Minimising L^p Distortion for Mappings Between Annuli

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ABSTRACT

When deforming or distorting one material object into another, for various physical reasons the final deformation is expected to minimise some sort of energy functional. Classically, the theory of quasiconformal mappings provides us with a theory of distortion, yielding some limited results concerning minimising the maximal distortion. The calculus of variations is aimed at extremising certain kinds of functionals (such as the integral of the gradient squared or of distortion over a region in the complex plane). This thesis investigates quasiconformal and related mappings between annuli, introduces some novel results, and outlines some conjectures for further research.

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CONTENTS

LIST OF FIGURES

INTRODUCTION

If we look at simply connected domains, then the Riemann Mapping Theorem (see Section 1.5) tells us that any simply connected region (which is not equal to the whole complex plane $\mathbb{C}$, or the extended complex plane (the Riemann sphere) $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$) is conformally equivalent to the unit disc $\mathbb{D}$. The exceptions to this theorem are obvious: the complex plane, because a bounded analytic function that is entire on the whole complex plane is constant (by Liouville's Theorem), and thus the image of the complex plane cannot be the unit disc; and the extended complex plane because it is both open and closed.

In light of Riemann's Mapping Theorem, a theorem by Schottky (see Section 2.3) and some other well-known results in quasiconformal theory, the problem of finding minimisers of distortion functionals (between doubly connected regions) can be simplified to looking at annuli.

Classical measures of distortion are investigated thoroughly, and a new way of calculating distortion is examined in some cases. In particular, the calculus of variations is applied to the problem, and gives sharp results for the classical distortion measures (with exception of some cases—see Section 3.2) as well as limited results for the new distortion measure. Some interesting results arise; in particular, there are cases where there are no minimisers, as well as cases where minimisers are difficult to find.

Throughout, the notation $\dot{\rho}$ will be used for the real derivative of ρ, to distinguish from the complex derivative, which uses the notation ρ'. Also, note that if a proposition or theorem number begins with ".0" instead of the number of the chapter that it is in, then it may be found in the appendix.

We have attempted to provide all information necessary to understand this thesis, and where not able to do so, provide reference to materials where further details may be found.

1. CONFORMAL MAPPINGS AND THE RIEMANN MAPPING THEOREM

The concept of a conformal mapping is built up through the basic ideas of analytic functions and the Cauchy-Riemann derivatives. Riemann's Mapping Theorem is examined in detail, and a proof given. The use of this important theorem is to specify precisely the first case where topology plays a role in finding extrema of distortion functionals. Other theorems, such as Schottky's Theorem on conformal mappings between annuli, will then be used to further simplify and isolate the problem.

1.1 Analytic functions

Definition 1.1.1. *Let $\Omega \subset \mathbb{C}$ be an open set. Then $f : \Omega \to \mathbb{C}$ is called* ***(complex-)differentiable*** *at $z_0 \in \Omega$ if*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists or, equivalently, if

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. Furthermore, f is said to be ***analytic*** *if f is complex-differentiable at each $z \in \Omega$; and f is said to be* ***analytic at*** *$z_0 \in \Omega$ if it is analytic on a neighborhood of z_0. If f is analytic on $\mathbb{C}$, then f is said to be* ***entire****.*

The next few propositions and theorems will be given without proof; it is assumed the reader is familiar with these properties.

Proposition 1.1.2. *If $f'(z_0)$ exists, then f is continuous at z_0.*

Proposition 1.1.3. *The following derivative rules hold:*

- $(cf)'(z_0) = cf'(z_0)$, *where c is any constant;*
- $(f+g)'(z_0) = f'(z_0) + g'(z_0)$, *the* ***sum rule*** *for derivatives;*
- $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$, *the* ***product rule*** *for derivatives;*
- *Provided $g(z_0) \neq 0$,* $\left(\dfrac{f}{g}\right)' = \dfrac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{[g(z_0)]^2}$, *the* ***quotient rule*** *for derivatives.*

Proposition 1.1.4. *(The* ***Chain Rule*** *for complex differentiation)*
Let $f : \Omega \to \mathbb{C}$ and $g : \Omega' \to \mathbb{C}$ be such that $f(\Omega) \subset \Omega'$. For any $z_0 \in \Omega$, if f is differentiable at z_0 and g is differentiable at $f(z_0)$, then $g \circ f$ is differentiable at z_0, and

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

It follows from the rules for differentiation that if f and g are analytic functions on some open domain $\Omega \subset \mathbb{C}$ that cf (where c is some constant), $f+g$, and fg are analytic on Ω, and that f/g is analytic in $\Omega \setminus \{z : g(z) = 0\}$ if this is not empty. Likewise the composition $g \circ f$, if $f(\Omega)$ is contained in the domain of g, is analytic on Ω.

Notable in the definition of the derivative of a function f at z_0 in some open region Ω is that the existence of

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

contains no constraint on how z_0 is approached; differentiability of f means that this limit is the same regardless of what sequence or curve we follow to get to z_0. This leads us to the formulation of the Cauchy-Riemann Equations.

The Cauchy-Riemann equations

For ease of notation, we denote each complex number in the form $z = x + iy$, where x and y are real numbers, and we denote the partial derivative $\partial(f)/\partial(x)$, for example, by f_x. Suppose a function $f(z) = f(x, y) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$. Then by holding the imaginary component of z constant ($y = y_0$) and approaching z_0 along a line parallel to the real axis, we obtain (Palka, 1991, p. 69):

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{f(x + iy_0) - f(x_0 + iy_0)}{x - x_0} \\ &= \lim_{z \to z_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} \\ &= \lim_{z \to z_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i \lim_{z \to z_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0} \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0) \\ &= f_x(z_0). \end{aligned}$$

Similarly, if we hold the real component of z constant ($x = x_0$) and approaching z_0 along a line parallel to the imaginary axis, we get

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{f(x_0 + iy) - f(x_0 + iy_0)}{i(y - y_0)} \\ &= v_y(x_0, y_0) - iu_y(x_0, y_0) \\ &= -if_y(z_0). \end{aligned}$$

Combining the results from the previous paragraph, we conclude that

$$u_x = v_y \quad , \quad v_x = -u_y,$$

or, equivalently,

$$f_x = -if_y,$$

at (x_0, y_0). These famous equations are known as the *Cauchy-Riemann equations*. A necessary condition for f to be complex-differentiable at z_0 is that f

satisfies the Cauchy-Riemann equations there. However, this is not in general a sufficient condition for differentiability; see, for example, Palka (1991, p. 75, Example 3.3). But this need not significantly reduce the utility of these equations: when there is more information available concerning f we can sometimes establish differentiability using the Cauchy-Riemann equations.

Theorem 1.1.5. *(The **Cauchy-Riemann Theorem**)*
Let $f = u + iv$ be defined in an open region $\Omega \subset \mathbb{C}$, and suppose that the partial derivatives of u and v with respect to x and y exist everywhere in Ω. If each of u_x, u_y, v_x, v_y is continuous at $z_0 \in \Omega$ and if the Cauchy-Riemann equations are satisfied at z_0, then f is differentiable at z_0 and $f'(z_0) = f_x(z_0)$.

Proof. (Bak and Newman, 1997, pp. 36–37) Let $h = \xi + i\eta$. We are required to show that

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f_x(z_0).$$

By the Mean Value Theorem for functions of a real variable (Theorem .0.1),

$$\begin{aligned}
\frac{u(z_0 + h) - u(z_0)}{h} &= \frac{u(x_0 + \xi, y_0 + \eta) - u(x_0, y_0)}{\xi + i\eta} \\
&= \frac{u(x_0 + \xi, y_0 + \eta) - u(x_0 + \xi, y_0)}{\xi + i\eta} \\
&\quad + \frac{u(x_0 + \xi, y_0) - u(x_0, y_0)}{\xi + i\eta} \\
&= \frac{\eta}{\xi + i\eta} u_y(x + \xi, y + t_1\eta) \\
&\quad + \frac{\xi}{\xi + i\eta} u_x(x + t_2\xi, y + \eta)
\end{aligned}$$

and, similarly,

$$\begin{aligned}
\frac{v(z_0 + h) - v(z_0)}{h} &= \frac{\eta}{\xi + i\eta} v_y(x + \xi, y + t_3\eta) \\
&\quad + \frac{\xi}{\xi + i\eta} v_x(x + t_4\xi, y + \eta)
\end{aligned}$$

for some t_k with $0 < t_k < 1$ and $k \in \{1, 2, 3, 4\}$. Hence

$$\begin{aligned}\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} &= \frac{\eta}{\xi+i\eta}[u_y(x+\xi, y+t_1\eta)+iv_y(x+\xi, y+t_3\eta)] \\ &\quad +\frac{\xi}{\xi+i\eta}[u_x(x+t_2\xi, y+\eta)+iv_x(x+t_4\xi, y+\eta)].\end{aligned}$$

Since *ex hypothesi* $f_x = -if_y$, we can write $f_x(z_0)$ in the form

$$f_x(z_0) = \frac{\eta}{\xi+i\eta}f_y(z_0)+\frac{\xi}{\xi+i\eta}f_x(z_0).$$

Subtracting this from both sides we obtain

$$\begin{aligned}\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} - f_x(z_0) &= \frac{\eta}{\xi+i\eta}[(u_y(x+\xi, y+t_1\eta)-u_y(x_0, y_0)) \\ &\quad +i(v_y(x+\xi, y+t_3\eta)-v_y(x_0, y_0))] \\ &\quad +\frac{\xi}{\xi+i\eta}[(u_x(x+t_2\xi, y+\eta)-u_x(x_0, y_0)) \\ &\quad +i(v_x(x+t_4\xi, y+\eta)-v_y(x_0, y_0))].\end{aligned}$$

But, since $\xi, \eta \to 0$ as $h \to 0$, each of the bracketed expressions tends to 0. Furthermore,

$$\left|\frac{\eta}{\xi+i\eta}\right|, \left|\frac{\xi}{\xi+i\eta}\right| \leqslant 1,$$

so that

$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} - f_x(z_0) = 0$$

as required. □

Before departing this section, it should be mentioned that there is a certain formalism associated with the preceding discussion. The *formal z- and $\bar{z}$-partial derivatives of f at z_0* (also known as the Cauchy-Riemann derivatives of f at z_0), denoted $f_z(z_0)$ and $f_{\bar{z}}(z_0)$ respectively, are defined by the formulae

$$f_z(z_0) = \frac{1}{2}\left(f_x(z_0) - if_y(z_0)\right)$$

and

$$f_{\bar{z}}(z_0) = \frac{1}{2}\left(f_x(z_0) + if_y(z_0)\right).$$

If f satisfies the Cauchy-Riemann equations at z_0, then it follows that $f_{\bar{z}}(z_0) = 0$; if f is differentiable at z_0, it is immediate that $f'(z_0) = f_z(z_0)$. However, it must be emphasized the existence of $f_z(z_0)$ does not place the same strength of demands on f as does the existence of $f'(z_0)$.

1.2 Conformal mappings

Definition 1.2.1. *A domain Ω in $\mathbb{C}$ is called* ***simply connected*** *if $\widehat{\mathbb{C}} \setminus \Omega$ is connected.*

Assuming that Ω is a region in the complex plane and that $f : \Omega \to \mathbb{C}$ is a member of the class $C^1(\Omega)$, we define the continuous real-valued function J_f on Ω by

$$J_f(z) = \det Df(z) = u_x(z)v_y(z) - u_y(z)v_x(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2.$$

Definition 1.2.2. *A mapping $f : \Omega \to \mathbb{C}$ is called* ***conformal at*** *$z_0 \in \Omega$ if there exist $r > 0$ and $\theta \in [0, 2\pi)$ such that for any curve $\gamma(t)$ that is differentiable at $t = 0$, with $\gamma(t) \in \Omega$ and $\gamma(0) = z_0$, which satisfies $\gamma'(0) \neq 0$, the curve $\sigma = f \circ \gamma$ is differentiable at $t = 0$, we have*

$$|\sigma'(0)| = r|\gamma'(0)|$$

and

$$\arg \sigma'(0) = \arg \gamma'(0) + \theta \mod 2\pi.$$

A mapping $f : \Omega \to \mathbb{C}$ is called ***locally conformal*** *if it is conformal at every point of Ω. If, in addition to being locally conformal, f is one-to-one, then we say f is* ***conformal*** *on Ω.*

See Marsden and Hoffman (1987, p. 71). Intuitively, being a simply connected domain means that there are no 'holes' in the domain; and to be a conformal map is to preserve angles between tangent vectors of curves. In practice, at least in two dimensions the property of being conformal boils down to being analytic with a nonzero derivative. During our discussion, we shall use whichever definition of conformality is most convenient at the time.

Proposition 1.2.3. *(The **Conformal Mapping Theorem**)*
Let $f : \Omega \to \mathbb{C}$ be analytic, and $f'(z_0) \neq 0$. Then f is conformal at z_0, with $\theta = \arg f'(z_0)$ and $r = |f'(z_0)|$.

Proof. Let $\gamma(t)$ be a curve in Ω, differentiable at $t = 0$, $\gamma(0) = z_0$, and $\gamma'(0) \neq 0$. Then by the chain rule, $\sigma'(0) = (f \circ \gamma)'(0) = f'(\gamma(0)) \cdot \gamma'(0) = f'(z_0) \cdot \gamma'(0)$. Therefore, taking $r = |f'(z_0)|$ and $\theta = \arg f'(z_0) \mod 2\pi$,

$$|\sigma'(0)| = |f'(z_0) \cdot \gamma'(0)| = |f'(z_0)||\gamma'(0)| = r|\gamma'(0)|$$

and

$$\arg(\sigma'(0)) = \arg f'(z_0) + \arg \gamma'(0) = \arg \gamma'(0) + \theta \mod 2\pi$$

as required. □

Proposition 1.2.4. *Suppose $f : \Omega \to \mathbb{C}$ is a conformal mapping. Then $f^{-1}(w)$ exists at each point $w \in f(\Omega)$, $(f^{-1})' = \frac{1}{f'}$, and f^{-1} is conformal on $f(\Omega)$.*

Proof. Since f is conformal, $f' \neq 0$ on Ω. Since f is bijective (one-to-one and onto) on $f(\Omega)$, f^{-1} exists. By the Inverse Function Theorem (.0.3), f^{-1} is one-to-one and analytic, with derivative given by $\frac{1}{f'}$, on $f(\Omega)$. Furthermore, since f' is nonzero and finite on Ω, $(f^{-1})'$ is nonzero and finite on $f(\Omega)$. Thus f^{-1} is conformal on $f(\Omega)$. □

1.3 Normal families

Assuming familiarity with pointwise convergence, we define the following.

Definition 1.3.1. *A sequence of functions $f_n : \Omega \to \mathbb{C}$ is said to* ***converge uniformly*** *to a function f if to each $\varepsilon > 0$ there corresponds a natural number N such that $|f_n(z) - f(z)| < \varepsilon$ for all $z \in \Omega$ whenever $n \geqslant N$. Also, a sequence of functions $(f_n)_{n \geqslant 1}$ in Ω is said to* ***converge normally*** *to a function f if (f_n) converges pointwise to f and if (f_n) converges uniformly to f on each compact set in Ω.*

Uniform convergence is obviously a stronger condition than pointwise convergence. Normal convergence sometimes goes by other names in the literature, for example 'locally uniform convergence' or 'uniform convergence on compacta in Ω'. Also, to check for normal convergence it is not necessary that we check for uniform convergence on every compact set contained in Ω; it is sufficient that (f_n) converges uniformly on each closed disk contained in Ω. For a proof, see Palka (1991, p. 247).

Definition 1.3.2. *If Ω is an open subset of $\mathbb{C}$, a set $\mathscr{F}$ of analytic functions on Ω is called a* ***normal family*** *if every sequence of functions in $\mathscr{F}$ has a subsequence which converges uniformly on closed disks in Ω.*

Definition 1.3.3. *A family of functions $\mathscr{F}$ defined on a region Ω is said to be* ***pointwise bounded in*** *Ω if for each fixed $z \in \Omega$ the set of values $\{f(z) : f \in \mathscr{F}\}$ is a bounded set of complex numbers. A family of functions $\mathscr{F}$ defined on a region Ω is called* ***locally bounded*** *if its members are uniformly bounded on each compact set in Ω.*

The latter means that for each compact $A \subset \Omega$ there exists a constant $m(A)$ with the property that $|f(z)| \leqslant m(A)$ for each $f \in \mathscr{F}$ and $z \in A$.

Definition 1.3.4. *A family of continuous functions $\mathscr{F}$ defined on some region Ω is called* ***normal in*** *Ω, or* ***pre-compact in*** *Ω, provided each sequence $(f_n)_{n \geqslant 1}$ from $\mathscr{F}$ has at least one subsequence $(f_{n_k})_{k \geqslant 1}$ that converges normally in Ω.*

Definition 1.3.5. *A family of continuous functions $\mathscr{F}$ defined on a region Ω is said to be* ***equicontinuous at*** *$z_0 \in \Omega$ if to each $\varepsilon > 0$ there corresponds* *a $\delta > 0$ such that $|z - z_0| < \delta$ implies that $|f(z) - f(z_0)| < \varepsilon$ for each $f \in \mathscr{F}$. The family $\mathscr{F}$ is called* **equicontinuous on** *Ω (or just equicontinuous) if it is equicontinuous at every point of Ω.*

Some texts, for example Marsden and Hoffman (1987, p. 225), also use the term *uniformly equicontinuous* to describe an equicontinuous family of functions. The next theorem is essential in proving the Arzelà-Ascoli Theorem, a necessary ingredient in proving the Riemann Mapping Theorem.

Theorem 1.3.6. *(Palka, 1991, p. 279) Let (f_n) be a sequence from an equicontinuous family of functions $\mathscr{F}$ defined on a region Ω. Suppose that this sequence converges pointwise in Ω. Then it converges normally in Ω.*

Proof. Let f be the pointwise limit of (f_n) in Ω, and choose an arbitrary compact set K in Ω. It is required to show that (f_n) converges uniformly to f on K. In view of Theorem .0.5, we need to show that (f_n) is a uniform Cauchy sequence on K. Assume, on the contrary, that (f_n) is not uniformly Cauchy. Then there must exist some number $\varepsilon > 0$ such that there is no integer N for which

$$|f_m(z) - f_n(z)| < \varepsilon$$

for every $z \in K$ and all $m > n \geqslant N$. In particular, choose $N = k$ for some integer k. To this k, there must correspond some integers m_k, n_k and some point $z_k \in K$ with the property that

$$|f_{m_k}(z) - f_{n_k}(z)| \geqslant \varepsilon. \tag{1.1}$$

From this we obtain the sequence of points $(z_k)_{k\geqslant 1}$ in K, which has at least one accumulation point, z_0, in K because K is compact. The family of functions $\mathscr{F}$ is equicontinuous at z_0 (since it is equicontinuous everywhere in Ω), so we can select $\delta > 0$ such that for each n,

$$|f_n(z) - f_n(z_0)| < \frac{\varepsilon}{3} \tag{1.2}$$

whenever $|z - z_0| < \delta$. Note that as $k \to \infty$, both $m_k, n_k \to \infty$ and hence

$$|f_{m_k}(z_0) - f_{n_k}(z_0)| \to |f(z_0) - f(z_0)| = 0.$$

Thus we can find k_0 with the property that

$$|f_{m_k}(z_0) - f_{n_k}(z_0)| < \frac{\varepsilon}{3} \tag{1.3}$$

whenever $k \geqslant k_0$. Furthermore, as z_0 is an accumulation point of (z_k), we can choose an index $k \geqslant k_0$ such that $|z_k - z_0| < \delta$. Equations (1.1), (1.2), and (1.3), together with the triangle inequality yield, for this k:

$$\begin{aligned} \varepsilon &\leqslant |f_{m_k}(z_k) - f_{n_k}(z_k)| \\ &\leqslant |f_{m_k}(z_k) - f_{m_k}(z_0)| + |f_{m_k}(z_0) - f_{n_k}(z_0)| + |f_{n_k}(z_0) - f_{n_k}(z_k)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

which is absurd. Hence, in fact, (f_n) is a uniform Cauchy sequence in K, and therefore $f_n \to f$ uniformly on K. □

The next section deals with some crucial results on which the proof of Riemann's Mapping Theorem rests.

1.4 The Arzelà-Ascoli Theorem and Montel's Theorem

Before establishing this piece of the Riemann Mapping Theorem jigsaw, we require two preliminary lemmata.

Lemma 1.4.1. *(Palka, 1991, p. 248) Suppose that each function in a sequence $(f_n)_{n \geqslant 1}$ is continuous in an open set Ω and that the sequence converges normally in Ω to the limit function f. Then f is continuous in Ω, and*

$$\int_\gamma f(z)dz = \lim_{n \to \infty} \int_\gamma f_n(z)dz$$

for every piecewise smooth path γ in Ω.

Proof. Fix a point $z_0 \in \Omega$. We verify that $f : \Omega \to \mathbb{C}$ is continuous at z_0. Given $\varepsilon > 0$, we must find a $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $z \in \Omega$ and $|z - z_0| < \delta$. By the triangle inequality, for any point $z \in \Omega$ and any index n,

$$|f(z) - f(z_0)| \leqslant |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)|. \quad (1.4)$$

Note that, since Ω is open, we can find $r > 0$ such that $\overline{D}(z_0, r) \subset \Omega$. Using the normal convergence of (f_n) on Ω, we find that (f_n) converges uniformly on $\overline{D}(z_0, r)$ by definition. Thus we can find an index n such that

$$|f_n(z) - f(z)| < \frac{\varepsilon}{3}$$

for each $z \in \overline{D}(z_0, r)$. In particular,

$$|f(z) - f_n(z)| + |f_n(z_0) - f(z_0)| < \frac{2\varepsilon}{3}$$

for every $z \in \overline{D}(z_0, r)$. Furthermore, since f_n is continuous on Ω for each n, it is by definition possible to choose $\delta > 0$ such that if $|z - z_0| < \delta$ then

$$|f_n(z) - f_n(z_0)| < \frac{\varepsilon}{3}.$$

Plugging these results into (1.4), we conclude that

$$|f(z) - f(z_0)| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

whenever $|z - z_0| < \delta$, which establishes the continuity of f on an arbitrary point z_0 in, and hence the entirety of, Ω.

To prove the second part of the lemma, let γ be a piecewise smooth path in Ω. Given $\varepsilon > 0$, using the normal convergence of (f_n), we can find N such that

$$|f_n(z) - f(z)| < \frac{\varepsilon}{\ell(\gamma) + 1}$$

holds whenever $n \geqslant N$ for γ restricted to each compact set in Ω. Hence

$$\left|\int_\gamma f_n(z)dz - \int_\gamma f(z)dz\right| = \left|\int_\gamma (f_n(z) - f(z))dz\right|$$

$$\leqslant \int_\gamma |(f_n)(z) - f(z)dz|\,|dz| \leqslant \int_\gamma \frac{\varepsilon}{\ell(\gamma)+1}|dz| = \frac{\varepsilon\ell(\gamma)}{\ell(\gamma)+1} < \varepsilon.$$

whenever $n \geqslant N$, for γ restricted to compacta in Ω. As γ is piecewise smooth (smooth in a finite number of compacta in Ω), this proves the desired result. □

Lemma 1.4.2. *(Palka, 1991, pp. 281–282) Let $(f_n)_{n\geqslant 1}$ be a sequence from an equicontinuous family of functions $\mathscr{F}$ defined on Ω. Suppose that the sequence $(f_n(\xi))$ is convergent (to $f(\xi)$) for every ξ belonging to a dense subset Σ of Ω. Then (f_n) converges normally in Ω.*

Proof. Given Theorem 1.3.6, we are required to show that (f_n) converges pointwise in Ω. To this end, fix $z \in \Omega$, and choose $\varepsilon > 0$. By the equicontinuity of $\mathscr{F}$ at z, we can choose $\delta > 0$ such that

$$|f_n(w) - f_n(z)| < \frac{\varepsilon}{3}$$

for all indices n, whenever $|w - z| < \delta$. Since Σ is dense in Ω, we can find $\zeta \in \Sigma$ such that $|\zeta - z| < \delta$. The sequence $(f_n(\zeta))$ converges to $f(\zeta)$, by hypothesis, and hence is a Cauchy sequence. Therefore, there is an integer N such that

$$|f_m(\zeta) - f_n(\zeta)| < \frac{\varepsilon}{3}$$

whenever $m > n \geqslant N$. Hence

$$\begin{aligned} |f_m(z) - f_n(z)| &\leqslant |f_m(z) - f_m(\zeta)| + |f_m(\zeta) - f_n(\zeta)| + |f_n(\zeta) - f_n(z)| \\ &\leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

whenever $m > n \geqslant N$, and hence $(f_n(z))$ is a Cauchy sequence. Since the choice of $z \in \Omega$ was arbitrary, the pointwise convergence of (f_n) in Ω is established, thus proving the lemma. □

Theorem 1.4.3. *(The* ***Arzelà-Ascoli Theorem****)*
A family $\mathscr{F}$ of functions that are defined and continuous on some region Ω is a normal family if and only if it is both equicontinuous and pointwise bounded in Ω.

Proof. (Palka, 1991, pp. 282–284) Assume that $\mathscr{F}$ is equicontinuous and pointwise bounded in Ω. Note that the set $S = \{z \in \Omega : \Re(z) \in \mathbb{Q}, \Im(z) \in \mathbb{Q}\}$ is dense in Ω and that, being countable, it is possible to list the elements of this set in a sequence. Let (z_n) be such a listing. Consider the sequence $(f_n(z_1))$. Since $\mathscr{F}$ is pointwise bounded, this sequence is bounded; by the Bolzano-Weierstrass Theorem (Theorem .0.4), there exists at least one accumulation point of $(f_n(z_1))$. Call this w_1. The sequence $(f_n(z_1))$ has a subsequence converging to w_1. That is, there exists a sequence of indices $m_{1,1} < m_{1,2} < m_{1,3} < \cdots$ such that

$$\lim_{k\to\infty} f_{m_{1,k}}(z_1) = w_1.$$

Note that the sequence of integers $(m_{1,k})$ is associated with just z_1 (hence the subscript, $1, k$). Now, the sequence $(f_{m_{1,k}}(z_2))_{k\geqslant 1}$ is also a bounded sequence of complex numbers. Take one of its accumulation points, w_2, and extract another subsequence of integers $m_{2,1} < m_{2,2} < m_{2,3} < \cdots$ from the sequence we already had, $(m_{1,k})$, with the property that

$$\lim_{k\to\infty} f_{m_{2,k}}(z_2) = w_2.$$

Repeating this process, to each positive integer l we assign a strictly increasing sequence of positive integers $(m_{l,k})$ such that

$$\lim_{k\to\infty} f_{m_{l,k}}(z_l) = w_l$$

and such that $(m_{l+1,k})$ is a subsequence of $(m_{l,k})$. For $k \geqslant 1$, set $n_k = m_{k,k}$. By construction $n_1 < n_2 < \cdots$. For fixed l, the subsequence (f_{n_k}) thus obtained is also a subsequence of $(f_{m_{l,k}})$, with the possible exception of the first $l-1$ terms. Therefore, for each l the sequence $(f_{n_k}(z_l))$ converges to w_l; and hence the sequence $(f_{n_k}(\xi))$ has a limit point for each $\xi \in \Sigma$. By Lemma 1.4.2, (f_{n_k}) converges normally in Ω. Hence, by definition, $\mathscr{F}$ is normal in Ω. □

An important consequence of the Arzelà-Ascoli Theorem, and the result that is commonly used in proving Riemann's Mapping Theorem, is the following theorem due to Paul Montel (1876-1975):

Theorem 1.4.4. ***(Montel's Theorem)***
(Palka, 1991, p. 285) Let $\mathscr{F}$ be a family of functions that are analytic in an open set Ω. Suppose that $\mathscr{F}$ is locally bounded in Ω. Then $\mathscr{F}$ is a normal family in this set.

Proof. Since the family of functions $\mathscr{F}$ is locally bounded in Ω, it is pointwise bounded in Ω. We prove that $\mathscr{F}$ is equicontinuous in Ω; for then, by the Arzelà-Ascoli Theorem, it is normal in Ω. Fix $z_0 \in \Omega$. Choose $r > 0$ such that the closed disk $K = \overline{D}(z_0, 2r) \subset \Omega$. Since $\mathbb{F}$ is locally bounded, there exists $m = m(K) > 0$ such that $|f(\zeta)| \leqslant m$ whenever $f \in \mathscr{F}$ and $\zeta \in K$. Now, for $z \in D(z_0, r)$, we use Cauchy's integral formula, together with the fact that $|\zeta - z_0| = 2r$ implies that $|\zeta - z| \geqslant r$ for $r \in D$, to obtain the estimate

$$\begin{aligned}
|f(z) - f(z_0)| &= \left| \frac{1}{2\pi i} \int\limits_{|\zeta - z_0| = 2r} \frac{f(\zeta)d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int\limits_{|\zeta - z_0| = 2r} \frac{f(\zeta)d\zeta}{\zeta - z_0} \right| \\
&= \frac{|z - z_0|}{2\pi} \left| \int\limits_{|\zeta - z_0| = 2r} \frac{f(\zeta)d\zeta}{(\zeta - z)(\zeta - z_0)} \right| \\
&\leqslant \frac{|z - z_0|}{2\pi} \int\limits_{|\zeta - z_0| = 2r} \frac{|f(\zeta)||d\zeta|}{|\zeta - z||\zeta - z_0|} \\
&\leqslant \frac{m|z - z_0|}{r}.
\end{aligned}$$

Given $\varepsilon > 0$, set $\delta = \min\{r, \frac{r\varepsilon}{m}\}$. Then the above estimate gives us that

$|f(z) - f(z_0)| < \varepsilon$ for all $f \in \mathscr{F}$ whenever $|z - z_0| < \delta$. Hence $\mathscr{F}$ is equicontinuous at z_0. Since z_0 was arbitrary, the Arzelà-Ascoli Theorem now shows that $\mathscr{F}$ is normal in Ω. □

1.5 The Riemann Mapping Theorem

Theorem 1.5.1. *(The **Riemann Mapping Theorem**)*
Let Ω be a simply connected region such that $\Omega \neq \mathbb{C}, \widehat{\mathbb{C}}$, and choose $z_0 \in \Omega$. Then there exists a unique conformal mapping $f : \Omega \to \mathbb{D}$, from Ω onto the unit disk $\mathbb{D}$, such that $f(z_0) = 0$ and $f'(z_0) > 0$.

See Marsden and Hoffman (1987, p. 347). Before we establish the proof of this very useful theorem, there is one more preliminary lemma that needs to be proved; another lemma, which will be used, is cited without proof.

Lemma 1.5.2. *Let Ω be a simply connected region properly contained in the complex plane and let z_0 be a point of Ω. Then there exists a conformal mapping $f : \Omega \to \mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.*

Proof. It suffices to show that we can map Ω into the unit disc conformally. For, once that is accomplished, we need only compose with a linear fractional transformation that takes z_0 to 0 and then multiply by a constant $e^{i\theta}$ chosen such that the derivative of the resulting map at z_0 is positive. If Ω is bounded, say $|z - z_0| < \rho$ for all $z \in \Omega$, then the map $z \mapsto (z - z_0)/\rho$ will do.

If Ω is not bounded, then there is at least one point a that it omits. The translation f_1 under which $z \mapsto (z - a)$ takes Ω to a simply connected region Ω_1 not containing 0. We can now take any branch f_2 of $\log z$ in Ω_1; there is guaranteed to be at least one such branch by Proposition .0.6. Moreover, by definition of the logarithm, f_2 is a univalent analytic function which takes Ω_1 to a simply connected region Ω_2. Fix $w_0 \in \Omega_2$, together with a radius ρ, such that $\overline{D}(w_0, \rho) \subset \Omega_2$. Setting $\widetilde{w}_0 = w_0 + 2\pi i$, note that $\overline{D}(\widetilde{w}_0, \rho)$ is disjoint from Ω_2. For, supposing there exists a point $\widetilde{w}$ in $\overline{D}(\widetilde{w}_0, \rho) \cap \Omega_2$, then $\widetilde{w} = f_2(\widetilde{z})$ for some point $\widetilde{z}$ in Ω_1, and also $\widetilde{w} = w + 2\pi i$ for some

$w \in \overline{D}(w_0, \rho)$. Furthermore there would exist a point $z \in \Omega_1$ such that $w = f_2(z)$, implying that

$$\widetilde{z} = e^{f_2(\widetilde{z})} = e^{\widetilde{w}} = e^{w+2\pi i} = e^{w} = e^{f_2(z)} = z,$$

which leads to $w = f_2(z) = f_2(\widetilde{z}) = \widetilde{w} = w + 2\pi i$, an absurdity. Since $|z - \widetilde{w}_0| > \rho$ for each $z \in \Omega_2$, the Möbius transformation given by $f_3(z) = \rho/(z - \widetilde{w}_0)$ is a conformal mapping that takes Ω_2 to Ω_3, a region contained in $\mathbb{D}$. The composition $f = f_3 \circ f_2 \circ f_1$ is thus a conformal map taking Ω into the unit disc. □

A consequence of this lemma is the following:

Lemma 1.5.3. *Let Ω be a simply connected domain not equal to $\mathbb{C}, \widehat{\mathbb{C}}$, let $z_0 \in \Omega$ and $f : \Omega \to \mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$. If $f(\Omega) \neq \mathbb{D}$, then there exists a conformal mapping $g : \Omega \to \mathbb{D}$ with $g(z_0) = 0$ and $g'(z_0) > f'(z_0)$.*

The existence of f in this lemma is guaranteed by Lemma 1.5.2. For a proof, see Palka (1991, pp. 418–419). We are now in a position to prove the Riemann Mapping Theorem.

Proof of the Riemann Mapping Theorem

Proof. Given a simply connected region Ω properly contained in $\mathbb{C}$ and given $z_0 \in \Omega$, we must show that there is exactly one analytic function on Ω which maps Ω onto $\mathbb{D}$ in a one-to-one fashion, with $f(z_0) = 0$ and $f'(z_0) > 0$. To this end, define

$$\mathscr{F} = \{f : \Omega \to \mathbb{D} | f \text{ is analytic and one-to-one on } \Omega, f(z_0) = 0, \text{ and } f'(z_0) > 0\}.$$

It is obvious that $\mathscr{F}$ is locally bounded in Ω. Lemma 1.5.2 ensures that $\mathscr{F}$ is nonempty. Suppose that $r > 0$ has the property that $D(z_0, r) \subset D$. Then

Cauchy's estimate (Theorem .0.9) yields

$$f'(z_0) = |f'(z_0)| \leqslant r^{-1}$$

for every member of $\mathscr{F}$. Hence, the set

$$S = \{f'(z_0) : f \in \mathscr{F}\}$$

is bounded. Denote the supremum of S by s. For each positive integer n, select $f_n \in \mathscr{F}$ such that

$$s - \frac{1}{n} \leqslant f_n'(z_0) \leqslant s.$$

Now we use Montel's Theorem (1.4.4) to extract a subsequence (f_{n_k}) from (f_n) that converges normally in Ω to some limit function f, analytic in Ω. Note that, in particular,

$$f(z_0) = \lim_{k\to\infty} f_{n_k}(z_0) = 0 \qquad \text{and} \qquad f'(z_0) = \lim_{k\to\infty} f'_{n_k}(z_0) = s > 0.$$

Therefore f is non-constant in Ω, and hence (with a little work, using Hurwitz's Theorem (Theorem .0.11)) f is univalent in Ω. Since $f(\Omega) \subset \overline{\mathbb{D}}$, by the Open Mapping Theorem (Theorem .0.10) $f(\Omega)$ is a subset of $\mathbb{D}$, and hence $f \in \mathscr{F}$. Now, suppose that $f(\Omega) \neq \mathbb{D}$. Then, by Lemma 1.5.3, we could find $g \in \mathscr{F}$ with $g'(z_0) > f'(z_0) = s$, which is absurd. Hence f maps Ω onto the unit disk.

For uniqueness, suppose that g is a second such mapping. Consider $\varphi : \mathbb{D} \to \mathbb{D}$ given by $\varphi = g \circ f^{-1}$. This is a conformal mapping of $\mathbb{D}$ onto itself, with $\varphi(0) = 0$ and $\varphi'(0) = g'(0)/f'(0) > 0$. The only conformal mappings from $\mathbb{D}$ onto itself are rotations about the origin (this is a well-established result in the theory of conformal mappings), and since $\varphi'(0) > 0$, we must have $\varphi(z) = z$. Therefore, $f(z) = g(z)$ for each $z \in \Omega$, and uniqueness is established. □

Remarks

The foregoing proof of Riemann's Mapping Theorem suggests a general strategy for obtaining minima for certain classes of problems. Namely, we first establish equicontinuity in a certain class of maps (for instance a minimising sequence). The Arzelà-Ascoli Theorem then provides a limiting map. We must then establish the regularity of the minimising map, so establishing it belongs to the class of maps under consideration.

We will see that quite restrictive hypotheses are necessary to guarantee this equicontinuity (quasiconformality is enough, but having integrable distortion is not). Thus the ideas around quasiconformal maps and equicontinuity lead to the existence of mappings of smallest maximal distortion (see next section). When we look at integrable distortion we must find new tools since equicontinuity has not been established except for mappings with distortion which is exponentially integrable (Martin and Iwaniec, 2001).