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MASSEY UNIVERSITY

The $(5, p)$ -Arithmetic Hyperbolic Lattices in Three Dimensions

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To my dear parents
Galavizh, Ahmad and my brother Saman
and my dear
Somayeh Sadeghi

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ABSTRACT

The group $Isom^+(\mathbb{H}^3) \cong PSL(2, \mathbb{C})$ contains an unlimited number of lattices of orientation-preserving isometries of hyperbolic 3-space (equivalently Kleinian groups of finite co-volume) that may be produced by using two elements of finite orders p and q as generators. For example, all but a finite number of $(p, 0)$ - $(q, 0)$ orbifold Dehn surgery on any of the infinite number of hyperbolic two-bridge links (or knots if $p = q$) would have (orbifold) fundamental groups that are such uniform (co-compact) lattices. However, it was demonstrated in [33] that, up to conjugacy, two elements of finite order could generate only a finite number of arithmetic lattices. In fact, it is proved in [33] that there are only a finite number of *nearly arithmetic* groups, that is groups generated by two elements of finite order that are discrete subgroups of arithmetic groups and are not free on the two generators.

The main result of this thesis is the determination of all the finitely many arithmetic lattices Γ in the orientation preserving isometry group of hyperbolic 3-space \mathbb{H}^3 generated by an element of order 5 and an element of order $p \geq 2$, along with the determination of all the associated nearly arithmetic groups. These groups Γ will have a presentation of the form

$$\Gamma \cong \langle f, g : f^5 = g^p = w(f, g) = \cdots = 1 \rangle$$

In this Thesis, we find that necessarily

- $p \in \{2, 3, 4, 5\}$
- The total degree of the invariant trace field

$$k\Gamma = \mathbb{Q}(\{tr^2(h) : h \in \Gamma\})$$

is at most 6 and at most 4 for lattices.

- Each orbifold is either a two bridge link of slope r/s surgered with $(5, 0)$, $(p, 0)$ orbifold Dehn surgery or a Heckoid group with rational slope $r/s \in [0, 1]$ and $w(f, g) = (w_{r/s})^r$ with $r \in \{2, 3, 4, 5\}$, and $w_{r/s}$ is a Farey word - described later.

For each such group, we find a discrete and faithful representation in $PSL(2, \mathbb{C})$, identify the rational slope r/s and identify the associated number theoretic data.

Chapter 1

Introduction

In this introductory chapter we give a brief history of the subject and then introduce some elementary results, such as the classification of elements and their geometric descriptions, before moving on to matters which will concern us. These include introducing the complex parameters which determine a group generated by two elements up to conjugacy and will underpin the arithmetic description of the groups we are looking for. Various symmetries of these parameter spaces (which are concrete realisations of deformation spaces when restricted to the class of discrete groups) are discussed. Using these symmetries will enable us to search for groups as they effectively reduce the search space four-fold. Ultimately we will need to understand pictures such as Figure 1.1 which is a one complex dimensional slice through the deformation space of discrete groups with two generators and also to understand the terms in its description we give below. In this case, the generators have orders of 4 and 5. The lines in Figure 1.1 are *pleating or bending rays* (as

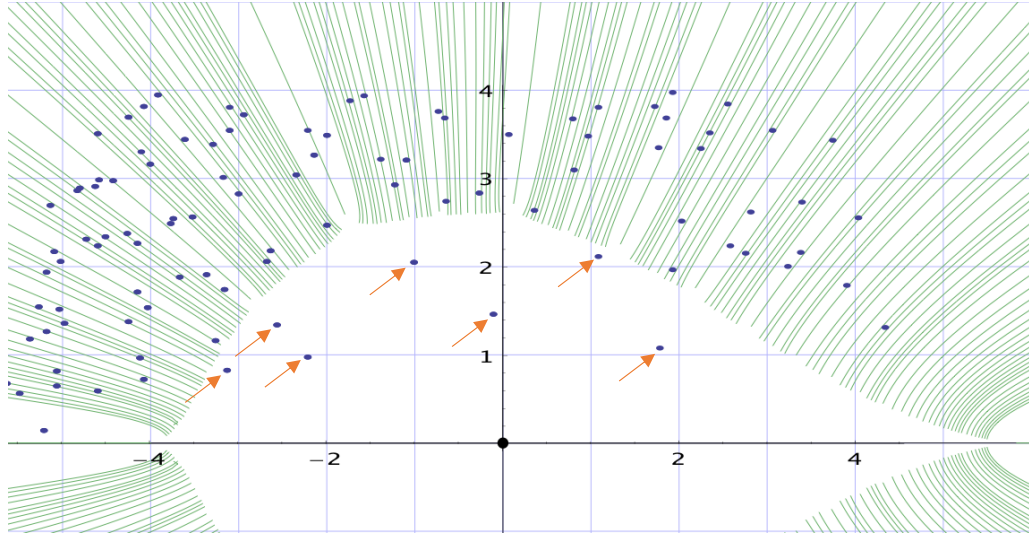


Figure 1.1: A concrete realisation of the set of Kleinian groups generated by elements of order 4 and 5

identified by Keen and Series in their studies of the Riley slice) and land at a dense set of the boundary at *cusps* groups by McMullen’s theorem. This describes the free part of this space and represents a concrete description of the quasiconformal deformation space of the abstract group $\mathbb{Z}/5\mathbb{Z} * \mathbb{Z}/4\mathbb{Z}$ freely generated by elements of order 4 and 5. The points illustrated in Figure 1.1 are Kleinian groups that satisfy the arithmetic criteria we develop which imply that they are subgroups of arithmetic groups and which are not “obviously” free as they do not have disjoint isometric circles.

However, most of the points we find are still freely generated and so we must implement additional known results about approximations to the complement of the non-free part. We therefore develop a method to certify that these groups are free. We are lucky in that we can obtain this certification – it did not have to be like

that as some of these points may well have been very close to the boundary (or even on the boundary as happens in the two parabolic generator case) and therefore may have defeated a computational approach which cannot describe the geometrically infinite groups which constitute most of the boundary. Next, the points identified by arrows are groups that are (apparently) not freely generated and are subgroups of arithmetic groups. We must then find and describe these groups in terms of a canonical relator (establishing they are not free), their arithmetic data and co-volume should it be finite – again it does not need to be as we find several interesting examples of web groups whose limit set in the Riemann sphere is a circle packing (similar to the Apollonian circles).

Therefore this chapter also contains some basic facts about Kleinian groups, such as the “ping-pong” lemma which will give us a first cut at finding the groups we are interested in. This lemma is used to bound a larger region than that illustrated in Figure 1.1. We also discuss discreteness inequalities, such as Jørgensen’s inequality, which we will use to cut down the size of some of the search spaces we consider.

1.1 What is hyperbolic geometry, and why study it?

Hyperbolic (non-Euclidean) geometry studies geometry on spaces of constant negative curvature. In dimension 2, surfaces of constant curvature are distinguished by whether their curvature K is positive, zero, or negative. If the space is complete and simply connected and if $K > 0$, then it is a sphere (with $K = 1/\text{radius}$); if $K = 0$, it is the Euclidean plane; and if $K < 0$, it is the hyperbolic plane, also

called 2-dimensional hyperbolic space. Such surfaces look the same at every point and in every direction and so ought to have lots of symmetries. The geometry of the sphere and the plane are familiar; however, hyperbolic geometry is the more general case. Hyperbolic space has many interesting features; some are similar to Euclidean geometry but some are quite different. In particular, hyperbolic space has a wealthy group of isometries, allowing various crystallographic symmetry patterns. This makes the geometry both rigid and flexible at the same time. Its properties and symmetries are closely related to tree-like growth patterns and fractals. We recount some of the basic history here. There are of course much fuller accounts that we leave the reader to explore, we are more interested in the geometric account, (e.g. [61]).

1.1.1 Historical importance

Another way to describe hyperbolic geometry is to say that it is the geometry of space in which Euclid's parallel axiom fails. One way to state the parallel assumption is that for every line L , and point P not on L , there is a unique line L_1 through P which does not meet L , even if extended infinitely far in both directions. Historically, hyperbolic geometry had enormous importance. Attempts to deduce the parallel axiom from Euclid's other axioms led to many developments within Euclidean geometry. Realization slowly dawned that all attempts to conclude the parallel axiom failed because one could develop a consistent system of geometry that was not Euclidean [54], in which a similar assumption failed while the other hypotheses remained true. This had enormous scientific and philosophical impor-

tance because it showed that, mathematically, there was nothing absolute about Euclidean space.

1.1.2 Hyperbolic geometry and surfaces

Any topological surface can be endowed with a geometric structure. This means that one can find a metric on the surface that in small regions looks like one of the three above types of geometry – spherical, Euclidean and hyperbolic, and such that the ‘overlap maps’ (arising from the definition of a manifold) are isometries of the appropriate geometry. Thus, for instance, a torus carries a Euclidean structure. At the same time, as we know, every surface of negative Euler characteristic (particularly, every closed surface of genus at least 2) carries a hyperbolic structure. In fact, in all cases except the triply marked sphere, arising from the action of a triangle group on the hyperbolic plane, the surfaces in question have not one but many such metrics.

Putting a hyperbolic metric on a surface allows one to study many surface features precisely. This applies especially to studying diffeomorphisms on surfaces, where hyperbolic geometry is crucial. Revolutionary work by William Thurston in the 1980s opened up the possibility of a similar description of 3-manifolds [62]. He showed that many 3 dimensional manifolds are endowed with a natural geometrical structure of one of a few kinds, and it turns out that in a very precise way, the most common geometry is hyperbolic. Thurston conjectured (apparently in 1982) that any topological 3-manifold could be canonically cut into pieces based on the

topological information alone, such that each piece carries one of 8 unique geometries, among them hyperbolic. This geometrization conjecture has now been proved through the remarkable work of Perelman, [46, 51, 52, 53]. Thurston's insights led to a great revolution in the study of hyperbolic 3 manifolds, making it a significant and very active area of mathematics in which spectacular progress has been made in recent years.

1.2 Summary of main Result.

Below is a summary of our main results of this Thesis. The terminology used is explained in the body of text.

Theorem 1.2.1. *[One generator order 5] Let $\Gamma = \langle f, g \rangle$ be an arithmetic lattice generated by f of order 5 and g of order $p \geq 2$. Then $p \in \{2, 3, 4, 5\}$ and the degree of the invariant trace field*

$$k\Gamma = \mathbb{Q}(\{\text{trace}^2(h) : h \in \Gamma\})$$

is at most 6.

1. *If $p = 2$, then there are seventeen groups.*
2. *If $p = 3$, then there are three groups.*
3. *If $p = 4$, then there are two groups.*
4. *If $p = 5$, then there are nine groups.*

Further, each group has a unique generating pair for the given orders.

The groups appearing in Theorem 1.2.1 all fall into the pattern of Heckoid groups; the singular graph of \mathbb{H}^3/Γ in \mathbb{S}^3 is one of the following four types:

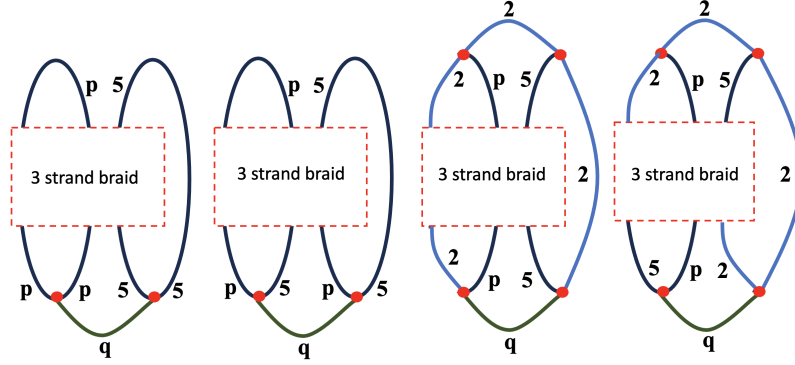


Figure 1. *The Heckoid orbifolds we encounter, $q \geq 1$. If $q = 1$, which does occur, we have a two-bridge link or two-bridge knot and must have $p = 5$ in that case. If $q \geq 2$ and $p \geq 6$, we always have a hyperfinite vertex (associated with a triangle subgroup) and the orbifold cannot have finite volume, so these do not occur (as lattices). This is discussed in more detail below.*

More precise information, including the arithmetic data, a co-volume approximation and a nontrivial relator are given in the tables of §§2.3 – 2.6 below. In fact these tables include all the two generator groups, generated by an element of 5 and $p \geq 2$ which is not freely generated by these two elements. For these groups there are no more values of p than those found in the lattice cases, but they can have degree up to 6 for the invariant trace field. However in order to present all this information we must introduce some notation, and terminology. In the body of the text we explicitly identify these groups and their Orbifold quotient.

1.2.1 Kleinian groups.

Kleinian groups are discrete subgroups of $PSL(2, \mathbb{C})$ - the group of all 2×2 matrices with complex entries and determinant one and defined up to sign. It's well known that the isomorphism classes of Kleinian groups Γ that don't contain elliptics elements are in one to one correspondence with hyperbolic 3-manifolds given as quotients \mathbb{H}^3/Γ . We are interested in those groups that have finite co-volume, that is the orbit space \mathbb{H}^3/Γ has finite volume. Arithmetic Kleinian groups are an important family of finite co-volume Kleinian groups. Very roughly these groups "look like" lattices of $n \times n$ groups of matrices with integer entries $GL(n, \mathbb{Z})$ and up to finite index. The reader is referred to the excellent book of Maclachlan and Reid [39] for detailed definitions and explanations of arithmetic groups and their relationship to 3-dimensional hyperbolic geometry. We briefly present the following definition as it relates to our work.

1.2.2 Arithmetic groups

Let k be a number field with exactly one complex place and let A be a quaternion algebra over k which is ramified at all real places. Let ρ be a k -embedding of A into $M_2(\mathbb{C})$ and let \mathcal{O} be an order of A . Then a subgroup Γ of $PSL(2, \mathbb{C})$ is an arithmetic Kleinian group if it is commensurable with some $P_\rho(\mathcal{O}_1)$ where \mathcal{O}_1 are the elements of \mathcal{O} of norm 1.

Remark 1.2.2. The field k and the algebra A in the definition are not so mysterious.

In fact $k = k\Gamma$, the invariant trace field, and

$$\rho(A) = A\Gamma = \{\sum a_i \gamma_i, a_i \in k\Gamma, \gamma_i \in \Gamma^2\}.$$

Definition 1.2.3. Let A be a quaternion algebra over a number field k and let R be the ring of integers in k . An order $O \subset A$ is an **R – lattice** in A that contains

1. see [39]

Arithmetic lattices have profound connections to number theory, automorphic forms, and the Langlands program.

We now return to more mundane matters and describe some basic elements of the theory of Kleinian groups.

1.2.3 Classification of elements

Let $\Gamma, \tilde{\Gamma} \subset Isom^+(\mathbb{H}^3)$ be two subgroups. We say that Γ and $\tilde{\Gamma}$ are conjugate if there is a Möbius transformation (or isometry) f such that

$$\tilde{\Gamma} = f^{-1} \circ \Gamma \circ f. \tag{1.2.1}$$

Most things of interest in the geometry of discrete groups are conjugacy invariants, and in effect, the conjugating map merely changes coordinates by an isometry in the quotient space. Thus we will almost always only be interested in information up to conjugacy.

The relationship between the matrix group $PSL(2, \mathbb{C})$ and the group of hyper-

bolic isometries is effected by the relationship

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2, \mathbb{C}) \leftrightarrow f(z) = \frac{az + b}{cz + d}, \quad ad - bc = 1$$

Where the linear fractional transformation f are the boundary values on the sphere at infinity of a unique hyperbolic isometry, see Beardon [7]. Without always saying so we use this identification and speak of $PSL(2, \mathbb{C})$ as the (orientation preserving) isometry group of hyperbolic 3-space.

The trace is a natural conjugacy invariant of matrices on $SL(2, \mathbb{C})$, but unfortunately, it is not well defined on $PSL(2, \mathbb{C})$. It is well defined up to sign. Therefore we make the following definition. For

$$f(z) = \frac{az + b}{cz + d}, \quad ad - bc = 1 \tag{1.2.2}$$

we set $tr(f) = a + d$ and set

$$\beta(f) = tr^2(f) - 4 \tag{1.2.3}$$

Then $\beta : Isom^+(\mathbb{H}^3) \rightarrow \mathbb{C}$ is a well defined conjugacy invariant of elements. We note the following easy consequences of the same results for matrices. $\beta(Id) = 0, \beta(f) = \beta(f^{-1}), \beta(g \circ f \circ g^{-1}) = \beta(f), \beta(g \circ f) = \beta(f \circ g)$. Given a linear fractional transformation, we can solve the fixed point equation $f(z) = z$, to find $cz^2 + (d - a)z - b = 0$ and the solutions are

$$z = \frac{q}{2c}((a - d) \pm \sqrt{(d - a)^2 + 4bc}), c \neq 0$$

$$z = \infty, -b/(a - 1/a), c = 0, a \neq 1$$

$$z = \infty, c = 0, a = d = 1.$$

The discriminant here is

$$(d - a)^2 - 4bc = a^2 + d^2 - 2ad + 4bc = (a + d)^2 - 4 = \beta(f).$$

If a linear fractional transformation f has two fixed points, say z_0, z_1 , then we can choose another linear fractional transformation g with $g(z_0) = 0$ and $g(z_1) = \infty$, (if there is one fixed point z_0 , then choose $g(z_0) = \infty$ and follow these arguments). Now $h(z) = (g \circ f \circ g^{-1})(z)$ fixes 0 and ∞ . Thus $h(z) = \lambda z, \lambda \in \mathbb{C} - \{0\}$. There is an obvious dichotomy in the dynamics of the map h acting on $\hat{\mathbb{C}}$ here. If $|\lambda| = 1$, then the group $\langle h \rangle$ is a compact group of rotations. This leads to the following definitions, for $f \neq \text{identity}$.

1. **Elliptic:** f is elliptic if $\beta(f) \in [-4, 0)$. In this case f is conjugate to a rotation $z \mapsto \eta z, |\eta| = 1$.
2. **Loxodromic:** f is loxodromic if $\beta(f) \notin [-4, 0]$. In this case f is conjugate to a dilation $z \mapsto \lambda z, 0 < |\lambda| \neq 1$.
3. **Parabolic:** f is parabolic if $\beta(f) = 0$, and $f \neq Id$. In this case f is conjugate to the translation $z \mapsto z + 1$.

A couple of $\beta(f)$ values are worth observing. When $\beta(f) \in [-4, 0)$, f is conjugate to a rotation. This rotation may have a certain period p the *order* of f and this order is denoted $o(f) = p$.

1. If $\beta(f) = -4$, then f is elliptic of order two.
2. If $\beta(f) = -3$, then f is elliptic of order three.

3. If $\beta(f) = -2$, then f is elliptic of order four.
4. If $\beta(f) = -1$, then f is elliptic of order six.
5. If $\beta(f) = -4 \sin^2(\frac{p\pi}{n})$, with $(p, n) = 1$, then f is elliptic of order n .

We further say that f is a primitive elliptic of order n if $\beta(f) = -4 \sin^2(\frac{\pi}{n})$, so both f and f^{-1} are simultaneously primitive or not.

1.3 The geometry of elements.

We now discuss the action of each of the elements in hyperbolic space. If $\beta(f) \neq 0$, then f has two fixed points, say z_1, z_2 . The hyperbolic line whose endpoints are z_1 and z_2 is called the axis of f , denoted $axis(f)$. The axis of f is clearly left invariant under the map f , $f(axis(f)) = axis(f)$ while under conjugacy it is elementary to see that

$$axis(g \circ f \circ g^{-1}) = g(axis(f)).$$

In fact, up to conjugacy, we could assume $z_1 = 0$ and $z_2 = \infty$. Then the axis would be the t -axis, $\{(0, 0, t) : t > 0\}$, and we would have f conjugate to $g : z \mapsto \lambda^2 z$. We put $\lambda^2 = re^{i\theta}$, $r \geq 1$ (note that g and g^{-1} are conjugate, so we can assume $r \geq 1$) and see that with $j = (0, 0, 1) \in \mathbb{H}^3$,

$$g(j) = r(j), \tau(g) = \rho_{\mathbb{H}^3}(j, g(j)) = \int_1^r \frac{dt}{t} = \log(r)$$

so g translates along its axis by a hyperbolic distance $\tau = \tau(g)$. This, of course, must be the same for f acting along its axis. We call this distance the translation

length and denote it by $\tau(f)$. As g translates along the t -axis, it also rotates around it by some angle (recall loxodromic means screwlike). This angle, $\theta = 2\text{Arg}(\lambda)$, is called the holonomy of g , denoted by $\theta(g)$. Of course this is a conjugacy invariant and $\theta(g) = \theta(f)$. Once we notice

$$g(z) = \lambda^2 z = r e^{i\theta} z \longleftrightarrow \begin{pmatrix} \sqrt{r} e^{\frac{i\theta}{2}} & 0 \\ 0 & \frac{e^{-\frac{\theta}{2}}}{\sqrt{r}} \end{pmatrix}$$

Therefore

$$\beta(f) = \beta(g) = \left(\sqrt{r} e^{\frac{i\theta}{2}} + \frac{e^{\frac{i\theta}{2}}}{\sqrt{r}} \right)^2 - 4 = -4 \sinh^2 \left(\frac{\tau}{2} + i \frac{\theta}{2} \right).$$

We record this as a lemma.

Lemma 1.3.1. *Let $f \in PSL(2, \mathbb{C})$ be a hyperbolic isometry and let $\tau_f = \tau(f)$ be the translation length of f along its axis, and $\theta_f = \theta(f)$ the holonomy of f about its axis. Then*

$$\beta(f) = 4 \sinh^2 \left(\frac{\tau_f}{2} + i \frac{\theta_f}{2} \right) = 2(\cosh(\tau_f + i\theta_f) - 1). \quad (1.3.1)$$

Knowing β identifies both $\tau(f)$ and $\theta(f)$, mod 2π .

1.4 Two-generator groups

We now define, for $f, g \in PSL(2, \mathbb{C}) \cong \text{Isom}^+(\mathbb{H}^3)$,

$$\gamma(f, g) = \text{tr}[f, g] - 2. \quad (1.4.1)$$

Here $\text{tr}[f, g] = \text{tr}(fgf^{-1}g^{-1})$ is defined unambiguously for $f, g \in PSL(2, \mathbb{C})$.

The parameters defined above conveniently encode various other geometric quantities. One of the most important is the complex hyperbolic distance $\delta + i\theta$ between

the axes of two non-parabolic Möbius transformations f and g . Here δ is the hyperbolic distance between the lines $axis(f)$ and $axis(g)$ and θ is the dihedral angle between the following. If α denotes the common perpendicular between $axis(f)$ and $axis(g)$, then one plane contains $axis(f)$ and α while the other contains $axis(g)$ and α . The easiest way to see the angle α is to use a conjugacy to arrange things so that the common perpendicular α lies on the t -axis. Then the angle α is simply the angle between the vertical projections to $\hat{\mathbb{C}} = \partial\mathbb{H}^3$ of $axis(f)$ and $axis(g)$ at the origin. Alternatively, θ is the holonomy of the transformation whose axis contains the common perpendicular and moves $axis(f)$ to $axis(g)$. If f and g are each elliptic or loxodromic, with translation lengths τ_f and τ_g , and holonomies η_f and η_g , then we have the following identities, [42].

$$\beta(f) = 4 \sinh^2\left(\frac{\tau_f + i\eta_f}{2}\right), \quad \beta(g) = 4 \sinh^2\left(\frac{\tau_g + i\eta_g}{2}\right) \quad (1.4.2)$$

$$\gamma(f, g) = \frac{\beta(f)\beta(g)}{4} \sinh^2(\delta + i\theta) \quad (1.4.3)$$

where $\delta + i\theta$ is the complex distance between the axes of f and g . From the previous equations, we derive the following useful formulas.

$$\cosh(\tau_f) = \frac{|\beta(f) + 4| + |\beta(f)|}{4} \quad (1.4.4)$$

$$\cosh(\eta_f) = \frac{|\beta(f) + 4| - |\beta(f)|}{4} \quad (1.4.5)$$

$$\cosh(2\delta) = \left| \frac{4\gamma(f, g)}{\beta(f)\beta(g)} + 1 \right| + \left| \frac{4\gamma(f, g)}{\beta(f)\beta(g)} \right| \quad (1.4.6)$$

$$\cos(2\theta) = \left| \frac{4\gamma(f, g)}{\beta(f)\beta(g)} + 1 \right| - \left| \frac{4\gamma(f, g)}{\beta(f)\beta(g)} \right| \quad (1.4.7)$$

We are often concerned with the case where one of the isometries, say g , is of order 2, in which case $\beta(g) = -4$, and (1.4.6) and (1.4.7) take the simpler form

$$\cosh(2\delta) = |1 - \gamma(f, g)/\beta(f)| + |\gamma(f, g)/\beta(f)| \quad (1.4.8)$$

$$\cos(2\theta) = |1 - \gamma(f, g)/\beta(f)| - |\gamma(f, g)/\beta(f)|. \quad (1.4.9)$$

Notice that for fixed $\beta(f) \in \mathbb{C}$ and fixed $\cosh(2\delta)$, the set of possible values for $\gamma(f, g)$ form an ellipse, while for fixed $\cosh(2\theta)$, we get hyperbola. Thus δ and θ give very appealing geometric orthogonal coordinates on $\mathbb{C} \setminus [\beta, 0]$.

1.4.1 Complex parameters for two generator groups.

The discussion above allows us to view the space of all two-generator Kleinian groups (up to conjugacy) as a subset of the three complex dimensional space \mathbb{C}^3 via the map

$$\langle f, g \rangle \rightarrow (\gamma(f, g), \beta(f), \beta(g)). \quad (1.4.10)$$

The following theorem is quite elementary, the proof simply amounting to solving equations after normalizing by conjugacy. However, the restriction $\gamma \neq 0$ cannot be removed.

Theorem 1.4.1. *If $\gamma \neq 0$, then the complex triple (γ, β, β') uniquely determines a two-generator Möbius group $\langle f, g \rangle$ up to conjugacy. Further, we have $\beta(f) = \beta$, $\beta(g) = \beta'$ and $\gamma(f, g) = \gamma$.*

It is worth recording the following elementary but valuable facts concerning the exception $\gamma \neq 0$ in Theorem 1.4.1.

Lemma 1.4.2. *If f and g are Möbius transformations and if $\gamma(f, g) = 0$, then f and g share a common fixed point in the Riemann sphere $\hat{\mathbb{C}}$. In particular, if $\langle f, g \rangle$ is discrete and $\gamma(f, g) = 0$, then $\langle f, g \rangle$ is virtually abelian (and so elementary).*

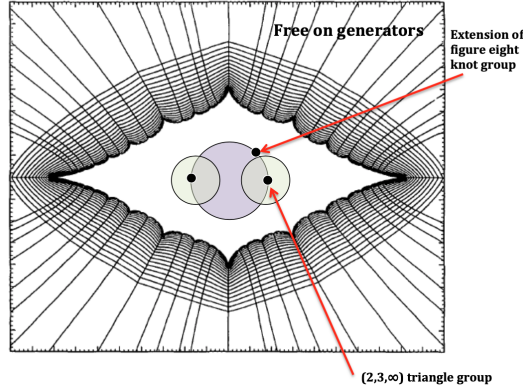


Figure 1.2: The Riley Slice

A fundamental problem is describing the space of all points in \mathbb{C}^3 corresponding to discrete two-generator groups. In this thesis, we will describe portions of this space, usually slices of co-dimension one or two. For instance, an important case will be the following. We fix p, q and let f and g be primitive elliptic of order p and q , respectively. Then any $\gamma \in \mathbb{C} \setminus \{0\}$ uniquely determines the conjugacy class of a two-generator group $\Gamma = \langle f, g \rangle$. For fixed p and q , it is an elementary consequence of a theorem of Jørgensen [33], which we outline below, that the set of all such γ values which correspond to discrete non-elementary groups are closed, and computer-generated pictures suggest that it is highly fractal. For instance, the well-known Riley slice groups generated by two parabolic generators, with our parameters, correspond to groups generated by an element of order $p = 2$ and a parabolic, $q = \infty$. The boundary of the space of such groups freely generated by two such elements is illustrated in Figure 1.2. This then is a picture of the slice $\beta(f) = 0, \beta(g) = -4, \gamma = \gamma(f, g) \in \mathbb{C}$. For γ outside the bounded region, the group Γ is freely generated by f , and g . The orbit space

$$F^2 = O(\Gamma)/\Gamma$$

is a marked sphere with two cone points of order two and two punctures. Here $O(\Gamma)$ is the *ordinary set* of Γ , the largest open region in $\hat{\mathbb{C}}$ on which Γ acts properly discontinuously.

We will find isolated γ values inside the bounded region, corresponding to γ values for discrete groups not freely generated by f and g . For instance, the value $\gamma = \frac{(1+i\sqrt{3})}{2}$ for the \mathbb{Z}^2 -an extension of the figure eight knot complement is identified in Figure 1.2. There we have also illustrated the unit disk in the centre; it is a consequence of the Shimitzu-Leutbecher inequality that there are no Kleinian groups with $|\gamma| < 1$. Polynomial trace identities extend this to further inequalities giving the two additional disks $|\gamma \pm 1| < r_0$, where $r_0^2(r_0+1) = 1$, and these additional disks also contain no Kleinian groups except for those γ values at their isolated centres $\gamma = \pm 1$ corresponding to the $(2, 3, \infty)$ triangle groups, and another extension of the $(3, 3, \infty)$ triangle group. Inside this region are infinitely many \mathbb{Z}_2 -extensions of the two-bridge knot, two-bridge links groups, as well as the $(2, p, \infty), p \geq 3$ triangle groups and other web groups.

Consider for a moment an example; f is parabolic and g elliptic of order 2 generating the $(2, p, \infty)$ triangle group. Then $h = fg$ is a primitive elliptic of order p . The axis of g and h are both perpendicular to a hyperplane on which a $(2, p, \infty)$ hyperbolic triangle is formed, so the angle between these axes is $\theta(g, h) = \theta = 0$. Next, a little hyperbolic trigonometry tells us that the distance $\delta = \delta(g, h)$ between the axis of g and h satisfies

$$\cosh(\delta) = \frac{1}{\sin(\frac{\pi}{p})}. \quad (1.4.11)$$

We see that.

$$\begin{aligned}\gamma(h, g) &= \frac{\beta(h)\beta(g)}{4} \sinh^2(\delta + i\theta) = -\beta(h) \sinh^2(\delta) \\ &= 4 \sin^2(\pi/p)(\cosh^2(\delta) - 1) = 4 - 4 \sin^2(\pi/p).\end{aligned}$$

Then

$$\gamma(f, g) = \gamma(fg, g) = \gamma(h, g). \quad (1.4.12)$$

This gives the infinite sequence of points

$$\{(4 - 4 \sin^2(\pi/p), 0, -4) : p \geq 3\} \quad (1.4.13)$$

lying in the bounded region of the space illustrated above. Notice that as $p \rightarrow \infty$, $4 - 4 \sin^2(\pi/p) \rightarrow 4$, a point lying in the boundary of this space, actually corresponding to the $(2, \infty, \infty)$ -triangle group which is free on these two-generators. We will see another of these spaces, and this generated picture in a bit when the generators are ordered two and five. Before that, we have to develop a bit more theory.

1.5 Projection to $(\gamma, \beta, -4)$

In what follows we will see a common feature or trying to describe the deformation spaces of discrete groups. That is the case where one generator has order two is hardest computationally. This is because all our bounds are weakest in that situation and the moduli space is largest (in a real sense). This case is, as we shall soon see, basically the same as the case that the generators have a common order. We therefore want to present some basic information about this case which will significantly cut down our searches. It generally does not apply in other cases,

however it does provide necessary conditions which we may use to eliminate (or find) specific examples.

This is an important projection from the three complex dimensional space of discrete groups (γ, β, β') to the two complex dimensional slices $\beta' = -4$, which preserves discreteness. This projection arises roughly because we can identify a closely related group that has one generator of order two. This goes as follows. First it shows that if f_1 and f_2 have the same trace, then there is an elliptic ϕ_1 of order two such that

$$f = \phi^{-1} \circ f_2 \circ \phi_1. \quad (1.5.1)$$

There are two such elliptic of order two, $\psi_i, i = 1, 2$, which are the two rotations of order 2 about the bisector of the common perpendicular between the axes. Then $g \circ f \circ g^{-1} = \phi_i \circ f \circ \phi_i^{-1} = 1, 2$ and it is easy to see that $\langle f, g \circ f \circ g^{-1} \rangle$ is an index-two subgroup of $\langle f, \phi_i \rangle, i = 1, 2$. We have not yet discussed discreteness closely, but that one group is of finite index in another will imply that if one group is discrete, so is the other. Next, we make a few simple calculations to see the following.

Group	Parameters
$\langle f, g \rangle$	$\leftrightarrow (\gamma, \beta, \beta')$
$\langle f, gfg^{-1} \rangle$	$\leftrightarrow (\gamma(\gamma - \beta), \beta, \beta)$
\cap	
$\langle f, \phi_i \rangle$	$\leftrightarrow (\gamma, \beta, -4), \text{ and } (\beta - \gamma, \beta, -4)$

We have the following lemma, which is very useful. For example, if we need to consider a loxodromic axis of f and its nearest translation, say under g . This gives us a group with two generators of the same trace f and gfg^{-1} . The two

groups the above constructions find are discrete and give the axial distances. So, with $\gamma = \gamma(f, g)$ we can go from $(\gamma, \beta(f), \beta(g))$ to $(\gamma, \beta(f), -4)$ and still retain the geometric information we might want.

Lemma 1.5.1. *Suppose (γ, β, β') are the parameters of a discrete group. Then so are $(\gamma, \beta, -4)$.*

$$(\gamma, \beta, \beta') \text{ discrete} \rightarrow (\gamma, \beta, -4) \text{ discrete.} \quad (1.5.2)$$

Further, away from a small finite set of exceptional parameters, this projection preserves the property of being non-elementary as well. This unique set occurs only in the presence of finite spherical and Euclidean triangle subgroups and motivates us to identify all the possible parameters for these groups. see [18]

1.6 Symmetries of the parameter space

A few natural symmetries of the parameter space will further cut down the region we have to investigate. The first obvious one is complex conjugation. Others can be deduced from the following two trace identities we often use. The first is well-known,

$$\text{tr}(fh) + \text{tr}(fh^{-1}) = \text{tr}(f)\text{tr}(h) \quad (1.6.1)$$

and the second is the Fricke identity

$$\text{tr}[f, h] = \text{tr}^2(f) + \text{tr}^2(g) + \text{tr}^2(fg) - \text{tr}(f)\text{tr}(g)\text{tr}(fg) - 2. \quad (1.6.2)$$

Thus

$$\gamma(f, g) = \beta(f) + \beta(g) + \beta(fg) - \text{tr}(f)\text{tr}(g)\text{tr}(fg) + 8 \quad (1.6.3)$$

and so if $g = hf^{-1}h^{-1}$, then

$$\begin{aligned}\gamma(f, hfh^{-1}) &= 2\beta(f) + ((\gamma(f, h) + 2)^2 - 4) - (\beta(f) + 4)(\gamma(f, h) + 2) + 8 \\ &= \gamma(f, h)(\gamma(f, h) - \beta(f)).\end{aligned}$$

We have already used this map $\gamma \mapsto \gamma(\gamma - \beta)$ above. Further, if g has order two, then $tr(g) = 0$ and $tr[f, g] - 2 = tr^2(f) + tr^2(fg) - 4$ and so

$$\gamma(f, g) = \beta(f) + \beta(fg) + 4. \quad (1.6.4)$$

This tells us that when g has ordered two

$$\beta(fg) = \gamma(f, g) - \beta(f) - 4. \quad (1.6.5)$$

Theorem 1.6.1. *Consider $\langle f^n, g \rangle \rightarrow \langle f, g \rangle$. Then*

$$\beta(f) = -4 \sinh^2\left(\frac{\tau + i\theta}{2}\right), \beta(f^n) = -4 \sinh^2\left(n\frac{\tau + i\theta}{2}\right) \quad (1.6.6)$$

and

$$(\gamma(f^n, g), \beta(f^n)) = \frac{\beta(f^n)}{\beta(f)}(\gamma, \beta). \quad (1.6.7)$$

That is

$$(\gamma, \beta) \mapsto T_n(\beta)(\gamma, \beta) \quad (1.6.8)$$

where S_n is a shifted Chebychev polynomial,

$$zS_n(z) = 2T_n\left(\frac{z}{2} + 1\right) - 2 \quad (1.6.9)$$

and T_n is the usual Chebychev polynomial, $T_n(\cosh(z)) = \cosh(nz)$.

1.7 Schottky groups and a combination theorem.

The first simple examples one meets of discrete non-elementary groups - perhaps after meeting Fuchsian groups are the Schottky groups. Here the limit set will be a

Cantor set in $\hat{\mathbb{C}}$, and every element in the group will be loxodromic. Let $f(z) = \frac{az+b}{cz+d}$. A natural way to construct these groups is through pairings of isometric circles. The isometric circles of f are

$$\{z \in \mathbb{C} : |cz + d| = 1\}, \{z \in \mathbb{C} : |cz - a| = 1\} \quad (1.7.1)$$

and f pairs these circles. Of course, the isometric circles of f and f^{-1} are the same. Let $U_1 = \{z \in \mathbb{C} : |cz + d| < 1\}$ and $U_2 = \{z \in \mathbb{C} : |cz - a| < 1\}$. The terminology arises as $|f'(z)| = 1$ on these circles. Given another Möbius transformation g , we can find a similar pair of disks V_1 and V_2 bounded by the isometric circles of g . The following theorem is often called the "ping-pong" lemma, a name suggested by its proof. The picture is illustrated below. An isometric disk is the (finite) region

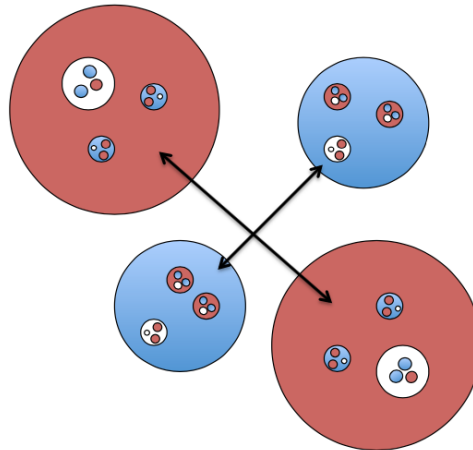


Figure 1.3: Ping- Pong picture for two elements

bounded by an isometric circle.

Theorem 1.7.1. *Suppose the four isometric disks U_1, U_2, V_1, V_2 are all disjoint. Then the group $\langle f, g \rangle$ is Kleinian and has a limit set a Cantor set. The group is algebraically isomorphic to the free group on two generators.*

This result is part of a much more general theory about the combinations of discrete groups. This includes free products and other amalgamations. This is explained in B. Maskit's book, [44]. We record the obvious generalization here.

Theorem 1.7.2 (Klein Combination Theorem.). *Let Γ_1 and Γ_2 be Kleinian groups acting on $\hat{\mathbb{C}}$ with fundamental domains U_1 and U_2 respectively. Suppose that*

$$\hat{\mathbb{C}}/U_2 \subset U_1, \hat{\mathbb{C}}/U_1 \subset U_2.$$

*Then the group $\langle \Gamma_1, \Gamma_2 \rangle$ generated by Γ_1 and Γ_2 is discrete and isomorphic to the free product $\Gamma_1 * \Gamma_2$.*

We want to use this theorem to describe various parameter spaces of discrete groups with elliptic generators. If f is elliptic with $f(\infty) \neq \infty$, then the two isometric circles of f intersect at the fixed points of f , and the exterior of the two bounded disks is a fundamental domain for f , as is the intersection of these two disks. Suppose Γ is a group generated by two primitive elliptic elements f, g of orders p, q . We normalise a choice of matrix representatives for the elements f, g as

$$f \sim \begin{pmatrix} \cos \pi/p & i \sin \pi/p \\ i \sin \pi/p & \cos \pi/p \end{pmatrix}, \quad g \sim \begin{pmatrix} \cos \pi/q & i\omega \sin \pi/q \\ i\omega \sin \pi/q & \cos \pi/q \end{pmatrix} \quad (1.7.2)$$

Here ω is a complex parameter which, by further conjugation, if necessary, we can assume lies in the unit disk, $|\omega| \leq 1$. Next, according to our discussion about fundamental domains for $\langle f \rangle$ and $\langle g \rangle$ and the Klein combination theorem, if the isometric circles of g lie inside the region bounded by the intersection of the isometric circles for f , then the group $\langle f, g \rangle$ is discrete and isomorphic to a free product $\langle f \rangle * \langle g \rangle \cong \mathbb{Z}_p * \mathbb{Z}_q$. Thus, with the normalization above if the inequality

$$|\sin(\pi/q) \cos(\pi/p) \pm \omega \cos(\pi/q) \sin(\pi/p)| + |\omega| \sin(\pi/p) \leq \sin(\pi/q) \quad (1.7.3)$$

holds for both choices of sign, then we have $G \cong \mathbb{Z}_p * \mathbb{Z}_q$. In particular, these hold if

$$|\omega| \leq \frac{\sin(\pi/p) \sin(\pi/q)}{((1 + \cos(\pi/p))(1 + \cos(\pi/q)))} \quad (1.7.4)$$

We then calculate that

$$\gamma(f, g) = (\omega - 1/\omega)^2 \sin^2(\pi/p) \sin^2(\pi/q). \quad (1.7.5)$$

A picture of the configurations of isometric circles for discrete groups freely generated by two elliptic is illustrated below in Figure 1.4 to get an idea.

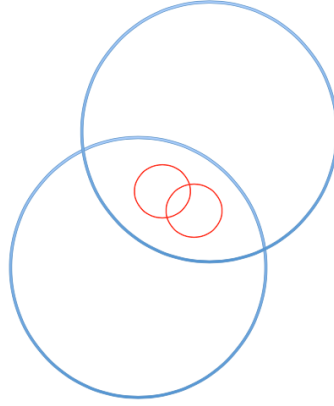


Figure 1.4: Two pairs of isometric disks, each pair lying in a fundamental region of the other. The group $\langle f, g \rangle$ is freely generated by f and g .

Now, putting these together, we obtain the following two results, [23].

Theorem 1.7.3. *Put*

$$\lambda_{p,q} = 4(\cos \pi/p + \cos \pi/q)^2 + 4(\cos \pi/p \cos \pi/q + 1)^2. \quad (1.7.6)$$

Let Γ be a Möbius group generated by primitive elliptics f and g of orders p and q respectively, $2 \leq p, q \leq \infty$ and not both equal to 2. Then with $\gamma = \gamma(f, g)$ the parameters for Γ are

$$(\gamma, -4 \sin^2(\pi/p), -4 \sin^2(\pi/q)).$$

Further, if γ lies outside of the open ellipse defined by the equation

$$\{z : |z| + |z + 4 \sin^2(\pi/p) \sin^2(\pi/q)| < \lambda_{p,q}\}, \quad (1.7.7)$$

then Γ is discrete and isomorphic to the free product of cyclic group. This result is sharp.

In particular, this theorem shows that Γ cannot be isomorphic to a finite co-volume lattice of $Isom^+(\mathbb{H}^3)$ when γ lies outside this ellipse. This is because a group of finite co-volume has Euler characteristic zero in three dimensions [62], so Γ cannot split into a free product of cyclic groups.

1.8 The space of Kleinian groups.

These days the first important universal constraint one meets when studying the geometry of discrete groups is Jørgensen's inequality [28]. This inequality is a generalization of the earlier Shimitzu-Leutbecher inequality, where one assumes that one of the generators is parabolic.

Theorem 1.8.1. (*Jørgensen's inequality*). *Let $\langle f, g \rangle$ be a Kleinian group. Then*

$$|\gamma(f, g)| + |\beta(f)| \geq 1. \quad (1.8.1)$$

This inequality is sharp

We give two important applications of Jørgensen's inequality.

Theorem 1.8.2. *A non-elementary subgroup Γ of $PSL(2, \mathbb{C})$ is discrete if and only if every two-generator subgroup is discrete.*

This basic argument is used quite a lot, as we will see. A sequence of n -generator Kleinian groups $\Gamma_i = \langle g_1^i, g_2^i, \dots, g_n^i \rangle$ is said to converge algebraically to $\Gamma = \langle g_1, g_2, \dots, g_n \rangle$ if for each $k = 1, 2, \dots, n$ we have $g_k^i \rightarrow g_k$. This convergence will be uniform in the spherical metric of the Riemann sphere $\hat{\mathbb{C}}$.

Theorem 1.8.3. *The space of n -generator discrete non-elementary groups is closed in the topology of algebraic convergence.*

The same arguments establish the following theorem.

Theorem 1.8.4. *Suppose that $\Gamma_i = \langle g_1^i, g_2^i, \dots, g_n^i \rangle$ converge algebraically to $\Gamma = \langle g_1, g_2, \dots, g_n \rangle$ as $i \rightarrow \infty$. Then the map back is an eventual homomorphism. That is for all sufficiently large i the map $\Gamma \rightarrow \Gamma_i$ given by $g_j \mapsto g_j^i$ extends to a homomorphism of the groups.*

We leave it to the reader to develop these results into theorems about the space of complex parameters. For instance

Theorem 1.8.5. *Let $(\gamma_i, \tilde{\beta}_i, \beta_i)$ be a sequence of parameters for discrete non-elementary (Kleinian) groups. Suppose that $\gamma_i \rightarrow \gamma$, $\beta_i \rightarrow \beta$, and $\tilde{\beta}_i \rightarrow \tilde{\beta}$. Then $(\gamma, \beta, \tilde{\beta})$ are the parameters for a Kleinian group.*

With a bit of work, one can deduce, using the eventual homomorphism back, the further implication that if $\beta = -4 \sin^2(m\pi/n)$ and $\tilde{\beta} = -4 \sin^2(\tilde{m}\pi/\tilde{n})$, (generators of finite order), then for all sufficiently large I , $\beta_i = \beta$ and $\tilde{\beta}_i = \tilde{\beta}$. This result implies that the spaces of Kleinian groups we investigate are closed.

Chapter 2

The $(5, p)$ -arithmetic hyperbolic lattices: Candidates

This chapter, and the two subsequent ones, contains the bulk of the new material in this thesis. The basic strategy for identifying the lattices in question remains similar to those we used in [43] with a few added complexities. However recent advances in computational number theory around provable results on discriminants and the degree of fields, which we discuss below, have made the problem accessible in the sense that the searches we are reduced to (basically searching through families of polynomials with integer coefficients) now only take between a few minutes and a couple of hours on an average desktop machine (we used Mathematica on a Mac.) as we can obtain stronger total degree bounds. Thus we identify, up to conjugacy, all of the finitely many arithmetic lattices generated by two elements of finite order 5 and p . We also allow $p = \infty$ to denote a parabolic generator.

2.1 What is known to date.

In two dimensions, a lattice in $Isom^+(\mathbb{H}^2)$ generated by two elements of finite order or parabolic is always a Fuchsian triangle group, and Takeuchi [60, Theorem 3] has identified all 82 of them. Here is a rundown of what is known in dimension three.

2.1.1 Two parabolic elements.

This case yields exactly 4 arithmetic lattices. Torsion is not present in any of these knots, and link complements. The explicit groups are listed in [24].

2.1.2 A parabolic element with an elliptic of order p , $2 \leq p < \infty$.

This case yields precisely 14 arithmetic lattices. Further $p \in \{2, 3, 4, 6\}$ is required. Six groups have $p = 2$, three have $p = 3$, and three have $p = 4$. The explicit groups are listed in [15].

2.1.3 An elliptic of order 4 and an elliptic of order $2 \leq p < \infty$.

There are exactly 97 arithmetic lattices generated by an elliptic element of order 4 and an elliptic of order $2 \leq p < \infty$. Necessarily $p \in \{2, 3, 4, 5, 6\}$. Fifty-four groups have $p = 2$, ten have $p = 3$, twenty-seven have $p = 4$, one has $p = 5$, and five have $p = 6$. The explicit groups can be found in [43].

2.1.4 Two elliptics of orders p and q with $p, q \geq 6$.

There are exactly 16 arithmetic lattices in this case and each case, $p = q$. Twelve groups have $p = q = 6$, and two have $p = q = 12$. The explicit groups can be found in [33].

2.2 The $(5, p)$ case.

The reader can see from the information above that with our result (summarised in the following theorem), one situation remains to deal with to complete the identification. That is, when one generator is elliptic with order 3 and $p = 2, 3$ or $p \geq 7$. We will address this problem in subsequent work – the most difficult case $p = 2$, where there are very many groups, is largely covered by the earlier PhD theses of Zhang and Cooper, [66, 13].

We recall here that a nearly arithmetic group is a discrete subgroup of an arithmetic group that is not free on its generators.

Theorem 2.2.1. *Let $\Gamma = \langle f, g \rangle$ be a nearly arithmetic group generated by f of order 5 and g of order $p \geq 2$, then $p \in \{2, 3, 4, 5, \infty\}$, and the degree of the invariant trace field $[\mathbb{Q}(\gamma) : \mathbb{Q}] \leq 4$.*

We will identify which of the nearly arithmetic groups are in fact arithmetic, that is they are finite co-volume lattices, in the tables below.

Theorem 2.2.2. *There are 12 arithmetic Kleinian groups (up to conjugacy) generated by two elliptic of order 5 and p .*

With this information on the degree and p , it is fairly straightforward to give the precise identification of the groups of Theorem (2.2.1). This identification is through the application of the conjecture that the classification theorem of [4, 5] for groups generated by two parabolics also holds for groups generated by two

primitive elements of finite order. That is that a Farey word (defined below) is a relator. We do not prove this conjecture but apply it by enumerating the Farey words and checking if they are elliptic or the identity. This turns out to be an effective strategy as there are $\pi^2 n^2/10$ Farey words of length less than or equal to n , whereas there are roughly 4^n possible words.

Our results are found in the tables below. Here

1. $\gamma_{5,i}$ is the trace of the commutator $tr[f, g] - 2$ when f and g are primitive generating pair;
2. $(Farey, order)$ gives the rational slope of a Farey word $W_{r/s}$, defined below, in the (primitive) generators f, g , which is either elliptic or the identity, and the order is given;
3. $polynomial$ gives the minimal polynomial for $\gamma_{5,i}$;
4. $words$ gives the Farey word;
5. $Disc.k\Gamma$ is the discriminant of the invariant trace field

$$k\Gamma = \mathbb{Q}(tr(h^2) : h \in \Gamma)$$

6. co-volume gives the volume of the quotient $\mathbb{H}^3/\Gamma_{5,i}$ to four decimal places.

Note that $\gamma = \gamma_{5,i}$ is an invariant of the Nielsen class of the generating pair f, g , and as we will see, arithmeticity implies that $k\Gamma = \mathbb{Q}(\gamma)$, if γ is not real. Note that γ real occurs in the case $p = 4$.

It is simple to generate a representation of the group in $PSL(2, \mathbb{C})$ uniquely up to conjugacy using the order of the generators and γ , [22]. The arithmeticity criteria we discuss below resolve the complicated general problem of demonstrating discreteness. As stated, column (*Farey, order*) shows that a particular word in the group $\langle f, g \rangle$ has a specific order (or is the identity). This term can be found in the following contexts. The *slope* of the simple closed curves on the 4 times punctured sphere \mathbb{S}_4 separating one pair of points from another is used to enumerate them. The Keen-Series-Maskit deformation theory [29, 31] then connects this slope (and a geodesic in the homotopy class) to a bending deformation along this geodesic of \mathbb{S}_4 that terminates on the boundary of the deformation space when the length of this curve falls to zero (so the associated word becomes parabolic).

When $tr(W_{r/s}(f, g)) = -2 \cos(2\pi/n)$ the cone angle along the particular set becomes $2\pi/n$ for a positive integer n , this bending yields cone manifolds, some of which turn out to be discrete lattices. Recent results of [4, 5] demonstrate that this process can characterize any discrete and faithful representations of groups created by two parabolic generators. We anticipate this conclusion for groups formed by two finite-order elements that are not free on their generators as well, and this, of course, motivated our approach. The deformation theory we explain elsewhere is

used to obtain a computational description of specific deformation spaces similar to the Riley Slice [29].

Chapter's Guide for the Reader

Each section in these chapters is split into some parts (which may correspond to one or more subsections): the first part of each section is a general introduction to some broad area important for that section (for example, hyperbolic knot theory, co-volume computation, Riley slice boundary).

1. Chapter 2 identifies a Kleinian group as a discrete group of Mobius transformations. It then identifies various arithmetic restrictions on the parameters (basically the traces of generators and commutators) for the $(5, p)$ group to be a non-elementary discrete subgroup arithmetic lattice. The main result here is the Identification Theorem 2.7.2. We then use this, with elementary arguments based on the combination theorems to find rough constraints on the possibilities for p and the total degree of the invariant trace field $k\Gamma = \mathbb{Q}(\{tr^2(h) : h \in \Gamma\})$. We then use some computational number theory results and other bounds to refine these rough bounds.
2. Chapter 3 initiates searches in each of the cases $p \in \{2, 3, 4, 5, 10\}$ to give a finite list of possible candidate values for $\gamma = tr[f, g] - 2$ when $\langle f, g \rangle$ is a discrete subgroup of an arithmetic Kleinian group generated by primitive elliptic elements of order p and 5.
3. Chapter 4 describes our methods to generate an approximation of the defor-

mation space (analogous to the Riley slice which we give a brief overview of) that is $\{\gamma \in \mathbb{C}; \langle f, g \rangle \text{ free}\}$. These are based on a generalisation of the Keen–Series description of the Riley slice [29] and other results generalising this to the elliptic case [18]. This is used as in [43] to decide whether a group we identify is free. This leaves us with a finite list of candidates which are not free and we then outline how to identify these groups efficiently. This is prefaced by a discussion of two bridge knots and links as the groups we identify are the very closely related Heckoid groups.

4. Chapter 5 runs quickly through the classical theory of Hyperbolic co-volume. Then explaining how the Borel system and Snap'Pea can give us co-volume bounds for alternating knots and Link complements.

This arrangement and the following flowcharts 2.1 give some flexibility regarding how the reader might approach this Chapter.

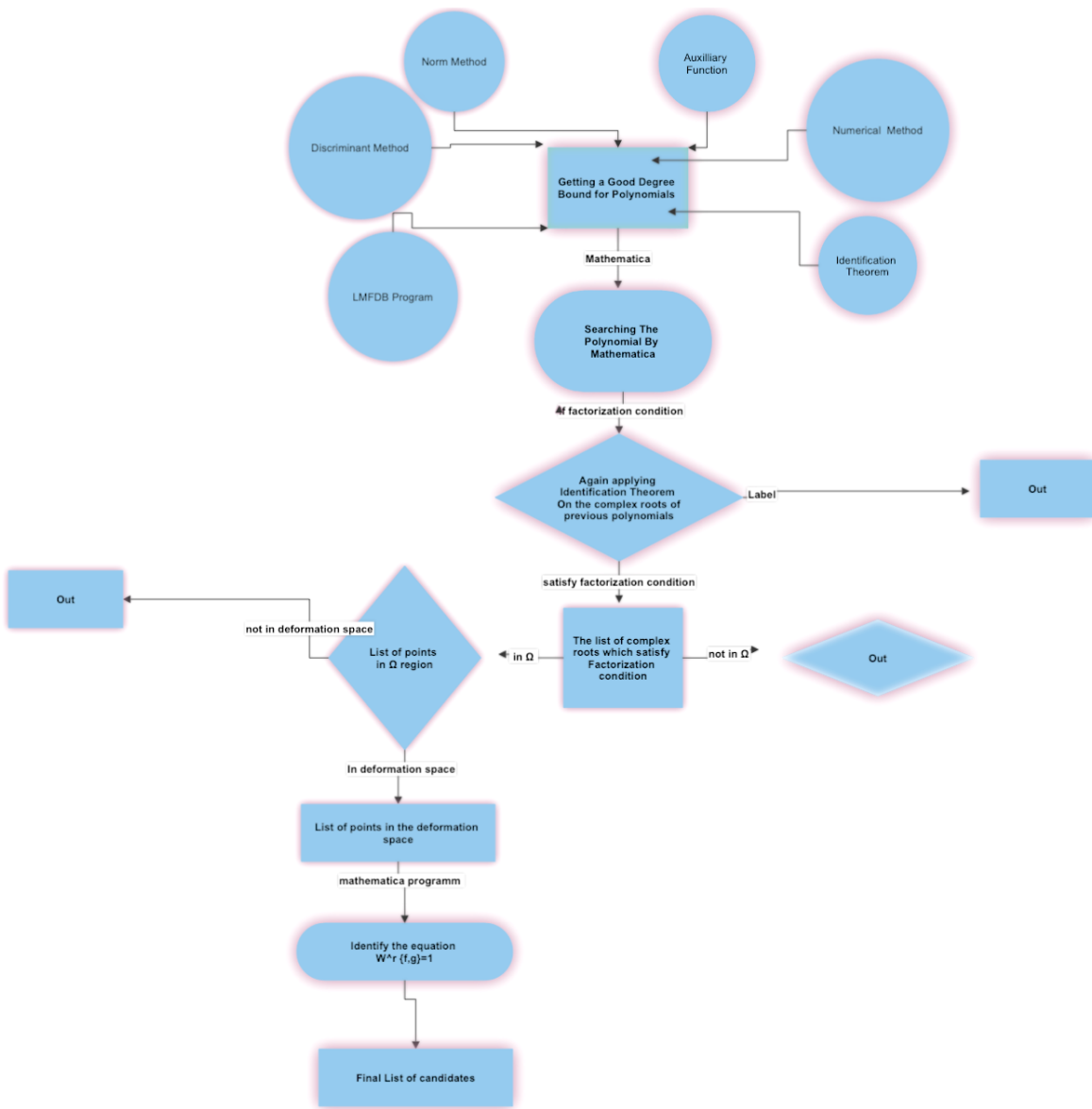


Figure 2.1: Chapter Flowcharts Guide

2.3 The (3, 5) arithmetic hyperbolic lattices.

No.	polynomial	(Farey,order)	γ	Disc. $k\Gamma$	covol.
1	$31 + 48z + 28z^2 + 8z^3 + z^4$	(19/30; 1)	$-2. + 2.0581i$	-400	4.593288
-	$1 + 4z + 2z^2 + z^3 + z^4$	(1/4; 3)	$0.3090 + 1.5882i$	-275	0.0781
-	$1 + 6z + 7z^2 + 4z^3 + z^4$	(1/3; 5)	$-1. + 1.2720i$	-400	∞
-	$5 + 5z + z^2$	$GT(3, 5; 5)$	-3.61803	-1375	∞
-	$1 + 3z + z^2$	$Tet(3, 5; 5)$	-2.61803	-1475	∞
-	$1 + 17z + 18z^2 + 7z^3 + z^4$	(2/5; 5)	$-2.3090 + 1.2339i$	-775	∞
-	$1 + 8z + 17z^2 + 6z^3 - 3z^4 + z^5 + z^6$	(1/5; 5)	$1.5343 + 1.5180i$	-104875	0.36288

The invariant trace field of $GT(3, 5; 5)$ is generated by the complex root of $z^4 - 3z^3 + 109z^2 + 518z + 556$ and of $Tet(3, 5; 5)$ by the complex root of $241 + 275z + 83z^2 + 5z^3 + z^4$. $GT(3, 5; 3)$ is not a subgroup of an arithmetic group as it's invariant trace field has two complex places. $Tet(3, 5; 3) \cong A_5$ and is not Kleinian.

2.3.1 Notes on the table.

- The first entry is $\gamma = -2. + 2.0581i$ with data (19/30; 1) is an orbifold Dehn surgery on the link $19/30 = [1, 1, 1, 2, 1, 2]$ as a continued fraction. This orbifold is therefore of finite co-volume and the co-volume value is provided by SnapPy. The complex distance between these axes is $\delta + i\theta = 1.14225 + 1.0925i$.
- The next entry is $\gamma = 0.3090 + 1.5882i$ with data (1/4; 3). The complex distance between the axes is $\delta + i\theta = 0.9726 + 0.5535i$. The polynomial factors

as

$$\left(z^2 + \frac{1 - \sqrt{5}}{2}z + \frac{3 + \sqrt{5}}{2}z^2\right)\left(z^2 + \frac{1 + \sqrt{5}}{2}z + \frac{3 - \sqrt{5}}{2}z^2\right)$$

We observe the invariant trace field is the same as the $3-5-3$ Coxeter group and so we might expect to identify this entry as a subgroup of that group.

Put

$$X = \begin{bmatrix} e^{\frac{i\pi}{3}} & 0 \\ \mu & e^{-\frac{i\pi}{3}} \end{bmatrix}, \quad Y = \begin{bmatrix} e^{\frac{i\pi}{5}} & 1 \\ 0 & e^{-\frac{i\pi}{5}} \end{bmatrix}, \quad x = X^{-1}, y = Y^{-1} \quad (2.3.1)$$

We choose $\mu = 2.32709 + 0.606658i$, and set $W = w_{1/4}$ and $w = W^{-1}$. Then $\text{Trace}W = 1$. We then set $Z = xWX$ and $z = Z^{-1}$. $Z^3 = -Id$. In what follows we work in $PSL(2, \mathbb{C})$ and so write this equation as $Z^3 = 1$. Then calculate that

$$(XZ)^2 = (YZ)^2 = (xZ)^2 = (yZ)^5 = [Y, Z]^5 = [X, Z]^2 = 1.$$

Thus $\langle Y, Z \rangle \cong A_5$ and $\langle X, Z \rangle \cong A_4$ are two spherical triangle groups on a common elliptic axis of order 3, see [22] for more data here. This is now enough to identify this group as a cocompact Heckoid group and in fact another presentation of $Tet(3, 3; 5)$.

- Here we present an alternative argument to identifying the group, more precisely the orbifold, in question. As noted above it works well when the denominator of the slope is small (no more than 5) as it considers all solutions to the polynomial equations. The entry is $\gamma = -1. + 1.2720i$ with data $(1/3; 5)$. The complex distance between the axes is $\delta + i\theta = 0.842476 + 0.955319i$. The polynomial factorises as

$$\left(z^2 + 2z + \frac{1}{2}(3 + \sqrt{5})\right)\left(z^2 + 2z + \frac{1}{2}(3 - \sqrt{5})\right)$$

There are two obvious candidates for the group. The $(1/3; 5)$ generalised triangle group, and the related Heckoid group. Both of these groups have

discrete and faithful representations in $PSL(2, \mathbb{C})$ by the orbifold theorem.

We have to show one of these groups is our group. With X, Y as above and

$W = X.Y.x.Y.X.y$, we compute

$$\begin{aligned} \text{trace}(W) &= \frac{1}{4}\mu\left(4\mu^2 - (\sqrt{5} + 3\sqrt{30 - 6\sqrt{5} + 1})\mu\right. \\ &\quad \left. - 3\sqrt{5} + \sqrt{6(\sqrt{5} + 5) + 19}\right) + (-1)^{7/15}(\sqrt[15]{-1} - 1) \\ [X, Y] &= \mu(\mu - \sqrt{3/2(5 - \sqrt{5})}) \\ [x.W.X.w] &= \mu^6 - 7.72648\mu^5 + 24.3651\mu^4 - 38.5069\mu^3 \\ &\quad + 30.1467\mu^2 - 9.61108\mu + 2. \end{aligned}$$

That $W^5 = 1$ gives us the four choices

$$W \in \left\{+2 \cos \frac{\pi}{5}, -2 \cos \frac{\pi}{5}, +2 \cos \frac{2\pi}{5}, -2 \cos \frac{2\pi}{5}\right\}.$$

The solution sets in order are

$$\mu \in \{0.827091, 1.20906, 1.82709\} \quad (2.3.2)$$

$$\mu \in \{-0.245872, 2.05456 \pm 1.22855i\} \quad (2.3.3)$$

$$\mu \in \{0.209057, 1.82709 \pm 0.78615i\} \quad (2.3.4)$$

$$\mu \in \{-0.0811417, 1.97219 \pm 1.07273i\} \quad (2.3.5)$$

The first set cannot occur as μ real gives us a $PSL(2, \mathbb{R})$ representation. The second set gives us the generalised triangle group (as W is primitive). We compute that the commutator parameter has minimal polynomial $-9 - 21z + 31z^2 + 46z^3 + 25z^4 + 7z^5 + z^6$ and that this group is not arithmetic (there are two complex places). Our group must be one of the two remaining candidates.

We compute that in the third and fourth cases

$$(x.W.X.w) = 1, (x.W.X.w) = 3$$

In the Heckoid group $(x.W.X.w)^3 = 1$, due to the A_5 vertex. So the Heckoid group has $\mu = 1.82709 + 0.78615i$ and we compute $\gamma(X, Y) = -1. + 1.2720i$ as we had hoped for. This group is not of finite co-volume due to the $(2, 5, 5)$ hyperfinite vertex.

- This next entry has $\gamma = -2.3090 + 1.2339i$ with data $(2/5; 5)$. The complex distance between the axes is $\delta + i\theta = 1.06127 + 1.2624i$. We quickly see $\langle X, W \rangle \cong A_5$ and that $\langle Y, W^2 \rangle$ is the $(5, 5, 5)$ triangle group as $(Y.W^2) = 2 \cos(\frac{\pi}{5})$. Further calculations reveal that $(w_{7/17}) = -1$. Finally the presence of the hyperfinite vertex assures us this group is Heckoid and not of finite co-volume.
- Let us now consider the last case in the $(3, 5)$ Table. We give yet another process to determine if the group we seek is a lattice. We have $\gamma = 1.5343 + 1.5180i$ is the complex root with positive real part of of

$$1 + 8z + 17z^2 + 6z^3 - 3z^4 + z^5 + z^6$$

and the complex distance between axes is $\delta + i\theta = 1.14241 + 0.323366i$. This degree 6 polynomial factors over $(\sqrt{5})$ as

$$\left(z^3 + \left(\frac{1}{2} - \frac{3\sqrt{5}}{2}\right)z^2 + 4z + 1\right) \left(z^3 + \left(\frac{1}{2} + \frac{3\sqrt{5}}{2}\right)z^2 + 4z + 1\right)$$

so one can easily check the root distribution condition of the Identification Theorem 2.7.3. Our relator is $w_{1/5} = W = X.Y.x.Y.x.Y.X.y.X.y$ and $W = 2 \cos \frac{2\pi}{5}$. Let X, Y be as at (2.3.1) with $\mu = 2.6848 + 0.45537i$. Then the group we wish to examine is conjugate to $\langle X, Y \rangle$. Put $A = W.X$ and $B = Y.x.W.X$. The group $\langle U, V \rangle$ identified as #38 in Table 3.8 of Cooper's Thesis [13] with U, V primitive elliptics of order three and two respectively,

and commutator value $\gamma^* = \gamma(U, V)$, γ^* complex and having minimal polynomial $z^6 + 8z^5 + 24z^4 + 35z^3 + 28z^2 + 12z + 1$ is identified as a lattice of co-volume $0.362887\dots$. The index two subgroups $\langle A, BAB^{-1} \rangle$ and $\langle U, VUV^{-1} \rangle$ are conjugate - they are both generated by primitive elliptics of order 3 and have the same commutator value $-1.91634 - 1.51803i$, [22]. Thus $\langle X, Y \rangle$ is a lattice. Using rewriting on Magma with the data above on A_5 subgroups shows $\langle A, B \rangle = \langle X, Y \rangle$. A bit of work shows this to be the Heckoid group $\langle x, y \rangle$ generated by two elements of order 3 with $(1/3; 5)$ and $(w_{1/3}) = -2 \cos \frac{2\pi}{5}$ and $\gamma(x, y)$ is the complex root of $1 + 11z + 37z^2 + 46z^3 + 27z^4 + 8z^5 + z^6$.

This last example has shown us that some of our groups may not be as expected, and that we might have identified a different generating pair for another Heckoid group. Fortunately this is rare as it is not easy to uncover.

2.4 The (4, 5) arithmetic hyperbolic lattices.

No.	polynomial	(Farey , order)	γ	Disc. $k\Gamma$	vol
1		$Tet(4, 5; 3)$	-1	-400	0.07177
2	$z^4 + 4z^3 + 7z^2 + 6z + 1$	(5/8; 3)	$-1. + 1.2720i$	-400	0.2870
3	$z^4 + 4z^3 + 14z^2 + 20z + 5$	(2/3; 4)	$-1. + 2.9106i$	-1600	∞

First, the cocompact lattice $Tet(4, 5; 3)$ has $\gamma = -1$. This group has presentation

$$\langle x, y : x^4 = y^5 = [x, y]^3 = (x[yx^{-1}])^2 = (x[y, x^{-1}]y)^2 = [y, x^{-1}]y^2 = 1 \rangle$$

and is the orientation preserving subgroup of the reflection group with Coxeter diagram $4 - 3 - 5$. The number field $k\Gamma$ has discriminant -400 . Next, with $\gamma = -1. + 2.9106i$, and X, Y as at (2.3.1) with the proviso that the entries $e^{i\pi/3}$ of X are $e^{i\pi/4}$, we compute that the $3/8$ word is

$$W = X.Y.x.y.X.y.x.Y.x.y.X.Y.x.Y.X.y$$

and that $(XW) = 0$, $[X, W] = 1$, so that $\langle X, W \rangle \cong S_4$. We then compute

$$\gamma(Y, W) = [Y, W] - 2 = -2 - 2.05817i,$$

with minimal polynomial $31 + 48z + 28z^2 + 8z^3 + z^4$. This group, generated by elements of order 3 and 5 is later identified as $(3, 0)$, $(5, 0)$, Dehn surgery on the link $19/30$ with data verified by SnapPy.

The last entry is a generalised triangle group - the trefoil with unknotting tunnel labelled 4 and bridges labelled 4 and 5. The precise values of traces at the vertex stabilisers are prescribed by Knapp's necessary and sufficient conditions on the traces of triangle groups, [?].

2.5 The nine (5, 5) arithmetic hyperbolic lattices.

No.	polynomial	(Farey,order)	γ	Disc. $k\Gamma$	covol.
1	$11 + 22z + 18z^2 + 7z^3 + z^4$	(3/5; 1)	$-1.1909 + 0.9815i$	-275	0.9372
2	$1 + z$	Tet(5,5;3)	-1	-475	0.5021
3	$11 + 28z + 23z^2 + 8z^3 + z^4$	(4/7; 1)	$-2 + 0.7861i$	-400	1.7224
4	$25 + 35z + 21z^2 + 7z^3 + z^4$	(5/8; 1)	$-1.1909 - 1.8925i$	-475	2.2398
5	$11 + 12z + 3z^2 + 2z^3 + z^4$	(2/7; 1)	$0.6180 + 2.0581i$	-400	2.8281
6	$1 + 10z + 32z^2 + 43z^3 + 26z^4 + 8z^5 + z^6$	(3/5; 3)	$-1.7653 + 1.5693i$	-254875	1.4007
7	$1 + 10z + 24z^2 + 23z^3 + 14z^4 + 6z^5 + z^6$	(1/3; 3)	$-0.3105 + 1.6058i$	-149875	0.62617
8	$1 + 8z + 20z^2 + 14z^3 - 4z^4 + z^5 + z^6$	(1/4; 3)	$1.6436 + 1.8277i$	-188875	0.9085
9	$1 + 3z + 3z^2 + 3z^3 + z^4$	(1/3; 5)	$-0.1909 + 0.9815i$	-275	0.07810

\mathbb{Z}_2 extensions of all of these groups we previously identified in [25] and their covolumes appear in Table $G_{5,i}$ from Cooper's thesis [13]. It had not been proven earlier that this list is complete, nor that the quotients had the orbifold structure described. This will be discussed later.

2.6 The (2, 5) arithmetic hyperbolic lattices.

All these groups arise as \mathbb{Z}_2 -extensions of groups generated by two elliptics of order 5 and therefore can be related to the previous table (just compare the discriminants).

No.	polynomial	(Farey,order)	γ	Disc. $k\Gamma$	covol.
1	$1 + 3z + 3z^2 + 3z^3 + z^4$	(1/5; 2)	$-0.1909 + 0.9815i$	-275	0.46860
2	$11 + 22z + 18z^2 + 7z^3 + z^4$	(3/10; 1)	$-1.1909 + 0.9815i$	-275	0.46860
3	$1 + 5z + 7z^2 + 5z^3 + z^4$	(1/4; 3)	$-0.6909 + 0.7228i$	-475	0.0933255
4	$11 + 14z + 12z^2 + 6z^3 + z^4$	(3/14; 1)	$-0.3819 + 1.2720i$	-400	0.86123
5	$1 + 6z + 7z^2 + 4z^3 + z^4$	(2/7; 2)	$-1. + 1.2720i$	-400	0.86123
6	$5 + 5z + 4z^2 + 4z^3 + z^4$	(3/16; 1)	$0.1180 + 1.169i$	-475	1.11927
7	$5 + 15z + 14z^2 + 6z^3 + z^4$	(5/16; 1)	$-1.5 + 1.1696$	-475	1.11927
8	$1 + 2z - 2z^2 + 2z^3 + z^4$	(1/7; 2)	$0.6180 + 0.7861i$	-400	0.861241
9	$11 + 28z + 23z^2 + 8z^3 + z^4$	(5/14; 1)	$-2. + 0.7861i$	-400	0.861241
10	$1 + z + 2z^2 + 4z^3 + z^4$	(3/14; 5)	$0.1180 + 0.6066i$	-275	0.03905
11	$1 + 9z + 12z^2 + 6z^3 + z^4$	(1/3; 5)	$-1.5 + 0.6066i$	-275	0.03905
12	$1 + 22z + 57z^2 + 59z^3 + 32z^4 + 9z^5 + z^6$	(3/10; 3)	$-1.30017 - 1.28803i$	-254875	0.700364
13	$1 + 8z + 17z^2 + 16z^3 + 12z^4 + 6z^5 + z^6$	(1/5; 3)	$-0.0817958 + 1.28803i$	-254875	0.700364
14	$1 + 13z + 42z^2 + 52z^3 + 31z^4 + 9z^5 + z^6$	(1/3; 3)	$-1.63475 - 0.850772i$	-149875	0.313085
15	$1 + 7z + 7z^2 + 8z^3 + 11z^4 + 6z^5 + z^6$	(1/6; 3)	$0.252783 + 0.850772i$	-149875	0.313085
16	$1 + 17z + 56z^2 + 68z^3 + 38z^4 + 10z^5 + z^6$	(3/8; 3)	$-2.2596 - 0.582598I$	-188875	0.4542714
17	$1 + 8z + z^2 - 8z^3 + 3z^4 + 5z^5 + z^6$	(1/8; 3)	$0.877637 + 0.582598I$	-188875	0.4542714

2.7 Total degree bounds.

Following on from [33], we explain the arithmetic conditions on the complex number $\gamma = \gamma(f, g)$ that are required for the group

$$\Gamma = \langle f, g : f^5 = g^p = \dots = 1 \rangle \quad (2.7.1)$$

to be a discrete subgroup of an arithmetic group. This is the Identification Theorem, which imposes tight requirements on $\gamma = \gamma(f, g)$. Next, for Γ to be a lattice in three dimensions, it must not be free on generators. The "disjoint isometric circles

test” and the Klein “ping-pong” lemma we have already discussed give crude upper bounds on $|\gamma|$ because if $|\gamma|$ are large, then the isometric circles of f and g are disjoint and $\langle f, g \rangle$ is freely generated by f and g .

The mathematical criterion and coarse bound, therefore, allow us to use the approach of [33] (also see Flammang and Rhin [21]) to achieve a total degree bound for the field $k\Gamma = \mathbb{Q}(\gamma)$. In particular, we ultimately prove the following theorem.

Theorem 2.7.1. *Assume that $\Gamma = \langle f, g \rangle$ is not freely generated by f of order 5 and g of order p and that Γ is a non-elementary discrete group which is a subgroup of an arithmetic Kleinian group. Set $\gamma = \gamma(f, g)$. Then $[\mathbb{Q}(\gamma) : \mathbb{Q}] \leq 6$.*

2.7.1 The standard representation

Define the two matrices

$$A = \begin{pmatrix} \cos \pi/p & i \sin \pi/p \\ i \sin \pi/p & \cos \pi/p \end{pmatrix}, \quad B = \begin{pmatrix} \cos \pi/q & iw \sin \pi/q \\ iw^{-1} \sin \pi/q & \cos \pi/q \end{pmatrix}. \quad (2.7.2)$$

Then if $G = \langle f, g \rangle$ is a non-elementary Kleinian group with $o(f) = p$ (where $o(f)$ denotes the order of f) and $o(g) = q$, where $p \geq q \geq 3$ then G can be normalized so that f, g are represented by the matrices $A, B \in PSL(2, \mathbb{C})$ respectively. The parameter γ is related to w by

$$\gamma = \sin^2 \frac{\pi}{p} \sin^2 \frac{\pi}{q} \left(w - \frac{1}{w}\right)^2. \quad (2.7.3)$$

Given γ , we can further normalise and choose w such that $|w| \leq 1$ and $\operatorname{Re}(w) \geq 0$.

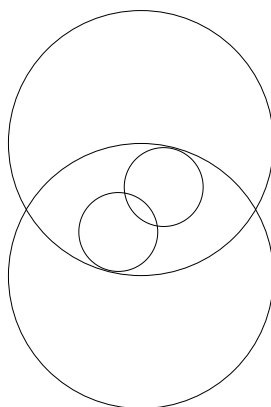


Figure 2.2: A pair of disjoint isometric circles with $p = 3$ and $w = \frac{1+i}{7}$

2.7.2 Isometric circles

The isometric circles of the Möbius transformations determined by (2.7.2) are

$$\left\{ z : \left| i \sin\left(\frac{\pi}{p}\right)z \pm \cos\left(\frac{\pi}{p}\right) \right|^2 = 1 \right\}, \quad \left\{ |iw^{-1} \sin\left(\frac{\pi}{q}\right)z \pm \cos\left(\frac{\pi}{q}\right)|^2 = 1 \right\}$$

These circles are paired by the respective Möbius transformations. With our normalization on w as above, these two pairs of circles are disjoint precisely when

$$|iw \cot \pi/q + i \cot \pi/p| + \frac{|w|}{\sin \pi/q} \leq \frac{1}{\sin \pi/p}. \quad (2.7.4)$$

We set $q = 5$ and want to describe this region in terms of γ for our later bounds.

We have

$$|w \cot \pi/5 + \cot \pi/p| + \frac{|w|}{\sin \pi/5} \leq \frac{1}{\sin \pi/p}. \quad (2.7.5)$$

As discussed earlier, see (1.7.2) and (1.7.3) the well-known Klein ping-pong argument quickly implies that should the isometric circles be pairwise disjoint or at

a worse tangent, the group generated by A and B is free on these two generators. It, therefore, can not be an arithmetic lattice. In fact in these circumstances (with $q = 5$ and $p \geq 3$) one can see from

$$\left[D\left(i, \cot \frac{\pi}{p}\right) \cap D\left(-i, \cot \frac{\pi}{p}\right) \right] \setminus \left[D\left(iw \cot\left(\frac{\pi}{5}\right), \frac{|w|}{\sin\left(\frac{\pi}{5}\right)}\right) \cup D\left(-iw \cot\left(\frac{\pi}{5}\right), \frac{|w|}{\sin\left(\frac{\pi}{5}\right)}\right) \right]$$

is a nonempty fundamental domain for the action of $\Gamma = \langle f, g \rangle$ on $\mathbb{C} \setminus \Lambda(\Gamma)$.

We now need to turn the inequality (2.7.5) into a condition on γ .

From (2.7.5) we see with $w = re^{i\theta}$, $0 \leq \theta \leq \pi/2$, that

$$\begin{aligned} |(r \cos(\theta) \cot \pi/5 + \cot \pi/p) + ir \sin(\theta) \cot \pi/5| + \frac{r}{\sin \pi/5} &\leq \frac{1}{\sin \pi/p} \\ (r \cos(\theta) \cot \pi/5 + \cot \pi/p)^2 + r^2 \sin^2(\theta) \cot^2 \pi/5 &\leq \left(\frac{1}{\sin \pi/p} - \frac{r}{\sin \pi/5} \right)^2 \end{aligned}$$

and the case of equality we want to find r as a function of θ . We have

$$5r^2 + 5 = 2\sqrt{5(2\sqrt{5} + 5)}r \cos(\theta) \cot\left(\frac{\pi}{p}\right) + 2\sqrt{10(\sqrt{5} + 5)}r \csc\left(\frac{\pi}{p}\right)$$

and hence

$$\begin{aligned} r &= \frac{1}{5} \left(\sqrt{5(2\sqrt{5} + 5)} \cos(\theta) \cot\left(\frac{\pi}{p}\right) \right. \\ &\quad \left. + \sqrt{\left(\sqrt{5(2\sqrt{5} + 5)} \cos(\theta) \cot\left(\frac{\pi}{p}\right) + \sqrt{10(\sqrt{5} + 5)} \csc\left(\frac{\pi}{p}\right) \right)^2 - 25} \right. \\ &\quad \left. + \sqrt{10(\sqrt{5} + 5)} \csc\left(\frac{\pi}{p}\right) \right) \end{aligned}$$

Then (2.7.3) gives us the value $\gamma = \sin^2 \frac{\pi}{p} \sin^2 \frac{\pi}{5} (re^{i\theta} - \frac{1}{re^{i\theta}})^2$ which defines the parametric equation bounding a region Ω_p . The regions Ω_p increase as $p \rightarrow \infty$.

This is illustrated below in Figure 2.3.

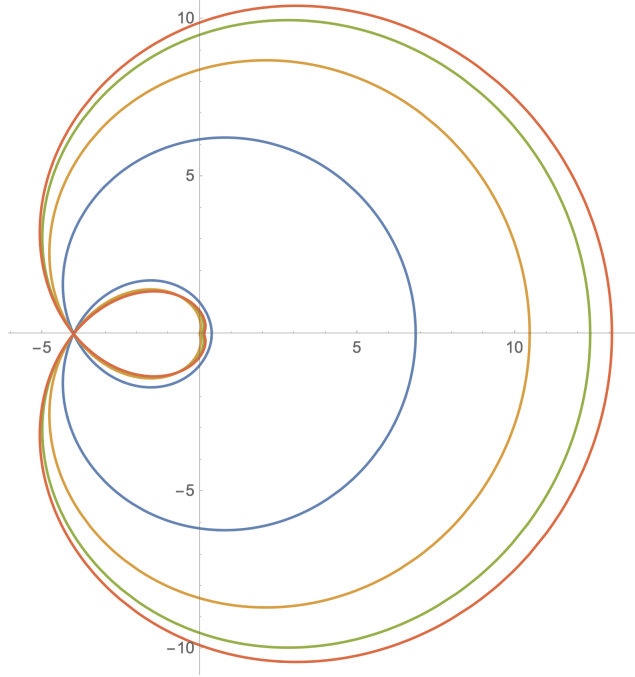


Figure 2.3: The region Ω_p . The exterior of this region contains the γ -values of groups which are discrete and free on generators because of the combination theorems.

The γ values for groups we seek are inside the curves Ω_p . However, as we will see, the regions Ω_p where isometric circles are disjoint are much larger than necessary to identify freely generated groups and generally provide poor bounds.

2.7.3 Real points

At this point we would like to remove the cases $\gamma \in \mathbb{R}$ from further consideration.

If $\gamma > 0$:

The group Γ is a Fuchsian triangle group or is freely generated, and, therefore not an arithmetic lattice. However, the $(5, p, q)$ - triangle groups $\langle a, b | a^5 = b^p = (ab)^q \rangle$ have parameters

$$\begin{aligned} (\gamma, \operatorname{tr}^2(f) - 4, \operatorname{tr}^2(g) - 4) &= (\gamma, -4 \sin^2(\frac{\pi}{5}), -4 \sin^2(\frac{\pi}{p})), \\ \gamma &= 4 \cos(\frac{\pi}{q}) (\sqrt{2} \cos(\frac{\pi}{p}) + \cos(\frac{\pi}{q})) + 2 \cos(\frac{2\pi}{p}) \end{aligned}$$

If $\gamma < 0$:

If $\langle f, g \rangle$ is not free on generators, then $-4 < \gamma < 0$. Otherwise, the circles are disjoint. It follows that $-2 < \operatorname{tr}[f, g] < 2$ and $[f, g]$ is elliptic. All groups generated by two elements of finite order and whose commutator has finite order along with their arithmeticity or otherwise were identified in [35]. As examples, the $(5, p, q)$ -generalised triangle groups $\langle a, b | a^5 = b^p = [a, b]^q \rangle$ have parameters

$$(\gamma, -4 \sin^2(\frac{\pi}{5}), -4 \sin^2(\frac{\pi}{p}))$$

However, there are other groups as well. which are arithmetic lattices.

2.7.4 Arithmetic restrictions on the commutator $\gamma(f, g)$

Having dispensed with the case that $\gamma = \gamma(f, g) \in \mathbb{R}$, we assume that γ is complex. We require the following preliminaries. Let G be any non-elementary finitely-generated subgroup of $PSL(2, \mathbb{C})$. Let $G^{(2)} = \langle g^2 \mid g \in G \rangle$ so that $G^{(2)}$ is a subgroup

of finite index in G . Define

$$\begin{aligned} kG &= \mathbb{Q}(\{tr(h) \mid h \in G^{(2)}\}) \\ AG &= \{\sum a_i h_i \mid a_i \in kG, h_i \in G^{(2)}\} \end{aligned} \tag{2.7.6}$$

where, with the usual abuse of notation, we regard elements of G as matrices, so that $A\Gamma \subset M_2(\mathbb{C})$. Then AG is a quaternion algebra over kG , and the pair (kG, AG) is an invariant of the commensurability class of G . If, in addition, G is a Kleinian group of finite co-volume, then kG is a number field.

We state the Identification Theorem of [33] as follows:

Theorem 2.7.2. *Suppose G is a subgroup of $PSL(2, \mathbb{C})$ which is finitely-generated and non-elementary. Then G is an arithmetic Kleinian group if and only if all the following conditions hold:*

1. kG is a number field with exactly one complex place,
2. for every $g \in G$, $tr(g)$ is an algebraic integer,
3. AG is ramified at all real places of kG .
4. G has finite co-volume.

It should be emphasized that the first three criteria imply that G is Kleinian, and in fact, G is a subgroup of an arithmetic Kleinian group. The first two requirements depend on the traces of elements of G . Furthermore, we may obtain a Hilbert symbol for AG in terms of the traces of elements of G so that the third condition is also dependent on the traces (for all this, see [39, Chap. 8]).

In our setting, the number field kG has an obvious real subfield, namely the field

$$L = \mathbb{Q}\left(\cos\left(\frac{2\pi}{5}\right), \cos\left(\frac{2\pi}{p}\right)\right) \quad (2.7.7)$$

We remind the reader that

$$\cos\left(\frac{2\pi}{5}\right) = \frac{1}{4}(-1 + \sqrt{5})$$

so that L has even degree over \mathbb{Q} . A proper subfield of a field with one complex place must be real, [39]. As we are now assuming $\gamma \in \mathbb{C} \setminus \mathbb{R}$ we must have $\mathbb{Q}(\gamma) = L(\gamma) = kG$.

We set

$$\begin{aligned} \beta_1 &= \text{tr}^2(f) - 4 = -4 \sin^2 \frac{\pi}{5} = \frac{1}{2}(-5 + \sqrt{5}) \\ \beta_2 &= \text{tr}^2(g) - 4 = -4 \sin^2 \frac{\pi}{p} \end{aligned}$$

We now discuss the ramification condition. A Hilbert symbol for the quaternion algebra can be found from any non-commuting pair of elements (assuming neither of which has order 2) following [25], [39, §3.6]. If $p \neq 2$, we may use f and g as such elements, and in the case $p = 2$, we may use f and gfg^{-1} which generate a subgroup of the index at most two (and replace β_2 by β_1 in the following calculation). The quaternion algebra then has the Hilbert symbol.

$$AG = \left(\frac{\beta_1(\beta_1 + 4), (\beta_1 + 4)(\beta_2 + 4)\gamma}{kG} \right) \quad (2.7.8)$$

Under all the real embeddings of kG , the term $\beta_1(\beta_1 + 4)$ is negative and $(\beta_1 + 4)(\beta_2 + 4)$ is positive. Thus AG is ramified at all real places of kG if and only if, under any real embedding τ of kG , we have

$$-4 \sin^2 \frac{k\pi}{p} \sin^2 \frac{2\pi}{5} < \tau(\gamma) < 0. \quad (2.7.9)$$

2.7.5 The factorisation condition

For an arithmetic group generated by elliptic elements of order 5 and p [36] observed that as a consequence of the Fricke identity, the number

$$\lambda = \operatorname{tr}(f) \operatorname{tr}(g) \operatorname{tr}(fg) \quad (2.7.10)$$

is an algebraic integer and satisfies the quadratic equation

$$x^2 - 16 \cos^2\left(\frac{\pi}{5}\right) \cos^2 \frac{\pi}{p} x + 16 \cos^2\left(\frac{\pi}{5}\right) \cos^2 \frac{\pi}{p} (-4 \sin^2\left(\frac{\pi}{5}\right) + 4 \cos^2 \frac{\pi}{p} - \gamma) = 0. \quad (2.7.11)$$

Thus

$$\begin{aligned} \lambda = & 8 \cos^2\left(\frac{\pi}{5}\right) \cos^2 \frac{\pi}{p} \\ & \pm 2 \cos\left(\frac{\pi}{5}\right) \cos \frac{\pi}{p} \sqrt{16 \cos^2\left(\frac{\pi}{5}\right) \cos^2 \frac{\pi}{p} - 4(-4 \sin^2\left(\frac{\pi}{5}\right) + 4 \cos^2 \frac{\pi}{p} - \gamma)} \end{aligned}$$

is an algebraic integer. Further, $2 \cos \frac{\pi}{5}$ is a unit and so

$$\begin{aligned} \alpha_p = & 4 \cos\left(\frac{\pi}{5}\right) \cos^2 \frac{\pi}{p} \\ & \pm \cos \frac{\pi}{p} \sqrt{16 \cos^2\left(\frac{\pi}{5}\right) \cos^2 \frac{\pi}{p} - 4(-4 \sin^2\left(\frac{\pi}{5}\right) + 4 \cos^2 \frac{\pi}{p} - \gamma)} \end{aligned} \quad (2.7.12)$$

is an algebraic integer. Since the field $\mathbb{Q}(\gamma)$ has one complex place we must have $\mathbb{Q}(\gamma) = \mathbb{Q}(\alpha_p)$.

Since $\mathbb{Q}(\gamma)$ has one complex place, the quadratic at (2.7.11) splits into linear factors over $L(\gamma)$. This implies that if τ is any real embedding of $L(\gamma)$, then the image of the discriminant of (2.7.11), which is $(4 + \beta_1)(4 + \beta_2)(\beta_1\beta_2 + 4\gamma)$, under τ must be positive. This is equivalent to requiring that

$$\tau(\beta_1\beta_2 + 4\gamma) > 0. \quad (2.7.13)$$

This factorisation criterion yielding $\mathbb{Q}(\gamma) = \mathbb{Q}(\alpha_p)$ is a powerful obstruction for an algebraic integer γ to satisfy when $p \neq 2$.

Putting this all together, we have the following theorem, which we will use to determine the possible γ values for the groups we seek.

Theorem 2.7.3 (The Identification Theorem). *Let $G = \langle f, g \rangle$ be a non-elementary subgroup of $PSL(2, \mathbb{C})$ with f of order 5 and g of order p , $p \geq 3$. Suppose $\gamma(f, g) = \gamma \in \mathbb{C} \setminus \mathbb{R}$. Then G is an arithmetic Kleinian group if and only if*

1. γ is an algebraic integer,
2. $\mathbb{Q}(\gamma) \supset L = \mathbb{Q}(\cos 2\pi/p, \cos 2\pi/q)$ and $\mathbb{Q}(\gamma)$ is a number field with exactly one complex place,
3. if $\tau : \mathbb{Q}(\gamma) \rightarrow \mathbb{R}$ is a real embedding such that $\tau|_L = \sigma$, then
$$-\sigma\left(\frac{\beta_1\beta_2}{4}\right) < \tau(\gamma) < 0, \tag{2.7.14}$$
4. the quadratic polynomial at (2.7.11) factorises over $\mathbb{Q}(\gamma)$,
5. G has finite co-volume.

In particular, we note the following result.

Corollary 2.7.4. *Let $\Gamma = \langle f, g \rangle$ be an arithmetic Kleinian group generated by elliptic elements of order 5 and p , with $p \neq 2$. Then the minimal polynomial for α_p (defined at 2.7.12) has the same degree as the minimal polynomial for $\gamma = \gamma(f, g)$.*

2.8 The possible values of p

If the generator f has order 5 and g has order p , then Theorem 2.7.3, implies first, that γ is an algebraic integer. Secondly, that $\mathbb{Q}(\gamma)$ has exactly one complex place and thirdly, that $\mathbb{Q}(\gamma)$ must contain

$$L = \mathbb{Q}(\cos 2\pi/5, \cos 2\pi/p).$$

Let

$$r = [\mathbb{Q}(\gamma) : L] \geq 2, \tag{2.8.1}$$

since the minimal polynomial for γ factors over L , then with $n = [\mathbb{Q}(\gamma) : \mathbb{Q}]$ and $\mu = [L : \mathbb{Q}]$ we have

$$n = r\mu = [\mathbb{Q}(\gamma) : \mathbb{Q}] \geq 4$$

2.8.1 First bounds

Now we apply the facts given above to identify possible triples $(5; p; r)$, where r is defined at (2.8.1), for which there may exist a γ -parameter corresponding to an arithmetic Kleinian group $\langle f, g \rangle$ with $o(f) = 5$ and $o(g) = p$ primitive elliptic and which is not freely generated by these two elements. Our first estimate is significant only because it reduces the number of triples we must look at more closely to a manageable list.

Let $P_\alpha(z)$ be the minimal polynomial for

$$\alpha = (\gamma + 1) \cdot 2 \cos(\pi/5) = (\gamma + 1) \left(\frac{1 + \sqrt{5}}{2} \right)$$

Note that $2 \cos(\pi/5) \in L$ is a unit, and α is an algebraic integer. Then $P_\alpha(z)$ has complex roots $\alpha, \bar{\alpha}$, and the fields $\mathbb{Q}(\alpha) \cong \mathbb{Q}(\gamma)$ since $2 \cos(\pi/5) \in \mathbb{Q}(\gamma)$ and both fields $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\gamma)$ must have one complex place as $\mathbb{Q}(\alpha) \subset \mathbb{Q}(\gamma)$.

We now use (2.7.14). For any real embedding $\tau : \mathbb{Q}(\gamma) \rightarrow \mathbb{R}$ with $\tau|_L = \sigma$ we compute that

$$\begin{aligned} \tau(\alpha) &= \tau\left((\gamma + 1)\left(\frac{1 + \sqrt{5}}{2}\right)\right) \\ &= (\tau(\gamma) + 1)\sigma\left(\frac{1 + \sqrt{5}}{2}\right) \end{aligned}$$

Now σ is a real embedding of the field L . When restricted to $\mathbb{Q}(\sqrt{5}) \subset L$ either

$$\sigma\left(\frac{1 + \sqrt{5}}{2}\right) = \frac{1 + \sqrt{5}}{2}$$

when $\sigma|_{\mathbb{Q}(\sqrt{5})}$ is the identity, or otherwise

$$\sigma\left(\frac{1 + \sqrt{5}}{2}\right) = \frac{1 - \sqrt{5}}{2}$$

We next recall from the Identification Theorem that

$$-\sigma\left(\frac{\beta_1\beta_2}{4}\right) < \tau(\gamma) < 0$$

Now $\beta_1\beta_2/4 = 4 \sin^2(\pi/5) \sin^2(\pi/p)$, and so we have

$$-\sigma(4 \sin^2(\pi/5)) < \tau(\gamma) < 0$$

Adding one throughout,

$$1 - \sigma(4 \sin^2(\pi/5)) < \tau(\gamma + 1) < 1 \tag{2.8.2}$$

When $\sigma|_{\mathbb{Q}(\sqrt{5})} = \text{identity}$ we can multiply through by the positive number $\frac{1 + \sqrt{5}}{2}$

to see

$$-4 \sin^2(\pi/5) \frac{1 + \sqrt{5}}{2} + \frac{1 + \sqrt{5}}{2} < \tau\left((\gamma + 1) \frac{1 + \sqrt{5}}{2}\right) = \tau(\alpha) < \frac{1 + \sqrt{5}}{2}$$

and hence

$$\frac{1 - \sqrt{5}}{2} < \tau(\alpha) < \frac{1 + \sqrt{5}}{2} \quad (2.8.3)$$

Next, when σ is not the identity on $\mathbb{Q}(\sqrt{5})$ we multiply (2.8.2) through by the negative number $\frac{1-\sqrt{5}}{2}$ and observe $\sigma(4 \sin^2(\pi/5)) = 4 \sin^2(2\pi/5)$ to see the following.

$$-4 \sin^2(2\pi/5) \frac{1 - \sqrt{5}}{2} + \frac{1 - \sqrt{5}}{2} > \tau(\gamma + 1) \sigma\left(\frac{1 + \sqrt{5}}{2}\right) = \tau(\alpha) > \frac{1 - \sqrt{5}}{2}$$

Hence

$$\frac{1 + \sqrt{5}}{2} > \tau(\alpha) > \frac{1 - \sqrt{5}}{2}$$

which is the same as (2.8.3). Thus we have found that

$$\frac{1}{2}(1 - \sqrt{5}) < \tau(\alpha) < \frac{1}{2}(1 + \sqrt{5}) \quad (2.8.4)$$

for all real embeddings τ .

This implies that all the real roots r_i , $i = 1, 2, \dots, n - 2$, of the polynomial $P_\alpha(z)$ of degree n belong to the interval $[-\frac{1}{2}(-1 + \sqrt{5}), \frac{1}{2}(1 + \sqrt{5})]$, this is approximately the interval

$$-0.618033989 \leq \tau(\alpha) \leq 1.618033989.$$

Auxillary functions.

We now achieve a degree bound from this information using the following *method of auxiliary functions*. Set

$$H_1(x) = x(x - 1), \quad H_2(x) = (x^2 - x - 1)$$

$P_\alpha(z)$, the minimal polynomial for α , and $H_i(x)$ share no roots. To see this, it is clear that H_1 and $P_\alpha(z)$ have no common roots as P_α is irreducible over \mathbb{Z} . The

roots of H_2 are the points $\frac{1}{2}(1 \pm \sqrt{5})$. By the irrational root theorem, if P has one of the roots of H_2 as a root, then it also has the conjugate, and H_2 is a divisor, and so P_α is not minimal degree. Thus these three polynomials share no roots. We may compute the resultant at 0. This will be a nonzero integer. Thus

$$1 \leq |\text{Resultant}[P_\alpha, H_1](0)| = |\alpha|^2 |\alpha - 1|^2 \prod_{i=1}^{n-2} |r_i| |1 - r_i| \quad (2.8.5)$$

$$1 \leq |\text{Resultant}[P_\alpha, H_2](0)| = |\alpha^2 - \alpha - 1|^2 \prod_{i=1}^{n-2} |r_i^2 - r_i - 1| \quad (2.8.6)$$

where

$$r_1, r_2, \dots, r_{n-2} \in \left[\frac{1}{2}(1 - \sqrt{5}), \frac{1}{2}(1 + \sqrt{5}) \right]$$

are the real roots of P_α .

We want to choose $a \geq 1$ so that the function

$$G(x) = |H_2(x)| |H_1(x)|^a = |x(x-1)|^a |x^2 - x - 1|$$

has its three maxima of the same value. The function G vanishes at 0, 1 and $\frac{1}{2}(1 \pm \sqrt{5})$. By symmetry, $\frac{1}{2}$ is a maximum, and with maximum value, $\frac{5}{4^{a+1}}$, and the other two maxima share the same maximum value. We can find this value on the interval $[\frac{1}{2}(1 - \sqrt{5}), 0]$ on which the G is negative, by differentiation. The derivative is

$$G'(x) = -(-1+x)^{-1+a} x^{-1+a} (-1+2x) ((-1+x)x + a(-1+(-1+x)x))$$

So we solve $(-1+x)x + a(-1+(-1+x)x) = 0$ to find

$$x = (1+a - \sqrt{1+6a+5a^2}) / (2+2a)$$

and

$$G(x) = -a^a (1+a)^{-1-a}$$

Equating this with $G(1/2)$ gives us the equation

$$5 = a^a \left(\frac{4}{1+a} \right)^{a+1}$$

which has a unique solution in the interval $[1, 1.5]$ and that value is approximately $a = 1.2945735620$ to ten decimal places. Then the maximum of G on this interval is less than

$$t_0 = 0.2077299. \quad (2.8.7)$$

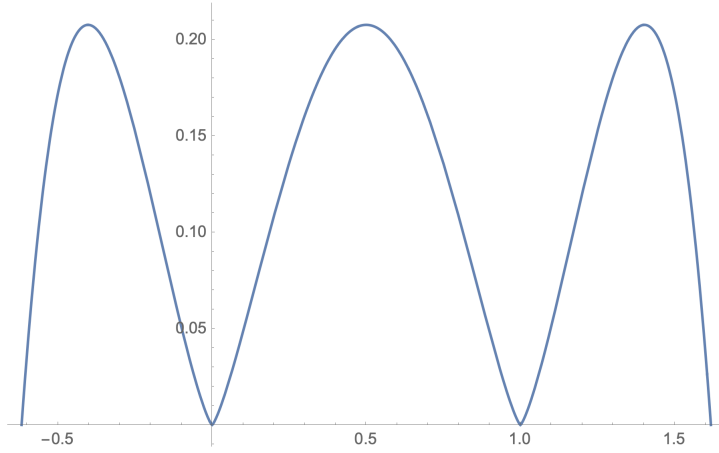


Figure 2.4: The graph of the function G on the interval $[\frac{1}{2}(1 - \sqrt{5}), \frac{1}{2}(1 + \sqrt{5})]$.

Now we combine (2.8.5) and (2.8.6) to get the inequality

$$\begin{aligned} 1 &\leq |Resultant[P_\alpha, H_1](0)|^a |Resultant[P_\alpha, H_2](0)| \\ &= |\alpha|^{2a} |\alpha - 1|^{2a} |\alpha^2 - \alpha - 1|^2 \prod_{i=1}^{n-2} |G(r_i)| \end{aligned} \quad (2.8.8)$$

Notice that the first term is, in fact, $|G(\alpha)|^2$. For arbitrary p , our disjoint isometric circle's criteria define a region in which γ must lie inside if the associated group is to be arithmetic. Then α lies inside a similar region, and we want to find the maximum value of $|G(\alpha)|^2$. This is easily seen to be when $\theta = 0$ (in the definition of γ) uses some elementary calculus. We have

$$|G(\alpha)|^2 \leq 2.35267 \times 10^{12} \quad (2.8.9)$$

We put these two pieces of information in (2.8.8) to find

$$1 \leq 2.35267 \times 10^{12} \times (0.2077299)^{n-2}. \quad (2.8.10)$$

This implies $n \leq 21$, but as we know, n is even, so we must have $n \leq 20$.

Since $\gamma = \gamma(f, g)$ is assumed complex and $\mathbb{Q}(\gamma) = L(\gamma)$ we must have the real base field $L = \mathbb{Q}(2 \cos(2\pi/p), 2 \cos(2\pi/5))$ has degree at most 10. We compute the degrees of these fields (it is typically the degree of $\mathbb{Q}(\cos(2\pi/LCM(5, p)))$ unless p is even, or a prime power).

This yields the following first bounds on p .

Theorem 2.8.1. *Let $\Gamma = \langle f, g \rangle$ be a subgroup of an arithmetic Kleinian group with $o(f) = 5$ and $o(g) = p$, which is not free on these two generators. Then one of the following occurs.*

1. $2 \leq p \leq 12$.
2. $p \in \{14, 15, 16, 18, 20, 22, 24, 25, 30, 40, 50, 60\}$.

Before considering the more sophisticated number theory that we will ultimately need, with specific values of p at hand, we can revisit the calculation above to obtain further information. The first is the poor bound on $|G(\alpha)|^2$, which we can now replace with a bound which depends on p .

If $p = 11$, then $|G(\alpha)|^2 = 1.58488 \times 10^{12}$ and $n = 20$ is impossible, and as $[L : \mathbb{Q}] = 10$ we find $p = 11$ is impossible.

If $p = 50$, then we may argue as follows. We recall from the Identification Theorem that for any real embedding τ of $\mathbb{Q}(\gamma) = \mathbb{Q}(\alpha)$, we have

$$-\sigma(4 \sin^2(\pi/5) \sin^2(\pi/50)) < \tau(\gamma) < 0$$

Our argument has considered the worst case when all the real roots r_i lie on the maxima of G , which is certainly impossible. However, we can choose a particular real embedding to get more information. For instance if we choose τ to be the identity on $\mathbb{Q}(\sqrt{5})$ and $\tau(\sin^2(\pi/50)) = \sin^2(3\pi/50)$ as we may, then

$$2 \cos(\pi/5)(1 - 4 \sin^2(\pi/5) \sin^2(3\pi/50)) < \tau(\alpha) < \cos(2\pi/5) \quad (2.8.11)$$

As $[\mathbb{Q}(\cos(2\pi/5), \cos(2\pi/50)) : \mathbb{Q}] = 10$ we see α is at most quadratic over L . Then (2.8.11) implies there is a pair of real roots in the interval $[1.53952, 1.61803]$. On this interval $|G| \leq 0.133214$ and when $p = 50$ we have $|G(\alpha)|^2 \leq 2.31111 \times 10^{12}$ and so the equation test now reads as

$$1 \leq 2.31111 \times 10^{12} \times 0.2077299^{n-4} \times 0.133214^2$$

and so $n \leq 18$ and $p = 50$ is impossible. This same argument with $p = 25$ and $\tau(\sin^2(\pi/25)) = \sin^2(2\pi/25)$ removes that case as well since we identify two roots in the interval

$$[2 \cos(\pi/5)(1 - 4 \sin^2(\pi/5) \sin^2(2\pi/25)), \cos(2\pi/5)].$$

This argument can be used when the degree r of γ over L is large. With $\tau|_L = \textit{identity}$ it implies there are $r - 2$ real roots in the interval

$$[2 \cos(\pi/5)(1 - 4 \sin^2(\pi/5) \sin^2(\pi/p)), 2 \cos(\pi/5)]$$

and so if $\cos(2\pi/5)(1 - 4 \sin^2(\pi/5) \sin^2(\pi/p))$ is larger than the maxima of G we gain new information. As an example, if $p = 12$ and $r = 5$, then

$$2 \cos(\pi/5)(1 - 4 \sin^2(\pi/5) \sin^2(\pi/12)) = 1.46825 \dots$$

which is larger than the maxima, so there are 3 real roots in the interval $[1.46825, 2 \cos(\pi/5)]$ where $|G| \leq 0.1923922$. Hence we have

$$1 \leq |G(\alpha_p)|^2 \times 0.2077299^{n-5} \times 0.1923922^3$$

which is not possible if $n = 20$, so $n \leq 16$. The same argument removes the additional cases $p = 15, 20, 30$.

At this point, we have the remaining cases to deal with, as shown in Table 2.1. These degree bounds are still too large for effective searches.

$[L : \mathbb{Q}]$	p	comment
2	3, 4, 5, 6, 10	total degree ≤ 18
4	8	total degree ≤ 20
4	12, 15, 20, 30	total degree ≤ 16
6	7, 9, 14, 18	γ quadratic or cubic over L
8	16, 24, 40, 60	γ quadratic over L
10	22	γ quadratic over L

Table 2.1: The First Total degree bound for γ .

For the last part of this section, we will try to decrease the total degree bounds on some of the remaining cases listed in Table 2.1. This will be done by choosing a different algebraic integer than α as at this point we have more specific information.

The case $p = 4$

Let $p = 4$. Suppose that $\alpha = \gamma + 1$. Then α is an algebraic integer, and our bounds show

$$|\alpha| \leq 1 + \sqrt{2} + \frac{\sqrt{5}}{2} + \sqrt{10} + \frac{7}{2} \approx 10.19453$$

Then all the real roots of the minimal polynomial $p(z)$ for α lie in the interval $[1 - 2\sin^2(2\pi/5), 1]$. Thus for all real embeddings τ ,

$$\frac{1}{4}(-\sqrt{5} - 1) \leq \tau(\alpha) \leq 1$$

where $\frac{1}{4}(-\sqrt{5} - 1) \approx -0.809017$. We will continue to denote these real roots as r_i , $i = 1, 2, \dots, n - 2$. Considering the parametric equation defining the region with γ must lie, and a little calculus gives

$$|\alpha|^2 |1 - \alpha^2| \leq 10697.3 \dots$$

This value is assumed when γ and hence α is real and positive. We then calculate that

$$\begin{aligned} 1 &\leq |p(0)|^2 |p(-1)p(1)| = |\alpha|^4 \prod_{i=1}^{n-2} r_i^2 \times |\alpha - 1|^4 \prod_{i=1}^{n-2} (1 - r_i^2) \\ &= 1.14429 \times 10^8 \prod_{i=1}^{n-2} r_i^2 (1 - r_i^2) < 1.14429 \times 10^8 \times \left(\frac{1}{4}\right)^{n-2} \end{aligned}$$

Where we have used the elementary fact that $0 \leq x^2(1 - x^2) \leq \frac{1}{4}$ on the interval $[-1, 1]$. The estimate (2.8.12) implies that $n \leq 15$, and as n is even, $n \leq 14$.

The case $p = 5$

Let $p = 5$. Set

$$\alpha = \gamma \times \left(\frac{3 + \sqrt{5}}{2}\right) + 1.$$

As $\frac{3 \pm \sqrt{5}}{2}$ is an algebraic integer and also a unit α is an algebraic integer.

As before we apply the embedding τ to α , note that $L = \mathbb{Q}(\sqrt{5})$ and if $\tau|_{\mathbb{Q}(\sqrt{5})} = \sigma$ we have

$$-\sigma(4 \sin^4(\pi/5)) < \tau(\gamma) < 0$$

And recall $4 \sin^4(\pi/5) = 5/8(-3 + \sqrt{5})$. If σ is the identity, we have

$$\begin{aligned} 1 + 5 \frac{-3 + \sqrt{5}}{8} \left(\frac{3 + \sqrt{5}}{2} \right) &< 1 + \tau(\gamma) \left(\frac{3 + \sqrt{5}}{2} \right) < 1 \\ -\frac{1}{4} &< \tau(\alpha) < 1, \end{aligned}$$

with a similar calculation when $\sigma(\sqrt{5}) = -\sqrt{5}$. Thus, following our earlier arguments, the real roots of the minimal polynomial for α lie in the interval $[\frac{1}{4}, 1]$. The obvious choice for an auxiliary function is $|x|^a(1-x)$ for suitable a . In this case, we want the value at $-1/4$ to be an absolute maximum on our interval and to coincide with the other maximum at $x = 1/(1_a)$. This gives us the equation

$$3/4^{-a-1} = a^a(a+1)^{-a-1}$$

The solution we seek is $a = 1.29457\dots$. Then the maximum value of the auxiliary function is approximately 0.20773.

$$\begin{aligned} 1 &\leq |p(0)|^a \leq |\alpha|^{2a} \prod_{i=1}^{n-2} |r_i|^a \\ 1 &\leq |p(1)| \leq |1 - \alpha|^2 \prod_{i=1}^{n-2} |1 - r_i| \\ 1 &\leq |\alpha|^{2a} |1 - \alpha|^2 \prod_{i=1}^{n-2} |r_i|^a |1 - r_i| \leq |\alpha|^{2a} |1 - \alpha|^2 (0.20773)^{n-2}. \end{aligned}$$

Just as before, we calculate that

$$|\alpha|^{2a} |1 - \alpha|^2 \leq 520476.$$

and hence

$$1 \leq |\alpha|^{2a} |1 - \alpha|^2 (0.20773)^{n-2}$$

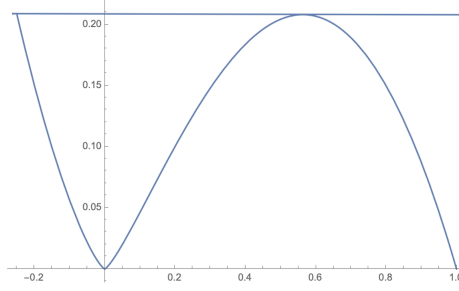


Figure 2.5: The graph of $|r_i|^a|1 - r_i|$, $a = 1.29457\dots$

and this implies $n \leq 10$.

The case $p = 6$

Let $p = 6$. Then γ is an algebraic integer with minimal polynomial P with real roots in $-1 \leq \tau(\gamma) \leq 0$. As before, we calculate that

$$|\gamma|^2|1 + \gamma|^2 \leq 18818.4$$

and hence

$$\begin{aligned} 1 &\leq |P(0)||P(-1)| = |\gamma|^2|1 + \gamma|^2 \prod_{i=1}^{n-2} |r_i||1 + r_i| \\ &\leq 18818.4 \times \left(\frac{1}{4}\right)^{n-2} \end{aligned}$$

This implies that $n \leq 8$.

The case $p = 8$

Let $\alpha = \gamma \times \frac{1}{2}(\sqrt{5} + 1)(1 + \sqrt{2})$. As

$$-4 \sin^2\left(\frac{\pi}{8}\right) \sin^2\left(\frac{\pi}{5}\right) = -\frac{1}{8}(\sqrt{2} - 2)(\sqrt{5} - 5)$$

for real embeddings τ ,

$$-\sigma\left(\frac{1}{8}(\sqrt{2} - 2)(\sqrt{5} - 5)\right) < \tau(\gamma) < 0$$

and hence

$$-\sigma\left(\frac{1}{8}(\sqrt{2} - 2)(\sqrt{5} - 5)\right) \frac{1}{2}(\sqrt{5} + 1)(1 + \sqrt{2}) < \tau(\alpha) < 0$$

Consideration of the four embeddings $\sqrt{2} \leftrightarrow -\sqrt{2}$ and $\sqrt{5} \leftrightarrow -\sqrt{5}$ implies that all real embeddings have

$$-0.790569415 \leq \tau(\alpha) \leq 0.790569415$$

with half the real roots positive and half negative. Recall that in this case $L = \mathbb{Q}(\sqrt{2}, \sqrt{5})$ and so the total degree is divisible by 4. Let $P(z)$ be a minimal polynomial of degree n for α . Set $H(x) = x^2(2x^2 - 1)^{1.3}$ as an auxiliary function. If $H(x)$ and $P(z)$ do share roots, then by irrational roots theorem, $2z^2 - 1$ is a divisor, and so $P(z)$ is not minimal degree. Thus $|\text{Resultant}(H, P)| = \prod_{P(z_i)=0}^n |H(z_i)| \geq 1$.

Respectively, the following relation must be satisfied

$$1 \leq |\alpha|^4 |2\alpha^2 - 1|^{2.6} \prod_{i=1}^{4r-2} |r_i|^2 |2r_i^2 - 1|^{1.3} \leq |\alpha|^4 |2\alpha^2 - 1|^{2.6} (.105)^{4r-2}$$

This states that $n \leq 16$.

Theorem 2.8.2. (*Detection of units*) Let $P(z)$ be a minimal polynomial for algebraic integer β , then β is unit iff the Norm $N(\beta) = P(0) = \pm 1$.

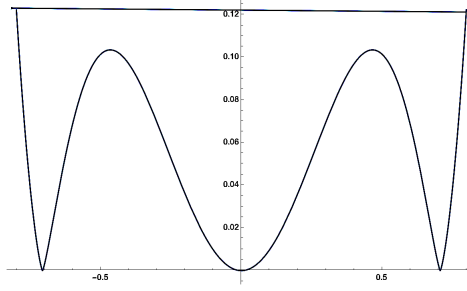


Figure 2.6: The graph of $|r_i|^2|1 - 2r_i^2|^{1.3}$

β	<i>Minimal Polynomial</i>
$4 \sin^2(\pi/10)$	$1 - 3z + z^2$
$4 \sin^2(\pi/12)$	$1 - 4z + z^2$
$4 \sin^2(\pi/14)$	$-1 + 6z - 5z^2 + z^3$
$4 \sin^2(\pi/15)$	$1 - 8z + 14z^2 - 7z^3 + z^4$
$4 \sin^2(\pi/18)$	$-1 + 9z - 6z^2 + z^3$
$4 \sin^2(\pi/20)$	$1 - 12z + 19z^2 - 8z^3 + z^4$

Table 2.2: The list of Units

Proof. Let β be an algebraic integer and unit, then β^{-1} is in the ring of the integer.

Thus

$$N(\beta\beta^{-1}) = N(\beta)N(\beta^{-1}) = \pm 1$$

This states that $N(\beta) = \pm 1$. □

At this point, we summarise what we have achieved so far.

Theorem 2.8.3. *Let $\Gamma = \langle f, g \rangle$ be a discrete group with $o(f) = 5$ and $o(g) = p \in \{10, 12, 14, 15, 18, 20\}$, which is not free on these two generators. Suppose $\gamma = \gamma(f; g)$. Then the total degree is less than 12.*

Proof. Set

$$\alpha = \frac{\gamma}{4 \sin^2(\pi/p)} + 1$$

For arbitrary p , following the Theorem 2.8.2, $4 \sin^2(\pi/p)$ is algebraic and a unit (see Table 2.2). Assume $P(z)$ is the minimal polynomial for α , then $P(z)$ has complex roots α and $\bar{\alpha}$, and $\mathbb{Q}(\alpha) \simeq \mathbb{Q}(\gamma)$. In addition, by applying real embedding τ on α ,

$$\tau(\alpha) = \frac{\tau(\gamma)}{\sigma(4 \sin^2(\pi/p))} + 1$$

We have at hand

$$\sigma(4 \sin^2(\pi/5))\sigma(\sin^2(\pi/p)) < \tau(\gamma) < 0$$

Therefore

$$0 \leq -\sin^2(2\pi/5) + 1 \leq \tau(\alpha) \leq 1.$$

Set $H(x) = x(x-1)(2x-1)$ as an auxiliary function that does not share roots with $P(z)$. Then as before, the $|Resultant(H(z), P(z))| \geq 1$. Respectively, the following inequality must be held for the roots z_i of $P(z)$.

$$\begin{aligned} 1 &\leq \prod_{P(z_i)=0}^n |H(z_i)| \\ &= |\alpha|^2 |\alpha-1|^2 |2\alpha-1|^2 \cdots \prod_{i=1}^{n-2} |r_i| |1-r_i| |1-2r_i| \\ &\leq |\alpha|^2 |\alpha-1|^2 |2\alpha-1|^2 (.0966)^{n-2} \end{aligned} \tag{2.8.12}$$

This gives us the following information.

1. If $p = 10$. Then the total degree is $n \leq 10$.
2. If $p = 12, 14, 18$. Then the polynomial must be quadratic over \mathbb{L} .

3. If $p = 15, 20$. Then the polynomial must be quadratic or cubic over L .

□

This calculation is summarised as follows

$[L : \mathbb{Q}]$	p	comment
2	3	total degree ≤ 18
2	4	total degree ≤ 14
2	5	total degree ≤ 10
2	6	total degree ≤ 8
6	7, 9	γ quadratic or cubic over L
4	8, 30	total degree ≤ 16
2	10	total degree ≤ 10
4	15, 20	γ quadratic or cubic over L
4	12, 14, 16, 22, 40, 60	γ quadratic over L

2.8.2 Discriminant bounds.

From the previous subsections, we now have better bounds for each case's total degree of the invariant trace field. In this subsection, we will use the *Discriminant method* to eliminate that p for which $\mu \geq 4$ (recall that $\mu = [L; \mathbb{Q}]$). Further, we will establish that if $\mu = 2$, the total degree is $n \leq 8$. We consider the discriminant

of the minimal polynomial for γ . This is

$$\text{disc}(\gamma) = |\gamma - \bar{\gamma}|^2 \prod_{i=1}^{n-2} |\gamma - r_i|^4 \prod_{1 \leq i < j \leq n-2} (r_i - r_j)^2 \quad (2.8.13)$$

where the $n - 2$ real roots

$$r_i \in [\min_{k,l} -4 \sin^2(\frac{k\pi}{5}) \sin^2(\frac{l\pi}{p}), 0].$$

Let

$$\mu = [L : \mathbb{Q}] = [\mathbb{Q}(\cos(\frac{2\pi}{5}), \cos(\frac{2\pi}{p})) : \mathbb{Q}]. \quad (2.8.14)$$

So that the total degree of the field $\mathbb{Q}(\gamma)$ is $n = r\mu$. For $n \geq 2$, let D_n denote the minimum absolute value of the discriminant of any field of degree n over \mathbb{Q} with exactly one complex place. For small values of n , the number D_n has been widely investigated ([12, 16, 17]), and lower bounds for D_n for all n can be computed ([47, 49, 56, 64]). In [49], the bound is given in the form $D_n > A^{n-2} B^2 \exp(-E)$ for varying values of A, B and E . Choosing, by experimentation, suitable values from this table, we obtain the bounds shown in previous Table. We will use more precise data later.

2.8.3 Schur's bound

We will need to use Schur's bound [63], which gives that, if $-1 \leq x_1 < x_2 < \dots < x_r \leq 1$ with $r \geq 3$ then

$$\prod_{1 \leq i < j \leq r} (x_i - x_j)^2 \leq M_r = \frac{2^2 3^3 \dots r^r 2^2 3^3 \dots (r-2)^{r-2}}{3^3 5^5 \dots (2r-3)^{2r-3}}. \quad (2.8.15)$$

Suppose $P(z)$ is a minimal polynomial for algebraic integer γ . Then the discriminant can be written as follows.

$$\text{disc}(\gamma) = |\gamma - \bar{\gamma}|^2 |\gamma - r_i|^{4(n-2)} \prod_{i=1}^{n-2} |r_i - r_j|^2. \quad (2.8.16)$$

Degree n	Bound	Degree n	Bound
2	3	3	27
4	275	5	4511
6	92779	7	2306599
8	68856875	9	$0.11063894 \times 10^{10}$
10	$0.31503776 \times 10^{11}$	11	$0.90315026 \times 10^{12}$
12	$0.25891511 \times 10^{14}$	13	$0.74225785 \times 10^{15}$
14	$0.21279048 \times 10^{17}$	15	$0.61002775 \times 10^{18}$
16	$0.17488275 \times 10^{20}$	17	$0.50135388 \times 10^{21}$
18	$0.14372813 \times 10^{23}$	19	$0.41203981 \times 10^{24}$
20	$0.11812357 \times 10^{26}$		

Table 2.3: Lower Bounds for The Discriminant

Following Table 2.3, the lower bound we have at hand gives $D_n \leq \text{disc}(\gamma)$. Let $\Delta_1 = \delta_{\mathbb{Q}(\gamma)|L}$, the relative discriminant of the field extension $\mathbb{Q}(\gamma) | L$, and let Δ denote the discriminant of the basis $1, \gamma, \gamma^2, \dots, \gamma^{r-1}$ over L . Then

$$|N_{L|\mathbb{Q}}(\Delta)| \geq |N_{L|\mathbb{Q}}(\Delta_1)|.$$

Otherwise

$$|N_{L|\mathbb{Q}}(\delta_{\mathbb{Q}(\gamma)|L})| = |\Delta_{\mathbb{Q}(\gamma)}|/\Delta_L^r. \quad (2.8.17)$$

Also, by knowing that $[L : \mathbb{Q}] = [\mathbb{Q}(\cos(\frac{2\pi}{5}), \cos(\frac{2\pi}{p})) : \mathbb{Q}]$, for $\mu \geq 4$, by a direct calculation through relative discriminant (2.8.13), we shall find there are no arithmetic groups. Before starting the analysis, let us mention the database NFDB,

which helps find the minimal polynomial for an algebraic integer γ in the region Ω_p .

p	$[L : \mathbb{Q}]$	Δ_p
3	2	5
4	2	5
5	2	5
6	2	5
7	6	$5^3 \cdot 7^2$
8	4	$5^2 \cdot 8^2$
9	6	$5^3 \cdot 9^2$
10	2	5
12	4	$5^2 \cdot 12^2$
14	4	$5^3 \cdot 49^2$
15	4	1125
16	8	$5^4 \cdot 2^{22}$
18	4	$5^3 \cdot 81^2$
20	4	2000
30	4	1125

Table 2.4: The Δ_p discriminant for field L

2.8.4 LMFDB Program

The LMFDB ¹ (the database of L-functions, modular forms, number fields, and related objects) is an extensive database of mathematical objects arising in Number Theory. We may search for objects with specific properties, browse categories, and explore individual plots or view distributions of various objects. The LMFDB makes visible the connections predicted by the Langlands program.

Here we mention how one can use the LMFDB.

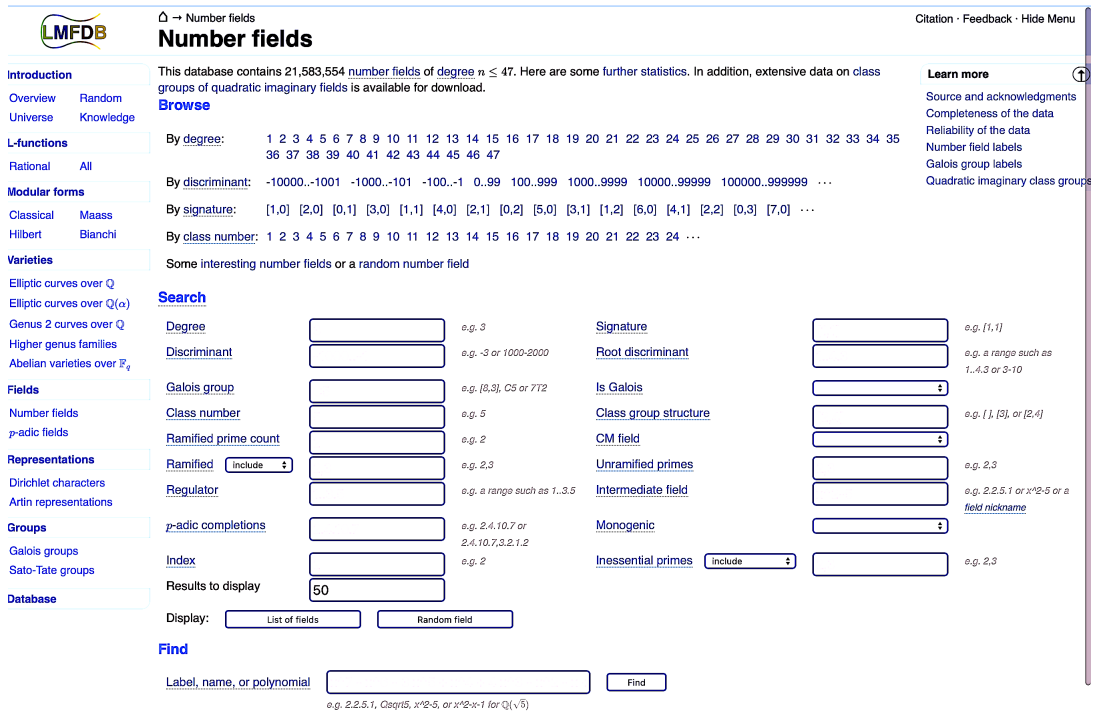


Figure 2.7: LMFDB environment

1. Browsing: One way to investigate information on the database is through browsing. You can browse each type of object in the database by distinguishing features. For example, you can browse by a degree or underlying object to look for information about L-functions.
2. Searching: Another way to investigate is through searching. You can search by using a label to find a specific object or by specifying certain characteristics

¹<http://www.lmfdb.org/NumberField/>

an object may have. For each of these types of search, there is an example in grey text and a description of possible inputs. A search may leave some fields blank (or with grey text).

3. Interesting: Each section of the LMFDB contains a link to a curated list of notable or interesting objects.
4. Learn more: Each section of the LMFDB contains a "Learn more about" box on the right side with links to additional information, including details regarding the source, extent, and reliability of the data.
5. Breadcrumbs: On the top left of the header of a given page, there is a description of your current navigation path. Clicking on an item in the path will take you back to that page.
6. Properties: On the upper right-hand side of a homepage for an object, there is often a box containing specific properties of the object. This box can be closed and opened by clicking on the vertical arrow.
7. Related Objects: Often below the properties box in the upper right-hand side of a homepage for an object is a box containing links to the homepages of directly related objects.

In LMFDB, we have just used the Number Field page drawn from proven sources (see the source and acknowledgments for number field pages in the program). Using the Identification Theorem, we mention by way of example to clarify how we can search for specific polynomials with one complex conjugate root associated with arithmetic conditions.

Explanation for searching Number Fields in the Program

Let's try to find the minimal polynomials with integer coefficients and complex root γ . The polynomial we are searching for has degree n , where $n - 2$ are real roots and

one complex conjugate pair. This implies that the program's signature is $[1, n - 2]$. We have an upper and lower bound for the discriminant from the Identification Theorem and the region Ω_p for γ . In addition, the field $L = \mathbb{Q}(\cos \pi/p, \cos \pi/5) \subset \mathbb{Q}(\gamma)$ can be presented as Intermediate Field in the program. As we know, using the discriminant bound, Δ_L^r is a factor of the discriminant. Then all primes which divide Δ_L^r , respectively divide the discriminant, are possible Ramified prime numbers in the program. Additional details for each condition can be found by clicking on the Number Field page in the program.

If $p \in \{12, 14, 16, 22, 40, 60\}$, Quadratic

First, assume $p = 12$. Then the minimal polynomial P for γ has complex conjugate pairs of roots γ and $\bar{\gamma}$ in Ω_{12} and 6 real roots. Thus, the discriminant is bounded by

$$|\gamma - \bar{\gamma}|^2 \cdot 2^{12} \sin^4\left(\frac{k\pi}{5}\right) \sin^4\left(\frac{l\pi}{12}\right) \cdot (12^2 \cdot 5^2)^2 \geq D_8.$$

Following the equation 2.8.17, $\Delta_L^2 = 2^8 \times 5^4 \times 3^4$ is a factor of discriminant, and 2, 3 and 5 are possibilities for ramified places. In addition, the discriminant has upper and lower bounds. Putting this information in the resource LMFDB gives us no candidate for the minimal polynomial of γ .

If $p=14$

Assume $p = 14$. Let the minimal polynomial P be a quadratic polynomial for γ over L , then the distribution of the real roots is two in $[-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{3\pi}{14}), 0]$, two

in $[-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{\pi}{14}), 0]$, two in $[-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{3\pi}{14}), 0]$, two in $[-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{5\pi}{14}), 0]$, and two in $[-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{5\pi}{14}), 0]$. Therefore

$$\begin{aligned} & |\gamma - \bar{\gamma}|^2 (-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{\pi}{14}))^2 (-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{3\pi}{14}))^2 (-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{3\pi}{14}))^2 \\ & (-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{5\pi}{14}))^2 (-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{5\pi}{14}))^2 \cdot (49^2 \cdot 5^3)^2 \\ & = |\gamma - \bar{\gamma}|^2 \cdot (0.1791)^2 \cdot (0.537224)^2 (1.121803)^2 (2.936920)^2 (1.40647284)^2 \\ & (49^2 \cdot 5^3)^2 \geq D_{12}. \end{aligned}$$

As we know $D_{12} \geq 0.25891511 \times 10^{14}$. Then from the previous inequality, it follows that $37.689689 \leq |\gamma - \bar{\gamma}| = 2\Im m(\gamma)$. This implies that $\gamma \notin \Omega_{12}$ contradicts the arithmetic condition at Identification Theorem.

If $p=16$

Next, let $p = 16$. And $\mathbb{Q}(\gamma)$ is a quadratic over \mathbb{Q} . Then the relative discriminant over $L = \mathbb{Q}(\cos(2\pi/5), \cos(2\pi/16))$, which has complex conjugate pairs of roots γ and $\bar{\gamma}$ and 10 real roots which come in pairs, two in $[-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{3\pi}{16}), 0]$, two in $[-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{5\pi}{16}), 0]$, two in $[-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{7\pi}{16}), 0]$, two in $[-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{5\pi}{16}), 0]$, two in $[-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{3\pi}{16}), 0]$, and two in $[-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{7\pi}{16}), 0]$, and two in

$[-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{\pi}{16}), 0]$, can be written as follows.

$$\begin{aligned}
& |\gamma - \bar{\gamma}|^2 (-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{3\pi}{16}))^2 (-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{5\pi}{16}))^2 (-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{7\pi}{16}))^2 \\
& (-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{5\pi}{16}))^2 (-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{3\pi}{16}))^2 (-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{7\pi}{16}))^2 \\
& (-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{\pi}{16}))^2 \cdot (5^4 \cdot 8^2)^2 \\
& = |\gamma - \bar{\gamma}|^2 \cdot (0.4265552573)^2 \cdot (0.9554107539)^2 \cdot (1.329368062)^2 (1.116736162)^2 \\
& (2.501297827)^2 \cdot (3.480330769)^2 \cdot (0.1377032193)^2 \cdot (5^4 \cdot 8^2)^2 \geq D_{16}.
\end{aligned}$$

we know $D_{16} \geq 0.17488275 \times 10^{20}$. Then the previous inequality follows that $154822.47 \leq |\gamma - \bar{\gamma}| = 2\Im m(\gamma)$. This implies that $\gamma \notin \Omega_{16}$, Which contradicts the arithmetic criteria of the Identification Theorem.

The calculation is the same for the remaining cases; there is nothing to report.

If $p \in \{7, 9, 15, 20\}$, Quadratic or Cubic.

Four cases have $\mu = 6, 4$ and still need to be eliminated. From Theorem 2.8.1, the minimal polynomial of $\gamma_p \in \Omega_p$ must be quadratic or cubic. Let us work through these cases, first with $p = 7$. Let $\mathbb{Q}(\gamma)$ be a quadratic over \mathbb{Q} . Then the distribution of real roots is, two in

$[-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{2\pi}{7}), 0]$, two in $[-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{\pi}{7}), 0]$, two in

$[-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{2\pi}{7}), 0]$, two in $[-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{3\pi}{7}), 0]$, and two in

$[-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{3\pi}{7}), 0]$. The discriminant is

$$\begin{aligned} & |\gamma - \bar{\gamma}|^2 (-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{2\pi}{7}))^2 (-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{\pi}{7}))^2 (-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{2\pi}{7}))^2 \\ & (-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{3\pi}{7}))^2 (-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{3\pi}{7}))^2 \cdot (7^2 \cdot 5^3)^2 \\ & = |\gamma - \bar{\gamma}|^2 \cdot (0.681113347)^2 \cdot (2.211561146)^2 \cdot (3.438884988)^2 (1.313537182)^2 \\ & (0.8447411894)^2 \cdot (7^2 \cdot 5^3)^2 \geq D_{12}. \end{aligned}$$

Noting $D_{12} \geq 0.25891511 \times 10^{14}$ in the previous inequality gives $143.73 \leq |\gamma - \bar{\gamma}| = 2\Im m(\gamma)$. Then $\gamma \notin \Omega_7$, which again contradicts the arithmetic condition at Identification Theorem.

If $p=9$

Let $p = 9$ and the minimal polynomial P of γ be quadratic over L . Then there is a complex conjugate pair of roots γ and $\bar{\gamma}$ and also 10 real roots which come in pairs, two in $[-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{2\pi}{9}), 0]$, two in $[-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{\pi}{9}), 0]$, two in $[-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{2\pi}{9}), 0]$, two in $[-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{5\pi}{9}), 0]$, and two in $[-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{5\pi}{9}), 0]$. Therefore the discriminant bound can be written as follows.

$$\begin{aligned} & |\gamma - \bar{\gamma}|^2 (-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{2\pi}{9}))^2 (-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{\pi}{9}))^2 (-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{2\pi}{9}))^2 \\ & (-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{5\pi}{9}))^2 (-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{5\pi}{9}))^2 \cdot (9^2 \cdot 5^3)^2 = |\gamma - \bar{\gamma}|^2 \cdot (0.5709)^2 \\ & \cdot (1.340294)^2 \cdot (3.50893)^2 (1.494884)^2 (0.42322)^2 \cdot (9^2 \cdot 5^3)^2 \geq D_{12}. \end{aligned}$$

With $D_{12} \geq 0.25891511 \times 10^{14}$ in the previous inequality gives

$$294.717909 \leq |\gamma - \bar{\gamma}| = 2\Im m(\gamma).$$

Hence $\gamma \notin \Omega_7$, which is a contradiction.

If p=15

If $p = 15$, the minimal polynomial P of γ is a quadratic over L . We have the following inequality using the lower bound for discriminant (Table 2.3).

$$|\gamma - \bar{\gamma}|^2 \cdot 2^{12} \sin^4\left(\frac{k\pi}{5}\right) \sin^4\left(\frac{l\pi}{15}\right) \cdot (1125)^2 \geq D_8.$$

Using Ω_{15} , we have the upper bound for γ . Then 10^9 will be the upper bound for the above discriminant. Following the LMFDB, there are 1236 possibilities for the minimal polynomials of γ . Knowing that 3 is a ramifying prime, and $\Delta_L^r = 5^6 \times 3^4$ is a factor of discriminant, then our possible cases reduce to 12 candidates. There are listed in Table 2.5. Each candidate has two real roots, with $|r_i - r_j| \geq 4$, which contradicts the arithmetic condition at the Identification Theorem.

If p=20

Let $p = 20$. The minimal polynomial P for γ is quadratic over L . Using the equation 2.8.17, the discriminant (2.8.16) states the following inequality.

$$|\gamma - \bar{\gamma}|^2 \cdot 2^{12} \sin^4\left(\frac{k\pi}{5}\right) \sin^4\left(\frac{l\pi}{20}\right) \cdot (2000)^2 \geq D_8$$

Using the region Ω_{20} , the upper bound for γ gives us the upper bound 9×10^9 for previous inequality. Following the LMFDB source, there are 2965 possible candidates. In addition, $\Delta_L^r = 2^8 \times 5^6$ is a factor of discriminant. This reduces our search to 12 cases. These candidates can be seen in Figure 2.8.

From Figure 2.8, each candidate has at least two real roots such that

Discriminant	Galois group	Polynomial
$-3^4 5^6 31$	T_{27}	$x^8 - 2x^7 - 2x^6 + 11x^5 - 15x^4 + x^3 + 13x^2 - 7x + 1$
$-3^6 5^6 31$	T_{27}	$x^8 - 2x^7 - 4x^6 + 4x^5 + 4x^4 + 2x^3 - x^2 - 4x + 1$
$-3^4 5^6 59$	T_{27}	$x^8 - 4x^7 + 5x^6 - x^5 - 6x^4 + 9x^3 - 4x + 1$
$-3^4 5^6 179$	T_{27}	$x^8 - 3x^7 + x^6 + x^5 - x^4 + 8x^3 - 6x^2 - x + 1$
$-3^4 5^7 151$	T_{27}	$x^8 - 2x^7 - 2x^6 + x^5 + 11x^3 - 2x^2 - 7x + 1$
$-3^4 5^6 239$	T_{27}	$x^8 - 3x^7 + 7x^5 - 11x^4 + 3x^3 + 10x^2 - 7x + 1$
$-3^4 5^6 359$	T_{27}	$x^8 - 2x^7 + 8x^5 - 16x^4 + 2x^3 + 15x^2 - 8x + 1$
$-3^4 5^6 419$	T_{27}	$x^8 - 3x^7 + 2x^6 - 5x^5 + 6x^4 + 10x^3 - 12x^2 + x + 1$
$-3^4 5^6 479$	T_{27}	$x^8 - 2x^7 - 5x^6 + 13x^5 - 6x^4 - 13x^3 + 25x^2 - 13x + 1$
$-3^4 5^6 599$	T_{27}	$x^8 2x^7 - 6x^6 + 12x^5 + 6x^4 - 20x^3 + 5x^2 + 10x - 5$
$-3^4 5^6 659$	T_{27}	$x^8 - x^7 - 5x^6 + 6x^5 + 4x^4 - 14x^3 + 9x + 1$
$-3^4 5^6 719$	T_{27}	$x^8 - x^7 - 6x^6 + 13x^5 - x^4 - 29x^3 - 36x^2 - 11x + 1$

Table 2.5: $p = 15$, List of polynomials from NFDB program

$|r_i - r_j| \geq 4$. This contradicts an arithmetic condition in Identification Theorem.

Following the previous method, the calculation is the same for the cubic case, with nothing to report.

Label	Polynomial	Discriminant	Galois group	Class group
8.6.6553600000.1	$x^8 - 2x^6 - 5x^4 + 6x^2 - 1$	$-2^{20} \cdot 5^4$	$((C_4 \times C_2) : C_2) : C_2$ (as 8T31)	trivial
8.6.2821440000.1	$x^8 - 8x^6 + 18x^4 - 8x^2 - 4$	$-2^{22} \cdot 5^4$	$((C_4 \times C_2) : C_2) : C_2$ (as 8T31)	trivial
8.6.4096000000.1	$x^8 - 6x^6 + 6x^4 + 4x^2 - 4$	$-2^{18} \cdot 5^6$	$(C_3 : C_2) : C_2$ (as 8T27)	trivial
8.6.4096000000.2	$x^8 - 2x^6 - 6x^4 + 12x^2 - 4$	$-2^{18} \cdot 5^6$	$(C_3 : C_2) : C_2$ (as 8T27)	trivial
8.6.16384000000.1	$x^8 - 10x^6 + 15x^4 + 50x^2 - 25$	$-2^{20} \cdot 5^6$	$((C_4 \times C_2) : C_2) : C_2$ (as 8T31)	trivial
8.6.20480000000.1	$x^8 - 10x^6 + 30x^4 - 20x^2 - 20$	$-2^{18} \cdot 5^7$	$(C_3 : C_2) : C_2$ (as 8T27)	trivial
8.6.20480000000.2	$x^8 - 20x^4 + 40x^2 - 20$	$-2^{18} \cdot 5^7$	$(C_3 : C_2) : C_2$ (as 8T27)	trivial
8.6.65536000000.1	$x^8 - 70x^4 + 200x^2 - 100$	$-2^{22} \cdot 5^6$	$((C_4 \times C_2) : C_2) : C_2$ (as 8T31)	trivial
8.6.65536000000.2	$x^8 - 8x^6 + 14x^4 + 8x^2 - 4$	$-2^{22} \cdot 5^6$	$(C_3 : C_2) : C_2$ (as 8T27)	trivial
8.6.65536000000.3	$x^8 - 4x^6 - 14x^4 + 16x^2 - 4$	$-2^{22} \cdot 5^6$	$(C_3 : C_2) : C_2$ (as 8T27)	trivial
8.6.81920000000.1	$x^8 - 10x^6 + 25x^4 - 10x^2 - 5$	$-2^{20} \cdot 5^7$	$(C_3 : C_2) : C_2$ (as 8T27)	trivial
8.6.81920000000.2	$x^8 - 20x^4 + 20x^2 - 5$	$-2^{20} \cdot 5^7$	$(C_3 : C_2) : C_2$ (as 8T27)	trivial

Figure 2.8: $p = 20$, List of polynomials from LMFDB program

If $p = 8, 30$, **Quadratic.**

Let $p = 8$, and the minimal polynomial P for γ be a quadratic over $L = \mathbb{Q}(\cos(2\pi/5), \cos(2\pi/8))$. Then it has complex conjugate pairs of roots γ and $\bar{\gamma}$ and 6 real roots which come in pairs, two in $[-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{3\pi}{8}), 0]$, two in $[-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{\pi}{8}), 0]$, and two in $[-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{3\pi}{8}), 0]$. Consequently, the relative discriminant's norm is constrained by

$$|\gamma - \bar{\gamma}|^2 \cdot 2^{12} \sin^4\left(\frac{k\pi}{5}\right) \sin^4\left(\frac{l\pi}{8}\right) \cdot (64 \cdot 25)^2 \geq D_8.$$

We use the Ω_8 region to have the upper bound for γ . We know that $\Delta_L^r = 2^{12} \times 5^4$ is a factor of the discriminant. In addition, 2 and 5 are ramified primes. Putting all previous conditions on our search in LMFDB gives us a special candidate. (See Figure 2.9)

From Figure 2.9, the polynomial $x^8 - 2x^7 - 5x^6 + 14x^5 - 18x^3 + 4x^2 + 6x - 1$ is a candidate which has γ in Ω_8 . This case has six real roots. Two of the real roots do not satisfy the distribution root condition $|4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{3\pi}{8})| \geq |r - s|$. This contradicts an arithmetic condition in Identification Theorem.

Label	Polynomial	Discriminant	Galois group	Class group
8.6.134560000.1	$x^8 - 3x^6 + 4x^2 - 1$	$-2^8 \cdot 5^4 \cdot 29^2$	$C_2 \wr C_2 \wr C_2$ (as 8T35)	trivial
8.6.134560000.2	$x^8 - 6x^4 + 5x^2 - 1$	$-2^8 \cdot 5^4 \cdot 29^2$	$C_2 \wr C_2 \wr C_2$ (as 8T35)	trivial
8.6.250210000.1	$x^8 - x^7 + x^6 - x^5 - 10x^4 + 3x^3 + 9x^2 - 1$	$-2^4 \cdot 5^4 \cdot 131 \cdot 191$	$S_4 \wr C_2$ (as 8T47)	trivial
8.6.292960000.1	$x^8 - 2x^7 - 4x^6 + 2x^5 + 7x^4 + 6x^3 - 4x^2 - 6x - 1$	$-2^8 \cdot 5^4 \cdot 1831$	$S_4 \wr C_2$ (as 8T47)	trivial
8.6.301960000.1	$x^8 - 3x^7 + x^6 + 7x^5 - 12x^4 + x^3 + 11x^2 - 4x - 1$	$-2^6 \cdot 5^4 \cdot 7549$	$S_4 \wr C_2$ (as 8T47)	trivial
8.6.316000000.1	$x^8 - 2x^7 + x^6 - 6x^5 - x^4 + 12x^3 - x^2 - 4x + 1$	$-2^8 \cdot 5^6 \cdot 79$	$((C_3 \wr C_2) : C_2) : C_2$ (as 8T27)	trivial
8.6.316840000.1	$x^8 - 2x^7 - 3x^6 + 3x^5 + 8x^4 - x^3 - 7x^2 + x + 1$	$-2^6 \cdot 5^4 \cdot 89^2$	$C_2 \wr C_2 \wr C_2$ (as 8T35)	trivial
8.6.358010000.1	$x^8 - 2x^7 - 5x^6 + 7x^5 + 10x^4 - 5x^3 - 7x^2 + x + 1$	$-2^4 \cdot 5^4 \cdot 35801$	$S_4 \wr C_2$ (as 8T47)	trivial
8.6.380000000.1	$x^8 - 4x^7 + 2x^6 + 8x^5 - 10x^4 + 2x^3 + 7x^2 - 6x + 1$	$-2^8 \cdot 5^7 \cdot 19$	$((C_3 \wr C_2) : C_2) : C_2$ (as 8T27)	trivial
8.6.380410000.1	$x^8 - x^7 - 5x^6 + 7x^5 - 2x^4 - 7x^3 + 7x^2 - 1$	$-2^4 \cdot 5^4 \cdot 109 \cdot 349$	$S_4 \wr C_2$ (as 8T47)	trivial
8.6.380960000.1	$x^8 - 5x^6 - 6x^5 - 7x^4 + 9x^2 + 6x + 1$	$-2^8 \cdot 5^4 \cdot 2381$	$S_4 \wr C_2$ (as 8T47)	trivial
8.6.386560000.1	$x^8 - 2x^7 - 5x^6 + 8x^5 + 5x^4 - 4x^3 - 3x^2 - 2x + 1$	$-2^{12} \cdot 5^4 \cdot 151$	$((C_4 \times C_2) : C_2) : C_2$ (as 8T31)	trivial
8.6.403360000.1	$x^8 - 2x^7 - 5x^6 + 14x^5 - 18x^3 + 4x^2 + 6x - 1$	$-2^8 \cdot 5^4 \cdot 2521$	$S_4 \wr C_2$ (as 8T47)	trivial

Figure 2.9: The list of polynomials for case $p = 8$

If $p=30$

For the last part, Let $p = 30$. The minimal polynomial P for γ is quadratic over L . The polynomial has a complex conjugate pair of roots γ and $\bar{\gamma}$ and a real root in γ and $\bar{\gamma}$ and 6 real roots. Therefore the discriminant (2.8.16) can be written as follows.

$$|\gamma - \bar{\gamma}|^2 \cdot 2^{12} \sin^4\left(\frac{k\pi}{5}\right) \sin^4\left(\frac{l\pi}{30}\right) \cdot (1125)^2 \geq D_8.$$

Using Ω_{30} , the upper bound for γ gives the upper bound 6×10^9 for the above discriminant. We have at hand $\Delta_L^r = 3^4 \times 5^6$ as a factor of the discriminant. Searching the minimal polynomial with these conditions in the LMFDB gives us 1856 cases. None of these possible polynomials satisfies the root restrictions we require. In particular, it implies no integral translation of any of these polynomials has all its authentic roots in an interval of length $[-4 \sin^2(\frac{k\pi}{5}) \sin^2(\frac{l\pi}{30}), 0]$.

If $p = 8, 30$, Cubic.

Let $p = 8$, and the minimal polynomial P of γ is a cubic over

$L = \mathbb{Q}(\cos(2\pi/5), \cos(2\pi/8))$. Then it has the complex conjugate pair of roots γ and $\bar{\gamma}$ and 10 real roots which, one real root in $[-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{\pi}{8}), 0]$, three in $[-4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{3\pi}{8}), 0]$, three in $[-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{\pi}{8}), 0]$, three in $[-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{3\pi}{8}), 0]$. Therefore the norm of the relative discriminant is bounded by

$$D_{12} \leq |\gamma - \bar{\gamma}|^2 |\gamma + 4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{\pi}{8})|^8 \cdot \prod^{k,l} (4 \sin^2(\frac{k\pi}{5}) \sin^2(\frac{l\pi}{8}))^6 \cdot (64 \cdot 25)^3.$$

Using Ω_8 , we have an upper bound for γ . We have the upper bound 2×10^{17} for the above discriminant. Then from the resource LMFDB, there are 712 such fields. We know that $\Delta_L^r = 2^{18} \times 5^6$ is a factor of discriminant, and 2 is the smallest ramified prime. From these conditions, among all possibilities, just the case $x^{12} - 7x^{10} + 14x^8 + x^6 - 28x^4 + 21x^2 - 1$ satisfies the conditions. This case does have discriminant field $2^{20} \times 5^6 \times 19^4$, and the Galois group T_{208} . This case has two real roots which do not satisfy the distribution root condition $|r-s| \leq 4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{3\pi}{8})$. This contradicts an arithmetic condition of the Identification Theorem.

If $p=30$

Next, assume $p = 30$. The minimal polynomial P of γ has the complex conjugate pair of roots γ and $\bar{\gamma}$ and real roots. Therefore the discriminant (2.8.16) is bounded by

$$D_{12} \leq |\gamma - \bar{\gamma}|^2 |\gamma + 4 \sin^2(\frac{\pi}{5}) \sin^2(\frac{\pi}{30})|^8 \prod^{k,l} (4 \sin^2(\frac{k\pi}{5}) \sin^2(\frac{l\pi}{30}))^6 \cdot (1125)^3$$

Using Ω_{30} region, the upper bound for γ gives us the upper bound 5×10^{22} for above discriminant. In addition, $\Delta_L^r = 5^6 \times 3^6$ is a factor of discriminant. Then from the resource LMFDB, there are 7 possibilities. These cases do not satisfy the distribution root condition $|r_i - r_j| \leq 3.461636252$. This contradicts an arithmetic condition in Identification Theorem.

2.8.5 The remaining cases.

What we have found so far is summarised in the following theorem.

Theorem 2.8.4. *Let $\Gamma = \langle f, g \rangle$ be a discrete group with $o(f) = 5$ and $o(g) = p$. Suppose that $\mu = [\mathbb{Q}(\gamma) : L] \geq 4$. Then Γ is free on these generators.*

Next, following the Discriminant Method, we shall try to show that the total degree of a monic and irreducible polynomial of $\gamma_p \in \Omega_p$, where $p \in \{2, 3, 4, 5, 10\}$, can be decreased to 8.

If $p = 3$

Let's suppose $p = 3$. We proved that the total degree for the minimal polynomial of $\gamma \in \Omega_3$ is $n \leq 18$. And the real roots are in $-3 \sin(\frac{2\pi}{5})^2 < \tau(\gamma) < 0$. Using the relative discriminant, the following inequality holds.

$$|\gamma - \bar{\gamma}|^2 |\gamma + 2|^{8(r-2)} (-3 \sin^2(\frac{2\pi}{5}))^{r(r-1)} (\frac{-3 \sin^2(\frac{\pi}{5})}{2})^{(r-2)(r-3)} \cdot M_{r-2} \cdot (5)^r \geq D_{2r}$$

We know that $\Delta_L^r = 5$ is a factor of the discriminant. Using the Ω_3 region, we have the upper bound for γ and then the upper bound for the discriminant. From the resource LMFDB, for each $[\mathbb{Q}(\gamma) : L] = r$, we can search all possible cases for the minimal polynomials of γ of degree $n = \mu r$, where the discriminant has a lower bound D_{2r} . Then the following results will appear.

1. If $r = 5$, Then there are 732 polynomials.
2. If $r = 6$, Then there are 16 polynomials.
3. If $r = 7$, Then there are 2 polynomials.
4. If $r = 8$, Then there are 53 polynomials.
5. If $r = 9$, Then there are 7 polynomials.

For each polynomial from the above list, by direct calculation, we have at least two real roots r and s such that $|r - s| \geq 3 \sin^2(\frac{\pi}{5})$. This means they do not satisfy the distribution root condition, which contradicts the arithmetic condition in Identification Theorem.

If $p = 4$

Let's suppose $p = 4$. We have proved that the total degree for a minimal polynomial of $\gamma \in \Omega_4$ is $n \leq 14$. The real roots are in $[-2 \sin(\frac{2\pi}{5})^2, 0]$. Following the relative discriminant, we have

$$|\gamma - \bar{\gamma}|^2 |\gamma + 2|^{8(r-2)} (-2 \sin^2(\frac{2\pi}{5}))^{r(r-1)} (\frac{-2 \sin^2(\frac{\pi}{5})}{2})^{(r-2)(r-3)} \cdot M_{r-2} \cdot (5)^r \geq D_{2r}.$$

We know that $\Delta_L^r = 5$ is a factor of the discriminant. Using Ω_4 , we have the upper bound for γ and then the upper bound for the discriminant. From the resource LMFDB, for each $[\mathbb{Q}(\gamma) : L] = r$, we can search all possible cases for the minimal polynomials of γ of degree $n = \mu r$, where the discriminant has a lower bound D_n . Thus, the following results will come up.

1. If $r = 5$, Then there are 732 polynomials.
2. If $r = 6$, Then there are 21 polynomials.
3. If $r = 7$, Then there are 4 polynomials.

For each polynomial from the above list, by direct calculation, we have at least two real roots r and s such that $|r - s| \geq 2 \sin^2(\frac{2\pi}{5})$. This means that they do not satisfy the distribution root condition. This contradicts our arithmetic condition. Therefore the total degree for the minimal polynomial of γ is at most 8.

If $p = 5$

Let's suppose $p = 5$. We have shown that the total degree for the minimal polynomial of $\gamma \in \Omega_5$ is $n \leq 10$. The real roots are in $[-4 \sin(\frac{2\pi}{5})^4, 0]$. Following the relative discriminant, we have

$$|\gamma - \bar{\gamma}|^2 |\gamma + 4 \sin^4(\frac{2\pi}{5})|^{8(r-2)} (-4 \sin^4(\frac{2\pi}{5}))^{r(r-1)} (\frac{-4 \sin^4(\frac{\pi}{5})}{2})^{(r-2)(r-3)}.$$

$$M_{r-2} \cdot (5)^r \geq D_{2r}.$$

$\Delta_L^r = 5$ is a factor of discriminant. We use the Ω_5 region to have the upper bound for γ and the upper bound for discriminant. From the resource LMFDB, for each $[\mathbb{Q}(\gamma) : L] = r$, we can search all possible cases for the minimal polynomials of γ of degree $n = \mu r$, where the discriminant has a lower bound D_n . Therefore the following result will come up.

1. If $r = 5$, Then there are 732 polynomials.

For each polynomial from the above list, by direct calculation, we have at least two real roots r and s such that $|r - s| \geq 4 \sin^4(\frac{2\pi}{5})$. This means that they do not satisfy the distribution root condition. This again contradicts our arithmetic condition. Therefore the total degree for the minimal polynomial of γ is at most 8.

If $p = 10$

Let's suppose $p = 10$. We have found that the total degree for the minimal polynomial of $\gamma \in \Omega_{10}$ is $n \leq 10$. The real roots are in $[-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{3\pi}{10}), 0]$. The relative discriminant for the minimal polynomial can be written as follows.

$$|\gamma - \bar{\gamma}|^2 |\gamma + 4 \sin^4(\frac{2\pi}{5})|^{8(r-2)} (-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{3\pi}{10}))^{r(r-1)} \cdot \left(\frac{-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{3\pi}{10})}{2}\right)^{(r-2)(r-3)} \cdot M_{r-2} \cdot (5)^r \geq D_{2r}$$

As we know that $\Delta_L^r = 5$ is a factor of the discriminant. We use Ω_{10} region to have the upper bound for γ . This gives us the upper bound for discriminant. From the resource LMFDB, for each $[\mathbb{Q}(\gamma) : L] = r$, we can search all possible cases for the minimal polynomials of γ of degree $n = \mu r$, where the discriminant has a lower bound D_n . The following results will come from the resource LMFDB.

1. If $r = 5$, Then there are 418 polynomials.

Each polynomial from the above list has at least two real roots r and s such that $|r - s| \geq |-4 \sin^2(\frac{2\pi}{5}) \sin^2(\frac{3\pi}{10})|$. This contradicts an arithmetic condition in Identification Theorem. This implies that the total degree for a minimal polynomial of γ is $n \leq 8$.

Chapter 3

Searching for γ -values:

$p \in \{2, 3, 4, 5, 10\}$, and total degree

$$n \leq 8.$$

At this point, we have very good information about the possibilities for p (in fact all but $p = 10$ have examples) and reasonable degree bounds. Thus we set about by using the Mathematica programme to find all possible candidates for the minimal polynomial of γ when $p \in \{2, 3, 4, 5, 10\}$, and the total degree is $n \leq 8$. Apart from the case $p = 10$ the real subfield has degree 2 and so the minimal polynomial has degree either 2, 3 or 4 over $\mathbb{Q}(-4 \sin^2(\frac{\pi}{5}))$.

Each search goes the same way:

- First we obtain bounds for the coefficients for the minimal polynomial for γ over \mathbb{Z} . These bounds are obtained from the following information:
 - $|\gamma| \leq M_p$ as γ must lie inside the region Ω_p . (Sometimes we may use the more refined information giving bounds on $\Im m(\gamma)$ as developed in the next chapter when the search is exceptionally large, but the Ω_p region is, as we have seen with the triangle groups, offers a sharp description of the real part).
 - From Jørgensen’s inequality, and related inequalities, we can also assume $|\gamma| \geq a_p$, [22].
 - All the real roots $r_i, i = 1, \dots, n - 2$ of this minimal polynomial lie in the interval $(-4 \sin^2 \frac{\pi}{p} \sin^2 \frac{\pi}{5}, 0)$ (again if $p \neq 10$).
 - Thus we can find bounds on the integer coefficients as they are elementary symmetric functions of things we have bounds for. (Note that we will usually search for $1 + \gamma$ to give better absolute value bounds on the real roots and so smaller ranges for the integer coefficients).
- The minimal polynomial for γ over \mathbb{Z} splits into two conjugate factors with coefficients in the integers of the field $\mathbb{Q}(-4 \sin^2(\frac{\pi}{5}))$. The coefficients of these factors are of the form $(a_i + ib_i\sqrt{5})/2$ with $a_i, b_i \in \mathbb{Z}$. Multiplying out and using the bounds on the integer coefficients we can inductively bound the a_i and b_i .
- Using these bounds we search through all possible polynomials removing those for which either
 - $|\gamma|$ is too large,

- has more than one pair of complex conjugate roots,
 - or finally that the real roots do not lie in the specified intervals.
- With the remaining list of polynomials we check the factorisation condition and the other criteria of the Identification Theorem.

In practice, we use some of the criteria from the Identification Theorem to avoid as many loops running as we can. For instance, if the degree is odd, then the value at 0 is positive and at $-4 \sin^2 \frac{\pi}{p} \sin^2 \frac{\pi}{5}$ is negative since all real roots lie in this interval. There are other elementary restrictions we build in but there is no substantive additional number theory used to limit searches. Such techniques can be found in [58, 21] the latter reference here identified all possible polynomials in the case of generators of order 2 and 3 in response to questions of Maclachlan and Martin.

Case $p = 3, r = 2$, Searching Code.

Let's suppose that $r = 2$, then the minimal polynomial has degree 4 and

$$(z^2 + ez + g)(z^2 + jz + i).$$

The first polynomial has roots γ and $\bar{\gamma}$, and the second has roots w , and s , Therefore

$$\begin{aligned} 0 \leq g = |\gamma|^2 &\leq 46.97; & -14 \leq e = -\gamma + \bar{\gamma} &\leq 8; \\ 0 \leq i = ws &\leq 7.36; & 0 \leq j = -w - s &\leq 5.42; \end{aligned}$$

Since we have assumed the first polynomial has a complex conjugate pair of roots,

so

$$\begin{aligned}g &= \frac{(p + q\sqrt{5})}{2}; & i &= \frac{(p - q\sqrt{5})}{2}; \\e &= \frac{(t + m\sqrt{5})}{2}; & j &= \frac{(t - m\sqrt{5})}{2};\end{aligned}$$

Then by putting bounds for p , q , m and t and following code

```

Degree2 = {}; Poly48 = {};

For[p = 0, p ≤ 28, p++,
For[q = Floor[(p - 14.72)/√5] + 1, q ≤ p/√5 + 1, q++,
For[t = -6, t ≤ 7, t++,
For[m = Floor[(t - 10.84)/√5] + 1, m ≤ t/√5 + 1, m++,
{g = (p + q√5)/2; i = (p - q√5)/2; .
e = (t + m√5)/2; j = (t - m√5)/2;
If[-4g + e² ≤ 0 && j² > 4i
&& Mod[t² - 5m², 4] == 0
&& Mod[p² - 5q², 4] == 0 &&
-5.42 ≤ -j - √(j² - 4i)
&& -j + √(j² - 4i) ≤ 0,
If[(Im[½(-e + √(-4g + e²))]/2.83)² + (Re[½(-e + √(-4g + e²))]/5)² ≤ 1, .
{Degree2 = Append[Degree2, ½(-e + √(-4g + e²))], .
Poly48 = Simplify[Append[Poly48, {(z² + ze + g)(z² + zj + i)}]}]]]]];

```

Using this search code, we find 44 candidates at hand, 39 of them are excluded as they do not lie in the Ω_3 region. Next, by applying Factorization conditions

$$\lambda = \cos\left(\frac{\pi}{5}\right) \pm \sqrt{\gamma + 3 \sin^2\left(\frac{\pi}{5}\right)}$$

for none of the remaining cases, except the polynomial $1 + 4z + 2z^2 + z^3 + z^4$, $\mathbb{Q}(\lambda)$ and $\mathbb{Q}(\gamma)$ have the same degree. This contradicts the Identification Theorem

apart from the one remaining case. This candidate is the group represented as

γ polynomial	λ polynomial
$1 + 4\gamma + 2\gamma^2 + \gamma^3 + \gamma^4$	$-9 + 9\lambda + 2\lambda^2 - 4\lambda^3 + \lambda^4$

Table 3.1: Case $p = 3$: List of minimal polynomials for γ and λ

$G = (5, 3; 3)$ with slope $1/4$. It is nearly arithmetic, but not of finite co-volume.

Case $p = 3$, $r = 3$, Searching Code

Suppose that $r = 3$, then the minimal polynomial for γ has the form

$$p(z) = (z^3 + a_2z^2 + a_1z + a_0)(z^3 + b_2z^2 + b_1z + b_0) \quad (3.0.1)$$

We continue to assume the first polynomial has a complex conjugate pair of roots.

We write, for odd p_i and q_i

$$2a_0 = p_0 + q_0\sqrt{5}, \quad 2b_0 = p_0 - q_0\sqrt{5}$$

$$2a_1 = p_1 + q_1\sqrt{5}, \quad 2b_1 = p_1 - q_1\sqrt{5}$$

$$2a_2 = p_2 + q_2\sqrt{5}, \quad 2b_2 = p_2 - q_2\sqrt{5}$$

The first polynomial factor has roots $\gamma, \bar{\gamma}, r$ with $r \in (-3 \sin^2 \frac{\pi}{5}, 0)$ and the second factor has roots $r_1, r_2, r_3 \in [-3 \sin^2 \frac{3\pi}{5}, 0]$. Thus

$$a_0 = -|\gamma|^2 r, \quad b_0 = -r_1 r_2 r_3.$$

$$a_1 = |\gamma|^2 + 2r \Re[\gamma], \quad b_1 = r_1 r_2 + r_2 r_3 + r_1 r_3.$$

$$a_2 = -2\Re[\gamma] - r, \quad b_2 = -(r_1 + r_2 + r_3).$$

From this we see that

$$\begin{aligned} 0 < a_0 < 23.1089436806, & \quad 0 < b_0 < 2.24303 \\ -1 < a_1 < 121, & \quad 0 < b_1 < 5.14058 \\ -22.3 < a_2 < 8.19, & \quad 0 < b_2 < 3.92705 \end{aligned}$$

Adding these inequalities provides bounds on p_i and q_i .

$$\begin{aligned} 0 < a_0 + b_0 = p_0 < 25.35 \\ -1 < a_1 + b_1 = p_1 < 126.14 \\ -22.3 < a_2 + b_2 = p_2 < 12.11 \end{aligned}$$

$$\begin{aligned} p_0 = 2n_0 + 1, \quad 0 \leq n_0 \leq 12, & \quad \left[\frac{2n_0 + 1 - 4.48}{\sqrt{5}} \right] + 1 \leq q_0 \leq \left[\frac{2n_0 + 1}{\sqrt{5}} \right], \\ p_1 = 2n_1 + 1, \quad 0 \leq n_1 \leq 62, & \quad \left[\frac{2n_1 + 1 - 10.28}{\sqrt{5}} \right] + 1 \leq q_1 \leq \left[\frac{2n_1 + 1}{\sqrt{5}} \right], \\ p_2 = 2n_2 + 1, \quad -11 \leq n_2 \leq 6, & \quad \left[\frac{2n_2 + 1 - 7.84}{\sqrt{5}} \right] + 1 \leq q_2 \leq \left[\frac{2n_2 + 1}{\sqrt{5}} \right]. \end{aligned}$$

This yields 138,662 cases to consider. Again we have a few conditions to satisfy. That the second polynomial has three real roots in $[-3 \sin^2 \frac{3\pi}{5}, 0]$, is our first target. Further, the first factor at (3.0.1) must have exactly one real root which lies in $[-3 \sin^2 \frac{\pi}{5}, 0]$. This certainly implies its value at $-3 \sin^2 \frac{\pi}{5}$ is negative. Finally, we must have the discriminant positive;

$$-27b_0^2 + b_1^2(-4b_1 + b_2^2) + 2b_0(9b_1b_2 - 2b_2^3) > 0.$$

We conclude there are no cases of a total degree of 6. For degree 8, the calculation is the same, and it follows that there is no non-free discrete group.

Case $p = 2, r = 2$, Searching Code

Let consider $r = 2$, then the minimal polynomial of $\gamma \in \Omega_2$ is quartic as follow;

$$(z^2 + ez + g)(z^2 + jz + i)$$

Assume that $\gamma, \bar{\gamma}$ are complex conjugate roots and r and s be real roots, then there are the following relations;

$$\begin{aligned} 0 \leq g = |\gamma|^2 \leq 16 & & -8 \leq e = -(\gamma + \bar{\gamma}) \leq 8; \\ 0 \leq i = st \leq 13.090169; & & 0 \leq j = -(s + t) \leq 7.236067977; \end{aligned}$$

By use of conjugacy of roots, we may express the integral coefficients as follows;

$$\begin{aligned} g &= \frac{(p + q\sqrt{5})}{2}; & i &= \frac{(p - q\sqrt{5})}{2}; \\ e &= \frac{(t + m\sqrt{5})}{2}; & j &= \frac{(t - m\sqrt{5})}{2}; \end{aligned}$$

With our bounds for p , q , t , and m , the following code gives 432 cases.

```

Poly49 = {};For[p = 0, p ≤ 15, p++,
For[q = Floor[ $\frac{-p}{\sqrt{5}}$ ], q ≤ Floor[ $\frac{32-p}{\sqrt{5}}$ ] + 1, q++,
If[p2 - 5q2 > 3, .For[t = -4, t ≤ 8, t++,
For[m = Floor[ $\frac{-t}{\sqrt{5}}$ ]
, m ≤ Floor[ $\frac{26.18033989-t}{\sqrt{5}}$ ] + 1, m++,
g =  $\frac{(p+q\sqrt{5})}{2}$ ; i =  $\frac{(p-q\sqrt{5})}{2}$ ;
e =  $\frac{(t+m\sqrt{5})}{2}$ ; j =  $\frac{(t-m\sqrt{5})}{2}$ ;
If[Discriminant[z2 + ez + g, z] ≤ 0
&&Discriminant[z2 + jz + i, z] > 0&&
Mod[p2 - 5q2, 4]==0
&&Mod[t2 - 5m2, 4]==0,
{Poly49 = Simplify[Append[Poly49, (z2 + e
z + g)(z2 + jz + i)]]}]]]]]]

```

In addition, these cases must satisfy distributed root conditions as follows.

```

Result99 = {};
For[i = 1, i ≤ Length[Poly49] + 1, i++,
For[j = 1, j ≤ Length[QQ] + 1, j++,
  {γ = Root[Poly49[[i]], 4];
  w = Root[Poly49[[i]], 1];
  y = Root[Poly49[[i]], 2];
  If[γ == QQ[[j]]&& -4sin[ $\frac{2\pi}{5}$ ]2 ≤ w ≤ 0&& -4sin[ $\frac{2\pi}{5}$ ]2 ≤ y ≤ 0, .
  {Result99 = Append[Result99, Poly49[[i]], Print[Poly49[[i]]]}]}]]

```

This decreases the number of cases to 12. These are below.

$-0.190983 + 0.981593i, -0.690983 + 0.722871i, -1. + 1.27202i,$
 $-0.690983 + 1.46307i, -1.5 + 0.606658i, -0.690983 + 1.23391i,$
 $-1.5 + 1.16963i, -0.690983 + 1.58825i, -1.19098 + 0.981593i,$
 $0.118034 + 0.606658i, 0.118034 + 1.16963i, -0.381966 + 1.27202i.$

Which all are mentioned in Tables.

Case $p = 2, r = 3$, Searching Code

Let suppose that $r = 3$, and the total degree be 6, then the minimal polynomial is

$$(z^3 + fz^2 + ez + g)(z^3 + kz^2 + jz + i).$$

The left cubic polynomial has two complex conjugate roots $\gamma, \bar{\gamma}$, and one real root r , and the second polynomial has three real roots. so

$$0 \leq g \leq 13; \quad -10 \leq e \leq 18; \quad -6 \leq f \leq 8;$$

$$0 \leq i \leq 48; \quad 0 \leq j \leq 39; \quad 0 \leq k \leq 11$$

Respectively, using the conjugates as before we obtain the following bounds;

$$\begin{aligned} g &= \frac{(p + q\sqrt{5})}{2}; & i &= \frac{(p - q\sqrt{5})}{2}; & k &= \frac{(b - n\sqrt{5})}{2} \\ e &= \frac{(t + m\sqrt{5})}{2}; & f &= \frac{(b + n\sqrt{5})}{2}; & j &= \frac{(t - m\sqrt{5})}{2} \end{aligned}$$

By searching through the following code,

```

Poly38 = {};

For[p = 0, p ≤ 30, p++,

For[q = Floor[ $\frac{-p}{\sqrt{5}}$ ] + 1, q ≤ Floor[ $\frac{26 - p}{\sqrt{5}}$ ] + 1, q++,

If[20 > p2 - 5q2 > 3, For[t = -5, t ≤ 30, t++,

For[m = Floor[ $\frac{-t - 20}{\sqrt{5}}$ ] + 1, m ≤ Floor[ $\frac{36 - t}{\sqrt{5}}$ ] + 1, m++, .

For[b = -3, b ≤ 10, b++,

For[n = Floor[ $\frac{-b - 12}{\sqrt{5}}$ ] + 1

, n ≤ Floor[ $\frac{16 - b}{\sqrt{5}}$ ] + 1, n++, .

{g =  $\frac{(p + q\sqrt{5})}{2}$ ; i =  $\frac{(p - q\sqrt{5})}{2}$ ; k =  $\frac{(b - n\sqrt{5})}{2}$ ; .

e =  $\frac{(t + m\sqrt{5})}{2}$ ; f =  $\frac{(b + n\sqrt{5})}{2}$ ; j =  $\frac{(t - m\sqrt{5})}{2}$ ;

-4sin[ $\frac{1\pi}{5}$ ]2 ≤ r ≤ 0; -4sin[ $\frac{2\pi}{5}$ ]2 ≤ s ≤ 0; -4sin[ $\frac{2\pi}{5}$ ]2 ≤ t ≤ 0;

-4sin[ $\frac{2\pi}{5}$ ]2 ≤ y ≤ 0; e =  $\frac{(t + m\sqrt{5})}{2}$ ; j =  $\frac{(t - m\sqrt{5})}{2}$ ; f =  $\frac{(b + n\sqrt{5})}{2}$ ;

If[Discriminant[z3 + kz2 + jz + i, z] > 0 && Mod[p2 - 5q2, 4] == 0 && .

Mod[b2 - 5n2, 4] == 0 && Mod[t2 - 5m2, 4] == 0 &&

1 + (f + k) + (j + e + kf) + (i + g + jf + ek) + (if + gk + ej) +

(jg + ie) + gi > 0 && Discriminant[z3 + fz2 + ez + g, z] ≤ 0,

{Poly38 = Simplify[Append[Poly38, z6 + (f + k)z5 + (j + e + kf)z4 +

(i + g + jf + ek)z3 + (if + gk + ej)z2 + (jg + ie)z + gi]]]]]]]]].

```

Using this code, we find 52 candidates, and none of them satisfied the Identification Theorem. For the calculation of degree 8, following the previous code, we

see that no candidate in the region satisfies the distribution root condition.

Case $p = 4, r = 2$, Searching Code

Let suppose that $r = 2$, then the minimal polynomial is

$$(z^2 + ez + g)(z^2 + jz + i). \quad (3.0.2)$$

The first polynomial has roots γ and $\bar{\gamma}$, and the second has roots w , and s , Therefore

$$\begin{aligned} 0 \leq g = |\gamma|^2 &\leq 84.53; & -18 \leq e = -(\gamma + \bar{\gamma}) &\leq 8; \\ 0 \leq i = ws &\leq 3.272542; & 0 \leq j = -w - s &\leq 3.618033; \end{aligned}$$

Let's assume the first polynomial has complex conjugate pair of roots, so

$$\begin{aligned} g &= \frac{(p + q\sqrt{5})}{2}; & i &= \frac{(p - q\sqrt{5})}{2}; \\ e &= \frac{(t + m\sqrt{5})}{2}; & j &= \frac{(t - m\sqrt{5})}{2}; \end{aligned}$$

Using our bounds on p , q , m , and t we can extract 12 cases from the following code.

```

Degree2 = {}; Poly58 = {};

For[p = 0, p ≤ 80, p++,
For[q = Floor[-p/√5] + 1, q ≤ p/√5 + 1, q++, .
For[t = -6, t ≤ 7, t++,
For[m = Floor[(t - 7.23606)/√5] + 1, m ≤ t/√5 + 1, m++, .
{g = (p + q√5)/2; i = (p - q√5)/2; .
e = (t + m√5)/2; j = (t - m√5)/2;
If[-4g + e² ≤ 0 && j² > 4i && Mod[t² - 5m², 4] == 0
&& Mod[p² - 5q², 4] == 0 && -5.42 ≤ -j - √(j² - 4i) && -j + √(j² - 4i) ≤ 0,
If[(Im[½(-e + √(-4g + e²))]/2.83)² + (Re[½(-e + √(-4g + e²))]/5)² ≤ 1, .
{Degree2 = Append[Degree2, ½(-e + √(-4g + e²))], .
Poly58 = Simplify[Append[Poly58, {(z² + ze + g)(z² + zj + i)}]]]]]]];

```

These cases are,

$$\begin{aligned}
&1 + 4z + 2z^2 + z^3 + z^4, 1 + 6z + 7z^2 + 4z^3 + z^4, 4 + 12z + 10z^2 + 4z^3 + z^4, \\
&1 + 12z - 2z^2 - 3z^3 + z^4, 1 + 8z + 3z^2 - 2z^3 + z^4, 1 + 11z + 6z^2 + z^3 + z^4, \\
&1 + 7z + 8z^2 + 2z^3 + z^4, 1 + 10z + 12z^2 + 5z^3 + z^4, 9 + 24z + 20z^2 + 7z^3 + z^4, \\
&5 + 15z + z^2 - 3z^3 + z^4, 5 + 15z + 9z^2 + z^3 + z^4, 5 + 20z + 14z^2 + 4z^3 + z^4.
\end{aligned}$$

Next, let's check the arithmeticity of these points. By applying the factorization

condition to these points

$$\lambda = 8 \cos^2\left(\frac{\pi}{5}\right) \cos^2 \frac{\pi}{4} \pm 2 \cos\left(\frac{\pi}{5}\right) \cos \frac{\pi}{4} \sqrt{16 \cos^2\left(\frac{\pi}{5}\right) \cos^2 \frac{\pi}{4} - 4(-4 \sin^2\left(\frac{\pi}{5}\right) + 4 \cos^2 \frac{\pi}{4} - \gamma)}$$

And this fact that γ must have the same total degree bound for a minimal polynomial of λ , we will have just two cases as follows.

$$1 + 6z + 7z^2 + 4z^3 + z^4, 5 + 20z + 14z^2 + 4z^3 + z^4.$$

These cases are two new nearly arithmetic, but not arithmetic groups represented by **Tet(4, 5; 3)** and **Tet(4, 5; 4)**, [35] where these are discussed. For total degrees 6 and 8 there is no arithmetic Kleinian group, see [43].

Case $p = 6, r = 2$, Searching Code

Let suppose that $r = 2$, then the minimal polynomial of $\gamma \in \Omega_6$ is

$$(z^2 + ez + g)(z^2 + jz + i). \tag{3.0.3}$$

The first polynomial has roots γ and $\bar{\gamma}$, and the second has roots w , and s , Therefore

$$\begin{aligned} 0 \leq g = |\gamma|^2 &\leq 125.95; & -8 \leq e = -(\gamma + \bar{\gamma}) &\leq 22.44; \\ 0 \leq i = ws &\leq .81813; & 0 \leq j = -w - s &\leq 1.8090; \end{aligned}$$

Suppose the first polynomial has complex conjugate pairs of roots, so the coefficients have the following relations.

$$\begin{aligned} g &= \frac{(p + q\sqrt{5})}{2}; & i &= \frac{(p - q\sqrt{5})}{2}; \\ e &= \frac{(t + m\sqrt{5})}{2}; & j &= \frac{(t - m\sqrt{5})}{2}; \end{aligned}$$

All possible candidates are obvious from the previous relations and the following code.

```

Degree2 = {}; Poly68 = {};

For[p = 0, p ≤ 63, p++,
For[q = Floor[-p/√5] + 1, q ≤ p/√5 + 1, q++, .
For[t = -4, t ≤ 12, t++,
For[m = Floor[-t - 16/√5] + 1, m ≤ t/√5 + 1, m++, .
{g = (p + q√5)/2; i = (p - q√5)/2; .
e = (t + m√5)/2; j = (t - m√5)/2;
If[-4g + e2 ≤ 0 && j2 > 4i && Mod[t2 - 5m2, 4] == 0
&& Mod[p2 - 5q2, 4] == 0 &&
-1.8090 ≤ -j - √(j2 - 4i) && -j + √(j2 - 4i) ≤ 0,
{Degree2 = Append[Degree2, 1/2(-e + √(-4g + e2))], .
Poly68 = Simplify[Append[Poly68, {z2 + ze + g)(z2 + zj + i)}]]]]]]].

```

We stop here at our continued progress and invite readers to see [36] for complete proof in this case.

Case $p = 10, r = 2$, Searching Code

Let consider $r = 2$, then the minimal polynomial of $\gamma \in \Omega$ is

$$(z^2 + ez + g)(z^2 + jz + i).$$

Let's suppose that w, s, γ , and $\bar{\gamma}$ be roots for polynomial. Then the coefficients and roots have relations as follows

$$0 \leq g = |\gamma|^2 \leq 144; \quad -24 \leq e = -(\gamma + \bar{\gamma}) \leq 8;$$

$$0 \leq i = ws \leq 5.6; \quad 0 \leq j = -w - s \leq 4.73;$$

More precisely, by conjugacy of roots, we have at hand

$$\begin{aligned} g &= \frac{(p + q\sqrt{5})}{2}; & i &= \frac{(p - q\sqrt{5})}{2}; \\ e &= \frac{(t + m\sqrt{5})}{2}; & j &= \frac{(t - m\sqrt{5})}{2}; \end{aligned}$$

The above relations give us useful bounds for p , q , t , and m . In addition, following the below code,

```

Degree2 = {}; Poly78 = {};

For[p = 0, p ≤ 70, p++,
For[q = Floor[ $\frac{p - 11.2}{\sqrt{5}}$ ] + 1, q ≤  $\frac{p}{\sqrt{5}}$  + 1, q++, .
For[t = -12, t ≤ 7, t++,
For[m = Floor[ $\frac{t - 9.46}{\sqrt{5}}$ ] + 1, m ≤  $\frac{t}{\sqrt{5}}$  + 1, m++, .
{g =  $\frac{(p + q\sqrt{5})}{2}$ ; i =  $\frac{(p - q\sqrt{5})}{2}$ ;
e =  $\frac{(t + m\sqrt{5})}{2}$ ; j =  $\frac{(t - m\sqrt{5})}{2}$ ;
If[-4g + e2 ≤ 0 && j2 > 4i && Mod[t2 - 5m2, 4] == 0
&& Mod[p2 - 5q2, 4] == 0 &&
-2.36 ≤ -j -  $\sqrt{j^2 - 4i}$  && -j +  $\sqrt{j^2 - 4i}$  ≤ 0,
If[( $\frac{Im[\frac{1}{2}(-e + \sqrt{-4g + e^2})]}{2.83}$ )2 + ( $\frac{Re[\frac{1}{2}(-e + \sqrt{-4g + e^2})]}{5}$ )2 ≤ 1, .
{Degree2 = Append[Degree2,  $\frac{1}{2}(-e + \sqrt{-4g + e^2})$ ], .
Poly78 = Simplify[Append[Poly78, {(z2 + ze + g)(z2 + zj + i)}]]]]]]]]].

```

Shows that there are two points $1.61803 + 2.05817i$, and $-0.5 + 2.56984i$, which fulfil two monic polynomials $1 + 8z + 3z^2 - 2z^3 + z^4$ and $1 + 7z + 8z^2 + 2z^3 + z^4$. These two choices must also meet the following factorization conditions as well.

$$\lambda = 8 \cos\left(\frac{\pi}{5}\right)^2 \cos\left(\frac{\pi}{10}\right)^2 \pm 2 \cos\left(\frac{\pi}{5}\right) \cos\left(\frac{\pi}{10}\right) \sqrt{16 \cos\left(\frac{\pi}{5}\right)^2 \cos\left(\frac{\pi}{10}\right)^2 - 4(-\gamma - 4 \sin\left(\frac{\pi}{5}\right)^2 + 4 \cos\left(\frac{\pi}{10}\right)^2)}.$$

γ polynomial	λ polynomial
$1 + 8\gamma + 3\gamma^2 - 2\gamma^3 + \gamma^4$	$275 + 300\lambda - 625\lambda^2 - 200\lambda^3 + 370\lambda^4 - 190\lambda^5 + 120\lambda^6 - 20\lambda^7 + \lambda^8$
$1 + 7\gamma + 8\gamma^2 + 2\gamma^3 + \gamma^4$	$1025 + 825\lambda - 2325\lambda^2 - 450\lambda^3 + 1545\lambda^4 - 610\lambda^5 + 160\lambda^6 - 20\lambda^7 + \lambda^8$

They do not satisfy the Theorem 2.7.4.

Case $p = 10, r = 3$, Searching Code

Suppose that $r = 3$, and the minimal polynomial for γ be

$$p(z) = (z^3 + a_2z^2 + a_1z + a_0)(z^3 + b_2z^2 + b_1z + b_0) \quad (3.0.4)$$

We continue to assume the first polynomial has a complex conjugate pair of roots.

We write, for odd p_i and q_i

$$\begin{aligned} 2a_0 &= p + q_0\sqrt{5}, & 2b_0 &= p - q_0\sqrt{5} \\ 2a_1 &= p_1 + q_1\sqrt{5}, & 2b_1 &= p_1 - q_1\sqrt{5} \\ 2a_2 &= p_2 + q_2\sqrt{5}, & 2b_2 &= p_2 - q_2\sqrt{5} \end{aligned}$$

The first polynomial factor has roots $\gamma, \bar{\gamma}, r$ with $r \in (-4 \sin^2 \frac{\pi}{10} \sin^2 \frac{\pi}{5}, 0)$ and the second factor has roots $r_1, r_2, r_3 \in [-4 \sin^2 \frac{3\pi}{10} \sin^2 \frac{3\pi}{5}, 0]$. Thus

$$\begin{aligned} a_0 &= -|\gamma|^2 r, & b_0 &= -r_1 r_2 r_3. \\ a_1 &= |\gamma|^2 + 2r \Re[\gamma], & b_1 &= r_1 r_2 + r_2 r_3 + r_1 r_3. \\ a_2 &= -2\Re[\gamma] - r, & b_2 &= -(r_1 + r_2 + r_3). \end{aligned}$$

From this, we see that

$$0 < a_0 < 20.25, \quad 0 < b_0 < 14$$

$$0 < a_1 < 150, \quad 0 < b_1 < 16$$

$$-9 < a_2 < 25, \quad 0 < b_2 < 8$$

Adding these inequalities provides bounds on p_i and q_i .

$$0 < a_0 + b_0 = p_0 < 18$$

$$-1 < a_1 + b_1 = p_1 < 84$$

$$-4 < a_2 + b_2 = p_2 < 17$$

This yields 5326 cases to consider. Again we have a few conditions to satisfy.

That the second polynomial has three real roots in $[-4 \sin^2 \frac{3\pi}{10} \sin^2 \frac{3\pi}{5}, 0]$ is our first target. Further, the first factor must have exactly one real root, which lies in $[-4 \sin^2 \frac{\pi}{10} \sin^2 \frac{\pi}{5}, 0]$. Finally, we must have the discriminant positive;

$$-27b_0^2 + b_1^2(-4b_1 + b_2^2) + 2b_0(9b_1b_2 - 2b_2^3) > 0.$$

We conclude there is no case of a total degree of 6. For the total degree of 8, the calculation is the same, and it can be seen there is no candidate.

Chapter 4

Finding the groups: Keen-Series theory in the elliptic case.

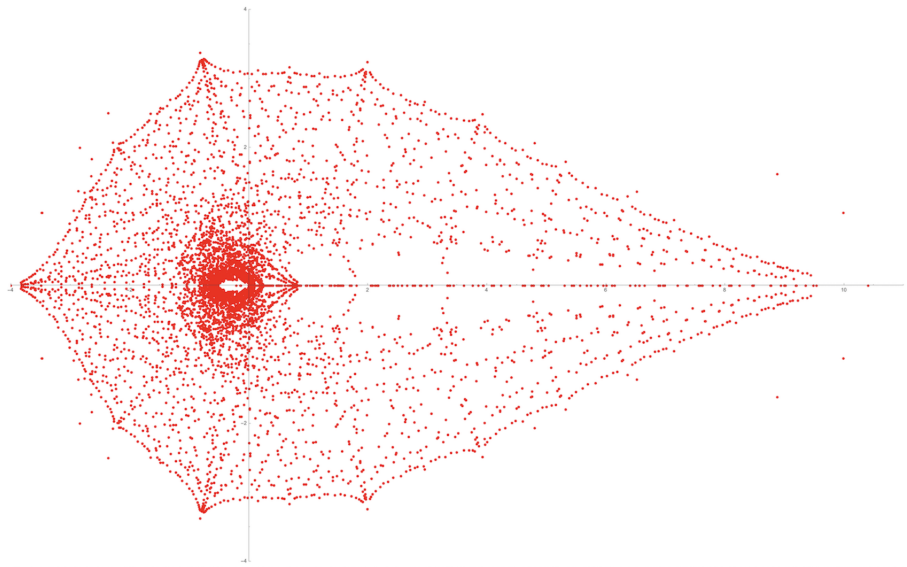


Figure 4.1: The exterior of the quasiconformal deformation space of groups generated by two elements of order 5 analogous to the Riley slice: the quasiconformal deformation space of groups generated by two parabolic elements.

At this point in the thesis, for each p , $2 \leq p \leq \infty$, we have shown that there are no nearly arithmetic $(5, p)$ groups if $p \geq 6$, and for each $2 \leq p \leq 5$ a (relatively) short list of candidate groups, that is the possible γ values for the group with parameters

$$\left(\gamma, -4 \sin^2 \frac{\pi}{p}, -4 \sin^2 \frac{\pi}{5}\right)$$

to be a subgroup of an arithmetic lattice that is not “obviously” free on its generators.

This chapter is more explanatory as, in effect, we simply carry out the partial algorithms alluded to in [19] to find computational descriptions of deformation spaces of groups generated by two primitive elliptic. We also motivate the (nowhere stated) conjecture that if there is a nontrivial relator in such a group, then there is a Farey word which is either the identity of elliptic through a discussion of two bridge knots and links and the important invariant of *rational slope*.

First, we need to review some of the Keen-Series theory associated with the Riley slice in the setting of groups generated by two elliptic, see [18, 19]. This is ultimately to remove some of our candidates which are free, and then identify those which are not.

Figure 4.2 is a computation description of the exterior of the quasiconformal deformation space of Kleinian groups free on two generators of order 5 and is an analogue of the Riley slice as given in [18, 19]. The black points are solutions of the equation $tr(W_{r/s}) = -2$, where $W_{r/s}$ is a Farey word (which we define after our discussion of two-bridge knots and links). For each slope r/s two of these points

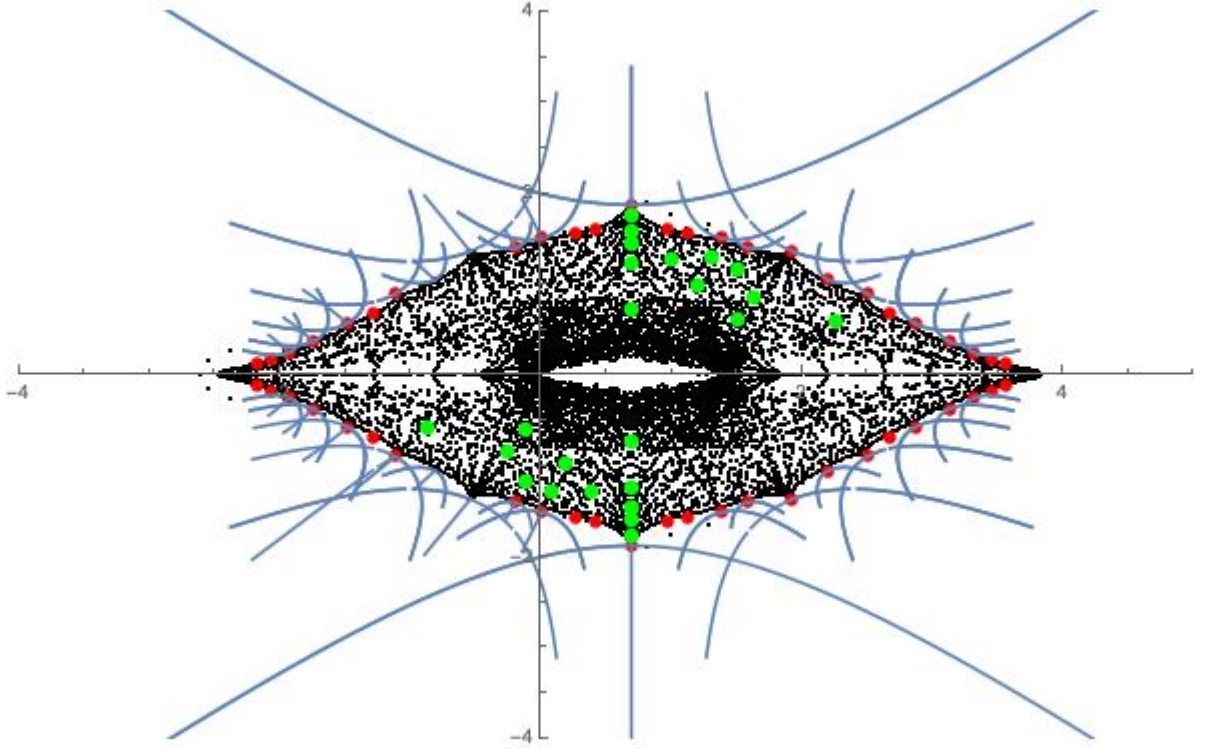


Figure 4.2: $\mathfrak{R}_{5,5}$ analogue of the Riley slice for groups generated by two elliptic of order 5. The points that we find by our searchers are the green points, while the red points are cusp groups.

will be cusp groups on the boundary and the remainder are (almost always) not discrete - the cusp groups r/s where $r = 1$ and $1 \leq s \leq 20$ are illustrated in red. It is clear these groups associated with solutions to $\text{tr}(W_{r/s}) = -2$ do not lie in the open set of discrete Schottky groups as $W_{r/s}$ is parabolic or the identity and so arbitrary small deformations will give elliptic elements of infinite order.

Also shown are the “half spaces” $\{\Re[\text{tr}(W_{r/s})] < -2\}$ where the peripheral subgroups have disjoint isometric circles and which lie entirely within the freely generated space.

The green points are the candidates we have found from our search in the $(5, 5)$ case.

We now need to explain the terms in the above description. This leads us to the topological description of the orbifolds \mathbb{H}^3/Γ for the groups we will find.

4.1 Two-bridge knots and links.

Let us begin by recalling some basic facts about knots and links. All knots and links with less than ten crossings are rational or may be produced by placing rational tangles into a small number of planar graphs, see [32]. Rational knots are expressed as follows:

1. Two-bridge knots. A 2-bridge knot is one with two separate arcs, each overpassing a sequential sequence of crossings, and each crossing in the figure is in one of these sequences. The two arcs are referred to as the diagram's bridges.
2. Closures of rational tangles in the numerator or denominator. Successive twists on adjacent lines form a logical tangle.
3. Four-strand braid plant closures.

All three representations are interchangeable. Conway introduced the concept of a tangle in his work enumerating and categorizing knots and links in 1967. He defined rational knots as numerator or denominator closures of rational tangles. Conway also described a rational tangle's fraction as a rational number or ∞ . He

discovered that given a rational tangle; this number equals a continuous fraction expression with all numerators equal to one and all denominators of the same sign, which can be read from an alternating standard form tangle diagram. Schubert

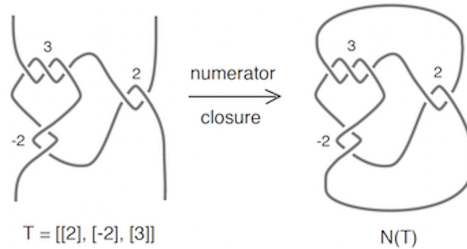


Figure 4.3: A rational tangle and a rational knot

first established the categorization of rational knots and links as 2-bridge links. He proposed the following theorem:

Theorem 4.1.1. *Assume you have rational tangles with fractions $\frac{r}{s}$ and $\frac{m}{n}$. Here r and s are relatively prime as are m and n . If $K(\frac{r}{s})$ and $K(\frac{m}{n})$ indicate the corresponding rational knots formed by taking numerator closures of these tangles, then these knots are isotopic if and only if*

1. $r = m$
2. Either $\text{Mod}[s, r] = n$ or $\text{Mod}[sn, r] = 1$.

Also, we have the following converse:

Theorem 4.1.2. *(Conway, 1975) Two rational tangles are isotopic if and only if they have the same fraction.*

Definition 4.1.3. A 2-tangle is a correct embedding of two un-oriented arcs and a finite number of circles in a 3-ball \mathbb{B}^3 , with the four ends inside the boundaries of

\mathbb{B}^3 . A rational tangle is a proper embedding of two un-oriented arcs α_1, α_2 in a \mathbb{B}^3 , with the four ends lying in the boundary of \mathbb{B}^3 and a pair homeomorphism:

$$h : (\mathbb{B}^3, \alpha_1, \alpha_2) \rightarrow (\mathbb{D}^2 \times \mathbb{I}, \{x, y\} \times \mathbb{I}).$$

This is analogous to asserting that rational tangles have specified representations that may be created by applying a finite number of successive twists of neighbouring endpoints beginning with two unknotted and unlinked arcs. For more details, see [57, 10, 32].

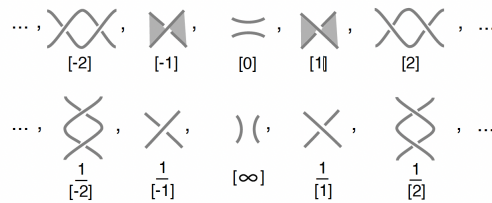


Figure 4.4: The elementary rational tangles and the types of crossings.

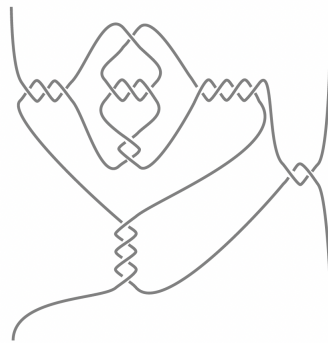


Figure 4.5: A rational tangle in twist form.

A rational tangle in standard form can be described algebraically by a continued fraction built from the integer tangles $[a_1], [a_2], \dots, [a_n]$ with all numerators equal to

1, By an expression of the type:

$$T = [[a_1], [a_2], \dots, [a_n]] := [a_1] + \frac{1}{[a_2] + \dots + \frac{1}{[a_{n-1}] + \frac{1}{[a_n]}}$$

For $a_2, \dots, a_n \in \mathbb{Z} - \{0\}$ and n even or odd. We allow $[a_1]$ to be the tangle $[0]$. This expression follows inductively from the following equation,

$$T * \frac{1}{[n]} = \frac{1}{[n] + \frac{1}{T}} \quad (4.1.1)$$

Then a rational tangle is said to be in continued fraction form. For $T = [[a_1], [a_2], \dots, [a_n]]$ the following statements are now straightforward.

1. $T + [\pm 1] = [[a_1 \pm 1], [a_2], \dots, [a_n]]$,
2. $\frac{1}{T} = [[0], [a_1], [a_2], \dots, [a_n]]$,
3. $-T = [[-a_1], [-a_2], \dots, [-a_n]]$,
4. $T = [[a_1], [a_2], \dots, [a_n - 1], [1]]$, if $a_n > 0$,
5. $T = [[a_1], [a_2], \dots, [a_n + 1], [-1]]$, if $a_n < 0$.

Definition 4.1.4. From the above discussion, it makes sense to assign to a rational tangle in standard form, $T = [[a_1], [a_2], \dots, [a_n]]$, for $a_1 \in \gamma$, $a_2, \dots, a_n \in \mathbb{Z} - \{0\}$ and n even or odd, the continued fraction

$$F(T) = [a_1, a_2, \dots, a_n] := a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_{n-1} + \frac{1}{a_n}}} \quad (4.1.2)$$

If $T \neq \infty$, and $F(\infty) := \infty = \frac{1}{0}$, as a formal expression. For any rational number r, s the following statements are straightforward.

1. $\frac{r}{s} + [\pm 1] = [[a_1 \pm 1], [a_2], \dots, [a_n]]$,

2. $\frac{s}{r} = [[0], [a_1], [a_2], \dots, [a_n]],$
3. $-\frac{r}{s} = [[-a_1], [-a_2], \dots, [-a_n]],$
4. $\frac{r}{s} = [[a_1], [a_2], \dots, [a_n - 1], [1]],$ if $a_n > 0,$
5. $\frac{r}{s} = [[a_1], [a_2], \dots, [a_n + 1], [-1]],$ if $a_n < 0.$

Example 4.1.5. Suppose that $[a_1, a_2, \dots, a_n]$ is a continued fraction for tangle, then



Figure 4.6: The ambiguity of the first crossing

4.2 Farey words and polynomials.

We now have the definition of the words we want to examine.

Definition 4.2.1. A Farey word $W_{r/s}$ is the relator coming from the Wirtinger presentation of a two-bridge knot or link with rational slope r/s . Thus $W_{r/s} = W_{r/s}(x, y)$ where x and y are (meridian) loops around the two bridges that generate the fundamental group.

We now bring these ideas back to our search.

Let

$$X = \begin{pmatrix} e^{i\pi/5} & 1 \\ 0 & e^{-i\pi/5} \end{pmatrix}, \quad Y = \begin{pmatrix} e^{i\pi/p} & 0 \\ z & e^{-i\pi/p} \end{pmatrix} \quad (4.2.1)$$

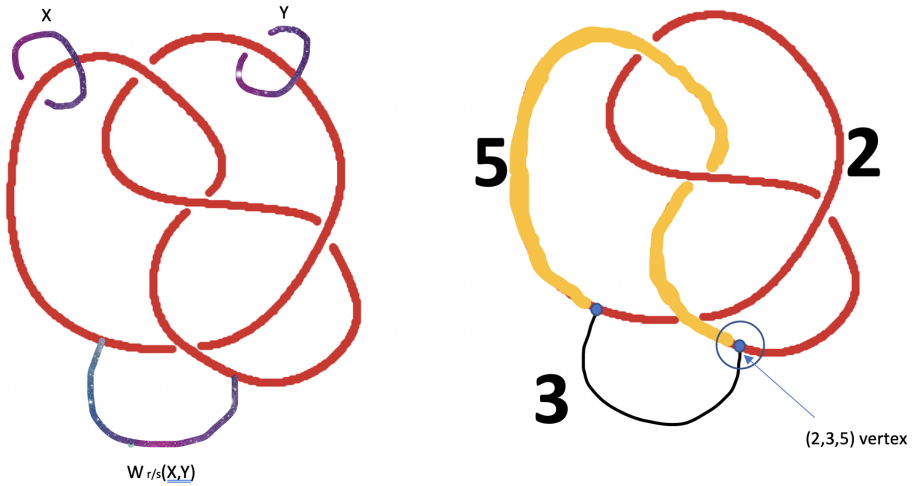


Figure 4.7: **Left:** 3/5 knot with generators X and Y for the fundamental group and $W_{r/s}(X, Y)$ represented by an unknotting tunnel. **Right:** The knot 3/5 filled by orbifold Dehn surgeries to give an orbifold with presentation $\langle X^5 = Y^2 = W_{r/s}^3 = 1 \rangle$. This orbifold is guaranteed hyperbolic by the Orbifold Theorem of Thurston (Geometrization of 3-orbifolds, [8]).

be elliptics of order 5 and p in $PSL(2, \mathbb{C})$.

We define the $5, p$ Farey polynomial with slope r/s as

$$F5p(z; r/s) = \text{tr}(W_{r/s}(X, Y)), \quad (4.2.2)$$

where X and Y are defined at (4.2.1) and $W_{r/s}$ is the relator for the two bridge knots or link of slope r/s . It is a polynomial of degree s in the variable z and has coefficients which are integers in $\mathbb{Q}(4 \sin^2 \frac{\pi}{p}, 4 \sin^2 \frac{\pi}{5})$. We record the following elementary calculation.

Lemma 4.2.2. *With X and Y as above we have*

$$\gamma(X, Y) = 2 - 4z \sin\left(\frac{\pi}{5}\right) \sin\left(\frac{\pi}{p}\right) + z^2.$$

The first connection we wish to observe is in the following result. It is in part a consequence of the orbifold theorem and standard facts about 2 bridge knots and links. We refer to Thurston's book [62] for the reader to discover what Dehn surgery is and the basic facts concerning it.

Remark 4.2.3. Let $K(r/s)$ be a two bridge link. Then performing $(5, 0)$ and $(p, 0)$ Dehn surgeries on different components of the link complement $\mathbb{S}^3 \setminus K(r/s)$ gives a hyperbolic 3 orbifold with base space \mathbb{S}^3 . This orbifold, say $\mathcal{O}_{5,p;r/s}$, has (orbifold) fundamental group with a presentation of the form

$$\langle f, g : f^5 = g^p = W_{r/s}(f, g) = 1 \rangle$$

and $F5p(z; r/s) = 2$, where

$$\gamma(X, Y) = 2 - 4z \sin\left(\frac{\pi}{5}\right) \sin\left(\frac{\pi}{p}\right) + z^2. \quad (4.2.3)$$

Conversely, if \tilde{f} and \tilde{g} are primitive elliptic elements of order 5 and p represented by X and Y at (4.2.1), if z is defined by (4.2.3) and has $F5p(z; r/s) = 2$ and if $\tilde{\Gamma} = \langle \tilde{f}, \tilde{g} \rangle$ is discrete and hyperbolic, then it is a factor of the group Γ , and hence has smaller co-volume. In fact if $m = |\text{Ker}(\phi)|$, where ϕ is the homomorphism induced by

$$f \xrightarrow{\phi} \tilde{f}, \quad g \xrightarrow{\phi} \tilde{g}$$

then

$$\text{Vol}(\mathbb{H}^3/\tilde{\Gamma}) = \text{Vol}(\mathcal{O}_{5,p;r/s})/m. \quad (4.2.4)$$

Most often this index is $m = 1$, but it is not always. See our earlier examples in Section 2.7.3 referencing [35] concerning the tetrahedral groups and the generalised triangle groups which both have generators of the same order and elliptic commu-

tators of the same order. The commutator is primitive in the generalised triangle groups.

We remark that there may be many solutions z to the equation $F5p(z) = -2 \cos \frac{\pi}{p}$ (in fact there are s counting possible multiplicity) and for each such solution $W_{r/s}(f, g)$ is an elliptic element of order p ($p = 1$ meaning the identity). An interesting problem here is determining which of these groups is discrete. We circumvent this problem with the arithmetic criteria.

The Farey polynomials are a class of polynomials that are also interesting as combinatorial objects in their own right and have strong links to hypergeometric functions, see [30, 20]. In [20] a combinatorial recursion algorithm is obtained which enables us to quickly find a Farey polynomial without having to go through the topological representations and explicitly find the word $W_{r/s}$.

4.2.1 The Riley slice.

At this point, we give a brief description of the Riley slice \mathfrak{R} which basically is the deformation space parameterizing the complex structures on the four-times punctured sphere \mathbb{S}_4^2 . More precisely, define a family Γ_μ for $\mu \in \mathbb{C} \setminus \{0\}$, of subgroups of $PSL(2, \mathbb{C})$ by $\Gamma_\mu = \langle f, g_\mu \rangle$ where f and g are represented by X and Y

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \quad (4.2.5)$$

are parabolic. The group Γ_μ acts on the Riemann sphere $\hat{\mathbb{C}}$, and there is the largest open (possibly empty) set $\Omega(\Gamma_\mu)$ on which this group acts discontinuously.

If $\Omega(\Gamma_\mu) \neq \emptyset$ the quotient $\Omega(\Gamma_\mu)/\Gamma_\mu$ is a Riemann surface. The Riley slice is defined by

$$\mathfrak{R} = \{\mu : \Omega(\Gamma_\mu)/\Gamma_\mu \text{ is homeomorphic to } \mathbb{S}_4^2\} \quad (4.2.6)$$

The Riley slice may alternatively be identified with the quotient of the Teichmüller space $\tau_{0,4}$ (genus 0 surface with 4 punctures) by a Dehn twist over a simple closed curve that divides one pair of punctures from another in \mathbb{S}_4^2 . It is the mapping class group because $\tau_{0,4}$ is simply connected, and the Dehn twist provides an infinite cyclic group working effectively and discretely on $\tau_{0,4}$. As a result, the Riley slice is a topological annulus with an intrinsic hyperbolic metric that agrees with the Teichmüller metric. The Riley slice has a foliation structure according to Keen and Series theory [29, 30, 31]. This structure is made up of a set of curves parameterized by the rationals \mathbb{Q} that radiate out from the slice's boundary and which are dense in the slice (the so-called rational pleating rays) and a natural completion (in the sense that we can add curves parameterized by \mathbb{R}/\mathbb{Q} to fill out the entire slice). The figure below depicts the Riley slice along with a number of pleating rays. The Riley slice's exterior contains all the discrete but not free groups Γ_μ , such as all hyperbolic two-bridge knot complements. These are found on or near the end of *extended* pleating rays (as illustrated below). Recently, [4, 5] comprehensively described all the discrete groups beyond the Riley slice as Heckoid groups and their close cousins. There are at most two Nielsen classes of parabolic generating pairs for any such group. Riley's boundary is a Jordan curve with outwardly pointed cusps. Non-discrete groups are generically free, but every neighbourhood of a non-discrete group contains a supergroup of any two parabolic groups — discrete or otherwise — plus a group with any stipulated number of different Nielsen classes.

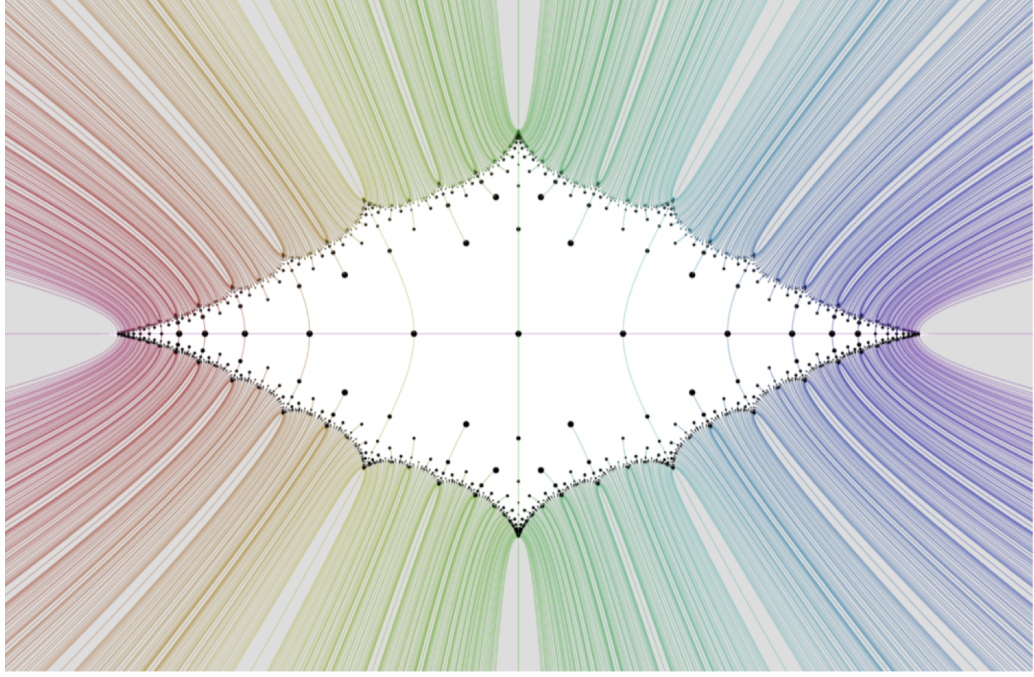


Figure 4.8: Riley's slice is the unbounded region that is foliated by 'rational pleating rays' (the coloured curves). This space's symmetries include complex conjugation and $\mu \leftrightarrow -\mu$ interchanging g with g^{-1} . Y. Yamashita provided the image.

Now, without proof, we review various theorems and lemmas. For more details, see [20].

Lemma 4.2.4. *Let p/q be the rational slope and $W_{p/q}$ a Farey word. Then $W_{p/q}$ has word length $2q$. Further:*

1. *If q is even, then there are $u, v \in \langle x, y \rangle$ that $W_{p/q} = xux^{-1}u^{-1} = v y v^{-1} y^{-1}$*
2. *If q is odd, then there are $u, v \in \langle x, y \rangle$ such that $W_{p/q} = xuyu^{-1} = vx^{-1}v^{-1}y^{-1}$*

In particular, if q is even, $W_{p/q}$ is a commutator in two different ways. The word

length of $W_{p/q}$ is $2q$.

We can view $W_{p/q}$ as a word $W_{p/q}(\mu)$ in Γ_μ by performing the substitution $x \rightarrow X$, and $y \rightarrow Y_\mu$:

$$W_{p/q} = \begin{bmatrix} a_{p/q}(\mu) & b_{p/q}(\mu) \\ c_{p/q}(\mu) & d_{p/q}(\mu) \end{bmatrix}$$

Such that $a_{p/q}(\mu)d_{p/q}(\mu) - c_{p/q}(\mu)b_{p/q}(\mu) = 1$. The entries of $W_{p/q}$ are polynomials of degree q in the symbol μ . In particular, the $\text{trace}(W_{p/q}(\mu))$ is a polynomial of degree q in μ and it is this we have earlier called the Farey polynomial of slope p/q

The groups on the boundary of the Riley slice for which $\Omega(\Gamma_\mu)/\Gamma_\mu$ is a disjoint union of triply punctured spheres are called cusp groups.

Keen and Series [29] study the boundary of the Riley slice by considering what happens for a fixed slope p/q as $\text{trace}(f_{p/q}) \rightarrow -2$, $\text{trace}(f_{p/q}) \in \mathbb{R}$. Keen and Series show that the Farey polynomial $P_{p/q}$ has a branch of its inverse so that the pleating ray $R_{p/q}$ is

$$R_{p/q} = P_{p/q}^{-1}(-\infty, 2]. \quad (4.2.7)$$

This ray lies entirely in the closure of the Riley slice and meets the boundary at a point μ corresponding to a cusp group where $P_{p/q}(\mu) = -2$. These cusp groups have a limit set consisting of a circle packing. A result of McMullen [33] shows these limits to be dense in the boundary of the Riley slice.

The main theorem we want to mention is the following theorem generalising Keen-Series' results which appears in [18].

Theorem 4.2.5. *Let $P_{p/q}$ be a Farey polynomial. Then there is a branch of the*

inverse of $P_{p/q}$ such that

$$P_{\frac{p}{q}}^{-1}(H_{-2}) \subset \mathfrak{R}, \quad H_{-2} = \Re(\gamma) < -2. \quad (4.2.8)$$

The bounds in the theorem are illustrated as Figure 4.9.

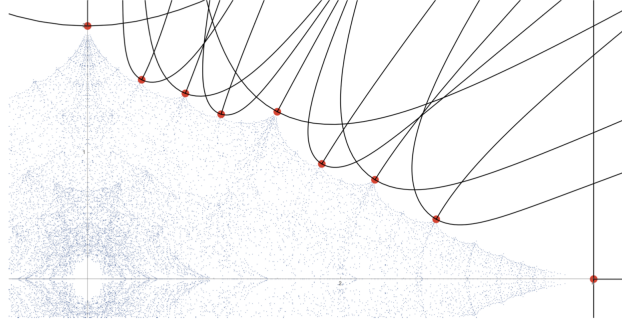


Figure 4.9: The Riley slice with neighbourhoods illustrated for our pleating ray values.

We are going to use the following generalisation of this result to the case of groups generated by two elliptic. In particular, in our case of groups generated by elliptic of order 5 and p , and the polynomials $F5p(z; r/s)$ we have

Theorem 4.2.6. *Let $r/s \in [0, 1]$ be a Farey fraction. There is a branch of the inverse of $p(z) = F5p(z; r/s)$ such that*

$$p^{-1}(H_{-2}) \subset \mathfrak{R}(5, p), \quad H_{-2} = \Re(\gamma) < -2. \quad (4.2.9)$$

Here, in analogy with the Riley slice, we have

$$\mathfrak{R}_{5,p} = \{ \mu : \Omega(\Gamma_\mu) / \Gamma_\mu \text{ is homeomorphic to } \mathbb{S}_{5,5,p}^2 \} \quad (4.2.10)$$

- $\mathbb{S}_{5,5,p}^2$ is the Riemann surface which is the Riemann sphere with 4-cone points. Two of which are cone points of order 5 and the other two are cone points of order p .

- The group Γ_μ is

$$\Gamma_\mu = \langle X, Y \rangle$$

where X and Y are defined by (4.2.1).

We recall Farey's addition. If r_1/s_1 and r_2/s_2 are *Farey neighbours*, that is if $r_1s_2 - r_2s_1 = \pm 1$, then

$$\frac{r_1}{s_1} \oplus \frac{r_2}{s_2} = \frac{r_1 + r_2}{s_1 + s_2}$$

and

$$\frac{r_1}{s_1} \ominus \frac{r_2}{s_2} = \frac{|r_1 - r_2|}{|s_1 - s_2|}$$

Let

$$\alpha = e^{i\pi/5}, \quad \beta = e^{i\pi/p} \tag{4.2.11}$$

The recursion formulas given in [20] are the following. If r_1/s_1 and r_2/s_2 are Farey neighbours, then

$$\begin{aligned} & F5p\left(z; \frac{r_1}{s_1} \oplus \frac{r_2}{s_2}\right) + F5p\left(z; \frac{r_1}{s_1}\right)F5p\left(z; \frac{r_2}{s_2}\right) + F5p\left(z; \frac{r_1}{s_1} \ominus \frac{r_2}{s_2}\right) \\ &= \begin{cases} 4 + 1/\alpha^2 + \alpha^2 + 1/\beta^2 + \beta^2 & s_1 + s_2 \text{ even} \\ 2(\alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}) & s_1 + s_2 \text{ odd} \end{cases} \end{aligned}$$

Then defining the initial polynomials,

1. **FareyPolynomial** $[0, 1, \alpha, \beta] := \frac{\alpha}{\beta} + \frac{\beta}{\alpha} - \gamma$;
2. **FareyPolynomial** $[1, 1, \alpha, \beta] = \alpha\beta + \frac{1}{\alpha\beta} + \gamma$;
3. **FareyPolynomial** $[1, 2, \alpha, \beta] = 2 + (\alpha\beta - \frac{\alpha}{\beta} - \frac{\beta}{\alpha} + \frac{1}{\alpha\beta})\gamma + \gamma^2$;

In addition, the recursion gives us the polynomials as follows.

```

FareyPolynomial[p_, q_, α_, β_] := Module[{p1, p2, q1, q2},
  {p1, q1} = LeftFraction[p, q];
  {p2, q2} = RightFraction[p, q];
  Expand[4 + 1/α2 + α2 + 1/β2 + β2 -
  FareyPolynomial[p1, q1, α, β] * FareyPolynomial[p2, q2, α, β] -
  FareyPolynomial[Abs[p2 - p1], Abs[q2 - q1], α, β]] / EvenQ[q].

```

```

FareyPolynomial[p_, q_, α_, β_] := Module[{p1, p2, q1, q2},
  {p1, q1} = LeftFraction[p, q];
  {p2, q2} = RightFraction[p, q];
  Expand[2(αβ +  $\frac{\alpha}{\beta}$  +  $\frac{\beta}{\alpha}$  +  $\frac{1}{\alpha\beta}$ ) - FareyPolynomial[p1, q1, α, β] *
  FareyPolynomial[p2, q2, α, β] - FareyPolynomial[Abs[p2 - p1], Abs[q2 - q1], α, β]
  OddQ[q];

```

If we now draw the pleating rays $p^{-1}((-\infty, -2])$ and the “half spaces” $p^{-1}(\{z : \Re(z) \leq -2\})$ given by Theorem 4.2.5 where $p(z) = F_5 p(z; \frac{r}{s})$ for $s \leq 9$ (where we have chosen the correct branches of p^{-1} which can be done in an ad hoc way when s is small) we obtain the following pictures. We have also added in the points which are roots $\{z : F_5 p(z; \frac{r}{s}) = 0 : s \leq 200\}$.

From $\mathfrak{R}_{5,3}$ one can now get a picture of what the space of $\gamma = \gamma(f, g)$ values look like. Similarly

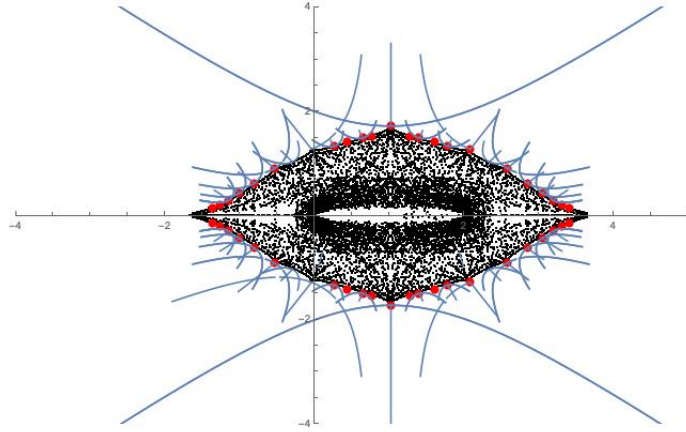


Figure 4.10: $\mathfrak{R}_{5,3}$ analogue of the Riley slice. Shown also are the half-space neighbourhoods of pleating rays lying entirely in the space of groups freely generated.

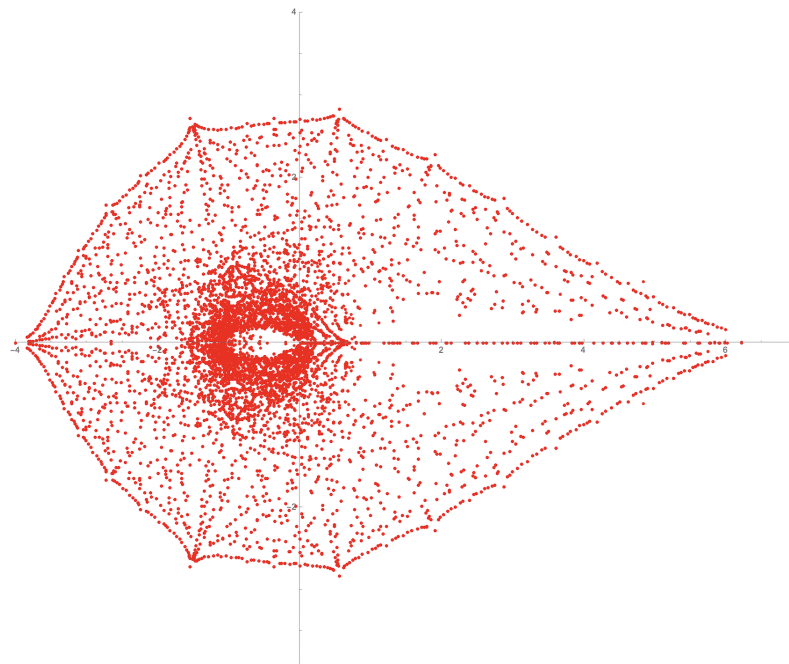


Figure 4.11: The complement of the set of commutators.

$\{\gamma : \langle f, g \rangle$ with f, g primitive elliptic of order 5 and 3 is discrete and freely generated}

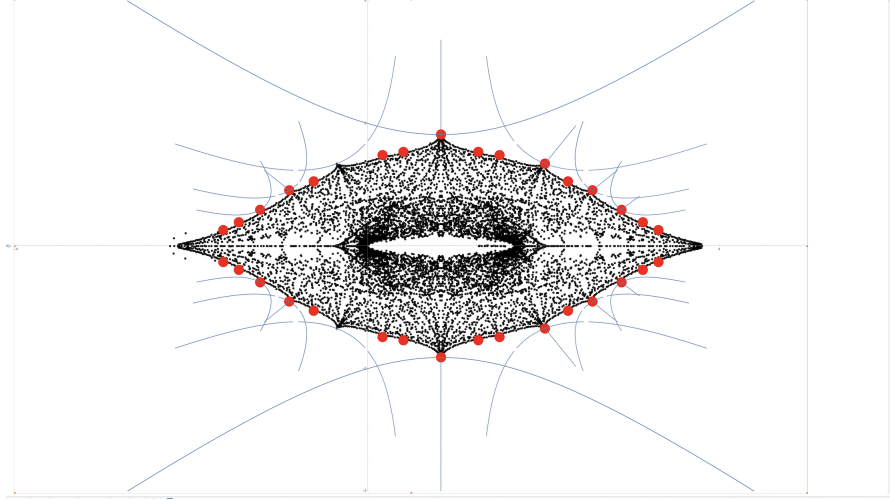


Figure 4.12: $\mathfrak{R}_{5,4}$

4.3 Wrapping up the searches.

In the previous chapter, we obtained lists of candidate $\gamma(f, g)$ values for those allowable values for p and degree. We now use our approximations to the spaces $\mathfrak{R}_{5,p}$ to capture points that are in the free part of these spaces. This is illustrated below in the case $\mathfrak{R}_{5,4}$. In this figure, the blue points are those within the region $\Omega_{5,4}$, and so are not freely generated, and also satisfy all the necessary arithmetic criteria. Now to eliminate most of these points, we first examine the 17 Farey polynomials and the pleating rays of low slopes where the denominator is small.

$$\{1/2, 3/5, 4/7, 6/7, 5/8, 5/9, 7/11, 8/13, 9/14, 11/17, 13/20, 16/21, 15/26\}. \quad (4.3.1)$$

The inverse image of the half-space $H = \{z = x + iy : x \leq -2\}$ under the branch of the Farey polynomial $P_{r/s} = F54(z; r/s)$ which yields the rational pleating ray r/s . For these low slopes, these preimages capture all but 7 points easily. There are some computation issues in drawing the figure below for the last 4 polynomial inverse images, but this is really only for visual confirmation. All we need to do is evaluate the associated polynomial with integer coefficients on the point in question,

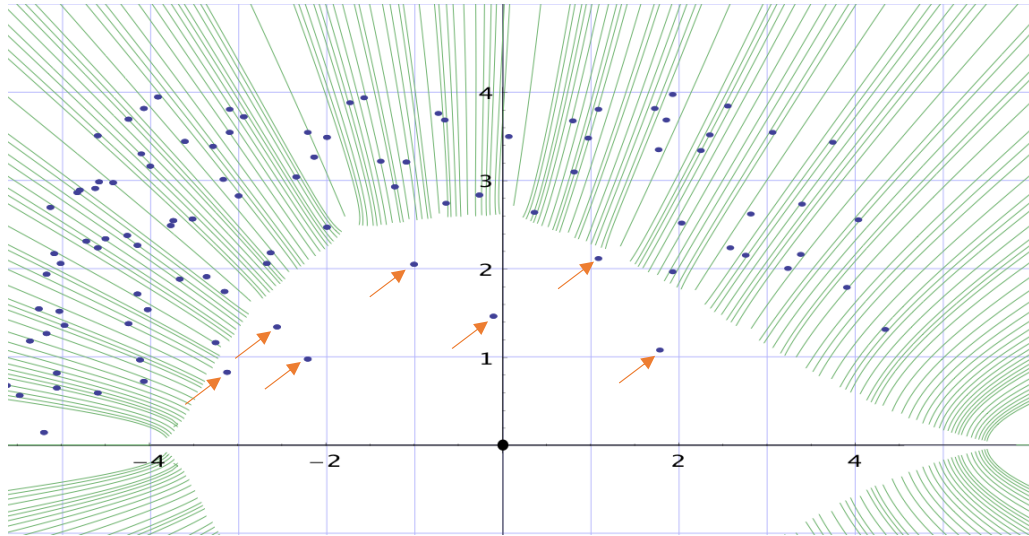


Figure 4.13: γ and pleating rays values found from $\mathfrak{R}_{5,4}$.

Figure 4.14: Neighbourhoods of the pleating rays with slopes given at (4.3.1) capturing all but 7 points.

show that the image lies in $\{z = x + iy : x \leq -2\}$, and then just check that we have the right branch - but this is determined by the pleating ray and path lifting which are direct to check as we can numerically identify the other roots and critical points.

There are now 7 points that remain and the associated groups are not going to be free on the two generators. see [43]

4.3.1 Relators.

The finiteness theorem of [36] states that there are only finitely many Kleinian groups $\langle f, g \rangle$ generated by two elements of finite order which are subgroups of an arithmetic Kleinian group and are not free on these two generators. Thus our classification should identify a nontrivial relator word in a candidate group. Briefly this word is found as follows.

The simple closed curves on the 4 times punctured sphere \mathbb{S}_4 separating one pair of points from another are enumerated via the deformation theory of Keen-Series-Maskit [29, 30, 31] which relates a *slope*, a rational fraction, a bending deformation along a homotopic geodesic of \mathbb{S}_4 . This deformation terminates on the boundary of moduli space as the length of this curve shrinks to zero where the associated word becomes parabolic. This bending locus is called a *pleating ray*. Bending further creates cone manifolds, some of which are discrete lattices when the cone angle becomes $2\pi/n$. It is through this process that we will find all of our examples.

In fact the recent results of [4, 5] show this process to describe all the discrete and faithful representations of groups generated by two parabolic elements which are not free. This slope is directly related to the Schubert normal form of a two-bridge knot or link, [10] and the Conway *slope* invariant from rational tangles. Here is an explicit example of a group we find. The rational slope is

$$3/5 = \frac{1}{\frac{5}{3}} = \frac{1}{1 + \frac{2}{3}} = \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}},$$

and continued fraction $3/5 = [1, 1, 2]$. We then present the two-bridge knot in normal form with crossings ordered downwards 2, 1, 1. The Wirtinger relation on

the meridian “bridge” generators f and g (both parabolic in this instance) is the word $w_{3/5}$ which “represents” the unknotting tunnel.

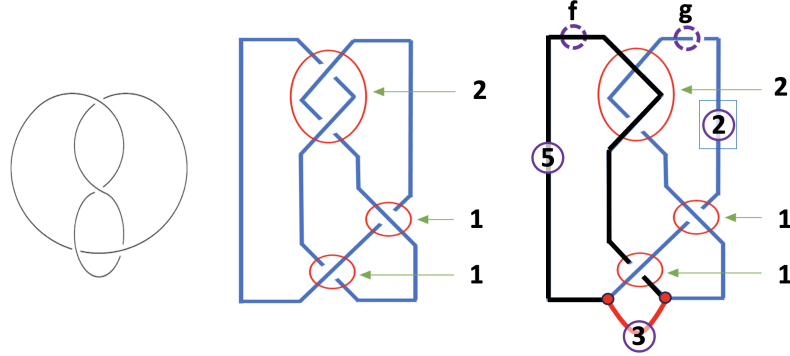


Figure 2. *Left: the 3/5, or figure eight, knot. Middle: figure eight in normal form. Right: A $(3/5; 3)$ group on bridge generators f of order 5 and g of order 2. This is a co-compact arithmetic generalised triangle group with presentation $\langle f^5 = g^2 = (w_{3/5})^3 = 1 \rangle$ and vertices stabilised by the spherical triangle group $(2, 3, 5) \cong A_5$.*

There is one slight issue here and that is that the exact choice of generators in $PSL(2, \mathbb{C})$ matters since we classify all Nielsen pairs and it is possible that there may be a different pair of the same orders generating the same group, and obviously they will have different relations. In fact this does not happen here, and probably never does unless the orbifold is a knot or link (as happens with the classification in the parabollic case, [4]).

There is a simple recipe for moving from a rational slope to a word. This is discussed in detail in [20] where a combinatorial recursion formula is found for both these words and their traces. For instance the first few rational slopes (greater than $\frac{1}{2}$) are $\{\frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, 1\}$ and in the same order the words are, with $x = X^{-1}$

and $y = Y^{-1}$,

$$\{XYxy, XYxyXYxYXyxYXy, XYxyXyxYXy, XYxyXyxYxyXYxYXy, \\ XYxYXy, XYxYXyXyxYxyXy, XYxYxyXy, XYxYxYXyXy, Xy\}$$

These are all the words we will need below. Now let us describe how to read the tables by considering an example. If we look at the 5th entry of the (2, 5)-arithmetic lattices table below we have $X = f$ with order 5 and $Y = g$ with order 2. We see $\gamma(f, g) \approx -1 + 1.2720i$, the complex root with positive real part of its minimal polynomial $1 + 6z + 7z^2 + 4z^3 + z^4$ (there is only ever one conjugate pair of complex roots – a consequence of Theorem 2.7.3).

The pair $(2/7; 2)$ indicates the word $w_{2/7} = XYxYXyXyxYxyXy$, (in fact the symmetry of the rational slopes give $\frac{2}{7} \cong \frac{5}{7}$) is elliptic of order two when X, Y have the form at (2.3.1) and μ has one of the two values. We calculate with the value $\mu = 0.250882 - 0.68781i$,

$$w_{2/7} = \begin{pmatrix} -0.0358965 - 0.253917i & 3.06848 + 0.494866i \\ -0.298497 + 0.0421989i & 0.0358965 + 0.253917i \end{pmatrix}$$

or when $\mu = 2.10026 + 0.68781i$

$$w_{2/7} = \begin{pmatrix} 0.762439 - 0.133799i & 0.0761256 - 1.73093i \\ -0.15729 - 0.896301i & -0.762439 + 0.133799i \end{pmatrix}$$

which in either case gives us

$$w_{2/7} \approx 0 \tag{4.3.2}$$

and the exact value 0 can be confirmed by using integer arithmetic here. This gives us an incomplete presentation

$$\langle f, g : f^5 = g^2 = (fgf^{-1}gfg^{-1}fg^{-1}f^{-1}gf^{-1}g^{-1}fg^{-1})^2 = \dots = 1 \rangle$$

It is not so obvious how to proceed to get a complete presentation from here. However the group $\langle f, h = fg^{-1} \rangle$, generated by two elements of order 5, has index two in $\langle f, g \rangle$ and so cannot split as a free product and must remain a subgroup of an arithmetic group. It therefore must appear on our (5, 5) table. With

$$\beta = -4 \sin^2 \frac{\pi}{5} = \frac{1}{2}(-5 + \sqrt{5})$$

We compute that

$$\gamma^* = \gamma(f, h) = \gamma(\gamma - \beta), \quad \gamma = \gamma(f, g). \quad (4.3.3)$$

The minimal polynomial for γ is $1 + 6z + 7z^2 + 4z^3 + z^4$ and so we may compute that the minimal polynomial for γ^* is

$$11 + 28z + 23z^2 + 8z^3 + z^4$$

and $\gamma^* \approx -2 + 0.786151i$. That these minimal polynomials have the same degree is a consequence of arithmeticity, see the Identification Theorem 2.7.3 below. This new group $\langle f, h \rangle$ is the entry # 3 in the $p = 5$ Table §2.5. Thus $\langle f, h \rangle$ arises from (5, 0) orbifold Dehn surgery on the two-bridge knot 4/7. This we can calculate from SnapPy to have the stated volume and presentation ($w_{4/7} = \textit{identity}$)

$$\langle f, h : f^5 = h^5 = fhf^{-1}h^{-1}fhf^{-1}hf^{-1}hf^{-1}f^{-1}hf^{-1} = 1 \rangle$$

SnapPy also identifies the symmetry group of this knot complement, and our original group is one of the two \mathbb{Z}_2 extensions of this group. Thus both entries on these tables are lattices.

Other groups here are arithmetic generalised triangle groups in the sense of [26], however they are usually not identified in that paper as that is restricted largely to the case $w_{1/2} = [X, Y]$ is elliptic.

When we know we have a lattice, we have also given a co-volume approximation. This is obtained from adapting the Poincaré subroutine in Week’s programme Snap to our setting of groups generated by two elliptic elements, see Cooper [13]. This approximation is enough to give the precise index of the the group $\langle f, g \rangle$ in its maximal order and the latter has an explicit co-volume formula due to Borel [9] if further refinement is needed.

A number of other groups identified are interesting infinite co-volume web-groups with a hyper-ideal vertex, or of finite index in the truncated tetrahedral reflection groups enumerated in [14] and which are also subgroups of an arithmetic lattice. Our list of such groups here is complete since we find all non-splitting groups, before we identify if they are a lattice or not. These groups are called “nearly arithmetic’ in [33]. Here the numbers are sufficiently small (the degrees of associated polynomials) that we can say definitively that a group is not a lattice. Let us give a first example of how that is achieved.

Example.

The first non-lattice entry of the (5, 5)-table at §2.5 with the polynomial $1 + 3z + 3z^2 + 3z^3 + z^4$. Let γ be the complex root of this polynomial with positive real part, then solve $\gamma = \mu(\mu - 4 \sin^2 \frac{\pi}{p})$. We find solutions

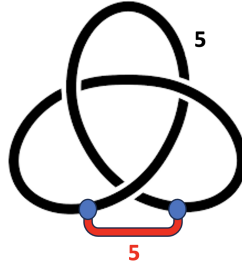
$$\mu_1 = -0.118034 - 0.606658i, \quad \mu_2 = 1.5 + 0.606658i$$

Then the Farey polynomial

$$F_{1/3}^{5,5}(\mu_2) := (w_{1/3}) = 1.61803 \dots = 2 \cos \frac{\pi}{5}.$$

Also $F_{2/3}^{5,5}(\mu_1) = (w_{2/3}) = 2 \cos \frac{\pi}{5}$. As the case may be, we find $w_{1/3}$ or $w_{2/3}$ is primitive elliptic of order 5.

From above, the $2/3$ word in X and Y is $XYxYXy$ and its trace is a degree 3 polynomial in μ . So $(XYxYXy) = 2 \cos(\pi/5)$ has one complex conjugate pair of roots which must be $\mu_1, \overline{\mu_1}$. Thurston's orbifold theorem, see [8] tells us that there is a complete hyperbolic structure of infinite co-volume on the trefoil knot $2/3$ labeled with singular set 5 and unknotting tunnel labeled 5, as illustrated below.



The two (hyperfinite) vertices are stabilised by the $(5, 5; 5)$ triangle group. The Wirtinger word giving the presentation of this group $\langle X^5 = Y^5 = W_{2/3}^5 \rangle$. The calculation of the orbifold fundamental group is explained in Ratcliffe's book [54, Chapter 13]. From this we see that $W_{2/3}$ must be primitive elliptic of order 5. This cubic equation determines μ_1 (or its conjugate) which in turn determines γ so that the data $f^5 = g^5 = 1$, both primitive, $w_{2/3}^5(f, g) = 1$ and primitive, determines the group uniquely up to conjugacy. Since $\gamma(f, g)$ uniquely determines the group, it follows that the entry we are considering is actually the generalised triangle group

$$\langle f, g | f^5 = g^5 = w_{2/3}^5 = 1 \rangle$$

Thus the table entries follow, as do the two \mathbb{Z}_2 extensions which are the first two non-lattice entries in the $(p = 2)$ Table §2.6.

There is an issue with applying this sort of argument in generality. We were lucky the the polynomial had degree 3 and so the solution we sought was essentially the only possibility. The real solution for μ gives $\gamma = \gamma(f, g)$ real and we will see in a moment why this is not possible, as it implies $[f, g]$ is elliptic (and so one of a classified finite number of groups [35]), or the group splits ($\gamma \leq -4$), or the group is Fuchsian ($\gamma \geq 0$). The problem is with higher degree allowing more possibilities for $\gamma(f, g)$ which we must eliminate. In the tables that now follow §§2.3-2.6, we consider other cases to show how to deal with these and related issues.

4.3.2 Identifying the groups.

We are now tasked with identifying the groups. This is quite straightforward. We enumerate the polynomials $F5p(z; \frac{r}{s})$ with numerator $s \leq 50$. Then, given a candidate γ value numerically, we solve (4.2.2) numerically to get two values of μ , call them μ_{\pm} .

$$\gamma = 2 - 4\mu_{\pm} \sin\left(\frac{\pi}{5}\right) \sin\left(\frac{\pi}{p}\right) + \mu_{\pm}^2.$$

We then find all rational slopes r/s so that for either $\mu = \mu_+$ or $\mu = \mu_-$ we have

- $-2.1 \leq \Re e(F5p(\mu; \frac{r}{s})) \leq 2.1$
- $|\Im m(F5p(\mu; \frac{r}{s}))| \leq 0.1.$

These conditions tell us it is likely $W_{r/s}$ is elliptic and $\Re e(F5p(\mu; \frac{r}{s}))$ tells us what its order is likely to be. Usually, this gives us one or two candidate values (most often just one) for r/s and the order. We then can use exact arithmetic to verify

that $W_{r/s}$ is elliptic of the given order.

Finally, if the orbifold $\mathcal{O}_{5,p;r/s}$ defined at (4.2.3) has finite volume, then, as discussed at (4.2.3) so does our group and hence we have identified a lattice. This is most often the case. There is an issue here if both $p \neq 5$ and r/s are not a link. However, our group will have finite co-volume as soon as all the vertices of the singular graph (a two-bridge knot with bridges 5 and p and an unknotting tunnel labelled by the order of $W_{r/s}$) are spherical or Euclidean triangle groups. See for instance Figure 4.2.1 earlier.

Next, we discuss how to compute the co-volumes we identify.

Chapter 5

Co-Volume Computation, $\mu(\Gamma)$

In this chapter, we shall look at the arithmetic and geometric approaches for calculating the volume of hyperbolic three-orbifolds to present the information on the co-volume of the lattices in our tables.

Co-Volume calculation may be traced back to Bianchi and Humbert. Consider the ring of integers R_d in a number field $\mathbb{Q}(\gamma)$, then $PGL_2(\mathbb{C}, R_d)$ is a discrete subgroup of $PGL_2(\mathbb{C})$. Let Γ be a torsion-free finite index subgroup in $PGL_2(\mathbb{C}, R_d)$. since $PGL_2(\mathbb{C})$ is the group of orientation preserving isometries of \mathbb{H}^3 , \mathbb{H}^3/Γ is an orientated hyperbolic three-manifold. This algebraic construction also yields unlimited hyperbolic orbifolds with co-volumes equal to rational, finite linear combinations of the Lobachevsky function.

$$\Theta(\theta) = \int_0^\theta -\ln |2 \sin(t)| dt \tag{5.0.1}$$

Below, we will remind two powerful methods for computing hyperbolic volumes.

1. Borel's system estimates the lower bound for $\mu(\Gamma)$.
2. Alternating (p, q) - links and knots complement (Snap Py and LMFDB number theory program).

5.0.1 Borel's Method

Borel's machinery is a powerful method for finding lower bounds for co-volume. It was first introduced by Borel (see[9]). Let us mention fixing the following notations.

1. N_ν : the norm of the prime ideal p_ν associated to the valuation ν of k .
2. d_k : the discriminant of k .
3. h_k : the class number of k ($=I(k)/P(k)$) where $I(k)$ (resp. $P(k)$) is the group of fractional (resp. principal) ideals of k .
4. $\zeta_k(s)$: the Dedekind zeta function of k .
5. R_f (resp. R_∞): the set of finite (resp. infinite) places of k where A is ramified.
6. O_k^* : the group of units of k .
7. $O_{R_f}^*$: the group of R_f -units of k .
8. $O_{R_f,+}^*$: the group of elements of units of $O_{R_f}^*$ which are positive at all real places.
9. $P(k, R_\infty)$: the principal ideals with a positive generator at all real places.

10. M_1 : the subgroup of $I(k)$ generated by $P(k, R_\infty)$ and the ideals p_ν such that $\nu \in R_f$.
11. J_2 : image of $P(k)$ in J_1 .
12. J_1 : $I(k)/M_1$.
13. ${}_2J_1$: kernel of the mapping $y \rightarrow y^2$ in J_1 .
14. e : number of places of k dividing two and not contained in R_f .
15. ϑ : Maximal order for ring A .

The proofs of the following theorems are given in ([9] -Theorem 7.3).

Theorem 5.0.1. *Let Γ be a subgroup of $PGL_2(\mathbb{C})$ commensurable with Γ_ϑ . Then the co-volume of \mathbb{H}^3/Γ is an integral multiple of $2^{-e}Vol(\mathbb{H}^3/\Gamma_\vartheta)$. It is equal to $Vol(\mathbb{H}^3/\Gamma_\vartheta)$ if Γ is a maximal arithmetic group and $Vol(\mathbb{H}^3/\Gamma_\vartheta)$ otherwise.*

Theorem 5.0.2. *Let ϑ be a maximal order of ring A . Then*

$$Vol(H^3/\Gamma_\vartheta) = \frac{|d_k|^{\frac{3}{2}} \zeta_k(2)}{(4\pi^2)^{[k:\mathbb{Q}]-1}} \prod_{\nu \in R_f} (N_\nu - 1) \times \frac{1}{[O_{R_f,+}^* : (O_{R_f}^*)]^2 [{}_2J_1 : J_2]} \quad (5.0.2)$$

Here we calculate an example for finding the lower bound for $\mu(\Gamma)$, which is mentioned in tables. In other cases, calculations are the same.

Example 5.0.3. Suppose that $\gamma = -0.690983 + 0.722871i$, so the irreducible monic polynomial is $1 + 5z + 7z^2 + 5z^3 + z^4$ and discriminant is -475 . Then there is one prime ideal $5\mathfrak{p}$, which is ramified. therefore by using Dirichlet's unit theorem ($R_f = \{5\mathfrak{p}\}$) $O_{R_f}^*$ is an abelian group of rank 1, so that $[O_{R_f,+}^* : (O_{R_f}^*)^2] = 2^{r_f+1} = 4$.

Also

$$\zeta_{\mathbb{Q}(\gamma)}(2) = \prod_{\mathfrak{p} \subset \gamma[\gamma]} \frac{1}{1 - (N_{\mathbb{Q}(\gamma)/\mathbb{Q}}(\mathfrak{p}))^{-2}} \geq 1 \quad (5.0.3)$$

In addition, for non-archimedean places $\prod_{\nu \in R_f} (N_\nu - 1) \geq 2$, using the LMFDB program as mentioned before, the class number is $h_{\mathbb{Q}(\gamma)} = 1$ so that $[{}_2J_1 : J_2] = 1$.

Then

$$\text{Vol}(\mathbb{H}^3/\Gamma_\vartheta) \geq \frac{|-475|^{\frac{3}{2}} \times (2)}{(4\pi^2)^3 \times 4} = 0.084126187$$

5.0.2 Volume for γ^*

[35] classifies all Kleinian groups with two elliptic generators and an elliptic commutator. There are three types of these groups. All Kleinian groups with two elliptic generators of orders p, q , and a commutator of order n are found. If p and q are 2, Γ is either elementary or contains a Fuchsian group of indices with a maximum of 2.

Let us recall some notable outcomes.

Theorem 5.0.4. *Let $\Gamma = \langle f, g \rangle$ be a non-elementary Kleinian group where f, g are elliptic elements with $o(f) = p, o(g) = q, o[f, g] = n$ and $p, q \geq 3$. Assume that f, g are represented by matrices X, Y with $\text{tr}(X) = 2 \cos \frac{\pi}{p}, \text{tr}(Y) = 2 \cos \frac{\pi}{q}$. Then one of the following must hold:*

1. $\text{tr}[X, Y] = -2 \cos \frac{\pi}{n}$ and $\Gamma \cong GT(p, q; n)$

2. $tr[X, Y] = -2 \cos \frac{2\pi}{n}$, n is odd, $(p, q; n) = (3, 3; 3), (3, 4; 3), (3, 5; 3), (4, 4; 3)$,
and $\Gamma \cong Tet(p, q; n)$

3. $tr[X, Y] = -2 \cos \frac{4\pi}{n}$, $p = q = n$ is odd and at least 7, and $\Gamma \cong \Gamma_n$

Theorem 5.0.5. *Suppose $\Gamma = \langle f, g \rangle$ is a Kleinian group where $o(f) = p$, $o(g) = q$, $o[f, g] = n$. Assume that Γ is not elementary and has no invariant hyperbolic plane. The only cocompact groups Γ are the groups $Tet(p, q; n)$ for $(3, 3; 5), (4, 5; 3), (5, 5; 3)$. These are arithmetic and pairwise non-commensurable. The only non-cocompact groups of finite co-volume are the groups $GT(3, 3; 3), GT(3, 4; 2), Tet(3, 6; 3), Tet(6, 6; 3), Tet(4, 6; 3)$, which are all arithmetic and commensurable with $PSL(2, O_3)$, $GT(4, 4; 2)$ which is arithmetic and commensurable with $PSL(2, O_1)$, and, finally, $Tet(5, 6; 3)$ which is non-arithmetic.*

Now by using theorems 5.0.4, for case $(5, 5, n)$, we have following results;

1. If $\gamma = -2.61803 + 0i$, then $\Gamma \cong Tet(5, 5; 5)$

2. If $\gamma = -2$, then $\Gamma \cong GT(5, 5; 2)$

3. If $\gamma = -1$, then $\Gamma \cong Tet(5, 5; 3)$

4. If $\gamma = -3$, then $\Gamma \cong GT(5, 5; 3)$

5. If $\gamma = -3.61803 + 0i$, then $\Gamma \cong GT(5, 5; 5)$

From the Theorem 5.0.5, the only case with finite co-volume is $\gamma = -1$.

5.0.3 Co-Volume for alternating 2-bridge links and knots

For the last part of this paper, First, we remind the reader of some background tools to draw alternating Knots and Links. Second, by putting them in Snap Pea's program, their volumes will be computed .

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Appendix A

Snap Py

A.1 What is the Snap Py?

Snap Py is a program for studying the topology and geometry of 3-manifolds, focusing on hyperbolic structures. It runs on Mac OS X, Linux, and Windows. It combines a link editor and 3D graphics for Dirichlet domains and cusp neighbourhoods with a powerful command-line interface based on Python. You can see it in action, learn how to install it, and watch the tutorial. Some clarifications related to Snap Py are listed as follows

- Intro and quick start: an 11-minute video on youtube ¹ with the basics: Part I ² and Part II ³.

¹<http://youtube.com/user/NathanDunfield>

²<https://www.youtube.com/watch?v=ezo19L-JTTI>

³<http://www.youtube.com/watch?v=Js4qwyIs-Oo>

- An hour-long demo Practical computation with hyperbolic 3-manifolds, recorded at the Thurston Memorial Conference.
- SnapPy, SageMath, and Docker are friends with associated materials (4 pages), including homework.

Here, using continuing fractions, we create several more connections and knots linked to slopes in tables 1, 2, and 3. To get a co-volume $\mu(\Gamma)$ associated with these links and knots, just put them into Snap Py's program, and their co-volumes will be computed using (p, q) - Dehn surgery on borders of $\mathbb{S}^3 - K(p, q)$.

The following alternating knots came from continued fractional defined in [4.1.3](#).



Hi. It's SnapPy.
SnapPy is based on the SnapPea kernel, written by Jeff Weeks.
Type "Manifold?" to get started.

In[1]: Manifold?

Init signature: Manifold(self, /, *args, **kwargs)
Docstring:
Manifold(spec=None)

A Manifold is a Triangulation together with a geometric structure. That is, a Manifold is an ideal triangulation of the interior of a compact 3-manifold with torus boundary, where each tetrahedron has been assigned the geometry of an ideal tetrahedron in hyperbolic 3-space. A Dehn-filling can be specified for each boundary component, allowing the description of closed 3-manifolds and some orbifolds. Here's a quick example:

```
>>> M = Manifold('9_42')
>>> M.volume() # doctest: +NUMERIC6
4.05686022
>>> M.cusp_info('shape') # doctest: +NUMERIC6
[-4.278936315 + 1.95728679*I]
```

A Manifold can be specified in a number of ways, e.g.

- Manifold('9_42') : The complement of the knot 9_42 in S^3 .
- Manifold('m125(1,2)(4,5)') : The SnapPea census manifold m125 where the first cusp has Dehn filling (1,2) and the second cusp has filling (4,5).
- Manifold() : Opens a link editor window where can you specify a link complement.

In general, the specification can be from among the below, with information on Dehn fillings added.

- SnapPea cusped census manifolds: e.g. 'm123', 's123', 'v123'.
- Link complements:
 - + Rolfsen's table: e.g. '4_1', '04_1', '5^2_6', '6_4^7', 'L20935', 'l104001'.
 - + Hoste-Thistlethwaite Knotscape table: e.g. '11a17' or '12n345'
 - + Callahan-Dean-Weeks-Champanerkar-Kofman-Patterson knots: e.g. 'K6_21'.
 - + Dowker-Thistlethwaite code: e.g. 'DT:[(6,8,2,4)]'
- Once-punctured torus bundles: e.g. 'b++LLR', 'b+-lLR', 'bo-RRL', 'bn+LRLR'
- Fibered manifold associated to a braid: 'Braid[1,2,-3,4]'

Here, the braid is thought of as a mapping class of the punctured disc, and this manifold is the corresponding mapping torus. If you want the braid closure, do (1,0) filling of the last cusp.

- From mapping class group data using Twister:

```
'Bundle(S_{1,1}, [a0, B1])' or 'Splitting(S_{1,0}, [b1, A0], [a0,B1])'
```

See the help for the 'twister' module for more.

- A SnapPea triangulation or link projection file: 'filename'

The file will be loaded if found in the current directory or the path given by the shell variable SNAPPEA_MANIFOLD_DIRECTORY.

- A string containing the contents of a SnapPea triangulation or link projection file.

File: ~/Desktop/SnapPy.app/Contents/Resources/lib/python3.10/snappy/__init__.py
Type: type
Subclasses:

In[2]: Manifold()

Starting the link editor.
Select Tools->Send to SnapPy to load the link complement.

Figure A.1: Snap Py environment

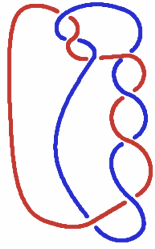


Figure A.2: 13/16 link

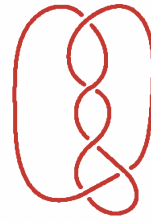


Figure A.3: 4/7 knot

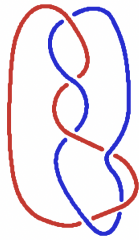


Figure A.4: 7/10 link

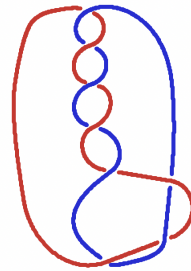


Figure A.5: 11/16 links

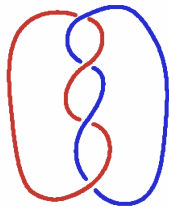


Figure A.6: 4/5 link



Figure A.7: 5/7 link