FUNDAMENTALS OF RIEMANNIAN GEOMETRY AND ITS EVOLUTION

A thesis presented in partial fulfilment of the requirements for the degree of

Master of Science
in
Mathematics

at Massey University, Palmerston North,
New Zealand.

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2000

Abstract

In this thesis we study the theory of Riemannian manifolds: these are smooth manifolds equipped with Riemannian metrics, which allow one to measure geometric quantities such as distances and angles.

The main objectives are:

(i) to introduce some of the main ideas of Riemannian geometry, the geometry of curved spaces.

(ii) to present the basic concepts of Riemannian geometry such as Riemannian connections, geodesics, curvature (which describes the most important geometric features of universes) and Jacobi fields (which provide the relationship between geodesics and curvature).

(iii) to show how we can generalize the notion of Gaussian curvature for surfaces to the notion of sectional curvature for Riemannian manifolds using the second fundamental form associated with an isometric immersion. Finally we compute the sectional curvatures of our model Riemannian manifolds - Euclidean spaces, spheres and hyperbolic spaces.

Acknowledgements

I am privileged here in expressing my deep gratitude and appreciation to my supervisor, Dr. Gillian M. Thornly, for her excellent guidance, professional assistance, encouragement, valuable comments and giving her time, knowledge and experience freely from the beginning of my thesis until the final review of the manuscript.

My special thanks to Dr. Marijcke Vlieg-Hulstman for her voluntary assistance given to me in word processing mathematical formulas.

I would like to thank Phil Etheridge for helping me in my computing work.

Many thanks are also extended to my fellow graduate assistants, Deborah, Kathir, Simon, Patrick and Paul for their continuous support and encouragement.

My profound thanks to all of the staff in the Department of Mathematics at Massey University for providing me all facilities and creating a friendly environment to carry out my studies successfully.

I would like to express my sincere gratitude and appreciation to the Massey University for the financial assistance given to me during the course of my study.

Finally I would like to thank my husband, Asoka and my daughter, Nadeesha for their continuous encouragement, patience and love until the completion of my thesis.

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Chapter 0

Introduction

0.1 The Evolution of Riemannian Geometry

Geometry is the branch of mathematics that deals with the relationships, properties and measurements of solids, surfaces, lines and angles. It also considers spatial relationships, the theory of space and figures in space. The name comes from Greek words meaning, "land" and "to measure". Geometry was first used by the Egyptians to measure lands. Later it was highly developed by the great Greek mathematicians.

About 300 B.C, Euclid was a Greek mathematician. Elements of Euclid is a scientific work containing the foundations of ancient mathematics: elementary geometry, number theory, algebra, the general theory of proportion and a method for the determination of areas and volumes. The geometry based on the assumptions of Euclid and dealing with the study of plane and solid or space geometry is called Euclidean geometry. In the 19^{th} century, new kinds of geometry, called Non-Euclidean geometry, were created. Any kind of geometry not based upon Euclid's assumptions is called Non-Euclidean geometry. E.g:- Differential geometry (Surface geometry), Hyperbolic geometry, Riemannian geometry, etc. Classical differential geometry consisted of the study of curves and surfaces (embedded in three-dimensional Euclidean space) by means of the differential and integral calculus.

The founders of Non-Euclidean geometry were Gauss, Riemann, Bolyai and Lobachevski. All of them investigated the possibilities of changing Euclid's parallel postulate, which said that one and only one line parallel to a given line could be drawn through a point outside that line. Until the 19^{th} century, this was accepted as a " self -evident truth ". The replacement of this postulate led to new geometries. In the early part of the 19^{th} century, Carl Friedrich Gauss (1777-1855) was considered to be one of the most original mathematicians living in Germany. He was a pioneer in Non-Euclidean geometry, statistics and probability. He developed the theory of functions and the geometry of curved surfaces. Gauss defined a notion of curvature (Gaussian curvature) for surfaces, which measures the amount that the surface deviates from its tangent plane at each point on the surface.

Towards the end of his life (1855) Gauss was fortunate to have an excellent student, Gerg Friedrich Riemann (1826-1866), who was the founder of Riemannian geometry. Riemann's life was short but marvelously creative. He took up the ideas of Gauss. On June 10^{th} in 1854, he delivered his inaugural lecture, entitled " On the Hypotheses that lie at the foundations of geometry ". Several vital concepts of modern mathematics appeared for the first time from his lecture. In particular, he

1. Introduced the concept of a manifold.
2. Explained how different metric relations could be defined on a manifold.
3. Extended Gauss's notion of curvature of a surface to higher dimensional manifolds.

The concepts of Riemannian geometry played an important role in the formulation of the general theory of relativity. Riemannian geometry is a special geometry, the geometry of curved spaces, associated with differentiable manifolds and has many applications to Physics. During the closing decades of the 19^{th} century, Levi-civita (1873-1941), took up the ideas of Riemann and contributed the concept of parallel displacement or parallel transport, which plays an important role in Riemannian geometry.

0.2 Generalization of Surface Theory to Riemannian Geometry

Surface is one of the basic concepts in geometry. The definitions of a surface in various fields of geometry differ substantially. In elementary geometry, one considers planes, multifaceted surfaces, as well as certain curved surfaces (for example, spheres). Each curved surface is defined in a special way, very often as a set of points or lines. The general concept of surface is only explained, not defined, in elementary geometry: one says that a surface is the boundary of a body, or the trace of a moving line, etc. In analytic and algebraic geometry, a surface is considered as a set of points the coordinates of which satisfy equations of a particular form. In three-dimensional Euclidean space, $\Re^3$, a surface is obtained by deforming pieces of the plane and arranging them in such a way that the resulting figure has no sharp points, edges, or self-intersection. We must require that a surface be smooth and two-dimensional, so that the usual notions of calculus can be extended to it. A surface is defined by means of the concept of a surface patch, which is a homeomorphic image of a square in $\Re^3$. A surface is understood to be a connected set, which is the union of surface patches (for example, a sphere is the union of two hemispheres, which are surface patches). Usually, a surface is specified in $\Re^3$ by a vector function

$$r = r(x(u,v), y(u,v), z(u,v)), \text{ where } 0 \le u,\ v \le 1.$$

The first example of a manifold, is a regular surface in $\Re^3$.

0.2.1 Definition

A subset $S \subset \Re^3$ is a *regular surface,* if, for every point $p \in S$, there exists a neighborhood V of p in $\Re^3$ and a mapping x: $U \subset \Re^2 \to V \cap S$ of an open set $U \subset \Re^2$ onto $V \cap S$, such that:

(a) x is a differentiable homeomorphism;

(b) The differential $(dx)_q : \Re^2 \to \Re^3$ is injective for all $q \in U$

The mapping x is called a *parametrization* of S at p. The neighborhood $V \cap S$ of p in S is called a *coordinate neighborhood.*

A major defect of the definition of regular surface is its dependence on $\Re^3$. This situation gradually became clear to the mathematicians of 19$^{\text{th}}$ century. Riemann drew the correct conclusion, which says that there must exist a geometrical theory of surfaces completely independent of $\Re^3$. His idea was to replace the dot product by a arbitrary inner product on each tangent plane of S. He observed that all the notions of the intrinsic geometry (for example, Gaussian curvature) only depended on the choice of an inner product on each tangent plane of S. Next we will introduce the notion of abstract surface which is an outgrowth of the definition of the regular surface.

Historically, it took a long time to appear due to the fact that the fundamental role of the change of parameters in the definition of a surface in $\Re^3$ was not clearly understood.

0.2.2 Definition

An *abstract surface* (differentiable manifold of dimension 2) is a set S together with a family of one-to-one mappings $x_\alpha : U_\alpha \to S$ of open sets $U_\alpha \subset \Re^2$ into S such that

(i) $\bigcup_\alpha x_\alpha(U_\alpha) = S$.

(ii) For each pair α, β with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \phi$, we have that $x_\alpha^{-1}(W), x_\beta^{-1}(W)$ are open sets in $\Re^2$, and $x_\alpha^{-1} \circ x_\beta, x_\beta^{-1} \circ x_\alpha$ are differentiable mappings.

The pair (U_α, x_α) with $p \in x_\alpha(U_\alpha)$ is called a *parametrization* of S around p. $x_\alpha(U_\alpha)$ is called a *coordinate neighborhood* at p. The family $\{U_\alpha, x_\alpha\}$ is called a *differentiable structure* for S.

Shifting then from surfaces in $\Re^3$ to abstract surfaces and, from the dot product to arbitrary inner products, we get the following definition.

0.2.3 Definition

A *geometric surface* is an abstract surface furnished with an inner product $\langle\ ,\ \rangle$, on each of its tangent planes. This inner product is required to be differentiable in the sense that if V and W are differentiable vector fields on S then $\langle V, W\rangle$ is a differentiable real-valued function on S.

We emphasize that each tangent plane T_pS of S has its own inner product. An assignment of inner products to tangent planes as in the above definition is called a *geometric structure* (or metric tensor or "ds^2") on S. We emphasize that the same surface furnished with two different geometric structures gives rise to two different geometric surfaces.

If we look back to the definition of abstract surface, we see that the number 2 has played no essential role. Thus, we can extend that definition to an arbitrary n and this may be useful in future.

0.2.4 Definition

A *differentiable manifold* of dimension n is a set M and a family of injective mappings $x_\alpha : U_\alpha \subset \Re^n \to M$ of open sets U_α of $\Re^n$ into M such that

(I) $\bigcup_\alpha x_\alpha(U_\alpha) = M$

(II) For any pair α, β with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \phi$, the sets $x_\alpha^{-1}(W), x_\beta^{-1}(W)$ are open sets in $\Re^n$ and the mappings $x_\alpha^{-1} \circ x_\beta, x_\beta^{-1} \circ x_\alpha$ are differentiable.

(III) The family $\{ (U_\alpha, x_\alpha) \}$ is maximal relative to the conditions (I) and (II).

The pair (U_α, x_α) with $p \in x_\alpha(U_\alpha)$ is called a *parametrization* of M around p. $x_\alpha(U_\alpha)$ is called a *coordinate neighborhood* at p. A family $\{ (U_\alpha, x_\alpha) \}$ satisfying (I) and (II) is called a *differentiable structure* on M

For example, curves are one-dimensional manifolds because every point of a curve can be located by a single parameter. Also surfaces are two-dimensional manifolds since for each piece of a surface, every point can be located by surface coordinates. Generalizing, we say that an n-dimensional manifold is a set, such that on every piece, of it, we can locate points by using n coordinates.

0.2.5 The metric coefficients of the surface

Gauss presented the most important formula in surface geometry in 1827. This appeared in his paper " General investigation of curved surface ".

$$ds^2 = a_{11}du_1^2 + 2a_{12}du_1du_2 + a_{22}du_2^2 \qquad (1)$$

It expresses the distance between two infinitesimally close points on the surface in terms of surface coordinates u_1, u_2. He considered that the geometry of a surface is Euclidean in infinitesimal neighborhoods. Thus, a surface can be regarded as an infinite collection of Euclidean spaces that are smoothly joined together. Another way of thinking about this is to regard the surface as the envelope of its tangent planes.

The proof of this formula for a surface in $\Re^3$ is briefly as follows.

Let p be any point on the surface and (u_1, u_2) be the surface coordinates of p. Let s be the value of arc length.

Then the rectangular Cartesian coordinates of p are $(x(u_1,u_2), y(u_1,u_2), z(u_1,u_2))$.

So $$dx = \frac{\partial x}{\partial u_1} du_1 + \frac{\partial x}{\partial u_2} du_2 \qquad \text{(a)}$$

$$dy = \frac{\partial y}{\partial u_1} du_1 + \frac{\partial y}{\partial u_2} du_2 \qquad \text{(b)}$$

$$dz = \frac{\partial z}{\partial u_1} du_1 + \frac{\partial z}{\partial u_2} du_2 \qquad \text{(c)}$$

We know that $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$ (Pythagorean formula)

Substituting from (a), (b) and (c),

$$(ds)^2 = \left(\frac{\partial x}{\partial u_1} du_1 + \frac{\partial x}{\partial u_2} du_2\right)^2 + \left(\frac{\partial y}{\partial u_1} du_1 + \frac{\partial y}{\partial u_2} du_2\right)^2 + \left(\frac{\partial z}{\partial u_1} du_1 + \frac{\partial z}{\partial u_2} du_2\right)^2$$

Simplifying the terms in brackets and taking,

$$a_{11} = \left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2$$

$$a_{12} = \frac{\partial x}{\partial u_1}\frac{\partial x}{\partial u_2} + \frac{\partial y}{\partial u_1}\frac{\partial y}{\partial u_2} + \frac{\partial z}{\partial u_1}\frac{\partial z}{\partial u_2} = a_{21}$$

$$a_{22} = \left(\frac{\partial x}{\partial u_2}\right)^2 + \left(\frac{\partial y}{\partial u_2}\right)^2 + \left(\frac{\partial z}{\partial u_2}\right)^2$$

So, $ds^2 = a_{11} du_1^2 + 2a_{12} du_1 du_2 + a_{22} du_2^2$. Hence the result.

The expression (1) appearing on the right hand side of the equation is called the *first fundamental form* and a_{11}, a_{12}, a_{22} are called the *metric coefficients*. They vary from point to point as one moves across the surface. But in the Euclidean plane we can choose coordinates so that the metric coefficients are constants.

Consider a horizontal plane lying in three-dimensional Euclidean space.

The equation of this plane is z = constant. We can choose the coordinates $u_1 = x, u_2 = y$ on the plane.

Then $\frac{\partial x}{\partial u_1} = 1, \frac{\partial x}{\partial u_2} = 0, \frac{\partial y}{\partial u_1} = 0, \frac{\partial y}{\partial u_2} = 1, \frac{\partial z}{\partial u_1} = 0, \frac{\partial z}{\partial u_2} = 0$. Therefore, we can show that $a_{11} = 1, a_{12} = 0, a_{22} = 1$. That is, the metric coefficients are constant for the plane. Therefore $ds^2 = du_1^2 + du_2^2$. (from (1))

Consider the sphere with radius *r*, centered at the origin. Let θ and ϕ be surface coordinates (except at the poles) of any point *p*, where $u_1 = \theta, u_2 = \phi$. The Cartesian coordinates of *p* can be expressed in terms of θ and ϕ as

$$x = r\cos\phi\cos\theta$$

$$y = r\cos\phi\sin\theta$$

$$z = r\sin\phi, \qquad \text{where } 0 \le \theta < 2\pi, \ \ -\pi/2 \langle \ \phi \ \langle \ \pi/2$$

Taking partial derivatives of the functions in these expressions,

$$\frac{\partial x}{\partial \theta} = -r\cos\phi\sin\vartheta, \qquad \frac{\partial x}{\partial \phi} = -r\sin\phi\cos\theta$$

$$\frac{\partial y}{\partial \theta} = r\cos\phi\cos\theta, \qquad \frac{\partial y}{\partial \phi} = -r\sin\phi\sin\theta$$

$$\frac{\partial z}{\partial \theta} = 0, \qquad \frac{\partial z}{\partial \phi} = r\cos\phi$$

Substituting these expressions into equations (a), (b), (c) and using the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, then $a_{11} = r^2\cos^2\phi$, $a_{12} = 0$, $a_{22} = r^2$.

Hence, equation (1) becomes

$$ds^2 = r^2\cos^2\phi d\theta^2 + r^2 d\phi^2.$$

This is the expression for the square of the length of an infinitesimal line element on the sphere. It is clear that the metric coefficient a_{11} varies with ϕ.

0.2.6 Generalization of metric coefficients of surfaces to Riemannian space

Generalizing the formula which Gauss obtained and extending it to n-dimensional manifolds, Riemann explained some basic concepts of a n-dimensional manifold. Consider a point p in an n-dimensional manifold and let $u_1, u_2, ..., u_n$ be its coordinates. Take a second point q whose coordinates $u_1 + du_1, u_2 + du_2, ..., u_n + du_n$ differ only infinitesimally from those of p. Riemann suggested that the square of the length ds of the line element joining p to q is given by

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}\, du_i\, du_j. \qquad (2)$$

where g_{ij} are functions of $u_1, u_2, ..., u_n$. This directly generalizes the formula (1) Gauss obtained for the line element of a surface. The expression on the right hand side of the equation (2) is a quadratic form in the variables $du_1, du_2, ..., du_n$, where ds^2 is positive unless q and p coincide. Therefore the quadratic form is said to be positive definite.

Using the expression (2) for determining length, he defined a *Riemannian metric* (see the definition in chapter 1) on the differentiable manifold. It provides the ability to calculate the length of paths in the manifold, and angles between tangent vectors in the same tangent space of the manifold. A manifold furnished with a Riemannian metric is called a *Riemannian manifold* or a *Riemannian space.*

For an example, in an *n*-dimensional Euclidean space, the square of the length of a line segment is given by the Pythagorean formula.

$$ds^2 = dx_1^2 + dx_2^2 + \ldots + dx_n^2 \ , \tag{3}$$

where $x_1, x_2, \ldots, x_n$ are rectangular Cartesian coordinates. It is clear that (3) is a special case of (2) with $g_{11} = 1, g_{22} = 1, \ldots, g_{nn} = 1$. Thus, Euclidean space is a special case of Riemannian space. Riemann called Euclidean spaces flat. A Riemannian space is locally Euclidean which means that an infinitesimal neighborhood of a point appears to be Euclidean. Just as the surface can be regarded as the envelope of its tangent planes, we may think of a Riemannian space as a collection of Euclidean spaces. We may say that a Riemannian space is infinitesimally flat or locally Euclidean.

0.2.7 Generalizing Gaussian curvature into Riemannian Geometry

In 1760, L. Euler described the curvature of a surface in space by two numbers at each point, called the *principal curvatures*. He defined the principal curvatures k_1 and k_2 of a surface by considering the curvature of curves, k_n, obtained by intersecting the surface with planes normal to the surface at an arbitrary point and taking $k_1 = \max k_n$ and $k_2 = \min k_n$. But at the time of Gauss, it was not clear that the principal curvatures would be an adequate definition of curvature. Gauss was the first to realize that surfaces have an intrinsic metric geometry that is independent of the surrounding space. More precisely, a property of surfaces in $\Re^3$ is called intrinsic if it is preserved by isometries. Even though the principal curvatures are not intrinsic, Gauss made the surprising discovery in 1827, that the product of the principal curvatures, now called the Gaussian curvature, is intrinsic. Gauss was amazed by his wonderful results and then named the theorem as Theorema Egregium, which is in colloquial American English can be translated roughly as "Totally Awesome Theorem". To get an idea of what Gaussian curvature tells us about surfaces, let's look at few examples. Simplest of all is the plane, which has both principal curvatures equal to zero and therefore has constant Gaussian curvature equal to zero. Another simple example is a sphere of radius r. Any normal planes intersect the sphere in great circles, which have radius r and therefore curvatures are $\pm 1/r$ (sign depends on whether we choose the outward pointing or inward pointing normal). Thus the principal curvatures are both equal to $\pm 1/r$, and the Gaussian curvature is equal to $1/r^2$ and always positive on the sphere.

The model spaces of surface theory are the surfaces with constant Gaussian curvature. We have discussed two of them: the Euclidean plane $\Re^2$ and the sphere of radius r. The third model is a surface of constant negative curvature, which is not so easy to visualize. Let's just mention that the upper half plane $\{(x, y): y > 0\}$ with the Riemannian metric $g = R^2(dx^2 + dy^2)/y^2$ has constant negative curvature $-1/R^2$, where R is a constant. In the special case $R = 1$ the curvature is -1. This is called the *hyperbolic plane.*

Here again generalizing the ideas of Gauss, Riemann defined the intrinsic geometry of a Riemannian space. Just as the notion of Gaussian curvature he thought that Riemannin curvature is a measure of the degree to which a Riemannian space differs from Euclidean space. In Euclidean space, he considered that the Riemannian curvature is zero everywhere. As with the surfaces, the basic geometric invariant is curvature. But the curvature becomes much more complicated quantity in higher dimensions because a manifold may curve in so many directions. The curvature can vary from point to point, but there are important special cases in which Riemann's measure is constant across the entire space. As with the surfaces, the model spaces of Riemannian geometry are the manifolds with constant sectional curvature (see chapter 3). In the end of the chapter 5, we introduce three classes of highly symmetric model Riemannian manifolds:- Euclidean spaces, spheres, and hyperbolic spaces. All most all of the properties of Riemannian geometry are related to the curvature. Therefore as in surface geometry, we can say that the curvature was the main source to develop Riemannian geometry.

The main objective of this thesis is to discuss more details about the curvature of the Riemannian manifold.