Res. Lett. Inf. Math. Sci.. (2001) **2**, 89-92
Available online at http://www.massey.ac.nz/wwiims/~rlims

Trifree Objects in E-Varieties of Locally E-Solid Semigroups

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Abstract
We construct a modification of Churchill and Trotter's trifree objects in e-varieties of locally E-solid semigroups, which have the property, for e-varieties of locally inverse semigroups and for e-varieties of E-solid semigroups, of being isomorphic to the bifree objects.

1. Introduction

It is shown in [7], bifree objects (called e-free objects in [7]) exist only in e-varieties of E-solid semigroups or e-varieties of locally inverse semigroups. In [2], Kadourek generalizes the concept of bifree objects to trifree objects for e-varieties of locally orthodox semigroups. Churchill and Trotter then use this concept to construct trifree objects for e-varieties of locally E-solid semigroups in [5]. In this paper, we construct a modification of Churchill and Trotter's trifree objects in e-varieties of locally E-solid semigroups, which have the property, for e-varieties of E-solid semigroups, of being isomorphic to the bifree objects.

2. Preliminaries

An e-variety is a class of regular semigroups closed under taking morphic images, regular subsemigroups, and direct products.

Let S be a regular semigroup. By an inverse unary operation on S we mean a unary operation $'$ on S, such that, for any $x \in S$, $xx'x = x$ and $x'xx' = x'$. A semigroup equipped with an inverse unary operation is called a regular unary semigroup. We denote by $\mathtt{RUS}$ the variety of all regular unary semigroups. For each e-variety $\mathtt{V}$ of regular semigroups, the class $V = \{ (S, \circ, ') \in \mathtt{RUS} : (S, \circ) \in \mathtt{V} \}$ is a variety of regular unary semigroups (see Hall [6]). In [5], Churchill and Trotter generalised the concept. Consider a set Λ and an e-variety $\mathtt{V}$. By a regular Λ-unary semigroup we mean a semigroup S together with a set $\{ (^{\lambda}) : \lambda \in \Lambda \}$ of inverse unary operations on S. The class $\mathtt{V}_\Lambda = \{ (S, \circ, (^{\lambda})_{\lambda \in \Lambda}) : (S, \circ) \in \mathtt{V} \}$ is then a variety of regular Λ-unary semigroups.

For a semigroup S, we denote the set of all idempotents of S by $E(S)$ and for $a, b \in S$ we denote the set of all inverses of a by $V(a)$ and the sandwich set of a, b by $S(a, b)$ (= $bV(ab)a$).

Result 2.1. ([5], Lemma 2.2, (ii))
Let S be a regular semigroup, $a, b \in S$, $a' \in V(a)$, and $b' \in V(b)$.

Then $S(a, b) = S(a'a, bb') \subseteq \left(\bigcap_{a' \in V(a),\, b' \in V(b)} V(a'abb') \right) \cap E(S)$.

By a **locally E-solid semigroup**, we mean a regular semigroup S in which all the local subsemigroups eSe, $e \in E(S)$, are E-solid.

Result 2.2. ([5], Lemma 4.1)
Let S be a regular semigroup. Then the following statements are equivalent.
1. S is locally E-solid.
2. For any $x, y \in S$, $a \in Sx$, $b \in yS$ and $k \in bV(b)S(x, y)V(a)a$, $V(b)S(k, k)V(a) \subseteq V(ab)$.

We use $^{-1}$ to mean an inverse unary operation on S such that s^{-1} is the group inverse of s whenever s is in a subgroup of S. Also, when s is in a subgroup H of S, we denote the identity of H by s^0.

Result 2.3. ([5], Corollary 4.2)
Let S be a locally E-solid semigroup and let a, b, x, y, k be as in Result 2.2. Then k is in a subgroup of S and $S(k, k) = \{ k^0 \}$.

Result 2.4. ([5], Lemma 2.3)
Let S be a regular semigroup and let k be an element in a subgroup of S. Suppose T is a regular subsemigroup of S containing k then k lies in a subgroup of T.
We say that a subset A of a semigroup is **regular** if $A \cap V(a) \neq \varnothing$ for all $a \in A$.

3. Trifree Objects

Let S be a locally E-solid semigroup and let A and C be subsets of S such that

① A is regular

② $C \subseteq \bigcup_{a, b \in A} S(a, b)$ ($\subseteq$ E, by Result 2.1)

and ③ for all $a, b \in A$, $C \cap S(a, b) \neq \varnothing$ if and only if $a'abb'$ is not in a subgroup of S for any choices of $a' \in V(a) \cap A$ and $b' \in V(b) \cap A$.

Note that, by ②, C is also regular.
We construct a subsemigroup R of S as follows.
Let $R_0 = A \cup C$, $R_1 = \langle R_0 \rangle$, and, in general, $R_{2i} = \{ r^{-1} \in S : r \in R_{2i-1} \text{ and } r\, H\, r^2 \} \cup R_{2i-1}$ (i.e. we add all group inverses of elements of R_{2i-1}) and $R_{2i+1} = \langle R_{2i} \rangle$. Then $R = \bigcup_{i \geq 0} R_{2i+1}$ is a subsemigroup of S.
We now show that, despite R being defined differently than in [5], we still have the following results.

Lemma 3.1

Let R be as above. For any $r \in R$, there are $a, b \in A$ such that $r \in Ra \cap bR$.
<u>Proof.</u> If $r \in A$ then, as A is regular, there is some $r' \in V(r) \cap A$ and so $r = rr'r \in Rr \cap rR$; if $r \in C$ then $r \in S(a, b)$ for some $a, b \in A$ and thus $r = bua$ for some $u \in V(ab)$. Then, for any $a' \in V(a) \cap A$ and $b' \in V(b) \cap A$, $r = bua = bb'buaa'a = bb'ra'a \in bR \cap Ra$. Hence the result holds for all $r \in A \cup C$ ($= R_0$) and so for all $r \in R_1$ ($= \langle R_0 \rangle$). Now suppose, for some $i \geq 1$, $r \in R_{2i-1}$ implies $r \in Ra \cap bR$ for some $a, b \in A$. If $r \in R_{2i}$ then either $r \in R_{2i-1}$ (and so the result holds) or $r = s^{-1}$ for some $s \in R_{2i-1}$ with $s\, H\, s^2$. For the later case, note that $r = s^{-1} = ss^{-3}s$. As $s \in R_{2i-1}$, by the inductive hypothesis, the result also holds. Therefore the result holds for all $r \in R_{2i+1}$ ($= \langle R_{2i} \rangle$).

Lemma 3.2
The semigroup R is regular and so R is the least regular subsemigroup of S containing $A \cup C$.
Proof. First we show that R is regular. Clearly $A \cup C$ is regular. Suppose R_{2i-2} is regular for some $i \geq 1$. Consider $r \in R_{2i}$ ($= R_{2i-1} \cup \{s^{-1} \in S : s \in R_{2i-1}$ and $s \mathcal{H} s^2\}$). If $r = s^{-1}$ for some $s \in R_{2i-1}$ with $s \mathcal{H} s^2$ then $s \in V(r) \cap R$ and so r has an inverse in R; if $r \in R_{2i-1} \setminus \{s^{-1} \in S : s \in R_{2i-1}$ and $s \mathcal{H} s^2\}$ then $r = r_1 r_2 \cdots r_n$ for some $r_1, r_2, \cdots, r_n \in R_{2i-2}$ (as $R_{2i-1} = \langle R_{2i-2} \rangle$). If $n = 1$ then $r = r_1 \in R_{2i-2}$. By the inductive hypothesis, r has an inverse in R. Now suppose $n \geq 2$ and suppose $r_1 r_2 \cdots r_{n-1}$ has an inverse in R. To simplify notation, let $t = r_1 r_2 \cdots r_{n-1}$ and $t' \in V(t) \cap R$. By Lemma 3.1, we may assume that $t \in Ra$ and $r_n \in bR$ for some $a, b \in A$. Now, if $a'abb'$ is in a subgroup of S for some $a' \in V(a) \cap A$ and $b' \in V(b) \cap A$ we take $k = r_n r_n' bb'(a'abb')^{-1} a'at't$; otherwise, we take $k = r_n r_n' ct't$ where $c \in S(a, b) \cap C$ ($\neq \varnothing$). In either case, $k \in r_n V(r_n) S(a, b) V(t) t \cap R$. Hence, by Result 2.3, k is in a subgroup of R and so $k^{-1} \in R$. Therefore $k^0 = kk^{-1} \in R$. Also, by Result 2.3, $S(k, k) = \{k^0\}$. Now consider $r_n' k^0 t'$ ($\in R$). We have $r_n' k^0 t' \in V(r_n) S(k, k) V(t) \subseteq V(tr_n) = V(r)$, by Result 2.2. Hence $V(r) \cap R \neq \varnothing$ and so R is regular by induction. Now the fact that R is the least of such semigroups follows from Result 2.4.

Let V be an e-variety of regular semigroups. Let X be a nonempty set and let $X' = \{x' : x \in X\}$ be a disjoint copy of X. Let $\overline{X} = X \cup X'$. As in [5], we consider $\Lambda = X \cup \overline{X}^2$ and let $(F_{V_\Lambda}(X), ({}^x)_{x \in X}, ({}^{yz})_{y, z \in \overline{X}})$ be the free object on X in V_Λ. Each $x' \in X'$ can be identified with x^x and so we may regard X' as a subset of $F_{V_\Lambda}(X)$. Let $s(y, z) = z(yz)^{yz}y$ for any $y, z \in \overline{X}$ and write $\overline{X}_1 = \overline{X} \cup \{ s(y, z) : y, z \in \overline{X},\ y^y yzz^z$ is not in a subgroup of $F_{V_\Lambda}(X)\}$.

We define **tied mapping** and **trifree object** as in [2].
Let S be a regular semigroup. A mapping $\phi : \overline{X}_1 \to S$ is called tied if

(i) $x'\phi \in V(x\phi)$ for every $x \in X$

and (ii) $s(y, z)\phi \in S(y\phi, z\phi)$ for all $s(y, z) \in \overline{X}_1$ where $y, z \in \overline{X}$.

We say that a pair $(TF_V(X), \phi)$ is a trifree object on X in a class V of regular semigroups if (i) $\phi : \overline{X}_1 \to TF_V(X)$ is a tied mapping, (ii) $TF_V(X)$ is a member of V, and (iii) for any $S \in \mathrm{V}$ and for any tied mapping $\theta : \overline{X}_1 \to S$, there is a unique morphism $\psi : TF_V(X) \to S$ such that $\phi \psi = \theta$.

Now, consider the e-variety $\mathtt{LES}$ of locally E-solid semigroups. Let V be a sub e-variety of $\mathtt{LES}$ and $(F_{V_\Lambda}(X), ({}^x)_{x \in X}, ({}^{yz})_{y, z \in \overline{X}})$ be the free object on X in V_Λ. Then $F_{V_\Lambda}(X)$ is locally E-solid and so there exists a least regular subsemigroup R of $F_{V_\Lambda}(X)$ containing $\overline{X}_1$, as constructed in Lemma 3.2.

Let $\iota : \overline{X}_1 \to R$ be the natural injection. Then clearly ι is tied.
Now, we have the desired result.

Theorem 3.3 (R, ι) is the trifree object on X in V.
Proof. Clearly ι is tied and R is a member of V. Let S be a member of V and let $\phi : \overline{X}_1 \to S$ be a tied mapping. Let ϕ' be the restriction of ϕ to X. For every $x \in X$, we can choose an inverse unary operation $({}^x)$ on S such that $(x\phi)^x = x^x\phi$. For every $s(y, z) \in \overline{X}_1$ where $y, z \in \overline{X}$, we have $s(y, z)\phi \in S(y\phi, z\phi) = z\phi V(y\phi z\phi) y\phi$. Hence there is some $w_{yz} \in V(y\phi z\phi)$ such that $s(y, z)\phi = z\phi w_{yz} y\phi$. We can choose a unary inverse operation $({}^{yz})$ on S such that $(y\phi z\phi)^{yz} = w_{yz}$. For every $y, z \in \overline{X}$ with $s(y, z) \notin \overline{X}_1$, we simply take any unary inverse operation $({}^{yz})$ on S. Then $(S, ({}^x)_{x \in X}, ({}^{yz})_{y, z \in \overline{X}})$ is a member of V_Λ. Then, by freeness of $F_{V_\Lambda}(X)$, there is a Λ-unary semigroup morphism

φ' : $F_{V_\Lambda}(X) \to S$ extending ϕ'. Let φ be the restriction of φ' to R. Then φ extends ϕ: if $x \in X$ then $x\varphi = x\varphi' = x\phi' = x\phi$ and $x^x\varphi = x^x\varphi' = (x\varphi')^x = (x\phi)^x = x^x\phi$; and if $s(y, z) \in \overline{X}_1$ where $y, z \in \overline{X}$ then $s(y, z)\varphi = s(y, z)\varphi' = (z(yz)^{yz}y)\varphi' = z\varphi'(yz\varphi')^{yz}y\varphi' = z\varphi'(y\varphi'z\varphi')^{yz}y\varphi' = z\phi(y\phi z\phi)^{yz}y\phi = z\phi w_{yz}y\phi = s(y, z)\phi$. The uniqueness of φ can be shown in exactly the same way as in [5]. Hence (R, ι) is the trifree object on X in V.

Remark. The trifree object we construct here is equal to the one constructed in [5] when V = LES. This is because the only way to get y^yyzz^z in a subgroup of $F_{LES_\Lambda}(X)$ is to have $z = y^y$. However this then implies that $X \cap X' \neq \emptyset$, a contradiction. Hence y^yyzz^z is never in a subgroup of $F_{LES_\Lambda}(X)$. In other words, in the case that V = LES, we remove no element from Churchill and Trotter's generating set $\overline{X}_1$ at all. For e-subvarieties of LES, of course the trifree objects are in many cases different to those of Churchill and Trotter [5].

The advantage of the trifree object we construct here is that:
the trifree object is isomorphic to the bifree object when $V \subseteq LI$ or $V \subseteq ES$ where LI is the e-variety of all locally inverse semigroups and ES is the e-variety of all E-solid semigroups.
To see this, if $V \subseteq LI$ then s(y, z) is always in the bifree object for any $y, z \in \overline{X}$ and hence it is only a routine manner to show that the trifree object is isomorphic to the bifree object; if $V \subseteq ES$ then y^yyzz^z is always in a subgroup of $F_{V_\Lambda}(X)$. Hence $\overline{X}_1 = \overline{X}$ and so the trifree object (which is generated by $\overline{X}_1$) is isomorphic to the bifree object (which is generated by $\overline{X}$).

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