)^{k}) \end{aligned} \tag{4.9}$$

for $k = 1, 2, \ldots$. ♦

There is no need to apply this test when there is a unique solution (Lemma 4.5). Similarity transformations can be used to transform A and B into Jordan canonical form. These transformations make separate Sylvester equations for each pair of eigenvalues ($\lambda_i(A)$ and $\lambda_j(B)$). As the Sylvester equation has a unique solution if $\lambda_i(A) \neq \lambda_j(B)$ the test in Theorem 4.7 only needs to be applied when $\lambda_i(A) = \lambda_j(B)$.

If nonunique solutions exist, then the solution has the form:

$$X = X_0 + X_d \tag{4.10}$$

where X_0 is a particular solution to (4.5) and X_d is the general solution to the equation:

$$AX_d = X_dB \tag{4.11}$$

The solution to (4.11) is studied in Gantmacher (1959, 1:215-224). Non-trivial solutions to (4.11) exist only if $\lambda_i(A) = \lambda_j(B)$ for some i and j.

4.4. PARTIAL FRACTION EXPANSION

The traditional method for partial fraction expansion for scalar transfer functions involves the calculation of residues using the cover up rule (or similar). There are a number of techniques which use state-space representations of the transfer function: Vandermonde matrix methods (Leyva-Ramos 1991a,1991b), matrix fractions methods (Leyva-Ramos 1991c), and dyadic forms (Shaked 1976a). All these methods calculate the residues for each pole. In the partial fraction expansion step in Wiener-Hopf methods only some of the terms of the expansion are required. If a residue method for partial fraction expansion is used, the residues of the desired poles have to be

calculated. These terms then have to be combined into a single rational function. An approach which avoids the need to calculate the residues individually is the state-space method based on a Sylvester equation (Frame 1964). Another advantage of using this Sylvester equation method is that there are no difficulties with dealing with repeated roots.

The Sylvester equation method for partial fraction expansion is presented in Section 4.4.1. The nature of partial fraction expansion when the Sylvester equation does not have a unique solution is then investigated. This partial fraction expansion method is applied to the partial fraction expansion step in the method of Austin (1979). Nonunique partial fraction expansion is used to investigate removing Assumption 3.14 from the method of Austin (1979); this discussion is limited to considering the LQG problem in Example 3.1.

4.4.1. Sylvester Equation Method for Partial Fraction Expansion

The essence of the Sylvester equation partial fraction expansion method is that a state-space form with a quasi-upper triangular $\mathcal{A}$ matrix can be transformed into a state-space form with the same structure as the form for the sum of two transfer functions (2.10). It should be noted that the $\mathcal{A}$ matrix can always be transformed into a quasi-upper triangular matrix using an equivalence transformation (2.11). This method is summarised in the following theorem:

Theorem 4.8 Sylvester Equation Method for Partial Fraction Expansion

A state-space system with a quasi-upper triangular $\mathcal{A}$ matrix can be expanded as:

$$\left[\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, [\, C_1 \,,\, C_2 \,], D\right]$$
$$= C_1(sI-A_{11})^{-1}(B_1+XB_2) + (C_2-C_1X)(sI-A_{22})^{-1}B_2 + D \qquad (4.12)$$

where X is a solution to the Sylvester equation:

$$XA_{22}-A_{11}X+A_{12} = 0 \qquad (4.13)$$

Proof

Consider a transformation on the states of equation (4.12) of the form:

$$T = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \tag{4.14}$$

Using this transformation on the left hand side of (4.12) gives:

$$\left[\begin{bmatrix} A_{11} & XA_{22}-A_{11}X+A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1+XB_2 \\ B_2 \end{bmatrix}, [\, C_1 \,,\, C_2-C_1X \,], D\right] \tag{4.15}$$

Now to separate the two parts (A_{11} and A_{22}), X needs to be chosen to satisfy the Sylvester equation (4.13). This Sylvester equation has a unique solution if and only if $\lambda_i(A_{22}) \neq \lambda_j(A_{11})$ for all i and j. When this condition is not satisfied, either a solution does not exist in which case the particular expansion cannot be performed, or the Sylvester equation has a nonunique solution; partial fraction expansion for Sylvester equations with nonunique solutions will be considered in Section 4.4.2. The state-space form in (4.15) now has the same structure as the form for the sum of two transfer functions (2.10), and so the desired expansion has been performed.

$$\begin{aligned} &\left[\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1+XB_2 \\ B_2 \end{bmatrix}, [\, C_1 \,,\, C_2-C_1X \,], D\right] \\ &\quad = [\, A_{11} \,,\, B_1+XB_2 \,,\, C_1 \,,\, 0] + [\, A_{22} \,,\, B_2 \,,\, C_2-C_1X \,,\, 0] + D \\ &\quad = C_1(sI-A_{11})^{-1}(B_1+XB_2) + (C_2-C_1X)(sI-A_{22})^{-1}B_2 + D \end{aligned} \tag{4.16}$$

♦

The state-space representation of the partial fraction expansion (4.12) can be written down directly once the solution to the Sylvester equation (4.13) is obtained. This method has the advantage that there is no need for the intermediate step of calculating the residues and then combining some of the terms into a single rational transfer function.

An important example of a quasi-upper triangular A matrix is the formula for a product of two transfer functions (2.9). This form has particular application to Wiener-Hopf methods.

Corollary 4.9 Partial Fraction Expansion for Products

The partial fraction expansion of a product of two transfer functions, with the state-space form given in (2.9), is:

$$[A_1, B_1, C_1, D_1] * [A_2, B_2, C_2, D_2]$$
$$= [A_1, B_1D_2+XB_2, C_1, 0] + [A_2, B_2, D_1C_2-C_1X, 0] + D_1D_2$$
$$= C_1(sI-A_1)^{-1}(B_1D_2+XB_2) + (D_1C_2-C_1X)(sI-A_2)^{-1}B_2 + D_1D_2 \quad (4.17)$$

where X satisfies the Sylvester equation:

$$XA_2-A_1X+B_1C_2 = 0 \quad (4.18)$$

♦

4.4.2. Nonunique Partial Fraction Expansion

One of the advantages of using the Sylvester equation method for partial fraction expansion presented in the previous section is that it can be used to split up the common poles into two separate terms provided a solution to the Sylvester equation (4.13) exists. By common poles it is meant that $\lambda_i(A_{11}) = \lambda_j(A_{22})$ for some i and j in which case Theorem 4.7 can be used to determine whether a solution exists. The existence of a solution to such a Sylvester equation means that partial fraction expansion is not uniquely defined. The nature of this nonunique partial fraction expansion is discussed in this section.

Lemma 4.10 Nonunique Partial Fraction Expansion

For an expansion of the form described in Theorem 4.8 where A_{11} and A_{22} have at least one common eigenvalue and the Sylvester equation (4.13) has a solution $X=X_0+X_d$, the nonunique part of the partial fraction expansion is given by:

$$C_1(sI-A_{11})^{-1}X_dB_2 = C_1X_d(sI-A_{22})^{-1}B_2 \quad (4.19)$$

Proof

Suppose that in (4.12) there exists a common eigenvalue in A_{11} and A_{22} (that is, $\lambda_i(A_{11}) = \lambda_j(A_{22})$ for some i and j) and the Sylvester equation (4.13) has a solution. From Theorem 4.8 the partial fraction expansion can then be written as:

$$\left[\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, [\, C_1 \,,\, C_2 \,], D\right]$$

$$= C_1(sI-A_{11})^{-1}(B_1+(X_0+X_d)B_2) + (C_2-C_1(X_0+X_d))(sI-A_{22})^{-1}B_2+D$$

$$= C_1(sI-A_{11})^{-1}(B_1+X_0B_2) + (C_2-C_1X_0)(sI-A_{22})^{-1}B_2+D$$

$$+ C_1(sI-A_{11})^{-1}X_dB_2 - C_1X_d(sI-A_{22})^{-1}B_2 \qquad (4.20)$$

The first two terms of (4.20) correspond to the particular solution, X_0, of the Sylvester equation and the last two terms to the nonunique solution, X_d. Manipulation of (4.11) with A:=A_{11} and B:=A_{22} leads to:

$$X_dA_{22} = A_{11}X_d$$

$$X_d(sI-A_{22}) = (sI-A_{11})X_d$$

$$X_d(sI-A_{22})^{-1} = (sI-A_{11})^{-1}X_d \qquad (4.21)$$

Therefore the terms in the expansion (4.20) due to the nonunique part of X, X_d satisfy (4.19). ♦

This means that the total expansion for a common pole remains constant (as expected). The nonuniqueness of the partial fraction expansion pertains to how the residue of the common pole is split between the two parts of the expansion ($(sI-A_{11})^{-1}$ and $(sI-A_{22})^{-1}$).

4.4.3. Partial Fraction Expansion in Wiener-Hopf Methods

The Wiener-Hopf solution as derived by Austin (1979) and Shaked (1976b) requires the right half plane poles of $\Delta T\Gamma$ to be fixed as a requirement for stability ($[\Delta T\Gamma]_0 = M_0$). The stable part of $\Delta T\Gamma$ can then be optimised to give the optimal controller ($[\Delta T\Gamma]_+ = M_+$). These two tasks can be done in a single step using the Sylvester equation approach for partial fraction expansion. First it is necessary to write M in a state space form. While $(\Delta^*)^{-1}$ and $(\Gamma^*)^{-1}$ are not always proper, the transfer functions $(\Delta^*)^{-1}(P^*Q+S^T)$ and $E(\Phi_{dn}+\Phi_dG_d^*)(\Gamma^*)^{-1}$ are always proper (Lemma 4.4) and they can be represented by state-space representations:

$$(\Delta^*)^{-1}(P^*Q+S^T) = [\, A_1 \,,\, B_1 \,,\, C_1 \,,\, D_1]$$ (4.22)

$$E(\Phi_{dn}+\Phi_dG_d^*)(\Gamma^*)^{-1} = [\, A_2 \,,\, B_2 \,,\, C_2 \,,\, D_2]$$ (4.23)

A state-space representation for the transfer function M (3.30) can be constructed using the state-space product formula (2.9) and the state-space forms (4.22) and (4.23).

$$
\begin{aligned}
M &= (\Delta^*)^{-1}(P^*Q+S^T)(sI-A)^{-1}E(\Phi_{dn}+\Phi_d G_d^*)(\Gamma^*)^{-1} \\
&= [A_1, B_1, C_1, D_1](sI-A)^{-1}[A_2, B_2, C_2, D_2] \\
&= \left[\begin{bmatrix} A_1 & B_1 & 0 \\ 0 & A & C_2 \\ 0 & 0 & A_2 \end{bmatrix}, \begin{bmatrix} 0 \\ D_2 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1, D_1, 0 \end{bmatrix}, 0\right]
\end{aligned} \tag{4.24}
$$

The solution to the Wiener-Hopf problem requires the partial fraction expansion of the open loop system poles of M, $M_{\oplus}$. From the state-space form of M in (4.24), the open loop poles which are the eigenvalues of A need to be separated from the poles of A_1 and A_2. Two Sylvester Transformations are needed to perform this expansion: one to split A_1 from A, and the other to split A_2 from A. In terms of the state-space representation of M in (4.24), the appropriate state equivalence transformation (2.11) is:

$$
\begin{aligned}
T &= \begin{bmatrix} I & 0 & 0 \\ 0 & I & Y \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & X & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\
&= \begin{bmatrix} I & X & 0 \\ 0 & I & Y \\ 0 & 0 & I \end{bmatrix}
\end{aligned} \tag{4.25}
$$

where X and Y satisfy the Sylvester equations:

$$XA - A_1X + B_1 = 0 \tag{4.26}$$

$$YA_2 - AY + C_2 = 0 \tag{4.27}$$

Applying this transformation to (4.24) leads to:

$$
M = \left[\begin{bmatrix} A_1 & 0 & XC_2 \\ 0 & A & 0 \\ 0 & 0 & A_2 \end{bmatrix}, \begin{bmatrix} XD_2 \\ YB_2+D_2 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1, D_1-C_1X, C_1XY-D_1Y \end{bmatrix}, 0\right] \tag{4.28}
$$

This transformation allows the open loop poles (eigenvalues of A) to be separated from the other poles to give the required expansion:

$$M_{\oplus} = [A, YB_2+D_2, D_1-C_1X, 0] \tag{4.29}$$

$$
M_{\theta} = \left[\begin{bmatrix} A_1 & XC_2 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} XD_2 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1, C_1XY-D_1Y \end{bmatrix}, 0\right] \tag{4.30}
$$

This method for forming the partial fraction expansion of M is summarised in the following theorem:

Theorem 4.11 Sylvester Method for Partial Fraction Expansion in Wiener-Hopf Methods

The partial fraction expansion $M = M_{\oplus} + M_{\theta}$ can be formed from the state-space representations for $(\Delta^*)^{-1}(P^*Q+S^T)$ and $E(\Phi_{dn}+\Phi_d G_d^*)(\Gamma^*)^{-1}$ given by (4.22) and (4.23) respectively, and the Sylvester equations (4.26) and (4.27). The terms $M_{\oplus}$ and M_{θ} are given by (4.28) and (4.29) respectively. ♦

The Sylvester equations (4.26) and (4.27) have unique solutions if and only if $\lambda_i(A) \neq \lambda_j(A_1)$ and $\lambda_i(A) \neq \lambda_j(A_2)$. As the eigenvalues of A_1 and A_2 are at the poles of $(\Delta^*)^{-1}$ and $(\Gamma^*)^{-1}$ respectively the conditions for unique solutions to (4.26) and (4.27) are the same as Assumption 3.14. In Example 3.1 all of the residue of the common pole (s=2) was assigned to $M_{\oplus}$ whereas in the stabilising solution, given by the modern Wiener-Hopf method (Example 3.2), the residue of this pole was split between $M_{\oplus}$ and M_{θ}. The nonunique partial fraction expansion presented in the previous section also splits the residue of the common pole between the two parts of the expansion (Theorem 4.10). This property of the Sylvester equation method is used in the remainder of this section as a possible means of removing Assumption 3.14 from the method of Austin (1979).

The major problem with using this method for partial fraction expansion is the need to form the state-space representations (4.22) and (4.23). One way of doing this is to use the transfer function form of the various factors and then to convert to a state-space representation. The following example illustrates some of the difficulties which occur when dealing with nonunique solutions to the Sylvester equations.

Example 4.1

Consider the LQG problem in Example 3.1. A state-space representation of the open loop system is:

$$G(s) = G_d(s) = \left[\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \end{bmatrix}, 0\right]$$

The transfer functions (4.22) and (4.23) for this problem are:

$$(\Delta^*)^{-1}G^*Q_o = \frac{-s+2}{(-s+2)(-s+1)}$$

$$= \Phi_d G_d^*(\Gamma^*)^{-1} \qquad (4.31)$$

These transfer functions contain a pole-zero cancellation. Cancelling this pole would remove the common pole from the Sylvester equations and result in unique solutions. To obtain nonunique Sylvester equations it is necessary to leave this pole in the transfer function (and the state-space form). There are several state-space representations to this transfer function. For the transfer function $(\Delta^*)^{-1}G^*Q_o$ the following state-space representation is used:

$$(\Delta^*)^{-1}G^*Q_o = \left[\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix}, 0\right] \tag{4.32}$$

The Sylvester equation (4.26) for this problem is:

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

That is:

$$\begin{bmatrix} 0x_{11} & -2x_{12} \\ 1x_{21} & -1x_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$$

which has a nonunique solution of the form:

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} v & 0 \\ -2 & -1 \end{bmatrix}$$

where v is an arbitrary real number.

A different state-space representation of (4.31) is used for $\Phi_d G_d^*(\Gamma^*)^{-1}$:

$$\Phi_d G_d^*(\Gamma^*)^{-1} = \left[\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \end{bmatrix}, 0\right] \tag{4.33}$$

The Sylvester equation (4.27) for this representation is:

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

That is:

$$\begin{bmatrix} 0y_{11} & 2y_{12} \\ -1y_{21} & 1y_{22} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$$

which has a nonunique solution of the form:

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} w & -\frac{1}{2} \\ 0 & -1 \end{bmatrix}$$

where w is an arbitrary real number. From Theorem 4.11 $M_\oplus$ is given by:

$$\begin{aligned} M_\oplus &= \left[\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} w & -\frac{1}{2} \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, -\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} v & 0 \\ -2 & -1 \end{bmatrix}, 0 \right] \\ &= \left[\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} w-\frac{1}{2} \\ -1 \end{bmatrix}, \begin{bmatrix} 2-2v & 1 \end{bmatrix}, 0 \right] \\ &= \frac{-1}{s} + \frac{2wv+v+2w-1}{s-2} \end{aligned}$$

Any w,v such that 2wv+v+2w-1=18 will produce the stabilising solution to the LQG problem (Example 3.2). However, the problem of finding particular w and v which result in a stabilising solution makes this method of partial fraction expansion impractical.

There is another even more serious flaw with this method of partial fraction expansion. The transfer functions $\Phi_d G_d^*(\Gamma^*)^{-1}$ and $(\Delta^*)^{-1}G^*Q_o$ are both given by (4.31). In producing a solution to this problem, different state-space representations were used. The representations are not equivalent in the sense of (2.11). For $(\Delta^*)^{-1}G^*Q_o$ given by (4.32) the mode s=2 is uncontrollable while for $\Phi_d G_d^*(\Gamma^*)^{-1}$ given by (4.33) it is unobservable. If the state-space representation (4.33) is used instead of (4.32), the Sylvester equation (4.26) is:

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

That is:

$$\begin{bmatrix} 0x_{11} & -2x_{12} \\ 1x_{21} & -1x_{22} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}$$

This equation has no solution as there is no x_{11} such that $0x_{11} = -2$. A similar problem occurs if the state-space representation (4.32) instead of (4.33) in the Sylvester equation (4.27). In this example the way in which the state-space representations were chosen affects whether or not the partial fraction can be formed using the Sylvester method.

It was previously noted that the transfer function (4.31) contains a pole-zero cancellation at s=2. This cancellation was not performed previously so the effect of nonunique partial fraction expansion could be investigated. The effect of performing the cancellation is shown in the following:

$$(\Delta^*)^{-1}G^*Q_o = [\,1\,,\,1\,,\,-1\,,\,0] \tag{4.34}$$

The Sylvester equation (4.27) for this representation is:

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} - 1\begin{bmatrix} x_1 & x_2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

which has the unique solution: $\begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \end{bmatrix}$. Using (4.34) the unique solution to the Sylvester equation (4.27) is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The partial fraction expansion given by (4.28) for this expansion is:

$$\begin{aligned} M_{\oplus} &= \left[\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \end{bmatrix}, 0\right] \\ &= \frac{s+2}{s(s-2)} \end{aligned}$$

This solution for $M_{\oplus}$ is the same as the residue-based method in Example 3.1. In this example, the pole-zero cancellation in (4.31) is a vital part of the problem. The way in which this cancellation is handled affects whether or not the stabilising solution is among the solutions (if any exist). ♦

The Sylvester equation method outlined in this section provides a simple, direct method of forming $M_{\oplus}$. This approach can readily be used when Assumption 3.14 is satisfied as the Sylvester equations will have unique solutions. The possibility of using nonunique solutions to the Sylvester equations as a means of removing Assumption 3.14 was investigated in Example 4.1. In this example the particular state-space forms of (4.22) and (4.23) used affected whether or not there was a solution to the Sylvester equations. When nonunique solutions existed the stabilising solution to the Wiener-Hopf problem was contained among the solutions. However, there was no obvious way of selecting the stabilising solution from amongst all the solutions. These shortcomings with partial fraction expansion mean that the Sylvester method cannot be used to remove Assumption 3.14 from the method of Austin (1979).

The basic Sylvester equation approach introduced in this section will be used to investigate the partial fraction expansion step further in the next chapter. The actual way in which the expansion will be performed makes use of the state-space form of the spectral factors introduced in the next chapter and is conceptually different than the method presented in this section.

4.5. CONTOUR INTEGRATION

The evaluation of quadratic expressions by contour integrals was considered in the Section 3.3. The LQG performance index can be expressed as a contour integral (3.65) which can be evaluated using residue methods, provided the integral converges (Kreyszig 1983, Ch.14). In this section, some new state-space methods for evaluating contour integrals are presented.

Integrals of the type:

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} C(sI-A)^{-1}B\, ds \tag{4.35}$$

converge if there are no poles on the imaginary axis, and the integrand has numerator degree at least two less than the denominator degree. The integral is then evaluated by summing the residues of the left half plane poles. The following lemma gives a simple condition for the order of a transfer function.

Lemma 4.12 Order of a Transfer Function

A rational function has numerator degree at least two less than the denominator degree if its state-space form has CB=0.

Proof

Consider first a scalar rational function with state-space form $C(sI-A)^{-1}B$. From Lemma 2.5, the maximum number of zeros is n-m-d where n is the number of states (or denominator degree), m is the number of outputs (m=1 for a scalar transfer function) and d is the rank deficiency of CB. If CB=0, then the maximum number of zeros is n-2 and so the numerator degree is at least two less than the denominator degree.

The numerator degree of a matrix rational function is at least two less than the denominator degree if each element of the matrix is. That is, $C_{i,\cdot}B_{\cdot,j}=0$ for all i and j where $C_{i,\cdot}$ is the ith row of C and $B_{\cdot,j}$ is the jth column of B. Therefore the numerator degree is at least two less than the denominator degree if CB=0. ♦

This result can also be derived using the state-space method of determining the row order of a transfer function given in Section 4.2. Once it has been determined that the integral converges, the integral can be evaluated by summing the residues of the left

half plane poles. A state-space method for evaluating the sum of residues of a transfer function is given in the following lemma.

Lemma 4.13 Sum of Residues

The sum of the residues of a transfer function with a state-space description $C(sI-A)^{-1}B$ is CB.

Proof

$$C(sI-A)^{-1}B = CV^{-1}(sI-\Lambda)^{-1}VB$$

where Λ is the Jordan canonical form of A. Consider a general Jordan block of size m, $\mathcal{J}_m(\lambda)$. Now:

$$(sI-\mathcal{J}_m(\lambda))^{-1} = \begin{bmatrix} \frac{1}{s-\lambda} & \frac{-1}{(s-\lambda)^2} & \cdots & \frac{(-1)^{m-1}}{(s-\lambda)^m} \\ 0 & \frac{1}{s-\lambda} & \frac{-1}{(s-\lambda)^2} & \cdots \\ & 0 & \frac{1}{s-\lambda} & \cdots \\ & & \cdots & \frac{-1}{(s-\lambda)^2} \\ 0 & & 0 & \frac{1}{s-\lambda} \end{bmatrix} \tag{4.36}$$

The residue of this matrix at $s=\lambda$ is the identity matrix. The residue of $(sI-\Lambda)^{-1}$ is therefore the identity matrix. This fact can be used to evaluate the sum of the residues as:

$$\begin{aligned} \sum_{\text{Res}} C(sI-A)^{-1}B &= \sum_{\text{Res}} CV^{-1}(sI-\Lambda)^{-1}VB \\ &= CV^{-1}\left[\sum_{\text{Res}}(sI-\Lambda)^{-1}\right]VB \\ &= CV^{-1}IVB \\ &= CB \end{aligned} \tag{4.37}$$

♦

The product CB is central to the evaluation of contour integrals as it is used to determine the order of the transfer function (Lemma 4.12) and the sum of residues (Lemma 4.13). These two uses of the product CB can be combined to give the following result:

Lemma 4.14 A Zero Contour Integral

Suppose the integral in (4.35) converges, and the integrand in (4.35) has all its poles in the left half plane (or all in the right half plane). Then the integral is zero.

Proof

From Lemma 4.12, CB=0 if the integral converges. The integral is evaluated by summing all the left half plane residues (that is, all the residues, because the integrand is stable). From Lemma 4.13, the integral is given by CB which is zero. ♦

This lemma provides an elegant state-space justification of the steps in (3.48).

The results of Corollary 4.9 and Lemma 4.13 can be used to evaluate a quadratic expression in the frequency domain.

Theorem 4.15 Evaluation of Contour Integrals

For a stable transfer function $C(sI-A)^{-1}B$, the quadratic integral:

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} C(sI-A)^{-1}BB^T(sI+A^T)^{-1}(-C^T)\,ds = CXC^T \qquad (4.38)$$

where X satisfies: $XA+A^TX+BB^T = 0$.

Proof

The integrand can be shown to have numerator degree at least two less than the denominator degree using the state-space product formula, (2.9) and Lemma 4.12. The integral is evaluated by summing the residues of the left half of the complex plane. A partial fraction expansion of the form in Corollary 4.9 can be performed on the integrand of (4.38). The integrand becomes:

$$C(sI-A)^{-1}BB^T(sI+A^T)^{-1}(-C^T) = C(sI-A)^{-1}XC^T - CX(sI+A^T)^{-1}C^T \qquad (4.39)$$

As the system is stable, this partial fraction expansion has split the integrand into its stable and unstable parts. From Lemma 4.13, the sum of the left half plane residues is CXC^T. ♦

4.6. SOME USEFUL DESCRIPTOR FORMS

In this section, two useful descriptor forms are derived for the product of a state-space form and an inverse of a state-space form. These products of transfer functions occur in the solution of the Wiener-Hopf problem and will be used in Chapter 6 to derive descriptor forms of the LQG controller.

Lemma 4.16 Descriptor Form for Right Inverse

The product $(sI-A)^{-1}B\left[D+C(sI-A)^{-1}B\right]^{-1}$ has the descriptor form:

$$(sI-A)^{-1}B\left[D+C(sI-A)^{-1}B\right]^{-1} = \left[(I-BB^{L}+BDB^{L})(sI-A)+BC\right]^{-1}B$$

provided B has full column rank. The poles of this transfer function are exactly the zeros of $D+C(sI-A)^{-1}B$.

Proof

$$\begin{aligned}
&(sI-A)^{-1}B\left[D+C(sI-A)^{-1}B\right]^{-1} \\
&\quad = (sI-A)^{-1}B\left[I-B^{L}B+DB^{L}B+C(sI-A)^{-1}B\right]^{-1} \\
&\quad = \left[(I-BB^{L}+BDB^{L})(sI-A)+BC\right]^{-1}B
\end{aligned} \tag{4.40}$$

which is a descriptor form. The key to this derivation is the use of the identity $(I+AB)^{-1}A = A(I+BA)^{-1}$ in the second line of (4.40) (see Lemma 2.22). The addition the term $I-B^{L}B=0$ enables this identity to be applied. B^{L}, the left pseudo-inverse of B exists if B has full column rank. The poles of this system satisfy $\det((I-BB^{L}+BDB^{L})(sI-A)+BC)=0$. Some standard manipulation of determinants leads to:

$$\begin{aligned}
&\det\left[(I-BB^{L}+BDB^{L})(sI-A)+BC\right] \\
&\quad = \det(sI-A)\det\left[I-BB^{L}+BDB^{L}+BC(sI-A)^{-1}\right] \\
&\quad = \det(sI-A)\det\left[I-B^{L}B+DB^{L}B+C(sI-A)^{-1}B\right] \\
&\quad = \det\left(\begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix}\right) = 0
\end{aligned} \tag{4.41}$$

The last line of equation (4.40) is the definition of zeros of a transfer function $D+C(sI-A)^{-1}B$ (Definition 2.4). ♦

It follows from this lemma that g, the number of finite poles in the descriptor form (4.39), is the same as the number of finite zeros of the transfer function: $C(sI-A)^{-1}B+D$. From Definition 2.11, the number of impulsive modes in the descriptor form is f-g where $f=\text{rank}(I-BB^{L}+BDB^{L})$.

A similar result can be derived for the transfer function $\left[D+C(sI-A)^{-1}B\right]^{-1}C(sI-A)^{-1}$:

Lemma 4.17 Descriptor Form for Left Inverse

The product $\left[D+C(sI-A)^{-1}B\right]^{-1}C(sI-A)^{-1}$ has the descriptor form

$$\left[D+C(sI-A)^{-1}B\right]^{-1}C(sI-A)^{-1} = C\left[(sI-A)(I-C^{R}C+C^{R}DC)+BC\right]^{-1}$$

provided C has full row rank. The poles of this transfer function are exactly the zeros of $D+C(sI-A)^{-1}B$.

Proof

The proof is similar to the proof of Lemma 4.16. ♦

For the descriptor form (4.39) to exist it is necessary that B has full column rank. It is only necessary for the augmented matrix $\begin{bmatrix}B\\D\end{bmatrix}$ to have full column rank for the inverse in $(sI-A)^{-1}B\left[D+C(sI-A)^{-1}B\right]^{-1}$ to exist (Lemma 2.6). In this case a matrix F can be defined such that B-FD has full column rank. Using this matrix as the feedback gain matrix in a left fractional form of $D+C(sI-A)^{-1}B$, given in Section 3.2.2, the following descriptor forms can be derived:

Lemma 4.18 Fractional Descriptor Forms

The transfer function $(sI-A)^{-1}B\left[D+C(sI-A)^{-1}B\right]^{-1}$ has the following descriptor form:

$$(sI-A)^{-1}B\left[D+C(sI-A)^{-1}B\right]^{-1} = \left[E_F(sI-A)+(B_F+E_FF)C\right]^{-1}(B_F+E_FF) \qquad (4.42)$$

where

$B_F = B\text{-}FD$ has full column rank

$$E_F = I\text{-}B_F B_F^L + B_F D B_F^L \tag{4.43}$$

A similar result exists for the transfer function in Lemma 4.17. In this case:

$$\left[D+C(sI\text{-}A)^{-1}B\right]^{-1}C(sI\text{-}A)^{-1} = (C_F+FE_{F1})\left[(sI\text{-}A)E_{F1}+B(C_F+FE_{F1})\right]^{-1} \tag{4.44}$$

where

$C_F = C\text{-}DF$ has full row rank

$$E_{F1} = I\text{-}C_F^R C_F + C_F^R D C_F \tag{4.45}$$

Proof

The relationship in (4.42) will be derived; (4.44) can be derived using similar methods.

First, consider the following expansion:

$$\begin{aligned} B\left[D+C(sI\text{-}A)^{-1}B\right]^{-1} &= B\left[D+C(sI\text{-}A)^{-1}B\right]^{-1} - F + F \\ &= \left(B - F\tilde{D}_g^{-1}\tilde{N}_g\right)\tilde{N}_g^{-1}\tilde{D}_g + F \end{aligned} \tag{4.46}$$

The fractional representation of $D+C(sI\text{-}A)^{-1}B = \tilde{D}_g^{-1}\tilde{N}_g$ is used with $\tilde{D}_g = I+C(sI\text{-}A+FC)^{-1}B$ and $\tilde{N}_g = D+C(sI\text{-}A+FC)^{-1}(B\text{-}FD)$ (Lemma 2.9). Using this fractional representation the first term in (4.46) can be expressed as:

$$\begin{aligned} \left(B - F\tilde{D}_g^{-1}\tilde{N}_g\right)\tilde{N}_g^{-1}\tilde{D}_g &= \left(B - (I+FC(sI\text{-}A)^{-1})F\tilde{N}_g\right)\tilde{N}_g^{-1}\tilde{D}_g \\ &= \left(B\text{-}F\tilde{N}_g\right)\tilde{N}_g^{-1}\tilde{D}_g - FC(sI\text{-}A)^{-1}F\tilde{D}_g \\ &= \left(B\text{-}FD\text{-}FC(sI\text{-}A+FC)^{-1}(B\text{-}FD)\right)\tilde{N}_g^{-1}\tilde{D}_g - FC(sI\text{-}A)^{-1}F\tilde{D}_g \\ &= \left(I\text{-}FC(sI\text{-}A+FC)^{-1}\right)(B\text{-}FD)\tilde{N}_g^{-1}\tilde{D}_g - FC(sI\text{-}A)^{-} \\ &= (sI\text{-}A)(sI\text{-}A+FC)^{-1}(B\text{-}FD)\tilde{N}_g^{-1}\tilde{D}_g - FC(sI\text{-}A)^{-1}F\tilde{D}_g \end{aligned} \tag{4.47}$$

As B-DF has full column rank the descriptor form in Lemma 4.16 can be applied to the term $(sI\text{-}A+FC)^{-1}(B\text{-}FD)\tilde{N}_g^{-1}$.

$$(sI-A+FC)^{-1}(B-FD)\tilde{N}_g^{-1} = (sI-A+FC)^{-1}(B-FD)\left[D+C(sI-A+FC)^{-1}(B-FD)\right]^{-1}$$
$$= \left[E_F(sI-A+FC)+B_FC\right]^{-1}B_F \qquad (4.48)$$

where B_F and E_F are defined in (4.43). Substituting the descriptor form (4.48) into (4.47), and then (4.46) leads to:

$$B\left[D+C(sI-A)^{-1}B\right]^{-1} = B\left[D+C(sI-A)^{-1}B\right]^{-1} - F + F=$$
$$(sI-A)\left[E_F(sI-A+FC)+B_FC\right]^{-1}B_F\tilde{D}_g - FC(sI-A)^{-1}F\tilde{D}_g + F$$
$$= (sI-A)\left[E_F(sI-A+FC)+B_FC\right]^{-1}B_F\tilde{D}_g - F\left(\tilde{D}_g^{-1} - I\right)\tilde{D}_g + F$$
$$= (sI-A)\left[E_F(sI-A+FC)+B_FC\right]^{-1}B_F\tilde{D}_g + F\tilde{D}_g$$

Since from (2.11) and (2.17) it follows that $\tilde{D}_g^{-1} = I+C(sI-A)^{-1}F$:

$$(sI-A)\left[E_F(sI-A+FC)+B_FC\right]^{-1}B_F\tilde{D}_g + F\tilde{D}_g =$$
$$(sI-A)\left[E_F(sI-A+FC)+B_FC\right]^{-1}\left(B_F+\left[E_F(sI-A+FC)+B_FC\right](sI-A)^{-1}F\right)\tilde{D}_g$$
$$= (sI-A)\left[E_F(sI-A+FC)+B_FC\right]^{-1}(B_F+E_FF)\left(I+C(sI-A)^{-1}F\right)\tilde{D}_g$$
$$= (sI-A)\left[E_F(sI-A)+(B_F+E_FF)C\right]^{-1}(B_F+E_FF) \qquad (4.49)$$

Multiplying (4.49) on the left by $(sI-A)^{-1}$ gives the final descriptor form in (4.42). ♦

The descriptor forms presented in this section are used in Chapter 6 to study singular LQG controllers. While these descriptor forms have been derived using algebraic methods, they possess a feedback structure which allows the structure of singular LQG controllers to be established. This interpretation is presented as part of the LQG controller solution in Chapter 6.

4.7. SUMMARY

The purpose of this chapter has been to introduce some state-space techniques to study the solution to Wiener-Hopf problems. These included: tests for the order of a rational function; the use of the Sylvester or Lyapunov equation for partial fraction expansion; methods to evaluate contour integrals; and some descriptor forms to represent the product of a transfer function and an inverse transfer function. These techniques were applied to the Wiener-Hopf problem to establish that the transfer function M(s) (3.30) is

strictly proper and to make a partial fraction expansion of M(s) using a straightforward application of the Sylvester equation method. Nonunique solutions to Sylvester equations were used to investigate the removal of Assumption 3.14 from the general transfer function method of Austin (1979) but were found to be unsatisfactory as illustrated by Example 4.1.

These techniques will be used in subsequent chapters to develop a state-space approach to Wiener-Hopf problems. Further results relating to the partial fraction expansion which lead to the resolution of the problems inherent in the partial fraction step are presented in the next chapter.

CHAPTER 5

THE LINEAR MATRIX INEQUALITY AND THE WIENER-HOPF SOLUTION

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5.1. INTRODUCTION

In the previous chapter state-space methodology was introduced as a tool for solving the LQG problem using Wiener-Hopf methods. This theme is developed further in this chapter. First of all a state-space form of spectral factors which is central to the development of the results in this and subsequent chapters is developed. In fact, as will be shown in Chapter 6, the spectral factor defines the solution of the LQG problem completely. The key defining equation for the state-space form of the spectral factor is the Linear Matrix Inequality (LMI); this is shown to be a generalised form of a Riccati equation. In this chapter the role of the LMI in determining the Wiener-Hopf solution for the closed loop transfer function T(s) (2.59) is established.

A second method for partial fraction expansion is developed in this chapter. Like the methodology in the last chapter, Sylvester equations are used to perform the expansion. The difference is that the new method uses the LMI to expand the transfer function M(s). The main advantage of using LMI is that they allow the partial fraction expansion step to be related to the spectral factorisation step. The LQG solution for T(s) is completely specified by the solution to the LMI. That is, the step of partial fraction expansion is redundant. Furthermore, this partial fraction expansion is shown to result in a closed loop system which is stable without the need for Assumption 3.14.

The LQG performance index is evaluated using the contour integral (3.65). The LMI are used to evaluate this integral leading to a direct generalisation of the Riccati equation expressions for nonsingular LQG problems. In developing an expression for the performance index the concept of equivalent LQG problems is introduced. This concept considers a class of LQG problems (that is a set of state/input weightings and noise intensities) which result in the same controller, but with a different value of the performance index. The implications of equivalent LQG problems for the selection of weightings and noise intensities in the design of LQG controllers are discussed.

Throughout the development of these results it is assumed that the spectral factors are invertible. The structure of the solution to the Wiener-Hopf problem when this assumption is not satisfied is studied using the 'squaring-down' method of Austin (1979) and Halevi and Palmor (1986). The state feedback control problem which is an important class of LQG problems where the spectral factors are not invertible is used to illustrate this technique.

The techniques developed in this chapter to solve the LQG control problem can also be used to solve the minimum variance state estimation problem. This problem is

formulated and solved using Wiener-Hopf techniques with emphasis on the role of the LMI.

5.2. SPECTRAL FACTORS AND THE LINEAR MATRIX INEQUALITY

The spectral factors Δ and Γ (3.17) and (3.18) are the central feature of the frequency domain LQG methods. Their importance is analogous to that of the Riccati equation in time domain. However spectral factors are more general in that they can be used for singular LQG problems.

The results which are derived in this section have been presented in studies of singular LQG/H_2 control problems (Schumacher 1983; Willems *et al.* 1986; Stoorvogel 1992). The main purpose of this section is to develop these results in the context of spectral factorisation. Where it has not been possible to derive all of the results in this context, the relevant results are quoted from the literature. The properties of spectral factors are defined for Δ . The properties of Γ are exactly the same, with the obvious difference in the order of multiplication (that is $\Gamma\Gamma^*$ instead of $\Delta^*\Delta$).

5.2.1. State-Space Representations of Spectral Factors

State-space representations are used in this section to study the spectral factorisation problem (3.17). A state-space representation of (3.17) is:

$$\begin{aligned}\Delta^*\Delta &= (-B^T)(sI+A^T)^{-1}Q(sI-A)^{-1}B + S^T(sI-A)^{-1}B + (-B^T)(sI+A^T)^{-1}S+R \\ &= \left[(-B^T)(sI+A^T)^{-1}\,,\, I\right]\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}\begin{bmatrix}(sI-A)^{-1}B \\ I\end{bmatrix} \\ &= \left[\begin{bmatrix}-A^T & Q \\ 0 & A\end{bmatrix},\begin{bmatrix}S \\ B\end{bmatrix},\left[-B^T\,,\,S^T\right],R\right] \qquad (5.1)\end{aligned}$$

The following assumptions are made about the open loop system:

Assumption 5.1 Controllability and Pole Positions

(i) The pair (A,B) is completely controllable.
(ii) $\lambda_i(A) \neq \lambda_j(-A^T)$ for all i and j where $\lambda_i(A)$ is an eigenvalue of A. ♦

These assumptions are more than is strictly necessary for the development of this section. Assumption 5.1(i) can be reduced to (A,B) stabilisable and Assumption 5.1(ii)

can be removed. The following result is used in developing state-space forms of spectral factors:

Lemma 5.2 Similar State-Space Forms

Suppose $[A, B_1, C_1, D] = [A, B_2, C_2, D]$, then $B_1 = TB_2$ and $C_1T = C_2$ where T is a nonsingular solution of:

$$AT = TA \tag{5.2}$$

This lemma follows directly from definition of equivalent state-space representations (2.11). The general solution to (5.2) is discussed in Gantmacher (1959, vol. 1).

Spectral factorisation determines Δ which from Definition 3.5 can be represented in state-space form as:

$$\Delta = [A, B_\Delta, C_\Delta, D_\Delta]. \tag{5.3}$$

In the following lemma a particular state-space from of Δ is derived.

Lemma 5.3 A State-Space Form for Spectral factors

If Assumption 5.1(ii) holds, one particular state-space representation of the form given in (5.3) is:

$$\Delta = [A, B, K, D] \tag{5.4}$$

where $D^TD = R$.

Proof

Using (2.9) and (5.3), (5.1) can be written as:

$$\begin{aligned}\Delta^*\Delta &= \left[\begin{bmatrix}-A^T & C_\Delta^T C_\Delta \\ 0 & A\end{bmatrix}, \begin{bmatrix}C_\Delta^T D_\Delta \\ B_\Delta\end{bmatrix}, \begin{bmatrix}-B_\Delta^T, D_\Delta^T C_\Delta\end{bmatrix}, D_\Delta^T D_\Delta\right] \\ &= \left[\begin{bmatrix}-A^T & Q \\ 0 & A\end{bmatrix}, \begin{bmatrix}S \\ B\end{bmatrix}, \begin{bmatrix}-B^T, S^T\end{bmatrix}, R\right]\end{aligned} \tag{5.5}$$

The last equality follows from (5.1). It is immediate that $D_\Delta^T D_\Delta = R$.

The approach used in proving this lemma is to transform the $\mathcal{A}$ matrix on the right hand side of (5.5) so it is equal to the $\mathcal{A}$ matrix on the left hand side. The $\mathcal{B}$ and $\mathcal{C}$ matrices can then be related using Lemma 5.2. To make the $\mathcal{A}$ matrices equal the right hand side of (5.5) is transformed by $T = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$.

$$\Delta^*\Delta = \left[\begin{bmatrix} -A^T & XA+A^TX+Q \\ 0 & A \end{bmatrix}, \begin{bmatrix} XB+S \\ B \end{bmatrix}, [-B^T \; B^TX+S^T] \; , R \right] \tag{5.6}$$

with X chosen such that:

$$XA+A^TX+Q = C_\Delta^T C_\Delta \tag{5.7}$$

For a given C_Δ, there exists a unique X satisfying (5.7) if and only if $\lambda_i(A) \neq \lambda_j(-A^T)$ for all i and j where $\lambda_i(A)$ is an eigenvalue of A. If $\lambda_i(A) = \lambda_j(-A^T)$ for some i and j then a solution to (5.7) may or may not exist (Lemma 4.5). To proceed, it is assumed that there is a unique solution to (5.7) (Assumption 5.1(ii)). The $\mathcal{A}$ matrices of the state-space representation of the spectral factor product $\Delta^*\Delta$ and of the state-space representation (5.1) are now equal and so Lemma 5.2 can be applied. The transformation matrix T is partitioned to conform with the partitioning of the $\mathcal{A}$ matrix:

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \tag{5.8}$$

Applying the identity in (5.2) leads to:

$$-T_{11}A^T = -A^TT_{11} + C_\Delta^T C_\Delta T_{21} \tag{5.9a}$$

$$T_{11}C_\Delta^T C_\Delta + T_{12}A = -A^TT_{12} + C_\Delta^T C_\Delta T_{22} \tag{5.9b}$$

$$T_{21}A = -A^TT_{21} \tag{5.9c}$$

$$T_{21}C_\Delta^T C_\Delta + T_{22}A = AT_{22} \tag{5.9d}$$

Assumption 5.1(ii) implies that $T_{21} \equiv 0$ from (5.9c). Simplifying (5.9a) and (5.9d) leads to nonsingular matrices T_{11} and T_{22} such that $T_{11}A^T = A^TT_{11}$ and $T_{22}A = AT_{22}$. Applying this transformation to the $\mathcal{B}$ and $\mathcal{C}$ matrices of (5.6) gives:

$$B_\Delta = T_{22}B \text{ and } T_{11}^T B_\Delta = B \tag{5.10}$$

Using this relationship the state-space representation of Δ in (5.3) can therefore be written as

$$\begin{aligned}\Delta &= [\,A\,,\,T_{22}B\,,\,C_\Delta\,,\,D_\Delta] \\ &= [\,(T_{22})^{-1}AT_{22}\,,\,(T_{22})^{-1}T_{22}B\,,\,C_\Delta T_{22}\,,\,D_\Delta] \\ &= [A\,,\,B\,,\,K\,,\,D] \end{aligned} \tag{5.11}$$

Hence the claim of the lemma is justified. ♦

A specific state-space representation has been derived for the spectral factor Δ. The advantage of using this representation is that only two constant matrices (K and D) have to be determined to specify Δ. As a consequence of Definition 3.5(iii), it is necessary that (A,B) is stabilisable and (A,K) is detectable for Δ to be a spectral factor otherwise there would exist right half plane decoupling zeros (Definition 2.4(ii)). The requirement that the spectral factor is invertible implies that the matrix [K D] has full row rank and $\begin{bmatrix} B \\ D \end{bmatrix}$ has full column rank. It should be noted that this is a necessary but not a sufficient condition for the invertibility of Δ (Lemma 2.6). The following important result about the rank of spectral factors was shown by Schumacher (1983):

Lemma 5.4 Rank of Spectral Factors
(Schumacher 1983)

$$\text{rank}(\Delta) = \text{rank}(C_1(sI-A)^{-1}B+D_1)$$

where C_1 and D_1 are defined by the H_2 formulation in Definition 3.2 ♦

Using the state-space representation of Δ in (5.4) leads to the following result:

Theorem 5.5 Linear Matrix Inequality
(Willems 1971)

Given a state-space representation of the spectral factor having the form shown in (5.4), then subject to Assumption 5.1, there exists a symmetric matrix X such that:

$$XA+A^TX+Q-K^TK = 0 \tag{5.12a}$$

$$D^TK = B^TX+S^T \tag{5.12b}$$

The relationships in (5.12) can be written in an augmented matrix known as the Linear Matrix Inequality:

$$LMI(X) = \begin{bmatrix} A^TX+XA+Q & XB+S \\ B^TX+S^T & R \end{bmatrix} = \begin{bmatrix} K^T \\ D^T \end{bmatrix} \begin{bmatrix} K , D \end{bmatrix} \geq 0 \qquad (5.13)$$

The converse of this theorem is also true. (that is, given that a symmetric X exists such that (5.13) is satisfied, then $\Delta = D+K(sI-A)^{-1}B$ satisfies the spectral factor equation (5.1) (Willems 1971).

Proof

Following a similar procedure to the proof in Lemma 5.3, (5.12a) can be derived and (5.10) becomes:

$$B = T_{22}B \quad \text{and} \quad T_{11}^T B = B \qquad (5.14)$$

The symmetry of X is immediate as both X and X^T are solutions for (5.12a). Multiplying (5.14) on the left by A^j leads to :

$$A^jB = A^jT_{22}B \quad \text{for } j \geq 0 \qquad (5.15)$$

Using the commutativity of A and T_{22} in (5.9d) leads to :

$$A^jB = T_{22}A^jB \qquad (5.16)$$

Augmenting (5.15) for j = 0 to n-1 leads to:

$$\begin{bmatrix} B , AB , A^2B , \ldots , A^{n-1}B \end{bmatrix} = T_{22}\begin{bmatrix} B , AB , A^2B , \ldots , A^{n-1}B \end{bmatrix} \qquad (5.17)$$

Controllability of the pair (A,B) implies that $\begin{bmatrix} B , AB , A^2B , \ldots , A^{n-1}B \end{bmatrix}$ has full row rank (Definition 2.2). This implies that $T_{22} = I$. Similar arguments can be used to show that $T_{11} = I$.

(5.9b) now has the solution $T_{12} \equiv 0$ and therefore T = I. The $\mathcal{B}$ and $\mathcal{C}$ matrices of the left side of (5.5) and right hand side of (5.6) can therefore be equated:

$$\begin{bmatrix} K^TD \\ B \end{bmatrix} = \begin{bmatrix} S+XB \\ B \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -B^T , D^TK \end{bmatrix} = \begin{bmatrix} -B^T , B^TX+S^T \end{bmatrix} \qquad (5.18)$$

which gives (5.12b) ♦

Shaked (1976a) derived the LMI for the special case of R=0 and S=0 for the dual filtering problem. Willems (1971) also developed this relationship from the Dissipation

Inequality. This relationship has also been used in work on singular optimal control problems (Bell and Jacobson 1975; Clements and Anderson 1978). The approach used in this chapter is different from these approaches in that spectral factors were used as the starting point.

When R=0, (5.12b) reduces to $B^TX= -S^T$ and $XB= -S$ from which it follows that $S^TB = B^TS$. This condition is well known as a necessary condition for the LQG control problem to have a solution (Bell and Jacobson 1975).

Assumption 5.1(ii), that $\lambda_i(A) \neq \lambda_j(-A^T)$, means A cannot have an eigenvalue λ, such that λ and $-\lambda$ are both eigenvalues of A. One consequence of this restriction is that A can have no eigenvalues on the imaginary axis. Assumption 5.1(ii) is removed in the next section.

5.2.2. Fractional Representations of Spectral Factors

The generalised spectral factor was related to the spectral factor definition used by Youla (1960) in Definition 3.7. This definition of spectral factors together with fractional representations are used in this section to remove Assumption 5.1(ii) from Lemma 5.3 and Theorem 5.5. Consider a fractional representation based on A-BF being stable (Lemma 2.9) with:

$$\begin{aligned} P &= N_p(D_p)^{-1} \\ &= \left[(sI-A+BF)^{-1}B\right]\left[I-F(sI-A+BF)^{-1}B\right]^{-1}. \end{aligned}$$

Applying this fractional form to the spectral factor Δ leads to:

$$\begin{aligned} \Delta^*\Delta &= (D_p^*)^{-1}\left[N_p^*QN_p+D_p^*RD_p+D_p^*S^TN_p+N_p^*SD_p\right](D_p)^{-1} \\ &= (D_p^*)^{-1}\left[\Lambda^*\Lambda\right](D_p)^{-1} \end{aligned} \tag{5.19}$$

The 'inner' spectral factor Λ is a spectral factor based on stable transfer functions. Using the result in Lemma 5.3, the state-space form of Λ is:

$$\begin{aligned} \Lambda &= K(sI-A+BF)^{-1}B+D \\ &= \left[K\ ,\ D\right]\begin{bmatrix}(sI-A+BF)^{-1}B \\ I\end{bmatrix}. \end{aligned} \tag{5.20}$$

As the poles of Λ are in the left half of the complex plane Λ is unique up to product, from the left, by a unitary matrix (Lemma 3.6). Therefore $\left[K\ ,\ D\right]$ is unique up to

product, from the left, by a unitary matrix. Note that the unitary matrix on the left of $\begin{bmatrix} K , D \end{bmatrix}$ does not make any difference to the LMI (5.13). The generalised spectral factor Δ can be determined from Λ as follows:

$$\Delta = \Lambda(D_p)^{-1} = (K+DF)(sI-A)^{-1}B+D \tag{5.21}$$

It follows from (5.12) and the fractional representations for Δ that:

$$X(A-BF)+(A-BF)^TX+Q+F^TRF+F^TS^T+SF-K^TK = 0 \tag{5.22a}$$

$$D^TK= B^TX+S^T-RF \tag{5.22b}$$

which give the LMI for Λ:

$$\begin{bmatrix} X(A-BF)+(A-BF)^TX+Q+F^TRF+F^TS^T+SF & XB+S-F^TR^T \\ B^TX+S^T-RF & R \end{bmatrix} = \begin{bmatrix} K^T \\ D^T \end{bmatrix} \begin{bmatrix} K , D \end{bmatrix} \tag{5.23}$$

There exists a unique symmetric X which satisfies the Lyapunov equation (5.22a). However, using (5.22b) to obtain an expression for B^TX and substituting this into (5.22a) gives the original LMI (5.13) with K+DF as the 'K' matrix. The interesting point from this analysis is that the same X satisfies the LMI for Λ and Δ. While there may not be a unique solution to the Lyapunov equation for Δ, there always is for Λ and therefore there exists a solution to the Lyapunov equation for Δ in (5.12a). Assumption 5.1(ii) is therefore not a necessary condition for Lemma 5.3 and Theorem 5.5.

The spectral factor Λ (5.20) depends on the particular feedback gains F used in forming the fractional representation. The relationship between the spectral factors and the feedback gains is considered in the following lemma.

Lemma 5.6 Fractional Forms for Spectral Factors

Suppose the spectral factor for the fractional representation based on F_1 is Λ_1 and is defined by K_1. Then the spectral factor, Λ_2 for the fractional representation based on F_2 and defined by K_2 is related to Λ_1 by:

$$K_2 = K_1 + D(F_1 - F_2)$$

Proof

For simplicity, the case S= 0 is considered. From the LMI for Λ_1 (5.23):

$$X(A-BF_1)+(A-BF_1)^TX+Q+F_1^TRF_1-K_1^TK_1 = 0 \quad (5.24a)$$

$$D^TK= B^TX-RF_1 \quad (5.24b)$$

Choose another set of feedback gains F_2 such that $A-BF_2$ is stable.

$$F_1 = F_2 + (F_1 - F_2) = F_2 + \nabla F \quad (5.25)$$

Using (5.24b) and (5.25) to substitute for XBF_1 , $F_1^TB^TX$ and F_1 in (5.24a) gives:

$$X(A-BF_2)+(A-BF_2)^TX+Q+F_2^TRF_2-(K_1+D\nabla F)^T(K_1+D\nabla F) = 0$$

$$D^T(K_1+D\nabla F)= B^TX-RF_2 \quad (5.26)$$

which form the LMI for Λ_2 with $K_2 = (K_1+D\nabla F)$. ♦

5.2.3. The Linear Matrix Inequality and the Riccati Equation

It is known that the Riccati equation is related to the spectral factor if R is positive definite (Macfarlane 1970; Willems 1971). In fact, LMI (5.13) was shown to be a more general form of the Riccati equation by Willems (1971). If R is positive definite then D is full rank, and so from (5.12b):

$$K = (D^T)^{-1}(B^TX+S^T) \quad (5.27)$$

This expression is then used in the Lyapunov equation in (5.12a) to give the Riccati equation:

$$XA+A^TX+Q-(XB+S)R^{-1}(B^TX+S^T) = 0 \quad (5.28)$$

which is associated with the optimal state feedback problem. In this case $K= DK_c$ where

$$K_c = D^{-1}K$$

$$= R^{-1}(B^TX+S^T) \quad (5.29)$$

is the optimal state feedback gain matrix. A full discussion of the solutions of the Riccati equation can be found in Willems (1971). As D is full rank, the state-space inverse formula in (2.14) can be used to calculate the zeros of Δ.

$$\text{Zeros}(\Delta) = \lambda(A\text{-}BD^{-1}K) = \lambda(A\text{-}BK_c) \tag{5.30}$$

It is well-known that there is a unique positive semidefinite matrix X satisfying the Riccati equation (5.27) such that $A\text{-}BD^{-1}K$ is stable provided that (A,B) is stabilisable and there are no modes on the imaginary axis which are undetectable from K.

It is only necessary that (A,B) is stabilisable for the Riccati equation (5.27) to have a solution. The assumption that (A,B) is completely controllable in Theorem 5.5 can therefore be relaxed to (A,B) stabilisable if R is positive definite.

5.2.4. Disturbance Spectral Factor

In this section the analogous results for the disturbance spectral factor Γ which satisfies (3.18) are presented. These results can be derived using similar techniques to those used in developing Lemma 5.3 and Theorem 5.5 for Δ. The main difference between Γ and Δ is the order of multiplication (Γ is a left spectral factor).

Lemma 5.7 State-Space Form of the Spectral Factor Γ

The state-space representation of Γ can be written in the form:

$$\Gamma = [A\ ,\ L\ ,\ C\ ,\ V] \tag{5.31}$$

where $\Phi_n = VV^T$. ♦

The pairs (A,C) and (A,L) are required to be detectable and stabilisable respectively for Γ to have no zeros in the right half plane. The LMI for Γ is given by:

Theorem 5.8 LMI for Γ

Given a state-space representation of the spectral factor Γ having the form shown in (5.31) there exists a matrix $Y \geq 0$ such that:

$$\begin{bmatrix} YA^T+AY+E\Phi_dE^T & YC^T+E\Phi_{dn} \\ CY+\Phi_{dn}^TE^T & \Phi_n \end{bmatrix} = \begin{bmatrix} L \\ V \end{bmatrix} [L^T, V^T] \tag{5.32}$$

♦

When the spectral density matrix of the measurement noise Φ_n is full rank the Riccati equation:

$$YA^T + AY + E\Phi_dE^T - (YC^T+E\Phi_{dn})\Phi_n^{-1}(\Phi_{dn}^TE^T+CY) = 0 \tag{5.33}$$

can be used instead of the LMI (5.32). The Kalman filter gains are:

$$\begin{aligned} K_f &= LV^{-1} \\ &= (YC^T+E\Phi_{dn})\Phi_n^{-1} \end{aligned} \tag{5.34}$$

5.2.5. Further Results on the Linear Matrix Inequality

The presentation of the LMI has to this point been based on spectral factorisation. This relationship has also been investigated from a number of other perspectives. One ~~of the~~ recent perspective is based on Geometric Theory (Willems *et al.* 1986; Stoorvogel 1992). The following theorem is from Stoorvogel (1992a):

Theorem 5.9 Geometric Results for the LMI
(Stoorvogel 1992a)

Assume (A,B) is stabilisable and (A,C) is detectable. There exists matrices X and Y satisfying:

(i) $\text{LMI}(X) \geq 0$, $\text{rank}(\text{LMI}(X)) = \text{rank}(C_1(sI-A)^{-1}B+D_1)$

(ii) $\text{rank}\begin{bmatrix} sI-A & , & -B \\ \text{LMI}(X) & & \end{bmatrix} = n + \text{rank}(C_1(sI-A)^{-1}B+D_1)$ for all s such that $\text{Re}(s) \geq 0$

(iii) $\text{LMI}(Y) \geq 0$, $\text{rank}(\text{LMI}(Y)) = \text{rank}(C(sI-A)^{-1}B_1+V_1)$

(iv) $\text{rank}\left(\begin{bmatrix} \begin{bmatrix} sI-A \\ -C \end{bmatrix} , \text{LMI}(Y) \end{bmatrix}\right) = n + \text{rank}(C(sI-A)^{-1}B_1+V_1)$

for all s such that $\text{Re}(s) \geq 0$

where LMI(X) and LMI(Y) are defined by (5.13) and (5.32) respectively. X and Y are uniquely defined by the above conditions and are positive semidefinite. Moreover, X and Y are the largest solutions of their respective LMI (that is, if $LMI(X_1) \geq 0$ and $LMI(Y_1) \geq 0$ then $X \geq X_1$ and $Y \geq Y_1$).

There are several important properties of the LMI introduced in the above theorem. Firstly, it is only necessary that (A,B) is stabilisable and (A,C) is detectable; the stronger assumptions of controllability and observability are not required. Secondly, the solutions X and Y of the LMI are positive semidefinite. Conditions (ii) and (iv) of the above theorem are equivalent to the requirement that the spectral factors Δ and Γ are minimum phase (have all their zeros in the left half of the complex plane). The positions of the zeros of the spectral factor Δ can be determined from the H_2 formulation of the spectral factor problem (Definition 3.8) using the following result:

Lemma 5.10 Zeros of Spectral Factors
(Chen *et al.* 1993)

Δ has a total of $n-n_f$ finite invariant zeros where n_f is the number of infinite zeros of $\Delta_1 = C_1(sI-A)^{-1}B+D_1$ in (3.25). The invariant zeros of Δ are given by:

(i) the left half plane invariant zeros of Δ_1 ,

(ii) the invariant zeros of Δ_1 on the imaginary axis,

(iii) the mirror image in the imaginary axis of the right half plane invariant zeros of Δ_1 ,

(iv) zeros due to the 'squaring down' of Δ_1 . These zeros can be calculated from a Riccati equation and the Special Coordinate Basis of Saberi and Sannuti (1990).

From Condition (ii) of the above result the invariant zeros on the imaginary axis are the same for Δ_1 and Δ. For this reason only LQG problems for which Δ_1 have no zeros on the imaginary axis are considered.

A reduced-order Riccati equation can be used to calculate X (Stoorvogel and Trentelman 1990) from which K and D can be calculated. Soroka and Shaked (1988b) present another reduced-order Riccati equation method for calculating the solution to

the LMI for the dual problem (that is, for the spectral factor Γ). These reduced-order Riccati equation methods require an appropriate choice of basis of the state vector. It may not always be possible to form the state basis in a numerically reliable way. Another state-space method based on Hermitian pencils has recently been presented by Clements and Glover (1989); further work remains to be done on producing efficient numerical routines for this method.

Solving the LMI (5.13) is equivalent to solving the spectral factorisation problem (3.17). The advantages of using the LMI, rather than rational transfer function methods, to study the Wiener-Hopf solution are considered in the remainder of this chapter.

5.3. PARTIAL FRACTION EXPANSION AND CLOSED LOOP STABILITY

The requirement that the closed loop system is stable led to the partial fraction expansion step in Wiener-Hopf methods (Theorem 3.15). However, in order to derive this in Theorem 3.15 it was necessary to assume that $(\Delta^*)^{-1}$ and $(\Gamma^*)^{-1}$ have no poles in common with the open loop disturbance transfer function P_d (Assumption 3.14). This condition is not required in any other of the LQG solution methods and therefore it would be desirable to eliminate these conditions from Theorem 3.15.

A state-space method of performing the partial fraction expansion step using two Sylvester equations was presented in Section 4.4.2. It was not possible to remove the Assumption 3.14 using this technique. In this section, a second method of performing the partial fraction expansion is presented which uses the LMI to relate the partial fraction expansion to the standard state-space form of the spectral factors introduced in Section 5.2. Again, a Sylvester equation approach is used although it is applied to the transfer function M in a different way. The minimum fuel LQG problem is presented as an example of an LQG problem when Assumption 3.14 is not satisfied. The effect of these conditions on the partial fraction expansion is demonstrated. It is then shown that using this method of partial fraction expansion completely resolves the issue of closed loop stability without the need for Assumption 3.14.

5.3.1. Partial Fraction Expansion and the Linear Matrix Inequality

One of the main problems with the Sylvester equation method in Theorem 4.11 was the need to form the state-space representations (4.54) and (4.55). The method presented in this section avoids this problem by transforming the internal part of M (3.30), $((P^*Q+S^T)P_d(\Phi_d G_d^*+\Phi_{dn}))$ to $\Delta^* C_m(sI-A)^{-1}B_m\Gamma^*$; this transformation removes the terms

$(\Delta^*)^{-1}$ and $(\Gamma^*)^{-1}$ from M. The key to this method is to use the LMI (5.13) and (5.32) to expand M leading to the following state-space form of $M_\oplus$:

Theorem 5.11 Partial Fraction Expansion Theorem

A state-space representation of $M_\oplus$, the partial fraction expansion of the system poles in M (3.30), is:

$$M_\oplus = K(sI-A)^{-1}L \tag{5.35}$$

where K and L are defined by the Linear Matrix Inequalities (5.13) and (5.32).

Proof

Using the formula for the product of two state-space forms (2.9), the transfer function M (3.30) is given by:

$$\begin{aligned} M &= (\Delta^*)^{-1}(P^*Q+S^T)P_d(\Phi_d G_d^*+\Phi_{dn})(\Gamma^*)^{-1} \\ &= (\Delta^*)^{-1}\left[\begin{bmatrix} -A^T & Q \\ 0 & A \end{bmatrix}, \begin{bmatrix} 0 \\ E \end{bmatrix}, \begin{bmatrix} -B^T , S^T \end{bmatrix}, 0\right](\Phi_d G_d^*+\Phi_{dn})(\Gamma^*)^{-1} \end{aligned} \tag{5.36}$$

Now consider a transformation $T = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$ on the state-space representation of the second factor of (5.36) with X such that:

$$XA+A^TX+Q = K^TK \tag{5.37}$$

This is the Lyapunov equation associated with the LMI (5.13) for which a solution exists, regardless of whether $\lambda_i(A) = \lambda_j(-A^T)$ for some i and j (this was shown in Section 5.2.3). The state-space representation is then transformed to:

$$M = (\Delta^*)^{-1}\left[\begin{bmatrix} -A^T & K^TK \\ 0 & A \end{bmatrix}, \begin{bmatrix} XE \\ E \end{bmatrix}, \begin{bmatrix} -B^T , B^TX+S^T \end{bmatrix}, 0\right](\Phi_d G_d^*+\Phi_{dn})(\Gamma^*)^{-1}$$

Now as (5.36) is the Lyapunov equation associated with the LMI for Δ (5.13), $B^TX+S^T = D^TK$. The state-space representation can be recognised as containing the product of Δ^* and $K(sI-A)^{-1}E$ (see (2.9)).

$$\begin{aligned} M &= (\Delta^*)^{-1}\left(-B^T(sI+A^T)^{-1}XE + \Delta^*K(sI-A)^{-1}E\right)(\Phi_d G_d^*+\Phi_{dn})(\Gamma^*)^{-1} \\ &= -(\Delta^*)^{-1}B^T(sI+A^T)^{-1}XE(\Phi_d G_d^*+\Phi_{dn})(\Gamma^*)^{-1} + K(sI-A)^{-1}E(\Phi_d G_d^*+\Phi_{dn})(\Gamma^*)^{-1} \end{aligned} \tag{5.38}$$

The first term in (5.38) has no terms in $(sI-A)^{-1}$ and so is not part of $M_{\oplus}$. A similar reduction can be performed on the second term of (5.38).

$$\begin{aligned} M_2 &= K(sI-A)^{-1}E(\Phi_d G_d^* + \Phi_{dn})(\Gamma^*)^{-1} \\ &= \left[\begin{bmatrix} A & E\Phi_d E^T \\ 0 & -A^T \end{bmatrix}, \begin{bmatrix} E\Phi_{dn} \\ -C^T \end{bmatrix}, \begin{bmatrix} K & 0 \end{bmatrix}, 0\right](\Gamma^*)^{-1} \end{aligned} \tag{5.39}$$

Now consider a transformation on the state-space representation in (5.39) of the form $T = \begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix}$ with Y chosen such that:

$$YA^T + AY + E\Phi_d E^T = LL^T \tag{5.40}$$

This transformation leads to:

$$\begin{aligned} M_2 &= \left[\begin{bmatrix} A & LL^T \\ 0 & -A^T \end{bmatrix}, \begin{bmatrix} YC^T + E\Phi_{dn} \\ -C^T \end{bmatrix}, \begin{bmatrix} K & KY \end{bmatrix}, 0\right](\Gamma^*)^{-1} \\ &= \left[\begin{bmatrix} A & LL^T \\ 0 & -A^T \end{bmatrix}, \begin{bmatrix} LV^T \\ -C^T \end{bmatrix}, \begin{bmatrix} K & KY \end{bmatrix}, 0\right](\Gamma^*)^{-1} \end{aligned} \tag{5.41}$$

Now as (5.40) is the Lyapunov equation associated with the LMI for Γ (5.32), $YC^T + E\Phi_{dn} = LV^T$. The state-space representation can be recognised as containing the product of $K(sI-A)^{-1}L$ and Γ^*.

$$\begin{aligned} M_2 &= K(sI-A)^{-1}L\Gamma^*(\Gamma^*)^{-1} - KY(sI+A^T)^{-1}C^T(\Gamma^*)^{-1} \\ &= K(sI-A)^{-1}L - KY(sI+A^T)^{-1}C^T(\Gamma^*)^{-1} \end{aligned} \tag{5.42}$$

The transfer function M can therefore be expanded as

$$M = K(sI-A)^{-1}L - KY(sI+A^T)^{-1}C^T(\Gamma^*)^{-1} - (\Delta^*)^{-1}B^T(sI+A^T)^{-1}XE(\Phi_d G_d^* + \Phi_{dn})(\Gamma^*)^{-1} \tag{5.43}$$

The last two terms of M in (5.43) have no terms in $(sI-A)^{-1}$; therefore $M_{\oplus}$ is given by the first term as claimed in the statement of the theorem. ♦

A method has now been developed to form $M_{\oplus}$ from the state-space forms of Δ and Γ, (5.4) and (5.31). This result generalises the results of Shaked (1976b), in which the relationship between the partial fraction expansion step and the Riccati equations is established, to singular LQG problems. In fact, although the derivation above looks like partial fraction expansion, there is a subtle difference between this method and standard partial fraction expansion. The difference is that the poles due to terms in $(sI-A)^{-1}$ can

be distinguished from the other poles of M. The difference only becomes significant when there are poles of $P_d(s)$ which are also poles of $(\Delta^*)^{-1}$ or $(\Gamma^*)^{-1}$. The particular definition of $M_\oplus$ given in Theorem 5.11 ensures that the closed loop system is stable; this will be demonstrated in Section 5.3.3. First, an example is considered which illustrates some of the subtleties of this definition of partial fraction.

5.3.2. Minimum Fuel LQG Problems

In this section the LQG problem of finding the controller which minimises the amount of control action (fuel) used. The performance index is:

$$J = \mathcal{E}\left[\mathbf{u}^T(t)R\mathbf{u}(t)\right] \tag{5.44}$$

(That is Q = 0 and S = 0). For this control problem it is necessary that R is positive definite otherwise the spectral factor Δ will not be invertible. The transfer function M (3.30) is clearly equal to zero for this problem. Following a traditional approach to partial fraction expansion, one would conclude that $M_\oplus = 0$, and hence T=0 and H=0 (that is, the best thing to do if you wish to minimise the amount of control action is to do nothing). There is no problem with this conclusion if the open loop system is stable. To resolve this problem it is necessary to consider the partial fraction expansion method in Theorem 5.11. From (5.43):

$$M_\theta = -KY(sI+A^T)^{-1}C^T(\Gamma^*)^{-1} - (\Delta^*)^{-1}B^T(sI+A^T)^{-1}XE(\Phi_d G_d^* + \Phi_{dn})(\Gamma^*)^{-1} \tag{5.45}$$

As R > 0, from the Riccati equation (5.27), $K = (D^T)^{-1}B^TX$ and:

$$(sI+A^T-K^T(D^T)^{-1}B^T)X - X(sI-A) = 0$$

$$\text{or} \quad (sI+A^T-K^T(D^T)^{-1}B^T)^{-1}X = X(sI-A)^{-1} \tag{5.46}$$

the following identity holds:

$$\begin{aligned}(\Delta^*)^{-1}B^T(sI+A^T)^{-1}X &= (D^T)^{-1}B^T(sI+A^T-K^T(D^T)^{-1}B^T)^{-1}X \\ &= (D^T)^{-1}B^TX(sI-A)^{-1} \\ &= K(sI-A)^{-1}\end{aligned} \tag{5.47}$$

Applying this identity to the second term in (5.45) and using the expansion in (5.39) to (5.42) gives:

$$M_\theta = - KY(sI+A^T)^{-1}C^T(\Gamma^*)^{-1} - K(sI-A)^{-1}E(\Phi_d G_d^* + \Phi_{dn})(\Gamma^*)^{-1}$$

$$= - KY(sI+A^T)^{-1}C^T(\Gamma^*)^{-1} - K(sI-A)^{-1}L + KY(sI+A^T)^{-1}C^T(\Gamma^*)^{-1}$$

$$= - K(sI-A)^{-1}L \qquad (5.48)$$

Therefore:

$$M = M_\oplus + M_\theta = 0 \qquad (5.49)$$

as indicated by the preceding discussion. From this analysis, it is clear that the partial fraction expansion method used in Section 5.3.1 is subtly different than the standard partial fraction expansion. The difference is that only the terms in $(sI-A)^{-1}$ are considered to be part of $M_\oplus$.

The reason for defining $M_\oplus$ as in Theorem 5.11 is that it leads to a stable closed loop system. This is shown in the next section.

5.3.3. Stability of the Closed Loop System

It remains to be shown that the partial fraction expansion of M presented in Theorem 5.11 leads to a stabilizing controller. A controller is said to be stabilising if the resulting closed loop system is stable. The following lemma is used in proving the main stability result.

Lemma 5.12 Descriptor Form for $G\Delta^{-1}$

The product $G\Delta^{-1}$ has the descriptor form $C\left[(I-BB^L+BDB^L)(sI-A)+BK\right]^{-1}B$ provided B has full column rank. This transfer function is stable.

Proof

This lemma follows directly from Lemma 4.16 as:

$$G\Delta^{-1} = C(sI-A)^{-1}B\left[D+K(sI-A)^{-1}B\right]^{-1} \qquad (5.50)$$

From Lemma 4.16 the poles of the descriptor form are the same as the zeros of Δ. Therefore $G\Delta^{-1}$ is stable. ♦

Using similar arguments it can also be shown that $\Gamma^{-1}G_d$ is stable given that C has full row rank. Alternatively the stability of $G\Delta^{-1}$ and $\Gamma^{-1}G_d$ can be shown using fractional representations without the assumptions about the rank of B and C.

The main stability result is now stated.

Theorem 5.13 Stability Theorem

If (A,B) is stabilisable, (A,C) is detectable, and Δ and Γ have no zeros on the imaginary axis, the LQG problem with spectral factors $\Delta = D+K(sI-A)^{-1}B$ and $\Gamma = V+C(sI-A)^{-1}L$ has the stabilising solution $T = \Delta^{-1}M_{\oplus}\Gamma^{-1}$ where $M_{\oplus} = K(sI-A)^{-1}L$.

Proof

P_d-PTG_d is stable if and only if G_d-GTG_d $(=C(P_d-PTG_d))$ is. Since $P\Delta^{-1}$ and $\Gamma^{-1}G_d$ are stable (Lemma 3.13), the condition that the four transfer functions in Definition 2.23 are all stable is equivalent to the condition that $\Gamma-G\Delta^{-1}M_{\oplus}$, $\Delta^{-1}M_{\oplus}$ and $M_{\oplus}\Gamma^{-1}$ are all stable.

For simplicity, only the case where B has full column rank is considered allowing the descriptor representation for $G\Delta^{-1}$ in Lemma 5.12 to be used. If B does not have full column rank the fractional descriptor forms of Lemma 4.18 can be used. Now:

$$\begin{aligned}\Gamma-G\Delta^{-1}M_{\oplus} &= V+C(sI-A)^{-1}L-C\left[E'(sI-A)+BK\right]^{-1}BK(sI-A)^{-1}L \\ &= V+C\left[E'(sI-A)+BK\right]^{-1}\left(E'(sI-A)+BK-BK\right)(sI-A)^{-1}L \\ &= V+C(E'(sI-A)+BK)^{-1}E'L \qquad (5.51)\end{aligned}$$

where:

$$E' = I-BB^{L}+BDB^{L}. \qquad (5.52)$$

From Lemma 5.12, $[E'(sI-A)+BK]^{-1}$ only has left half plane poles. Therefore $\Gamma-G\Delta^{-1}M_{\oplus}$ is stable.

By definition spectral factors have all their invariant zeros in the left half plane. As invariant zeros include input and output decoupling zeros (Macfarlane and Karcanias 1976), (A,B) must be stabilisable and (A,K) must be detectable. Therefore there exists a

matrix F such that $\tilde{A}_0$ = A-FK is stable. Using the fractional representations introduced in Section 2.2.3:

$$\Delta = \left[I-K(sI-\tilde{A}_0)^{-1}F\right]^{-1}\left[K(sI-\tilde{A}_0)^{-1}(B-FD)+D\right] \tag{5.53}$$

$$M_{\oplus} = \left[I-K(sI-\tilde{A}_0)^{-1}F\right]^{-1}\left[K(sI-\tilde{A}_0)^{-1}L\right] \tag{5.54}$$

$$\Delta^{-1}M_{\oplus} = \left[K(sI-\tilde{A}_0)^{-1}(B-FD)+D\right]^{-1}\left[K(sI-\tilde{A}_0)^{-1}L\right] \tag{5.55}$$

Now since Δ only has zeros in the left half plane, and these zeros are invariant under feedback (Macfarlane and Karcanias 1976), $\left[K(sI-\tilde{A}_0)^{-1}(B-FD)+D\right]^{-1}$ only has poles in the left half plane. Hence $\Delta^{-1}M_{\oplus}$ is stable.

Similar arguments can be used to show that $M_{\oplus}\Gamma^{-1}$ is stable: choosing F such that A_0 = A-LF is stable gives:

$$M_{\oplus} = K(sI-A_0)^{-1}L\left[I - F(sI-A_0)^{-1}L\right]^{-1}$$

$$\Gamma = \left[V + (C-VF)(sI-A_0)^{-1}L\right]\left[I - F(sI-A_0)^{-1}L\right]^{-1}$$

$$M_{\oplus}\Gamma^{-1} = K(sI-A_0)^{-1}L\left[V + (C-VF)(sI-A_0)^{-1}L\right]^{-1} \tag{5.56}$$

Following similar lines of argument as for $\Delta^{-1}M_{\oplus}$, it follows that $M_{\oplus}\Gamma^{-1}$ is stable. Therefore, the closed loop system is stabilised by the LQG controller. ♦

This result shows that the expansion of M in Theorem 5.11 ($M_{\oplus}$ = $K(sI-A)^{-1}L$) leads to a stable closed loop system. In deriving this result no use was made of Assumption 3.14 (the open loop poles do not coincide with the poles of $(\Delta^*)^{-1}$ or $(\Gamma^*)^{-1}$). This argument is in fact the reverse of the argument in Theorem 3.15 (that is, the solution $T=\Delta^{-1}M_{\oplus}\Gamma^{-1}$ implies that the closed loop system is stable). The results in this section have resolved the difficulties with the general transfer function approach to LQG control problems of Shaked (1976b) and Austin (1979) which were discussed in Section 3.4.2.

The state-space approach used in this chapter has led to a simplified method to solve LQG problems with Wiener-Hopf methods. The partial fraction expansion step is redundant, since it was shown in Theorem 5.11 to be defined by the spectral factors. This relationship between the spectral factors and the partial fraction expansion step has been established for nonsingular LQG problems for each of the Wiener-Hopf methods in Section 3.4 (Shaked 1976; Park and Youla 1992; Grimble 1987). All of these

derivations use the Riccati equations (5.27) and (5.33). The use of the LMI (5.13) and (5.32) in this section has allowed these results to be extended to include singular LQG problems.

5.4. THE PERFORMANCE INDEX

The LQG controller is designed to minimise the performance index (3.1). The minimal performance index can be evaluated using the contour integral (3.65):

$$\begin{aligned} J_{min} &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(\mathrm{Tr}\left\{M_\theta M_\theta^* - MM^*\right\} + \mathrm{Tr}\left\{QP_d\Phi_d P_d^*\right\}\right)ds \\ &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(\mathrm{Tr}\left\{QP_d\Phi_d P_d^*\right\} - \mathrm{Tr}\left\{M_\theta M_\oplus^* + M_\oplus M_\theta^* + M_\oplus M_\oplus^*\right\}\right)ds \end{aligned} \qquad (5.57)$$

This integral can be evaluated by summing all the residues in the left half of the complex plane. The existence of poles on the imaginary axis in the integrand of (5.57) implies that the integral does not converge. One disadvantage of using (5.57) is that M, $M_\oplus$ and P_d have poles on the imaginary axis if the open loop plant has. However, in all examples tried, when all the terms of the integrand are taken into account, any poles on the imaginary axis vanish. It is necessary to establish this is always the case.

The performance index for nonsingular LQG problems is given by:

$$\begin{aligned} J_{min} &= \mathrm{Tr}\left\{XE\Phi_d E^T + YK^TK\right\} \\ &= \mathrm{Tr}\left\{YQ + XLL^T\right\} \end{aligned} \qquad (5.58)$$

where $K^TK = K_c^TR^{-1}K_c$ and $LL^T = K_f\Phi_n^{-1}K_f^T$, and X and Y are the solutions to the state feedback control and Kalman filter Riccati equations (Kwakernaak and Sivan 1972, 394-396). These formulae have the advantage that they can be evaluated directly from the solutions to the Riccati equations, (5.27) and (5.33). Unfortunately the derivation of these formulae requires nonsingular R and Φ_n which limits their application to nonsingular LQG problems.

Recently Stoorvogel (1992a) used a decomposition method to derive an equivalent of (5.58) for singular H_2 control problems. The derivation presented in this section uses a similar decomposition to Stoorvogel (1992a) applied to (5.57). The use of this decomposition in a Wiener-Hopf context has the advantage that no assumption about properness, or even existence, of the controller is necessary. All that is required is closed loop stability and that the integrand in (5.57) has numerator degree at least two

less than the denominator degree. Stability is guaranteed by Theorem 5.13 and the order of the integrand is established in Lemma 4.4. The state-space methods for evaluating contour integrals which were introduced in Section 4.5 are used in this section to evaluate (5.57).

Before deriving a state-space expression for the performance index, the concept of an equivalent LQG problem is introduced.

5.4.1. Equivalent LQG Problems

The performance index (3.1) is the sum of weighted variances/covariances of the states and inputs. As such, the performance index must be nonnegative (Willems 1971). The weightings are chosen to produce a desired closed loop response. There are a number of criteria that need to be considered when selecting the weightings. The first term of the performance index, $\mathcal{E}\left[\mathbf{x}^T(t)Q\mathbf{x}(t)\right]$, constrains the deviations of the states; the second term, $\mathcal{E}\left[\mathbf{u}^T(t)R\mathbf{u}(t)\right]$, constrains the amount of control input; and the third and fourth terms, $\mathcal{E}\left[\mathbf{x}^T(t)S^T\mathbf{u}(t)+\mathbf{u}^T(t)S\mathbf{x}(t)\right]$, constrain products of inputs and states. The three cases, summarised in Table 5.1, are generally extreme cases which would not be implemented, but would set the limits on the state, input and power variances/covariances that could be achieved. A typical LQG design would be a trade off between these three competing factors.

The weighting matrices Q, R and S can arise out of a natural specification for controlled performance or can be treated as design parameters. The resulting controller is optimal in the least squares sense.

Another specification of closed loop response could be in terms of response time or the position of the closed loop poles. The weightings Q, R and S can be chosen place the closed loop poles in a desired region of the complex plane. A discussion of some of the techniques available for pole placement can be found in Grimble and Johnson (1986, Ch.5). Another method is the Loop Transfer Recovery technique of Doyle and Stein (1979). This method uses weightings to improve the closed loop system's robustness to changes in the model of the open loop system. The Loop Transfer Recovery technique will be discussed further in Section 6.6.2.

The noise intensities $\begin{bmatrix} E\Phi_d E^T & E\Phi_{dn} \\ \Phi_{dn}^T E^T & \Phi_n \end{bmatrix}$ are defined by the external disturbances acting on the open loop system. A white noise process is often a good representation of these

disturbances and so the noise intensities are physically meaningful parameters. The modern synthesis methods of Doyle and Stein (1981) treat these as design parameters.

Table 5.1

Minimum Variance	(R=0 , S=0) that is minimise the deviations of the outputs from the setpoint.
Minimum Power	(Q=0 , R=0). (for example an electrical circuit which has a variable input voltage, V and the current,I being measured could have a controller design specification to minimise the power V*I)
Minimum Fuel	(Q=0 , S=0): Minimise the amount of control action (fuel) used.

A difficulty with using the weightings as design parameters to determine the closed loop response is that there is not a unique weighting which corresponds with a particular closed loop response. To illustrate this point consider two LQG problems with weightings $W_0 = \begin{bmatrix} Q_0 & S_0 \\ S_0^T & R_0 \end{bmatrix}$ and $W_1 = \begin{bmatrix} Q_1 & S_1 \\ S_1^T & R_1 \end{bmatrix}$ where:

$$\begin{bmatrix} Q_1 & S_1 \\ S_1^T & R_1 \end{bmatrix} = \begin{bmatrix} Q_0+A^TX'+X'A & S_0+X'B \\ S_0^T+B^TX' & R_0 \end{bmatrix} \tag{5.59}$$

and X' is a symmetric matrix. The two weighting matrices W_0 and W_1 are required to be positive semidefinite. The maximal solutions to the LMI, denoted X_0 and X_1, for the two LQG problems are related by:

$$X_1 = X_0 + X'$$

The spectral factor Δ is the same for both sets of weightings as their LMI are given by:

$$\text{LMI}(W_0 , X_0) = \left[K , D\right]^T\left[K , D\right]$$

$$\begin{aligned} LMI(W_1 , X_1) &= LMI(W_1 , X_0+X') \\ &= LMI(W_0 , X_0) \\ &= \left[K , D\right]^T \left[K , D\right] \end{aligned}$$

An LQG problem defined by W_0 is said to be equivalent to one defined by W_1 if W_0 and W_1 are related by (5.59). Theorem 5.9 can be used to determine suitable X' such that W_0 and W_1 are positive semidefinite.

A similar relationship exists for the noise intensities. The spectral factor Γ is the same for both noise intensities, $\Phi_0 = \begin{bmatrix} \Phi_{D0} & \Phi_{Dn0} \\ \Phi_{Dn0}^T & \Phi_{n0} \end{bmatrix}$ and $\Phi_1 = \begin{bmatrix} \Phi_{D1} & \Phi_{Dn1} \\ \Phi_{Dn1}^T & \Phi_{n1} \end{bmatrix}$ if:

$$\begin{bmatrix} \Phi_{D1} & \Phi_{Dn1} \\ \Phi_{Dn1}^T & \Phi_{n1} \end{bmatrix} = \begin{bmatrix} \Phi_{D0}+AY'+Y'A^T & \Phi_{Dn0}+Y'C^T \\ \Phi_{Dn0}^T+CY' & \Phi_{n0} \end{bmatrix} \tag{5.60}$$

where Y' is symmetric matrix (here **D** = E**d**). The two noise intensities Φ_0 and Φ_1 are required to be positive semidefinite.

Definition 5.14 Equivalent LQG Problems

For the open loop system (3.2), LQG problems with state/input weightings $W = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$ and noise intensities $\Phi = \begin{bmatrix} \Phi_D & \Phi_{Dn} \\ \Phi_{Dn}^T & \Phi_n \end{bmatrix}$ are equivalent when the spectral factors Δ and Γ are invariant. ♦

The concept is related to "W-transformation" used by Popov (1964) for optimal state feedback control problems and "signal model equivalence" used by Schumacher (1985) for optimal filtering problems. Together these concepts are termed "equivalent LQG problems" in this thesis. It was mentioned in Section 3.2 that permitting cross-weightings between the states and inputs and cross-correlation between the disturbance and measurement noise provides a more 'complete' picture of the LQG problem. The sense in which this LQG problem is more 'complete' is that equivalent LQG problems can be defined as above: equivalent LQG problems cannot be defined without cross-weighting and cross-correlation.

It is obvious that the same T(s) is the solution to all of the members of a particular equivalent LQG class as T(s) only depends on A, B, C, K, D, L and V.

5.4.2. Evaluation of the Performance Index

In this section the concept of equivalent LQG problems is used to evaluate the performance index (5.57). Before explicitly evaluating the performance index the difference in the value of the performance index between two equivalent LQG problems is evaluated. It is then shown that every LQG problem has an equivalent LQG problem with a zero performance index. This provides a method of evaluating the performance index.

Theorem 5.15 The Performance Index for Equivalent LQG Problems

If the conditions of the Theorem 5.13 are satisfied, then the performance index for the LQG problem $J(W_0,\Phi_0)$ is given by:

$$\begin{aligned} J(W_0,\Phi_0) &= J(W_1,\Phi_1) + \mathrm{Tr}\left\{X'\Phi_{D0} + Y'Q_1\right\} \\ &= J(W_1,\Phi_1) + \mathrm{Tr}\left\{Y'Q_0 + X'\Phi_{D1}\right\} \end{aligned} \tag{5.61}$$

where W_0 and W_1 are related by (5.59) and Φ_0 and Φ_1 are related by (5.60).

Proof

The proof of this theorem is quite lengthy. The approach is to rearrange the performance index (5.57) for the original problem $J(W_0,\Phi_0)$ into two parts: one containing the performance index for the equivalent LQG problem $J(W_1,\Phi_1)$, and the other part containing the rest of the terms. This manipulation of the performance index takes two stages:

(i) reduce the LQG problem to one with state/input weightings, W_1

(ii) reduce the LQG problem to one with noise intensities, Φ_1

Step (i) is performed in Lemma 5.16 and Step (ii) in Lemma 5.17.

Lemma 5.16 Step (i): Transformation of Weightings W_0 to W_1

The performance index of an LQG problem $J(W_0,\Phi_0)$ is given by:

$$J(W_0,\Phi_0) = J(W_1,\Phi_0) + \mathrm{Tr}\left\{X'\Phi_{D0}\right\} \tag{5.62}$$

Proof

For the original LQG problem with the performance index $J(W_0,\Phi_0)$:

$$M(W_0,\Phi_0) = (\Delta^*)^{-1}\left[(-B^T)(sI+A^T)^{-1}Q_0+S_0^T\right](sI-A)^{-1}\left[\Phi_{D0}(sI+A^T)^{-1}(-C^T)+\Phi_{Dn0}\right](\Gamma^*)^{-1}$$

where $M(W_0,\Phi_0)$ is the transfer function M for the LQG problem with weightings W_0 and noise intensities Φ_0. $M(W_0,\Phi_0)$ can be expanded using the same method as in the proof of Theorem 5.11 ((5.36) to (5.38)) but using (5.59) instead of the LMI (5.13).

$$\begin{aligned} M(W_0,\Phi_0) &= (\Delta^*)^{-1}\left[(-B^T)(sI+A^T)^{-1}Q_1+S_1^T\right](sI-A)^{-1}\left[\Phi_{D0}(sI+A^T)^{-1}(-C^T)+\Phi_{Dn0}\right](\Gamma^*)^{-1} \\ &\quad + (\Delta^*)^{-1}(-B^T)(sI+A^T)^{-1}X'\left[\Phi_{D0}(sI+A^T)^{-1}(-C^T)+\Phi_{Dn0}\right](\Gamma^*)^{-1} \\ &= M(W_1,\Phi_0) + [\bar{M}]_\theta \end{aligned} \tag{5.63}$$

$[\bar{M}]_\theta$ is the part of $M(W_0,\Phi_0)$ which does not belong to $M(W_1,\Phi_0)$ (that is $[\bar{M}]_\theta = M(W_0,\Phi_0) - M(W_1,\Phi_0)$). The subscript θ is used as $[\bar{M}]_\theta$ has no terms in $(sI-A)^{-1}$.

As the two LQG problems are equivalent:

$$M(W_0,\Phi_0)_\oplus = M(W_1,\Phi_0)_\oplus = K(sI-A)^{-1}L$$

The second term of the integrand in (5.57) can therefore be expressed as:

$$\begin{aligned} &Tr\left\{M_\theta M_\oplus^* + M_\oplus M_\theta^* + M_\oplus M_\oplus^*\right\} \\ &\quad = Tr\left\{M(W_1,\Phi_0)_\theta M_\oplus^* + M_\oplus M(W_1,\Phi_0)_\theta^* + M_\oplus M_\oplus^*\right\} + Tr\left\{[\bar{M}]_\theta M_\oplus^* + M_\oplus [\bar{M}]_\theta^*\right\} \end{aligned} \tag{5.64}$$

The expansion in (5.64) has split the terms in $Tr\left\{M_\theta M_\oplus^* + M_\oplus M_\theta^* + M_\oplus M_\oplus^*\right\}$ associated with $J(W_1,\Phi_0)$ from those not associated with $J(W_1,\Phi_0)$. The rest of the integrand (5.57) can be split in a similar way.

$$Tr\left\{Q_0(sI-A)^{-1}(-\Phi_{D0})(sI+A^T)^{-1}\right\} = Tr\left\{(sI+A^T)^{-1}Q_0(sI-A)^{-1}(-\Phi_{D0})\right\} \tag{5.65}$$

Now from (5.59):

$$(sI+A^T)X' - X'(sI-A) + Q_0 = Q_1$$

or

$$(sI+A^T)^{-1}Q_0(sI-A)^{-1} = (sI+A^T)^{-1}Q_1(sI-A)^{-1} + (sI+A^T)^{-1}X' - X'(sI-A)^{-1} \tag{5.66}$$

Applying this identity to (5.65) leads to:

$$\begin{aligned} \mathrm{Tr}\left\{Q_0(sI-A)^{-1}(-\Phi_{D0})(sI+A^T)^{-1}\right\} &= \mathrm{Tr}\left\{(sI+A^T)^{-1}Q_1(sI-A)^{-1}(-\Phi_{D0})\right\} \\ &+ \mathrm{Tr}\left\{X'(sI-A)^{-1}\Phi_{D0} - (sI+A^T)^{-1}X'\Phi_{D0}\right\} \end{aligned} \tag{5.67}$$

Using (5.64) and (5.67), the performance index is given by:

$$\begin{aligned} J(W_0,\Phi_0) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Big(& \mathrm{Tr}\left\{Q_1(sI-A)^{-1}(-\Phi_{D0})(sI+A^T)^{-1}\right\} \\ & - \mathrm{Tr}\left\{M(W_1,\Phi_0)_\theta M_\oplus^* + M_\oplus M(W_1,\Phi_0)_\theta^* + M_\oplus M_\oplus^*\right\} \\ & + \mathrm{Tr}\left\{X'(sI-A)^{-1}\Phi_{D0} - (sI+A^T)^{-1}X'\Phi_{D0}\right\} - \mathrm{Tr}\left\{\bar{M}_\theta M_\oplus^* + M_\oplus \bar{M}_\theta^*\right\}\Big) ds \\ = J(W_1,\Phi_0) + \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} & \left(\mathrm{Tr}\left\{X'(sI-A)^{-1}\Phi_{D0} - \Phi_{D0}(sI+A^T)^{-1}X'\right\} - \mathrm{Tr}\left\{\bar{M}_\theta M_\oplus^* + M_\oplus \bar{M}_\theta^*\right\}\right) ds \end{aligned} \tag{5.68}$$

This expansion has split the performance index into two parts: $J(W_1,\Phi_0)$ and a contour integral. It is necessary to establish the convergence of the contour integral before evaluating it: it is necessary to establish that the integrand is analytic on the imaginary axis and is of the required order.

Consider the following terms of the integrand:

$$\begin{aligned} & \mathrm{Tr}\left\{ X'(sI-A)^{-1}\Phi_{D0}\right\} - \mathrm{Tr}\left\{ M_\oplus[\bar{M}]_\theta^*\right\} \\ & \quad = \mathrm{Tr}\left\{ X'(sI-A)^{-1}\Phi_{D0} - X'(sI-A)^{-1}B\Delta^{-1}M_\oplus\Gamma^{-1}(C(sI-A)^{-1}\Phi_{D0} + \Phi_{Dn0}^T)\right\} \\ & \quad = \mathrm{Tr}\left\{ X'\left[(sI-A)^{-1} - PTC(sI-A)^{-1}\right]\Phi_{D0}\right\} + \mathrm{Tr}\left\{ X'PT\Phi_{Dn0}^T\right\} \end{aligned} \tag{5.69}$$

The last line uses $T = \Delta^{-1}M_\oplus\Gamma^{-1}$. The transfer functions $((sI-A)^{-1} - PTC(sI-A)^{-1})$ and PT (note that the first term is similar to $P_d - PTG_d$) are guaranteed to be stable by Theorem 5.13. The other terms in the integrand in (5.68) form the adjoint of (5.69) and therefore only have poles in the right half of the complex plane. Therefore there are no poles on the imaginary axis. The integrand in (5.68) has numerator degree at least two less than the denominator as it is the difference between the integrands associated with the performance indices, $J(W_0,\Phi_{D0})$ and $J(W_1,\Phi_{D0})$, which are both of the required order. Therefore the contour integral in (5.68) is finite.

It is necessary to check that the terms for the left half plane poles in (5.69) are strictly proper before applying the state-space techniques of Section 4.5. First it is shown that $X'(sI-A)^{-1}B\Delta^{-1}$ is strictly proper. Consider:

$$(-sI-A^T)^{-1}\left(Q_0(sI-A)^{-1}B+S_0\right)\Delta^{-1}$$
$$= (-sI-A^T)^{-1}\left(Q_1(sI-A)^{-1}B+S_1\right)\Delta^{-1} + X'(sI-A)^{-1}B\Delta^{-1} \qquad (5.70)$$

This expansion is obtained using (5.59). The term on the left hand side and first term of the right hand side are strictly proper as $\left(Q_0(sI-A)^{-1}B+S_0\right)\Delta^{-1}$ and $\left(Q_1(sI-A)^{-1}B+S_1\right)\Delta^{-1}$ are proper (Lemma 4.4). Therefore $X'(sI-A)^{-1}B\Delta^{-1}$, which is the difference between two strictly proper transfer functions, is also strictly proper.

Secondly, $\Gamma^{-1}\left(C(sI-A)^{-1}\Phi_{D0}+\Phi_{Dn0}^T\right)$ is proper (Lemma 4.4).

The transfer function $X'(sI-A)^{-1}B\Delta^{-1}M_\oplus\Gamma^{-1}\left(C(sI-A)^{-1}\Phi_{D0}+\Phi_{Dn0}^T\right)$ is therefore strictly proper as it is the product of two strictly proper and a proper transfer function ($X'(sI-A)^{-1}B\Delta^{-1}$, $M_\oplus$ and $\Gamma^{-1}\left(C(sI-A)^{-1}\Phi_{D0}+\Phi_{Dn0}^T\right)$). This transfer function is given the following state-space representation:

$$X'(sI-A)^{-1}B\Delta^{-1}M_\oplus\Gamma^{-1}\left(C(sI-A)^{-1}\Phi_{D0}+\Phi_{Dn0}^T\right) = C_p(sI-A_p)^{-1}B_p$$

By using the formula for multiplication of two state-space forms (2.9), it can be shown that $C_pB_p=0$. From the formula for addition of two state-space representations, (2.10), the expression in (5.69) has the following state-space representation:

$$X'(sI-A)^{-1}\Phi_{D0} - X'(sI-A)^{-1}B\Delta^{-1}M_\oplus\Gamma^{-1}\left(C(sI-A)^{-1}\Phi_{D0}+\Phi_{Dn0}^T\right)$$
$$= \left[\begin{bmatrix} A & 0 \\ 0 & A_p \end{bmatrix}, \begin{bmatrix} \Phi_{D0} \\ B_p \end{bmatrix}, \begin{bmatrix} X' , -C_p \end{bmatrix}, 0\right]$$

Note that this state-space form is not minimal, but this is not important. From Lemma 4.13, the sum of the residues, given by the product of the $\mathcal{C}$ and $\mathcal{B}$ matrices, is:

$$X'\Phi_{D0} - C_pB_p = X'\Phi_{D0}$$

Therefore the performance index can be expanded using (5.62). ♦

Lemma 5.17 Step (ii): Transformation of Noise Intensities Φ_0 to Φ_1

The performance index $J(W_1,\Phi_0)$ in Lemma 5.16 is given by:

$$J(W_1,\Phi_0) = J(W_1,\Phi_1) + \mathrm{Tr}\{Y'Q_1\}$$

Proof

This reduction is performed in a similar way to step (i), and therefore the details are omitted. The terms $M(W_1,\Phi_0)$ and $\mathrm{Tr}\{Q_1(sI-A)^{-1}(-\Phi_{D0})(sI+A^T)^{-1}\}$ are expanded using an expansion of the same type as in the proof of Theorem 5.11 ((5.39) to (5.41)), but using (5.60) instead of the LMI (5.32). ♦

Combining the results of Lemmas 5.16 and 5.17 gives:

$$J(W_0,\Phi_0) = J(W_1,\Phi_1) + \mathrm{Tr}\{X'\Phi_{D0} + Y'Q_1\}$$

The second expression for $J(W_1,\Phi_1)$ in (5.61) can be derived in a similar way by reducing the LQG problem first with respect to Y' and then with respect to X'. This completes the proof of Theorem 5.15. ♦

The main purpose of Theorem 5.15 is to show that even though the resulting controller is invariant for equivalent LQG problems, the performance index changes.

If an equivalent LQG problem with $J(W,\Phi) = 0$ can be found, Theorem 5.15 can be used to determine the value of the performance index. The next lemma shows that such an LQG problem exists.

Lemma 5.18 Zero Cost LQG Problem

Consider an LQG problem with weightings $W = \begin{bmatrix} K^TK & K^TD \\ D^TK & D^TD \end{bmatrix}$ and noise intensities $\Phi = \begin{bmatrix} LL^T & LV^T \\ VL^T & VV^T \end{bmatrix}$, then the performance index for LQG problem is $J(W,\Phi) = 0$.

Proof

For this LQG problem, the transfer function M (3.30) is:

$$\begin{aligned} M &= (\Delta^*)^{-1}(-B^T(sI+A^T)^{-1}K^TK+D^TK)(sI-A)^{-1}(LV^T-LL^T(sI+A^T)^{-1}C^T)(\Gamma^*)^{-1} \\ &= (\Delta^*)^{-1}\Delta^* K(sI-A)^{-1}L\,\Gamma^*(\Gamma^*)^{-1} \\ &= M_\oplus \end{aligned} \tag{5.71}$$

from Theorem 5.11. Therefore $M_\theta = 0$.

The performance index (5.57) is therefore given by:

$$\begin{aligned} J_{min} &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left(\mathrm{Tr}\left\{K^TK(sI-A)^{-1}(-LL^T)(sI+A^T)^{-1}\right\} - \mathrm{Tr}\left\{M_\oplus M_\oplus^*\right\} \right) ds \\ &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left(\mathrm{Tr}\left\{K(sI-A)^{-1}L\,(-L^T(sI+A^T)^{-1}K^T)\right\} - \mathrm{Tr}\left\{M_\oplus M_\oplus^*\right\} \right) ds \\ &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left(\mathrm{Tr}\left\{M_\oplus M_\oplus^*\right\} - \mathrm{Tr}\left\{M_\oplus M_\oplus^*\right\} \right) ds \\ &= 0 \end{aligned} \tag{5.72}$$

♦

For this LQG problem the solutions to the LMI (5.13) and (5.32) are X=0 and Y=0 respectively.

As the original weightings $W = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$ and noise intensities $\Phi = \begin{bmatrix} E\Phi_dE^T & E\Phi_{dn} \\ \Phi_{dn}^TE^T & \Phi_n \end{bmatrix}$ always have an equivalent LQG problem with zero cost of the form in Lemma 5.18, the performance index can be evaluated using the LMI (5.13) and (5.32) and Theorem 5.17. This step is summarised in the next theorem.

Theorem 5.19 Evaluation of the Performance Index

If the conditions of Theorem 5.13 are satisfied, then the performance index for an LQG problem with $W = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$ and $\Phi = \begin{bmatrix} E\Phi_dE^T & E\Phi_{dn} \\ \Phi_{dn}^TE^T & \Phi_n \end{bmatrix}$ is given by:

$$\begin{aligned} J(W,\Phi) &= \mathrm{Tr}\left\{XE\Phi_dE^T + YK^TK\right\} \\ &= \mathrm{Tr}\left\{YQ + XLL^T\right\} \end{aligned} \tag{5.73}$$

where X and K are defined by the LMI for Δ (5.13) and Y and L are defined by the LMI for Γ (5.32). ♦

The expressions for the performance index in (5.73) were also derived by Stoorvogel (1992a). The main contribution of the derivation in this section is to show how the LMI can be used to evaluate the contour integral expressions for the performance index.

A number of methods for selecting the weightings (and noise intensities) were briefly described in Section 5.4.1. In Theorem 5.15 it was shown that an equivalent LQG problem can be used to produce exactly the same closed loop response, but with a different value for the performance index. Therefore the actual value of the performance index for a particular closed loop response is not unique.

The value of the noise intensities are generally related to physical noise processes affecting the plant. In this case, the weightings W can be chosen such that $W = \begin{bmatrix} K^TK & K^TD \\ D^TK & D^TD \end{bmatrix}$ (in which case X=0 is the solution to the LMI (5.13)). The performance index is then given by:

$$J_{min} = Tr\{YK^TK\}. \tag{5.74}$$

When the noise intensities are not the physical quantities the performance index can be made to be zero from Lemma 5.18! In conclusion, unless the weightings and noise intensities have physical meaning, the actual value of the performance index is meaningless.

The performance index has been used to compare the performance of different control strategies. Grimble and Johnson (1986) devote a chapter to comparisons of performance of controllers based on the value of the LQG performance index. Austin (1979) used the value of the performance index to compare the performance of feedback, feedforward and feedforward/feedback control laws. The performance index has also been used to compare the performance of LQG controllers with other non-optimal controllers. Of course, it is known *a priori* that the LQG controller has the smaller performance index, but the question these researchers are interested in is "By how much is the performance index smaller?". The fact that the performance index for the optimal controller can be made to be zero by appropriate choice of weightings and noise intensities means that performance degradation resulting from using a non-optimal controller cannot be expressed as a percentage. One could talk about the difference between the performance indices, but it is not clear if this difference remains constant over the class of equivalent LQG problems.

5.4.3. Summary

A method to evaluate the performance index using the solutions to LMI (5.13) and (5.32) has been derived in this section. This result generalises the well-known formulae for nonsingular LQG problems (5.58) to include singular LQG problems. In developing this result the key concept was that of equivalent LQG problems (Definition 5.14). Equivalent LQG problems comprise of sets of weights and noise intensities which result in the same solution. While the closed loop response is the same for equivalent LQG problems the value of the performance index changes. An LQG problem with zero cost was presented in Lemma 5.18. This LQG problem depends only on the state-space form of the spectral factors Δ and Γ ((5.4) and (5.31)). Therefore all LQG problems have an equivalent LQG problem with zero cost. This equivalent LQG problem was used in conjunction with Theorem 5.15 to obtain the expressions for the performance index in (5.73).

As well as providing a means of evaluating the performance index, equivalent LQG problems bring into question the interpretation of the value of the performance index. In cases when the performance index is not physically meaningful, that is, when the weightings and noise intensities are used as design parameters to produce a desired closed loop response, an equivalent LQG problem can be chosen such that the performance index has a different value; from Lemma 5.18 it could even be zero! The implications of this result are that the value of the performance index and comparisons of different strategies based on the performance index are suspect unless the performance index has a physical interpretation.

5.5. NONUNIQUE SOLUTIONS TO WIENER-HOPF PROBLEMS

The requirement that the spectral factors Δ and Γ are invertible has been central to the development of the results in this thesis: the stabilising solution to the Wiener-Hopf problem is given by $T=\Delta^{-1}M_{\oplus}\Gamma^{-1}$. Obviously such a solution is not valid when the spectral factors are not invertible. However the requirement that the spectral factors are invertible is not necessary for a stabilising solution to exist. It was mentioned in passing in Section 3.3.2 that the requirement that the spectral factors are invertible can be relaxed to left invertibility for Γ and right invertibility for Δ. This extension is considered in this section.

From Theorem 5.9, Δ will be singular when $\text{rank}(\text{LMI}(X)) = r_1 < r$ and Γ will be singular when $\text{rank}(\text{LMI}(Y)) = m_1 < m$. In either case non-square spectral factors can be

formed (r_1 x r for Δ and m x m_1 for Γ); Δ will be right invertible and Γ will be left invertible.

The inverses of the spectral factors were introduced into the frequency domain expression in (3.28) as a means of standardising the quadratic integrand. The partial fraction expansion was introduced in Section 3.4.2 to ensure that the closed loop system was stable. In this chapter a state-space method for partial fraction expansion was presented. This method can be used whether or not the spectral factors are invertible; consider the following expansion:

$$\begin{aligned}
&\left[B^T(-sI-A^T)^{-1}Q+S^T\right](sI-A)^{-1}\left[E\Phi_d E^T(-sI-A^T)^{-1}C^T+E\Phi_{dn}\right] \\
&\quad = \Delta^*K(sI-A)^{-1}L\Gamma^* + \Delta^*KY(-sI-A^T)^{-1}C^T + B^T(-sI-A^T)^{-1}X\left[E\Phi_d E^T(-sI-A^T)^{-1}C^T+E\Phi_{dn}\right] \\
&\quad = \Delta^*M_\oplus\Gamma^* + \Delta^*KY(-sI-A^T)^{-1}C^T + B^T(-sI-A^T)^{-1}X\left[E\Phi_d E^T(-sI-A^T)^{-1}C^T+E\Phi_{dn}\right] \\
&\quad = \Delta^*M_\oplus\Gamma^* + M_\vartheta
\end{aligned} \tag{5.75}$$

This expansion was used in Theorem 5.11 to cancel $(\Delta^*)^{-1}$ and $(\Gamma^*)^{-1}$ in M; here it is used to produce a standardised quadratic integrand:

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\Big\{\,\mathrm{Tr}\Big\{\,\Delta T\Gamma\Gamma^*T^*\Delta^* - M_\oplus\Gamma^*T^*\Delta^* - \Delta T\Gamma M_\oplus^* - TM_\vartheta^* + M_\vartheta T^*\,\Big\} + \mathrm{Tr}\Big\{QP_d\Phi_dP_d^*\Big\}\Big\}\,ds \tag{5.76}$$

Completing the square on this integrand leads to:

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\Big\{\,\mathrm{Tr}\Big\{\,(\Delta T\Gamma - M_\oplus)(\Delta T\Gamma - M_\oplus)^* - M_\oplus M_\oplus^* - TM_\vartheta^* + M_\vartheta T^*\,\Big\} + \mathrm{Tr}\Big\{QP_d\Phi_dP_d^*\Big\}\Big\}\,ds$$

Provided

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\mathrm{Tr}\Big\{TM_\vartheta^*\Big\}\,ds = 0$$

the performance index has a minimal value when:

$$\Delta T\Gamma = M_\oplus \tag{5.77}$$

This solution is the same as when the spectral factors are invertible. The general solution to this equation has the form:

$$T = \Delta^R M_\oplus \Gamma^L + Z - \Delta^R \Delta Z \Gamma \Gamma^L \tag{5.78}$$

where $\Delta\Delta^R = I$, $\Gamma^L\Gamma = I$ and Z is arbitrary.

While this solution ensures that the resulting closed loop system is causal it does not ensure stability: to ensure closed loop stability Δ^R, Γ^L and Z are required to be stable rational functions.

One important case when the spectral factor Γ is not invertible is when there is no measurement noise (Φ_n=0, Φ_{dn}=0) and there are less disturbance inputs than measurements; this case includes that of state feedback. For this reason attention will be focused on the case when the spectral factor Γ is singular. Similar techniques can be used if the spectral factor Δ is singular although as the weightings Q, S, and R are design parameters they can be chosen to ensure Δ is nonsingular.

One method of constructing a stable left inverse of Γ is to 'square-down' the transfer function Γ so that it is invertible. This idea has been used by Austin (1979) and Halevi and Palmor (1986) to study singular LQG problems. The process of squaring-down a transfer function has been studied extensively by Saberi and Sannuti (1987,1988,1990).

The application of this squaring-down matrix needs some care as it is possible to choose a matrix which results in a different LQG problem (specifically this matrix could introduce other non-minimum phase zeros into the disturbance transfer function). The solution to the LMI (5.32) is required to be the same for the original and squared-down spectral factorisation problems. In this case, it follows that the spectral factors for the original and squared-down problems, Γ and Γ_χ , satisfy:

$$\Gamma_\chi \Gamma_\chi^* = \chi \Gamma \Gamma^* \chi^* \tag{5.79}$$

where

$$\Gamma_\chi = \chi C(sI-A)^{-1}L_\chi + V_\chi$$

is invertible and minimum phase. The squaring-down matrix χ can be constant or dynamic (Saberi and Sannuti 1988). Only problems where χ is constant is considered in this section. The existence of a constant χ is equivalent to the existence of a stabilising, output feedback controller (Saberi and Sannuti 1988).

The LMI for the squared-down spectral factor Γ_χ is:

$$\begin{bmatrix} AY_\chi+Y_\chi A^T+E\Phi_d E^T & (Y_\chi C^T+E\Phi_{dn})\chi^T \\ \chi(CY_\chi+\Phi_{dn}^T E^T) & \chi\Phi_n\chi^T \end{bmatrix} = \begin{bmatrix} L_\chi \\ V_\chi \end{bmatrix}\begin{bmatrix} L_\chi^T , V_\chi^T \end{bmatrix}$$

$$= \begin{bmatrix} L \\ V \end{bmatrix}\begin{bmatrix} L^T , V^T \end{bmatrix} \qquad (5.80)$$

The solution to this LMI Y_χ is required to be equal to Y, the solution to the original LMI (5.32) for the original problem.

The squaring-down process is represented in the closed loop block diagram in Figure 5.1. In this method the closed loop transfer function from $\mathbf{y}_\chi$ to **u**, T_χ , is calculated using the standard methods. The closed loop transfer function from **y** to **u** is then given by:

$$T = T_\chi\chi$$

The performance index (3.28) for the squared-down problem is given by:

$$J_\chi = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Big(\mathrm{Tr}\Big\{\Delta^*\Delta T_\chi\chi\Gamma\Gamma^*\chi^T T_\chi^* - (P^*Q+S^T)P_d(\Phi_d G_d^*+\Phi_{dn})\chi^T T_\chi^* - T_\chi\chi(G_d\Phi_d+\Phi_{dn}^*)P_d^*(QP+S)\Big\} + \mathrm{Tr}\Big\{QP_d\Phi_d P_d^*\Big\}\Big)\, ds \qquad (5.81)$$

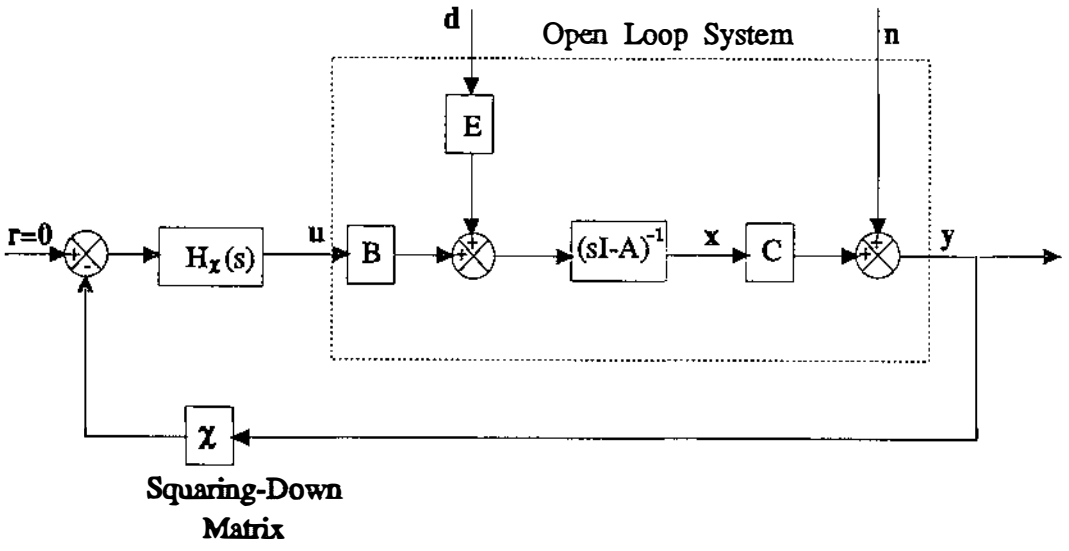

Figure 5.1
Squared-Down Controller Configuration

Using the methods of Section 3.3 and (5.79) the performance index (5.81) can be manipulated into the following form:

$$J_\chi = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left[\mathrm{Tr}\left\{(\Delta T_\chi \Gamma_\chi - M_\chi)(\Delta T_\chi \Gamma_\chi - M_\chi)^* - M_\chi M_\chi^*\right\} + \mathrm{Tr}\left\{QP_d\Phi_d P_d^*\right\}\right] ds \qquad (5.82)$$

where

$$M_\chi = (\Delta^*)^{-1}(P^*Q+S^T)P_d(\Phi_d G_d^*+\Phi_{dn})\chi^T(\Gamma_\chi^*)^{-1} \qquad (5.83)$$

This Wiener-Hopf problem now has a unique solution and so the results of Sections 5.3 and 5.4. apply. The value of the performance index for this reduced problem is:

$$J_\chi = \mathrm{Tr}\left\{XE\Phi_d E^T + Y_\chi KK^T\right\} \qquad (5.84)$$

As $Y_\chi = Y$, the performance index has not been changed by the squaring-down process. The methodology of constructing a solution for singular spectral factors is summarised in the following theorem:

Theorem 5.20 Wiener-Hopf Solutions for Singular Spectral Factors

If the spectral factor Γ is singular a solution can be constructed by choosing a squaring-down matrix χ such that Γ_χ is nonsingular and satisfies (5.79). The LMI for the squared-down problem (5.80) has the same solution as that of the LMI for the original problem. The controller for the original problem, H, is related to the controller for the squared-down problem, H_χ, by $H=H_\chi\chi$. A similar methodology can be used if the spectral factor Δ is singular.

5.5.1. The Singular Linear Quadratic State Control Problem

The Linear Quadratic (LQ) state controller design problem is usually the starting point for the development of LQG theory. The reason for this is that the state control problem is simpler, involving only one Riccati equation. Wiener-Hopf methods are not so convenient for studying the state control problem. Although it is relatively easy to derive the constant gain controller for nonsingular LQ state control problems, this solution is only one of many possible solutions. This problem fits into the class of problems considered in the previous section where the spectral factor Γ is singular. There are also difficulties in determining a controller from the Wiener-Hopf solution

for T; this problem is deferred until the next chapter. In this section the singular LQ state controller will be derived using the Wiener-Hopf techniques of this thesis.

For simplicity of notation it is assumed, without loss of generality, that Φ_d=I. The performance index for the state feedback LQ problem is:

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Big(\mathrm{Tr}\Big\{ (\Delta T(sI-A)^{-1}E-M_s)(\Delta T(sI-A)^{-1}E-M_s)^* - M_sM_s^* \Big\} + \mathrm{Tr}\Big\{ Q(sI-A)^{-1}EE^T(-sI-A^T)^{-1} \Big\} \Big)\, ds \qquad (5.85)$$

where

$$\begin{aligned} M_s &= (\Delta^*)^{-1}(-B^T(sI+A^T)^{-1}Q+S^T)(sI-A)^{-1}E \\ &= K(sI-A)^{-1}E + (\Delta^*)^{-1}B^T(-sI-A^T)^{-1}XE \end{aligned} \qquad (5.86)$$

This expansion of M_s is performed with the LMI (5.13) and the methods of Theorem 5.12. The spectral factor Γ is given by:

$$\Gamma = (sI-A)^{-1}E \qquad (5.87)$$

as this transfer function has no finite zeros (Lemma 2.5). The squared-down spectral factor is given by:

$$\Gamma_\chi = \chi(sI-A)^{-1}E$$

This squared-down spectral factor can only be formed if (A,E) is stabilisable; this is not a necessary condition in nonsingular state LQ problems (Kwakernaak and Sivan 1972). The minimum performance index occurs when:

$$(\Delta T - K)(sI-A)^{-1}E = 0 \qquad (5.88)$$

One solution to this problem is obviously $\Delta T = K$. However, this is not the only solution to (5.88). A generalised inverse solution to (5.88) is:

$$T = \Delta^{-1}\Big[K(sI-A)^{-1}E\Big((sI-A)^{-1}E\Big)^L + Z\Big[I - (sI-A)^{-1}E\Big((sI-A)^{-1}E\Big)^L\Big]\Big] \qquad (5.89)$$

This general solution to (5.88) does not however guarantee that the closed loop system is stable. Two solutions to this equation will be considered: first the simple solution

$$T = \Delta^{-1}K \qquad (5.90)$$

which corresponds to Z = K; the second solution considered is

$$T = \Delta^{-1}K(sI-A)^{-1}E\left[(sI-A)^{-1}E\right]^{L} \tag{5.91}$$

which corresponds to Z=0. For this second solution it is necessary to restrict the left inverses of $(sI-A)^{-1}E$ to those for which:

$$(sI-A)^{-1}E\left[(sI-A)^{-1}E\right]^{L} \tag{5.92}$$

is a stable transfer function. This condition is necessary to guarantee that Definition 2.23 is satisfied. In terms of the squaring-down method outlined in the previous solution:

$$\left[(sI-A)^{-1}E\right]^{L} = \left[\chi(sI-A)^{-1}E\right]^{-1}\chi \tag{5.93}$$

where χ is chosen such that $\chi(sI-A)^{-1}E$ is invertible and minimum phase (that is, Γ_{χ} is a spectral factor) and the solution to the LMI (5.80) is $Y_{\chi}=0$. It is necessary that (A,E) is stabilisable for this squaring-down process to be possible. Now (5.92) is given by:

$$\begin{aligned}(sI-A)^{-1}E\left[(sI-A)^{-1}E\right]^{L} &= (sI-A)^{-1}E\left[\chi(sI-A)^{-1}E\right]^{-1}\chi \\ &= \left[(I-EE^{L})(sI-A)+E\chi\right]^{-1}E\chi\end{aligned} \tag{5.94}$$

which from Lemma 4.16 is a stable descriptor form.

The advantage of using the simple solution (5.90) is that the state feedback controller does not depend on the disturbance process, as the solution (5.91) does. However, as will be shown in Section 6.4.1, the simple solution (5.90) cannot be used if the weighting on the control input is singular. To use the solution (5.91) it is necessary that (A,E) is stabilisable. There are many other possible solutions to (5.89); the consequences of choosing these solutions have not yet been investigated. These results are summarised in the following theorem:

Theorem 5.21 Singular State Feedback Control

The general Wiener-Hopf solution to the state feedback control problem is:

$$T = \Delta^{-1}\left[K(sI-A)^{-1}E\left[(sI-A)^{-1}E\right]^{L} + Z\left[I - (sI-A)^{-1}E\left[(sI-A)^{-1}E\right]^{L}\right]\right] \tag{5.89}$$

where Z is an arbitrary stable rational function and $\left[(sI-A)^{-1}E\right]^L$ is a stable left inverse of $(sI-A)^{-1}E$. Two specific solutions are:

$$T = \Delta^{-1}K \tag{5.90}$$

and, if (A,E) is stabilisable,

$$T = \Delta^{-1}M_{\oplus}\Gamma_{\chi}^{-1}\chi \tag{5.91}$$

where

$$\Gamma_{\chi} = \chi(sI-A)^{-1}E$$

and the solution to the LMI associated with Γ_{χ} is $Y_{\chi}=0$.

5.6. THE MINIMUM VARIANCE ESTIMATOR

Wiener-Hopf methodology has been used in this thesis to study the LQG controller design problem. In this section Wiener-Hopf techniques are used to solve the minimum variance estimation problem in cases where the measurement noise intensity Φ_n may be singular. It will be shown in the next chapter that the LQG controller contains a minimum variance estimator.

The solution to the singular minimum variance estimator problem has a Luenburger reduced-order observer structure (Luenburger 1971). The reduced-order nature of the observer was first shown using the methods of Bryson and Johansen (1965). In this solution the outputs are differentiated to obtain a nonsingular filtering problem. The structure and properties of the resulting filter have been investigated by Schumacher (1985), Soroka and Shaked (1988), and Halevi and Palmor (1986). The method of Bryson and Johansen (1965) is applicable to finite time as well as the infinite time problems considered in this thesis. Leondes and Yonezawa (1980), and O'Reilly and Newman (1975) considered the special case when only one differentiation is necessary to produce a nonsingular estimation problem. Another technique is the asymptotic approach which considers a singular estimation problem to be the limiting solution to a nonsingular estimation problem. In this approach the limiting solution of a Riccati equation is studied (Kwakernaak and Sivan 1972; Halevi and Palmor 1986).

The Wiener-Hopf methods which have been used in this thesis were originally used by Wiener (1949) for a least squares filtering problem. These methods have been applied

to the linear estimation problem by Shaked (1976a), Grimble (1978), and Shaked and Soroka (1987). The work of Soroka and Shaked (1987) is used as a starting point for this section as they provide a transfer function form of the optimal estimator.

The optimal estimator for the system (3.2) has the form:

$$\hat{\mathbf{x}}(s) = \Theta(s)\mathbf{y}(s) + \Theta_1(s)\mathbf{u}(s) \tag{5.95}$$

where

$$\Theta_1(s) = (sI-A)^{-1}B - \Theta(s)C(sI-A)^{-1}B \tag{5.96}$$

The reason for defining $\Theta_1(s)$ as in (5.96) is that the estimation error is independent of the control inputs $\mathbf{u}(s)$. The estimation error $\mathbf{e}(s) = \mathbf{x}(s)-\hat{\mathbf{x}}(s)$ is given by:

$$\mathbf{e}(s) = (sI-A)^{-1}E\mathbf{d}(s) - \Theta(s)(C(sI-A)^{-1}E\mathbf{d}(s)+\mathbf{n}(s)) \tag{5.97}$$

The minimum variance estimator is designed to minimise the variance of estimation error $\mathbf{e}(t)$ which is given by:

$$\begin{aligned}\mathcal{E}\left[\mathbf{e}(t)\mathbf{e}^T(t)\right] &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(\Theta\Gamma\Gamma^*\Theta^* - \Theta\Gamma M_e^* - M_e\Gamma^*\Theta^* + (sI-A)^{-1}E\Phi_d E^T(-sI-A^T)^{-1}\right)ds \\ &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left((\Theta\Gamma-M_e)(\Theta\Gamma-M_e)^* - M_eM_e^* + (sI-A)^{-1}E\Phi_d E^T(-sI-A^T)^{-1}\right)ds\end{aligned} \tag{5.98}$$

where Γ is the disturbance spectral factor as defined for the LQG control problem in (3.18) and the transfer function $M_e(s)$ is defined by:

$$\begin{aligned}M_e(s) &= (sI-A)^{-1}\left(E\Phi_d E^T(-sI-A^T)^{-1}+E\Phi_{dn}\right)(\Gamma^*)^{-1} \\ &= (sI-A)^{-1}L + Y(-sI-A^T)^{-1}C^T(\Gamma^*)^{-1}\end{aligned} \tag{5.99}$$

The expansion of M_e is performed using the LMI for Γ (5.32) and the partial fraction expansion methods used in Theorem 5.11. The estimator is required to be asymptotically stable, that is, the transfer functions from $\mathbf{d}(s)$ to $\mathbf{e}(s)$ and $\mathbf{n}(s)$ to $\mathbf{e}(s)$ in (5.97) are required to be stable. Restricting these transfer functions to be stable, the optimal estimator is given by:

$$\Theta(s) = [M_e]_{\oplus}\Gamma^{-1} = (sI-A)^{-1}L\Gamma^{-1} \tag{5.100}$$

The optimal estimator is therefore given by:

$$\hat{\mathbf{x}}(s) = (sI-A)^{-1}L\Gamma^{-1}\mathbf{y}(s) + \left((sI-A)^{-1}B-(sI-A)^{-1}L\Gamma^{-1}C(sI-A)^{-1}B\right)\mathbf{u}(s) \qquad (5.101)$$

It is also necessary to show that the optimal solution (5.100) results in a finite estimation error variance. This can be shown using similar methods to those used in evaluating the performance index in Section 5.4.

$$\begin{aligned}\mathcal{E}\left[\mathbf{e}(t)\mathbf{e}^T(t)\right] &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\Big((sI-A)^{-1}E\Phi_d E^T(-sI-A^T)^{-1} \\ &\qquad - [M_e]_{\oplus}[M_e]_{\theta}^* - [M_e]_{\theta}[M_e]_{\oplus}^* - [M_e]_{\oplus}[M_e]_{\oplus}^*\Big)\,ds \\ &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\Big(\left((sI-A)^{-1} - (sI-A)^{-1}L\Gamma^{-1}C(sI-A)^{-1}\right)Y \\ &\qquad + Y\left((sI-A)^{-1} - (sI-A)^{-1}L\Gamma^{-1}C(sI-A)^{-1}\right)^*\Big)\,ds\end{aligned}$$

The transfer function $\left((sI-A)^{-1} - (sI-A)^{-1}L\Gamma^{-1}C(sI-A)^{-1}\right)Y$ is stable, by design and the sum of residues of $(sI-A)^{-1}L\Gamma^{-1}C(sI-A)^{-1}Y$ can be shown to be zero using similar arguments to those in Lemma 5.16. Therefore the variance of the estimation error is given by:

$$\mathcal{E}\left[\mathbf{e}(t)\mathbf{e}^T(t)\right] = Y \qquad (5.102)$$

This gives a physical interpretation the maximal solution of the LMI (5.32).

These results are summarised in the following theorem:

Theorem 5.22 Minimum Variance Estimators

The minimum variance estimator is given by (5.101). The variance of the estimation error is Y, the solution to the LMI (5.32).

5.7. SUMMARY

A state-space approach to the Wiener-Hopf technique of Austin (1979) has been presented in this chapter. The major tool in this study was the Linear Matrix Inequality (LMI) which replaces the Riccati equation in singular LQG problems. The LMI were derived from consideration of spectral factors and were also shown to determine the partial fraction expansion step. This relationship was shown to resolve the difficulties

with Austin's method: specifically, Assumption 3.14 was removed, and it was shown that this solution to the Wiener-Hopf problem guarantees that the resulting closed loop system is stable. The LMI were also used to evaluate the contour integral for the LQG performance index leading to direct generalisations of the Riccati equation based expressions for nonsingular LQG problems. The Wiener-Hopf solution depends on the invertibility of the spectral factors. When this condition is not satisfied, the solution is nonunique. The state feedback control problem was presented as an example of such a problem. The minimum variance estimation problem was solved using Wiener-Hopf techniques and an LMI.

Throughout this chapter only the Wiener-Hopf solution for the closed loop transfer function T(s), defined in (2.59), is used: there has been no explicit use of the controller. In the next chapter a new descriptor form of the controller will be determined from the transfer function T(s).

CHAPTER 6

A DESCRIPTOR APPROACH TO SINGULAR LQG CONTROLLER DESIGN

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6.1. INTRODUCTION

The Wiener-Hopf method defines the optimal closed loop transfer function from the outputs **y** to the inputs **u**, T(s). It is necessary to use (3.64) to calculate the controller H(s). The problem with using (3.64) alone to calculate the controller is that it produces a nonminimal form for the controller. This form could even have zeros cancelling the unstable system poles as illustrated by Grimble and Johnson (1986, 1:111-114). Another problem with singular LQG problems is that there does not necessarily exist a controller which will produce the closed loop response specified by the Wiener-Hopf solution T(s). This was recently discussed by Stoorvogel (1992a) who used a disturbance decoupling approach to study this problem. However, apart from raising some interesting questions about the singular LQG control problem and providing an example which illustrated some of the difficulties, a complete classification of this problem was not offered. The problem of calculating the controller from the Wiener-Hopf solution and implications for the existence and uniqueness of the controller is discussed in this chapter.

It is well-known that the solution to the LQG problem for nonsingular control weightings and measurement noise intensity is specified by a state feedback controller and a Kalman filter. This result is known as the separation theorem and leads to a simple state-space formula for the controller (Kwakernaak and Sivan 1972). The nonsingular LQG controller is derived from the Wiener-Hopf solution. The purpose of providing this derivation is to highlight some of the standard, well-known properties of nonsingular LQG controllers so as to be able subsequently to compare and contrast the solution to the singular LQG control problem.

The main contribution in this chapter is the derivation of some new formulae which are valid for singular and nonsingular LQG controllers. Singular LQG controllers are not necessarily proper transfer functions and hence a state-space approach is not sufficient for this problem. Traditionally a rational transfer function approach is adopted to this problem (using the solution for T and simplifying the resulting controller). A new approach in this thesis is the development of explicit descriptor forms for LQG controllers which depend only on the open loop system and the state-space forms of the spectral factors Δ and Γ described in the previous chapter. The descriptor form of the LQG controller has a similar structure to the nonsingular LQG controller allowing properties of nonsingular LQG controllers to be extended to the singular case. The particular descriptor forms were derived as an algebraic tool in manipulating certain transfer functions. A feedback interpretation of these descriptor forms is given which further shows the similarity in structure of singular and nonsingular LQG controllers.

At present it is not known whether the separation theorem extends to singular LQG problems (Stoorvogel 1992). The descriptor forms developed in this chapter are used to establish the separation theorem for singular LQG problems. First, the descriptor approach is used to study the LQ state feedback control and the minimum variance estimation problems. It is then shown that the singular LQG controller is composed of an LQ state feedback controller and a minimum variance estimator. Two approaches to the separation theorem are taken: firstly, it is shown that the descriptor form is composed of a minimum variance estimator and an LQ state feedback controller; and secondly, it is shown that an LQG control problem can be decomposed into a minimum variance estimator problem and an LQ state feedback control problem.

While the number of descriptor states in the controller is the same as the number of states in the open loop system, the actual order of the controller is determined by the rank of the $\mathcal{E}$ matrix in the descriptor form $(sE-A)^{-1}$. When $\mathcal{E}$ is singular the descriptor form may contain finite and infinite modes. The infinite frequencies correspond to the impulsive or derivative nature of the controller. The properties of descriptor forms introduced in Chapter 2 are used to study the order of the controller. Particular emphasis is placed on deriving conditions under which an LQG problem leads to a proper controller. When these conditions are satisfied a reduced-order state-space form of the controller may be derived. Two special cases are considered: no measurement noise, and no weighting on the control input. For these two cases, some simple state-space forms are derived when the controller is proper. The order of the controller is also studied using multivariable zeros for LQG problems when either the weighting on the control input or the measurement noise intensity is full rank. The minimality of the descriptor forms is also discussed.

The descriptor approach to LQG controller design presented in this chapter has implications for the process of designing singular LQG controllers. The design of LQG controllers with proper closed loop transfer functions is discussed. The Loop Transfer Recovery technique of Doyle and Stein (1979) is discussed using the descriptor approach. To conclude this chapter the subject of existence and uniqueness of the LQG controller is studied using the descriptor approach.

6.2. CALCULATION OF THE CONTROLLER FROM THE WIENER-HOPF SOLUTION

Before proceeding to derive expressions for LQG controllers some comments are in order concerning the methodology required to determine the controller from the Wiener-Hopf solution. One of the advantages of Wiener-Hopf techniques is that the

closed loop transfer function T (2.59) is calculated as an intermediary step. The solution to the Wiener-Hopf problem was presented in the previous chapter and was shown to be defined by the solution to two LMI (5.13) and (5.31). This solution was shown to result in a stable closed loop system (Theorem 5.13) and a finite performance index (Theorem 5.19).

Throughout this chapter it is assumed that the spectral factors Δ and Γ are invertible and therefore the Wiener-Hopf solution is unique. If this is not the case, then a procedure similar to that in Section 5.5 could be adopted to transform the problem into one with a unique solution. Once the calculation of $M_{\oplus} = K(sI-A)^{-1}L$ has been performed, the Wiener-Hopf solution, $T = \Delta^{-1}M_{\oplus}\Gamma^{-1}$, can be manipulated to form the controller H. The transfer function T(s) was defined in (2.59) to be the closed loop transfer function:

$$T = [I+HG]^{-1}H = H[I+HG]^{-1} \tag{6.1}$$

The controller H is recovered from this equation by solving the following linear equations:

$$(I-TG)H = T \quad \text{or} \quad H(I-GT) = T \tag{6.2}$$

If (I-TG) and (I-GT) are invertible the controller (3.64) is given by

$$H = (I-TG)^{-1}T = T(I-GT)^{-1} \tag{6.3}$$

and hence the controller is unique. There are several alternative forms of the controller which can be derived from the Wiener-Hopf solution (3.63):

$$\begin{aligned} H(s) &= (\Delta-M_{\oplus}\Gamma^{-1}G)^{-1}M_{\oplus}\Gamma^{-1} \\ &= \Delta^{-1}M_{\oplus}(\Gamma-G\Delta^{-1}M_{\oplus})^{-1} \end{aligned} \tag{6.4}$$

However the inverses in (6.3) do not necessarily exist in which case the controller is either nonunique or does not exist. As (6.2) is a linear system of equations, a solution exists if and only if:

$$\text{rank}\left(\left[(I-TG)\ T\right]\right) = \text{rank}\left((I-TG)\right)$$

or

$$\text{rank}\left(\begin{bmatrix} I-GT \\ T \end{bmatrix}\right) = \text{rank}\left((I-GT)\right) \tag{6.5}$$

Only the case where the solution to (6.2) is unique will be considered in detail in this chapter. Some remarks about nonunique solutions will be made in Section 6.7.

Another drawback with using (6.3) or (6.4) to calculate the controller is that they yield nonminimal controller structures: they contain common factors in the numerator and denominators which need to be cancelled. The forms in (6.4) are preferable to those of (6.3), as (6.3) contains unstable common factors. This problem with forming the controller from T lead to criticism of the Wiener-Hopf method by Grimble and Johnson (1986, 111-114).

6.3. NONSINGULAR LQG CONTROLLERS

In this section the state-space form of the nonsingular LQG controller is derived from the Wiener-Hopf solution and the separation structure is discussed. This presentation provides the background for the solution and properties of singular LQG controllers which are developed in the rest of the chapter. The nonsingular LQG controller can be derived from the Wiener-Hopf solution as follows:

Theorem 6.1 Nonsingular LQG Controllers

The LQG controller for a nonsingular LQG problem is given by:

$$H(s) = K_c\left[sI-A+BK_c+K_fC\right]^{-1}K_f \tag{6.6}$$

where K_c and K_f are the optimal state feedback and Kalman filter gains.

Proof

The Wiener-Hopf solution for nonsingular LQG problems can be expanded using (5.29) and (5.34) to give:

$$\begin{aligned} T &= \Delta^{-1}M_{\oplus}\Gamma^{-1} \\ &= (D+K(sI-A)^{-1}B)^{-1}K(sI-A)^{-1}L(V+C(sI-A)^{-1}L)^{-1} \\ &= K_c(sI-A+BK_c)^{-1}(sI-A)(sI-A+K_fC)^{-1}K_f \end{aligned} \tag{6.7}$$

From (6.3) the controller is given by:

$$\begin{aligned} H &= (I-TG)^{-1}T \\ &= \left[I-K_c(sI-A+BK_c)^{-1}(sI-A)(sI-A+K_fC)^{-1}K_fC(sI-A)^{-1}B\right]^{-1}T \\ &= K_c\left[\,(sI-A+K_fC)(sI-A)^{-1}(sI-A+BK_c) - K_fC(sI-A)^{-1}BK_c\,\right]^{-1}K_f \end{aligned} \tag{6.8}$$

Since:

$$(sI-A+K_fC)(sI-A)^{-1}(sI-A+BK_c) = sI-A+BK_c+K_fC + K_fC(sI-A)^{-1}BK_c \tag{6.9}$$

the controller is given by (6.6). ♦

This result has been derived from Wiener-Hopf methods by a number of researchers: Shaked (1976b) who used manipulations on the Riccati equations; Park and Youla (1992) who used the method of Youla *et al.* (1976b); and Grimble (1987) who derived this result using the method of Kucera (1980).

It is well-known that the nonsingular LQG controller possesses a separation structure in which the states $\mathbf{x}(t)$ are estimated from the outputs $\mathbf{y}(t)$ by the Kalman filter:

$$\dot{\hat{\mathbf{x}}}(t) = A\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + K_f(\mathbf{y}(t)-\hat{\mathbf{y}}(t)) \tag{6.10}$$

These estimates $\hat{\mathbf{x}}(t)$ are then used in the state feedback controller:

$$\mathbf{u}(t) = -K_c\hat{\mathbf{x}}(t) \tag{6.11}$$

to determine the control inputs. The controller is represented by the block diagram in Figure 6.1. A useful state-space form of the closed loop system is that given by taking the states to be: $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ and $\hat{\mathbf{x}}(t)$:

$$\begin{bmatrix}\dot{\mathbf{e}}(t)\\ \dot{\hat{\mathbf{x}}}(t)\end{bmatrix} = \begin{bmatrix} A-K_fC & 0 \\ K_fC & A-BK_c\end{bmatrix}\begin{bmatrix}\mathbf{e}(t)\\ \hat{\mathbf{x}}(t)\end{bmatrix} + \begin{bmatrix} E & -K_f \\ 0 & K_f\end{bmatrix}\begin{bmatrix}\mathbf{d}(t)\\ \mathbf{n}(t)\end{bmatrix} \tag{6.12}$$

This form is useful in proving the separation theorem, as it can readily be shown (Kwakernaak and Sivan 1972, 389-402) that $\mathbf{e}(t)$ and $\hat{\mathbf{x}}(t)$ are independent, that is:

$$\mathcal{E}[\,\hat{\mathbf{x}}(t)\mathbf{e}^T(t)] = 0 \tag{6.13}$$

Writing $\mathbf{x}(t) = \hat{\mathbf{x}}(t)+\mathbf{e}(t)$ and applying (6.13) to the performance index (3.1) leads to:

$$J = \mathcal{E}\left[\begin{bmatrix}\hat{\mathbf{x}}^T(t) , \mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix} Q & S^T \\ S & R\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}(t)\\ \mathbf{u}(t)\end{bmatrix}\right] + Tr\left\{Q\mathcal{E}\left[\mathbf{e}(t)\mathbf{e}^T(t)\right]\right\} \tag{6.14}$$

where

$$\dot{\hat{\mathbf{x}}}(t) = A\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + \eta(t) \qquad \text{and} \qquad \Phi_\eta = K_f \Phi_n K_f^T = LL^T \tag{6.15}$$

As was shown in Section 5.6, the estimation problem $\mathcal{E}\left[\mathbf{e}(t)\mathbf{e}^T(t)\right]$ can be solved independently of the input $\mathbf{u}(t)$ to provide $\hat{\mathbf{x}}(t)$. The estimate which satisfies (6.15) is then used in the state feedback problem:

$$J = \mathcal{E}\left[\left[\hat{\mathbf{x}}^T(t)\,,\,\mathbf{u}^T(t)\right]\begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}(t) \\ \mathbf{u}(t)\end{bmatrix}\right]$$

which results in the state feedback controller (6.11).

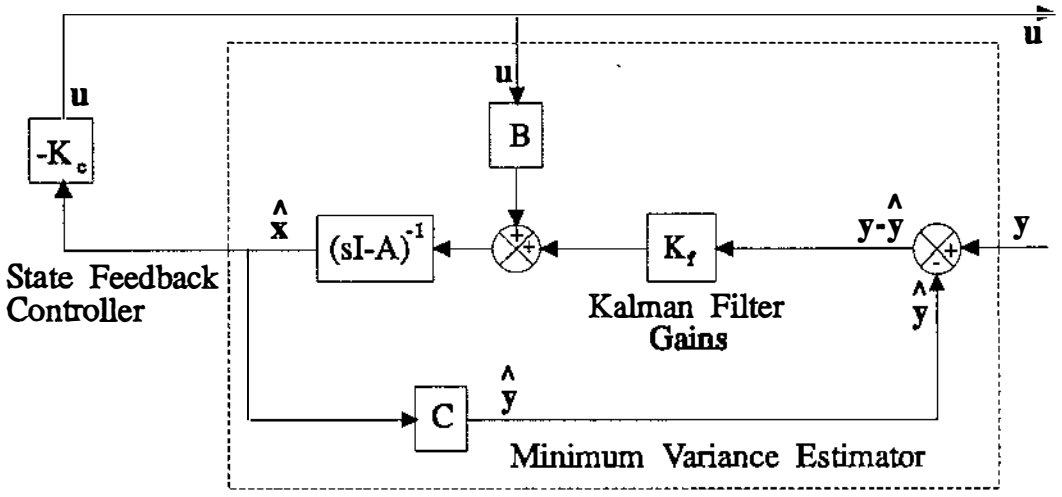

Figure 6.1
Structure of Nonsingular LQG Controllers

6.4. DESCRIPTOR FORMS FOR LQG CONTROLLERS

The state-space description of nonsingular LQG controllers presented in the previous section is not valid for singular LQG controllers, as singular LQG controllers can be improper, and if they are, cannot be described using state-space forms. In the derivation of the nonsingular LQG controller the invertibility of the matrices D and V ($R=D^TD$ and $\Phi_n=VV^T$) was used; therefore this methodology cannot be used for singular LQG problems. In this section a new form for LQG controllers which valid for singular and

nonsingular LQG problems is derived. This form is then used to establish the separation theorem for singular and nonsingular LQG controllers.

In calculating the controller Wiener-Hopf methods traditionally use rational transfer functions to allow for the possibility of improper controllers. Another possibility which has apparently been ignored in the literature is a descriptor or a generalised state-space approach. The descriptor forms in Section 4.6 were derived as alternative representations of the transfer functions:

$$(sI-A)^{-1}B\left[C(sI-A)^{-1}B+D\right]^{-1} \text{ and } \left[C(sI-A)^{-1}B+D\right]^{-1}C(sI-A)^{-1}$$

These forms arise in the Wiener-Hopf solution developed in the last chapter due to the factors of T:

$$\begin{aligned} T &= \Delta^{-1}M_{\oplus}\Gamma^{-1} \\ &= \left[K(sI-A)^{-1}B+D\right]^{-1}K(sI-A)^{-1}L\left[C(sI-A)^{-1}L+V\right]^{-1}. \end{aligned} \tag{6.16}$$

In order to apply the descriptor forms of Lemmas 4.16 and 4.17 to (6.16) it is necessary that K and L have full row and column rank respectively. In most problems of interest, this will be the case. This is a necessary condition if R = 0 and $\Phi_n = 0$ (otherwise the spectral factor will not be invertible). If R and/or Φ_n are positive definite, it is possible to derive forms of the controller without the rank requirement for K and L using the nonsingular techniques of the previous section. The only cases in which these rank requirements may not be met are partially singular LQG problems (where $R \geq 0$ and/or $\Phi_n \geq 0$). However, in this case the requirement that the spectral factors Δ and Γ are invertible ensures that the fractional descriptor forms of Lemma 4.18 can be used.

Describing the Wiener-Hopf solution using the descriptor forms of Section 4.6 leads to the following forms for LQG controllers:

Theorem 6.2 Descriptor Forms of LQG Controllers

If the K and L matrices from the spectral factors have full row and column rank respectively, then if the LQG controller is well-defined, it is given by:

$$H(s) = K\left[\, E_2(sI-A)E_1 + LCE_1 + E_2BK \,\right]^{-1}L \tag{6.17}$$

where

$$E_1 = I\text{-}K^RK+K^RDK \tag{6.18}$$

and

$$E_2 = I\text{-}LL^L+LVL^L. \tag{6.19}$$

If the K and/or L matrices from the spectral factors do not have full row and column rank respectively, then if the LQG controller is well-defined, it is given by:

$$H(s) = (K_F+F_1E_{F1})\left[E_{F2}(sI-A)E_{F1}+(L_F+F_2E_{F2})CE_{F1}+E_{F2}B(K_F+F_1E_{F1})\right]^{-1}(L_F+F_2E_{F2}) \tag{6.20}$$

where F_1 is chosen such that:

$$K_F = K\text{-}DF_1 \qquad \text{has full row rank}$$
$$E_{F1} = I\text{-}K_F^RK_F+K_F^RDK_F \tag{6.21}$$

and F_2 is chosen such that:

$$L_F = L\text{-}F_2V \qquad \text{has full column rank}$$
$$E_{F2} = I\text{-}L_FL_F^L+L_FVL_F^L \tag{6.22}$$

The controller is said to be well-defined (that is, the solution to (6.2) is unique) if the descriptor form (6.20) is regular for all F_1 and F_2 such that K_F and L_F have full row and column rank.

Proof

First consider the case when K and L have full row and column rank. Using the descriptor forms in Lemmas 4.16 and 4.17 the Wiener-Hopf solution for T is given by:

$$\begin{aligned} T &= \left(D+K(sI-A)^{-1}B\right)^{-1}K(sI-A)^{-1}L\left(V+C(sI-A)^{-1}L\right)^{-1} \\ &= K\left((sI-A)E_1+BK\right)^{-1}(sI-A)\left(E_2(sI-A)+LC\right)^{-1}L \end{aligned} \tag{6.23}$$

where $E_1 = I\text{-}K^RK+K^RDK$ and $E_2 = I\text{-}LL^L+LVL^L$. Using this expression for T and the identity (2.55) the controller can be calculated from (6.3):

$$\begin{aligned} H &= (I-TG)^{-1}T \\ &= \left[I - K\left[(sI-A)E_1+BK\right]^{-1}(sI-A)\left[E_2(sI-A)+LC\right]^{-1}LC(sI-A)^{-1}B\right]^{-1}T \\ &= K\left[\left[E_2(sI-A)+LC\right](sI-A)^{-1}\left[(sI-A)E_1+BK\right] - LC(sI-A)^{-1}BK\right]^{-1}L \end{aligned} \tag{6.24}$$

Since:

$$\left[E_2(sI-A)+LC\right](sI-A)^{-1}\left[(sI-A)E_1+BK\right] = E_2(sI-A)E_1+LCE_1+E_2BK+LC(sI-A)^{-1}BK \tag{6.25}$$

the controller has the descriptor form presented in (6.17).

The descriptor form (6.20) is derived using similar algebra, but using the fractional descriptor forms of Lemma 4.18; (6.17) can be considered as a special case of (6.20) with F_1 and F_2 zero. The derivation of (6.17) and (6.20) only used algebraic manipulations on (6.3) and so is valid only if (I-TG) is invertible. This requirement is equivalent to the descriptor forms in (6.17) or (6.20) being regular. ♦

The descriptor forms for LQG controllers, (6.17) and (6.20), were derived without needing to assume that R and Φ_n are full rank. They were derived specifically to study the form of singular LQG controllers. However, when R and Φ_n are full rank the nonsingular LQG controller (6.6) can be recovered from the descriptor form (6.20).

For a nonsingular LQG problem, the matrices E_{F1} and E_{F2} are nonsingular. Therefore:

$$\begin{aligned} (K_F+F_1E_{F1})E_{F1}^{-1} &= \left[I - K_FK_F^R(I-D)\right]^{-1}K_F + F_1 \\ &= D^{-1}(K-DF_1) + F_1 \\ &= K_c \end{aligned} \tag{6.26}$$

Similarly:

$$\begin{aligned} E_{F2}^{-1}(L_F+E_{F2}F_2) &= (L-F_2V)V^{-1} + F_2 \\ &= K_f \end{aligned} \tag{6.27}$$

Using these identities the nonsingular LQG controller (6.6) can be recovered from the descriptor form (6.20). The algebra used to derive this result is similar to that used to derive the nonsingular LQG controller in Theorem 6.1.

The descriptor form for singular LQG controllers can be used to determine the positions of the closed loop poles as follows:

Theorem 6.3 Closed Loop Poles

The closed loop poles are at the zeros of the spectral factors Δ and Γ.

Proof

Using the descriptor form for the controller (6.17) the closed loop system is described by:

$$\begin{bmatrix} I & 0 \\ 0 & E_2E_1 \end{bmatrix}\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{x}}_c(t) \end{bmatrix}=\begin{bmatrix} A & -BK \\ LC & E_2AE_1-LCE_1-E_2BK \end{bmatrix}\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}_c(t) \end{bmatrix}+\begin{bmatrix} E & 0 \\ 0 & L \end{bmatrix}\begin{bmatrix} \mathbf{d}(t) \\ \mathbf{n}(t) \end{bmatrix} \tag{6.28}$$

where $\mathbf{x}(s)$ is the vector of states of the system G(s) and $\mathbf{x}_c(s)$ is the vector of descriptor states of the controller H(s). The closed loop poles are given by:

$$\begin{aligned} &\det\left(\begin{bmatrix} sI-A & BK \\ -LC & E_2(sI-A)E_1+LCE_1+E_2BK \end{bmatrix}\right) \\ &\quad = \det(sI-A)\det\left(E_2(sI-A)E_1 + LCE_1 + E_2BK + LC(sI-A)^{-1}BK \right) \\ &\quad = \det(sI-A)\det\left(((sI-A)E_1+BK)(sI-A)^{-1}(E_2(sI-A)+LC) \right) \\ &\quad = 0 \end{aligned}$$

from (6.25).

$$= \det\left((sI-A)E_1+BK\right)\det\left(E_2(sI-A)+LC\right) \tag{6.29}$$

which from Lemmas 4.16 and 4.17 are the zeros of $\Delta=K(sI-A)^{-1}B+D$ and $\Gamma=C(sI-A)^{-1}L+V$. This derivation was presented using (6.17) for the controller. If (6.20) is used, the same derivation holds with the additional fact that zeros are invariant under feedback (Macfarlane and Karcanias 1976). ♦

This result is a generalisation of the familiar result for nonsingular LQG controllers where the closed loop poles are given by the eigenvalues of $A-BK_c$ and $A-K_fC$; these are the zeros of Δ and Γ (Section 5.2.3).

The descriptor approach to LQG control problems results in a formula for the LQG controller, (6.17), which is valid for singular and nonsingular LQG problems. The

descriptor form has a similar structure to the nonsingular LQG controller (6.6) which allows some of the properties of nonsingular LQG controllers to be extended to singular LQG controllers. The first property to be investigated is the separation structure of the LQG controller. For singular LQG problems, the LQ state feedback controller and minimum variance estimator structures for the nonsingular LQG problem are no longer valid. The structures of state feedback controllers and minimum variance estimators for singular problems are considered in the next two sections. The LQG controller is then shown to be composed of an LQ state feedback controller and a minimum variance estimator.

6.4.1. Descriptor Forms for LQ State Feedback Controllers

The Wiener-Hopf solution to the LQ state feedback control problem was considered in Section 5.5.1. It was shown that this Wiener-Hopf problem has a nonunique solution. The simple solution to this problem $T = \Delta^{-1}K$ given in (5.90) does not always lead to a well-defined controller when D is singular as:

$$\begin{aligned} H &= (I-TG)^{-1}T \\ &= (I-\Delta^{-1}K(sI-A)^{-1}B)^{-1}\Delta^{-1}K \\ &= (\Delta-K(sI-A)^{-1}B)^{-1}K \\ &= D^{-1}K \end{aligned} \tag{6.30}$$

For nonsingular control problems, the state feedback controller is given by the familiar constant feedback gain controller:

$$\mathbf{u}(s) = -K_c\mathbf{x}(s)$$

However for singular LQ problems, the controller H(s) is not defined for the solution $T = \Delta^{-1}K$. In this case it is possible to use the nonuniqueness of the Wiener-Hopf solution to find a well-defined controller. A second solution to the state-feedback problem is given by (5.91). The method of constructing this solution is to choose a χ so that:

$$\Gamma_\chi = \chi(sI-A)^{-1}E$$

is a nonsingular spectral factor. The controller is then specified by the descriptor forms in Theorem 6.2:

$$H(s) = K\left[E_2(sI-A)E_1+E_2BK+E\chi E_1\right]^{-1}E\chi \tag{6.31}$$

where E_1 is given by (6.18) and $E_2 = I - EE^L$. The controller is well-defined if the descriptor form (6.31) is regular. The existence of such a solution is a subject for further research. Willems *et al.* (1986) use geometric techniques to show how the nonuniqueness of the state feedback controller arises. However the link between their solution and the Wiener-Hopf solutions in this thesis is not clear. An example is given to indicate how one may calculate the LQ state feedback controller using Wiener-Hopf methods.

Example 6.1 - Calculation of a Singular LQ State Feedback Controller

Consider an LQ state feedback control problem where:

$$\Delta = \frac{s+2}{s(s-2)} = K(sI-A)^{-1}B = \left[\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \end{bmatrix}, 0\right]$$

$$P_d = (sI-A)^{-1}E = \left[\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \end{bmatrix}\right]$$

The reduction matrix χ is chosen to place the zero of Γ_χ at -1. This results in:

$$\chi = \begin{bmatrix} \frac{3}{4} & \frac{-1}{2} \end{bmatrix}$$

and

$$\Gamma_\chi = \frac{s+1}{s(s-2)} = \chi(sI-A)^{-1}E = \left[\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{3}{4} & \frac{-1}{2} \end{bmatrix}, 0\right]$$

To apply the descriptor formula (6.31) it is first necessary to calculate the matrices E_1 and E_2.

$$E_1 = I - K^R K = I - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$

$$E_2 = I - EE^L = I - \begin{bmatrix} 2 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$$

The controller can now be evaluated from (6.31):

$$H = \begin{bmatrix} -1 & 2 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} s - \begin{bmatrix} 3 & -4 \\ 4 & -5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{2}, \frac{3}{4} \end{bmatrix}$$
$$= \begin{bmatrix} -3 & 2 \end{bmatrix}$$ ♦

Generally singular state feedback controllers contain derivative action. In the above example the constant gain controller arises as $E_2E_1=0$. The order of LQG controllers will be discussed further in Section 6.5.

Most research into singular LQ state feedback control problems has not been concerned with calculating the controller, but rather in determining the closed loop response of the system (Francis 1979; Bell and Jacobson 1975; Clements and Anderson 1978). The closed loop transfer functions from disturbances to states and disturbances to inputs can be described using the descriptor approach of this chapter. Consider first the case where K has full row rank. Using the general solution to the state feedback problem (5.89) and the descriptor form in Lemma 4.17 for $\Delta^{-1}K(sI-A)^{-1}$, the states $\mathbf{x}(s)$ are given by:

$$\begin{aligned} \mathbf{x}(s) &= (P_d - PTP_d)\,\mathbf{d}(s) \\ &= \left(I - P\Delta^{-1}K\right)P_d\,\mathbf{d}(s) \\ &= (sI-A)^{-1}\left(I - BK\left((sI-A)E_1+BK\right)^{-1}\right)E\,\mathbf{d}(s) \\ &= E_1\left((sI-A)E_1+BK\right)^{-1}E\,\mathbf{d}(s) \end{aligned} \tag{6.32}$$

and the inputs $\mathbf{u}(s)$ by:

$$\begin{aligned} \mathbf{u}(s) &= -TP_d\,\mathbf{d}(s) \\ &= -K\left((sI-A)E_1+BK\right)^{-1}E\,\mathbf{d}(s) \end{aligned} \tag{6.33}$$

Defining:

$$\mathbf{x}_{cl}(s) = \left((sI-A)E_1+BK\right)^{-1}E\,\mathbf{d}(s)$$

allows (6.32) and (6.33) to be written as:

$$\mathbf{u}(s) = -K\mathbf{x}_{cl}(s) \tag{6.34}$$

and

$$\begin{aligned}\mathbf{x}(s) &= E_1\mathbf{x}_{cl}(s)\\ &= (I-K^RK+K^RDK)\mathbf{x}_{cl}(s)\\ &= \mathbf{x}_{cl}(s) + K^R(I-D)\mathbf{u}(s)\end{aligned} \tag{6.35}$$

By multiplying (6.35) by -K on the left and using (6.34) it follows that:

$$D\mathbf{u}(s) = -K\mathbf{x}(s) \tag{6.36}$$

This is the fundamental relationship for LQ state feedback control. (6.34) and (6.35) allow the state controller to be represented in block diagram form (Figure 6.2). The loop enclosed in dashed lines replaces the constant gain matrix of the nonsingular LQ controller. When D is nonsingular this loop can be eliminated, however it cannot be interpreted in isolation from the rest of the block diagram when D is singular.

When K does not have full row rank the descriptor forms (6.32) and (6.33) are replaced by the fractional descriptor forms:

$$\begin{aligned}\mathbf{x}(s) &= \left(I - P\Delta^{-1}K\right)P_d\,\mathbf{d}(s)\\ &= (sI-A)^{-1}\left[I - B(K-DF_1+F_1E_{F1})\left((sI-A)E_{F1}+B(K-DF_1+F_1E_{F1})\right)^{-1}\right]E\,\mathbf{d}(s)\\ &= E_{F1}\left((sI-A)E_{F1}+B(K-DF_1+F_1E_{F1})\right)^{-1}E\,\mathbf{d}(s)\end{aligned} \tag{6.37}$$

and

$$\mathbf{u}(s) = -(K-DF_1+F_1E_{F1})\left((sI-A)E_{F1}+B(K-DF_1+F_1E_{F1})\right)^{-1}E\,\mathbf{d}(s) \tag{6.38}$$

Defining:

$$\mathbf{x}_{cf}(s) = \left((sI-A)E_{F1}+B(K-DF_1+F_1E_{F1})\right)^{-1}E\,\mathbf{d}(s) \tag{6.39}$$

leads to:

$$\mathbf{u}(s) = -(K-DF_1+F_1E_{F1})\mathbf{x}_{cf}(s) \tag{6.40}$$

and

$$\begin{aligned}\mathbf{x}(s) &= E_{F1}\mathbf{x}_{cf}(s)\\ &= \mathbf{x}_{cf}(s) - (K-DF_1)^R(I-D)\mathbf{u}_f(s)\end{aligned} \tag{6.41}$$

where

$$\mathbf{u}_f(s) = -(K-DF_1)\mathbf{x}_{cf}(s) \tag{6.42}$$

Note that this definition allows the input $\mathbf{u}(s)$ to be written as:

$$\mathbf{u}(s) = \mathbf{u}_f(s) - F_1\mathbf{x}(s) \tag{6.43}$$

This fractional form can therefore be interpreted as having a preliminary state feedback F_1 as shown in the block diagram in Figure 6.3.

Premultiplying (6.41) by -K and noting from (6.21) that:

$$\begin{aligned} KE_{F1} &= (K-DF_1+DF_1)E_{F1} \\ &= (K-DF_1)\left[I-(K-DF_1)^R(I-D)(K-DF_1)\right] + DF_1E_{F1} \\ &= D((K-DF_1) + F_1E_{F1}) \end{aligned} \tag{6.44}$$

it follows that the relationship between $\mathbf{x}(s)$ and $\mathbf{u}(s)$ in (6.36) holds for the fractional form as well.

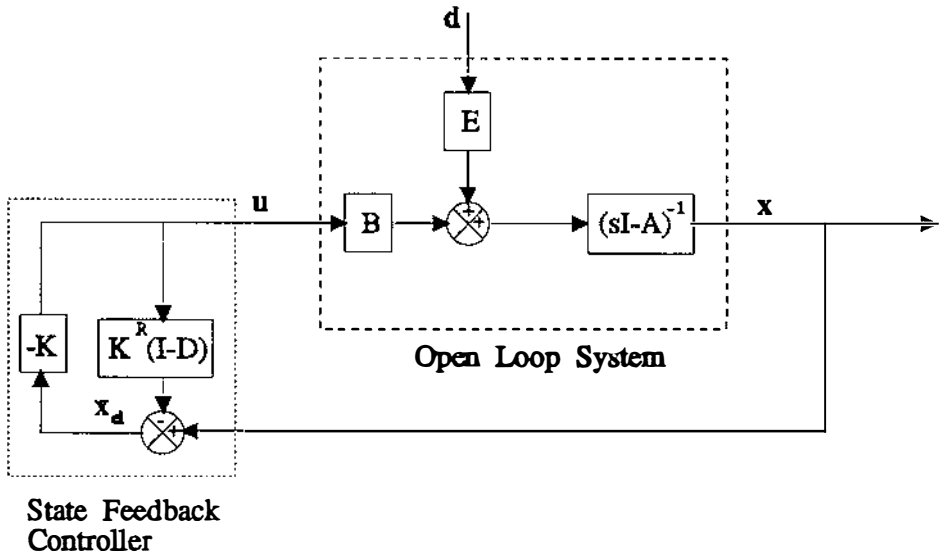

Figure 6.2
Structure of Singular State Feedback LQG Controllers

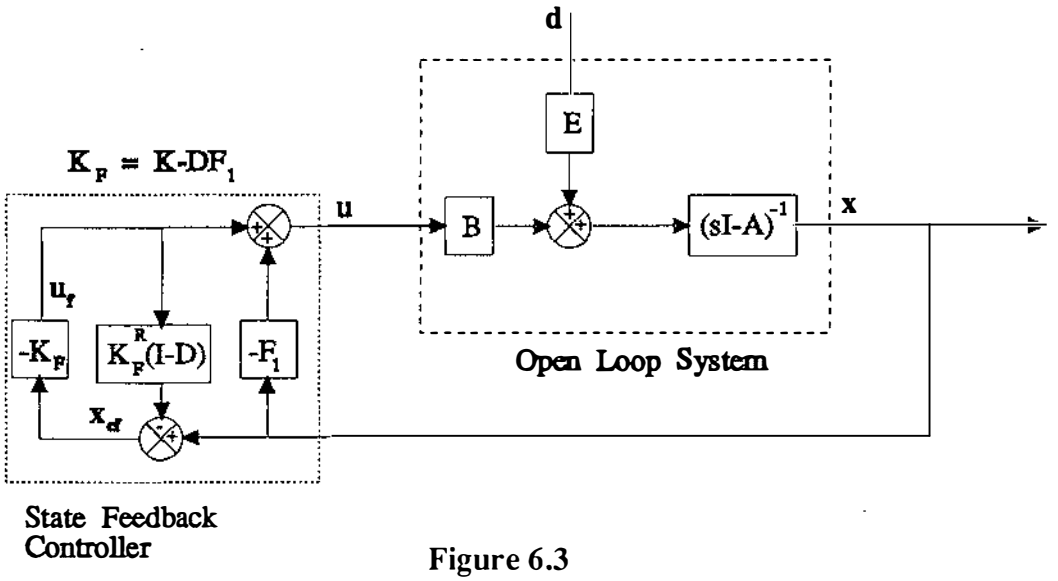

Figure 6.3
Structure of the Fractional Form
for Singular State Feedback LQG Controllers

This results of this section are summarised in the following theorem:

Theorem 6.4 Block Diagram Interpretation of Singular State Feedback

The LQ state feedback controller satisfies the relationship:

$$D\mathbf{u}(t) = -K\mathbf{x}(t) \tag{6.36}$$

The descriptor forms of the closed loop transfer functions have the block diagram representations shown in Figures 6.2 and 6.3. The fractional descriptor form can be interpreted as applying some preliminary feedback $F_1\mathbf{x}(t)$ as shown in Figure 6.3.

The descriptor forms used in this thesis were initially presented as a means to manipulate the form of the controller algebraically. However, the study of the LQ state feedback control problem has lead to a feedback (or block diagram) interpretation of the descriptor form. Specifically, the descriptor form in Lemma 4.17 is equivalent to applying a singular feedback $D\mathbf{u}(t) = -K\mathbf{x}(t)$ to the original system. This approach does not allow the explicit state feedback controller to be determined, but does permit the nature of the closed loop response to be determined. When the fractional descriptor forms of Lemma 4.18 are used, a similar analysis showed that this is equivalent to

applying some preliminary feedback $\mathbf{u}(t) = \mathbf{u}_f(t)+F_1\mathbf{x}(t)$; the relationship $D\mathbf{u}(t) = -K\mathbf{x}(t)$ is still satisfied for this descriptor form.

6.4.2. Descriptor Forms for Minimum Variance Estimators

The Wiener-Hopf solution to the minimum variance estimation problem was studied in Section 5.6. The descriptor approach used in this chapter can be applied to the estimator solution (5.101) without the requirement that the measurement noise intensity has full rank. From the estimator solution (5.101):

$$\begin{aligned}\hat{\mathbf{x}}(s) &= (sI-A)^{-1}L\Gamma^{-1}\mathbf{y}(s) + \left[(sI-A)^{-1}B-(sI-A)^{-1}L\Gamma^{-1}C(sI-A)^{-1}B\right] \\ &= \left[E_2(sI-A)+LC\right]^{-1}L\mathbf{y}(s) + \left[(sI-A)^{-1}B-\left[E_2(sI-A)+LC\right]^{-1}LC(sI-A)^{-1}B\right]\mathbf{u}(s) \\ &= \left[E_2(sI-A)+LC\right]^{-1}\left[L\mathbf{y}(s)+E_2B\mathbf{u}(s)\right] \end{aligned} \tag{6.45}$$

or in the time domain as:

$$E_2\dot{\hat{\mathbf{x}}}(t) = E_2\left[A\hat{\mathbf{x}}(t)+B\mathbf{u}(t)\right] + L(\mathbf{y}(t) - \hat{\mathbf{y}}(t)) \tag{6.46}$$

This descriptor form requires that L has full column rank. In the previous section a feedback interpretation of the descriptor forms was considered. A similar interpretation can be given to the descriptor forms used to describe the minimum variance estimator.

The minimum variance estimator (6.45) can be expanded as:

$$s\hat{\mathbf{x}}(s) = A\hat{\mathbf{x}}(s)+B\mathbf{u}(s) + L\left[(\mathbf{y}(s)-\hat{\mathbf{y}}(s)) + (I-V)L^{\perp}\mathbf{x}_L(s)\right] \tag{6.47}$$

where

$$\mathbf{x}_L(s) = (sI-A)\hat{\mathbf{x}}(s)-B\mathbf{u}(s) \tag{6.48}$$

This representation can be shown in block diagram form as in Figure 6.4. The portion of the block diagram in the dashed box satisfies the relationship:

$$E_2\mathbf{x}_L(s) = L(\mathbf{y}(s)-\hat{\mathbf{y}}(s)) \tag{6.49}$$

If E_2 is nonsingular:

$$\begin{aligned}\mathbf{x}_L(s) &= E_2^{-1}L(\mathbf{y}(s)-\hat{\mathbf{y}}(s)) \\ &= \left[I - L(I-V)L^{\perp}\right]^{-1}L(\mathbf{y}(s)-\hat{\mathbf{y}}(s))\end{aligned}$$

$$= LV^{-1}(\mathbf{y}(s)-\hat{\mathbf{y}}(s))$$
$$= K_f(\mathbf{y}(s)-\hat{\mathbf{y}}(s))$$

and so the internal loop can be interpreted as the Kalman filter gains. These results are dual to the results presented in Section 6.4.1 for the state feedback problem. They are not however quite as elegant as the state feedback case. If L does not have full column rank the fractional descriptor form of Lemma 4.18 may be used.

$$\hat{\mathbf{x}}(s) = \left[E_{F2}(sI-A)+(L_F+E_{F2}F_2)C\right]^{-1}\left[(L_F+E_{F2}F_2)\mathbf{y}(s)+E_{F2}B\mathbf{u}(s)\right] \tag{6.50}$$

where L_F and E_{F2} are defined in (6.22). This fractional form has the feedback interpretation shown in Figure 6.5 where the following relationships hold:

$$E_{F2}\mathbf{x}_L(s) = (L-F_2V+E_{F2}F_2)(\mathbf{y}(s)-\hat{\mathbf{y}}(s))$$
$$E_{F2}\mathbf{x}_{LF}(s) = (L-F_2V)(\mathbf{y}(s)-\hat{\mathbf{y}}(s))$$
$$\mathbf{x}_L(s) = \mathbf{x}_{LF}(s) + F_2(\mathbf{y}(s)-\hat{\mathbf{y}}(s)) \tag{6.51}$$

The results of this section are summarised in the following theorem:

Theorem 6.5 Descriptor Form for the Minimum Variance Estimator

If L has full column rank the minimum variance estimator is given by the descriptor form:

$$\hat{\mathbf{x}}(s) = \left[E_2(sI-A)+LC\right]^{-1}\left[L\mathbf{y}(s)+E_2B\mathbf{u}(s)\right] \tag{6.45}$$

otherwise it is given by the fractional descriptor form:

$$\hat{\mathbf{x}}(s) = \left[E_{F2}(sI-A)+(L_F+E_{F2}F_2)C\right]^{-1}\left[(L_F+E_{F2}F_2)\mathbf{y}(s)+E_{F2}B\mathbf{u}(s)\right] \tag{6.50}$$

where L_F and E_{F2} are defined in (6.22). The descriptor form can be represented in block diagram form as shown in Figure 6.4 or Figure 6.5 for the fractional descriptor form.

The descriptor form (6.45) has a similar structure to the full order estimator used for the nonsingular case in (6.10); this is the main difference between the descriptor approach and the traditional reduced-order observer approach of Bryson and Johansen (1965).

Although the descriptor form has n descriptor states, the singularity of E_2 can be used to reconcile the two approaches. Methods to produce the reduced-order form of the estimator (and of the LQG controller) will be presented in Section 6.5.

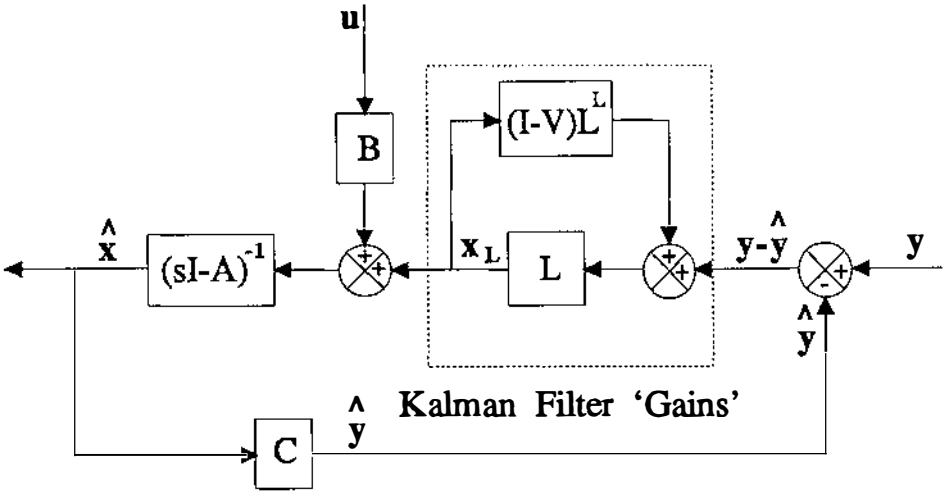

Figure 6.4
Structure of Minimum Variance Estimators

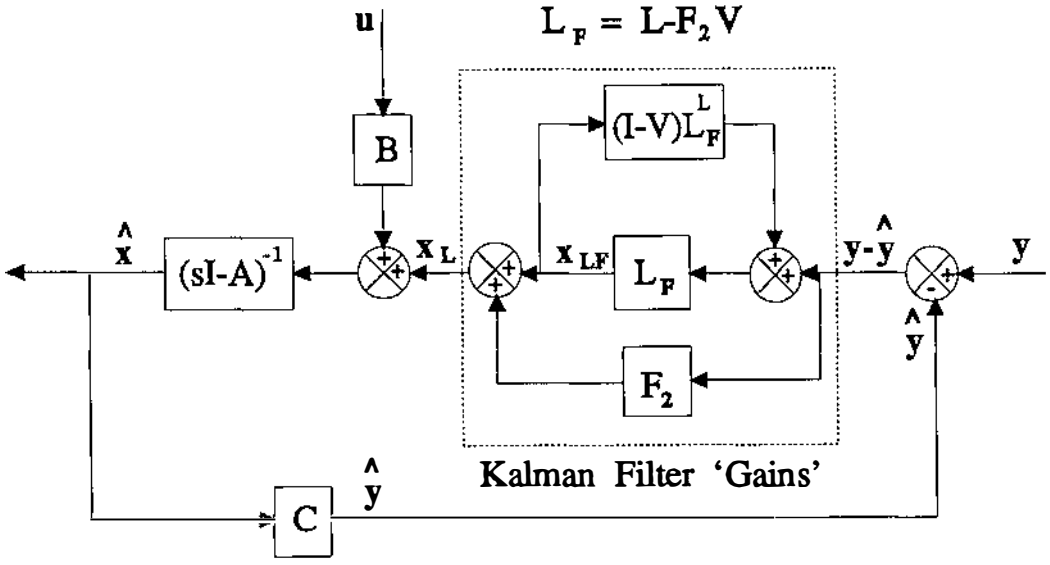

Figure 6.5
Structure of the Fractional Form for Minimum Variance Estimators

From (5.97), (5.100) and (6.45) the estimation error, $\mathbf{e}(s) = \mathbf{x}(s)-\hat{\mathbf{x}}(s)$ is described by:

$$\mathbf{e}(s) = \left[E_2(sI-A) + LC\right]^{-1}\left[E_2\mathbf{E}\mathbf{d}(s) - L\mathbf{n}(s)\right] \tag{6.52}$$

If there is no measurement noise the spectral factor $\Gamma=C(sI-A)^{-1}L$. Then from (5.101) the estimate of the outputs $\hat{\mathbf{y}}(t) = C\hat{\mathbf{x}}(t)$ is the same as the outputs from the system $\mathbf{y}(t)$. That is, the estimator does not filter the data. This is reasonable as the measurement signal is not corrupted by noise meaning that the states corresponding to $\mathbf{y}(t)$ do not need to be estimated as they are known perfectly. This does not however mean that the descriptor form (6.46) can be written as:

$$E_2\dot{\hat{\mathbf{x}}}(t) = E_2\left[A\hat{\mathbf{x}}(t)+B\mathbf{u}(t)\right] \tag{6.53}$$

as this descriptor form is not regular. The term LC is needed to make the descriptor form regular. It is possible to eliminate some of the descriptor states to form a reduced-order estimator. Methods for forming a reduced-order estimator from the descriptor form will be presented in Section 6.5.1.

6.4.3. The Separation Structure of LQG Controllers

The minimum variance estimator and LQ state control problems have been studied in the previous two sections. In this section the LQG controller (6.17) will be shown to be composed of an LQ state feedback controller and a minimum variance estimator for singular and nonsingular LQG problems.

Theorem 6.6 Separation Structure of LQG Controllers

The LQG controller given by (6.17) or (6.20) is composed of an LQ state feedback controller and a minimum variance estimator and can be represented in block diagram forms in Figures 6.6 and 6.7.

Proof

From the descriptor form of the controller (6.17) the control inputs $\mathbf{u}(s)$ are given by:

$$\mathbf{u}(s) = -K\mathbf{x}_c(s) \tag{6.54}$$

where controller states $\mathbf{x}_c(s)$ satisfy:

$$E_2\left[\mathbf{x}_c(s) + K^R(I\text{-}D)\mathbf{u}(s)\right]s = (E_2A\text{-}LC)\left[\mathbf{x}_c(s) + K^R(I\text{-}D)\mathbf{u}(s)\right] + E_2B\mathbf{u}(s) + L\mathbf{y}(s)$$

Rearranging this equation gives:

$$\begin{aligned}\mathbf{x}_c(s) &= \left[E_2(sI\text{-}A)+LC\right]^{-1}\left[E_2B\mathbf{u}(s)+L\mathbf{y}(s)\right] - K^R(I\text{-}D)\mathbf{u}(s)\\ &= \hat{\mathbf{x}}(s) - K^R(I\text{-}D)\mathbf{u}(s) \qquad (6.55)\end{aligned}$$

where

$$\hat{\mathbf{x}}(s) = \left[E_2(sI\text{-}A)+LC\right]^{-1}\left[E_2B\mathbf{u}(s)+L\mathbf{y}(s)\right] \qquad (6.56)$$

is the minimum variance estimator (6.45). Multiplying (6.55) on the left by -K gives:

$$D\mathbf{u}(s) = -K\hat{\mathbf{x}}(s) \qquad (6.57)$$

This is the same as the relationship for the LQ state feedback controller (6.36) with the estimate of the states being used instead of the actual states. The controller is therefore composed of the LQ state feedback controller which uses the estimates of the states provided by the minimum variance estimator. The structure of this controller is shown in block diagram form in Figure 6.6. The fractional form (6.20) can be decomposed similarly leading to the block diagram in Figure 6.7. ♦

An alternative understanding of the separation structure of the LQG controller can be obtained directly from the solution to the minimum variance estimator problem (5.101) and the LQ state feedback relationship (6.36) as follows:

Using the solution to minimum variance estimator (5.101), $-K\hat{\mathbf{x}}(s)$ is given by:

$$\begin{aligned}-K\hat{\mathbf{x}}(s) &= -K(sI\text{-}A)^{-1}L\Gamma^{-1}\mathbf{y}(s) + -K\left[(sI\text{-}A)^{-1}B\text{-}(sI\text{-}A)^{-1}L\Gamma^{-1}C(sI\text{-}A)^{-1}B\right]\mathbf{u}(s)\\ &= -M_{\oplus}\Gamma^{-1}\mathbf{y}(s) - \left[K(sI\text{-}A)^{-1}B\text{-}M_{\oplus}\Gamma^{-1}G\right]\mathbf{u}(s) \qquad (6.58)\end{aligned}$$

Now using the LQ state feedback relationship (6.36) with state estimates leads to

$$\begin{aligned}-M_{\oplus}\Gamma^{-1}\mathbf{y}(s) &= \left[D+K(sI\text{-}A)^{-1}B\text{-}M_{\oplus}\Gamma^{-1}G\right]\mathbf{u}(s)\\ &= \left[\Delta\text{-}M_{\oplus}\Gamma^{-1}G\right]\mathbf{u}(s) \qquad (6.59)\end{aligned}$$

or

$$\mathbf{u}(s) = -\left[\Delta\text{-}M_{\oplus}\Gamma^{-1}G\right]^{-1}M_{\oplus}\Gamma^{-1}\,\mathbf{y}(s)$$

provided $(\Delta - M_{\oplus}\Gamma^{-1}G)$ is invertible. This expression is the same as (6.4), one of the expressions for the LQG controller.

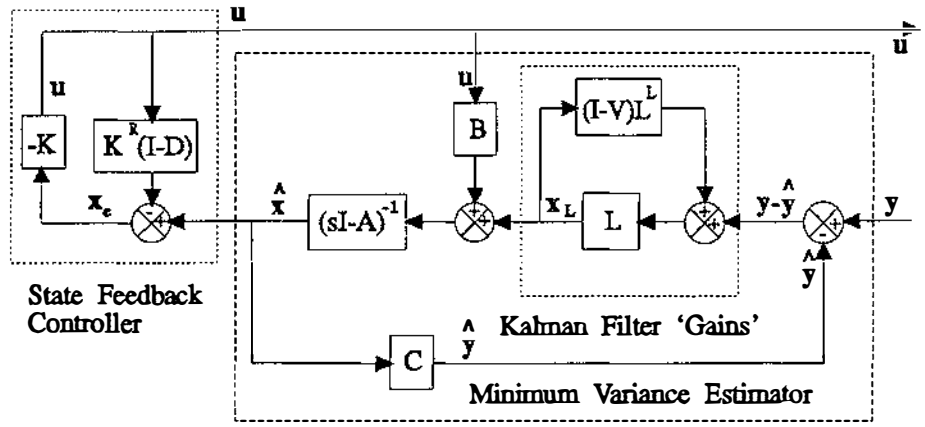

Figure 6.6
Structure of Singular LQG Controllers

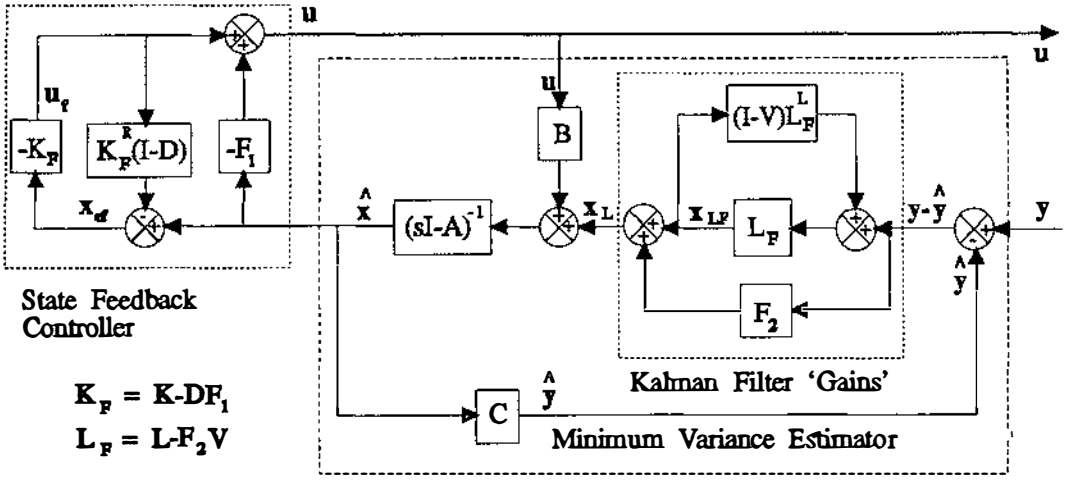

Figure 6.7
Structure of Fractional Form of Singular LQG Controllers

Traditional presentations of the separation theorems (Kwakernaak and Sivan 1972; Bryson and Ho 1975) show how the LQG problem can be decomposed into two problems: a minimum variance estimation problem and an LQ state feedback problem. This approach to the separation theorem is now extended to include singular LQG problems. The first step is to describe the closed loop system in terms of the states $[\mathbf{e}^T(t) \ , \ \hat{\mathbf{x}}^T(t)]^T$, or similar, to obtain the descriptor equivalent to (6.12). Next, it is necessary to establish the independence of $\mathbf{e}(t)$ and $\hat{\mathbf{x}}(t)$. For singular LQG problems it is necessary to ensure that the covariance between these two variables is finite. Finally, the performance index can be split in a similar way as in the nonsingular LQG problem.

Defining $\mathbf{e}(s) = \mathbf{x}(s) - \hat{\mathbf{x}}(s)$, it follows from (6.52) leads to:

$$\mathbf{e}(s) = \left(E_2(sI-A)+LC\right)^{-1}\left(E_2E\mathbf{d}(s)-L\mathbf{n}(s)\right) \tag{6.60}$$

From (6.54) and (6.55):

$$\begin{aligned} E_1\mathbf{x}_c(s) &= \hat{\mathbf{x}}(s) \\ &= \mathbf{x}(s) - \mathbf{e}(s) \\ &= (sI-A)^{-1}\left(B\mathbf{u}(s)+E\mathbf{d}(s)\right) - \mathbf{e}(s) \end{aligned} \tag{6.61}$$

from which:

$$\mathbf{x}_c(s) = \left((sI-A)E_1+BK\right)^{-1}\left(E\mathbf{d}(s) - (sI-A)\mathbf{e}(s)\right) \tag{6.62}$$

Combining (6.60) and (6.62) into one transfer function leads to:

$$\begin{bmatrix}\mathbf{e}(s)\\ \mathbf{x}_c(s)\end{bmatrix} = \left[\begin{bmatrix} E_2(sI-A)+LC & 0 \\ sI-A & (sI-A)E_1+BK\end{bmatrix}\right]^{-1}\begin{bmatrix} E_2E & -L \\ E & 0\end{bmatrix}\begin{bmatrix}\mathbf{d}(s)\\ \mathbf{n}(s)\end{bmatrix} \tag{6.63}$$

or in the time domain:

$$\begin{bmatrix} E_2 & 0 \\ I & E_1\end{bmatrix}\begin{bmatrix}\dot{\mathbf{e}}(t)\\ \dot{\mathbf{x}}_c(t)\end{bmatrix} = \begin{bmatrix} E_2A-LC & 0 \\ A & AE_1-BK\end{bmatrix}\begin{bmatrix}\mathbf{e}(t)\\ \mathbf{x}_c(t)\end{bmatrix} + \begin{bmatrix} E_2E & -L \\ E & 0\end{bmatrix}\begin{bmatrix}\mathbf{d}(t)\\ \mathbf{n}(t)\end{bmatrix} \tag{6.64}$$

This form is the descriptor equivalent of (6.12) from which it is immediate that the closed loop poles are given by:

$$\begin{aligned} &\det\left(\begin{bmatrix} E_2(sI-A)+LC & 0 \\ sI-A & (sI-A)E_1+BK\end{bmatrix}\right) \\ &\quad = \det((sI-A)E_1+BK)\det(E_2(sI-A)+LC) \end{aligned}$$

as shown in Theorem 6.3.

For nonsingular LQG control problems, the stochastic independence of the estimates $\hat{\mathbf{x}}(t)$ and the estimation error $\mathbf{e}(t)$ is a key property, that is, $\mathcal{E}[\hat{\mathbf{x}}(t)\mathbf{e}^T(t)]=0$. This is not true for singular LQG problems as the covariance $\mathcal{E}[\hat{\mathbf{x}}(t)\mathbf{e}^T(t)]$ is not necessarily finite. However, the covariance $\mathcal{E}[K\hat{\mathbf{x}}(t)\mathbf{e}^T(t)]$ is, as is shown in the following lemma:

Lemma 6.7 Independence of State Estimates and Estimation Error

$$\mathcal{E}[-K\hat{\mathbf{x}}(t)\mathbf{e}^T(t)] = \mathcal{E}[D\mathbf{u}(t)\mathbf{e}^T(t)] = 0$$

Proof

From (6.60) and (6.61):

$$\mathcal{E}[-K\hat{\mathbf{x}}(t)\mathbf{e}^T(t)] = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(-KE_1\left((sI-A)E_1+BK\right)^{-1}\left(-L\Gamma^{-1}\Gamma\Gamma^*(\Gamma^*)^{-1}L^T(-sI-A^T)^{-1}+LM_e^*\right)\right)ds \tag{6.65}$$

Now:

$$LM_e^* = LL^T(-sI-A^T)^{-1} + L\Gamma^{-1}C(sI-A)^{-1}Y \tag{6.66}$$

from the expansion of M_e in (5.99).

$$\mathcal{E}[-K\hat{\mathbf{x}}(t)\mathbf{e}^T(t)] = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(-KE_1\left((sI-A)E_1+BK\right)^{-1}L\Gamma^{-1}C(sI-A)^{-1}Y\right)ds \tag{6.67}$$

As $KE_1 = DK$ and using Lemma 4.16:

$$\begin{aligned}\mathcal{E}[-K\hat{\mathbf{x}}(t)\mathbf{e}^T(t)] &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(-D\Delta^{-1}K(sI-A)^{-1}L\Gamma^{-1}C(sI-A)^{-1}Y\right)ds \\ &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(-D\Delta^{-1}M_{\oplus}\Gamma^{-1}C(sI-A)^{-1}Y\right)ds \\ &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(-DTC(sI-A)^{-1}Y\right)ds\end{aligned} \tag{6.68}$$

The transfer function $TC(sI-A)^{-1}Y$ is stable as $TC(sI-A)^{-1}$ is stable (Theorem 5.13). As $D\Delta^{-1}$ is proper, and as $M_{\oplus}$ and $\Gamma^{-1}C(sI-A)^{-1}Y$ are strictly proper (from (5.99)), the transfer function $DTC(sI-A)^{-1}Y$ satisfies the conditions of Lemma 4.14 and therefore $\mathcal{E}[-K\hat{\mathbf{x}}(t)\mathbf{e}^T(t)]=0$. From (6.57) $D\mathbf{u}(t) = -K\hat{\mathbf{x}}(t)$ and so $\mathcal{E}[D\mathbf{u}(t)\mathbf{e}^T(t)]$ is also zero. ♦

This result is used to expand the performance index (3.1) in the following theorem:

Theorem 6.8 Decomposition of the Performance Index

The LQG controller, given by (6.17) or (6.20), allows the performance index to be expanded as:

$$\mathcal{E}\left[\begin{bmatrix}\mathbf{x}^T(t) , \mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix} Q & S^T \\ S & R\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{bmatrix}\right]$$
$$= \mathcal{E}\left[\begin{bmatrix}\hat{\mathbf{x}}^T(t) , \mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix} K^TK & K^TD \\ D^TK & D^TD\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}(t)\\ \mathbf{u}(t)\end{bmatrix}\right] + Tr\left\{ XE\Phi_d E^T\right\} + Tr\left\{K^TK\mathcal{E}\left[\mathbf{e}(t)\mathbf{e}^T(t)\right]\right\} \tag{6.69}$$

The first two terms in this expansion is an LQ state feedback control problem using the estimates of the states provided by the third term, a minimum variance estimation problem.

Proof

The performance index can be expanded as:

$$\mathcal{E}\left[\begin{bmatrix}\mathbf{x}^T(t) , \mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix} Q & S^T \\ S & R\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{bmatrix}\right]$$
$$= \mathcal{E}\left[\begin{bmatrix}\mathbf{x}^T(t) , \mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix} K^TK & K^TD \\ D^TK & D^TD\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{bmatrix}\right] + Tr\left\{ XE\Phi_d E^T\right\} \tag{6.70}$$

using the equivalent set of weights $\begin{bmatrix} K^TK & K^TD \\ D^TK & D^TD\end{bmatrix}$ and Lemma 5.16.

Substituting $\mathbf{x}(t) = \mathbf{e}(t) + \hat{\mathbf{x}}(t)$ into (6.70) and noting from Lemma 6.7 that:

$$\mathcal{E}\left[\mathbf{x}^T(t)K^TK\mathbf{x}(t)\right] = \mathcal{E}\left[(\mathbf{e}(t)+\hat{\mathbf{x}}(t))^TK^TK(\mathbf{e}(t)+\hat{\mathbf{x}}(t))\right]$$
$$= \mathcal{E}\left[\hat{\mathbf{x}}^T(t)K^TK\hat{\mathbf{x}}(t)\right] + Tr\left\{K^TK\mathcal{E}\left[\mathbf{e}(t)\mathbf{e}^T(t)\right]\right\} \tag{6.71}$$

and

$$\begin{aligned}\mathcal{E}\left[\mathbf{x}^T(t)K^TD\mathbf{u}(t)\right] &= \mathcal{E}\left[(\mathbf{e}(t)+\hat{\mathbf{x}}(t))^TK^TD\mathbf{u}(t)\right] \\ &= \mathcal{E}\left[\hat{\mathbf{x}}^T(t)K^TD\mathbf{u}(t)\right]\end{aligned} \tag{6.72}$$

gives:

$$\begin{aligned}&\mathcal{E}\left[\begin{bmatrix}\mathbf{x}^T(t) , \mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix}Q & S^T\\ S & R\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{bmatrix}\right] \\ &\quad = \mathcal{E}\left[\begin{bmatrix}\hat{\mathbf{x}}^T(t) , \mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix}K^TK & K^TD\\ D^TK & D^TD\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}(t)\\ \mathbf{u}(t)\end{bmatrix}\right] + Tr\left\{XE\Phi_dE^T\right\} + Tr\left\{K^TK\mathcal{E}\left[\mathbf{e}(t)\mathbf{e}^T(t)\right]\right\}\end{aligned} \tag{6.73}$$

The LQG problem has been decomposed into two parts: an LQ state feedback control problem; and an estimation problem. The estimation problem was shown to be independent of the control inputs in Section 5.6 and so can be optimised first to give the state estimates $\hat{\mathbf{x}}(t)$ which can be used in the LQ state feedback control problem:

$$\mathcal{E}\left[\begin{bmatrix}\hat{\mathbf{x}}^T(t) , \mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix}K^TK & K^TD\\ D^TK & D^TD\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}(t)\\ \mathbf{u}(t)\end{bmatrix}\right] \tag{6.74}$$

Finally, consider the disturbance input in the LQ state feedback control problem. From (6.62) and (6.56) it follows that $\hat{\mathbf{x}}(t)$ is described by the following differential equation:

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= A\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + E\mathbf{d}(t) - (\dot{\mathbf{e}}(t)-A\mathbf{e}(t)) \\ &= A\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + \eta(t)\end{aligned} \tag{6.75}$$

where

$$\eta(t) = E\mathbf{d}(t) - (\dot{\mathbf{e}}(t)-A\mathbf{e}(t)) \tag{6.76}$$

Using the expression for $\mathbf{e}(s)$ in (5.90) leads to the following form for $\eta(s)$:

$$\begin{aligned}\eta(s) &= E\mathbf{d}(s) - (sI-A)\left[(sI-A)^{-1}E\mathbf{d}(s)-(sI-A)^{-1}L\Gamma^{-1}\left[C(sI-A)^{-1}E\mathbf{d}(s)+\mathbf{n}(s)\right]\right] \\ &= L\Gamma^{-1}\left[C(sI-A)^{-1}E\mathbf{d}(s)+\mathbf{n}(s)\right]\end{aligned} \tag{6.77}$$

From Theorem 2.20 the autocovariance of $\eta(t)$ is therefore given by:

$$\mathcal{E}[\eta(t)\eta^T(\tau)] = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} L\Gamma^{-1}\left[C(sI-A)^{-1}\ \ I\right]\begin{bmatrix} E\Phi_d E^T & E\Phi_{dn} \\ \Phi_{dn}E^T & \Phi_{dn}\end{bmatrix}\left[C(sI-A)^{-1}\ \ I\right]^*(\Gamma^*)^{-1}L^T e^{s(t-\tau)}\,ds$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} L\Gamma^{-1}\Gamma\Gamma^*(\Gamma^*)^{-1}L^T e^{s(t-\tau)}\,ds$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} LL^T e^{s(t-\tau)}\,ds$$

$$= LL^T\delta(t-\tau) \qquad (6.78)$$

That is, $\eta(t)$ is a white noise process with intensity LL^T. The disturbance driving force $\eta(t)$ in (6.75) is therefore the same as for the nonsingular LQG problem in (6.15).

The LQ state feedback control problem (6.74) where state estimates $\hat{\mathbf{x}}(t)$ satisfy (6.75) therefore has the same solution as the state feedback problem with the true plant states $\mathbf{x}(t)$. As the weights of the state feedback are specified by K and D, the solution to the LMI (5.13) is X=0. ♦

As noted by Bryson and Ho (1975, 416), the decomposition of the performance index in Theorem 6.8 does not constitute a proof of the separation theorem as it assumes that the controller is composed of a state feedback controller and an estimator (not necessarily the optimal ones). Theorem 6.2 shows that the LQG controller, if it exists, has the required form.

For nonsingular LQG problems the separation theorem is used to calculate the controller. The state feedback controller and the estimator are calculated separately and are then combined into a single controller (6.6). For singular LQG problems the separation theorem shows that this approach could also be taken. However, if the weighting on the control inputs R is singular, it was shown in Section 6.4.1 that the problem of calculating the state feedback controller was difficult. The descriptor approach introduced in this section circumvents this problem by specifying the controller directly from the spectral factors. This form of the controller can be decomposed (Figures 6.6 and 6.7) to show the estimator and 'state feedback like' parts of the controller. The 'state feedback like' part of the controller merely satisfies the relationship $D\mathbf{u}(t) = -K\hat{\mathbf{x}}(t)$ as opposed to being a well-defined function. The method of constructing an LQ state feedback controller presented in Section 6.4.1 amounts to designing a 'perfect' estimator (that is, an estimator which estimates the states without error). This estimator is designed by specifying some measurements, $\mathbf{y}(t) = \chi\mathbf{x}(t)$, from which the states are estimated. In the design of an LQG controller from outputs, there is no need to design this perfect estimator as the states are estimated from the measurements $\mathbf{y}(t) = C\mathbf{x}(t) + \mathbf{n}(t)$. The advantage of using the descriptor forms presented in this chapter is that the LQG controller can be calculated directly from the

solutions to the LMI without the need to explicitly solve the LQ state feedback control problem.

The standard state-space approach outlined in Section 6.3 is normally used for nonsingular LQG problems. However, the descriptor approach introduced in this section could prove beneficial when the weighting on the controls (R) or the measurement noise intensities (Φ_n) are nearly singular as the inverses of R and Φ_n are not required in the descriptor forms. It is necessary to form a right inverse of K and a left inverse of L in the descriptor form. However, the fractional descriptor form (6.20) can be used to improve the condition of the matrices to be inverted.

6.5. ORDER OF THE CONTROLLER

In this section the response of the descriptor form of the controller in (6.17) is studied using the general descriptor theory presented in Section 2.2.4. Of particular interest is determining the order of the controller. First, the generalised order of the controller (Definition 2.11) is determined. Some general results on the number of finite modes are also derived. It will be of particular interest to determine whether the controller has no impulsive behaviour, that is, whether the controller is proper.

The order of the controller is taken to be the order of $(sE-A)^{-1}$ rather than the order of the transfer function $C(sE-A)^{-1}B$. This definition is adopted to avoid the difficulty of accounting for uncontrollable and unobservable modes. The subject of the presence of unobservable and uncontrollable modes is discussed in Section 6.5.3 when the minimality of the controller is discussed.

Only the descriptor form of the controller in (6.17) is considered in this section. All the results which are derived also hold for the fractional descriptor form (6.20) with a few modifications. However, the algebra needed to derive these results is more involved and so the details are omitted. First, some useful transformations on the descriptor states of (6.17) are presented.

Lemma 6.9 A Transformation of the Descriptor States

The matrix E_1 (6.18) is similar to the matrix $\begin{bmatrix} I_{n-r} & 0 \\ 0 & D \end{bmatrix}$ and the matrix E_2 (6.19) is similar to the matrix $\begin{bmatrix} I_{n-m} & 0 \\ 0 & V \end{bmatrix}$. The rank of the matrices E_1 and E_2 is $n-d_R$ and $n-d_{\Phi_n}$ respectively where d_R and d_{Φ_n} are the rank deficiency of R and Φ_n respectively.

Proof

The matrix E_1 can be transformed using the pair $\begin{bmatrix} M^T \\ K \end{bmatrix}$ and $[M\ K^R]$ where $KM=0$ $M^TM=I$ and $M^TK^R=0$. (Note the Moore-Penrose Pseudo Inverse, $K^R = K^T(KK^T)^{-1}$, is one possible choice.) That is, $\begin{bmatrix} M^T \\ K \end{bmatrix}$ is the inverse of $[M\ K^R]$. This transformation leads to:

$$\begin{aligned}\begin{bmatrix} M^T \\ K \end{bmatrix} E_1 [M\ K^R] &= \begin{bmatrix} M^T \\ K \end{bmatrix} \left(I - K^R(I-D)K \right) [M\ K^R] \\ &= I - \begin{bmatrix} 0 & 0 \\ 0 & I-D \end{bmatrix} \\ &= \begin{bmatrix} I_{n-r} & 0 \\ 0 & D \end{bmatrix} \end{aligned} \tag{6.79}$$

Similarly, E_2 can be transformed using the pair $\begin{bmatrix} N \\ L^L \end{bmatrix}$ and $[N^T\ L]$ where $NL = 0$, $NN^T=I$ and $L^LN^T=0$. That is, $\begin{bmatrix} N \\ L^L \end{bmatrix}$ is the inverse of $[N^T\ L]$. This transformation leads to:

$$\begin{aligned}\begin{bmatrix} N \\ L^L \end{bmatrix} E_2 [N^T\ L] &= \begin{bmatrix} N \\ L^L \end{bmatrix} \left(I - L(I-V)L^L \right) [N^T\ L] \\ &= I - \begin{bmatrix} 0 & 0 \\ 0 & I-V \end{bmatrix} \\ &= \begin{bmatrix} I_{n-m} & 0 \\ 0 & V \end{bmatrix} \end{aligned} \tag{6.80}$$

One immediate consequence of this lemma is that E_1 is nonsingular if and only if $R=D^TD$ is nonsingular. Similarly, E_2 is nonsingular if and only if $\Phi_n=VV^T$ is nonsingular. ♦

These transformations of E_1 and E_2 can be used to determine the generalised order of the LQG controller.

Lemma 6.10 Generalised Order of the Controller

The generalised order (Definition 2.11) of the LQG controller (6.17) is $\text{rank}(E_2E_1)$ and furthermore:

$$\text{rank}(E_2E_1) \le \min(n-d_R\ ,\ n-d_{\Phi_n}) \tag{6.81}$$

Proof

From Lemma 6.9 the rank of E_1 is n-d_R and the rank of E_2 is n-d_{Φ_n}. The generalised order of the controller is given by rank(E_2E_1). As the maximum rank of a product of matrices is the minimum rank of the two matrices (Barnett 1991), (6.81) follows. ♦

As the general formula for the singular LQG controller derived in Theorem 6.2 is a descriptor form it is possible that the controller contains an impulsive response (or derivative action). The controller will be strictly proper if and only if E_2E_1 is nonsingular. This will be true if and only if both E_2 and E_1 are nonsingular, that is, the LQG problem is nonsingular . If either E_1 or E_2 are singular, the controller will be proper if the number of finite modes of the descriptor form (6.17) is the same as the generalised order of the descriptor form (6.17) (Definition 2.11). In the next section some simple conditions are derived to determine whether the controller is proper.

6.5.1. Proper LQG Controllers

LQG problems for which the controller is proper, but not necessarily strictly proper, are studied in this section. First, some general conditions are derived to determine whether the LQG controller is proper are derived. When the controller is proper the descriptor representations can be reduced to state-space representations. A general method for this reduction is presented as well as two special cases: perfect measurements (R full rank, Φ_n=0), and minimum variance (R=0, Φ_n full rank). The reason for the emphasis on proper controllers is that in practice derivative control is not desirable due to the sensitivity of the controller to high frequency signals.

The following theorem gives some simple conditions for determining if a singular LQG controller is proper.

Theorem 6.11 Proper Controllers

(i) If rank(E_2E_1) = n-d_R then the controller (6.17) is proper if and only if:

$$\text{rank}\left(\begin{bmatrix} D & KBN_D \end{bmatrix}\right) = r \text{ and } E_2 \text{ is nonsingular} \qquad (6.82)$$

where $DN_D = 0$.

(ii) If rank(E_2E_1) = n-d_{Φ_n} then the controller (6.17) is proper if and only if:

$$\text{rank}\left(\begin{bmatrix} V \\ N_V CL \end{bmatrix}\right) = m \text{ and } E_1 \text{ is nonsingular} \tag{6.83}$$

where $N_V V=0$.

(iii) If $\text{rank}(E_2E_1) < \min(n-d_R\ ,\ n-d_{\Phi_n}) = n-d_{\Phi_n}-d_1 = n-d_R-d_2$ then the controller is proper only if:

$$\text{rank}\left(\begin{bmatrix} V \\ N_V CL \end{bmatrix}\right) \geq m-d_1 \tag{6.84}$$

and

$$\text{rank}\left(\begin{bmatrix} D & KBN_D \end{bmatrix}\right) \geq r-d_2 \tag{6.85}$$

Proof

This theorem follows from the application of Lemma 2.16 to the descriptor form of the controller. The following result about null spaces from Cullen (1972, Theorem 2.5) is used:

$$Ker\, E_1 \subseteq Ker\, E_2E_1 \quad \text{with equality if } E_2 \text{ is nonsingular.} \tag{6.86}$$

The three cases above are related to the bound on the rank of E_2E_1 in (6.81). The first case (6.82) deals with the possibility that $\text{rank}(E_2E_1)= n-d_R$. In this case the null space of E_2E_1 is given by:

$$E_2E_1N_E = E_2(I-K^R(I-D)K)N_E$$

where $N_E = Ker\, E_1$. A simple calculation reveals that:

$$N_E = K^RN_D$$

where $N_D = Ker\, D$. Applying Condition (ii) of Lemma 2.16 leads to the condition that the controller is proper if and only if:

$$\text{rank}\left(\begin{bmatrix} E_2E_1 & E_2BN_D \end{bmatrix}\right) = \text{rank}\left(E_2\begin{bmatrix} E_1 & BN_D \end{bmatrix}\right) = n \tag{6.87}$$

As E_2 is nonsingular (6.87) is satisfied if $\begin{bmatrix} E_1 & BN_D \end{bmatrix}$ has rank n. Applying the transformation of descriptor states used in (6.79) leads to:

$$\left[E_1\ BN_D\right] \sim \begin{bmatrix} I & 0 & MBN_D \\ 0 & D & KBN_D \end{bmatrix}$$

which will be full rank only if $\text{rank}\left(\left[D\ KBN_D\right]\right) = r$.

The second case where $\text{rank}(E_2E_1) = n\text{-}d_{\Phi_n}$ is similar. Noting that:

$$N_V L^L E_2 = 0$$

where $N_V V=0$. From Lemma 2.16, Condition (iii), the controller will be proper if and only if:

$$\text{rank}\left(\begin{bmatrix} E_2E_1 \\ N_VCE_1 \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} E_2 \\ N_VC \end{bmatrix}E_1\right) = n \tag{6.88}$$

As E_1 is nonsingular (6.88) is satisfied if $\begin{bmatrix} E_2 \\ N_VC \end{bmatrix}$ has rank n. As:

$$\begin{bmatrix} E_2 \\ N_VC \end{bmatrix} \sim \begin{bmatrix} I & 0 \\ 0 & V \\ N_VCN & N_VCL \end{bmatrix}$$

using the transformation in (6.80), the matrix will have rank n if $\begin{bmatrix} V \\ N_VCL \end{bmatrix}$ has rank m.

The third case deals with the case when the product of E_2 and E_1 is rank deficient. In this case the null space of E_1 only forms part of the null space of E_2E_1 and hence the inequality in (6.85) arises. It should be noted that while the conditions (6.84) and (6.85) are necessary for the controller to be proper, they are not sufficient (Example 6.1 is such an LQG problem). In this case the null space of E_2E_1 could be formed numerically using a computer package such as MATLAB and used directly to determine the null spaces to perform the test for properness in Lemma 2.16. ♦

The above theorem gives some simple conditions which guarantee that the controller is proper. If E_1 and/or E_2 are singular a reduced-order, state-space description can be derived using a transformation on the descriptor states. This reduction process, however may not be numerically stable (Bunse-Gerstner *et al.* (1992)). To form a reduced-order state-space description find a nonsingular transformation U,W such that:

$$W(E_2E_1)U = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \tag{6.89}$$

and

$$W(E_2AE_1\text{-}E_2BK\text{-}LCE_1)U = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{6.90}$$

$$KU = [K_1 \;\; K_2] \tag{6.91}$$

$$WL = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \tag{6.92}$$

If the conditions of Theorem 6.11 are satisfied then A_{22} will be nonsingular. In this case, the controller has the reduced-order form:

$$H(s) = K_r(sI\text{-}A_r)^{-1}L_r + D_r \tag{6.93}$$

where

$$A_r = \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)$$
$$K_r = \left(K_1 - K_2A_{22}^{-1}A_{21}\right)$$
$$L_r = \left(L_1 - A_{12}A_{22}^{-1}L_2\right)$$
$$D_r = -K_2A_{22}^{-1}L_2$$

Next, two special cases are considered for which there exist explicit, reduced-order state-space forms.

1 A State-Space Form for Perfect Measurement Controllers

The first case is perfect measurement control where the weighting on the control inputs is full rank and the measurements are made without any noise (that is, $\Phi_n = 0$). Using (6.26) the descriptor form of the controller (6.17) is given by:

$$H(s) = K_c\left(E_2(sI\text{-}A+BK_c)+LC\right)^{-1}L$$

Using the transformation in (6.80), the differential form of the controller becomes:

$$\begin{bmatrix} I_{n-m} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_{r1}(t) \\ \dot{x}_{r2}(t) \end{bmatrix} = \begin{bmatrix} N(A\text{-}BK_c)N^T & N(A\text{-}BK_c)L \\ -CN^T & -CL \end{bmatrix}\begin{bmatrix} x_{r1}(t) \\ x_{r2}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix}y(t)$$
$$u(t) = -K_c\begin{bmatrix} N^T & L \end{bmatrix}\begin{bmatrix} x_{r1}(t) \\ x_{r2}(t) \end{bmatrix} \tag{6.94}$$

The states for $\mathbf{x}_{r2}$ form the algebraic constraint: $CL\mathbf{x}_{r2}(t) = -CN^T\mathbf{x}_{r1}(t) + \mathbf{y}(t)$. Noting that V=0 for the perfect measurement case, it follows that N_v=I, and therefore from Theorem 6.11, (6.83), the controller is proper if and only if CL is full rank. Subject to

this condition, the states $\mathbf{x}_{r2}(t)$ can therefore be eliminated to give the following reduced-order state-space representation of the controller:

$$H(s) = K_c\left[(I-L(CL)^{-1}C)N^T\left[sI-N(A-BK_c)(I-L(CL)^{-1}C)N^T\right]^{-1}N(A-BK_c) + I\right]L(CL)^{-1} \tag{6.95}$$

The requirement that CL is full rank is used explicitly in (6.95).

Since K_c , the state feedback gain, appears as a factor to the left of the LQG controller in (6.95), it follows that the other terms act as an observer. It should be noted that the observer in (6.95) only has n-m states and hence it is a reduced-order observer. Writing the controller in differential form leads to:

$$\dot{\mathbf{x}}_{r1}(t) = NA\hat{\mathbf{x}}(t) + NB\mathbf{u}(t) \tag{6.96a}$$

$$\hat{\mathbf{x}}(t) = (I-L(CL)^{-1}C)N^T\mathbf{x}_{r1}(t) + L(CL)^{-1}\mathbf{y}(t) \tag{6.96b}$$

$$\mathbf{u}(t) = -K_c\hat{\mathbf{x}}(t) \tag{6.96c}$$

where (6.96a) and (6.96b) describe the observer and (6.96c) describes the state feedback controller.

2 A State-Space Form for Minimum Variance Controllers

The second case is the minimum variance controller with noisy measurements. That is, there is no weighting on the control inputs, and the noise intensity is full rank. This case is dual to the perfect measurement case in the sense that these results may be obtained by transposing (6.95). Using (6.27) the descriptor form of the controller is given by:

$$H(s) = K\left[(sI-A+K_fC)E_1+BK\right]^{-1}K_f$$

Using the transformation in (6.79), the differential form of the controller becomes:

$$\begin{bmatrix} I_{n-r} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\mathbf{x}}_{c1}(t) \\ \dot{\mathbf{x}}_{c2}(t) \end{bmatrix} = \begin{bmatrix} M^T(A-K_fC)M & -M^TB \\ K(A-K_fC)M & -KB \end{bmatrix}\begin{bmatrix} \mathbf{x}_{c1}(t) \\ \mathbf{x}_{c2}(t) \end{bmatrix} + \begin{bmatrix} M^T \\ K \end{bmatrix}K_f\mathbf{y}(t)$$

$$\mathbf{u}(t) = -\begin{bmatrix} 0 , I \end{bmatrix}\begin{bmatrix} \mathbf{x}_{c1}(t) \\ \mathbf{x}_{c2}(t) \end{bmatrix}$$

$$= -\mathbf{x}_{c2}(t) \tag{6.97}$$

The states for $\mathbf{x}_{c2}(t)$ form the algebraic constraint: $KB\mathbf{x}_{c2}(t) = K(A-K_fC)\mathbf{x}_{c1}(t) + KK_f\mathbf{y}(t)$. Noting that D=0 for the minimum variance case, it follows that N_D=I, and therefore

from Theorem 6.11, (6.82), the controller is proper if and only if KB is full rank. Subject to this condition, the states $\mathbf{x}_{c2}(t)$ can therefore be eliminated to give the following reduced-order state-space representation of the controller:

$$H(s) = (KB)^{-1}K\left[(A-K_fC)M\left[sI-M^T(I-B(KB)^{-1}K)(A-K_fC)M\right]^{-1}M^T(I-B(KB)^{-1}K) + I\right]K_f \tag{6.98}$$

This transfer function has a dual structure to the proper, perfect measurement controller in (6.95), although it is not as simple to separate the estimator and the state controller components of the controller as it was for the perfect measurement case. From (6.61) the state estimates are given by:

$$\begin{aligned}\hat{\mathbf{x}}(t) &= E_1\mathbf{x}_c(t)\\ &= \begin{bmatrix} M , 0 \end{bmatrix}\begin{bmatrix}\mathbf{x}_{c1}(t)\\ \mathbf{x}_{c2}(t)\end{bmatrix}\\ &= M\mathbf{x}_{c1}(t)\end{aligned} \tag{6.99}$$

Using this relationship between $\hat{\mathbf{x}}(t)$ and $\mathbf{x}_{c1}(t)$ leads to the following description of the controller:

$$\dot{\mathbf{x}}_{c1}(t) = M^T(A-K_fC)M\mathbf{x}_{c1}(t) + M^TB\mathbf{u}(t) + M^TK_f\mathbf{y}(t) \tag{6.100a}$$

$$= M^T\left(A\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + K_f(\mathbf{y}(t)-\hat{\mathbf{y}}(t))\right) \tag{6.100b}$$

$$= M^T\dot{\hat{\mathbf{x}}}(t)$$

where the control inputs are given by:

$$\mathbf{u}(t) = -(KB)^{-1}K\left((A-K_fC)M\mathbf{x}_{c1}(t) + K_f\mathbf{y}(t)\right) \tag{6.101a}$$

$$= -(KB)^{-1}K\left((A-K_fC)\hat{\mathbf{x}}(t) + K_f\mathbf{y}(t)\right) \tag{6.101b}$$

There are two ways in which this controller could be implemented. First, a full order estimator could be built and the 'state' controller (6.101b) could be implemented using the estimates $\hat{\mathbf{x}}(t)$. The term 'state' controller is used in quotes as the controller has a direct feedthrough term from the output $\mathbf{y}(t)$ as well as the state estimates $\hat{\mathbf{x}}(t)$. However, the fact that minimum variance control is being used means that it is not necessary to implement a full order observer. The reduced-order 'estimator', (6.100a), which has n-r states, could be implemented in conjunction with the state controller (6.101a).

6.5.2. The Order of the Controller and Multivariable Zeros

In the previous section descriptor system theory was used to determine the order of the controller. When either E_1 or E_2 is nonsingular, the order of the controller can also be determined by the number of finite zeros of certain transfer functions.

The case when R is full rank, and hence E_1 is nonsingular, is considered first. From Lemma 6.10, the generalised order of the controller is $f = \text{rank}(E_2) = n\text{-}d_{\Phi_n}$ where d_{Φ_n} is the rank deficiency of Φ_n. The number of finite frequencies of the controller is given by the degree of $\det (I\text{-}L(I\text{-}V)L^{L})(sI\text{-}A+BK_c)+LC)$. From Lemma 4.16:

$$\det ((I\text{-}L(I\text{-}V)L^{L})(sI\text{-}A+BK_c)+LC) = \det\left(\begin{bmatrix} sI\text{-}A+BK_c & -L \\ C & V \end{bmatrix}\right) \tag{6.102}$$

The right hand side of (6.102) is satisfied by the zeros of the transfer function $\xi(s) = C(sI\text{-}A+BK_c)^{-1}L+V$. Therefore the number of finite frequencies of (6.17) is the same as the number of finite zeros of $\xi(s) = C(sI\text{-}A+BK_c)^{-1}L+V$.

For the perfect measurement case Lemma 2.5 can be used to obtain a lower bound on the order of the controller. Applying this result to the LQG controller reveals that the controller contains at least d impulsive modes, where d is the rank deficiency of CL.

This bound on the order of the controller can also be determined from the spectral factor $\Gamma = C(sI\text{-}A)^{-1}L$ where:

$$\Gamma\Gamma^{*} = G_d(s)\Phi_d G_d^{*} \tag{6.103}$$

From Lemma 5.10, Γ has the same number of infinite zeros as $G_d(s)\phi_d$. These results are summarised in the following theorem:

Theorem 6.12 Order of Controller for Singular Measurement Noise Intensities

The order of the LQG controller when the weighting on the control inputs is full rank is determined by the number of finite zeros of the transfer function:

$$\xi(s) = C(sI\text{-}A+BK_c)^{-1}L+V \tag{6.104}$$

The controller will be proper if $\xi(s)$ has $n\text{-}d_{\Phi_n}$ finite zeros. If the measurements are 'perfect' ($\Phi_n=0$), the controller has at least d impulsive modes where d is the rank deficiency of CL. The controller is proper if CL is full rank. This will be true if $G_d(s)\phi_d$ has m infinite zeros; when $G_d(s)\phi_d$ is a square transfer function, this condition is satisfied when $\text{rank}(CE\phi_d) = m$.

The result about the number of zeros of $G_d\phi_d$ for a square system was also derived by Halevi and Palmor (1986). The results given here are however more general in that firstly, non-square transfer functions, $G_d\phi_d$ are considered and secondly, cases in which the controller has derivative action are considered.

If the spectral density of the measurement noise Φ_n is full rank, the following dual result holds:

Theorem 6.13 Order of Controller for Singular Control Weightings

The order of the LQG controller when the spectral density of the measurement noise is full rank is determined by the number of finite zeros of the transfer function:

$$\zeta(s) = K(sI-A+K_fC)^{-1}B+D \tag{6.105}$$

The controller will be proper if $\zeta(s)$ has $n\text{-}d_R$ zeros. If there is no weighting on the control inputs, the controller has at least d impulsive modes where d is the rank deficiency of KB. The controller is proper if KB is full rank. This will be true if $Q^{1/2}P(s)$ has r infinite zeros; when $Q^{1/2}P(s)$ is a square transfer function, this condition is met when $\text{rank}(Q^{1/2}B) = r$.

6.5.3. Minimality of the Controller

One of the main purposes in developing the descriptor forms of LQG controllers was to produce minimal expressions for the controller. The extent to which this aim has been achieved is discussed in this section.

A descriptor system is said to be minimal if the system is completely controllable and observable at finite and infinite modes and the descriptor form $(sE-A)^{-1}$ does not contain nondynamic modes (Definition 2.15). Since the state-space form of the nonsingular LQG controller (6.6) may have uncontrollable and/or unobservable modes the presence

of finite uncontrollable or unobservable modes in the controller should not therefore be of too much concern.

Generally the descriptor form will contain nondynamic modes unless A_{22}=0 in (6.90). The elimination of these modes when A_{22} is nonsingular was dealt with in Section 6.5.1. If A_{22} is singular, but not zero, then the nondynamic modes can be reduced by transforming the states using the method of Grimm (1988).

If E_1 or E_2 is singular it is possible that the descriptor forms of the controller contain unobservable or uncontrollable modes at infinity. From Definition 2.14, the descriptor form will be controllable at infinity if and only if:

$$\begin{aligned}
&\operatorname{rank}\left(\begin{bmatrix} E_2E_1 & L \end{bmatrix}\right) \\
&\quad = \operatorname{rank}\left(\begin{bmatrix} NE_1 & 0 \\ VL^LE_1 & I_m \end{bmatrix}\right) \\
&\quad = \operatorname{rank}\left(\begin{bmatrix} N[M\ K^RD] & 0 \\ VL^L[M\ K^RD] & I_m \end{bmatrix}\right) = n
\end{aligned} \tag{6.106}$$

which will have full row rank only if NE_1 or $N[M\ K^RD]$ has full row rank. The strong equivalence relationship $[M\ K^R]$ followed by $\begin{bmatrix} N \\ L^L \end{bmatrix}$ as defined in Lemma 6.9 was used in (6.106). The observability of the descriptor form at infinity can be determined similarly.

$$\begin{aligned}
&\operatorname{rank}\left(\begin{bmatrix} K \\ E_2E_1 \end{bmatrix}\right) \\
&\quad = \operatorname{rank}\left(\begin{bmatrix} 0 & I_r \\ E_2M & E_2K^RD \end{bmatrix}\right) \\
&\quad = \operatorname{rank}\left(\begin{bmatrix} 0 & I_r \\ \begin{bmatrix} N \\ VL^L \end{bmatrix}M & \begin{bmatrix} N \\ VL^L \end{bmatrix}K^RD \end{bmatrix}\right) = n
\end{aligned} \tag{6.107}$$

which will have full column rank only if E_2M or $\begin{bmatrix} N \\ VL^L \end{bmatrix}M$ has full column rank. The following result is useful in determining the rank of NM:

Lemma 6.14 Kouvaritakis and Shaked (1976, Lemma 4.1)

The rank deficiency of NM equals d, the rank deficiency of KL where KM=0 and NL=0.

Kouvaritakis and Shaked (1976) only derived this result for the square case (r=m) although the same argument can be used for the non-square case. This result can be used to show that NM will have full row rank (n-m) if rank(KL)=r and NM will have full column rank (n-r) if rank(KL)=m. The results for determining the controllability and observability of the descriptor form at infinity are summarised in the following lemma:

Lemma 6.15 Minimality of Controller

The descriptor form of the LQG controller (6.17) is controllable at infinity if and only if $N[M\ K^R D]$ has full row rank. It is observable at infinity if and only if $\begin{bmatrix} N \\ VL^L \end{bmatrix} M$ has full column rank. The following conditions are useful tests:

(i) If rank(KL)=r then the LQG controller is controllable at infinity.

(ii) If rank(KL)=m then the LQG controller is observable at infinity.

If R = 0 then Condition (i) is necessary. If Φ_n=0 then Condition (ii) is necessary.

One of the problems with the above analysis is that controllability at infinity does not distinguish between impulsive and nondynamic modes. It is possible to eliminate these modes using the algorithm of Grimm (1988). It should however be noted that this algorithm is not numerically stable. A MATLAB routine by Miminis (1992) for reducing finite and infinite uncontrollable modes is available.

In this section the minimality of the descriptor form of the LQG controller has been studied. It has been shown that this form does not necessarily satisfy any of the criteria for minimality of a descriptor form, that is, that it should be controllable and observable at finite and infinite modes, and it should have no nondynamic modes, and hence may be nonminimal. However, in forming the descriptor form in Theorem 6.2 the common factor $(sI-A)^{-1}$, which lead to the criticism of Grimble and Johnson (1986), was cancelled. The descriptor form of the controller has the advantage that it is a general form, valid for all LQG problems. It also has the structural advantage of allowing singular and nonsingular LQG controllers to be understood in a unified manner as was shown in Section 6.4.

6.6. SOME DESIGN ISSUES FOR SINGULAR LQG CONTROLLERS

The descriptor forms of the LQG controller (6.17) and (6.20) have been used to study the structure and order of the controller. There are also some issues that need to be considered when designing singular LQG controllers. Two design issues are considered in this section: the admissibility of the closed loop system (Halevi and Palmor 1988), and Loop Transfer Recovery techniques (Doyle and Stein 1979).

6.6.1. Admissibility of LQG Controllers

In practice, there is a reluctance on the part of a control engineer to use derivative action, due to sensitivity to high frequency signals. However, the analysis in Section 6.5 shows that the controller solution to singular LQG problems can involve derivative action. In some cases, the controller will be inadmissible (Halevi and Palmor 1988) in that the sensitivity function $(I+HG)^{-1}$ is improper. In this case the closed loop system will possess a zero gain margin. Such controllers cannot be implemented as an arbitrarily small change in the model can result in instability.

There are several procedures that can be adopted if the LQG controller is inadmissible. The first procedure is to design low pass filters before and after the controller (Soroka and Shaked (1988a). These filters make the return ratio H(s)G(s) strictly proper.

$$P_1(s)H(s)P_2(s)G(s)$$

where $P_1(s)$ and $P_2(s)$ are the low pass filters resulting in a non-zero gain margin. The design of the low pass filters is outside the scope of LQG theory and is not considered here.

Halevi and Palmor (1988) suggest that the weights on the control inputs be extended to include weights on the derivatives of the control inputs. This extension constrains the impulsive response of the control inputs. The main disadvantage of this extension is that the weighting on the derivatives of the control inputs cannot be handled within the state-space approach presented in this thesis. Halevi and Palmor (1988) only considered the perfect measurement LQG problem. If the weighting on the control input is singular the number of derivatives of $\mathbf{u}(t)$ that need to be weighted may need to be increased above the bound in Halevi and Palmor (1988, Theorem 6.4).

A third procedure for modifying the LQG problem to ensure that the control system is admissible is to modify the weightings and the disturbance model. One obvious way to do this is to add terms weighting the control inputs and fictitious measurement noise,

turning a singular LQG problem into a nonsingular one and hence the closed loop system will be admissible. Another approach is to modify the weightings on the states Q and the disturbance model $E\Phi_d E^T$ to change the LQG problem into an admissible one. This technique generally works best if either R or Φ_n is nonsingular as the controller is generally improper if both are singular (Theorem 6.11). If R is nonsingular, the disturbance input can be modified to:

$$E_{new}E_{new}^T = E\Phi_d E^T + \varepsilon^2 E_{mod}E_{mod}^T \tag{6.108}$$

such that the modified spectral factor $\Gamma_{new} = C(sI-A)^{-1}L_{new}+V_{new}$ leads to the rank condition (6.83) being satisfied. In this case the LQG controller will be proper (Theorem 6.11) and therefore admissible. The design parameter ε allows a trade-off between disturbance rejection and admissibility to be made. It should be noted that while a small ε might make the gain margin non-zero, it still might be unacceptably low. The choice of E_{mod} is also important in determining the gain margin of the system. If $G(s)=C(sI-A)^{-1}B$ is minimum phase and has enough finite zeros (n-m), then the loop transfer recovery method of Doyle and Stein (1979) could be used with $E_{mod} = B$ to improve the gain margin. This method is discussed in more detail in the next section.

The use of a proper LQG controller guarantees that the gain margin will be greater than zero. It should be noted that the requirement that the controller is proper is a sufficient, but not a necessary condition for the controller to be admissible. It was noted in the discussion of Theorem 6.11 that the LQG controller is not likely to be proper if R and Φ_n are both singular. Unless there are good physical reasons for a singular weighting on the control inputs, R should be full rank in which case the LQG controller is proper if and only if the rank condition (6.83) is satisfied. If R is singular, the controller can be made to be proper if Φ_n is full rank. The perfect measurement case is of more practical importance than the minimum variance case.

The relationship between the order of the controller and the number of finite zeros was considered in Section 6.5.2 (Theorems 6.12 and 6.13). In practice continuous systems rarely have enough zeros to ensure that the resulting LQG controller is proper. This is particularly so when the system arises from second order differential equations. As an example consider a system arising from Newton's second law with the displacement being measured. It is straightforward to show that for this system CE=0 and therefore from Theorem 6.11 the controller will be improper. Consequently, the controller for most unmodified LQG problems will be improper (that is, contain derivative action).

As well as being used to design minimum variance estimators the descriptor forms of this chapter can also be used to construct deterministic Luenburger reduced-order observers as follows:

(i) Choose L to place n-m zeros of $C(sI-A)^{-1}L$ (the algorithm of Berger *et al.* (1991) could be used to perform this task),
(ii) The descriptor form of the observer can then be constructed using (6.45),
(iii) To obtain a state-space representation of the reduced-order observer the reduction process using (6.89) to (6.93) could be performed.

This observer can then be used in conjunction with a state feedback controller to give a proper output feedback controller.

6.6.2. Loop Transfer Recovery Techniques

The nonsingular LQ state feedback controller (when S=0 and $R = \rho I$) has been shown to have a gain margin of at least 6dB and a phase margin of at least 60° (Anderson and Moore 1990). These are desirable qualities for a control system. However, if an observer is used to estimate the states these robustness qualities are lost (Doyle 1978). Doyle and Stein (1979) suggested an asymptotic method of designing an observer to recover the gain and phase margins of the state feedback controller. This method involves designing a Kalman filter with a fictitious disturbance noise and letting the measurement noise tend to zero. In this section the descriptor approach to singular LQG controller design is used to investigate the limiting estimator of the Loop Transfer Recovery method.

The Loop Transfer Recovery (LTR) method considers a break in the loop at the point (i) in Figure 6.8. The gain and phase margins of the state feedback controller are recovered if the loop transfer functions from $\mathbf{u}''(s)$ to $\mathbf{x}(s)$ is the same as that from $\mathbf{u}'(s)$ and $\mathbf{u}''(s)$ to $\hat{\mathbf{x}}(s)$, that is, the system descriptions:

$$\mathbf{x}(s) = (sI-A)^{-1}B\mathbf{u}''(s) \tag{6.109}$$

and:

$$\hat{\mathbf{x}}(s) = \left[E_2(sI-A)+LC\right]^{-1}LC(sI-A)^{-1}B\mathbf{u}''(s) + \left[E_2(sI-A)+LC\right]^{-1}LE_2B\mathbf{u}'(s) \tag{6.110}$$

are the same. These transfer functions will be identical if B=LW (note that it is necessary for $G(s) = C(sI-A)^{-1}B$ to be minimum phase and W may be singular, or even nonsquare!). In this case $E_2 = I-LL^{L}$ and $E_2B=0$ and since the identity:

$$\left[E_2(sI-A)+LC\right]^{-1}\left[E_2(sI-A)+LC\right] = I$$

can be rearranged as:

$$\left[E_2(sI-A)+LC\right]^{-1}LC = I - \left[E_2(sI-A)+LC\right]^{-1}E_2(sI-A)$$

it follows that:

$$\hat{\mathbf{x}}(s) = (sI-A)^{-1}B\mathbf{u}''(s) \tag{6.111}$$

which is identical to (6.109).

It should be noted that it is necessary to use a nonsingular state controller with this recovery procedure otherwise the controller is not well-defined. From (6.57) the control inputs $\mathbf{u}'(s)$ are related to the outputs $\mathbf{y}(s)$ by:

$$D\mathbf{u}'(s) = -K\left[E_2(sI-A)+LC\right]^{-1}L\mathbf{y}(s)$$

If D is nonsingular then the controller is given by:

$$\mathbf{u}'(s) = -K_c\left[E_2(sI-A)+LC\right]^{-1}L\mathbf{y}(s) \tag{6.112}$$

This controller can be decomposed into a state controller and an estimator where the estimator is given by:

$$E_2\dot{\hat{\mathbf{x}}}(t) = E_2A\hat{\mathbf{x}}(t) + L(\mathbf{y}(t)-\hat{\mathbf{y}}(t)) \tag{6.113}$$

The estimator in this controller, unlike the general form of the estimator (6.45), makes no use of the control inputs $\mathbf{u}'(s)$ in estimating the states. Note that this controller will always exist as $(E_2(sI-A)+LC)$ is regular if and only if Γ is invertible.

Another place at which the loop could be broken in Figure 6.8 is at (ii). This leads to the sensitivity recovery approach of Kwakernaak (1969). The estimator is assumed to be nonsingular in order to ensure the controller exists. In this case the internal loop transfer function at (ii)' is given by:

$$C(sI-A)^{-1}K_f \tag{6.114}$$

and the loop transfer function at (ii) is given by:

$$C(sI-A)^{-1}BK\left[(sI-A)E_1+BK\right]^{-1}K_f \tag{6.115}$$

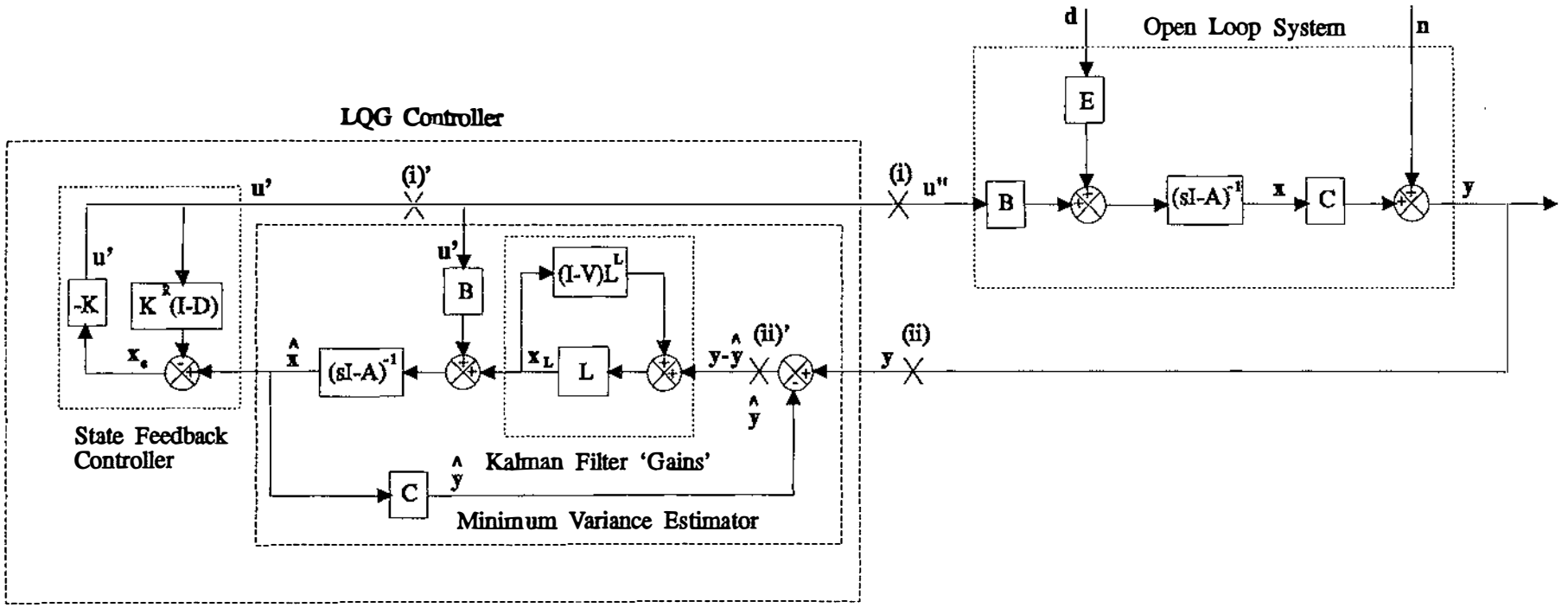

Figure 6.8
Loop Transfer Recovery Breakpoints

If D=0 and C=UK for some U, this transfer function can be made to be equal to (6.114) by designing a state controller such that $-K\mathbf{x}(t)=\mathbf{0}$. Then $E_1 = I-K^RK$, $CE_1=0$ and since:

$$BK\left[(sI-A)E_1+BK\right]^{-1} = I - (sI-A)E_1\left[(sI-A)E_1+BK\right]^{-1}$$

the loop transfer functions (6.114) and (6.115) are identical. The controller is given by:

$$\mathbf{u}(s) = -K\left[(sI-A)E_1+BK\right]^{-1}K_f\mathbf{y}(s) \tag{6.116}$$

Noting from (6.61) that $E_1\mathbf{x}_c(s) = \hat{\mathbf{x}}(s)$, the estimates satisfy the differential equation:

$$\dot{\hat{\mathbf{x}}}(t) = A\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + K_f\mathbf{y}(t) \tag{6.117}$$

For this controller the estimator is driven by the output $\mathbf{y}(t)$ instead of the output error $\mathbf{y}(t)-\hat{\mathbf{y}}(t)$ as in the general form of the nonsingular estimator in (6.10).

At first sight it might seem tempting to combine the Loop Transfer Recovery and Sensitivity Recovery methods to improve the robustness at both (i) and (ii) simultaneously. However, as was mentioned in the Loop Transfer Recovery section, the state controller cannot be singular and similarly the estimator cannot be singular in the sensitivity recovery procedure. Therefore combining the two procedures will not yield a well-defined controller.

6.7. NONUNIQUE CONTROLLERS

In Section 6.2 it was shown that the LQG controller could be determined from the Wiener-Hopf solution by solving the system of linear equations (6.2) which has a unique solution if and only if (I-TG) and (I-GT) are nonsingular. This condition was shown to be equivalent to the condition that the descriptor forms (6.17) and (6.20) are regular (Theorem 6.2). This result is preferable to the requiring the invertibility of rational matrices, due to the existence of results for the regularity of descriptor forms (Özçaldiran and Lewis 1990). If the inverses do not exist, then a controller exists only if a solution exists to the linear system of equations in (6.2). In this section the possibility of nonunique LQG controllers is investigated using the descriptor approach.

In the derivation of the descriptor forms in Lemmas 4.16, 4.17 and 4.18, the key is the use of the matrix identity (2.55). Indeed, this identity is central to all the results in this thesis. However, it only holds if the appropriate matrices are invertible. To proceed if (I-TG) is singular assume H(s)=KH'(s). Then from (6.2) it follows that:

$$\left(I - KT'LG\right)KH'(s) = KT'L$$
$$KT'\left(T'^{-1} - LGK\right)H'(s) = KT'L$$
$$KT'\left(E_2(sI-A)E_1+E_2BK+LCE_2\right)H'(s) = KT'L \qquad (6.118)$$

where

$$T' = ((sI-A)E_1+BK)^{-1}(sI-A)(E_2(sI-A)+LC) \qquad (6.119)$$

(6.118) can be rewritten as:

$$KT'\left(\left(E_2(sI-A)E_1+E_2BK+LCE_2\right)H'(s) - L\right) = 0$$

which has a solution if:

$$\text{rank}\left\{\left[E_2(sI-A)E_1+E_2BK+LCE_2\ ,\ L\right]\right\} = \text{rank}\left(E_2(sI-A)E_1+E_2BK+LCE_2\right) \qquad (6.120)$$

This condition is not necessary. All that is necessary is that there exists a H'(s) such that:

$$\left(E_2(sI-A)E_1+E_2BK+LCE_2\right)H'(s) - L$$

belongs to the null space of K (since T' is nonsingular, the null spaces of K and KT' are the same). The solution H'(s) can be written as:

$$H'(s) = \left(E_2(sI-A)E_1+E_2BK+LCE_2\right)^{\#}L \qquad (6.121)$$

where $\left(E_2(sI-A)E_1+E_2BK+LCE_2\right)^{\#}$ is a generalised inverse of $\left(E_2(sI-A)E_1+E_2BK+LCE_2\right)$.

The solution (6.121) shows that it has the form: H(s)=H''(s)L. Using this as a starting point leads to:

$$H''(s)L\left(I - GKT'L\right) = KT'L$$
$$H''(s)\left(T'^{-1} - LGK\right)T'L = KT'L$$
$$H''(s)\left(E_2(sI-A)E_1+E_2BK+LCE_2\right)T'L = KT'L$$

This equation has a solution if:

$$\text{rank}\left\{\begin{bmatrix} K \\ E_2(sI-A)E_1+E_2BK+LCE_2 \end{bmatrix}\right\} = \text{rank}\left(E_2(sI-A)E_1+E_2BK+LCE_2\right) \qquad (6.122)$$

A solution exists if and only if there exists an H'' such that L belongs to the null space of:

$$H''(s)\left[E_2(sI-A)E_1+E_2BK+LCE_2\right] - K$$

The solution of this equation can be written in terms of generalised inverses as:

$$H''(s) = K\left[E_2(sI-A)E_1+E_2BK+LCE_2\right]^{\#}$$

This solution is consistent with the initial assumption that H(s)=KH'(s).

The following example illustrates these points.

Example 6.2

For the Loop Transfer Recovery method presented in Section 6.6.2 when R=0 and Φ_n=0 an irregular descriptor form of the controller results. The controller, if it exists, has the following form:

$$H(s) = KH'(s)$$

where

$$\left[E_2(sI-A)E_1\right]H'(s) - L$$

belongs to the null space of K. Using the transformations in Lemma 6.9, a solution exists if:

$$\begin{bmatrix} N(sI-A)M & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} H_1(s) \\ H_2(s) \end{bmatrix} - \begin{bmatrix} 0 \\ I \end{bmatrix} = \begin{bmatrix} N(sI-A)MH_1(s) \\ -I \end{bmatrix}$$

belongs to the null space of [0 , I]. This condition is clearly not satisfied and therefore an LQG controller does not exist for this LQG problem. ♦

The conditions (6.120) and (6.122) are similar to the controllability and observability conditions for a descriptor system in Definition 2.14. The difference here is that instead of requiring the rank of the appropriate matrices to be the same as the size of the descriptor system, the ranks are only required to be equal to the rank of (sE-A). Proceeding along these lines conditions (6.120) and (6.122) can be interpreted as uncontrollable or unobservable irregular modes of a descriptor form. Further research is required to classify these phenomena more precisely. The concept of invariant subspaces used by Wonham (1979) could also prove useful. At this stage the only example of a nonunique controller which has been identified has a singular spectral factor. The link between the results in Section 5.5 and this section should be investigated.

6.8. SUMMARY

A new descriptor form for LQG controllers has been derived in this chapter which has the advantage that it is valid for singular as well as nonsingular LQG problems. The structure of the descriptor form has a structure similar to the nonsingular LQG controller and allows the separation structure of the singular LQG controller to be established. That is, the singular LQG controller, if it exists, is composed of an LQ state feedback controller and a minimum variance estimator. The difficulty in constructing an LQ state feedback controller for singular weightings on the control input was demonstrated by an example. The advantage of the descriptor form is that it is not necessary to construct the LQ state feedback controller explicitly. The controller can be written down directly from the solution to two LMI, (5.13) and (5.31). Given that an LQG controller has the structure of the descriptor form (6.17), it was shown that an LQG problem can be decomposed into a minimum variance estimation problem and an LQ state feedback problem which uses the estimates of the states provided by a minimum variance estimator.

Another advantage of the descriptor form of the controller is that some of the results available for the response of descriptor forms can be applied to (6.17). It is well-known that singular LQG problems can result in improper controller. Standard descriptor systems theory was applied to the descriptor form of the LQG controller (6.17) to determine the order of the controller. Particular emphasis was placed on identifying LQG problems which result in proper controllers. The two special cases of perfect measurement and minimum variance control were considered. The role of multivariable zeros in determining the order of the controller for certain LQG problems and the minimality of the descriptor form of the controller were considered.

The problem of designing admissible LQG controllers was discussed. In the terminology of Halevi and Palmor (1988) an admissible controller is one for which the sensitivity function is proper. Several methods of modifying an LQG problem to make it admissible were considered. The use of Loop Transfer Recovery techniques in designing LQG controllers with acceptable gain and phase margins was discussed using the descriptor approach. It was demonstrated that the loop transfer recovery technique of Doyle and Stein (1979) cannot be used in conjunction with the sensitivity recovery technique of Kwakernaak (1969) as the resulting controller is not well-defined.

The possibility of nonunique LQG controllers was briefly investigated. This is an area which needs further research.

CHAPTER 7

CONCLUSIONS AND FUTURE WORK

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7.1. CONCLUSIONS

The Wiener-Hopf method of Austin (1979) has been studied in this thesis. There were three initial aims in this thesis: to show that the solution resulting from the method of Austin results in a stable closed loop system, to remove assumptions restricting the positions of open loop poles, and to produce minimal expressions for controllers.

State-space methods were used to investigate the different steps of Wiener-Hopf methods. Linear Matrix Inequalities (LMI) were used to describe the spectral factors. It was then shown that the partial fraction expansion step is related to the LMI. Previously the connection between partial fraction expansion and Riccati equations for nonsingular LQG problems was known (Shaked 1976b). The results derived in this thesis extend this result to include singular LQG problems. The LMI was also used to evaluate the contour integral for the minimum value of the performance index. Using the LMI to determine the partial fraction expansion enabled the restriction on open loop poles to be removed and established that the solution found by the method of Austin results in a stable closed loop system.

In attempting to produce minimal expressions for LQG controllers a new descriptor approach was developed. This approach allowed some explicit formulae to be derived for singular and nonsingular LQG controllers which enabled the separation structure to be established for singular LQG controllers. The difference between singular and nonsingular controllers is that singular feedback loops replace the constant gain state controllers and Kalman filter gains in nonsingular LQG problems. The descriptor approach does not require that the state feedback controller and minimum variance estimator are calculated explicitly and then combined to form the LQG controller: the LQG controller is formed directly from the LMI. The descriptor form of the LQG controllers also proved useful in establishing the order of the controller; some simple conditions were derived to determine whether an LQG controller contains derivative action. When the descriptor form of the controller was derived the open loop poles were cancelled, thus providing a form of the controller which does not have the difficulties mentioned by Grimble and Johnson (1986). However, investigation of the descriptor form showed that it is not necessarily minimal as there may be finite and infinite modes which are uncontrollable or unobservable and there may also be nondynamic modes.

One of the main difficulties with singular LQG control problems is that there does not necessarily exist a controller which results in the desired closed loop system. In terms of the descriptor approach the existence of controllers can be related to the regularity of the descriptor forms for the controller.

7.2. SUMMARY OF THE DESCRIPTOR METHOD FOR CALCULATING LQG CONTROLLERS

A major contribution of this thesis is the development of a new method for calculating LQG controllers which is valid for singular and nonsingular problems. This methodology is summarised in this section. The solution to the LQG problem:

$$J = \mathcal{E}\left\{\begin{bmatrix}\mathbf{x}^T(t) , \mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix} Q & S \\ S^T & R\end{bmatrix}\begin{bmatrix}\mathbf{x}(t) \\ \mathbf{u}(t)\end{bmatrix}\right\}$$

where

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) + E\mathbf{d}(t)$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + \mathbf{n}(t)$$

and

$$\mathcal{L}\left\{\mathcal{E}\left\{\begin{bmatrix}\mathbf{d}^T(t) , \mathbf{n}^T(t)\end{bmatrix}\begin{bmatrix}\mathbf{d}(t) \\ \mathbf{n}(t)\end{bmatrix}\right\}\right\} = \begin{bmatrix} \Phi_d & \Phi_{dn} \\ \Phi_{dn}^T & \Phi_n\end{bmatrix}$$

can be found as follows:

Step 1 - Solution of the LMI

Find the maximal solution of the LMI (Theorem 5.9):

$$LMI(X) := \begin{bmatrix} A^TX+XA+Q & XB+S \\ B^TX+S^T & R\end{bmatrix} = \begin{bmatrix}K^T \\ D^T\end{bmatrix}\begin{bmatrix} K , D\end{bmatrix}$$

and

$$LMI(Y) := \begin{bmatrix} YA^T+AY+E\Phi_dE^T & YC^T+E\Phi_{dn} \\ CY+\Phi_{dn}^TE^T & \Phi_n\end{bmatrix} = \begin{bmatrix} L , V\end{bmatrix}\begin{bmatrix}L^T \\ V^T\end{bmatrix}$$

Routines for solving the LMI include the Hermitian pencil method of Clements and Glover (1989), and the reduced-order Riccati equation methods of Stoorvogel (1992b) and Soroka and Shaked (1988b).

Step 2 - Calculate the Minimum Value of the Performance Index

The minimum value of the performance index is given by:

$$J = Tr\left\{XE\Phi_dE^T + YK^TK\right\}$$

$$= Tr\left\{XLL^T + YQ\right\}$$

Caution is needed in interpreting the value of the performance index as an equivalent LQG problem (Definition 5.14) can be chosen which results in the same controller but has a different value of the performance index (Theorem 5.15). The value of the performance index should only be used if the weights and noise intensities for the LQG problem are physically meaningful.

Step 3 - Check Invertibility of Spectral Factors

Check that the spectral factors:

$$\Delta = K(sI-A)^{-1}B+D$$

and

$$\Gamma = C(sI-A)^{-1}L+V$$

are invertible. If the spectral factors are not invertible the squaring-down procedure outlined in Section 5.5 may be used. This squaring-down procedure results in nonunique solutions to the LQG problem.

Step 4 - Calculate Controller

The controller can then be formed as follows:

If K and L have full rank calculate K^R and L^L (the Moore-Penrose pseudo-inverses $K^R = K^T(KK^T)^{-1}$ and $L^L = (L^TL)^{-1}L^T$ can be used). Then the controller is given by:

$$H(s) = K\left[E_2(sI-A)E_1+E_2BK+LCE_1\right]^{-1}L$$

where

$$E_1 = I - K^R(I-D)K$$

$$E_2 = I - L(I-V)L^L$$

Otherwise choose F_1 and F_2 such that $K_F = (K-DF_1)$ and $L_F = (L-F_2V)$ have full rank and then calculate K_F^R and L_F^L. The controller is given by:

$$H(s) = (K_F+F_1E_{F1})\left[E_{F2}(sI-A)E_{F1}+E_2B(K_F+F_1E_{F1})+(L_F+F_2E_{F2})CE_{F1}\right]^{-1}(L_F+F_2E_{F2})$$

where

$$E_{F1} = I - K_F^R(I-D)K_F$$

$$E_{F2} = I - L_F(I-V)L_F^L$$

The controller is well-defined if the descriptor forms for the controller are regular.

It is not necessary to use the descriptor approach for nonsingular LQG problems as state-space representations can be used. However, when the weighting on the controls or the measurement noise intensities are nearly singular the descriptor approach could prove beneficial as their inverses are not required. It is also possible to use the fractional descriptor form to improve the condition of the K and L matrices before calculating their pseudo-inverses.

7.3. FUTURE WORK

The main outstanding issue with regard to the work presented in this thesis concerns nonunique solutions to LQG problems. Two possible types of nonuniqueness were mentioned: nonunique closed loop transfer functions which result when the spectral factors are singular; and nonunique controllers. An important LQG problem with a nonunique solution is state feedback control. This problem is generally considered to be the most basic optimal control problem. However, the output feedback LQG control problem is much simpler to solve using Wiener-Hopf methods. Further research into using Wiener-Hopf methods for state feedback control problems is necessary.

The descriptor approach developed in this thesis has provided a basic structure with which to study singular and nonsingular LQG control problems. The work presented here is, in the author's opinion, complete but is restricted to continuous time, regulator LQG control problems. Some other LQG problems which could benefit from using descriptor forms are discussed in this section.

The descriptor forms for LQG controllers arose naturally from using LMI to study Wiener-Hopf methods. The solution method should not however affect the solution. It could prove beneficial to revisit other solution methods for LQG problems to investigate how the descriptor forms arise.

The descriptor approach developed in this thesis should prove useful in studying the H_∞ control problem. The current solution to the H_∞ control problem is to solve an induced LQG or H_2 problem (Doyle *et al.* 1989). Singular H_∞ problems have been studied by Stoorvogel (1992b) using an extension of the LMI, the Quadratic Matrix Inequality and a disturbance decoupling approach. A practical singular H_∞ problem arises when the model uncertainty is multiplicative (Stoorvogel 1992b).

The results in this thesis are restricted to LQG problems where the open loop system has strictly proper transfer functions. Mehrmann (1989) considered LQG problems where the open loop system is described by a descriptor system. Clements (1993) has provided a state-space method for spectral factorisation for descriptor systems. Using

these results it should be possible to extend the partial fraction expansion step (Theorem 5.11) and the descriptor representations of LQG controllers (Chapter 6).

Only the continuous time LQG problem has been presented in this thesis. The solution to discrete singular LQG problems can also be studied using Wiener-Hopf methods but is fundamentally different due to the stability region being a closed region in the complex plane (the unit circle). This means that closed loop transfer functions must be proper (no poles at ∞). As the z-transform acts as forward shift operator the controller must also be proper otherwise it will be noncausal. Another difference between the continuous and discrete LQG control problems occurs in the definition of causality. If causality is defined such that $f(t)=0$ for all $t < 0$ a predictor observer results, but if it is defined such that $f(t)=0$ for all $t \leq 0$ a current time observer results. The first definition of causality leads to a partial fraction expansion of the discrete version of the transfer function M(s) (3.30) similar to the continuous time problem (Theorem 5.11) while the second definition includes the constant term in the expansion of M(s). The use of a current time observer results in a smaller value of the performance index. A discrete version of the LMI exists. However, unlike the continuous time problem, the Riccati equations are defined for singular problems and hence the descriptor approach is not necessary for singular LQG problems. The Riccati equations can be used to calculate the optimal state feedback controller and the full order Kalman filter. It is possible to use the descriptor approach to produce reduced-order controllers for perfect measurement or minimum variance LQG problems. At the time of writing this thesis only a preliminary study had been made of discrete LQG problems. These initial results, described briefly above, are very encouraging and will be published at a later date.

The regulator LQG control problem was studied in detail in this thesis. It is also possible to consider other controller configurations such as feedforward/feedback, tracking problems and three-degree-of-freedom configurations (Park and Bongiorno 1990). It would be interesting to see if LMI can be used to extend the Riccati equation based methods of Park and Youla (1992) to include singular problems.

APPENDIX

The following paper was presented at the 12th World Congress of the International Federation of Automatic Control, Sydney, Australia, 18-23 July 1993 and is found in Volume 2, pp475-778 of the Preprints of Papers. This paper represents some of the preliminary work in descriptor forms for LQG controllers presented in Chapter 6.

LINEAR QUADRATIC GAUSSIAN CONTROLLERS FOR PERFECT MEASUREMENTS

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Abstract. The Wiener-Hopf method is used to solve the Linear Quadratic Gaussian (LQG) controller design problem for continuous-time, time-invariant linear systems when the measurements are free of white noise. An explicit descriptor form for this controller is derived. This formula depends only on the spectral factors. Using the formula, conditions concerning the order and propemess of the controller are studied. When the controller is proper, it is shown to be comprised of a state feedback controller and a reduced order observer.

Keywords. Singular Control , Descriptor Systems , Reduced Order Observers

1. INTRODUCTION

Methods for designing LQG controllers using Riccati equations are well known. For output feedback control such methods presume that there is noise present in the output measurements and that all the control inputs are weighted. In this paper the form of the LQG controller is determined and examined for some of the situations in which the standard methods of LQG controller design do not yield a solution, because inverses of singular matrices are required. Specifically, the paper considers the perfect measurement noise case: there is no noise present in the measurements.

The Wiener-Hopf technique of Shaked (1976b) and Austin (1979) is used in this paper to derive a closed form of the LQG controller for measurements with no noise. With Wiener-Hopf techniques, the key step is spectral factorisation, whereas with time domain techniques the key step is the solution of Riccati equations. The central role of spectral factorisation in Wiener-Hopf techniques is discussed; explicit partial fraction expansion is not required in determining the controller. The analysis yields a descriptor form of the controller. There is no need to differentiate the outputs to obtain a reduced order model as required for example by Bryson and Johansen (1965), since the descriptor form contains the necessary reductions.

It is of particular interest to know when the perfect measurement LQG controller does not require derivative action in its implementation. Some simple conditions are derived which guarantee that the controller is proper. Under these conditions a state-space form of the LQG controller is derived from the Wiener-Hopf solution. It is shown that this controller can be separated into a state feedback controller and a reduced order observer.

The paper is organised as follows. Some notation is presented in Section 2 along with the particular descriptor form used in this paper. The Wiener-Hopf method for calculating LQG controllers for perfect measurements is presented in Section 3. The structure of the LQG controller is then studied in Section 4; the study focuses particularly on proper controllers.

2. PRELIMINARIES

2.1. *Notation*

The following notation is used in this paper:

- A^T is the matrix congugate transpose;
- B^L is the left Moore-Penrose pseudo inverse of B, $B^L = (B^TB)^{-1}B^T$ such that $B^LB = I$;
- $G^*(s)$ is defined as $G^T(-s)$;
- $\det(\cdot)$ is the matrix determinant;
- $\mathrm{Tr}\{\cdot\}$ is the matrix trace;
- $\mathcal{E}[\cdot]$ is the expected value over time, $t \in [0,\infty)$.

2.2. *A Useful Descriptor Form*

In this section a useful descriptor form is derived.

Lemma 1. The product $(sI-A)^{-1}B[C(sI-A)^{-1}B]^{-1}$ has the descriptor form $[(I-BB^L)(sI-A)+BC]^{-1}B$ and the poles of this transfer function are exactly the same as the zeros of $C(sI-A)^{-1}B$.

Proof. $(sI-A)^{-1}B\left[C(sI-A)^{-1}B\right]^{-1}$

$$= (sI-A)^{-1}B\left[I-B^LB+C(sI-A)^{-1}B\right]^{-1}$$

$$= \left[(I-BB^L)(sI-A)+BC\right]^{-1}B \qquad (1)$$

The inverse, B^L always exists as B is required to have full column rank for the transfer function $C(sI-A)^{-1}B$ to be invertible. The poles of this system satisfy $\det((I-BB^L)(sI-A)+BC)=0$. Some standard manipulation of determinants leads to:

$$\det\left[(I-BB^L)(sI-A)+BC\right] = \det\left(\begin{bmatrix} sI-A & -B \\ C & 0 \end{bmatrix}\right) \qquad (2)$$

The right hand side of equation (2) is the definition used by MacFarlane and Karcanias (1976) of zeros of a transfer function $C(sI-A)^{-1}B$. ♦

3. THE WIENER-HOPF METHOD FOR LQG CONTROLLER DESIGN

In this section the Wiener-Hopf method for LQG controller design of Shaked (1976a) and Austin (1979), is presented for the case when there is no measurement noise. The system to be controlled is described by the state-space representation:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) + E\mathbf{d}(t) \quad , \quad \mathbf{x}(0) = \mathbf{0}.$$
$$\mathbf{y}(t) = C\mathbf{x}(t) \tag{3}$$

where $\mathbf{y}(t)$ is the system output, $\mathbf{u}(t)$ is the control input and $\mathbf{d}(t)$ is the disturbance input. The pair (A,B) is assumed to be stabilisable and the pair (A,C) is assumed to be detectable. The transfer function description of equation (3) is given by:

$$\mathbf{x}(s) = P\mathbf{u}(s) + P_d\mathbf{d}(s)$$
$$\mathbf{y}(s) = G\mathbf{u}(s) + G_d\mathbf{d}(s)$$

where

$$P(s) = (sI-A)^{-1}B \quad , \quad P_d(s) = (sI-A)^{-1}E$$
$$G(s) = C(sI-A)^{-1}B \quad , \quad G_d(s) = C(sI-A)^{-1}E. \tag{4}$$

The controller H(s) is defined such that :

$$\mathbf{u}(s) = -H(s)\mathbf{y}(s). \tag{5}$$

The LQG controller minimises the following performance index :

$$J = \mathcal{E}\left\{ \begin{bmatrix} \mathbf{x}^T(t) , \mathbf{u}^T(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} \right\} \tag{6}$$

The performance index in equation (6) can be written in the frequency domain (Austin 1979) as :

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left(\mathrm{Tr}\left\{ (\Delta T\Gamma - M)(\Delta T\Gamma - M)^* - MM^* \right\} + \mathrm{Tr}\left\{ QP_d\Phi_dP_d^* \right\} \right) ds \tag{7}$$

In equation (7):

(i) The transfer function, T(s) is defined by:

$$T = (I+HG)^{-1}H \tag{8}$$

(ii) the disturbance d(t) is assumed to be a white noise process. It is described using the spectral density function:

$$\Phi_d = \phi_d\phi_d^T. \tag{9}$$

(iii) the generalised spectral factors Δ and Γ that are associated with the LQG problem are defined in Shaked (1976a). They satisfy:

$$\Delta^*\Delta = P^*QP + S^TP + P^*S + R \tag{10}$$

and

$$\Gamma\Gamma^* = G_d\Phi_dG_d^* \tag{11}$$

The spectral factors are required to be invertible and minimum phase (zeros strictly in the left half of the complex plane).

(iv) The transfer function, M is defined by:

$$M = (\Delta^*)^{-1}(P^*Q+S^T)P_d\Phi_dG_d^*(\Gamma^*)^{-1} \tag{12}$$

The weighting matrices Q and R are assumed to be positive semidefinite and positive definite respectively.

The spectral factor Δ can be represented in state-space form (MacFarlane, 1971):

$$\Delta = D(I+K_c(sI-A)^{-1}B) \text{ where } D^TD = R \tag{13}$$

and K_c is determined from the solution to the Riccati equation:

$$A^TX + XA + Q - (S+XB)R^{-1}(B^TX+S^T) = 0$$
$$K_c = R^{-1}(B^TX+S^T) \tag{14}$$

The spectral factor Γ can be represented in state-space form (Schumacher, 1985):

$$\Gamma = C(sI-A)^{-1}L \tag{15}$$

The spectral factor Δ is always invertible as R is full rank. The invertibility of Γ requires that C and L have full row and column rank respectively. One method for calculating L is to use the expressions for the minimum phase image of $G_d\phi_d$ in Shaked (1989). If the spectral factor is not invertible (this will happen if the number of disturbance inputs is less than the number of outputs), the solution to the LQG problem is not unique (Schumacher, 1985). The case of non-unique solutions to the LQG problem is considered in Noell (1994).

The controller H appears in equation (7) only in terms of T; the optimal controller H can be found from the optimal T. Constraining the solution to yield a stable closed loop system leads to (Noell, 1993):

$$\Delta T\Gamma = M_\oplus = DK_c(sI-A)^{-1}L \tag{16}$$

Hence the optimal T is:

$$T = \Delta^{-1}M_\oplus\Gamma^{-1} \tag{17}$$

For this solution, the integral in equation (7) can be shown to converge (Noell, 1994). The LQG controller from equation (8) is then given by :

$$\begin{aligned} H(s) &= (I-TG)^{-1}T \\ &= \Delta^{-1}M_\oplus\left(\Gamma - G\Delta^{-1}M_\oplus \right)^{-1} \\ &= K_c(sI-A+BK_c)^{-1}L\left(C(sI-A+BK_c)^{-1}L \right)^{-1} \\ &= K_c\left(E'(sI-A+BK_c) + LC \right)^{-1}L \end{aligned} \tag{18}$$

where E' = I-LL^{L}. The last line of equation (18) follows from Lemma 1. From equation (18) the controller will not exist if $C(sI-A+BK_c)^{-1}L$ is not singular. Equivalently, the controller does not exist if $\det(E'(sI-A+BK_c)+LC) \equiv 0$ (See Stoorvogel (1992) for an example for which a controller does not exist.)

The significance of the descriptor form in equation (18) is that once the K_c and L matrices for the spectral factors are determined, the controller is completely specified since L has full column rank.

It is now shown that the closed loop poles are at the zeros of Δ and Γ. It follows from equations (3), (5) and (18) that the closed loop system is described in differential form by:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ E'\dot{\mathbf{x}}_c(t) \end{bmatrix} = \begin{bmatrix} A & -BK_c \\ LC & E'(A-BK_c)-LC \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}_c(t) \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} \mathbf{d}(t) \tag{19}$$

where $\mathbf{x}(t)$ is the vector of states of the system G(s) and $\mathbf{x}_c(t)$ is the vector of states of the controller H(s). Thus the closed loop poles are given by:

$$\det\left(s\begin{bmatrix} I & 0 \\ 0 & E' \end{bmatrix} - \begin{bmatrix} A & -BK_c \\ LC & E'(A-BK_c)-LC \end{bmatrix} \right)$$
$$= \det(sI-A+BK_c)\det(E'(sI-A)+LC) \tag{20}$$

The zeros of Δ are at the eigenvalues of $A-BK_c$ and the zeros of Γ are at $\det(E'(sI-A)+LC) = 0$ from Lemma 1. These results are summarised in the following theorem.

Theorem 1. If the disturbance spectral factor Γ is invertible, then the LQG controller exists if $C(sI-A+BK_c)^{-1}L$ is non-singular. In this case, the controller is given by the descriptor form in equation (18). The closed loop poles are at the zeros of the spectral factors Δ and Γ. ♦

A more general derivation of the LQG problem in the presence of measurement noise and/or singular control weighting R is given in Noell (1994).

4. PROPERTIES OF PERFECT MEASUREMENT LQG CONTROLLERS

The descriptor form of the controller derived in Section 3 is a convenient form with which to study the properties of the LQG controller for perfect measurements. The order of the controller is related to the number of finite zeros in the spectral factor Γ. In cases where the controller is proper, a reduced order state-space controller is derived.

4.1. *Order of the Controller*

In order to determine the order of the controller it is necessary to recall a few properties of descriptor forms from Verghese *et al* (1981): for the descriptor form in equation (18), the generalised order of the system is f = rank(E') = n-m where L is a nxm matrix. The number of finite frequencies is given by g = degree($\det(E'(sI-A+BK_c)+LC)$). From Lemma 1:

$$\det(E'(sI-A+BK_c)+LC) = \det\left(\begin{bmatrix} sI-A+BK_c & -L \\ C & 0 \end{bmatrix}\right) \tag{21}$$

Therefore the number of finite frequencies of (18) is the same as the number of finite zeros of $\xi(s) = C(sI-A+BK_c)^{-1}L$. The number of impulsive modes is the f-g. From Kouvaritakis and Shaked (1976), the number of finite zeros of a transfer function is given by:

Theorem 2. The zeros of a mxm matrix transfer function $C(sI-A)^{-1}B$ are given by the solution to the eigenvalue problem sNM-NAM = 0 where NB=0 and CM=0. From this property of zeros, when rank(CB)=m-d, the maximum number of finite zeros is n-m-d where n is the dimension of A. If d=0 (that is CB is full rank), then the number of finite zeros is precisely n-m. ♦

Applying this theorem to the transfer function $\xi(s)$, the following conclusions about the order of the controller can be made:

Theorem 3. The controller in equation (18) will contain at least d impulsive modes where d is the rank deficiency of CL. The controller will be proper if CL is full rank. ♦

It should be noted that the above conditions for the properness of the controller are only sufficient conditions. It is possible that the impulsive modes are uncontrollable, or unobservable. Of particular interest is the case when the LQG controller is proper. This case is considered in the next section.

4.2. *Proper LQG Controllers for Systems with Perfect Measurement*

In the previous section it was shown that the LQG controller will be proper if the spectral factor Γ has n-m zeros. A simple state-space formula for the LQG controller is derived in this section. This controller is then shown to be comprised of a state feedback controller and a reduced order observer.

When a descriptor system is proper it can be represented in a state-space form. The properties of the matrix (I-LL^{L}) allow simplifications to be made. Consider a transformation on the states of the

descriptor form in equation (18) such that:

$$U^T(I-LL^L)U = \begin{bmatrix} I_{n-m} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{where } U^TU = I \qquad (22)$$

It can readily be shown that U has the form:

$$U = [\, N^T, L\chi^{-1} \,] \qquad (23)$$

where N forms an orthonormal null space of L (that is $NL = 0$, $NN^T = I_{n-m}$) and χ is a Cholesky factorisation of L^TL. When this transformation is applied to the descriptor form of the controller, certain states give rise to an an algebraic constraint. Eliminating these leads to the state-space representation of the LQG controller:

$$\dot{\mathbf{x}}_c(t) = NA\hat{\mathbf{x}}(t) + NB\mathbf{u}(t) \qquad (24a)$$

$$\hat{\mathbf{x}}(t) = (I-L(CL)^{-1}C)N^T\mathbf{x}_c(t) + L(CL)^{-1}\mathbf{y}(t) \qquad (24b)$$

$$\mathbf{u}(t) = -K_c\hat{\mathbf{x}}(t) \qquad (24c)$$

The requirement that CL is full rank is used explicitly in equation (24). The reason for writing the state-space form in the form given in (24) is to emphasise that the controller is composed of the optimal state feedback gain K_c and a reduced order observer. Specifically, equations (25a) and (25b) describe the observer and equation (25c) describes the state feedback controller. It should be noted that the observer in (24) is areduced order observer, having only has n-m states. These results are summarised in the following theorem:

Theorem 4. The LQG controller is proper if Γ has n-m zeros. In this case, the controller can be represented in the state-space form in equation (24). The controller can be separated into two parts: a) the optimal state feedback controller; and b) the optimal reduced order observer. ♦

5. CONCLUSIONS

An explicit descriptor system formula for LQG controllers when there is no measurement noise has been derived. This controller only depends on the model of the system under consideration and the spectral factors. The descriptor formula has the advantage that any derivative control action required is contained within the description of the controller.

The order of the controller was studied using properties of descriptor systems and multivariable zeros. It was found that the order of the controller depended on the number of finite zeros of the spectral factor Γ. The controller was shown to be proper if the spectral factor had the maximum number of finite zeros (n-m) in which case the state-space form in equation (24) could be used. This form was shown to be a combination of the optimal state feedback controller and the optimal reduced order observer.

The descriptor form developed in this paper can be used an alternative to the transfer function representation of the minimum variance estimator in Shaked and Soroka (1987). The descriptor form has the advantage that it has a similar structure to a full order observer. This issue is more fully discussed in Noell (1994); much of this material will be submitted for publication in the open literature.

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