Optimal harvesting strategies for fisheries: A differential equations approach

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Abstract

The purpose of fisheries management is to achieve a sustainable development of the activity, so that future generations can also benefit from the resource. However, the optimal harvesting strategy usually maximizes an economically important objective function formed by the harvester which can lead to the extinction of the resource population. Therefore, sustainability has been far more difficult to achieve than is commonly thought; fish populations are becoming increasingly limited and catches are declining due to overexploitation.

The aim of this research is to determine an optimal harvesting strategy which fulfills the economic objective of the harvester while maintaining the population density over a prespecified minimum viable level throughout the harvest. We develop and investigate the harvesting model in both deterministic and stochastic settings. We first employ the Expected Net Present Value approach and determine the optimal harvesting policy using various optimization techniques including optimal control theory and dynamic programming. Next we use real options theory, model fish harvesting as a real option, and compute the value of the harvesting opportunity which also yields the optimal harvesting strategy. We further extend the stochastic problem to include price elasticity of demand and present results for different values of the coefficient of elasticity.

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Chapter 1

Introduction

Fisheries management is a complex process and requires the integration of resource biology and ecology with socioeconomic and institutional factors affecting the behaviour of harvesters and policy-makers (ecologists). According to the Food and Agriculture Organization of the United Nations (FAO, 2007), in the year 2005, about 50% of the fish stock under observation experienced overexploitation or depletion. These statistics re-iterate the fact that fisheries need to be managed with an effective and carefully-defined objective in order to prevent overfishing and to allow the depleted stock to replenish.

1.1 Harvesting: deterministic viewpoint

There are several conflicting objectives at play in fisheries management. As pointed out by Hilborn & Walters (1992), excluding recreational fishing, these objectives can be classified as: biological and economic, and both can be explained on the basis of classic deterministic models, as described below.

Biological objective

The traditional objective for biologists has been the *Maximum Sustainable Yield* (or MSY). The concept of MSY can be illustrated most simply via a biological population undergoing deterministic growth as

$$\frac{dx}{dt} = F(x), \tag{1.1}$$

where $x = x(t)$ denotes the biomass of the fish population at time t and $F(x)$ is the biological net growth rate. For the logistic growth, the function $F(x)$ is given by

$$F(x) = rx\left(1 - \frac{x}{K}\right),$$

where $K > 0$ is the *environmental carrying capacity* and $r > 0$ is the *intrinsic growth rate* (Clark, 1990 provides a rigorous treatment of logistic growth function along with its solution). For logistic growth, Equation (1.1) transforms to

$$\frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t)}{K}\right). \tag{1.2}$$

Equation (1.2) can be integrated using separation of variables as follows:

$$\begin{aligned} \int \frac{dx(t)}{rx(t)\left(1 - \frac{x(t)}{K}\right)} &= \int dt \\ \Rightarrow \int \frac{dx(t)}{x(t)\left(K - x(t)\right)} &= \int \frac{r}{K} dt, \end{aligned}$$

and the solution is obtained as

$$x(t) = \frac{K}{1 + \left(\frac{K - x(0)}{x(0)}\right) e^{-rt}}. \tag{1.3}$$

Furthermore, Equation (1.2) possesses two equilibrium points: 0 and K, where 0 corresponds to unstable equilibrium and K corresponds to asymptotically stable equilibrium. When the stock level is below K, $\frac{dx}{dt} > 0$ and the population rises towards K. When the stock level is above K, $\frac{dx}{dt} < 0$ and the population declines towards K. The growth rate $F(x)$ is maximum at $x = K/2$ and the maximum value for $F(x)$ is $rK/4$, which is obtained by substituting $x = K/2$ in Equation (1.2).

Figure 1.1 shows the logistic growth curve $F(x)$ and Figure 1.2 illustrates the solution for the logistic growth equation; the population increases (or decreases) asymptotically towards its carrying capacity K depending on the initial level.

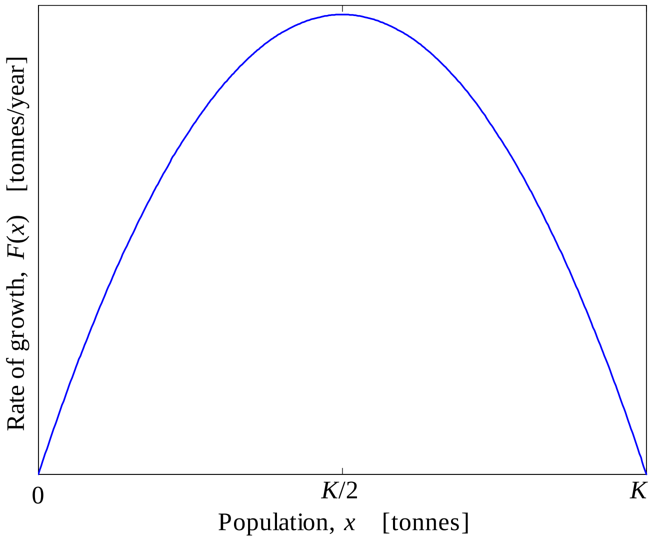


Figure 1.1: Logistic growth curve, $F(x)$: the growth rate is maximum when the population level is at half its carrying capacity.

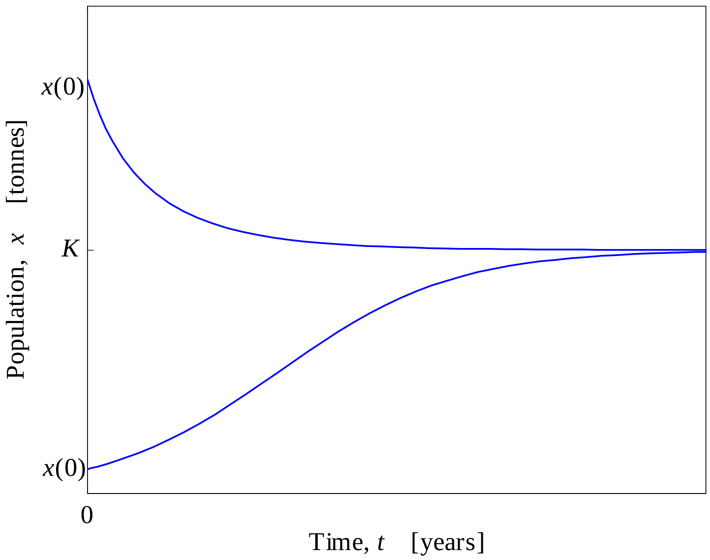


Figure 1.2: Logistic solution curve: the stock approaches its carrying capacity K asymptotically. If the initial population level $x(0)$ is below K then the stock increases towards K, whereas, if $x(0)$ is above K then the stock decreases towards K.

When the population is harvested at a rate $h(t)$, the growth equation (1.1) takes the form

$$\frac{dx}{dt} = F(x) - h(t, x). \tag{1.4}$$

If the population is harvested at a constant rate (or yield) h then the logistic growth model (1.2) becomes

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - h. \tag{1.5}$$

There are two equilibria for Equation (1.5), given by

$$\left.\begin{aligned} x_{\max} &= \frac{K+\sqrt{K^2-\frac{4hK}{r}}}{2} > \frac{K}{2} \\ x_{\min} &= \frac{K-\sqrt{K^2-\frac{4hK}{r}}}{2} < \frac{K}{2} \end{aligned}\right\}, \text{ provided that } h < \frac{rK}{4}. \tag{1.6}$$

Figure 1.3 illustrates the population growth under a constant harvesting rate h, where we have plotted three scenarios for the harvest rate: $h < \frac{rK}{4}$; $h = \frac{rK}{4}$; and $h > \frac{rK}{4}$. We now investigate the behaviour of the equilibrium points $x_{\max}$ and $x_{\min}$ given by (1.6) for $h < \frac{rK}{4}$.

- If at any time the population level $x(t) < x_{\min}$ then $\frac{dx}{dt} < 0$ and the population will become zero (the species would become extinct) in finite time.
- If at any time the population level $x(t)$ is such that $x_{\min} < x(t) < x_{\max}$ then the population will approach the stable equilibrium $x_{\max}$.
- If at any time the population level $x(t) > x_{\max}$ then $\frac{dx}{dt} < 0$ and the population will decline towards $x_{\max}$.

The above analysis shows that $x_{\min}$ is an unstable equilibrium point while $x_{\max}$ is a stable equilibrium point.

If $h = \frac{rK}{4}$ then $x_{\max} = x_{\min} = \frac{K}{2}$, thus there is only one equilibrium point in this case; $F(x)$ is also at its maximum, equal to $\frac{rK}{4}$. The equilibrium point $K/2$ is semistable and there are again three possibilities:

- If at any time the population level $x(t) < K/2$ then the population becomes zero.
- If at any time the population level $x(t) = K/2$ then the population always stays at that level.

- If at any time the population level $x(t) > K/2$ then the population declines towards $K/2$.

If $h > \frac{rK}{4}$ then $\frac{dx}{dt} < 0$, always, and the population will become zero in finite time.

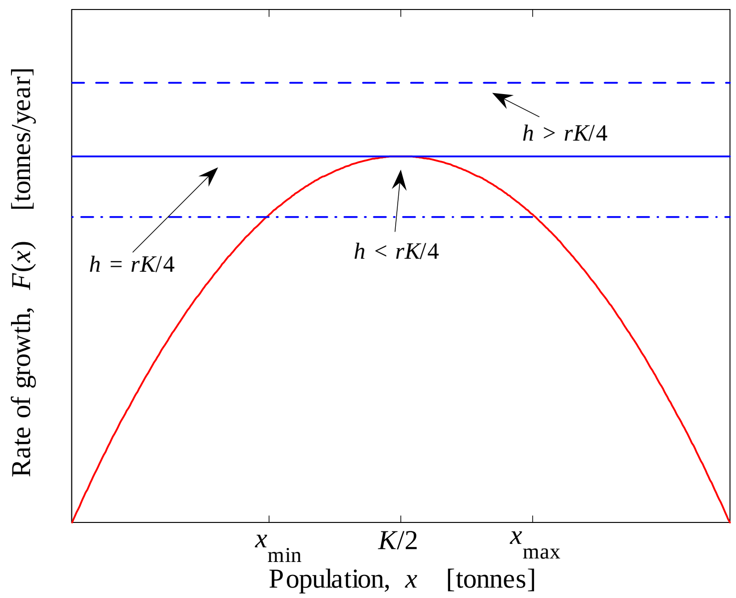


Figure 1.3: The population growth under a constant rate of harvest, h. There are two non-zero equilibria when $h < rK/4$: $x_{\min}$ (unstable equilibrium) and $x_{\max}$ (stable equilibrium); the fish-stock collapses if it falls below $x_{\min}$. There is one semi-stable non-zero equilibrium when $h = rK/4$, that is at $x = K/2$; here the biological growth is at its maximum, equal to $rK/4$. There is no non-zero equilibrium when $h > rK/4$; $dx/dt < 0$ always and the resource stock eventually becomes extinct.

Harvesting at rate $\frac{rK}{4}$, when the resource stock is at exactly half its carrying capacity $\left(\frac{K}{2}\right)$ and at its maximum growth rate $\left(\frac{rK}{4}\right)$, allows the population to remain at that level indefinitely since $\frac{dx}{dt} = 0$. This particular yield value is the MSY, denoted by h_{MSY}. Thus,

$$h_{MSY} = \max F(x) = \frac{rK}{4}.$$

The foregoing mathematical analysis reflects the viewpoint of most biologists who consider a resource to be overexploited when the population size has been reduced to a level below the population level at MSY.

In general, the rate of harvest $h(t)$ depends on the capability of the fishing fleet. This includes fishing gear, boats, and crew, to name a basic few. These components have to be combined and there are several factors that can be considered (Clark, 1985). In this thesis we will focus on the following two:

- **Effort** $E(t)$: The number of standardized fishing vessels fishing at a given time, with unit SFU (*Standardized Fishing Unit*).
- **Catchability** q: The ratio of fish caught per SFU per unit time, with unit $(\mathrm{SFU})^{-1}$(unit of time)$^{-1}$. This represents environmental factors that limit (or enhance) the ability to harvest the resource.

The ratio of yield (or catch) to effort, i.e. $\frac{h(t)}{E(t)}$, is an indication of the number of fish caught by the fleet, i.e. $qx(t)$. Consequently,

$$h(t) = qE(t)x(t). \tag{1.7}$$

Substituting for $h(t)$ from Equation (1.7) into Equation (1.4) we obtain the growth dynamics under harvesting as

$$\frac{dx}{dt} = F(x) - qE(t)x(t). \tag{1.8}$$

For logistic growth, Equation (1.8) becomes

$$\frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t)}{K}\right) - qE(t)x(t). \tag{1.9}$$

The model given by Equation (1.9) is known as the *Schaefer model* after the biologist M. B. Schaefer (Schaefer, 1957) who first proposed it as a realistic representation of fish growth under harvesting.

As discussed in Eide *et al.* (2003), some empirical works (such as those on Northeast Arctic cod harvest) employ the Cobb-Douglas function:

$$h = qE^{\alpha}x^{\beta},$$

where α denotes the effort-output elasticity giving % increase of catch h with 1% increase of fishing effort; β denotes stock-output elasticity giving % increase of catch h with 1% increase of stock biomass. For cod fisheries, $\alpha \geq 0$ and $0 \leq \beta \leq 1$.

Considering E to be a constant in the Schaefer model, Equation (1.9) gives a non-zero equilibrium point at

$$x_{eqm} = K\left(1 - \frac{qE}{r}\right) \text{ where } E < \frac{r}{q} \,.$$

If $E > \frac{r}{q}$ then $x_{eqm} < 0$ and the population is driven to extinction. The equilibrium harvest, denoted by h_{SY}, is called the *sustainable yield* and is given by the harvest at the equilibrium point $x = x_{eqm}$, i.e.

$$h_{SY} = qEx_{eqm} = qKE\left(1 - \frac{qE}{r}\right). \tag{1.10}$$

The effort maximizing the sustainable yield h_{SY} is obtained as $E_{MSY} = \frac{r}{2q}$; the Maximum Sustainable Yield is again $h_{MSY} = h_{SY}|_{E=E_{MSY}} = \frac{rK}{4}$. Figure 1.4 illustrates the plot of the sustainable yield h_{SY} versus the effort E. It is a parabolic curve, given by $h_{SY} = qEK - \frac{q^2E^2K}{r}$ with the MSY at $\left(\frac{r}{2q}, \frac{rK}{4}\right)$. The sustainable yield increases with effort up to the point of Maximum Sustainable Yield, $E = r/2q$, falling thereafter as fishing effort increases. However, the rise in E above $r/2q$ does not cause an immediate drop in the sustainable yield h_{SY}. The decline in h_{SY} (as observed in the graph) is due to the long-run effect of the increased effort which results in decreased fish population and, consequently, diminished harvest.

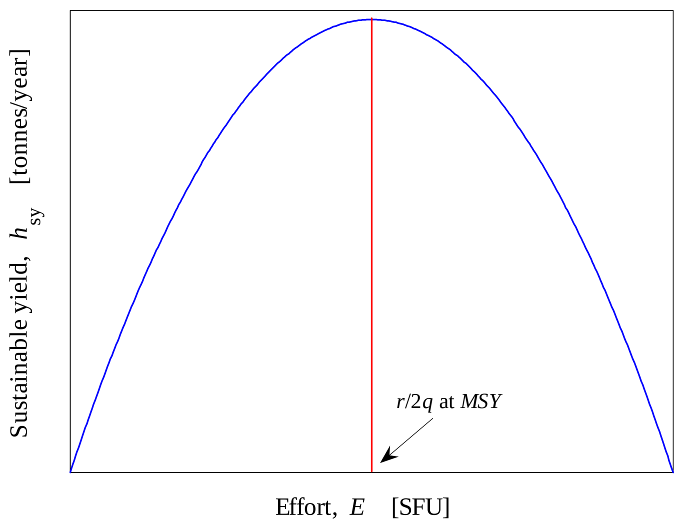


Figure 1.4: Sustainable yield h_{SY} versus effort E; h_{SY} increases with E while $E \leq E_{MSY}$, a further increase in E results in a drop in h_{SY}.

The logistic growth model, $F(x) = rx\left(1 - \frac{x}{K}\right)$, is a *pure compensation model* (Clark, 1990), that is, $\frac{F(x)}{x}$ is a decreasing function of x; here Equation (1.8) exhibits a unique non-zero stable equilibrium. Models for which $\frac{F(x)}{x}$ is an increasing function of x, at least for certain values of x, are known as *depensation models.* In this case, Equation (1.8) exhibits multiple equilibria; there exists an unstable equilibrium population level, such that, if the stock level falls below this unstable equilibrium point then the population declines to zero. However, there is a possibility to revive the stock to a sustainable level by reducing effort substantially. The depensation models which exhibit irreversibility in population growth are called *critical depensation models.* These models are characterized by a critical population threshold, such that, if the stock falls below this threshold then it undergoes an irreversible decline (to zero) and cannot be recovered even if harvesting is ceased. This critical threshold is called the *minimum viable population level.* A good description of depensation models can be found in Levy *et al.* (2006) and Clark (1990).

Economic objective

The incorporation of economic considerations into resource harvesting models leads to the subject called *bioeconomics.* One of the first economic models was developed by the Canadian economist Gordon (1954), based on Schaefer's model; that model introduces the concept of economic overfishing in open-access fisheries. An *open-access* fishery is one in which there is no regulation and fishing is uncontrolled. Gordon's model establishes that the net revenues (or *Sustainable Economic Rent*) derived from fishing are a function of Total Sustainable Revenues (TSR) and Total Costs (TC), given by

$$\begin{aligned} \textit{Sustainable Economic Rent} &= TSR - TC \\ \text{or alternatively,} \quad \textit{Sustainable Economic Rent} &= ph_{SY} - cE, \end{aligned} \tag{1.11}$$

where p is the (constant) price per unit harvest, h_{SY} is the sustainable yield and c is the (constant) cost per unit effort. The cost per unit effort includes fixed costs, variable costs and opportunity costs of labour and capital. Fixed costs are independent of fishing operations (depreciation, administration and insurance costs), whereas variable costs are incurred when fishers go fishing (fuel, bait, food and beverages, etc.). Opportunity costs are the net benefits that could have been achieved in the next best economic activity, i.e. other regional fisheries, capital investment or alternative employment, and thus must be integrated into cost estimations.

Gordon assumes an economic equilibrium, in addition to a biological equilibrium, to obtain the long-term yield of the fishery. This equilibrium occurs when TSR equals TC and thus *Sustainable Economic Rent* is zero, implying that there will be no stimulus for entry or exit to the fishery. The yield thus established provides a simultaneous equilibrium in both an economic and a biological sense, leading to bioeconomic (*bionomic*) equilibrium (BE). The name *bionomic* is due to both biological and economic parameters present in the model. The stock level x_{BE} corresponding to the *bionomic* equilibrium is obtained by equating the *Sustainable Economic Rent* in Equation (1.11) to zero, to give

$$x_{BE} = \frac{c}{pq}.$$

Since the biomass is at equilibrium, the corresponding *bionomic* effort E_{BE} can be determined by equating the growth rate (given by Equation (1.9)) to zero and using $x(t) = x_{BE}$; this yields

$$E_{BE} = \frac{r}{q}\left(1 - \frac{c}{pqK}\right).$$

It follows that if the fishing cost to price ratio is such that $\frac{c}{p} > qK$, the fishery will not be exploited at all.

The *Maximum Economic Yield*, denoted by h_{MEY}, is the sustainable yield maximizing the *Sustainable Economic Rent*. We now derive an expression for h_{MEY}. Substituting for h_{SY} from Equation (1.10) into Equation (1.11) we get

$$TSR - TC = pqEK\left(1 - \frac{qE}{r}\right) - cE. \tag{1.12}$$

From Equation (1.12), the maximum *Sustainable Economic Rent* occurs at fishing effort

$$E_{MEY} = \frac{r}{2q}\left(1 - \frac{c}{pqK}\right) = \frac{E_{BE}}{2}.$$

Now *Maximum Economic Yield* is the sustainable yield with effort equal to E_{MEY} (maximizing *Sustainable Economic Rent*). Substituting E in Equation (1.10) with E_{MEY} we obtain the *Maximum Economic Yield* as

$$h_{MEY} = h_{SY}|_{E=E_{MEY}} = qE_{MEY}K\left(1 - \frac{qE_{MEY}}{r}\right).$$

Hence the long-term sustainable biomass and yield of the fishery can be built by specifying the corresponding levels of fishing efforts: E_{BE}, E_{MSY}, and E_{MEY}.

The plot of TSR versus effort E is a parabolic curve, given by $TSR = pqKE\left(1 - \frac{qE}{r}\right)$, and the plot of TC $(= cE)$ versus E is a straight line (see Figure 1.5). Where they intersect is the bionomic equilibrium effort, E_{BE}. The effort at MSY is also shown in the middle of the E axis $\left(E_{MSY} = \frac{r}{2q}\right)$, as well as the effort at MEY $\left(\frac{E_{BE}}{2}\right)$.

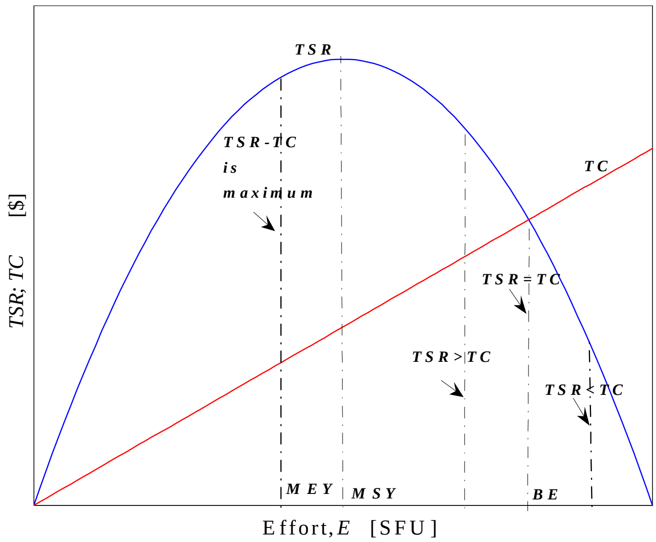


Figure 1.5: The evolution of total sustainable revenue TSR and total costs TC with effort E. The intersection of TSR curve and TC line corresponds to bionomic equilibrium, BE.

Gordon's basic argument is that in an open-access fishery the effort tends to approach an equilibrium effort E_{BE}, called the *bionomic equilibrium* effort, at which $TSR = TC$. A brief analysis shows that if $E > E_{BE}$ then $TSR < TC$ (*Sustainable Economic Rent* is negative) and if $E < E_{BE}$ then $TSR > TC$ (*Sustainable Economic Rent* is positive).

The former case implies that some fisheries are losing money and therefore drop out of the market, thus decreasing the total effort. Whereas, if $E < E_{BE}$, fishing is profitable which encourages new fisheries to join the market. Thus there should be a tendency to approach the bionomic equilibrium. Bionomic equilibrium describes the scenario in which *economic overfishing* occurs and the *Sustainable Economic Rent* is precisely zero. But if fishing effort were to be reduced, the *Sustainable Economic Rent* would become positive. The fishing effort maximizing the *Sustainable Economic Rent*, E_{MEY}, is precisely half of the effort corresponding to bionomic equilibrium, E_{BE}. Figure 1.5 shows E_{BE} to be higher than E_{MSY}. In practice, E_{BE} can be higher than or lower than E_{MSY}, depending on fishing costs and fish price (Clark, 2006). When $E_{BE} > E_{MSY}$, the harvest rate exceeds h_{MSY} and the resulting population level x_{BE} is below $K/2$; this situation is referred to as *biological overfishing* and leads to the depletion of the resource.

The economic model developed by Gordon is also called the Gordon-Schaefer model as it takes into account the following assumptions made by Schaefer:

- The population is at equilibrium.
- The catch per unit of effort ($CPUE$) is a relative index of population abundance: $CPUE = \frac{h}{E} = qx$. This implies that, for a fixed fishing effort, an increase in stock biomass leads to an increase in the catch at the same rate.
- The stock is constrained by a constant carrying capacity of the environment.
- The stock will respond immediately to variations in the magnitude of effort exerted; time-delays are ruled out.
- Fishing technology is constant; this is reflected primarily in the catchability-coefficient q which is assumed to be a constant.
- Unit prices and costs are constant and therefore independent of the level of effort exerted.

In addition to the assumptions listed above, Gordon made another assumption as follows:

- TC are proportional to effort and thus a change in the slope of the TC curve will determine changes in BE and MEY levels.

There are certain limitations to the Gordon-Schaefer model:

- All processes affecting stock productivity (e.g. fish growth, recruitment and mortality) are subsumed in the effective relationship between effort and catch. The fish stock x in the production function qEx lumps together young recruits and more mature fish; factors related to age are not taken into account.

- The catchability coefficient q is not always constant and may differ according to the available resources of the fishing fleet. Improvement in technology and fishing power determines that q often varies through time.

- $CPUE$ is not always an unbiased index of stock abundance, i.e., the average $CPUE$ does not always coincide with its estimated value. This is especially relevant for resources with patchy distribution (non-uniform spatial distribution) and without the capacity of redistribution in the fishing ground once fishing effort is exerted.

- Variations in the spatial distribution of the stock and differential allocation of fishing effort in the short term are not usually taken into account. The spatial distribution of the stock is assumed to be homogeneous which results in uniform allocation of the fishing effort. However, a differential allocation of fishing effort is more realistic as the resource stock is patchily distributed; biological parameters are extremely dependent on environmental conditions and change values even within small distances.

- Technological interdependencies can arise from the activity of fishing fleets with different fishing power, and even different effort costs, over a single fish stock; these are not accounted for.

- It becomes difficult to distinguish whether population fluctuations are due to fishing pressure or natural processes. In some fisheries, fishing effort could be exerted at levels greater than twice the optimum (Clark, 1985).

Critique of MSY

There is substantial literature in the field of MSY and the related economic theory. Recall that in an open-access fishery the fishing effort approaches the bionomic equilibrium effort, E_{BE}, and the corresponding population level is maintained at x_{BE}; when biological overfishing occurs, $x_{BE} < K/2$ (the population level at MSY). According to the theory of bionomic equilibrium, the fishing industry should gain by reducing effort. As the effort is reduced and the population is allowed to grow from x_{BE} to $\frac{K}{2}$, the catch grows accordingly with lesser effort and the consumer benefits by paying less. Reduction of effort, however, is not possible in an open-access fishery. If an open-access fishery is being harvested at MSY, new entrants can join the fishery with a belief that they can either make individual gains or share the loss with other entrants. Hardin (1968) in his famous essay called this situation "tragedy of the commons". Such a situation arises not only in fisheries but in any exploited biological resource where open access is granted. In order to alleviate this problem, Roughgarden & Smith (1996) recommend a target stock at $\frac{3K}{4}$ and taxation of revenues from any fish caught when the stock is below the target figure.

Larkin (1977) argued for the abandonment of MSY by challenging its goal on several grounds: it focuses on the fishery's target species ignoring other species and puts the other less productive species at a risk of extinction; it is based on the benefits of fishing and does not concentrate on the costs; it does not account for spatial variability in productivity (also noted by Botsford *et al.*, 1997).

Furthermore, as pointed out by Levy *et al.* (2006), MSY guidelines are based on average year stock growth rates and are inflexible since they fail to take into consideration stock fluctuations, which are sometimes caused by MSY policies themselves. This renders fish harvest vulnerable to the "ratchet effect". The ratchet effect occurs when, after a sequence of relatively stable years in fishing, a few good years (with above average stock growth) provoke the harvesters to invest more in their fishing capacity (e.g. specialized boats and gear). However, when the fishing conditions and stock growth return to normal or below normal, the fishing industry seeks help from the government in terms of additional quotas and increased allowable catches . There is no disinvestment of fishing capacity due to subsidies offered by the government and this encourages overharvesting during poor periods (Ludwig *et al.*, 1993). Inevitably, overfishing reduces biological populations substantially.

An age-structured model

We now discuss the *Beverton-Holt model*, which is a deterministic age-structured model introduced by Beverton & Holt (1957). Let $x(t)$ denote the number of fish at time t and suppose that a constant recruitment of fish, R, occurs at time $t = \tau$. Recruitment refers to the fish that have reached harvestable (mature) age. Thus age-structure enters the model through recruitment. Assuming the natural fish mortality to be a constant M, we get the growth dynamics as

$$\begin{aligned} \frac{dx}{dt} &= -Mx, \\ x(\tau) &= R. \end{aligned}$$

If a variable mortality due to fishing, $F(t)$, is also considered then the growth equation becomes

$$\begin{aligned} \frac{dx}{dt} &= -(M + F(t))x, \\ \text{along with} \quad x(\tau) &= R. \end{aligned}$$

Drawing an analogy with the Gordon-Schaefer model, $F(t) = qE$. Thus the function $F(t)$ is now the control variable specifying the intensity of fishing. Let $w(t)$ represent the average fish mass at time t. The total biomass at time t is then $B(t) = x(t)w(t)$ and the yield in biomass is $Y(t) = F(t)x(t)w(t) = F(t)B(t)$.

Next we describe the economic optimization of the net revenue according to Clark *et al.* (1973) and Hannesson (1975). Let p denote the fixed fish price per unit mass, ρ the discount rate and c the cost per unit harvesting. The objective is to maximize the net present value:

$$\int_0^\infty e^{-\rho t} F(t) \left[pB(t) - c \right] dt.$$

The state variable in this case is $B(t)$ and the state equation is

$$\frac{dB}{dt} = B(t)\left(\frac{\dot{w}}{w} - M - F(t)\right).$$

The constraint on fishing intensity is given by

$$0 \leq F(t) \leq F_{\max}.$$

The solution to this problem is of *bang-bang* type (see Chapter 2 for a description of this type of solutions). At time $t = 0$ there is not enough fish of mature size which renders fishing unprofitable. Therefore $F(t) = 0$ and no fishing is carried out until fish grow big enough to make harvesting profitable. Then at some later time, before the biomass reaches its maximum, fishing commences at maximum intensity. It appears then that introducing age structure in the bioeconomic models leads to the conclusion that periodic (*pulse*) fishing might be a preferable strategy for a fishery.

The independence of stock and recruitment in the Beverton-Holt model is one major unrealistic assumption that limits its utility as a bioeconomic model. In practice, estimates of the weight of the fish, $w(t)$, as well as the total mortality, $M + F(t)$, are available through relevant data. However the separate estimates of $F(t)$ and M, requiring data on varying levels of harvesting, are not easily available.

Other literature

A large body of literature has focused on the optimal harvesting policies when the resource stock follows deterministic growth (summarized in Clark, 1990). The Schaefer model and the economic model developed by Gordon, discussed in Section (1.1), are classic examples of deterministic models based on density-dependent growth.

Considerable research has gone into deterministic models with density-dependent growth and linear harvesting costs. Under general assumptions, including a profit function that is linear in harvest, the profit-maximizing policy is obtained as *bang-bang* which recommends harvesting at maximal rate when the population is above a critical threshold and at zero rate if the population falls below the critical level (Clark, 1990). Similar conclusions were drawn by Spence & Starrett (1975) for optimal control problems where the solution approaches some stationary value for the state variable. The approach to this stationary value could be more rapid or less rapid depending on the functions and parameters associated with the problem. For some problems, this approach is most rapid and the corresponding solution path is called *most rapid approach path* (*mrap*). The necessary condition for the existence of a *mrap* solution is that the objective function should be linear in the time-derivative of the state variable. As an example, the authors maximized the present value of the profit earned by a firm harvesting a renewable natural resource stock. Assuming the cost function to be linear in effort, they found that the solution was a *mrap* to an optimal equilibrium for the stock level.

The deterministic optimal harvesting problem with linear costs was also investigated by Clark (1979), Clark & Munro (1975), Reed (1979) and Clark *et al.* (1979). The computation of the optimal harvesting policy for an objective function which is linear in state (stock-level) and control (fishing-effort) is illustrated in Chapter 2.

1.2 Harvesting: stochastic viewpoint

The literature discussed in the previous section concentrates on the optimal harvesting problem in a deterministic framework. The evolution of natural stocks, however, is seldom deterministic; it is subject to stochastic perturbations due to environmental and other factors. Consequently, management strategies for these natural resources based upon deterministic population dynamics models are oversimplified. Moreover, in many models, the associated cost structure assumes the price to be either fixed or a prescribed function and this is not always realistic. These issues led to research in the area of stochastic growth and price dynamics. We now discuss the literature associated with harvesting in a stochastic environment.

The optimal rate of extraction when the resource stock follows stochastic growth has been studied extensively. The book by Mangel (1985) discusses natural resource optimization in a random environment. Reed (1974) considered a discrete-time optimization model and determined an optimal harvesting policy in a stochastic growth environment. The selling price was assumed to be fixed, and successive unharvested population levels were assumed to form a Markov chain. The cost of harvesting consisted of a fixed one-time set-up cost and a marginal density-dependent harvest cost. An optimal harvesting policy maximizing the expected net present value of the total profit earned over an infinite time horizon was sought. It was shown that there existed an optimal policy of type (S, s) with $S < s$, where S was a critical threshold for the population. Initiating harvest when the population level was above S resulted in non-negative revenues, however, the revenues could not cover the set-up cost unless the stock-level was well above S. The level s was the smallest population level above S at which the harvesting could be initiated profitably. Thus the optimal policy allowed a harvest in any period if and only if the population level exceeded s; in that case, a harvest down to level S could be made. If the set up cost was zero then harvesting could be initiated profitably once the population level exceeded S, and therefore S was equal to s.

Gleit (1978) investigated a continuous-time harvesting model based on stochastic growth, assuming the selling price to be a prescribed function independent of the harvest size. Instead of directly maximizing profit, the present value of the utility received from the profit levels was maximized. It was concluded that, keeping the population level fixed, the optimal harvest increased with an increase in the variance of the growth rate whilst the expected utility of the entire profit stream decreased.

Ludwig (1979) examined the effect of small amounts of noise in the population growth on the optimal harvesting policy. The price per unit harvest was assumed to be a fixed constant and perturbation methods were employed to study the problem. The results obtained in that paper were supplemented with numerical calculations in Ludwig & Varah (1979).

Lewis (1981) examined a discrete-time model for optimal harvesting with random growth and stochastic price. Pindyck (1984) discussed the effects of uncertainty in the growth rate of renewable resources assuming that the price was given endogenously by a downward sloping market demand curve; three different growth functions were used for analysis. It was found that, depending upon the growth function, the overall effect of an increase in the variance of stock fluctuations on the extraction rate could be to increase the extraction rate, decrease it, or leave it unchanged.

In most of the above-mentioned models, the price was considered either to be fixed or to be a known function, whereas in real-life situations the price evolves randomly. Anderson (1982) investigated a continuous-time optimal harvesting problem with logistic growth and random price dynamics; the expected utility of the profit was maximized. It was shown that an increase in the variance of the price results in diminished fishing effort.

Hanson & Ryan (1998) performed a numerical study of the effect of random fluctuations in price and population growth on the optimal solution, assuming costs to be quadratic in fishing effort. They introduced random price fluctuations through a multiplicative random process that included both small continuous-time fluctuations (modelled by a Wiener process) and the possibility of occasional, large random changes (characterized by Poisson processes). However, while performing simulations, the coefficients of the Wiener increments were assumed to be zero. They maximized the expected present value of the flow of profit and found that the random price fluctuations had a significant impact on the optimal return whilst the optimal effort levels were relatively unaffected.

Furthermore, harvesting in the presence of stochastic perturbations may reduce the stock below a critical level (called *minimum viable population level*) from which recovery is impossible; this phenomenon is called critical depensation (described in Section 1.1). Next we discuss the literature concerned with minimum viable population level.

Minimum viable population level and threshold harvesting

A survey of the sources of uncertainty in the stock growth together with an assessment of a minimum viable population size is due to Shaffer (1981). In that paper, a tentative definition of minimum viable population level was proposed as:

"*A minimum viable population level for any given species in any given habitat is the smallest isolated population having* 99% *chance of remaining extant for* 1000 *years despite the foreseeable effects of demographic, environmental and genetic stochasticity, and natural catastrophes*".

In a series of papers (Lande *et al.*, 1995; Saether *et al.*, 1996; Lande *et al.*, 1997) the authors considered two optimization criteria, the first being to maximize the expected cumulative yield (denoted by Y) before extinction, and the second to maximize the mean annual yield, $\frac{Y}{T}$, where T denoted the mean time to extinction. They found that, in both cases, the optimal policy was a generalized form of the *bang-bang* strategy. The first optimization criterion recommended the carrying capacity as the optimal threshold, irrespective of the form of expected dynamics and the magnitude of stochastic effects; the second recommended an optimal threshold depending upon the form of expected dynamics and the magnitude of stochastic effects. In another publication (Lande *et al.*, 1994), the same authors argued against economic discounting in the development of optimal strategies for sustainable use of biological resources.

Theoretical studies of estimates of threshold management policies to be set as targets of fisheries management are due to Mace (1994) and Quinn *et al.* (1990); a similar study based on some real data is due to Myers *et al.* (1994). McDonald *et al.* (2002) analyzed a harvesting model assuming the fishing costs to be inversely proportional to the stock level; the unit price was assumed to be a linear function of harvest rate so that the resulting revenue function was quadratic in rate of harvest. When the stock was above the minimum viable level, the optimal policy corresponding to stochastic growth was seen to recommend a more conservative harvest as compared with the optimal policy associated with deterministic growth.

Ludwig (1998) assumed a stochastic model for the resource population and compared various management strategies in the presence of critical depensation. The conclusion was that a strategy involving abrupt adjustment in harvest size, in accordance with fluctuations in the stock density, results in a lower probability of early extinction as compared with other strategies. This result seems to be in agreement with the *bang-bang* strategy of maximizing the discounted net return from harvesting if the stock was above a critical threshold and not harvesting at all otherwise, as shown in Clark (1990) and Reed (1979).

1.3 Overview of the thesis

In this thesis, we study a bio-economic model of a fishery that is being harvested continuously by a sole harvester, assuming that the population consists of a single species of fish and that the growth is only density-dependent. The work is based on the Schaefer model as it effectively incorporates the biological features of fish population, and has been widely used in literature to represent density-dependent growth. The world has witnessed many fisheries collapsing due to over-exploitation. We try to mitigate this problem by maintaining the population above a minimum viable level throughout the harvesting period. The layout of the thesis is as follows:

This chapter provides an introduction to fish harvesting and discusses the key papers associated with this research. Chapter 2 presents some preliminary concepts from stochastic calculus and optimization; these are required to formulate and solve the optimal harvesting problem developed in later chapters. An optimal harvesting problem, linear in the control variable, is also illustrated.

Chapter 3 investigates a deterministic model for harvesting with constant price per unit harvest and costs quadratic in fishing effort. The deterministic model is extended to its stochastic version in Chapter 4. The model with random growth and constant price is treated separately from the model with random growth and random price dynamics. In Chapters 3 and 4, the (expected) net present value of the total profit is maximized.

Chapter 5 formulates the optimal harvesting problem using real options theory. The results obtained using real options approach are compared with the solutions determined by employing the net present value approach in Chapters 3 and 4.

In Chapter 6, we introduce the concept of elasticity and reformulate the stochastic model (from Chapter 4) to include price elasticity of demand. Finally, Chapter 7 concludes the thesis with a summary of the work done; some directions for future research are also suggested. Throughout the study, we perform sensitivity analyses of the optimal solution with respect to various parameters present in the model. As we will note, the optimization problem being studied from Chapter 3 onwards cannot be solved analytically and therefore we have to resort to numerical methods. Appendices A, B and C include the working of the finite-difference approximations for the numerical solution of the partial differential equations obtained in this study. The numerical solution is obtained using MATLAB. The code can be made available by contacting the author.

The emphasis of this thesis is on the profit-making aspect of fisheries. It is a thorough study of the optimal harvesting policy and the profit earned by harvesting, focusing on quadratic costs and conservation of fish population by constraining the latter to always stay above a critical threshold. The prime reason for using quadratic costs is that it allows us to derive an analytical expression for the optimal effort; the resulting solution is different from the *bang-bang* solution which is usually obtained in the case of a linear cost function. Further justification for this assumption is provided in Chapter 3. The correlation between the fish growth and the price is usually ignored when modelling them as random variables; it is either not considered or equated to zero while performing simulations. We model the correlation explicitly and include it in the study while analyzing the final results. This thesis:

- Draws its basis from the literature discussed in this chapter and the literature associated with real options (discussed in Chapter 5),
- Extends the work to cover some aspects which have not been explored by the existing literature in full detail, and
- Presents new research, especially in the field of mathematical finance (real options and elasticity-modelling) and optimal harvesting.

Chapter 2

Background theory and application to linear control in harvesting

In this chapter we collect some important material needed for solving the optimal harvesting problem discussed in the next few chapters. The chapter proceeds as follows: Section 2.1 serves as an introduction to probability theory and stochastic calculus. In Section 2.2 we discuss the calculus of variations and in Section 2.3 we describe optimal control theory, both in a deterministic environment. We also solve a deterministic optimal harvesting problem, with linear harvesting costs, using the above-mentioned approaches. In Section 2.4 we explain the continuous-time dynamic programming technique in both deterministic and stochastic settings. Finally, we demonstrate a solution to the stochastic optimal harvesting problem with the costs linear in fishing effort; the unit price is considered to be fixed and the growth is random.

2.1 Fundamentals of Stochastic Calculus

In this section we provide an introduction to random variables and stochastic differential equations. A more rigorous treatment of this subject is provided by Billingsley (1995) and Chung & AitSahila (2003).

Measure and Probability Theory

A collection $\mathcal{I}$ of subsets of a sample space Ω is called a *σ-algebra* (or a σ-field) if:

(i) $\phi, \Omega \in \mathcal{I}$ where ϕ is the empty set.

(ii) If $A \in \mathcal{I}$ then $A' \in \mathcal{I}$ where A' is the complement of A.

(iii) $\mathcal{I}$ is closed under countable unions and intersections.

Any element of $\mathcal{I}$ is called a *measurable set* and $(\Omega, \mathcal{I})$ is called a *measurable space*.

A *measure* on a σ-algebra $\mathcal{I}$ is defined as a non-negative, extended, real-valued function m on $\mathcal{I}$ satisfying:

(i) $m(\phi) = 0$.

(ii) $m(A) = \sum_n m(A_n)$

where $\{A_n\}$ is a countable collection of pairwise disjoint sets in $\mathcal{I}$ such that

$$A = \bigcup_n A_n \in \mathcal{I}.$$

It can be shown that $m(A)$ is independent of choice of A_n. If $m(\Omega) = 1$ then m is called a *probability measure* on Ω and $(\Omega, \mathcal{I}, m)$ is called a *probability space*.

A *Borel σ-algebra* is a σ-algebra generated by open sets and its elements are called *Borel sets*. A function X from $(\Omega, \mathcal{I})$ to $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-field of real numbers, is called *measurable* if for any Borel set $B \in \mathcal{B}$ the set $\{\omega : X(\omega) \in B\} \in \mathcal{I}$. Equivalently, for all $x \in \mathbb{R}$, the set $\{\omega : X(\omega) \leq x\} \in \mathcal{I}$. This function X is called a *random variable* on $(\Omega, \mathcal{I})$. The *variance* of a random variable $X(t)$ quantifies the dispersion of $X(t)$ and is defined as

$$var\,[X(t)] = \mathcal{E}\,[X(t) - \mathcal{E}\,[X(t)]]^2\,,$$

where $\mathcal{E}$ denotes the expectation operator. The square root of variance is called *volatility*; it measures the deviation of $X(t)$ from its expected value. The covariance of two random variables $X(t)$ and $Y(t)$ is defined as

$$cov\,[X(t), Y(t)] = \mathcal{E}\,[X(t)Y(t)] - \mathcal{E}\,[X(t)]\,\mathcal{E}\,[Y(t)]\,.$$

The *correlation coefficient* between $X(t)$ and $Y(t)$ is given by

$$\rho_{XY} = \frac{cov\,[X(t), Y(t)]}{\sqrt{var\,[X(t)]}\sqrt{var\,[Y(t)]}}$$

The correlation coefficient ρ_{XY} is a dimensionless quantity, lying between $[-1,\ 1]$. The magnitude of ρ specifies the degree of interdependence between $X(t)$ and $Y(t)$. If ρ is zero then $X(t)$ and $Y(t)$ are said to be uncorrelated.

A *stochastic process* $X(t)$ is a collection of time-dependent random variables. Consider a stochastic process $X(t)$ which can assume values from a countable state space Ω. Let I_t denote the information available about $X(t)$ up to time t, i.e., $I_t = \sigma(\{X(s), 0 \le s \le t\})$. Evidently, $I_t \subset I_{t+1}$ for all t. The family of information sets $\{I_t : t \in [0, \infty)\}$, modelling the flow of information, is called a *filtration*. The stochastic process $X(t)$ is called *adapted* to the filtration I_t if its values up to and including time t are determined by the information provided by I_t. A stochastic process is called *non-anticipative* if it is independent of the future, i.e., its value at time t depends only on the information available up to that time. Throughout this discussion we will assume the stochastic process $X(t)$ to be defined on a probability space $(\Omega, \mathcal{I}, p)$, where p is a probability measure on Ω.

A continuous-time *martingale* is a stochastic process $X(t)$, adapted to a filtration I_t, such that for all $t > 0$ the following hold:

(i) $X(t)$ is integrable, i.e., $\mathcal{E}[|X(t)|] < \infty$.

(ii) $\mathcal{E}[X(t)|I_s] = X(s)$ for all $s < t$ almost surely (with probability 1).

A stochastic process $W(t)$ is called a *Wiener process* with respect to a filtration I_t if:

(i) $W(0) = 0$.

(ii) $W(t)$ is continuous over time.

(iii) $W(t)$ is a square integrable martingale with $\mathcal{E}[(W(t) - W(s))^2] = t - s, \;\; s \le t$.

The precise mathematical formulation for a Wiener process was carried out by Norbert Wiener in the year 1931.

Next we define *Brownian motion*. The term derives its origin from the name of the botanist Robert Brown who, in the nineteenth century, described the random movement of a particle of pollen suspended in fluid. The random movement was argued to be a consequence of bombardment of the particle by the molecules in thermal motion. Mathematically, Brownian motion is defined as a stochastic process $B(t)$ with the following properties:

(i) $B(0) = 0$.

(ii) $B(t)$ is continuous over time.

(iii) The process has independent increments which are normally distributed with mean zero and variance given by the time lag, i.e.,

$$(B(t) - B(s)) \sim N\left(0, \mid t - s \mid\right).$$

Although the definitions for a Wiener process and Brownian motion appear to be different in some respects, the famous Lévy theorem (stated below) proves that the two processes are exactly the same (Durrett, 1996).

Theorem 1 *If $X(t)$ is a continuous local martingale with $X(0) = 0$ and $var(X(t)) = t$ then $X(t)$ is a one-dimensional Brownian motion.*

Since a Wiener process satisfies all the requirements mentioned in the Lévy theorem (see Durrett, 1996 for further explanation), it follows that a Wiener process is a Brownian motion. Another important characteristic of Brownian motion is the Markov property. A stochastic process $X(t)$, adapted to a filtration I_t, is said to possess the *Markov property* if

$$p\left\{X(t+u) = j | X(s); 0 \leq s \leq t\right\} = p\left\{X(t+u) = j | X(t)\right\}.$$

Thus the future state of the process is independent of all the past states and depends only on the current state. Any process satisfying the Markov property is called a *Markov process*.

Limits of random variables

Consider a sequence of random variables $\{X_1, X_2, \ldots\}$, all defined on the same probability space. The sequence can converge to a random variable X in various ways:

(i) If $p\left(\lim_{n\to\infty} X_n = X\right) = 1$ then $X_n \to X$ *almost surely*.

(ii) If $\mathcal{E}\left[|X_n|^2\right] < \infty$, $\forall\ n$ and $\lim_{n\to\infty} \mathcal{E}\left[|X_n - X|^2\right] = 0$ then the convergence is called *Mean-square convergence* ($\mathcal{L}_2$ convergence).

(iii) If $\lim_{n\to\infty} p(X_n \leq a) \to p(X \leq a)\ \forall a \in \mathcal{R}$ then the convergence is called *Convergence in distribution.*

Kolmogorov's Strong Law of Large Numbers is a statement about almost sure convergence of the sequence of the average partial sum of independent and identically distributed random variables, $\{X_1, X_2, \ldots\}$. Let S_n denote the average partial sum consisting of the first n terms of this sequence of random variables. That is,

$$S_n = \frac{\sum\limits_{i=1}^{n} X_i}{n}.$$

If the expected value of all X_i*'s* is finite, *say* $\mathcal{E}[X_1] = \mathcal{E}[X_2] = ... = \mu < \infty$, then the Strong Law of Large Numbers states that

$$\lim_{n\to\infty} S_n \to \mu.$$

The *Central Limit Theorem* is a statement about convergence in distribution of the sum of independent and identically distributed random variables $\{X_1, X_2, \ldots\}$, each having expected value $\mathcal{E}[X_i] = \mu < \infty$ and variance $\mathcal{V}[X_i] = \sigma^2 < \infty$. According to the Central Limit Theorem, the random variable

$$\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{n}(X_i - \mu),$$

converges in distribution to a standard normal random variable as $n \to \infty$. Equivalently,

$$\lim_{n\to\infty} p\left(a \leq \frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{n}(X_i - \mu) \leq b\right) = \int\limits_{a}^{b} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

Mathematical formulation of Brownian motion

To introduce Brownian motion consider a stochastic process $w(t)$, that undergoes up/down jumps of size Δx at discrete time intervals of duration $\Delta t = \frac{1}{n}$, where n is some unspecified positive integer. These jumps occur with equal probability, i.e. $\frac{1}{2}$, throughout the time interval $0 \leq t \leq T$. After k steps, at time $t = k(\Delta t)$, the stochastic process will be at

$$\begin{aligned} w(t) = w(k\Delta t) &= w((k-1)\Delta t) \pm \Delta x \\ &= w((k-2)\Delta t) \pm \Delta x \pm \Delta x, \end{aligned}$$

which finally gives

$$w(k\Delta t) = \pm\Delta x \pm \Delta x \cdots \pm\Delta x, \tag{2.1}$$

where the right hand side contains k terms and we have assumed that $w(0) = 0$.

Let $\{x_1, \ldots, x_n, \ldots\}$ be an infinite sequence of random variables, each with $p(x_i = 1) = p(x_i = -1) = \frac{1}{2}$, where x_i specifies whether the jump was up or down between $(i-1)\Delta t$ and $i(\Delta t)$. These random variables are identically distributed with the expected value $\mathcal{E}[x_i] = 0$ and Variance $\mathcal{V}[x_i] = 1$. Using these $x_i' s$, Equation (2.1) becomes

$$w(k\Delta t) = (\Delta x) \sum_{i=1}^{k} x_i.$$

We wish to analyze the behaviour of the stochastic process $w(t)$ when the time duration between two consecutive jumps becomes negligible, i.e. as $\Delta t \to 0$, or equivalently, as $n \to \infty$. Since $t = k\Delta t = \frac{k}{n}$, $n \to \infty$ would imply that $k \to \infty$ (in order to keep t finite).

If we choose $\Delta x = \Delta t \left(= \frac{1}{n}\right)$,

$$w(t) \equiv w(k\Delta t) = \frac{1}{n} \sum_{i=1}^{k} x_i = \frac{k}{n} \sum_{i=1}^{k} \frac{x_i}{k} = t \sum_{i=1}^{k} \frac{x_i}{k}. \tag{2.2}$$

Now $n \to \infty$ implies $k \to \infty$ and utilizing the Strong Law of Large Numbers along with $\mathcal{E}[x_i] = 0$, Equation (2.2) yields $\lim\limits_{k\to\infty} w(t) = 0$, thus $\lim\limits_{n\to\infty} w(t) = 0$. Therefore, no genuine random behaviour in the limit is observed in this case.

Picking $\Delta x = \frac{1}{\sqrt{n}}$ instead, we get

$$w(t) \equiv w(k\Delta t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} x_i = \sqrt{t} \frac{\sum\limits_{i=1}^{k} x_i}{\sqrt{k}}.$$

By the Central Limit Theorem we obtain $\lim\limits_{n\to\infty} w(t) \longrightarrow z\sqrt{t}$, where $z \sim N(0,1)$ (here we have again used: $n \to \infty \Rightarrow k \to \infty$). Hence $w(t)$ is a normal random variable with mean 0 and variance t. Therefore,

$$p(a \leq w(t) \leq b) = p\left(\frac{a}{\sqrt{t}} \leq z \leq \frac{b}{\sqrt{t}}\right) = \int_{\frac{a}{\sqrt{t}}}^{\frac{b}{\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \int_{a}^{b} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx.$$

If $w(0) = 0$ then at time t,

$$\begin{aligned} \mathcal{E}[w(t)] &= 0, \\ \mathcal{V}[w(t)] &= \mathcal{E}[w^2(t)] - (\mathcal{E}[w(t)])^2 = t \ for \ t \geqslant 0. \end{aligned}$$

Furthermore, due to the property of independent increments, for $t > s$ we have

$$\mathcal{E}[w(t)w(s)] = \mathcal{E}[(w(t) - w(s))w(s) + w(s)^2] = \mathcal{E}[(w(t) - w(s))w(s)] + \mathcal{E}[w^2(s)] = 0 + s = s.$$

In general,

$$\mathcal{E}[w(t)w(s)] = \min(s, t).$$

The sample paths of Brownian motion are continuous but nowhere differentiable. To see this, given any interval (a, b) divide it into subintervals $a = t_1 < t_2 < \cdots < t_n = b$. Then

$$\sum_{k=1}^{n-1} |w(t_{k+1}) - w(t_k)|^2 \leq \max_{k \in \{1,2,\ldots,n-1\}} (|w(t_{k+1}) - w(t_k)|) \sum_{k=1}^{n-1} |w(t_{k+1}) - w(t_k)|. \tag{2.3}$$

$$\begin{aligned} \text{Now} \quad \mathcal{E}\left[\sum_{k=1}^{n-1} |w(t_{k+1}) - w(t_k)|^2\right] &= \sum_{k=1}^{n-1} \mathcal{E}\left[|w(t_{k+1}) - w(t_k)|^2\right] \\ &= \mathcal{E}\left[|w(t_2) - w(t_1)|^2\right] + \mathcal{E}\left[|w(t_3) - w(t_2)|^2\right] + \ldots \\ &\quad \ldots + \mathcal{E}\left[|w(t_n) - w(t_{n-1})|^2\right] \\ &= t_2 - t_1 + t_3 - t_2 + \ldots + t_n - t_{n-1} \\ &= t_n - t_1 = b - a. \end{aligned}$$

Therefore in terms of the expected value, as $n \longrightarrow \infty$, $\sum_{k=1}^{n-1} |w(t_{k+1}) - w(t_k)|^2 \longrightarrow (b - a)$ and $\max_{k \in \{1,2,\ldots,n-1\}} (|w(t_{k+1}) - w(t_k)|) \to 0$ (since $\mathcal{E}\left[|w(t_{k+1}) - w(t_k)|\right] = 0 \ \forall \ k$). Consequently, the inequality in (2.3) can hold only if $\sum_{k=1}^{n-1} |w(t_{k+1}) - w(t_k)| \to \infty$. Thus the sample paths of $w(t)$ have unbounded total variation in any interval.

Stochastic differential equations

The concept of stochastic differential equations can be understood on the basis of a typical example which we now present. Consider a stochastic process $x(t)$ which solves the generalized Brownian motion or *Itô process*

$$dx(t) = \mu(x(t), t)dt + \sigma(x(t), t)dw(t). \tag{2.4}$$

In Equation (2.4), $\mu(x(t), t)$ is called the *drift* parameter, and $\sigma(x(t), t)$, the *diffusion* parameter. Note that over any time interval dt, the change in $x(t)$ (represented by $dx(t)$) is normally distributed with expected value $\mathcal{E}[dx(t)] = \mu(x(t), t)dt$ and variance $\mathcal{V}[dx(t)] = \sigma^2(x(t), t)dt$. Equation (2.4) is a differential representation of the Itô process; the same equation can be represented in the integral form as

$$x(t) = x(0) + \int_0^t \mu(x(s), s)ds + \int_0^t \sigma(x(s), s)dw(s), \tag{2.5}$$

where the second integral in Equation (2.5), $\int_0^t \sigma(x(s), s)dw(s)$, is a stochastic or *Itô integral* defined by the limiting process

$$\begin{aligned} \int_0^t \sigma(x(s), s)dw(s) &= \lim_{\Delta t \to 0} \sum_{k=1}^{n-1} \sigma(x(t_k), t_k)\Delta w(t_k) \\ &= \lim_{\Delta t \to 0} \sum_{k=1}^{n-1} \sigma(x(t_k), t_k)\left[w(t_{k+1}) - w(t_k)\right], \end{aligned} \tag{2.6}$$

where $0 = t_1 < t_2 < \cdots < t_n = t$ is a uniform partition of the interval $[0,\ t]$ and $\Delta t = t_{k+1} - t_k$. For the Itô integral to exist, the two conditions imposed on the diffusion term are:

- $\sigma(x(t), t)$ is independent of the future or *non-anticipative.*
- $\mathcal{E}\left[\int_0^t \sigma^2(x(s), s)ds\right] < \infty.$

The limit present in Equation (2.6) is the *mean square limit* which is defined as

$$\mathcal{E}\left[\left(\int_0^t \sigma(x(s), s)dw(s) - \sum_{k=1}^{n-1} \sigma(x(t_k), t_k)\left[w(t_{k+1}) - w(t_k)\right]\right)^2\right] \to 0 \text{ as } n \to \infty.$$

In addition to the Itô integral there is the *Stratonovich integral* which uses the difference $\Delta w(t_k) = \frac{1}{2}\left[w(t_{k+1}) - w(t_{k-1})\right]$. The work done in this thesis is based on the Itô calculus and all the stochastic differential equations are defined using an Itô process.

Itô's lemma

Itô's lemma replaces the chain rule in a stochastic environment. Consider a stochastic process $x(t)$ satisfying

$$dx(t) = \mu(x(t),t)dt + \sigma(x(t),t)dw(t).$$

Let $\phi(x(t),t)$ be any smooth deterministic function. The total differential for the function $\phi(x(t),t)$ is given by

$$d\phi(x(t),t) = \frac{\partial \phi}{\partial t}dt + \frac{\partial \phi}{\partial x(t)}dx(t) + \frac{1}{2}\frac{\partial^2 \phi}{\partial x(t)^2}(dx(t))^2 + \dots$$

In ordinary calculus, with deterministic $x(t)$, higher order terms vanish in the limit. To see what happens when $x(t)$ is random, consider $(dx(t))^2$ as follows:

$$\begin{aligned}(dx(t))^2 &= \mu^2(x(t),t)dt^2 + \sigma^2(x(t),t)dw^2 + 2\mu(x(t),t)\sigma(x(t),t)dtdw \\ &\approx \sigma^2(x(t),t)dt,\end{aligned}$$

because $\mathcal{E}\left[dw^2(t)\right] = dt$, and for infinitesimally small dt terms in $(dt)^2$ and $(dt)^{\frac{3}{2}}$ go to zero faster than dt. Likewise, $(dx)^3$ will contain terms in higher powers of dt and can be ignored.

Hence Itô's lemma gives the differential $d\phi$ as

$$\begin{aligned}d\phi(x(t),t) &= \frac{\partial \phi}{\partial t}dt + \frac{\partial \phi}{\partial x(t)}dx(t) + \frac{1}{2}\frac{\partial^2 \phi}{\partial x(t)^2}(dx(t))^2 \\ &= \left(\frac{\partial \phi}{\partial t} + \mu(x(t),t)\frac{\partial \phi}{\partial x(t)} + \frac{1}{2}\sigma^2(x(t),t)\frac{\partial^2 \phi}{\partial x(t)^2}\right)dt + \sigma(x(t),t)\frac{\partial \phi}{\partial x(t)}dw(t).\end{aligned}$$

The extension of Itô's lemma to a function $\phi(x_1(t), x_2(t), \dots, x_n(t), t)$ of time and n Itô processes, $x_1(t), x_2(t), \dots, x_n(t)$, contains the correlation coefficients between the n Wiener increments, $dw_1(t)$, $dw_2(t)$, ..., $dw_n(t)$. Let ρ_{ij} denote the coefficient of correlation between the Wiener increments $dw_i(t)$ and $dw_j(t)$, i.e., $\mathcal{E}[dw_i(t)dw_j(t)] = \rho_{ij}dt$. Then $d\phi$ is given as

$$\begin{aligned} d\phi &= \left(\frac{\partial \phi}{\partial t} + \sum_{i=1}^{n} \mu_i(x_1,\ldots,x_n,t)\frac{\partial \phi}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2(x_1,\ldots,x_n,t)\frac{\partial^2 \phi}{\partial x_i^2} \right. \\ &\quad \left. + \frac{1}{2}\sum_{i\neq j}^{n} \rho_{ij}\sigma_i(x_1,\ldots,x_n,t)\sigma_j(x_1,\ldots,x_n,t)\frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) dt \\ &\quad + \sum_{i=1}^{n}\sigma_i(x_1,\ldots,x_n,t)\frac{\partial \phi}{\partial x_i} dw_i. \end{aligned}$$

Geometric Brownian motion

A stochastic process $x(t)$ is said to follow a *geometric Brownian motion* if $\frac{dx(t)}{x(t)}$ (the fractional change in $x(t)$) is normally distributed with $\mathcal{E}\left[\frac{dx(t)}{x(t)}\right] = \mu dt$ and $\mathcal{V}\left[\frac{dx(t)}{x(t)}\right] = \sigma^2 dt$; μ and σ are constants. Since these are the changes in the natural logarithm of $x(t)$, i.e. $\ln x(t)$, therefore $dx(t)$ is said to be *lognormally distributed*. The stochastic differential equation followed by $x(t)$ is given by

$$dx(t) = \mu x(t)dt + \sigma x(t)dw(t), \tag{2.7}$$

with the initial condition

$$x(0) = x_0.$$

In ordinary calculus, Equation (2.7) would imply that $d(\ln x(t)) = \mu dt + \sigma dw(t)$, however, this is not the case here. If $\phi(x(t)) = \ln x(t)$ then by the Itô's lemma,

$$d\phi(x(t)) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dw(t). \tag{2.8}$$

The solution to Equation (2.8) is

$$\phi(x(t)) = \phi(x(0)) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma w(t). \tag{2.9}$$

Since $x(t) = e^{\phi(x(t))}$, Equation (2.9) gives $x(t)$ as

$$x(t) = x_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma w(t)}.$$

The expected value of $x(t)$ is

$$\mathcal{E}[x(t)] = x_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t}\mathcal{E}[e^{\sigma w(t)}]. \tag{2.10}$$

It remains to calculate $\mathcal{E}[e^{\sigma w(t)}]$. From Itô's lemma, with $\phi(w(t)) = e^{\sigma w(t)}$, we have

$$d\phi(w(t)) = \frac{1}{2}\sigma^2 e^{\sigma w(t)}dt + \sigma e^{\sigma w(t)}dw(t),$$

which has solution

$$\phi(w(t)) = \phi(w(0)) + \frac{\sigma^2}{2}\int_0^t e^{\sigma w(s)}ds + \sigma \int_0^t e^{\sigma w(s)}dw(s).$$

Taking the expectation on both sides we get

$$\mathcal{E}[\phi(w(t))] = \mathcal{E}[\phi(w(0))] + \frac{\sigma^2}{2}\mathcal{E}\left[\int_0^t e^{\sigma w(s)}ds\right] + \sigma\mathcal{E}\left[\int_0^t e^{\sigma w(s)}dw(s)\right]. \tag{2.11}$$

Now

$$\mathcal{E}[\phi(w(0))] = \mathcal{E}[e^{\sigma w(0)}] = 1 \quad (\text{since } w(0) = 0).$$

Furthermore, using $\mathcal{E}\left[\int_0^t f(w(s), s)dw(s)\right] = 0$ for any non-anticipative function $f(w(s), s)$ (Henderson & Plaschko, 2005, page 31), we obtain

$$\mathcal{E}\left[\sigma\int_0^t e^{\sigma w(s)}dw(s)\right] = 0.$$

Consequently, Equation (2.11) reduces to

$$\begin{aligned}\mathcal{E}[\phi(w(t))] &= 1 + \frac{\sigma^2}{2}\mathcal{E}\left[\int_0^t e^{\sigma w(s)}ds\right] = 1 + \frac{\sigma^2}{2}\mathcal{E}\left[\int_0^t \phi(w(s))ds\right] \\ &= 1 + \frac{\sigma^2}{2}\int_0^t \mathcal{E}[\phi(w(s))]ds. \end{aligned} \tag{2.12}$$

In Equation (2.12), the expectation is associated with the function $\phi(w(s))$ (since $w(s)$ is a random variable) while the integration is being performed with respect to a deterministic variable, that is t. Since expectation is a linear operator, we can move the expectation operator inside the integral sign .

Differentiating both the sides of Equation (2.12) with respect to t leads to

$$\frac{d\mathcal{E}[\phi(w(t))]}{dt} = \frac{\sigma^2}{2}\mathcal{E}[\phi(w(t))].$$

This is an ordinary differential equation with solution

$$\mathcal{E}[\phi(w(t))] = e^{\frac{\sigma^2 t}{2}}.$$

Therefore,

$$\mathcal{E}[e^{\sigma w(t)}] = \mathcal{E}[\phi(w(t))] = e^{\frac{\sigma^2 t}{2}}. \tag{2.13}$$

Substituting $\mathcal{E}\left[e^{\sigma w(t)}\right]$ in Equation (2.10) yields the expected value of $x(t)$ as

$$\mathcal{E}[x(t)] = x_0 e^{\mu t} \quad for \quad t \geq 0.$$

The variance of $x(t)$ can be calculated as

$$\begin{aligned}
\mathcal{V}[x(t)] &= \mathcal{E}[x^2(t)] - (\mathcal{E}[x(t)])^2 \\
&= \mathcal{E}\left[\left(x_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma w(t)}\right)^2\right] - \left(x_0 e^{\mu t}\right)^2 \\
&= \mathcal{E}\left[x_0^2 e^{2\mu t - \sigma^2 t} e^{2\sigma w(t)}\right] - x_0^2 e^{2\mu t} \\
&= x_0^2 e^{2\mu t - \sigma^2 t} \mathcal{E}\left[e^{2\sigma w(t)}\right] - x_0^2 e^{2\mu t} \\
&= x_0^2 e^{2\mu t}\left(e^{-\sigma^2 t}\mathcal{E}\left[e^{2\sigma w(t)}\right] - 1\right).
\end{aligned} \tag{2.14}$$

From Equation (2.13) we have $\mathcal{E}\left[e^{\sigma w(t)}\right] = e^{\frac{\sigma^2 t}{2}}$. Changing σ to 2σ yields

$$\mathcal{E}[e^{2\sigma w(t)}] = e^{2\sigma^2 t}.$$

Finally, substituting $\mathcal{E}[e^{2\sigma w(t)}]$ in Equation (2.14) gives the variance as

$$\mathcal{V}[x(t)] = x_0^2 e^{2\mu t}(e^{\sigma^2 t} - 1).$$

2.2 Optimization concepts: Calculus of Variations

The calculus of variations is a classical technique for solving dynamic optimization problems (see Kamien & Schwartz, 1991 and Chiang, 2000 for an in-depth treatment of this subject). The simplest problem of the calculus of variations is a maximization (minimization) problem where the objective

$$J(x) = \int_0^T g(t, x, \dot{x})dt \tag{2.15}$$

has to be maximized (minimized) with respect to x, subject to

$$\left.\begin{aligned} x(0) &= x_0 \\ x(T) &= x_T \end{aligned}\right\}, \tag{2.16}$$

where g is a twice differentiable function and dot denotes the derivative with respect to time. We have used the notation $x(t)$ for a certain state at a certain time t, whereas x without an argument denotes the entire path: $\{x(t) : t \in [0,\ T]\}$ when T is finite; and $\{x(t) : t \in [0,\ T)\}$ when T is infinite. Any x satisfying the boundary conditions (2.16) is said to be *admissible*. The calculus of variations is based on the analysis of infinitesimally small variations to an admissible x optimizing the objective function (2.15), and leads to the following *Euler-Lagrange equation* satisfied by all such solutions:

$$\frac{\partial g}{\partial x} = \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right).$$

We illustrate a solution to the optimal harvesting problem, with costs linear in harvest rate, using the calculus of variations approach (as discussed in Clark & Munro, 1975).

Optimal harvesting with linear costs: the calculus of variations approach

Following the approach of Clark and Munro (1975), we consider an open-access fishery exploited by a single individual (firm) whose sole motive is to maximize the long-term profit. The harvester's goal is to obtain the harvest level that maximizes the objective

$$\int_0^\infty e^{-\rho t}\left[p - c(x)\right] h dt.$$

The maximization has to be performed subject to

$$\begin{aligned} \dot{x} &\equiv \frac{dx}{dt} = f(x) - h, \\ x(0) &= x_0, \end{aligned} \tag{2.17}$$

where $x = x(t)$ denotes population, $f(x)$ is the recruitment function (biological growth), $h = h(t)$ is the rate of harvest, p (constant) is the unit price of the harvested resource and $c(x)$ is the unit cost of harvesting.

The maximized objective function can be expressed as

$$J^* = \max_h \int_0^\infty e^{-\rho t} \left[p - c(x)\right] h dt. \tag{2.18}$$

Thus the harvester wishes to control the harvest rate $h(t)$, at each time t, so as to maximize the discounted net profit over an infinite horizon. The profit is discounted with rate ρ for the reason that \$1 today is worth more than \$1 tomorrow, or equivalently, \$1 after t units in the future are worth $(1-\rho)^t$ today (i.e., the *present value* is $(1-\rho)^t$). If the discount is compounded n times per year then t units of time amount to nt discount periods. This gives the present value of \$1 obtained after t units in the future as $\left(1 - \frac{\rho}{n}\right)^{nt}$. By allowing for continuous compounding, the present value of \$1 is obtained as

$$\lim_{n \longrightarrow \infty} \left(1 - \frac{\rho}{n}\right)^{nt} = e^{-\rho t}.$$

Substituting for h from Equation (2.17) into the integral present in Equation (2.18) gives

$$J^* = \max_x \int_0^\infty e^{-\rho t}[p - c(x)][f(x) - \dot{x}]dt.$$

Utilizing the necessary *Euler-Lagrange* condition from the calculus of variations, $\frac{\partial g}{\partial x} = \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right)$, we obtain the following implicit equation for the population x:

$$f\,'(x) - \frac{c'(x)f(x)}{p - c(x)} = \rho, \tag{2.19}$$

where prime denotes differentiation with respect to x.

Equation (2.19) can be rewritten as

$$\frac{\partial}{\partial x}\left[(p - c(x))f(x)\right] = \rho[p - c(x)]. \tag{2.20}$$

Using Equation (2.20), the optimal population level x^*(maximizing the discounted net profit) can in principle be obtained by solving

$$\frac{\partial}{\partial x^*}\left[(p - c(x^*))f(x^*)\right] = \rho[p - c(x^*)]. \tag{2.21}$$

The final solution x^* can be inserted into the growth equation (2.17) to obtain the optimal harvest rate $h^* = f(x^*)$. In general, however, Equation (2.21) might be non-linear in x^* and possess more than one distinct root, in which case x^* may not be uniquely determined.

The case of Logistic growth

Assuming the growth for the resource as $f(x) = rx\left(1 - \frac{x}{K}\right)$, the harvest rate as $h = qEx$, and the cost function as $c(x) = \frac{c}{qx}$, where c is the constant cost per unit effort, the maximized objective (2.18) becomes

$$J^* = \max_h \int_0^\infty e^{-\rho t}\left(p - \frac{c}{qx}\right) h dt = \max_E \int_0^\infty e^{-\rho t}\left(pqx - c\right) E dt, \tag{2.22}$$

subject to

$$\begin{aligned} \dot{x} &= rx\left(1 - \frac{x}{K}\right) - qEx, \\ x(0) &= x_0. \end{aligned} \tag{2.23}$$

Utilizing Equation (2.23), effort E can be determined as

$$E = \frac{rx\left(1 - \frac{x}{K}\right) - \dot{x}}{qx}. \tag{2.24}$$

Substituting E from Equation (2.24) into the integrand present in Equation (2.22) leads to the following maximized objective:

$$J^* = \max_x \int_0^\infty e^{-\rho t}\left(p - \frac{c}{qx}\right)\left[rx\left(1 - \frac{x}{K}\right) - \dot{x}\right] dt.$$

Application of the Euler-Lagrange condition produces a quadratic equation in x that can be solved to give the optimal population level as

$$x^* = \frac{K}{4}\left[\left(1 + \frac{c}{pKq} - \frac{\rho}{r}\right) + \sqrt{\left(1 + \frac{c}{pKq} - \frac{\rho}{r}\right)^2 + \frac{8c\rho}{pKqr}}\right]. \tag{2.25}$$

The optimal harvesting policy is to drive the population level towards x^* as quickly as possible. Once the stock level reaches x^*, the rate of harvest is kept equal to the biological growth rate so that the population stays at x^*; $\dot{x}(t)$ is zero in this case and the optimal effort is obtained from Equation (2.24) as $E^*(t) = \frac{rx^*\left(1-\frac{x^*}{K}\right)}{qx^*}$. Thus, assuming that the effort E is constrained as $0 \leq E \leq E_{\max}$, the optimal harvesting policy in $[0,\ \infty)$ is

$$E^*(t) = \begin{cases} E_{\max}, & x(t) > x^*, \\ \frac{rx^*\left(1-\frac{x^*}{K}\right)}{qx^*}, & x(t) = x^*, \\ 0, & x(t) < x^*. \end{cases}$$

Next consider the finite time-horizon problem

$$J = \max_E \int_0^T e^{-\rho t} \left(pqx - c\right) E dt,$$

subject to

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - qEx,$$
$$x(0) = x_0,\ x(T) = x_T,$$
$$0 \leq E \leq E_{\max},$$

In this case, the population would have to be driven to x^* from some initial value $x_0 \neq x^*$. The path x^*, however, would have to be abandoned before T, say at time s, to meet the specified terminal condition x_T, producing the so-called turnpike behaviour in between s and T (Samuelson, 1965).

2.3 Optimization concepts: Optimal Control

In this section, we present a brief outline of optimal control theory; further details can be found in Kirk (1970) and Bryson & Ho (1975). Consider a dynamical system described as

$$\begin{aligned} \dot{x} &= f(x, u, t), \qquad (2.26) \\ x(0) &= x_0, \end{aligned}$$

with state vector $x \in \mathbb{R}^n$ and control input $u \in \mathbb{R}^m$.

The performance index (objective function) is defined as

$$J = \phi(x(T),T) + \int_0^T g(x,u,t)dt,$$

where $[0,T]$ is the time interval of interest. The optimal control problem is to find the input $u^*(t)$ in $[0,T]$, that drives the system along a trajectory $x^*(t)$, such that the objective function J is optimized and also the p terminal conditions

$$\psi(x(T),T) = 0,$$

are satisfied for a given function $\psi \in \mathbb{R}^p$.

We first construct the *Hamiltonian*

$$\mathcal{H}(x,u,t) = g(x,u,t) + \lambda^{\mathsf{T}} f(x,u,t),$$

where $\lambda \in \mathbb{R}^n$ is called the vector of *Lagrange multipliers* and the superscript T denotes transpose. In summary, the necessary optimality conditions are:

- $\dot{x} = f(x,u,t),\;$ n differential equations,
- $-\dot{\lambda} = \frac{\partial \mathcal{H}}{\partial x},\;$ n (*costate* or *adjoint*) differential equations,
- $\frac{\partial \mathcal{H}}{\partial u} = 0,\;$ m algebraic (*stationarity*) equations,
- $x(0) = x_0,\;$ n boundary conditions,
- $\lambda^{\mathsf{T}}(T) = \left(\frac{\partial \phi}{\partial x(T)} + \mu^{\mathsf{T}} \frac{\partial \psi}{\partial x(T)}\right),\;$ n boundary conditions,
- $\psi(x(T),T) = 0,\;$ p terminal conditions,

where $\mu \in \mathbb{R}^p$ is the constant vector of Lagrange multipliers associated with the p terminal conditions.

The stationarity conditions determine the control m-vector, $u(x,\lambda,t)$. The $2n$ differential equations with the $2n$ boundary conditions form a two-point boundary value problem, where the constant p-vector, μ, is to be found from the p terminal conditions. The stationarity condition that $\text{grad}_u \mathcal{H}$ $\left(\frac{\partial \mathcal{H}}{\partial u}\right) = 0$ is necessary but not sufficient for optimality. Sufficiency is guaranteed if, in addition, the Hessian matrix $\frac{\partial^2 \mathcal{H}}{\partial u^2}$ is positive-definite (negative-definite) for a minimum (maximum).

In the absence of the terminal conditions $\psi(x(T),T) = 0$, there is no μ vector to be determined. Consequently, the boundary conditions for the vector λ become

$$\lambda^{\mathsf{T}}(T) = \frac{\partial \phi}{\partial x(T)}.$$

If $\phi(x(T),T) = 0$ then the final value of the multiplier vector (*transversality condition*) is

$$\lambda^{\mathsf{T}}(T) = \mathbf{0}.$$

The time derivative of the Hamiltonian is

$$\dot{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial t} + (\dot{x})^{\mathsf{T}} \left(\frac{\partial \mathcal{H}}{\partial x}\right) + (\dot{u})^{\mathsf{T}} \left(\frac{\partial \mathcal{H}}{\partial u}\right) + (\dot{\lambda})^{\mathsf{T}} f. \tag{2.27}$$

Substituting for $\dot{x}$ from Equation (2.26) and $\dot{\lambda}$ from adjoint equations $-\dot{\lambda} = \frac{\partial \mathcal{H}}{\partial x}$, Equation (2.27) becomes

$$\dot{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial t} + (\dot{u})^{\mathsf{T}} \left(\frac{\partial \mathcal{H}}{\partial u}\right). \tag{2.28}$$

We let $\mathcal{H}^*$ denote the optimized Hamiltonian. The stationarity equations imply that $\frac{\partial \mathcal{H}^*}{\partial u} = 0$ on the optimal path, therefore Equation (2.28) reduces to

$$\dot{\mathcal{H}}^* = \frac{\partial \mathcal{H}^*}{\partial t}. \tag{2.29}$$

If f and g are not explicit functions of time then substituting Equation (2.26) in Equation (2.29) leads to

$$\dot{\mathcal{H}}^* = 0.$$

Hence for a time-invariant system and objective function the Hamiltonian is constant on the optimal trajectory. If the final time T is also unspecified then on the optimal path we also have

$$\mathcal{H}^* = 0.$$

When $T \rightarrow \infty$, the above problem is called an *infinite horizon optimal control problem.* In such problems, the terminal time is considered free. Consequently, $\mathcal{H}^*(t) = 0$ for all $t \in [0, \infty)$. Usually, infinite horizon problems involve a boundary condition at infinity so that $\lim\limits_{t \longrightarrow \infty} x(t) = x_\infty$ and, consequently, $\lim\limits_{t \longrightarrow \infty} \lambda(t) \neq 0$. Otherwise, if the terminal state is free then $\lim\limits_{t \longrightarrow \infty} \lambda(t) = 0$ (as in the finite horizon case). Another issue is the convergence of the integral $\int_0^\infty g(x,u,t)dt$. In most of the management problems the integrand involves a discount factor, $e^{-\rho t}$, which guarantees the convergence of the integral $\int_0^\infty e^{-\rho t} g(x,u,t)dt$ provided $\sup_u g(x,u,t) = G$ for all t. In that case,

$$\int_0^\infty e^{-\rho t} g(x,u,t)dt \leq \int_0^\infty e^{-\rho t} G dt = \frac{G}{\rho}.$$

The optimal control problem described here does not impose any constraints on the control variable u. If the control is constrained to lie in an admissible region such that $u_{\min} \leq u \leq u_{\max}$ then all of the optimality conditions outlined earlier still hold, the only exception being the stationarity condition $\frac{\partial \mathcal{H}}{\partial u} = 0$, which is replaced by the following more general (necessary, but again not sufficient) condition:

- $\mathcal{H}(x^*, u^*, \lambda^*, t) \leqslant \mathcal{H}(x^*, u, \lambda^*, t)$ for all admissible u, if the objective is to minimize J,
- $\mathcal{H}(x^*, u^*, \lambda^*, t) \geqslant \mathcal{H}(x^*, u, \lambda^*, t)$ for all admissible u, if the objective is to maximize J.

The optimality requirement is called *Pontryagin's minimum (maximum) principle* which can be stated as:

The Hamiltonian must be minimized (maximized) over all admissible u for optimal values of the state and costate.

This principle is broader than the first-order stationarity condition $\frac{\partial \mathcal{H}}{\partial u} = 0$ for it optimizes the Hamiltonian over the entire admissible control region $[u_{\min},\ u_{\max}]$, whereas, $\frac{\partial \mathcal{H}}{\partial u} = 0$ is valid only in the open interval $(u_{\min},\ u_{\max})$ which excludes optimal occurring at either $u_{\min}$ or $u_{\max}$ (corner solutions). If there is a finite time interval $[t_1,\ t_2]$ during which the Pontryagin condition provides no information about the relationship between $u^*(t)$, $x^*(t)$, and $\lambda^*(t)$ then the problem is *singular* and the interval $[t_1,\ t_2]$ is called a singular interval.

There is an economic interpretation associated with the Hamiltonian function (Dorfman, 1969) which also clarifies the reason for adopting the Hamiltonian as the function to be optimized: Consider a maximization problem where $J = \int_0^T g(x, u,t)dt$ represents cumulative profit earned during $[0,\ T]$ with the state dynamics given by Equation (2.26). The product of the associated Hamiltonian with dt is given by

$$\mathcal{H}dt = g(x,u,t)dt + \lambda^{\mathsf{T}} f(x,u,t)dt = g(x,u,t)dt + \lambda^{\mathsf{T}} dx. \tag{2.30}$$

For the time interval $[t, t+dt]$, the first term in Equation (2.30), i.e. $g(x, u,t)dt$, represents the direct profit contribution to J when the firm possesses capital $x(t)$ and the decision is to apply $u(t)$ over $[t, t+dt]$. The second term, $\lambda^{\mathsf{T}} dx$, represents the change in the capital over $[t,\ t+dt]$ which is equivalent to the value of the capital accumulated during the interval. Therefore $\lambda^{\mathsf{T}} dx$ is the indirect contribution to J in dollars. The decision is to choose $u(t)$ so as to make the total contribution to J, i.e. $g(x,u,t)dt + \lambda^{\mathsf{T}} f(x,u,t)dt$, as big as possible; maximizing J only would neglect the effect of the capital accumulation. This implies that the Hamiltonian must be optimized at each instant of time t.

The costate differential equations $-\dot{\lambda} = \frac{\partial \mathcal{H}}{\partial x}$ also admit an economic interpretation: $-\dot{\lambda}(t)$ is the rate at which a unit of capital depreciates over $[t,\ t+dt]$, or equivalently, $-\dot{\lambda}dt$ is the *marginal cost* of holding that capital, whereas $\frac{\partial \mathcal{H}(x(t),u(t),t)}{\partial x(t)}dt = \frac{\partial g(x(t),u(t),t)}{\partial x(t)}dt + \lambda(x(t), u(t), t)^{\mathsf{T}} \frac{\partial f(x(t),u(t),t)}{\partial x(t)}dt$ is the *marginal revenue* gained over $[t,\ t+dt]$. Hence the costate equations imply that marginal cost equals marginal revenue.

Recalling the optimal harvesting problem solved in Section 2.2, the analogy with harvesting is straightforward: x is the fish population (capital) and h is the decision to harvest, with the direct profit contribution embodied in the integrand $e^{-\rho t}(p - c(x))hdt$, and the indirect contribution coming from the change dx in the population level.

In economic applications of optimal control theory the integrand function usually contains a discount factor, $e^{-\rho t}$, and has the form: $e^{-\rho t} g(x, u,t)$. To avoid the complexities inherent in the differentiations involved, a new Hamiltonian is introduced which is free of the discount factor. This Hamiltonian is called the *current value Hamiltonian* (discussed in standard books on optimization, for instance Kamien & Schwartz, 1991), to emphasize its undiscounted nature. Because the Hamiltonian involves Lagrange multipliers, current value

Lagrange multipliers λ_c also have to be introduced as

$$\lambda_c(t) = e^{\rho t}\lambda(t).$$

The current value Hamiltonian takes the form

$$\mathcal{H}_c = \mathcal{H}e^{\rho t} = g(x, u,t) + \lambda_c^{\mathsf{T}} f(x, u,t).$$

The stationarity condition $\frac{\partial \mathcal{H}}{\partial u} = 0$ simply transforms to $\frac{\partial \mathcal{H}_c}{\partial u} = 0$. The costate equation, however, changes to

$$\dot{\lambda}_c = -\frac{\partial \mathcal{H}_c}{\partial x} + \rho\lambda_c.$$

In the absence of terminal boundary conditions on the state x, the transversality condition becomes

$$\lambda_c(T)e^{-\rho T} = 0.$$

which implies

$$\begin{aligned} \lambda_c(T) &= 0 \qquad \text{if } T \text{ is finite,} \\ \lim_{t\to\infty} \lambda_c(t) &= 0 \quad \text{if } T \text{ is infinite.} \end{aligned}$$

For a time-invariant system and objective function, the time-derivative of the current value Hamiltonian is given by

$$\dot{\mathcal{H}}_c = (\dot{x})^{\mathsf{T}} \left(\frac{\partial \mathcal{H}_c}{\partial x}\right) + (\dot{\lambda}_c)^{\mathsf{T}}\dot{x} = \left(\frac{\partial \mathcal{H}_c}{\partial x}\right)^{\mathsf{T}} \dot{x} + \left(-\frac{\partial \mathcal{H}_c}{\partial x} + \rho\lambda_c\right)^{\mathsf{T}} \dot{x} = \rho\lambda_c^{\mathsf{T}}\dot{x}$$

Thus $\mathcal{H}_c$ is not constant over time unless $\rho = 0$.

There is a special class of optimal control problems for which the Hamiltonian is linear in u implying that $\frac{\partial \mathcal{H}}{\partial u}$ is not necessarily 0 on the optimal path. When $\mathcal{H}$ is plotted against u, the plot is a straight line with the optimal control to be found at a boundary of u. When $\mathcal{H}$ versus u yields a horizontal line, there is no unique optimal control since any value for u yields the same value for $\mathcal{H}$; it then becomes a problem with singular control. Thus the form of the optimal control in the time interval $[0,\ T]$, for a minimization problem, is obtained as

$$u^*(t) = \begin{cases} u_{\max} \text{ if } \frac{\partial \mathcal{H}(x(t),u(t),t)}{\partial u(t)} < 0, \\ \text{singular if } \frac{\partial \mathcal{H}(x(t),u(t),t)}{\partial u(t)} = 0, \\ u_{\min} \text{ if } \frac{\partial \mathcal{H}(x(t),u(t),t)}{\partial u(t)} > 0. \end{cases} \tag{2.31}$$

On the other hand, for a maximization problem we get

$$u^*(t) = \begin{cases} u_{\min} \text{ if } \frac{\partial \mathcal{H}(x(t),u(t),t)}{\partial u(t)} < 0, \\ \text{singular if } \frac{\partial \mathcal{H}(x(t),u(t),t)}{\partial u(t)} = 0, \\ u_{\max} \text{ if } \frac{\partial \mathcal{H}(x(t),u(t),t)}{\partial u(t)} > 0. \end{cases} \tag{2.32}$$

The solutions (2.31) and (2.32) are mathematical statements of the *bang-bang principle.* The sign of the function $\frac{\partial \mathcal{H}}{\partial u}$ is instrumental in setting the control switches and is appropriately called the *switching function.*

Optimal harvesting with linear costs: the Hamiltonian approach

Consider again the optimal harvesting problem, with the Schaefer model (Section 1.1) for growth and fishing costs linear in effort, with the objective

$$\max_E \int_0^T e^{-\rho t}(pqx - c)E dt,$$

subject to

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - qEx,$$

$$x(0) = x_0,$$

$$0 \leq E \leq E_{\max}.$$

The associated Hamiltonian is

$$\begin{aligned} \mathcal{H} &= e^{-\rho t}(pqx - c)E + \lambda\left[rx\left(1 - \frac{x}{K}\right) - qEx\right] \\ &= \left[e^{-\rho t}(pqx - c) - \lambda qx\right]E + \lambda rx\left(1 - \frac{x}{K}\right). \end{aligned} \tag{2.33}$$

The *switching function* is the coefficient of the control E in Equation (2.33). Singular control occurs when $\frac{\partial \mathcal{H}}{\partial E}$, i.e. the switching function $e^{-\rho t}(pqx - c) - \lambda qx$, is zero over a time interval. The form of the optimal control in $[0,\ T]$ is then obtained as

$$E^*(t) = \begin{cases} E_{\max}, & e^{-\rho t}(pqx(t) - c) - \lambda qx(t) > 0, \\ \text{singular}, & e^{-\rho t}(pqx(t) - c) - \lambda qx(t) = 0, \\ 0, & e^{-\rho t}(pqx(t) - c) - \lambda qx(t) < 0. \end{cases}$$

It follows that if at a point $x(t)$ the following holds

$$e^{-\rho t}(pqx(t) - c) - \lambda qx(t) = 0$$

then the control is singular. To analyze this situation, suppose that the following holds over a time interval $[t_1,\ t_2]$:

$$e^{-\rho t}(pqx(t) - c) - \lambda(t)qx(t) = 0. \tag{2.34}$$

Equation (2.34) can be solved for $\lambda(t)$ as

$$\lambda(t) = e^{-\rho t}\left(p - \frac{c}{qx(t)}\right).$$

Differentiating $\lambda(t)$ with respect to t and equating $\dot{\lambda}(t)$ with $-\frac{\partial \mathcal{H}}{\partial x}$ leads to

$$r - 2r\frac{x(t)}{K} + \frac{cr\left(1 - \frac{x(t)}{K}\right)}{pqx(t) - c} = \rho. \tag{2.35}$$

The optimal state should satisfy Equation (2.35) when the control is singular. This equation is quadratic in $x(t)$ and can be solved to give the same optimal solution x^* as obtained using the calculus of variations; this x^* serves as a switching point for the control (as seen for the optimal solution obtained using the calculus of variations in Section 2.2).

2.4 Optimization concepts: Dynamic Programming

Dynamic programming presents an alternative approach to optimal control (see Ross, 1983 or Bertsekas, 1987 for a comprehensive description). It is a technique for solving sequential optimization problems by converting a sequential multistage problem to a series of single-stage problems. The term dynamic programming was coined in 1957 by Richard Bellman (Bellman, 1957) who based his approach on the *principle of optimality* which can be stated as:

From any point on the optimal trajectory, the remaining trajectory is optimal for the corresponding problem initiated at that point.

This principle introduces the concept of the optimal value function and allows the build up of solutions by progressing backwards in time.

We first consider deterministic settings. As a general example let us consider the system

$$\begin{aligned}\dot{x} &= f(x,u,t),\\ x(0) &= x_0,\end{aligned}$$

with state $x \in \mathbb{R}^n$ and control vector $u \in \mathbb{R}^m$. With this system, the performance index (or objective function) to be maximized is given by

$$J(x) = \phi(x(T),T) + \int_0^T g(x,u,t)dt.$$

We denote by $V(x(t),t)$ the optimal value of the objective function that could be obtained by starting at state $x(t)$ at time t. The function $V(x(t),t)$ is called the optimal value function. To develop a formula for $V(x(t),t)$ we assume that the function is known for $t+\Delta t$, where Δt is small, and then move backwards to t to obtain

$$V(x(t),t) = \max_{u(t)} \left[g(x,u,t)\Delta t + V(x(t){+}\Delta x(t), t+\Delta t)\right]. \tag{2.36}$$

where $\Delta x(t) = x(t+\Delta t) - x(t) \approx f(x(t),u(t),t)\Delta t$; to derive relation (2.36) it is assumed that a fixed u is applied between times t and $t+\Delta t$. This yields an immediate contribution of approximately $g(x,u,t)\Delta t$ from the integral term in the objective and, additionally, transfers the state $x(t)$ approximately to the point $x(t){+}f(x(t),u(t),t)\Delta t$ at time $t+\Delta t$; the optimal return is known from the point $t+\Delta t$. Assuming that $V(x(t),t)$ is a smooth function, a Taylor expansion of $V(x(t){+}\Delta x(t), t{+}\Delta t)$ around $(x(t),t)$ gives

$$\begin{aligned}V(x(t){+}\Delta x(t), t+\Delta t) &= V(x(t){+}f(x(t),u(t),t)\Delta t, t+\Delta t)\\ &= V(x(t),t) + \frac{\partial V(x(t),t)}{\partial t}\Delta t + \frac{\partial V(x(t),t)}{\partial x(t)} f(x(t),u(t),t)\Delta t + \mathcal{O}(\Delta t^2).\end{aligned}$$

Substituting the expansion for $V(x(t){+}\Delta x(t), t+\Delta t)$ into Equation (2.36) yields

$$\begin{aligned}V(x(t),t) &= \max_{u(t)} \left[g(x(t),u(t),t)\Delta t + V(x(t),t) + \frac{\partial V(x(t),t)}{\partial t}\Delta t\right.\\ &\qquad \left. + \frac{\partial V(x(t),t)}{\partial x(t)} f(x(t),u(t),t)\Delta t + \mathcal{O}(\Delta t^2)\right].\end{aligned}$$

Now $V(x(t),t)$ does not depend on $u(t)$ and can be taken outside the maximization where it then cancels out with the left-hand side. Also, $\frac{\partial V}{\theta t}\Delta t$ can be taken outside the maximization.

Now divide by Δt and let $\Delta t \to 0$. This yields the final result, known as the *Hamilton-Jacobi-Bellman* (HJB) *equation*

$$-\frac{\partial V(x(t),t)}{\partial t} = \max_{u(t)} \left[g(x(t),u(t),t) + \frac{\partial V(x(t),t)}{\partial x(t)} f(x(t),u(t),t) \right]. \tag{2.37}$$

The associated boundary condition is given by

$$V(x(T),T) = \phi(x(T),T),$$

which is the optimal value starting at the terminal time. The derivative $\frac{\partial V(x(t),t)}{\partial x(t)}$ is the per unit change in the objective function for a small change in $x(t)$, so in fact, it is identical to the adjoint variable $\lambda(t)$. We also recall that the Hamiltonian is $\mathcal{H}(x,u,t) = g(x,u,t) + \lambda f(x,t)$ and, therefore, the Hamilton-Jacobi-Bellman equation can be written as

$$-\frac{\partial V(x(t),t)}{\partial t} = \max_{u(t)} \left[\mathcal{H}(x(t),u(t),t) \right].$$

The Hamilton-Jacobi-Bellman equation is a non-linear partial differential equation for the optimal return $V(x(t),t)$ and is often difficult to solve analytically. However, the advantage of the dynamic programming approach is that it automatically determines the optimal control in feedback form. Next we derive the Hamilton-Jacobi-Bellman equation when there is a discount factor, $e^{-\rho t}$, present in the integrand of the objective function as

$$J(x) = \phi(x(T),T) + \int_0^T e^{-\rho t} g(x,u,t) dt.$$

Proceeding in the usual manner, the optimal value is

$$V(x(t),t) = \max_{u(t)} \left[e^{-\rho\Delta t} g(x(t),u(t),t)\Delta t + e^{-\rho\Delta t} V(x(t)+\Delta x(t), t+\Delta t) \right]. \tag{2.38}$$

When Δt is small, $e^{-\rho\Delta t} \approx (1-\rho\Delta t)$. Consequently, for small Δt, Equation (2.38) becomes

$$\begin{aligned} V(x(t),t) &\approx \max_{u(t)} [(1-\rho\Delta t) g(x(t),u(t),t)\Delta t + (1-\rho\Delta t) V(x(t)+\Delta x(t), t+\Delta t)] \\ &= \max_{u(t)} \left[g(x(t),u(t),t)\Delta t + (1-\rho\Delta t) \left\{ V(x(t),t) + \frac{\partial V(x(t),t)}{\partial t}\Delta t \right.\right. \\ &\qquad \left.\left. + \frac{\partial V(x(t),t)}{\partial x(t)} f(x(t),u(t),t)\Delta t + \mathcal{O}(\Delta t^2) \right\} \right]. \end{aligned}$$

Equivalently,

$$\begin{aligned} V(x(t),t) &= \max_{u(t)} [g(x(t),u(t),t)\Delta t + (1-\rho\Delta t)V(x(t),t) \\ &\qquad + \frac{\partial V(x(t),t)}{\partial t}\Delta t + \frac{\partial V(x(t),t)}{\partial x(t)} f(x(t),u(t),t)\Delta t + \mathcal{O}(\Delta t^2)\Big] . \end{aligned}$$

Dividing throughout by Δt and taking the limit as $\Delta t \longrightarrow 0$ gives the Hamilton-Jacobi-Bellman equation

$$-\frac{\partial V(x(t),t)}{\partial t} + \rho V(x(t),t) = \max_{u(t)} \left[g(x(t),u(t),t) + \frac{\partial V(x(t),t)}{\partial x(t)} f(x(t),u(t),t) \right], \qquad (2.39)$$

with the final condition $V(x(T),T) = \phi(x(T),T)$. Note that Equation (2.39) with $\rho \to 0$ gives Equation (2.37).

Stochastic dynamic programming

While obtaining Equations (2.37) and (2.39) the state variables were assumed to be known with certainty. If this were not the case, the state of the system would be a stochastic process and its evolution would be described by a stochastic differential equation. When a control is involved in the dynamic process, the problem becomes a stochastic optimal control problem. We now consider the optimality of the control when the state is disturbed by a random process and the system state becomes a Markov process. For a stochastic optimal control problem with n states, the following performance index is introduced:

$$J(x) = \mathcal{E}\left[\phi(x(T),T) + \int_0^T e^{-\rho t} g(x,u,t)dt | x(0) = x_0 \right],$$

subject to

$$\begin{aligned} dx &= f(x,u,t)dt + \sigma(x,u,t)dw, \qquad (2.40) \\ x(0) &= x_0, \end{aligned}$$

where dw is the n-vector of Wiener increments.

Proceeding as in the deterministic case leads to

$$V(x(t),t) = \max_{u(t)} \left[e^{-\rho\Delta t} g(x(t),u(t),t)\Delta t + e^{-\rho \Delta t}\mathcal{E}\left[V(x(t)+\Delta x(t), t+\Delta t)\right]\right]. \qquad (2.41)$$

Again for small Δt Equation (2.41) can be written as

$$\begin{aligned}
V(x(t),t) &\approx \max_{u(t)}[(1-\rho\Delta t)g(x(t),u(t),t)\Delta t + (1-\rho\Delta t)\mathcal{E}\left[V(x(t)+\Delta x(t),t+\Delta t)\right]] \\
&= \max_{u(t)}\left[g(x(t),u(t),t)\Delta t + (1-\rho\Delta t)\mathcal{E}\left[V(x(t),t) + \frac{\partial V(x(t),t)}{\partial t}\Delta t\right.\right. \\
&\qquad \left.\left. + \frac{\partial V(x(t),t)}{\partial x(t)}\Delta x(t) + \frac{1}{2}\frac{\partial^2 V(x(t),t)}{\partial x^2(t)}\Delta x^2(t) + \mathcal{O}(\Delta t^2)\right]\right] \\
&= \max_{u(t)}\left[g(x(t),u(t),t)\Delta t + (1-\rho\Delta t)V(x(t),t) + \frac{\partial V(x(t),t)}{\partial t}\Delta t\right. \\
&\qquad \left. + \frac{\partial V(x(t),t)}{\partial x(t)}\mathcal{E}[\Delta x(t)] + \frac{1}{2}\frac{\partial^2 V(x(t),t)}{\partial x^2(t)}\mathcal{E}[\Delta x^2(t)] + \mathcal{O}(\Delta t^2)\right]. \qquad (2.42)
\end{aligned}$$

From Equation (2.40) we have

$$\begin{aligned}
\Delta x^2(t) &= (f(x(t),u(t),t)\Delta t + \sigma(x(t),u(t),t)\Delta w(t))^2 \\
&= f(x(t),u(t),t)^2\Delta t^2 + \sigma(x(t),u(t),t)^2\Delta w(t)^2 \\
&\quad +2f(x(t),u(t),t)\sigma(x(t),u(t),t)\Delta t\Delta w(t).
\end{aligned}$$

Taking the expectation of both sides yields

$$\begin{aligned}
\mathcal{E}[\Delta x^2(t)] &= \mathcal{E}\left[f(x(t),u(t),t)^2\Delta t^2 + \sigma(x(t),u(t),t)^2\Delta w(t)^2\right. \\
&\quad \left. +2f(x(t),u(t),t)\sigma(x(t),u(t),t)\Delta t\Delta w(t)\right] \\
&= f(x(t),u(t),t)^2\Delta t^2 + \sigma(x(t),u(t),t)^2\mathcal{E}\left[\Delta w(t)^2\right] \\
&\quad +2f(x(t),u(t),t)\sigma(x(t),u(t),t)\Delta t\mathcal{E}\left[\Delta w(t)\right]. \qquad (2.43)
\end{aligned}$$

By definition (see Section 2.1) a Wiener increment is normally distributed with the expected value equal to zero and the variance equal to time-lag. Applying this to Equation (2.43) and ignoring *higher order terms* in Δt we obtain

$$\mathcal{E}[\Delta x^2(t)] = \sigma(x(t),u(t),t)^2\Delta t \qquad (2.44)$$

Substituting for $\mathcal{E}[\Delta x^2(t)]$ from Equation (2.44) into Equation (2.42) gives

$$\begin{aligned}
V(x(t),t) &= \max_{u(t)}\left[g(x(t),u(t),t)\Delta t + (1-\rho\Delta t)V(x(t),t) + \frac{\partial V(x(t),t)}{\partial t}\Delta t\right. \\
&\quad \left. + \frac{\partial V(x(t),t)}{\partial x(t)}f(x(t),u(t),t)\Delta t + \frac{1}{2}\frac{\partial^2 V(x(t),t)}{\partial x^2(t)}\sigma(x(t),u(t),t)^2\Delta t + \mathcal{O}(\Delta t^2)\right].
\end{aligned}$$

Proceeding as in the deterministic case, the Hamilton-Jacobi-Bellman equation for the stochastic optimal control problem is thus a deterministic partial differential equation:

$$\begin{aligned}\rho V(x(t),t) &= \max_{u(t)}\left[g(x(t),u(t),t)+\frac{\partial V(x(t),t)}{\partial t}+\frac{\partial V(x(t),t)}{\partial x(t)}f(x(t),u(t),t)\right.\\ &\left.\quad+\frac{1}{2}\frac{\partial^2 V(x(t),t)}{\partial x^2(t)}\sigma^2\right],\end{aligned} \tag{2.45}$$

with the final condition $V(x(T),T)=\phi(x(T),T)$.

Optimal harvesting with linear costs: the stochastic dynamic programming approach

We now examine an optimal harvesting problem with exponential biological growth, i.e. rx, and linear costs. Considering the growth to be stochastic, the growth dynamics can be specified as

$$\begin{aligned}dx &= (r-qE)xdt+\sigma x dw,\\ x(0) &= x_0.\end{aligned}$$

The effort is constrained as

$$0\le E\le E_{\max}.$$

The objective is to maximize the expected value of the discounted profit over an infinite horizon, given by

$$J(x)=\mathcal{E}\left[\max_E\int_0^\infty e^{-\rho t}\left(pqx-c\right)Edt\right],$$

where the discount rate ρ is a positive constant.

In finite-horizon problems, the value function $V(x(t),t)$ is a function of time and therefore the time derivative $\frac{\partial V(x(t),t)}{\partial t}$ appears in the Hamilton-Jacobi-Bellman equation. In infinite-horizon problems with time-invariant state equations and integrand, the value function is time-invariant and depends only on the initial state, so that, $V=V(x(t))$ and $\frac{\partial V(x(t))}{\partial t}=0$. Thus, utilizing Equation (2.45), the Hamilton-Jacobi-Bellman equation for this problem is

$$\begin{aligned}\rho V(x(t)) &= \max_{E(t)}\left[(pqx(t)-c)E(t)+\frac{\partial V(x(t))}{\partial x(t)}(r-qE(t))x(t)\right.\\ &\left.\quad+\frac{1}{2}\sigma^2 x(t)^2\frac{\partial^2 V(x(t))}{\partial x(t)^2}\right].\end{aligned} \tag{2.46}$$

Since the control $E(t)$ appears linearly in the objective function and the state-dynamics, the choice of optimal control is bang-bang with the switching function

$$pqx(t) - c - qx(t)\frac{\partial V(x(t))}{\partial x(t)}.$$

Solving the equation $pqx(t) - c - qx(t)\frac{\partial V(x(t))}{\partial x(t)} = 0$ provides the boundary between the two extreme controls: 0 and $E_{\max}$. This boundary is the critical population level, x^*.

Thus,

$$E^*(t) = \begin{cases} 0, & 0 \le x(t) < x^*, \\ E_{\max}, & x(t) \ge x^*. \end{cases} \tag{2.47}$$

Using the optimal control $E^*(t)$, Equation (2.46) can be rewritten as

$$\begin{aligned} \rho V(x(t)) &= (pqx(t) - c)E^*(t) + \frac{\partial V(x(t))}{\partial x(t)}(r - qE^*(t))x(t) \\ &\quad + \frac{1}{2}\sigma^2 x(t)^2 \frac{\partial^2 V(x(t))}{\partial x(t)^2}. \end{aligned} \tag{2.48}$$

To calculate x^*, we solve the following two equations arising from the application of extreme controls (0 and $E_{\max}$) to Equation (2.48):

$$\rho V(x(t)) = rx\frac{\partial V(x(t))}{\partial x(t)} + \frac{1}{2}\sigma^2 x(t)^2 \frac{\partial^2 V(x(t))}{\partial x(t)^2}, \quad \text{for } 0 \le x(t) < x^*, \tag{2.49}$$

$$\begin{aligned} \rho V(x(t)) &= (pqx(t) - c)E_{\max} + \frac{\partial V(x(t))}{\partial x(t)}(r - qE_{\max})x(t) \\ &\quad + \frac{1}{2}\sigma^2 x(t)^2 \frac{\partial^2 V(x(t))}{\partial x(t)^2}, \qquad \text{for } x(t) \ge x^*. \end{aligned} \tag{2.50}$$

Equation (2.49) is a *homogeneous Euler equation* and Equation (2.50) is a *nonhomogeneous Euler equation.* The analytical solutions to these type of equations are described in standard books on ordinary differential equations, e.g. Boyce & DiPrima (2003). The solution to Equation (2.49) is of the form

$$V(x(t)) = \alpha_1 x(t)^{\beta_1} + \alpha_2 x(t)^{\beta_2}, \tag{2.51}$$

where α_1, α_2 are constants and β_1 and β_2 are as follows:

$$\begin{aligned}\beta_1 &= \frac{(\sigma^2 - 2r) + \sqrt{(\sigma^2 - 2r)^2 + 8\rho\sigma^2}}{2\sigma^2} > 0, \\ \beta_2 &= \frac{(\sigma^2 - 2r) - \sqrt{(\sigma^2 - 2r)^2 + 8\rho\sigma^2}}{2\sigma^2} < 0,\end{aligned}$$

Since $V(0) = 0$ and $\beta_2 < 0$, Equation (2.51) can hold only if $\alpha_2 = 0$. The remaining task now is to determine α_1. At $x(t) = x^*$, $V(x(t))$ must satisfy the switching function

$$pqx(t) - c - qx(t)\frac{\partial V(x(t))}{\partial x(t)} = 0. \tag{2.52}$$

Substituting for $V(x(t))$ from Equation (2.51) and replacing $x(t)$ with x^*, Equation (2.52) becomes

$$pqx^* - c - qx^*\alpha_1\beta_1(x^*)^{\beta_1 - 1} = 0.$$

This yields α_1 as

$$\alpha_1 = \frac{(x^*)^{1-\beta_1}}{\beta_1}\left(p - \frac{c}{qx^*}\right),$$

so that

$$\begin{aligned}V(x(t)) &= \frac{1}{\beta_1}\left(px^* - \frac{c}{q}\right)\left(\frac{x(t)}{x^*}\right)^{\beta_1}, \\ \text{where } \beta_1 &= \frac{(\sigma^2 - 2r) + \sqrt{(\sigma^2 - 2r)^2 + 8\rho\sigma^2}}{2\sigma^2}, \qquad 0 \le x(t) < x^*.\end{aligned}$$

Now we concentrate on Equation (2.50). As noted earlier, it is a *nonhomogeneous Euler equation* and its solution consists of the superposition of its *particular solution* and the *complementary solution* corresponding to homogeneous equation

$$\frac{1}{2}\sigma^2 x(t)^2\frac{\partial^2 V(x(t))}{\partial x(t)^2} + \frac{\partial V(x(t))}{\partial x(t)}(r - qE_{\max})x(t) - \rho V = 0.$$

Similar to the solution of Equation (2.49), the complementary solution has the form

$$V_c(x) = \alpha_1 x^{\beta_1} + \alpha_2 x^{\beta_2},$$

where

$$\begin{aligned}\beta_1 &= \frac{[\sigma^2 - 2(r - qE_{\max})] + \sqrt{[\sigma^2 - 2(r - qE_{\max})]^2 + 8\rho\sigma^2}}{2\sigma^2} > 0, \\ \beta_2 &= \frac{[\sigma^2 - 2(r - qE_{\max})] - \sqrt{[\sigma^2 - 2(r - qE_{\max})]^2 + 8\rho\sigma^2}}{2\sigma^2} < 0,\end{aligned} \tag{2.53}$$

and α_1, α_2 are constants.

Using the standard *variation of parameters* method, the particular solution has the form

$$V_p(x(t)) = \gamma_1(x(t))x(t)^{\beta_1} + \gamma_2(x(t))x(t)^{\beta_2},$$

where $\gamma_1(x)$ and $\gamma_2(x)$ are functions whose first order derivatives can be determined from the following system:

$$\begin{aligned}
\gamma_1'(x(t))x(t)^{\beta_1} + \gamma_2'(x(t))x(t)^{\beta_2} &= 0,\\
\gamma_1'(x(t))\beta_1 x(t)^{\beta_1-1} + \gamma_2'(x(t))\beta_2 x(t)^{\beta_2-1} &= \frac{-2(pqx(t)-c)E_{\max}}{\sigma^2 x(t)^2}.
\end{aligned}$$

Solving this system gives

$$\gamma_1'(x(t)) = \frac{2(pqx(t)-c)E_{\max}}{\sigma^2(\beta_2-\beta_1)x(t)^{\beta_1+1}}, \qquad \gamma_2'(x(t)) = \frac{-2(pqx(t)-c)E_{\max}}{\sigma^2(\beta_2-\beta_1)x(t)^{\beta_2+1}}.$$

Integrating both equations leads to

$$\begin{aligned}
\gamma_1(x(t)) &= \frac{2E_{\max}x(t)^{-\beta_1}}{\sigma^2(\beta_2-\beta_1)}\left(\frac{pqx(t)}{1-\beta_1} + \frac{c}{\beta_1}\right),\\
\gamma_2(x(t)) &= \frac{2E_{\max}x(t)^{-\beta_2}}{\sigma^2(\beta_1-\beta_2)}\left(\frac{pqx(t)}{1-\beta_2} + \frac{c}{\beta_2}\right).
\end{aligned}$$

The particular solution is then

$$V_p(x(t)) = \frac{2E_{\max}}{\sigma^2(\beta_2-\beta_1)}\left(\frac{pqx(t)}{1-\beta_1} + \frac{c}{\beta_1}\right) + \frac{2E_{\max}}{\sigma^2(\beta_1-\beta_2)}\left(\frac{pqx(t)}{1-\beta_2} + \frac{c}{\beta_2}\right).$$

The general solution to Equation (2.50) is given by

$$\begin{aligned}
V(x(t)) &= V_c(x(t)) + V_p(x(t))\\
&= \alpha_1 x(t)^{\beta_1} + \alpha_2 x(t)^{\beta_2} + \frac{2E_{\max}}{\sigma^2(\beta_2-\beta_1)}\left(\frac{pqx(t)}{1-\beta_1} + \frac{c}{\beta_1}\right)\\
&\quad + \frac{2E_{\max}}{\sigma^2(\beta_1-\beta_2)}\left(\frac{pqx(t)}{1-\beta_2} + \frac{c}{\beta_2}\right).
\end{aligned} \tag{2.54}$$

From Equation (2.53), for β_1 to be greater than 1 we need

$$\begin{aligned}
\frac{[\sigma^2 - 2(r-qE_{\max})] + \sqrt{[\sigma^2 - 2(r-qE_{\max})]^2 + 8\rho\sigma^2}}{2\sigma^2} &> 1\\
\Leftrightarrow \sigma^2 - 2(r-qE_{\max}) + \sqrt{[\sigma^2 - 2(r-qE_{\max})]^2 + 8\rho\sigma^2} &> 2\sigma^2
\end{aligned}$$

$$\Leftrightarrow \sqrt{[\sigma^2 - 2(r-qE_{\max})]^2 + 8\rho\sigma^2} > \sigma^2 + 2(r-qE_{\max}).$$

Simplifying we get that the following inequality should hold in order to have $\beta_1 > 1$:

$$\rho > r - qE_{\max}. \tag{2.55}$$

If we fix $E_{\max} = r/q$, then the inequality (2.55) is equivalent to

$$\rho > 0,$$

which is true. Furthermore, the integral $\mathcal{E}\left[\max\limits_{E(t)} \int\limits_0^\infty e^{-\rho t}\left(pqx - c\right) E dt\right]$ is bounded from above by a function linear in x (Kolosov, 1999, page 134). Since $\beta_1 > 1$, the upper bound is violated unless $\alpha_1 = 0$. Therefore the value function for $x(t) \geq x^*$, given by Equation (2.54), takes the form

$$\begin{aligned} V(x(t)) &= \alpha_2 x(t)^{\beta_2} + \frac{2E_{\max}}{\sigma^2(\beta_2 - \beta_1)}\left(\frac{pqx(t)}{1-\beta_1} + \frac{c}{\beta_1}\right) \\ &\quad + \frac{2E_{\max}}{\sigma^2(\beta_1 - \beta_2)}\left(\frac{pqx(t)}{1-\beta_2} + \frac{c}{\beta_2}\right). \end{aligned} \tag{2.56}$$

The only unknown parameter is α_2 and it is determined by enforcing the switching condition

$$pqx(t) - c - qx(t)\frac{\partial V(x(t))}{\partial x(t)} = 0 \quad \text{at } x(t) = x^*.$$

Substituting for $V(x(t))$ from Equation (2.56) and putting $x(t) = x^*$ gives

$$\alpha_2 = \frac{(x^*)^{1-\beta_2}}{\beta_2}\left(p + \frac{2pqE_{\max}}{\sigma^2(1-\beta_1)(1-\beta_2)}\right) - \frac{c}{q\beta_2}(x^*)^{-\beta_2}.$$

Continuity at $x(t) = x^*$ implies that the two value functions are equal there. Equating both second order derivatives $\frac{\partial^2 V(x(t))}{\partial x(t)^2}$ at $x(t) = x^*$ yields

$$x^* = \frac{c\left(\frac{k_1+k_2-2qE_{\max}}{2\sigma^2}\right)}{pq\left(\frac{k_1+k_2-2qE_{\max}}{2\sigma^2} + \frac{4qE_{\max}}{\sigma^2+2(r-qE_{\max})-k_2}\right)},$$

where

$$\begin{aligned} k_1 &= \sqrt{(\sigma^2 - 2r)^2 + 8\rho\sigma^2}, \\ k_2 &= \sqrt{[\sigma^2 - 2(r - qE_{\max})]^2 + 8\rho\sigma^2}. \end{aligned}$$

Substituting $E_{\max} = r/q$, the expression for x^* finally simplifies to

$$x^* = \frac{c\left(\frac{k_1+k_2-2r}{2\sigma^2}\right)}{pq\left(\frac{k_1+k_2-2r}{2\sigma^2} + \frac{4r}{\sigma^2-k_2}\right)},$$

where

$$k_1 = \sqrt{(\sigma^2 - 2r)^2 + 8\rho\sigma^2}, \quad (2.57)$$

$$k_2 = \sqrt{\sigma^4 + 8\rho\sigma^2}. \quad (2.58)$$

Thus we have obtained the critical population level x^* which serves as the switching point for the optimal effort E^*. Now x^* can be rewritten as

$$x^* = \frac{c}{pq}\left(\frac{1}{1 - \frac{8\sigma^2 r}{(k_2-\sigma^2)(k_1+k_2-2r)}}\right). \quad (2.59)$$

Recall from Section 1.1 that when $x^* > c/pq$, *Sustainable Economic Rent* is positive. From Equation (2.59),

$$x^* > \frac{c}{pq} \quad \text{if} \quad \frac{8\sigma^2 r}{(k_2 - \sigma^2)(k_1 + k_2 - 2r)} > 0.$$

From Equation (2.58), $k_2 > \sigma^2$ is true. Therefore, in order to have a positive value for *Sustainable Economic Rent* we must have

$$k_1 + k_2 > 2r. \quad (2.60)$$

Thus if the parameter values $(r, \rho$ and $\sigma)$ are such that the inequality (2.60) is satisfied, *Sustainable Economic Rent* would be positive.

Chapter 3

Deterministic Environment

3.1 Introduction

In this chapter we present the optimal harvesting problem in a deterministic setting, based on the Schaefer model (discussed in Section 1.1) and quadratic costs. The layout of the chapter is as follows: Section 3.2 describes the formulation of our model which is then investigated using three different optimization techniques. Section 3.3 illustrates the dynamic programming method, Section 3.4 demonstrates the Hamiltonian method and Section 3.5 deals with a variational approach. Due to the complexity and non-linearity of the problem, numerical methods have to be applied to solve the system of differential equations obtained in each case. Section 3.6 presents a sensitivity analysis of the net present value of the flow of profit and Section 3.7 covers infinite horizon harvesting. Section 3.8 provides a summary.

3.2 Model formulation

Following the deterministic Schaefer model (see Section 1.1), the growth dynamics of the resource population can be specified as

$$\frac{dx(\tau)}{d\tau} = rx(\tau)\left(1 - \frac{x(\tau)}{K}\right) - qE(\tau)x(\tau), \tag{3.1}$$

where τ denotes time and the rest of the notation holds as before.

The fishing effort is constrained as

$$0 \leq E(\tau) \leq E_{\max} < \infty \text{ for all } \tau, \tag{3.2}$$

where $E_{\max}$ is a fixed constant. It is usual practice in the fisheries literature to fix r/q as an upper bound for fishing effort (Clark, 1990; Hanson & Ryan, 1998), the reason being that the growth equation (3.1) yields a negative equilibrium for the population when effort is expended at a level greater than r/q (see Section 1.1), and this amounts to the extinction of the resource stock.

Apart from the biological reason mentioned above, $E_{\max}$ is also required for representing an upper bound on the harvesting capacity of the harvester. The crew, gear and number of vessels that the harvester possesses can influence the total effort that the harvester is able to expend; in this case, it is possible for the harvester to have $E_{\max}$ different from r/q. An *ad hoc* value for $E_{\max}$, depending upon the maximum capacity of the fishing fleet and the amount of capital invested, has been considered by Nøstbakken (2006) and Clark *et al.* (1979).

We consider the fish price per unit harvest to be a constant, denoted by p. The cost of harvest is assumed to be of quadratic form, given by $c_1 E(\tau) + \frac{c_2}{2} E(\tau)^2$, where c_1 and c_2 are positive constants. This implies that the fishing cost does not increase linearly with effort and the marginal increase is of the form $c_1 + c_2 E(\tau)$. As noted by Hanson (2006), it can be due to the use of unspecialized gear and boats for additional effort when the best ones have already been employed. Sancho & Mitchell (1975) and Holt *et al.* (1960) also point out that a quadratic cost function appears to be more realistic. Furthermore, we suppose that harvesting starts at an initial time 0 and continues up to a (finite) terminal time T.

The net profit earned by the harvester at time τ is denoted by $\Pi(\tau)$, and is calculated as: unit fish price $\times$ harvest $-$ costs, so that,

$$\Pi(\tau) = p(qE(\tau)x(\tau)) - c_1 E(\tau) - \frac{c_2}{2} E(\tau)^2,$$

or,

$$\Pi(\tau) = \left(pqx(\tau) - c_1 - \frac{c_2}{2} E(\tau) \right) E(\tau). \tag{3.3}$$

The future profits are discounted at rate δ. Therefore the net present value of the total flow of profit earned between the initial time 0 and the terminal time T is given by

$$PV = \int_0^T e^{-\delta\tau}\Pi(\tau)d\tau. \tag{3.4}$$

Substituting for $\Pi(\tau)$ from Equation (3.3), we can rewrite the net present value defined in Equation (3.4) as

$$PV = \int_0^T e^{-\delta\tau}\left(pqx(\tau) - c_1 - \frac{c_2}{2}E(\tau)\right)E(\tau)d\tau.$$

We assume that the optimal harvesting strategy maximizes the net present value of total flow of profit, i.e. the total discounted profit, and define $J^*(x(t), t)$ as the optimal (maximized) value of the total discounted profit obtained by initiating the harvest at time t and state $x(t)$, where $0 \le t \le T$. Then we can write

$$J^*(x(t), t) = \max_{\substack{E(\tau)\\ t\le\tau\le T}} \left[\int_t^T e^{-\delta(\tau-t)}\left(pqx(\tau) - c_1 - \frac{c_2}{2}E(\tau)\right)E(\tau)d\tau\right],$$

so that, in particular,

$$J^*(x(T), T) = 0. \tag{3.5}$$

The problem of deriving an optimal harvesting policy, starting from the initial time $t = 0$, can now be formulated as an optimal control problem. The control variable is $E(\tau)$ and the payoff (value) function is given by

$$J^*(x(0), 0) = \max_{\substack{E(\tau)\\ 0\le\tau\le T}} \left[\int_0^T e^{-\delta\tau}\left(pqx(\tau) - c_1 - \frac{c_2}{2}E(\tau)\right)E(\tau)d\tau\right],$$

subject to the initial condition

$$x(0) = x_0,$$

and the constraints on effort

$$0 \le E(\tau) \le E_{\max} < \infty \text{ for all } \tau.$$

Furthermore, we wish to maintain the population density above a minimum viable level throughout the harvest. So we introduce a lower bound on the population level, denoted by $x_{\min}$.

In the sections that follow we analyze the deterministic model using three different methods: Dynamic Programming, Hamiltonian method, and a Variational method.

3.3 Dynamic Programming technique

We consider the constrained optimization problem, where the effort is bounded, and use the dynamic programming technique (discussed in Section 2.4) to derive the Hamilton-Jacobi-Bellman equation for the total discounted profit.

Derivation of the Hamilton-Jacobi-Bellman equation

We start with the current-valued payoff function at time t as follows:

$$J^*(x(t),t) = \max_{\substack{E(\tau)\\ t\le\tau\le T}} \int_t^T e^{-\delta(\tau-t)} \left(pqx(\tau) - c_1 - \frac{c_2}{2}E(\tau)\right) E(\tau)d\tau, \tag{3.6}$$

subject to the growth equation

$$\frac{dx(\tau)}{d\tau} = rx(\tau)\left(1 - \frac{x(\tau)}{K}\right) - qE(\tau)x(\tau). \tag{3.7}$$

Breaking down the integral in Equation (3.6) we get

$$\begin{aligned} J^*(x(t),t) = \max_{\substack{E(\tau)\\ t\le\tau\le T}} \Bigg[& \int_t^{t+dt} e^{-\delta(\tau-t)} \left(pqx(\tau) - c_1 - \frac{c_2}{2}E(\tau)\right) E(\tau)d\tau \\ & + \int_{t+dt}^T e^{-\delta[(\tau-t)+dt-dt]} \left(pqx(\tau) - c_1 - \frac{c_2}{2}E(\tau)\right) E(\tau)d\tau \Bigg], \end{aligned} \tag{3.8}$$

where dt is very small. Using the approximation $e^{-\delta dt} \approx (1-\delta dt)$, valid for small dt, in Equation (3.8) we obtain

$$\begin{aligned} J^*(x(t),t) = \max_{\substack{E(\tau)\\ t\le\tau\le T}} \Bigg[& \int_t^{t+dt} e^{-\delta(\tau-t)} \left(pqx(\tau) - c_1 - \frac{c_2}{2}E(\tau)\right) E(\tau)d\tau \\ & +(1-\delta dt) \int_{t+dt}^T e^{-\delta(\tau-t-dt)} \left(pqx(\tau) - c_1 - \frac{c_2}{2}E(\tau)\right) E(\tau)d\tau \Bigg]. \end{aligned} \tag{3.9}$$

Applying the principle of optimality (Bellman, 1957), Equation (3.9) can be rewritten as

$$
\begin{aligned}
J^*(x(t),t) = \max_{\substack{E(\tau)\\ t\le\tau\le t+dt}} \Bigg[& \int_t^{t+dt} e^{-\delta(\tau-t)} \left(pqx(\tau) - c_1 - \frac{c_2}{2}E(\tau)\right) E(\tau)d\tau \\
& +(1-\delta dt) \max_{\substack{E(\tau)\\ t+dt\le\tau\le T}} \int_{t+dt}^{T} e^{-\delta(\tau-(t+dt))} \left(pqx(\tau) - c_1 - \frac{c_2}{2}E(\tau)\right) E(\tau)d\tau \Bigg],
\end{aligned}
$$

which is equivalent to

$$
J^*(x(t),t) = \max_{E(t)} \left[\left(pqx(t) - c_1 - \frac{c_2}{2}E(t)\right) E(t)dt + (1-\delta dt) J^*(x(t+dt), t+dt) \right]. \quad (3.10)
$$

Now $x(t+dt)$ represents the population density at time $t+dt$, therefore it is equal to the sum of the population level, $x(t)$, at time t and the increment in population density, $dx(t)$, in time dt. In other words,

$$
x(t+dt) = x(t) + dx(t).
$$

This gives

$$
J^*(x(t+dt), t+dt) = J^*(x(t)+dx(t), t+dt).
$$

Expanding $J^*(x(t)+dx(t), t+dt)$ around $(x(t),t)$ yields

$$
\begin{aligned}
J^*(x(t)+dx(t), t+dt) \quad = \quad & J^*(x(t),t) + \frac{\partial J^*(x(t),t)}{\partial x(t)} dx(t) + \frac{\partial J^*(x(t),t)}{\partial t} dt \\
& + \textit{higher order terms in } dt. \qquad (3.11)
\end{aligned}
$$

Substituting $J^*(x(t)+dx(t), t+dt)$ from Equation (3.11) into Equation (3.10) and ignoring the *higher order terms in* dt we get

$$
\begin{aligned}
0 \quad = \quad \max_{E(t)} \Bigg[& \left(pqx(t) - c_1 - \frac{c_2}{2}E(t)\right) E(t) - \delta J^*(x(t),t) \\
& + \frac{\partial J^*(x(t),t)}{\partial x(t)} \frac{dx(t)}{dt} + \frac{\partial J^*(x(t),t)}{\partial t} \Bigg], \qquad (3.12)
\end{aligned}
$$

where we have divided throughout by dt.

Substituting $\frac{dx(t)}{dt}$ from Equation (3.7) into Equation (3.12) leads to

$$-\frac{\partial J^*(x(t),t)}{\partial t} = \max_{E(t)} \Big[\Big(pqx(t) - c_1 - \frac{c_2}{2}E(t) \Big) E(t) - \delta J^*(x(t),t) + \frac{\partial J^*(x(t),t)}{\partial x(t)} \Big\{ rx(t) \Big(1 - \frac{x(t)}{K} \Big) - qE(t)x(t) \Big\} \Big]. \tag{3.13}$$

Equation (3.13) is the required Hamilton-Jacobi-Bellman equation for the total discounted profit. Since dynamic programming uses backward induction, time t in Equation (3.13) is moving backwards from the terminal time T to the initial time 0 (standard approach in dynamic programming).

An analytic expression for the optimal effort

We let G denote the control-switching term for the maximization being carried out in Equation (3.13). Then G consists of all the terms present in Equation (3.13) which contain $E(t)$, i.e.

$$G = \Big(pqx(t) - c_1 - \frac{c_2}{2}E(t) \Big) E(t) + \frac{\partial J^*(x(t),t)}{\partial x(t)} \left(-qE(t)x(t) \right).$$

If $E^*_{uc}(t)$ denotes the solution with regard to the unconstrained maximum then $E^*_{uc}(t)$ is given by the solution to $\frac{dG}{dE} = 0$, which implies

$$pqx(t) - c_1 - c_2 E^*_{uc}(t) - \frac{\partial J^*(x(t),t)}{\partial x(t)} qx(t) = 0.$$

Rearranging terms we obtain

$$E^*_{uc}(t) = \left(p - \frac{\partial J^*(x(t),t)}{\partial x(t)} \right) \frac{q}{c_2} x(t) - \frac{c_1}{c_2}. \tag{3.14}$$

Using the constrained optimal effort, denoted by $E^*(t)$, in place of $E(t)$ we can rewrite the Hamilton-Jacobi-Bellman equation (3.13) as

$$-\frac{\partial J^*(x(t),t)}{\partial t} = \Big(pqx(t) - c_1 - \frac{c_2}{2}E^*(t) \Big) E^*(t) - \delta J^*(x(t),t) + \frac{\partial J^*}{\partial x} \Big\{ rx(t) \Big(1 - \frac{x(t)}{K} \Big) - qE^*(t)x(t) \Big\}, \tag{3.15}$$

where

$$E^*(t) = \begin{cases} 0, & E^*_{uc}(t) < 0, \\ \left(p - \frac{\partial J^*}{\partial x}\right) \frac{qx(t)}{c_2} - \frac{c_1}{c_2}, & 0 \le E^*_{uc}(t) \le E_{\max}, \\ E_{\max}, & E^*_{uc}(t) > E_{\max}. \end{cases} \tag{3.16}$$

The boundary conditions associated with the problem can be specified as follows:

- At the final time T, $J^* = 0$ by definition (see Equation (3.5)). Therefore the boundary condition associated with the temporal variable is

$$J^*(x, T) = 0. \tag{3.17}$$

- There is no harvesting when the population level is at $x_{\min}$, therefore the spatial boundary condition is

$$E^*(x_{\min}, t) = 0. \tag{3.18}$$

In order to obtain the optimal harvesting policy, Equations (3.15)-(3.18) have to be solved subject to the given initial stock level x_0, the fixed minimum viable population level $x_{\min}$ and the growth dynamics given by Equation (3.7). Due to the complicated nature of the partial differential equation (3.15), it is not possible to solve it analytically. Therefore we apply a (Crank-Nicolson) finite-difference method to work out a numerical solution; the procedure is described in Appendix A. The numerical scheme, to find the maximized total discounted profit and the optimal effort, is coded in MATLAB.

Numerical solution and discussion

In the optimal harvesting problem under consideration we have imposed constraints on the control variable $E(t)$, and therefore, in essence, we are solving a constrained optimization problem. Furthermore, Equation (3.16) indicates that the biological and economic parameters present in the problem have a significant impact on the optimal harvesting strategy. Before demonstrating the numerical solution, we discuss the effect of these parameters on the nature of the solution. If the parameter values are such that the optimal effort $E^*(t)$ stays within $[0,\ E_{\max}]$ then the constraints on effort are not binding. Consequently, the optimal solution stays the same even if the constraints are omitted. In other words, both constrained and unconstrained optimal harvesting problems yield the same optimal solutions in this case. However, if the resultant optimal effort is outside the permissible range then the constraints come into play and become binding. Under these circumstances, the optimal effort $E^*(t)$ is adjusted, and is set to its bound at the instant of constraint violation. We provide further explanation with the help of the simulations presented next. The parameter values used for simulation purposes are summarized in Table 3.1.

Table 3.1: Parameter values for the simulation of the Hamilton-Jacobi-Bellman equation

Parameter	Description	Value	Unit
r	Intrinsic growth rate	0.71	year^{-1}
δ	Discount rate	0.12	year^{-1}
q	Catchability coefficient	0.0001	SFU^{-1} year^{-1}
K	Biological carrying capacity	10^6	tonnes
$x_{\min}$	Minimum viable population level	$0.4K$	tonnes
p	Unit harvest price	0.5	\$ tonne^{-1}
T	Terminal time	1	year
$E_{\max}$	Maximum effort	r/q	SFU

Non-binding constraints

We first present an example where the constraints on the optimal effort are not violated. Figure 3.1 demonstrates the optimal effort path when the harvesting is initiated at the carrying capacity K; the cost parameters are fixed as $c_1 = \$0.01/\text{SFU}/\text{year}$ and $c_2 = \$0.01/\text{SFU}^2/\text{year}$. We note that the optimal effort stays between 0 and $E_{\max}$, so the constraints on effort are not binding in this case. Since it is biologically feasible and economically beneficial to harvest at positive levels, the optimal effort never falls below 0. However, it is not beneficial to harvest at full capacity as an increase in effort is accompanied by an increase in the harvesting cost, and this additional cost exceeds the revenue earned from additional harvest. Consequently, the optimal effort always stays below $E_{\max}$.

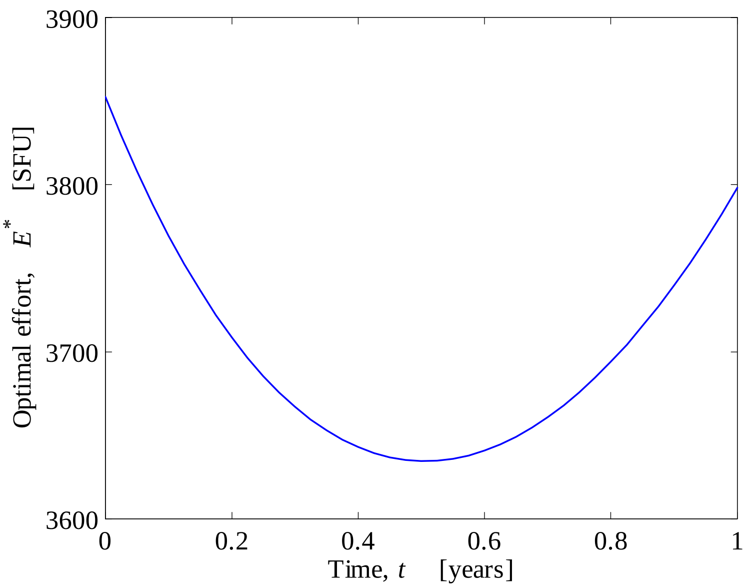


Figure 3.1: Numerical solution for the optimal effort strategy when the cost parameters are fixed as c_1 = \$0.01/SFU/year and c_2 = \$0.01/(SFU)^2/year with $x_0 = K$. The constraints on the optimal effort are not binding in this case.

Binding constraints

Next we illustrate the scenario where the unconstrained optimal effort falls out of the range $[0,\ E_{\max}]$. The cost coefficients c_1 and c_2 are now allowed to assume lower values, specifically c_1 = \$0.001/SFU/year and c_2 = \$0.001/SFU^2/year, and the rest of the parameters are left unchanged.

Since the costs are reduced, it now becomes profitable to raise the fishing effort and the recommended optimal effort calculated using Equation (3.14) results in a value greater that $E_{\max}$. The final optimal effort, however, is determined by Equation (3.16), where the constraints are exercised and the optimal effort is not allowed to exceed $E_{\max}$. Figure 3.2 presents the optimal effort solution obtained under these circumstances. We observe that, in this case, the constraint on effort is binding and the optimal effort stays at the maximum possible effort level, $E_{\max}$.

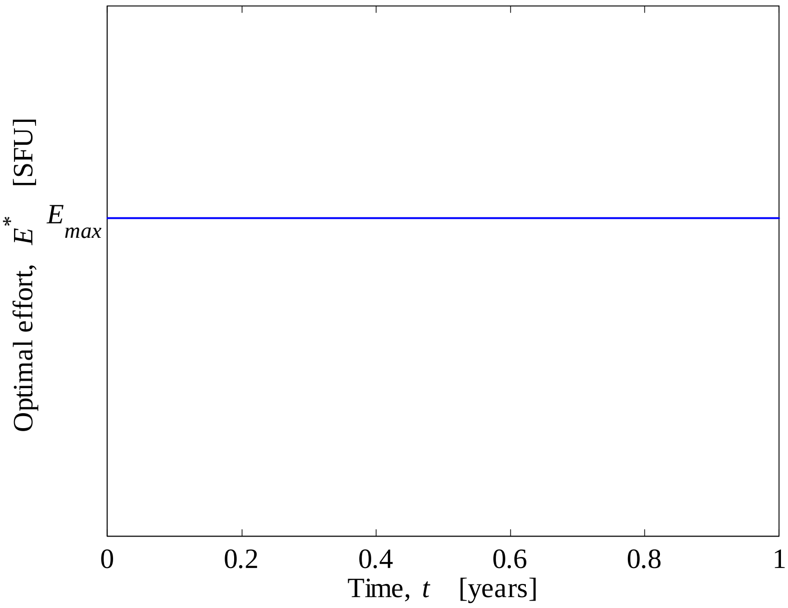


Figure 3.2: The optimal effort solution when the cost parameters are fixed as c_1 = \$0.001/SFU/year and c_2 = \$0.001/(SFU)^2/year with $x_0 = K$. The constraint on effort becomes binding in this case and the optimal effort stays at $E_{\max}$ throughout the harvest.

Minimum viable level

We wish to study the significance of the minimum viable level constraint which we have introduced on the population undergoing harvest. There are two circumstances under which the resource stock can be driven below the minimum viable level. We illustrate each one of these scenarios with the help of an example, and show how the constraint imposed over the minimum viable population level conserves the fish population. The parameter values utilized are again from Table 3.1.

The first case, where the population can fall below the minimum level, is when the initial stock level is low, i.e., x_0 is close to the minimum viable population level. We demonstrate this situation by fixing the minimum viable level at $0.4K$ and setting $x_0 = 0.5K$, $c_1 =$ \$0.01/SFU/year and $c_2 =$ \$0.01/SFU^2/year. Figure 3.3 presents the evolution of the population, under the influence of optimal effort, for these parameter values. The asterisks signify the case where the boundary condition given by Equation (3.18) is employed, i.e. $x_{\min}$ is set equal to $0.4K$, ensuring that the effort is reduced to zero if the harvested stock level drops to the minimum viable level. Thus the population is allowed to recover before harvesting is restarted. It can be seen that, in this case, the population under harvesting always stays above the minimum level. Next we consider the case where we have ignored the boundary condition specified by Equation (3.18); this amounts to the removal of the lower bound $x_{\min}$ from the population density by setting $x_{\min} = 0$. The consequence is visible via the solid line in Figure 3.3, which shows that, under these circumstances, the stock falls below the minimum viable level. While extinction does not occur till the terminal time T, there is a possibility that any remaining fish may not be able to survive.

The second scenario, where the resource stock can be driven below the minimum viable population level, is when $E_{\max}$ is very high and it is profitable to harvest using maximum effort. We demonstrate this situation by fixing $E_{\max} = 20000$ SFU $> r/q$, $x_0 = K, c_1 =$ \$0.001/SFU/year and $c_2 =$ \$0.001/SFU^2/year; the minimum viable level is again $0.4K$. Even though the initial population level is high in this case, still there is a possibility of biological over-fishing as is evident from Figure 3.4. The asterisks again represent the solution where the constraint $x_{\min} = 0.4K$ has been imposed and the solid line illustrates the solution obtained by ignoring the restrictions on the minimum viable level (here $x_{\min} = 0$). In the former case, the fish stock is conserved and the population level stays above the minimum viable level throughout harvesting, whereas in the latter case the stock level falls below the minimum viable level which can be fatal for the fish population.

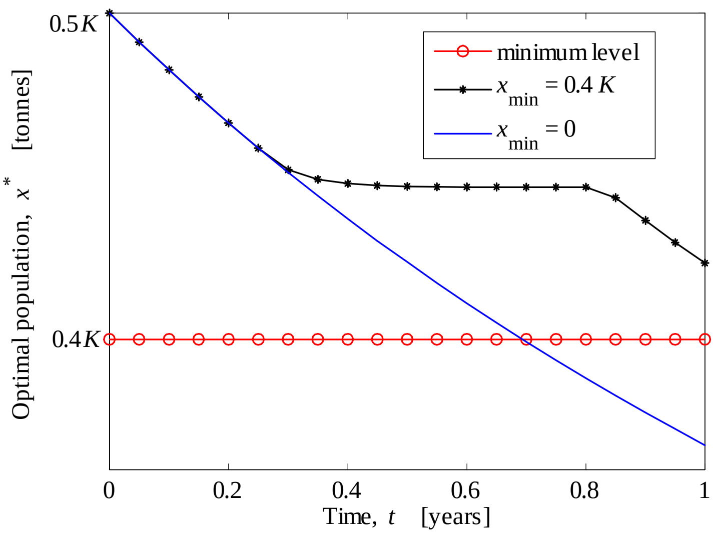


Figure 3.3: The population growth under the optimal harvest when the initial population level is fixed at $0.5K$*,* c_1 *= \$0.01/SFU/year and* c_2 *= \$0.01/(SFU)*^{2}*/year; the minimum viable level is fixed at* $0.4K$*. The final population drops below the minimum level when the constraint on the minimum viable population level is not actively enforced.*

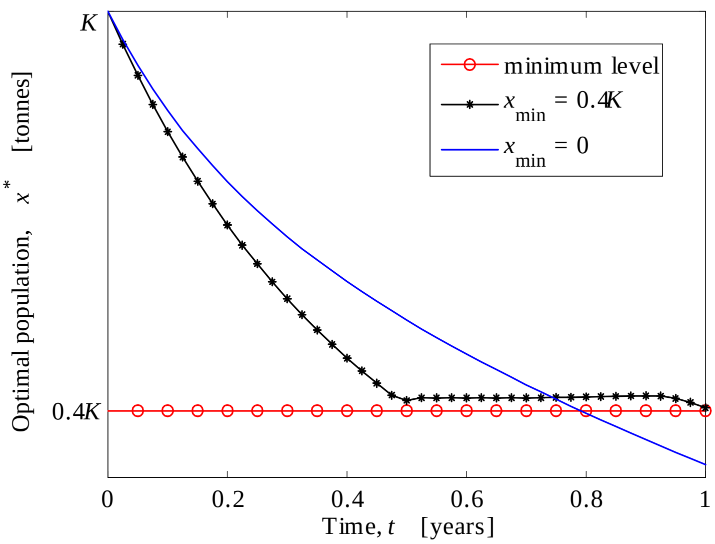


Figure 3.4: The population growth under the optimal harvest for high $E_{\max}$ *(= 20000 SFU) and low cost:* c_1 *= \$0.001/SFU/year and* c_2 *= \$0.001/(SFU)*^{2}*/year; the minimum viable level is again fixed at* $0.4K$*. The unconstrained fish stock drops below the minimum viable level, even though* x_0 *is high (fixed at the carrying capacity* K*).*

3.4 Hamiltonian method

In this section we aim to find the optimal solution using the Hamiltonian method which is based on Pontryagin's maximum principle (a brief introduction to the Hamiltonian method and the maximum principle is included in Section 2.3). Here we are solving the unconstrained optimal harvesting problem where we ignore the constraints on fishing effort given by Equation (3.2).

Derivation of the system of equations

We recall from Section 3.2 that the maximized objective function is

$$J^*(x(0),0) = \max_{\substack{E(\tau)\\0\leq\tau\leq T}} \int_0^T e^{-\delta\tau}\left(pqx(\tau) - c_1 - \frac{c_2}{2}E(\tau)\right)E(\tau)d\tau.$$

This objective function can be reformulated as

$$J^* = \max_{\substack{E(\tau)\\0\leq\tau\leq T}} \int_0^T e^{-\delta\tau} f(x(\tau),E(\tau))d\tau,$$

$$\text{where} \qquad f = \left(pqx(\tau) - c_1 - \frac{c_2}{2}E(\tau)\right)E(\tau). \tag{3.19}$$

We form the current-valued Hamiltonian as follows:

$$\begin{aligned}\mathcal{H}(E,x) &= f(x(\tau),E(\tau),\tau) + m(\tau)\frac{dx(\tau)}{d\tau}\\ &= \left(pqx(\tau) - c_1 - \frac{c_2}{2}E(\tau)\right)E(\tau) + m(\tau)\left\{rx(\tau)\left(1-\frac{x(\tau)}{K}\right) - qE(\tau)x(\tau)\right\},\end{aligned} \tag{3.20}$$

where $m(\tau)$ is a Lagrange multiplier, and $f(x(\tau),E(\tau),\tau)$ and $\frac{dx(\tau)}{d\tau}$ have been substituted with Equations (3.19) and (3.7) respectively.

In order to find the control $E(\tau)$ maximizing the objective function, we need to solve the following system of Euler-Lagrange equations:

$$\frac{\partial\mathcal{H}}{\partial E} = 0, \tag{3.21}$$

$$\frac{dm(\tau)}{d\tau} = \delta m(\tau) - \frac{\partial H}{\partial x}, \tag{3.22}$$

and the growth equation

$$\frac{dx(\tau)}{d\tau} = rx(\tau)\left(1 - \frac{x(\tau)}{K}\right) - qE(\tau)x(\tau), \tag{3.23}$$

subject to the initial condition

$$x(0) = x_0 \text{ (constant)}, \tag{3.24}$$

along with the transversality condition (see Section 2.3)

$$m(T) = 0. \tag{3.25}$$

Differentiating Equation (3.20) with respect to E and using Equation (3.21) gives

$$[pqx(\tau) - c_1 - c_2E(\tau)] - mqx(\tau) = 0,$$

which can be solved for effort as

$$E(\tau) = \frac{(p - m(\tau))qx(\tau) - c_1}{c_2}. \tag{3.26}$$

This $E(\tau)$ is the required (unconstrained) optimal effort.

Differentiating Equation (3.20) with respect to x we have

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial x} &= pqE(\tau) + rm(\tau) - \frac{2m(\tau)rx(\tau)}{K} - m(\tau)qE(\tau) \\ &= (p - m(\tau))\, qE(\tau) + rm(\tau) - \frac{2m(\tau)rx(\tau)}{K}. \end{aligned} \tag{3.27}$$

Substituting Equation (3.26) for $E(\tau)$ into Equation (3.27) yields

$$\frac{\partial \mathcal{H}}{\partial x} = (p - m(\tau))\, q\left[\frac{(p - m(\tau))qx(\tau) - c_1}{c_2}\right] + rm(\tau) - \frac{2m(\tau)rx(\tau)}{K}. \tag{3.28}$$

Using Equation (3.28) in Equation (3.22) we get

$$\frac{dm(\tau)}{d\tau} = \delta m(\tau) - rm(\tau) - (p - m(\tau))\, q\left[\frac{(p - m(\tau))qx(\tau) - c_1}{c_2}\right] + \frac{2m(\tau)rx(\tau)}{K},$$

which is equivalent to

$$\frac{dm(\tau)}{d\tau} = (\delta - r)\, m(\tau) + \frac{2rm(\tau)x(\tau)}{K} - \frac{(p - m(\tau))^2q^2x(\tau)}{c_2} + \frac{(p - m(\tau))qc_1}{c_2}. \tag{3.29}$$

Substituting Equation (3.26) for optimal $E(\tau)$ into Equation (3.23) leads to

$$\frac{dx(\tau)}{d\tau} = rx(\tau)\left(1 - \frac{x(\tau)}{K}\right) - \frac{1}{c_2}[(p - m(\tau))q^2x(\tau)^2] + \frac{qx(\tau)c_1}{c_2}. \tag{3.30}$$

Thus we have obtained a system comprising two ordinary differential equations, Equation (3.29) and Equation (3.30). We know the initial value of the population level, $x(0)$ (given by Equation (3.24)), and the terminal value of the Lagrange multiplier, $m(T)$ (given by Equation (3.25)). This indicates that we have a two-point boundary-value problem with split conditions.

We use the shooting method (Stanoyevitch, 2005) for determining the numerical solution for the above-mentioned boundary-value problem. Following this method we try several initial values for the Lagrange multiplier $m(\tau)$, solve the boundary-value problem as an initial-value problem, and compare the terminal value obtained for $m(\tau)$ in each case with the actual terminal value, which is 0. Using trial and error we wish to reduce the difference between the actual and the obtained terminal value below a pre-specified tolerance; this procedure yields the initial value $m(0)$. As $x(0)$ is given already, the system (3.29)-(3.30) can now be solved as an initial value problem. The solution obtained for $m(\tau)$ is substituted in Equation (3.26) and this gives the unconstrained optimal effort. The parameter values used for simulation purposes are same as those listed in Table 3.1 and the initial population level is fixed at the carrying capacity K.

Numerical results and discussion

We compare the solutions obtained by employing the Hamiltonian method with the solutions determined using dynamic programming. Figure 3.5 illustrates the optimal effort path produced by the Hamiltonian method where c_1 = \$0.01/SFU/year, c_2 = \$0.01/SFU^2/year and $x_0 = K$. We noted in Section 3.3 that, for these parameter values, the constraints on the optimal effort are not binding at any time. Hence the constrained optimal solution determined using dynamic programming, illustrated by Figure 3.1, is the same as its unconstrained counterpart obtained by using the Hamiltonian method, represented by Figure 3.5.

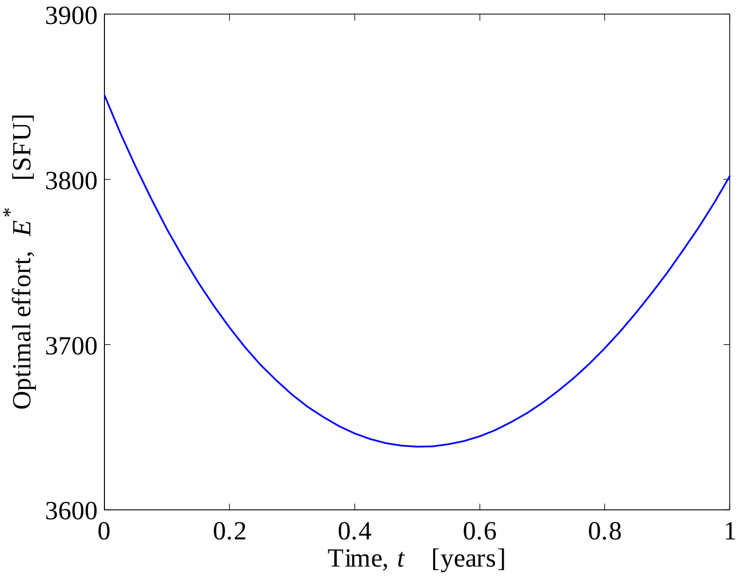


Figure 3.5: Numerical solution for the optimal effort yielded by the Hamiltonian method for c_1 = \$0.01/SFU/year and c_2 = \$0.01/(SFU)^2/year with $x_0 = K$. The solution obtained here coincides with the solution obtained using the dynamic programming technique for the same set of parameter values.

Next we fix $c_1 = \$0.001/\text{SFU/year}, c_2 = \$0.001/\text{SFU}^2/\text{year}$ and retain the same values for the remaining parameters. Figure 3.6 presents the optimal effort solution for this case. We observe that the optimal effort solution obtained here is different from the optimal effort path ($E^* = E_{\max}$) produced by the dynamic programming approach (Figure 3.2). It was noted in Section 3.3 that the constraints on the optimal effort become binding for this set of parameter values. Since we do not incorporate the constraints in the Hamiltonian method, the resultant optimal effort falls outside $[0,\ E_{\max}]$; here, the optimal effort stays above $E_{\max}$ throughout the harvesting period. On the other hand, the dynamic programming technique does not allow the optimal effort to exceed the upper bound $E_{\max}$. Consequently, the optimal effort determined using dynamic programming stays at $E_{\max}$.

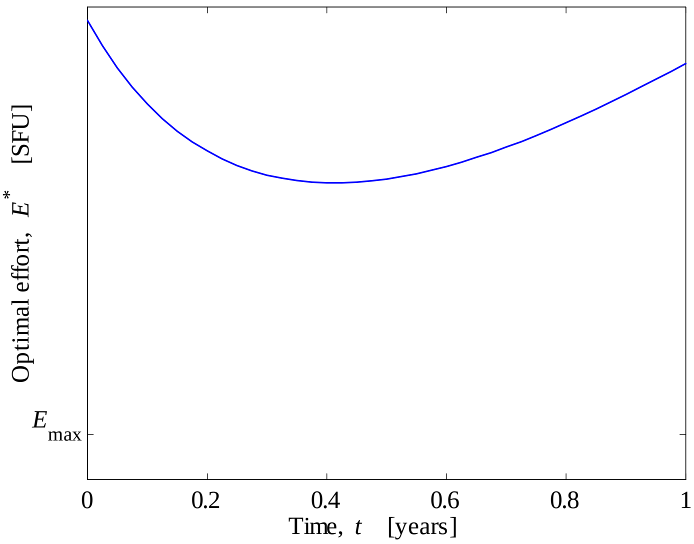


Figure 3.6: Numerical solution for the optimal effort produced by the Hamiltonian method when c_1 = \$0.001/SFU/year and c_2 = \$0.001/(SFU)^2/year with $x_0 = K$. Here the optimal effort stays above $E_{\max}$ throughout the harvest. This solution is different from the optimal solution obtained using the dynamic programming approach where the optimal effort was equal to $E_{\max}$ for identical parameter values.

3.5 Variational approach

We now investigate the unconstrained optimal harvesting problem using a variational approach based on the calculus of variations (the calculus of variations is discussed in Section 2.2; see also Chiang, 2000 for a description of the variational approach employed here).

Derivation of the system of equations

Again the performance index is given by

$$J(x(0),0) = \int_0^T e^{-\rho\tau} \left(p(\tau)qx(\tau) - c_1 - \frac{c_2}{2}E(\tau) \right) E(\tau)d\tau, \tag{3.31}$$

which has to be maximized with respect to the control $E(\tau)$, subject to the growth equation

$$\frac{dx(\tau)}{d\tau} = rx(\tau)\left(1 - \frac{x(\tau)}{K}\right) - qEx(\tau), \tag{3.32}$$

and the initial condition

$$x(0) = x_0. \tag{3.33}$$

Let $E^*(\tau)$ furnish an unconstrained maximum for the performance index given by Equation (3.31) and $x^*(\tau)$ denote the corresponding population growth. We introduce a small variation in $E^*(\tau)$ and $x^*(\tau)$ as follows:

$$\begin{aligned} E(\tau) &= E^*(\tau) + h(\tau), \\ x(\tau) &= x^*(\tau) + k(\tau), \end{aligned}$$

where $h(\tau)$ and $k(\tau)$ are arbitrary functions such that $|h| \ll 1$ and $|k| \ll 1$.

The first variation of J is then given by

$$\begin{aligned} \triangle J &= \int_0^T e^{-\rho\tau}\left(pq(x^* + k) - c_1 - \frac{c_2}{2}(E^* + h)\right)(E^* + h)d\tau \\ &\quad - \int_0^T e^{-\rho\tau}\left(pqx^* - c_1 - \frac{c_2}{2}E^*\right)E^* d\tau, \end{aligned} \tag{3.34}$$

where x^*, E^*, k and h are functions of time.

Simplifying Equation (3.34) and ignoring the second order terms in h and k yields

$$\triangle J = \int_0^T e^{-\rho\tau} h(pqx^* - c_1 - c_2 E^*)d\tau + \int_0^T e^{-\rho\tau} k(pqE^*)d\tau. \tag{3.35}$$

Replacing x by $(x^* + k)$ and E by $(E^* + h)$ in Equation (3.32) we get

$$\frac{d(x^* + k)}{d\tau} = r(x^* + k)\left(1 - \frac{(x^* + k)}{K}\right) - q(E^* + h)(x^* + k) \tag{3.36}$$

Simplifying Equation (3.36) leads to

$$\frac{dk}{d\tau} = r\left(1 - \frac{2x^*}{K}\right)k - qE^*k - qx^*h + \textit{higher order terms in } k \textit{ and } h. \tag{3.37}$$

Rearranging Equation (3.37) and ignoring the *higher order terms*, we obtain h as

$$h = \frac{r\left(1 - \frac{2x^*}{K}\right)k - \frac{dk}{d\tau} - qE^*k}{qx^*}. \tag{3.38}$$

As $E^*(\tau)$ and $x^*(\tau)$ correspond to a maximum for J, the first variation of J given by Equation (3.34) is zero, which amounts to

$$\int_0^T e^{-\rho\tau} h(pqx^* - c_1 - c_2E^*)d\tau + \int_0^T e^{-\rho\tau} k(pqE^*)d\tau = 0. \tag{3.39}$$

Substituting Equation (3.38) into Equation (3.39) we obtain

$$\begin{aligned}&\int_0^T e^{-\rho\tau} \left\{ (pqx^* - c_1 - c_2E^*) \left(\frac{r\left(1 - \frac{2x^*}{K}\right) - qE^*}{qx^*} \right) + pqE^* \right\} kd\tau \\ &- \int_0^T e^{-\rho\tau} \left(\frac{pqx^* - c_1 - c_2E^*}{qx^*} \right) \frac{dk}{d\tau} d\tau = 0\end{aligned} \tag{3.40}$$

Note that Equation (3.40) holds for all k. Integrating by parts in the second integral in Equation (3.40) yields

$$\begin{aligned}&\int_0^T e^{-\rho\tau} \Bigg[(pqx^* - c_1 - c_2E^*) \left(\frac{r\left(1 - \frac{2x^*}{K}\right) - qE^*}{qx^*} \right) + pqE^* \\ &\qquad -\rho \left(\frac{pqx^* - c_1 - c_2E^*}{qx^*} \right) - \frac{c_1}{q} \left(\frac{1}{x^*} \right)' - \frac{c_2}{q} \left(\frac{E^*}{x^*} \right)' \Bigg] kd\tau \\ &-e^{-\rho T} \left(p - \frac{c_1 - c_2E^*(T)}{qx^*(T)} \right) k(T) + \left(p - \frac{c_1 - c_2E^*(0)}{qx^*(0)} \right) k(0) = 0.\end{aligned} \tag{3.41}$$

For Equation (3.41) to hold, each of the three terms present in the equation should be identically zero. This implies

$$\begin{aligned}&\int_0^T e^{-\rho\tau} \Bigg[(pqx^* - c_1 - c_2E^*) \left(\frac{r\left(1 - \frac{2x^*}{K}\right) - qE^*}{qx^*} \right) + pqE^* \\ &\qquad -\rho \left(\frac{pqx^* - c_1 - c_2E^*}{qx^*} \right) - \frac{c_1}{q} \left(\frac{1}{x^*} \right)' - \frac{c_2}{q} \left(\frac{E^*}{x^*} \right)' \Bigg] kd\tau = 0,\end{aligned} \tag{3.42}$$

$$\left(p - \frac{c_1 - c_2E^*(T)}{qx^*(T)} \right) k(T) = 0, \tag{3.43}$$

and

$$\left(p - \frac{c_1 - c_2E^*(0)}{qx^*(0)} \right) k(0) = 0. \tag{3.44}$$

We first examine Equation (3.44). Since the initial state $x^*(0)$ is given as x_0, Equation (3.44) implies $k(0) = 0$.

Next we consider Equation (3.43). As the final state $x^*(T)$ is free therefore $k(T) \neq 0$. Consequently,

$$\left(p - \frac{c_1 - c_2 E^*(T)}{qx^*(T)}\right) = 0,$$

which leads to the boundary condition

$$E^*(T) = \frac{pqx^*(T) - c_1}{c_2}. \tag{3.45}$$

Finally, the integral in Equation (3.42) is equal to zero for any arbitrary $k(\tau)$; this amounts to the integrand being identically zero. Therefore,

$$\begin{aligned} e^{-\rho\tau} \Bigg[&(pqx^* - c_1 - c_2 E^*) \left(\frac{r\left(1 - \frac{2x^*}{K}\right) - qE^*}{qx^*} \right) + pqE^* \\ &-\rho \left(\frac{pqx^* - c_1 - c_2 E^*}{qx^*} \right) - \frac{c_1}{q} \left(\frac{1}{x^*} \right)' - \frac{c_2}{q} \left(\frac{E^*}{x^*} \right)' \Bigg] k = 0. \end{aligned} \tag{3.46}$$

Simplifying Equation (3.46) we get

$$\frac{dE^*}{d\tau} = r\frac{E^* x^*}{K} + \rho E^* + \frac{pq}{c_2}(r - \rho)x^* - \frac{2rpq}{c_2 K} x^{*2} + \frac{c_1}{c_2}\left(\rho + \frac{rx^*}{K}\right). \tag{3.47}$$

To summarize, we have obtained a system of ordinary differential equations comprising of the optimal effort dynamics (given by Equation (3.47)) and the corresponding population dynamics (given by Equation (3.32)). We rewrite them as follows:

$$\begin{aligned} \frac{dE^*}{d\tau} &= r\frac{E^* x^*}{K} + \rho E^* + \frac{pq}{c}(r - \rho)x^* - \frac{2rpq}{cK} x^{*2} + \frac{c_1}{c_2}\left(\rho + \frac{rx^*}{K}\right), \\ \frac{dx^*}{d\tau} &= rx^*\left(1 - \frac{x^*}{K}\right) - qE^* x^*. \end{aligned}$$

We use MATLAB to determine a numerical solution for this system of equations. As we have the final boundary condition for effort (given by Equation (3.45)) and the initial boundary condition for population (given by Equation (3.33)), we use the shooting method for simulation purposes. The initial population level is again fixed at the carrying capacity K and the remaining parameter values are those in Table 3.1.

Numerical results and discussion

We compare the optimal solution obtained by using the variational approach with the optimal solutions found by the dynamic programming technique and the Hamiltonian method. We first fix c_1 = \$0.01/SFU/year and c_2 = \$0.01/SFU^2/year; recall that the constraints on the optimal effort are not binding for these parameter values (see Section 3.3). Figure 3.7 illustrates the optimal effort path determined using the variational approach; we find that the optimal solution obtained here agrees with the optimal solution produced by the dynamic programming technique (Figure 3.1) and the Hamiltonian method (Figure 3.5).

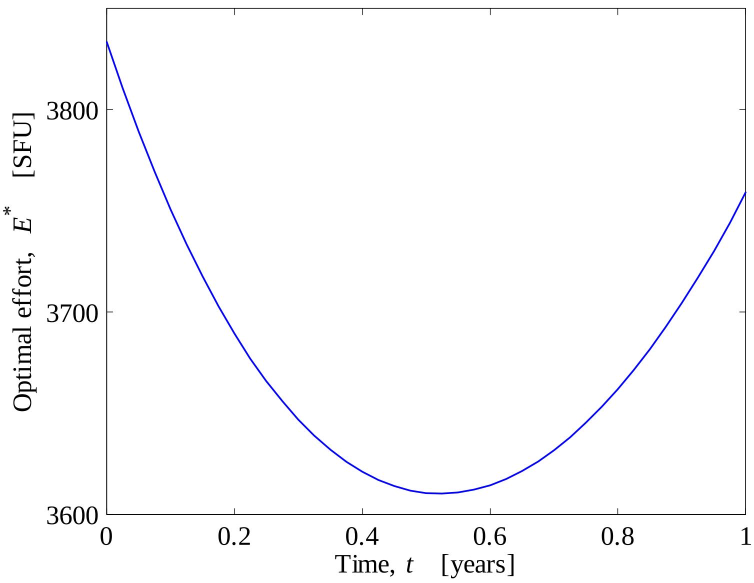


Figure 3.7: The optimal effort calculated with the variational approach for c_1 = \$0.01/SFU/year and c_2 = \$0.01/(SFU)^2/year with $x_0 = K$. This solution agrees with the optimal effort solution obtained using dynamic programming and the Hamiltonian method, for the same parameter values.

Next we study the behaviour of the optimal solution when the cost parameters c_1 and c_2 are low, demonstrated by fixing c_1 =\$ 0.001/SFU(year) and c_2 =\$ 0.001/(SFU)^2(year). Figure 3.8 illustrates the solution for optimal effort obtained in this case. We notice that the solution obtained here is the same as the solution produced by the Hamiltonian method (Figure 3.6), the reason being that the constraints on effort are not taken into account by any of these two approaches. As noted earlier in Sections 3.3 and 3.4, the dynamic programming approach results in a different solution where the optimal effort is forced to follow the constraints and to stay at $E_{\max}$ throughout the harvest.

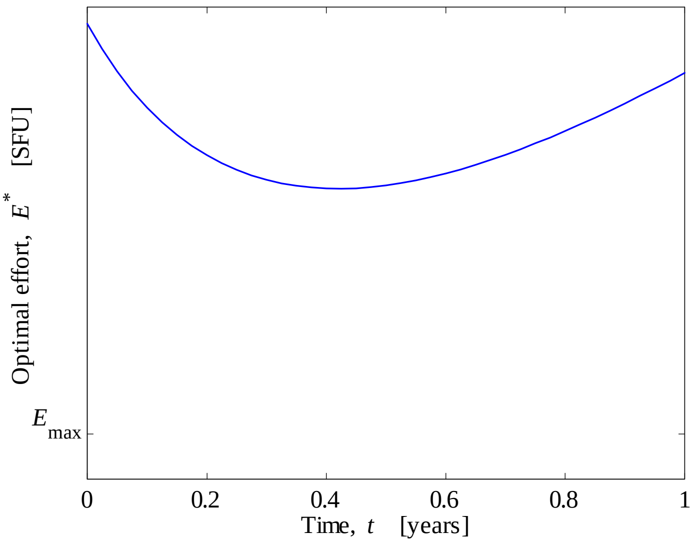


Figure 3.8: The optimal effort obtained using the variational approach when the cost parameters c_1 and c_2 are low (c_1 = \$0.001/SFU/year and c_2 = \$0.001/(SFU)^2/year. The unconstrained solution is same as the solution produced by the Hamiltonian method for the same set of parameter values, but different from the constrained optimal solution obtained using the dynamic programming technique.

3.6 Sensitivity analysis

In this section we consider the constrained optimal harvesting problem formulated using the dynamic programming technique, presented in Section 3.3. The expression $E^*(t)$ for the optimal effort, given by Equation (3.16), includes various parameters, e.g. the catchability coefficient q, the cost parameters c_1 and c_2, the stock size $x(t)$ and the price p. Therefore the simulated optimal harvesting strategy, and consequently the total discounted profit, is highly sensitive to the values used for these parameters. We investigate the sensitivity of the total discounted profit to different combinations of catchability and cost parameters.

We concentrate on those values of c_1 and c_2 for which harvesting is feasible and profitable. We fix $x_0 = K$, $c_1 =$ \$0.01/SFU/year and $c_2 =$ \$0.01/SFU^2/year as the base values, and record the present value of the total profit corresponding to a high and a low value of the catchability coefficient q; the low value for the catchability coefficient is demonstrated using $q = 0.0001$/SFU/year and the high value is demonstrated using $q = 0.1$/SFU/year. Then we introduce variations in the values of c_1 and c_2, and examine their effect on the total discounted profit. The observations are recorded in Table 3.2 and are based on the numerical simulations of the Hamilton-Jacobi-Bellman equation (3.15), using parameter values from Table 3.1.

For $q = 0.0001$/SFU/year, an increase in the value of c_1 incurs a minimal change in the total discounted profit, whereas the same increase in c_2 incurs a significant drop; hence the quadratic term in the cost function is dominant when the catchability is low. An explanation for this effect can be provided by studying Figure 3.9 where we observe that a rise in c_1 does not have a substantial effect on the optimal effort expended, whereas, a rise in c_2 results in a considerable decrease in the optimal effort and the overall effect is a pronounced decline in the total discounted profit. On the other hand, when $q = 0.1$/SFU/year, the total discounted profit is more or less constant; a change in the values of c_1 and c_2 does not have a significant effect on the total discounted profit. This can be attributed to the fact that the optimal effort stays at E_{max} for high catchability, as demonstrated by Figure 3.10.

Table 3.2: Sensitivity analysis of the total discounted profit with respect to different combinations of catchability and cost parameters.

q [SFU^{-1} year^{-1}]	c_1 [\$ SFU^{-1} year^{-1}]	c_2 [\$ SFU^{-2} year^{-1}]	$J^*(x(0) = K, 0)$ [\$]
0.0001	0.01	0.01	8.7141×10^4
	1	0.01	8.3689×10^4
	0.01	1	0.1184×10^4
0.1	0.01	0.01	2.5641×10^5
	1	0.01	2.5641×10^5
	0.01	1	2.5639×10^5

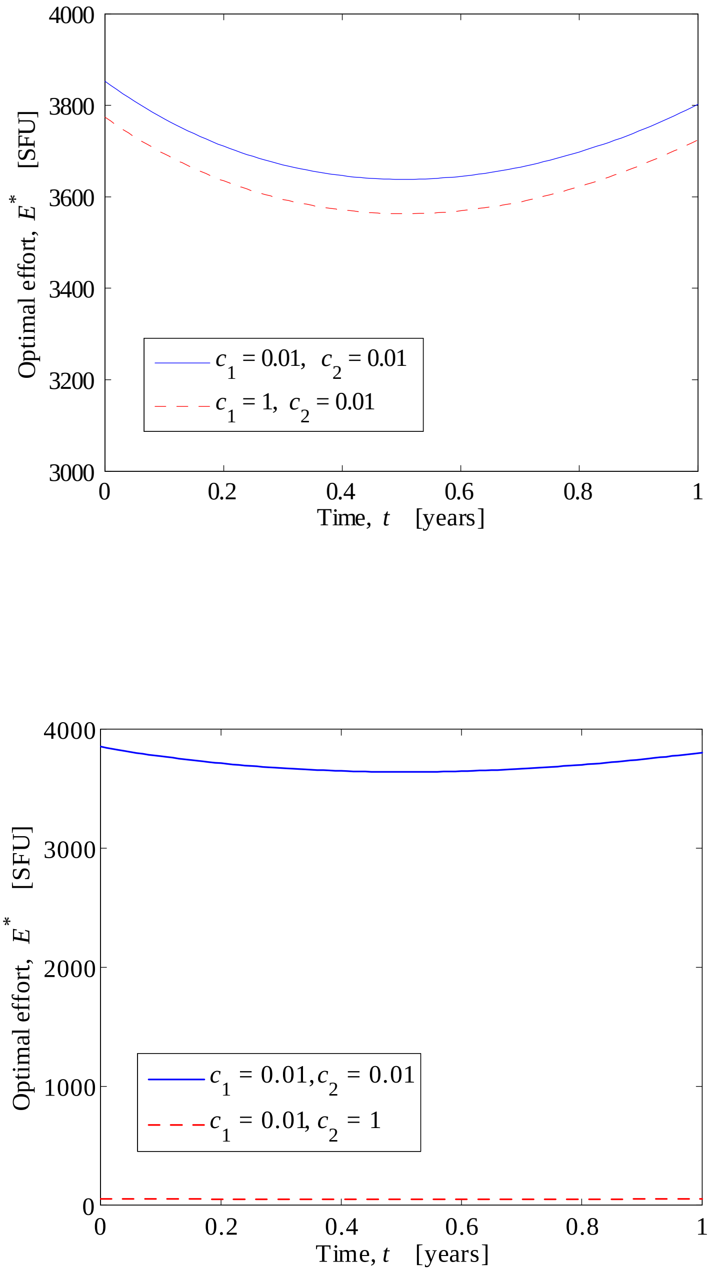
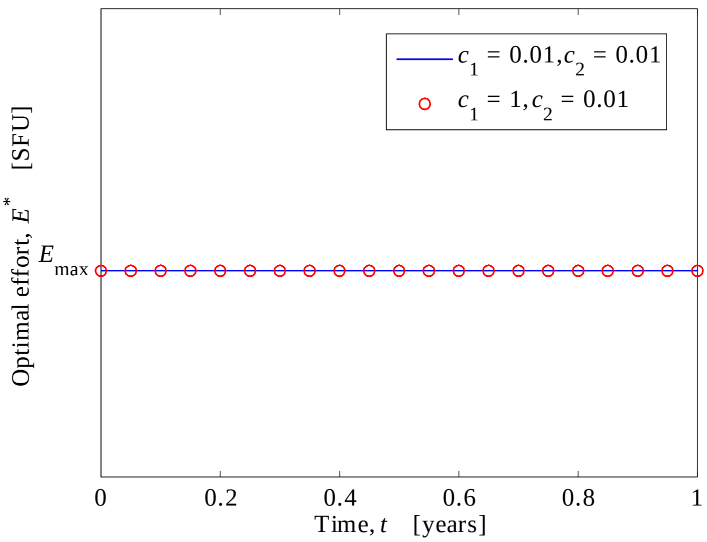


Figure 3.9: Effect of an increase in the cost parameters on the optimal effort policy for a low catchability, demonstrated by fixing $q = 0.0001$/SFU/year. Top: c_2 is fixed while c_1 is allowed to vary; Bottom: c_1 is fixed while c_2 is allowed to vary. An increase in c_2 has a much greater impact on the optimal effort than an increase in c_1.



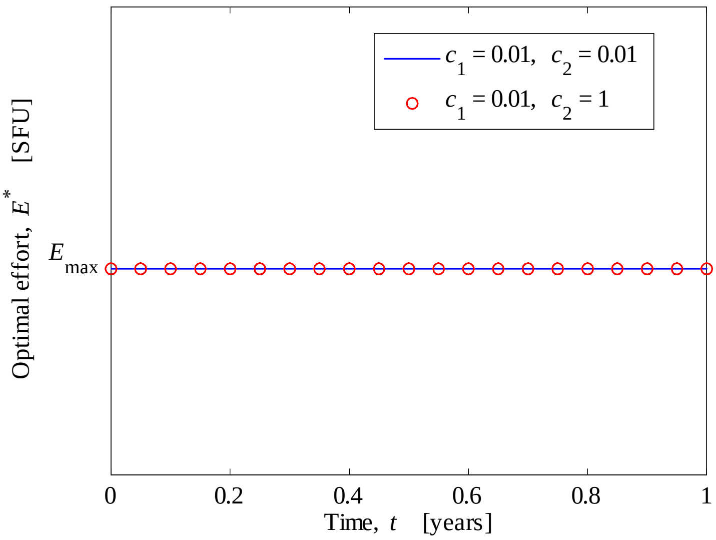


Figure 3.10: Effect of an increase in the cost parameters on the optimal effort policy for a high catchability, demonstrated by fixing $q = 0.1/SFU/year$. Top: c_2 is fixed while c_1 is allowed to vary; Bottom: c_1 is fixed while c_2 is allowed to vary. The optimal effort stays at $E_{\max}$ in all the cases illustrated here.

3.7 Infinite horizon

Thus far, we have considered harvesting for a finite time horizon where the terminal time T is fixed. The objective in this section is to study the behaviour of the optimal solution corresponding to harvesting for an infinite time horizon ($T \to \infty$).

As noted by Tsur & Zemel (2001), when the terminal time is infinite it is not always necessary to study the evolution of the optimal solution path. For a broad category of time-autonomous optimization problems, where the calendar time t is not explicitly present in the formulation of the problem (except in the discount factor), the optimal solution is time-invariant in the long run and converges to an equilibrium state. This idea of time-invariant optimal solution can also be understood by studying the Bellman equation (3.15) as explained below.

For a fixed terminal time, the Bellman equation contains the time derivative, $\frac{\partial J^*(x(t),t)}{\partial t}$, of the value function J^*. However, in the case of an infinite time horizon J^* depends only on the state variable, provided the state dynamics and the profit function are time-autonomous (Miranda, 2002), and therefore $\frac{\partial J^*(x(t),t)}{\partial t} = 0$. We notice that the time variable is not explicitly present in the growth equation (3.7) or in the profit function given by Equation (3.3). Therefore, for infinite horizon harvesting the Hamilton-Jacobi-Bellman equation (3.15) reduces to

$$\begin{aligned} 0 \;=\; \max_{E(t)} \Bigg[& \left(pqx(t) - c_1 - \frac{c_2}{2}E(t)\right) E(t) - \delta J^*(x(t),t) \\ & + \frac{\partial J^*(x(t),t)}{\partial x(t)} \left\{ rx(t)\left(1 - \frac{x(t)}{K}\right) - qE(t)x(t) \right\} \Bigg], \end{aligned}$$

and the optimal effort and population level converge to their respective optimal equilibrium states.

Determining long run optimal steady state solution

We use the method developed by Tsur & Zemel (2001) to find the optimal steady-state solution for an infinite-horizon harvesting and present a brief description of the method before applying it. Retaining their original notation, the infinite-horizon problem investigated by Tsur & Zemel (2001) was of the form

$$V(X_0) = \max_{c_t} \int_0^{\infty} e^{-rt} f(X_t, c_t) dt$$

subject to the state equation $\dot{X}_t \equiv dX_t/dt = g(X_t, c_t)$;

satisfying $\underline{X} \leq X_t \leq \bar{X}$, with initial state X_0, given $\underline{c} \leq c_t \leq \bar{c}$, where c_t is the control vector. The discount rate r was assumed to be positive and the functions f and g were time-autonomous and sufficiently smooth. The *evolution function* was introduced based on the idea that one is not better off by deviating from an optimal steady state and entering a nearby steady state. For a steady-state policy $c = R(X)$, the evolution function was denoted by $L(X)$ and was given by

$$L(X) \equiv r \left(\frac{f_c(X, R(X))}{g_c(X, R(X))} + W'(X) \right),$$

where the subscripts denoted partial derivatives and the prime denoted differentiation with respect to the state variable; $W(X) = f(X, R(X))/r$ was the steady state value so that $W(X) \leq V(X)$. The equality was obeyed only when the policy $R(X)$ was optimal. Using the variational method, it was shown that any optimal state must be a root of the evolution function. An optimal steady-state policy could then be determined by solving for the roots of the evolution function.

We now follow the same procedure as described above to formulate the infinite horizon optimal harvesting problem. To our knowledge, the infinite horizon problem with logistic growth and quadratic costs has never been examined using this approach. In this context, the value function is given by

$$\begin{aligned} J^*(x(0), 0) &= \max_{E(t)} \int_0^{\infty} e^{-\delta t} \left(pqx(t) - c_1 - \frac{c_2}{2} E(t) \right) E(t) dt \\ &= \max_{E(t)} \int_0^{\infty} e^{-\delta t} f(x(t), E(t)) dt \quad (say), \end{aligned} \tag{3.48}$$

subject to the growth equation

$$\begin{aligned} \frac{dx(t)}{dt} &= rx(t) \left(1 - \frac{x(t)}{K} \right) - qE(t)x(t) \\ &= g(x(t), E(t)) \quad (say), \end{aligned} \tag{3.49}$$

satisfying $0 \leq x(t) \leq K$, with $x(0) = x_0$ and constraints on effort as $0 \leq E(t) \leq E_{\max}$.

The associated evolution function can be specified as

$$L(X_{eqm}) \equiv \delta \left(\frac{f_E(X_{eqm}, E_{eqm})}{g_E(X_{eqm}, E_{eqm})} + W'(X_{eqm}) \right), \tag{3.50}$$

where X_{eqm} represents a steady state (not necessarily optimal) for the state variable $x(t)$, and $E_{eqm} \equiv E(X_{eqm})$ is the corresponding steady-state effort. At equilibrium the rate of growth of the fish stock is zero, therefore the growth equation (3.49) gives

$$\frac{dX_{eqm}}{dt} \equiv rX_{eqm}\left(1 - \frac{X_{eqm}}{K}\right) - qE_{eqm}X_{eqm} = 0. \tag{3.51}$$

Rearranging Equation (3.51), the equilibrium effort is obtained as

$$E_{eqm} = \frac{r}{q}\left(1 - \frac{X_{eqm}}{K}\right). \tag{3.52}$$

We now determine the partial derivatives present in the evolution function (3.50). From Equations (3.48) and (3.49) we have

$$\begin{aligned} f(x(t), E(t)) &= \left(pqx(t) - c_1 - \frac{c_2}{2}E(t)\right)E(t), \\ g(x(t), E(t)) &= rx(t)\left(1 - \frac{x(t)}{K}\right) - qE(t)x(t). \end{aligned}$$

Differentiating the functions f and g with respect to E and evaluating the derivatives at the equilibrium state yields

$$\begin{aligned} f_E(X_{eqm}, E_{eqm}) &= pqX_{eqm} - c_1 - c_2\frac{r}{q}\left(1 - \frac{X_{eqm}}{K}\right), \\ g_E(X_{eqm}, E_{eqm}) &= -qX_{eqm}. \end{aligned} \tag{3.53}$$

Furthermore,

$$\begin{aligned} W(X_{eqm}) &= \frac{f(X_{eqm}, E_{eqm})}{\delta} \\ &= \frac{1}{\delta}\left[pqX_{eqm}\frac{r}{q}\left(1 - \frac{X_{eqm}}{K}\right) - c_1\frac{r}{q}\left(1 - \frac{X_{eqm}}{K}\right) - \frac{c_2}{2}\frac{r^2}{q^2}\left(1 - \frac{X_{eqm}}{K}\right)^2\right] \end{aligned} \tag{3.54}$$

Differentiating Equation (3.54) with respect to X_{eqm} gives

$$W'(X_{eqm}) = pr - \frac{2prX_{eqm}}{K} + \frac{c_1 r}{qK} - \frac{c_2 r^2 X_{eqm}}{q^2K^2} + \frac{c_2 r^2}{q^2 K}. \tag{3.55}$$

Substituting the functions given by Equations (3.53) and (3.55) in the evolution function (3.50) and equating it to zero, we obtain the following equation (which is quadratic in X_{eqm}):

$$\left(\frac{2pr}{K}+\frac{c_2r^2}{q^2K^2}\right)q^2X_{eqm}^2+\left[\left(p+\frac{c_2r}{q^2K}\right)(\delta-r)-\frac{rc_1}{qK}\right]q^2X_{eqm}-\delta(c_1q+c_2r)=0. \quad (3.56)$$

We denote the optimal steady-state solutions for effort and population by E_∞ and X_∞ respectively. The quadratic equation (3.56) can be solved for X_{eqm}; the optimal equilibrium population level X_∞ is obtained as the positive root whilst the other root is negative and can be discarded. The steady-state, to which the optimal effort converges in the long run, can be determined by substituting X_{eqm} with X_∞ in Equation (3.52). We calculate the optimal equilibrium for the parameter values specified in Table 3.1, and find that under the optimal policy the population level stabilizes at

$$X_\infty = 0.6924 \times K, \quad (3.57)$$

and the effort approaches the equilibrium value

$$E_\infty = 2183.5 \text{ SFU}. \quad (3.58)$$

Simulations of Bellman equation for large values of terminal time

In the previous section, we determined the long-run steady states for the optimal effort and population level corresponding to an infinite horizon harvesting. We now wish to study the evolution of the optimal effort and the fish stock when finite-horizon harvesting is carried on for a long period; the objective is to investigate how the latter compares with the equilibrium states obtained for infinite-horizon harvesting. For this purpose we simulate the Hamilton-Jacobi-Bellman equation (3.15) obtained for finite-time harvesting, for large values of the terminal time T.

We use the parameter values given in Table 3.1 with c_1 = \$0.01/SFU/year and c_2 = \$0.01/SFU^2/year, and perform numerical simulations of Equation (3.15) for $T = 1$ year, $T = 10$ years and $T = 20$ years. Recall that, for this set of parameters, the equilibrium states (corresponding to infinite-horizon harvesting) for the optimal stock level and the optimal effort are obtained as $X_\infty = 0.6924K$ and $E_\infty = 2183.5$ SFU (see Equations (3.57) and

(3.58)). We first analyze the nature of the optimal solution for finite-horizon harvesting when the initial population level is above X_∞, specifically $x_0 = 0.7K$. Figure 3.11 illustrates the optimal effort path and Figure 3.12 presents the associated optimal population growth. We note that the steady state behaviour of the optimal solution is not visible when $T = 1$ year. However, as T increases, the optimal effort tends to stabilize at E_∞ and the optimal stock level settles at X_∞ before undergoing a change towards the end of the harvesting period.

During harvesting, there is a trade-off between harvesting today and letting the population grow for increased future harvest. But near the terminal time, there is no incentive for the harvester to allow the population to grow. Therefore the optimal effort displays an increase over a few stages before the final time and, consequently, the population level declines. If harvesting had carried on for an infinite time then the optimal effort and the optimal population would have stayed forever at E_∞ and X_∞ respectively.

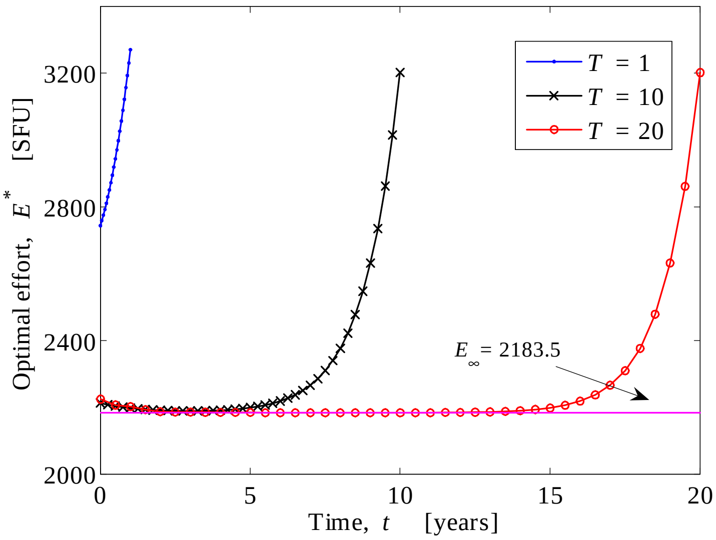


Figure 3.11: Comparison of the optimal effort, E^*, obtained using the Bellman equation for a long-term finite-horizon harvesting with the steady state, E_∞, corresponding to the infinite-horizon optimal solution; the initial stock level is above the corresponding optimal equilibrium value. As T increases, E^* stabilizes at E_∞ before increasing towards the end.

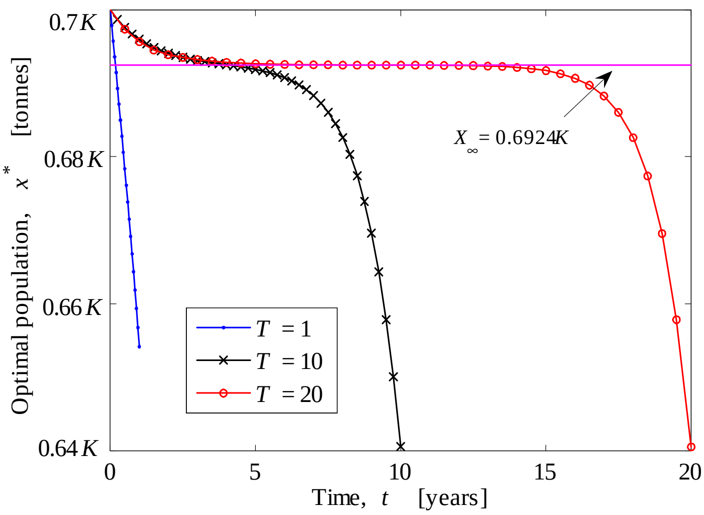


Figure 3.12: Comparison of the optimal population level, x^*, corresponding to a long-term finite-horizon harvesting with the optimal steady state, X_∞, associated with the infinite-horizon optimal solution. As T increases, x^* settles at X_∞; a decline is observed for a few time stages before the terminal time.

Next we analyze the case where harvesting is initiated below the steady state X_∞; to demonstrate this we fix $x_0 = 0.65K$ and retain the rest of the parameter values. Figures 3.13 and 3.14 show the optimal effort path and the optimal stock growth respectively. Under these circumstances, the optimal solutions corresponding to $T = 10$ years and $T = 20$ years portray the same qualitative behaviour whereas the optimal solution for $T = 1$ year traces a totally different path. For $T = 1$ year, the resource stock undergoes a decline from the beginning and continues to drop until harvesting ceases, whereas, for comparatively large values of T (illustrated using $T = 10$ years and $T = 20$ years), the optimal effort rises to E_∞ and the optimal population increases to X_∞; the steady states are maintained for a while before a change in the optimal effort drives the population level down in the final stages.

Thus the optimal effort policy corresponding to an infinite time-horizon recommends a stationary value for the effort in the long run, with the population stabilized at the corresponding steady state. If time is sufficiently long, the optimal solution associated with a finite-horizon harvesting stabilizes itself at the equilibrium state corresponding to the solution for an infinite horizon before re-adjusting itself close to the terminal time.

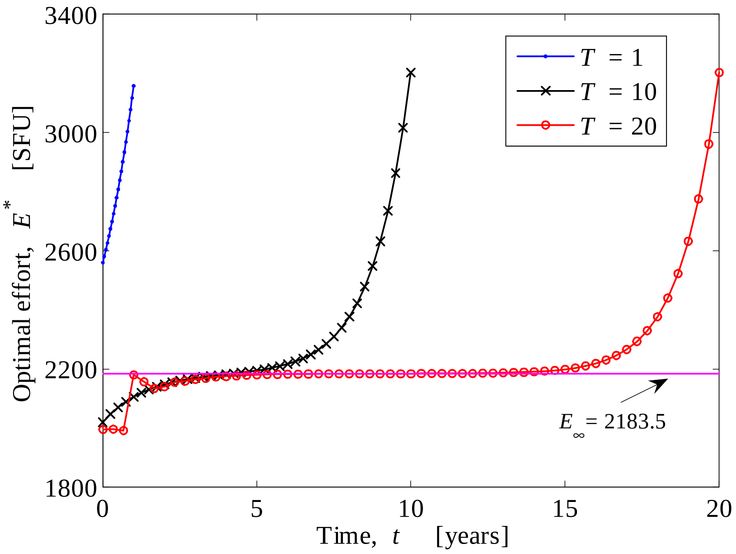


Figure 3.13: Comparison of the optimal effort solution, E^*, associated with a long-term finite-horizon harvesting, with the optimal equibrium value, E_∞, obtained for an infinite-horizon harvesting; the initial stock level is below the respective optimal steady state. E^* approaches E_∞ as T increases and stays there for a while before a change is observed in a few final time stages.

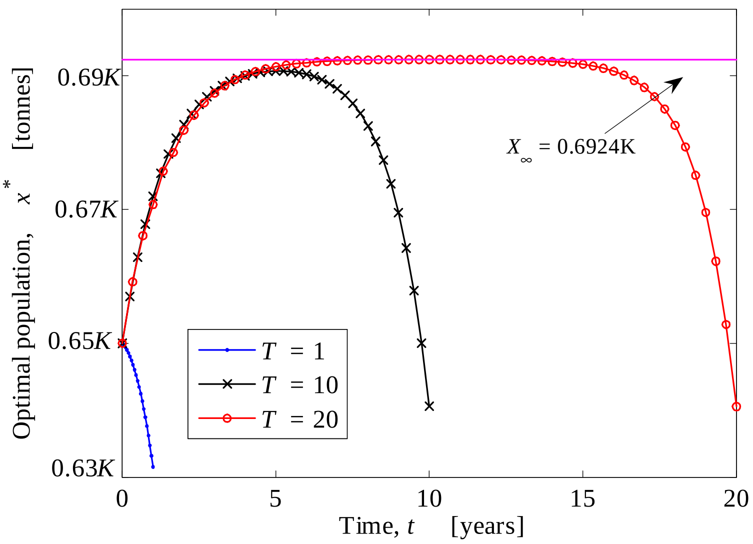


Figure 3.14: Comparison of the solution for a long-run finite-horizon optimal population level , x^*, with the optimal steady state, X_∞, corresponding to an infinite-horizon harvesting. As T increases, x^* stabilizes at X_∞ before undergoing a change towards the end of the harvesting period.

3.8 Summary

In this chapter, we discussed a deterministic model for fish harvesting with a constant non-zero price and quadratic costs. The optimal harvesting policy maximizing the net present value of flow of profit was sought, while the resource stock was maintained above a minimum viable level throughout the harvest. The model was examined using the following three methods:

- Dynamic programming: This technique resulted in the Hamilton-Jacobi-Bellman partial differential equation. The equation was solved using the Crank-Nicolson finite-difference method and at each step the effort was checked and forced to follow the constraints.
- Hamiltonian method: The method demonstrated in this chapter did not incorporate the constraints on effort and yielded a system of two ordinary differential equations. Moreover, this method introduced an additional variable (Lagrange multiplier) to the problem. As it was a boundary-value problem, the system had to be solved numerically with split boundary conditions using the shooting method.
- Variational approach: This method also led to a system of two ordinary differential equations and provided the optimal effort solution straightaway. It did not, however, include the constraints on effort. Furthermore, we again had to solve with split boundary conditions.

We observed that all of these methods resulted in the same optimal solution for the set of parameters corresponding to non-binding constraints on effort. In other words, when the parameter values were such that the constraints on the optimal effort were not binding and the optimal effort lay within $[0,\ E_{\max}]$, the optimal effort policy produced by all the methods was the same. However, when the constraints on the optimal effort were binding, the dynamic programming technique resulted in an optimal policy which was different from the optimal solution produced by the other two methods. Further, we found that the profit maximizing harvesting policy can drive the population to extinction, if the proper constraints are not enforced to ensure that the resource stock stays above the minimum viable level during harvesting.

Following that, we carried out a sensitivity analysis of the total discounted profit which suggested that, for a low catchability, the total discount profit is highly sensitive to a change in the quadratic coefficient of cost, c_2, whereas a change in the linear coefficient, c_1, does not have a significant impact. The total discounted profit was observed to be constant when the catchability coefficient was high.

Finally, we studied infinite-horizon harvesting where we determined the long-run steady states for the optimal effort and the population level. Then we simulated the optimal solution corresponding to large values of the terminal time for finite-horizon harvesting and found that this optimal solution tends to stabilize at the optimal steady-state solution associated with infinite-horizon harvesting. However, due to the finiteness of the terminal time, the former undergoes a change towards the end of the harvesting period. We further noted that harvesting can be performed at higher levels when the initial population level is well above the long-run steady state X_∞, whereas, an initial level lower that the long-run steady state value forces the optimal effort to adjust itself so that the population can rise to X_∞.

Chapter 4

Stochastic Environment

4.1 Introduction

This chapter concentrates on the optimal harvesting problem in a stochastic environment. The chapter is organized as follows: Section 4.2 develops a harvesting model for a fishery experiencing continuous disturbance in stock and price; the noise is represented by two different Wiener processes which may be correlated or uncorrelated. Section 4.3 presents the stochastic dynamic programming technique, which yields a Hamilton-Jacobi-Bellman partial differential equation describing the expected net present value of the total profit (or the expected total discounted profit). We first consider the fully stochastic problem, with random growth and random price, and then reduce it to the case of random growth and constant price.

In Section 4.4 we present the optimal solution corresponding to stochastic growth and constant price and Section 4.5 illustrates the optimal solution when both growth and price are stochastic. The optimal solutions corresponding to both short-term and long-term harvesting are illustrated and their sensitivity to the variance of growth and price is examined. Section 4.6 focuses on the sensitivity of the expected total discounted profit to the catchability and cost parameters and Section 4.7 discusses the correlation between the resource stock and the price. Section 4.8 summarizes the work done in this chapter.

4.2 Stochastic model formulation

The deterministic model (discussed in Section 3.2) is now extended to its stochastic version, with the growth dynamics of the resource population given by an Itô-type stochastic differential equation

$$dx(\tau) = \left\{ rx(\tau)\left(1 - \frac{x(\tau)}{K}\right) - qE(\tau)x(\tau) \right\} d\tau + \sigma_1 x(\tau) dW_1(\tau), \tag{4.1}$$

where σ_1 is a positive constant representing random growth effects (*growth volatility*) and $dW_1(\tau)$ is a standard Wiener increment.

The effort is again constrained as

$$0 \leq E(\tau) \leq E_{\max} < \infty \text{ for all } \tau, \tag{4.2}$$

where $E_{\max}$ is a fixed constant. We assume the fish price to be random; the price-dynamics also follow an Itô-type stochastic differential equation

$$dp(\tau) = \mu_p p(\tau) d\tau + \sigma_2 p(\tau) dW_2(\tau), \tag{4.3}$$

where $p(\tau)$ is the price per unit harvest at time τ, $dW_2(\tau)$ denotes a standard Wiener increment and σ_2 is a positive constant representing the magnitude of random price effects (*price volatility*). Here the drift in price, $\mu_p p(\tau)$, is assumed to be proportional to the spot price; μ_p is a positive constant. Note that we have assumed that the fishery is small so that it's harvest has no effect on the world prices.

The harvesting cost is again assumed to be quadratic in fishing effort, given by $c_1(\tau)E(\tau) + \frac{c_2}{2}E(\tau)^2$ where c_1 and c_2 are positive constants. The expected net value of the profit earned by the harvester at time τ is denoted by $\Pi(\tau)$ and is given by the difference between revenue and costs

$$\begin{aligned}\Pi(\tau) &= \mathcal{E}\left[p(\tau)qx(\tau)E(\tau) - c_1 E(\tau) - \frac{c_2}{2}E(\tau)^2\right] \\ &= \mathcal{E}\left[\left(p(\tau)qx(\tau) - c_1 - \frac{c_2}{2}E(\tau)\right)E(\tau)\right],\end{aligned}$$

where $\mathcal{E}$ is the expectation operator.

It is further assumed that harvesting is initiated at time 0 and continues up to a finite time T. The future profits are discounted at rate δ and the expected net present value of the total profit earned by the harvester is obtained as

$$PV = \mathcal{E}\left[\int_0^T e^{-\delta\tau}\Pi(\tau)d\tau\right].$$

We assume that the optimal harvesting strategy maximizes the expected net present value of the total profit. Thus the principle underlying the optimal harvesting stays the same as the deterministic case (see Section 3.2) but the calculations here are based on the expected values of the stochastic processes involved. We let $J^*(x(t), p(t), t)$ denote the maximized value of the expected total discounted profit when harvesting begins at time t and continues up to the terminal time T. Then we can write

$$\begin{aligned} J^*(x(t), p(t), t) &= \max_{\substack{E(\tau)\\ t\le\tau\le T}} \mathcal{E}\left[\int_t^T e^{-\delta(\tau-t)}\Pi(\tau)d\tau\right] \\ &= \max_{\substack{E(\tau)\\ t\le\tau\le T}} \mathcal{E}\left[\int_t^T e^{-\delta(\tau-t)}\left(p(\tau)qx(\tau) - c_1 - \frac{c_2}{2}E(\tau)\right)E(\tau)d\tau\right], \end{aligned}$$

so that, in particular,

$$J^*(x(T), p(T), T) = 0. \tag{4.4}$$

Furthermore, a minimum viable population level $x_{\min}$ is maintained throughout harvesting. Under these assumptions the problem of deriving an optimal harvesting strategy, for the period $[0, T]$, can be formulated as a stochastic optimal control problem where the control variable is $E(\tau)$ and the value function is given by

$$\begin{aligned} J^*(x(0), p(0), 0) &= \max_{\substack{E(\tau)\\ 0\le\tau\le T}} \mathcal{E}\left[\int_0^T e^{-\delta\tau}\Pi(\tau)d\tau\right] \\ &= \max_{\substack{E(\tau)\\ 0\le\tau\le T}} \mathcal{E}\left[\int_0^T e^{-\delta\tau}\left(p(\tau)qx(\tau) - c_1 - \frac{c_2}{2}E(\tau)\right)E(\tau)d\tau\right]. \end{aligned} \tag{4.5}$$

The optimal solution must be determined subject to the growth dynamics given by Equation (4.1), the price dynamics given by Equation (4.3), the constraints on effort given by (4.2), the minimum viable population level $x_{\min}$, and the initial conditions $x(0) = x_0$ and $p(0) = p_0$.

4.3 Stochastic dynamic programming

In this section we derive the Hamilton-Jacobi-Bellman equation, corresponding to the expected total discounted profit, by employing stochastic dynamic programming (see Section 2.4 for an introduction to stochastic dynamic programming).

Derivation of the Hamilton-Jacobi-Bellman equation

In order to derive the Hamilton-Jacobi-Bellman equation, we consider the current value form of Equation (4.5) as

$$J^*(x(t), p(t), t) = \max_{\substack{E(\tau) \\ t \le \tau \le T}} \mathcal{E} \left[\int_t^T e^{-\delta(\tau - t)} \left(p(\tau) q x(\tau) - c_1 - \frac{c_2}{2} E(\tau) \right) E(\tau) d\tau \right], \tag{4.6}$$

where t denotes the current time and $x(t)$ and $p(t)$ represent the population level and price respectively at time t.

The integral in Equation (4.6) can be broken down as

$$\begin{aligned} J^*(x(t), p(t), t) = \max_{\substack{E(\tau) \\ t \le \tau \le T}} \mathcal{E} \Bigg[& \int_t^{t+dt} e^{-\delta(\tau - t)} \left(p(\tau) q x(\tau) - c_1 - \frac{c_2}{2} E(\tau) \right) E(\tau) d\tau + \\ & \int_{t+dt}^T e^{-\delta(\tau - t - dt + dt)} \left(p(\tau) q x(\tau) - c_1 - \frac{c_2}{2} E(\tau) \right) E(\tau) d\tau \Bigg]. \end{aligned}$$

Utilizing the approximation $e^{-\delta dt} \approx (1 - \delta dt)$, valid for small dt, we get

$$\begin{aligned} J^*(x(t), p(t), t) = \max_{\substack{E(\tau) \\ t \le \tau \le T}} \mathcal{E} \Bigg[& \int_t^{t+dt} e^{-\delta(\tau - t)} \left(p(\tau) q x(\tau) - c_1 - \frac{c_2}{2} E(\tau) \right) E(\tau) d\tau + \\ & (1 - \delta dt) \int_{t+dt}^T e^{-\delta(\tau - (t+dt))} \left(p(\tau) q x(\tau) - c_1 - \frac{c_2}{2} E(\tau) \right) E(\tau) d\tau \Bigg]. \end{aligned} \tag{4.7}$$

Applying the principle of optimality (Bellman, 1957), Equation (4.7) becomes

$$\begin{aligned} J^*(x(t), p(t), t) = \max_{\substack{E(\tau) \\ t \le \tau \le t+dt}} \mathcal{E} \Bigg[& \int_t^{t+dt} e^{-\delta(\tau - t)} \left(p(\tau) q x(\tau) - c_1 - \frac{c_2}{2} E(\tau) \right) E(\tau) d\tau + \\ & (1 - \delta dt) \max_{\substack{E(\tau) \\ t+dt \le \tau \le T}} \int_{t+dt}^T e^{-\delta(\tau - (t+dt))} \left(p(\tau) q x(\tau) - c_1 - \frac{c_2}{2} E(\tau) \right) E(\tau) d\tau \Bigg], \end{aligned}$$

which is equivalent to

$$J^*(x(t),p(t),t) = \max_{E(t)} \mathcal{E}\left[\left((p(t)qx(t) - c_1 - \frac{c_2}{2}E(t)\right)E(t)dt \right.$$
$$\left. + (1-\delta dt)J^*(x(t)+dx(t),p(t)+dp(t),t+dt)\right]. \tag{4.8}$$

Expanding $J^*(x(t)+dx(t),p(t)+dp(t),t+dt)$ around $(x(t),p(t),t)$ we obtain

$$\begin{aligned} J^*(x(t)+dx(t),p(t)+dp(t),t+dt) = {} & J^*(x(t),p(t),t) + \frac{\partial J^*(x(t),p(t),t)}{\partial t}dt \\ & + \frac{\partial J^*(x(t),p(t),t)}{\partial x(t)}dx(t) + \frac{\partial J^*(x(t),p(t),t)}{\partial p(t)}dp(t) \\ & + \frac{1}{2}\frac{\partial^2 J^*(x(t),p(t),t)}{\partial x(t)^2}dx(t)^2 + \frac{1}{2}\frac{\partial^2 J^*(x(t),p(t),t)}{\partial p(t)^2}dp(t)^2 \\ & + \frac{\partial^2 J^*(x(t),p(t),t)}{\partial x(t)\partial p(t)}dx(t)dp(t) + \mathcal{O}(dt^2). \end{aligned} \tag{4.9}$$

Substituting for $dx(t)$ and $dp(t)$ from Equations (4.1) and (4.3) respectively, we can rewrite Equation (4.9) as

$$\begin{aligned} J^*(x(t)+dx(t),p(t)+dp(t),t+dt) = {} & J^*(x(t),p(t),t) + \frac{\partial J^*(x(t),p(t),t)}{\partial t}dt \\ & + \frac{\partial J^*(x(t),p(t),t)}{\partial x(t)}\left[\left\{rx(t)\left(1-\frac{x(t)}{K}\right) - qE(t)x(t)\right\}dt + \sigma_1 x(t)dW_1(t)\right] \\ & + \frac{\partial J^*(x(t),p(t),t)}{\partial p(t)}\left[\mu_p p(t)dt + \sigma_2 p(t)dW_2(t)\right] \\ & + \frac{1}{2}\frac{\partial^2 J^*(x(t),p(t),t)}{\partial x(t)^2}\left[\left\{rx(t)\left(1-\frac{x(t)}{K}\right) - qE(t)x(t)\right\}^2 dt^2 + \sigma_1^2 x(t)^2 dW_1(t)^2 \right. \\ & \left. +2\left\{rx(t)\left(1-\frac{x(t)}{K}\right) - qE(t)x(t)\right\}\sigma_1 x(t)dW_1(t)dt\right] \\ & + \frac{1}{2}\frac{\partial^2 J^*(x(t),p(t),t)}{\partial p(t)^2}[\mu_p^2 p(t)^2 dt^2 + \sigma_2^2 p(t)^2 dW_2(t)^2 + 2\mu_p p(t)\sigma_2 p(t)dW_2(t)dt] \\ & + \frac{\partial^2 J^*(x(t),p(t),t)}{\partial x(t)\partial p(t)}\sigma_1 x(t)\sigma_2 p(t)dW_1(t)dW_2(t) + \mathcal{O}(dt)^2 \end{aligned}$$

Rearranging we get

$$
\begin{aligned}
J^*(x(t)+dx(t),p(t)+dp(t),t+dt) = {} & J^*(x(t),p(t),t) + \frac{\partial J^*(x(t),p(t),t)}{\partial t}dt \\
& + \frac{\partial J^*(x(t),p(t),t)}{\partial x(t)} \left\{ rx(t)\left(1-\frac{x(t)}{K}\right) - qE(t)x(t) \right\} dt \\
& + \frac{\partial J^*(x(t),p(t),t)}{\partial x(t)} \sigma_1 x(t) dW_1(t) \\
& + \frac{\partial J^*(x(t),p(t),t)}{\partial p(t)} \mu_p p(t) dt + \frac{\partial J^*(x(t),p(t),t)}{\partial p(t)} \sigma_2 p(t) dW_2(t) \\
& + \frac{1}{2}\frac{\partial^2 J^*(x(t),p(t),t)}{\partial x(t)^2} \sigma_1^2 x(t)^2 dW_1(t)^2 + \frac{1}{2}\frac{\partial^2 J^*(x(t),p(t),t)}{\partial p(t)^2} \sigma_2^2 p(t)^2 dW_2(t)^2 \\
& + \frac{\partial^2 J^*(x(t),p(t),t)}{\partial x(t)^2} \left\{ rx(t)\left(1-\frac{x(t)}{K}\right) - qE(t)x(t) \right\} \sigma_1 x(t) dW_1(t) dt) \\
& + \frac{\partial^2 J^*(x(t),p(t),t)}{\partial p(t)^2} \mu_p p(t) \sigma_2 p(t) dW_2(t) dt \\
& + \frac{\partial^2 J^*(x(t),p(t),t)}{\partial x(t)\partial p(t)} \sigma_1 x(t) \sigma_2 p(t) dW_1(t) dW_2(t) + \mathcal{O}(dt^2).
\end{aligned}
\tag{4.10}
$$

Substituting $J^*(x(t)+dx(t),p(t)+dp(t),t+dt)$ from Equation (4.10) in Equation (4.8) gives

$$
\begin{aligned}
J^*(x(t),p(t),t) = \max_{E(t)} \mathcal{E} \Bigg[& \left(p(t)qx(t) - c_1 - \frac{c_2}{2}E(t) \right) E(t)dt + (1-\delta dt) J^*(x(t),p(t),t) \\
& + (1-\delta dt) \left(\frac{\partial J^*}{\partial t}dt + \frac{\partial J^*}{\partial x(t)} \left\{ rx(t)\left(1-\frac{x(t)}{K}\right) - qE(t)x(t) \right\} dt \right. \\
& \quad + \frac{\partial J^*}{\partial p(t)} \mu_p p(t) dt + \frac{\partial J^*}{\partial x(t)} \sigma_1 x(t) dW_1(t) + \frac{\partial J^*}{\partial p(t)} \sigma_2 p(t) dW_2(t) \\
& \quad + \frac{1}{2}\frac{\partial^2 J^*}{\partial x(t)^2} \sigma_1^2 x(t)^2 dW_1(t)^2 + \frac{1}{2}\frac{\partial^2 J^*}{\partial p(t)^2} \sigma_2^2 p(t)^2 dW_2(t)^2 \\
& \quad + \frac{\partial^2 J^*}{\partial x(t)^2} \left\{ rx(t)\left(1-\frac{x(t)}{K}\right) - qE(t)x(t) \right\} \sigma_1 x(t) dW_1(t) dt \\
& \quad + \frac{\partial^2 J^*}{\partial p(t)^2} \mu_p p(t) \sigma_2 p(t) dW_2(t) dt \\
& \left. + \frac{\partial^2 J^*}{\partial x(t)\partial p(t)} \sigma_1 x(t) \sigma_2 p(t) dW_1(t) dW_2(t) \right) + \mathcal{O}(dt^2) \Bigg].
\end{aligned}
\tag{4.11}
$$

Finally, simplifying Equation (4.11) leads to

$$
\begin{aligned}
0 = \max_{E(t)} \mathcal{E} \Bigg[& \left(p(t)qx(t) - c_1 - \frac{c_2}{2}E(t) \right) E(t)dt - \delta J^* dt + \frac{\partial J^*}{\partial t} dt \\
& + \frac{\partial J^*}{\partial x(t)} \left\{ rx(t) \left(1 - \frac{x(t)}{K} \right) - qE(t)x(t) \right\} dt + \frac{\partial J^*}{\partial p(t)} \mu_p p(t) dt \\
& + \frac{\partial J^*}{\partial x(t)} \sigma_1(t) dW_1(t) - \delta \frac{\partial J^*}{\partial x(t)} \sigma_1 x(t) dW_1(t) dt + \frac{\partial J^*}{\partial p(t)} \sigma_2 p(t) dW_2(t) \\
& - \delta \frac{\partial J^*}{\partial p(t)} \sigma_2 p(t) dW_2(t) dt + \frac{1}{2} \frac{\partial^2 J^*}{\partial x(t)^2} \sigma_1^2 p(t)^2 dW_1(t)^2 - \delta \frac{1}{2} \frac{\partial^2 J^*}{\partial x(t)^2} \sigma_1^2 x(t)^2 dW_1(t)^2 dt \\
& + \frac{1}{2} \frac{\partial^2 J^*}{\partial p(t)^2} \sigma_2^2 p(t)^2 dW_2(t)^2 - \delta \frac{1}{2} \frac{\partial^2 J^*}{\partial p(t)^2} \sigma_2^2 p(t)^2 dW_2(t)^2 dt \\
& + \frac{\partial^2 J^*}{\partial x(t)^2} \left\{ rx(t) \left(1 - \frac{x(t)}{K} \right) - qE(t)x(t) \right\} \sigma_1 x(t) dW_1(t) dt \\
& + \frac{\partial^2 J^*}{\partial p(t)^2} \mu_p p(t) \sigma_2 p(t) dW_2(t) dt + \frac{\partial^2 J^*}{\partial x(t) \partial p(t)} \sigma_1 x(t) \sigma_2 p(t) dW_1(t) dW_2(t) \\
& - \delta \frac{\partial^2 J^*}{\partial x(t) \partial p(t)} \sigma_1 x(t) \sigma_2 p(t) dW_1(t) dW_2(t) dt + \mathcal{O}(dt^2) \Big) \Bigg],
\end{aligned}
\tag{4.12}
$$

where J^* represents $J^*(x(t), p(t), t)$, i.e. the expected value of the maximized total discounted profit earned by initiating the harvest at time t and continuing up to the final time T.

The case of zero correlation

We first suppose that the two Wiener increments, $dW_1(t)$ and $dW_2(t)$, are uncorrelated so that $\mathcal{E}[dW_1(t)dW_2(t)] = 0$. Taking the expectation of Equation (4.12) and using $\mathcal{E}[dW_1(t)] = 0$; $\mathcal{E}[dW_2(t)] = 0$; $\mathcal{E}\left[dW_1(t)^2\right] = dt$; $\mathcal{E}\left[dW_2(t)^2\right] = dt$; and $\mathcal{E}[dW_1(t)dW_2(t)] = 0$ we get

$$
\begin{aligned}
0 = \max_{E(t)} \Bigg[& \left(p(t)qx(t) - c_1 - \frac{c_2}{2}E(t) \right) E(t)dt - \delta J^* dt + \frac{\partial J^*}{\partial t} dt \\
& + \left(\frac{\partial J^*}{\partial x(t)} \left\{ rx(t) \left(1 - \frac{x(t)}{K} \right) - qE(t)x(t) \right\} + \frac{\partial J^*}{\partial p(t)} \mu_p p(t) \right) dt \\
& + \frac{1}{2} \frac{\partial^2 J^*}{\partial x(t)^2} \sigma_1^2 x(t)^2 dt + \frac{1}{2} \frac{\partial^2 J^*}{\partial p(t)^2} \sigma_2^2 p(t)^2 dt + \mathcal{O}(dt^2) \Bigg].
\end{aligned}
\tag{4.13}
$$

Dividing Equation (4.13) throughout by dt, rearranging terms and letting $dt \to 0$ we obtain

$$\begin{aligned}
-\frac{\partial J^*}{\partial t} = \max_{E(t)} & \left[\left(p(t)qx(t) - c_1 - \frac{c_2}{2}E(t) \right) E(t) - \delta J^* \right. \\
& + \frac{\partial J^*}{\partial x(t)} \left\{ rx(t) \left(1 - \frac{x(t)}{K} \right) - qE(t)x(t) \right\} + \frac{\partial J^*}{\partial p(t)} \mu_p p(t) \\
& \left. + \frac{1}{2}\frac{\partial^2 J^*}{\partial x(t)^2} \sigma_1^2 x(t)^2 + \frac{1}{2}\frac{\partial^2 J^*}{\partial p(t)^2} \sigma_2^2 p(t)^2 \right],
\end{aligned} \tag{4.14}$$

which is the required partial differential equation for the expected total discounted profit when $dW_1(t)$ and $dW_2(t)$ are uncorrelated.

The case of non-zero correlation

Next we assume that the two Wiener increments, $dW_1(t)$ and $dW_2(t)$, are correlated with a correlation-coefficient $\rho \neq 0, -1 \leq \rho \leq 1$. Analytically, we suppose that $dW_1(t)$ is a linear combination of two uncorrelated Wiener processes, $dW_2(t)$ and $dW_3(t)$, so that we can write

$$dW_1(t) = a dW_2(t) + b dW_3(t), \tag{4.15}$$

where a and b are constants. Note that:

- Since the expected value of a Wiener increment is zero, $\mathcal{E}[dW_1(t)] = 0$; $\mathcal{E}[dW_2(t)] = 0$; $\mathcal{E}[dW_3(t)] = 0$.
- Since the variance of a Wiener increment is equal to the time-lag, $\mathcal{E}\left[dW_1(t)^2\right] = dt$; $\mathcal{E}\left[dW_2(t)^2\right] = dt; \mathcal{E}\left[dW_3(t)^2\right] = dt$.
- Since $dW_2(t)$ and $dW_3(t)$ are uncorrelated, $\mathcal{E}[dW_2(t)dW_3(t)] = 0$.

Using the above-mentioned expected values, the correlation-coefficient between $dW_1(t)$ and $dW_2(t)$ can be calculated as

$$\begin{aligned}
\rho &= Corr(dW_1(t), dW_2(t)) = \frac{\mathcal{E}[dW_1(t)dW_2(t)] - \mathcal{E}[dW_1(t)]\,\mathcal{E}[dW_2(t)]}{\sqrt{Var(dW_1(t)}\sqrt{Var\,(dW_2(t))}} \\
&= \frac{\mathcal{E}[dW_1(t)dW_2(t)] - 0}{\sqrt{dt}\sqrt{dt}} = \frac{\mathcal{E}[dW_1(t)dW_2(t)]}{dt},
\end{aligned}$$

which gives

$$\mathcal{E}[dW_1(t)dW_2(t)] = \rho dt. \tag{4.16}$$

From Equation (4.15)

$$\begin{aligned}
dW_1(t)dW_2(t) &= adW_2(t)^2 + bdW_3(t)dW_2(t) \\
\Rightarrow \mathcal{E}\left[dW_1(t)dW_2(t)\right] &= a\mathcal{E}\left[dW_2(t)^2\right] + b\mathcal{E}\left[dW_3(t)dW_2(t)\right] \\
&= adt. \qquad (4.17)
\end{aligned}$$

Comparing Equations (4.16) and (4.17) we obtain

$$a = \rho.$$

Furthermore,

$$\begin{aligned}
dt &= \mathcal{E}\left[dW_1(t)^2\right] = \mathcal{E}\left[(adW_2(t) + bdW_3(t))^2\right] \\
&= \mathcal{E}\left[a^2 dW_2(t)^2 + b^2 dW_3(t)^2 + 2abdW_2(t)dW_3(t)\right] \\
&= a^2\mathcal{E}\left[dW_2(t)^2\right] + b^2\mathcal{E}\left[dW_3(t)^2\right] + 2ab\mathcal{E}\left[dW_2(t)dW_3(t)\right] \\
&= a^2 dt + b^2 dt \\
\Rightarrow 1 &= a^2 + b^2 \\
\text{or } \; b &= \sqrt{1-a^2} = \sqrt{1-\rho^2} \qquad (\text{since } a = \rho).
\end{aligned}$$

Substituting $a = \rho$ and $b = \sqrt{1-\rho^2}$ in Equation (4.15) we get

$$dW_1(t) = \rho dW_2(t) + \sqrt{1-\rho^2}dW_3(t). \qquad (4.18)$$

We now derive the Hamilton-Jacobi-Bellman equation for this scenario where $dW_1(t)$ and $dW_2(t)$ are correlated, as specified by Equation (4.18); recall that ρ is the coefficient of correlation between $dW_1(t)$ and $dW_2(t)$. We take the expectation of Equation (4.12) and use $\mathcal{E}\left[dW_1(t)\right] = 0$; $\mathcal{E}\left[dW_2(t)\right] = 0$; $\mathcal{E}\left[dW_1(t)^2\right] = dt$; $\mathcal{E}\left[dW_2(t)^2\right] = dt$; $\mathcal{E}\left[dW_1(t)dW_2(t)\right] = \rho dt$ to obtain

$$\begin{aligned}
0 = \max_{E(t)} \Bigg[&\left(p(t)qx(t) - c_1 - \frac{c_2}{2}E(t)\right)E(t)dt - \delta J^* dt + \frac{\partial J^*}{\partial t}dt \\
&+ \left(\frac{\partial J^*}{\partial x(t)}\left\{rx(t)\left(1 - \frac{x(t)}{K}\right) - qE(t)x(t)\right\} + \frac{\partial J^*}{\partial p(t)}\mu_p p(t)\right)dt + \frac{1}{2}\frac{\partial^2 J^*}{\partial x(t)^2}\sigma_1^2 x(t)^2 dt \\
&+ \frac{1}{2}\frac{\partial^2 J^*}{\partial p(t)^2}\sigma_2^2 p(t)^2 dt + \frac{\partial^2 J^*}{\partial x(t)\partial p(t)}\rho\sigma_1 x(t)\sigma_2 p(t)dt + \mathcal{O}(dt)^2\Bigg]. \qquad (4.19)
\end{aligned}$$

Dividing Equation (4.19) throughout by dt, letting $dt \to 0$ and rearranging yields

$$\begin{aligned} -\frac{\partial J^*}{\partial t} = \max_{E(t)} \Bigg[& \left(p(t)qx(t) - c_1 - \frac{c_2}{2}E(t) \right) E(t) - \delta J^* \\ & + \frac{\partial J^*}{\partial x(t)} \left\{ rx(t) \left(1 - \frac{x(t)}{K} \right) - qE(t)x(t) \right\} + \frac{\partial J^*}{\partial p(t)} \mu_p p(t) \\ & + \frac{1}{2} \frac{\partial^2 J^*}{\partial x(t)^2} \sigma_1^2 x(t)^2 + \frac{1}{2} \frac{\partial^2 J^*}{\partial p(t)^2} \sigma_2^2 p(t)^2 + \frac{\partial^2 J^*}{\partial x(t) \partial p(t)} \rho \sigma_1 x(t) \sigma_2 p(t) \Bigg], \end{aligned} \tag{4.20}$$

which is the partial differential equation describing the expected total discounted profit for non-zero correlation between $dW_1(t)$ and $dW_2(t)$.

Comparing Equations (4.14) and (4.20), obtained for the case of zero correlation and non-zero correlation respectively, we note that the two equations are the same except one extra term (containing the correlation-coefficient ρ) present in the latter. If we put $\rho = 0$ then Equation (4.20) reduces to Equation (4.14). Therefore, the solution for Equation (4.14) can be obtained by solving Equation (4.20) with $\rho = 0$.

An analytic expression for the optimal effort

Denoting the control switching term in Equation (4.20) by D we can write

$$D = \max_{E(t)} \left[\left(pqx(t) - c_1 - \frac{c_2}{2}E(t) \right) E(t) - \frac{\partial J^*(x(t),t)}{\partial x(t)} qE(t)x(t) \right]. \tag{4.21}$$

Let $E^*_{uc}(t)$ correspond to the maximization being carried out in Equation (4.21). Then $E^*_{uc}(t)$ is obtained by solving $\frac{\partial D}{\partial E} = 0$ which gives

$$\begin{aligned} (pqx(t) - c_1 - c_2 E^*_{uc}(t)) - \frac{\partial J^*(x(t),t)}{\partial x(t)} qx(t) = 0 \\ \Longrightarrow E^*_{uc}(t) = \left(p - \frac{\partial J^*(x(t),t)}{\partial x(t)} \right) \frac{qx(t)}{c_2} - \frac{c_1}{c_2}. \end{aligned}$$

We denote the constrained optimal effort by $E^*(t)$, and replace $E(t)$ with $E^*(t)$ in Equation (4.20) to obtain

$$\begin{aligned}
-\frac{\partial J^*}{\partial t} &= \left(p(t)qx(t) - c_1 - \frac{c_2}{2}E^*(t)\right)E^*(t) - \delta J^* \\
&+ \frac{\partial J^*(x(t),p(t),t)}{\partial x(t)}\left\{rx(t)\left(1 - \frac{x(t)}{K}\right) - qE^*(t)x(t)\right\} + \frac{\partial J^*}{\partial p(t)}\mu_p p(t) \\
&+ \frac{\sigma_1^2 x(t)^2}{2}\frac{\partial^2 J^*}{\partial x(t)^2} + \frac{\sigma_2^2 p(t)^2}{2}\frac{\partial^2 J^*}{\partial p(t)^2} + \frac{\partial^2 J^*}{\partial x(t)\partial p(t)}\rho\sigma_1 x(t)\sigma_2 p(t). \qquad (4.22)
\end{aligned}$$

where

$$E^*(t) = \begin{cases} 0, & E^*_{uc}(t) < 0, \\ \left(p(t) - \frac{\partial J^*}{\partial x(t)}\right)\frac{qx(t)}{c_2} - \frac{c_1}{c_2}, & 0 \le E^*_{uc}(t) \le E_{\max}, \\ E_{\max}, & E^*_{uc}(t) > E_{\max}. \end{cases} \qquad (4.23)$$

The boundary conditions associated with the problem are as follows:

- There is no harvesting when the population is at the minimum viable level $x_{\min}$, therefore the boundary condition corresponding to the spatial variable x is

$$E^*(x_{\min}, p, t) = 0. \qquad (4.24)$$

- It is not profitable to harvest when the price is 0, consequently the boundary condition associated with the spatial variable p is

$$E^*(x, 0, t) = 0. \qquad (4.25)$$

- The boundary condition for the temporal variable t follows from Equation (4.4), i.e.

$$J^*(x(T), p(T), T) = 0. \qquad (4.26)$$

To find the optimal harvesting policy, we need to solve the system comprising Equations (4.22)-(4.26) subject to the growth and the price dynamics, the pre-specified minimum viable population level, and the given initial values for the fish stock and the price. As in the deterministic case (Chapter 3), we resort to numerical methods for determining the optimal solution.

Fixed price case

If we keep the price fixed at a constant value p and assume that only the population evolves stochastically then Equation (4.22) reduces to

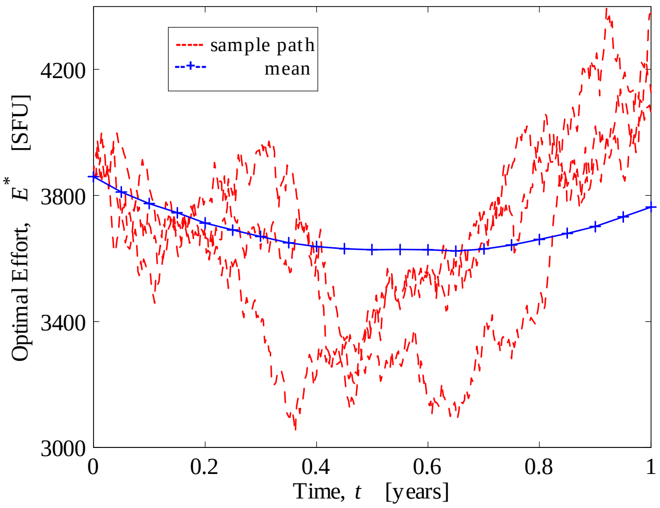
$$
\begin{aligned}
-\frac{\partial J^*(x(t),t)}{\partial t} = \max_{E(t)} \Bigg[& \left(pqx(t) - c_1 - \frac{c_2}{2}E(t)\right)E(t) - \delta J^*(x(t),t) \\
& + \frac{\partial J^*(x(t),t)}{\partial x(t)} \left\{ r(x(t))\left(1 - \frac{x(t)}{K}\right) - qE(t)x(t) \right\} \\
& + \frac{1}{2}\frac{\partial^2 J^*(x(t),t)}{\partial x(t)^2}\sigma_1^2 x(t)^2 \Bigg]. \qquad (4.27)
\end{aligned}
$$

Under these circumstances, the boundary condition (4.25) is no longer relevant and Equation (4.27) need only be solved with the remaining boundary conditions and constraints.

To summarize, the stochastic effects in the problem under consideration are due to the presence of two random variables: one related to stock growth and the other related to the evolution of price, where the two Wiener increments associated with the growth and the price dynamics can be either correlated or uncorrelated. Furthermore, assuming the price to be a fixed constant, the original problem can be reduced to include only one stochastic variable, i.e. the population level. The Hamilton-Jacobi-Bellman partial differential equation obtained in each case is non-linear and involves one temporal variable along with two random spatial variables (or one in the case of constant price). These complexities make the equation analytically intractable, therefore we use numerical methods to determine the optimal solution. We now investigate the optimal solution obtained in each of the above-mentioned cases.

4.4 The optimal solution: random growth and constant price

First, we consider the growth to be stochastic and the price per unit harvest to be a fixed constant. Equation (4.27) describes the expected total discounted profit corresponding to this case. In order to obtain the optimal solution, we perform numerical simulations of Equation (4.27) using a Crank-Nicolson finite-difference method. The computational procedure for the finite-difference scheme is included in Appendix B. Figure 4.1 shows the optimal solution for the effort $E^*(t)$ and the corresponding (optimal) population growth $x^*(t)$. Parameter values are summarized in Table 4.1. The dashed lines represent three different realizations for the Wiener process $dW_1(t)$ and the solid line represents the mean taken over 2000 such paths.



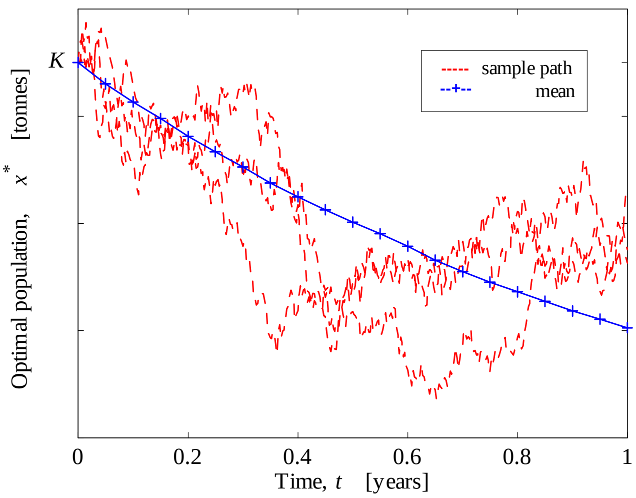


Figure 4.1: The optimal solution for random growth and constant price. Top: optimal effort; Bottom: population growth under the influence of optimal effort. In both graphs, the dashed lines illustrate three distinct sample paths and the solid line is the mean taken over 2000 such paths.

Table 4.1: Parameter values for the simulation of the Hamilton-Jacobi-Bellman equation associated with the case of random growth and constant price

Parameter	Description	Value	Unit
r	Intrinsic growth rate	0.71	year^{-1}
δ	Discount rate	0.12	year^{-1}
μ	Price drift	0.02	year^{-1}
q	Catchability coefficient	0.0001	SFU^{-1} year^{-1}
K	Biological carrying capacity	10^6	tonnes
$x_{\min}$	Minimum viable population level	$0.4K$	tonnes
p	Unit harvest price	0.5	\$ tonne^{-1}
c_1	Linear cost coefficient	0.01	\$ SFU^{-1} year^{-1}
c_2	Quadratic cost coefficient	0.01	\$ SFU^{-2} year^{-1}
σ_1	Growth volatility	0.2	year$^{-1/2}$

Effect of variance: short-term solution

In Chapter 3, we noted that the optimal solution for short-term harvesting was different from the solution associated with long-term and infinite-horizon harvesting. Here we wish to investigate the effect of growth volatility σ_1 on the optimal solution when the harvesting is carried out only for a short period of time. In this context, we fix $T = 1$ year and investigate the average optimal solution, where the average is taken over 2000 sample paths. Figure 4.2 presents the average optimal solution for three different values of σ_1; an increase in the value of σ_1 indicates that the stochastic fluctuations in the growth dynamics are rising in magnitude. The impact of pronounced random effects is visible in Figure 4.2 which shows that, on an average, increased volatility implies that the population level is at an increased risk of falling. Consequently, the optimal policy recommends conservative harvesting and, therefore, the optimal effort declines with a rise in the growth volatility.

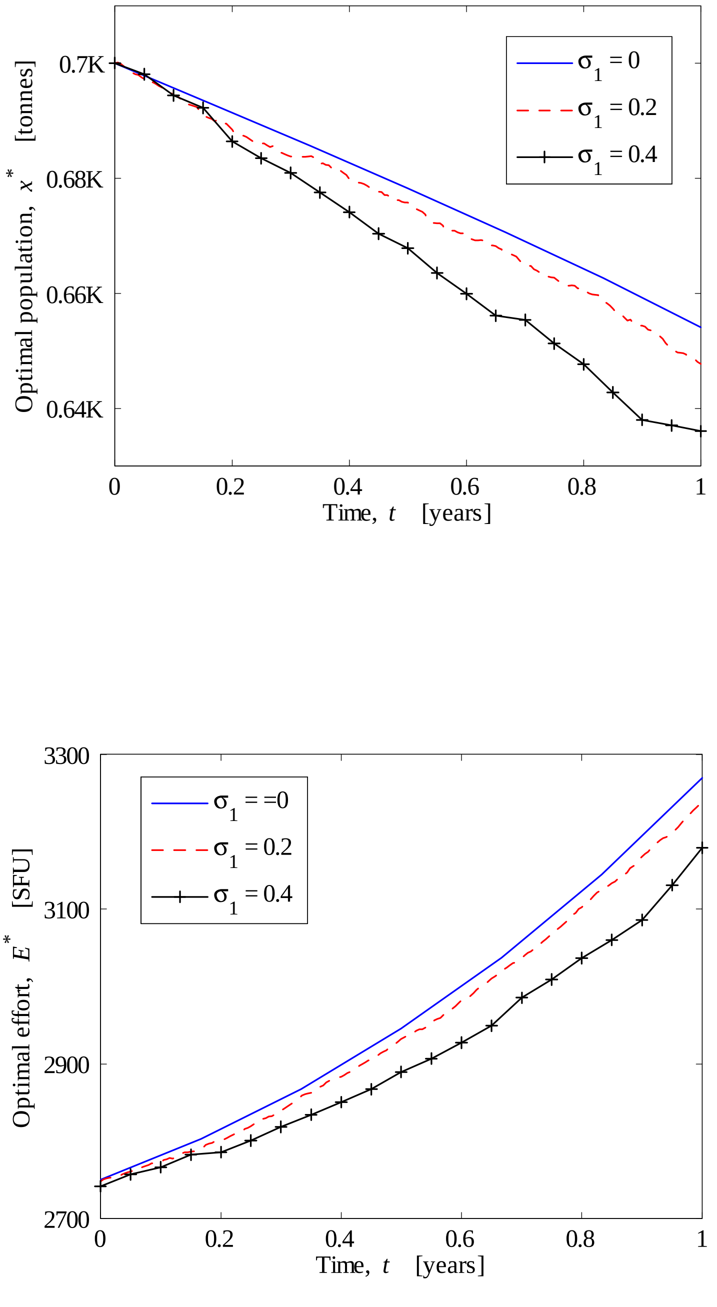


Figure 4.2: The effect of growth volatility σ_1 on a short-term optimal solution for the case of random growth and constant price. Top: optimal population path; Bottom: optimal effort. An increase in σ_1 results in diminished population growth and, consequently, decreased harvest.

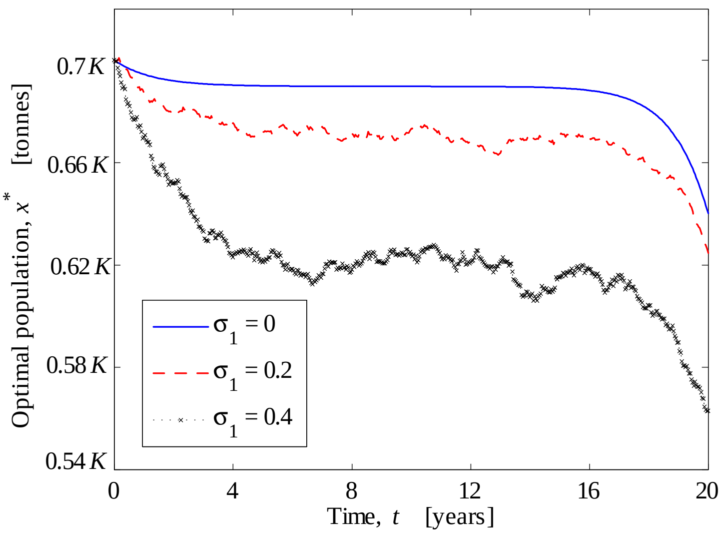
Effect of variance: long-term solution

In Section 3.7, we discussed infinite-horizon harvesting in a deterministic setting and observed that both optimal effort and the corresponding optimal population level approached a steady state; these steady states were denoted by E_∞ and X_∞ respectively. When the terminal time was infinite, the steady state was maintained eternally. For finite-horizon harvesting, provided the harvesting period was sufficiently long, the optimal effort and the optimal population level were seen to stabilize at their respective (above-mentioned) steady states for a while, before undergoing a change towards the end of the harvesting period. Thus the long-run optimal solution was different from the solution obtained for infinite-horizon only during the last few stages, and this difference was due to the finiteness of the terminal time in the former case. We now analyze the effect of growth volatility on the long-run optimal solution for finite-horizon harvesting. For this purpose, we fix $T = 20$ years and solve Equation (4.27), corresponding to stochastic growth and constant price, for different values of growth volatility σ_1.

Figure 4.3 demonstrates the mean optimal effort path and the mean optimal population growth for different values of σ_1; the parameter values are the same as given in Table 4.1, and the mean is taken over 2000 realizations of $dW_1(t)$. The initial population level is fixed at $0.7K$. The deterministic solution, which in essence corresponds to $\sigma_1 = 0$, is illustrated by the solid line. All of the other illustrations correspond to the optimal solution for the stochastic problem with different non-zero values for σ_1. Recall from Section 3.7 that the optimal steady-state solution associated with these parameter values is

$$X_\infty = 0.6924 \times K, \qquad E_\infty = 2183.5 \text{ SFU}.$$

In every solution for non-zero σ_1, the optimal effort and the optimal population appear to approach a steady state and stay more or less stable for a while. The optimal solution, however, is not completely stable and exhibits some fluctuations due to continuous disturbance (characterized by σ_1). As in the deterministic case, the optimal path is adjusted in the last few stages due to the fixed (finite) terminal time. Furthermore, the steady state solution associated with the stochastic problem changes as σ_1 changes. An increase in σ_1 lowers the steady-state value for the optimal stock level as well as for the optimal effort solution.



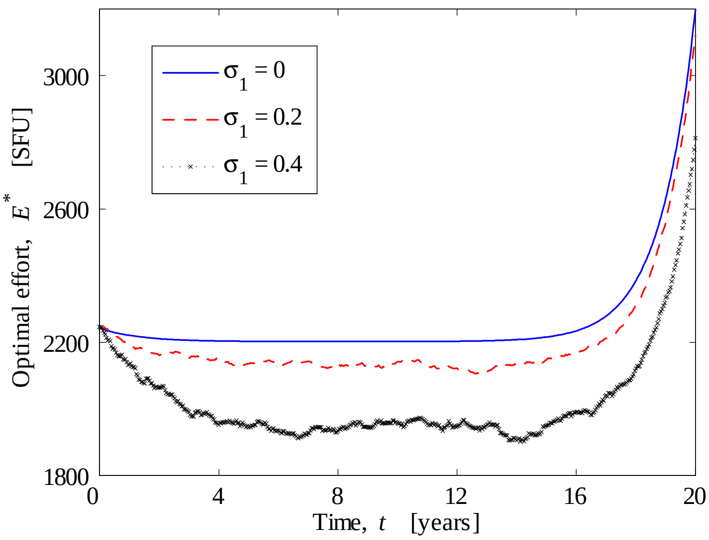


Figure 4.3: The effect of growth volatility σ_1 on a long-term optimal solution for the case of random growth and constant price. Top: optimal population growth; Bottom: optimal effort path. As σ_1 increases, the long-run steady-state values associated with the optimal effort and the optimal population level are lowered.

4.5 The optimal solution: random growth and random price

We now consider both fish growth and price per unit harvest to be stochastic. The optimal solution for this case is determined by simulating the Hamilton-Jacobi-Bellman equation (4.22). The numerical scheme is once again formulated using a Crank-Nicolson finite-difference method, as described in Appendix C. Figures 4.4 and 4.5 show the optimal solution for $\rho = 0$ and $\rho \neq 0$ respectively, with parameter values as given in Table 4.2. The dashed lines represent the optimal solution for three different realizations of Wiener processes, and the solid lines illustrate the average optimal solution paths for effort and population where the average is taken over 2000 realizations of $dW_1(t)$ and $dW_2(t)$.

Table 4.2: Parameter values for the simulation of the Hamilton-Jacobi-Bellman equation associated with the case of random growth and random price

Parameter	Description	Value	Unit
r	Intrinsic growth rate	0.71	year^{-1}
δ	Discount rate	0.12	year^{-1}
μ	Price drift	0.02	year^{-1}
q	Catchability coefficient	0.0001	SFU^{-1} year^{-1}
K	Biological carrying capacity	10^6	tonnes
$x_{\min}$	Minimum viable population level	$0.4K$	tonnes
p_0	Initial price	0.5	\$ tonne^{-1}
c_1	Linear cost coefficient	0.01	\$ SFU^{-1} year^{-1}
c_2	Quadratic cost coefficient	0.01	\$ SFU^{-2} year^{-1}
σ_1	Growth volatility	0.2	year$^{-1/2}$
σ_2	Price volatility	0.2	year$^{-1/2}$
ρ	Correlation between dW_1 and dW_2	-0.5	year^{-1}

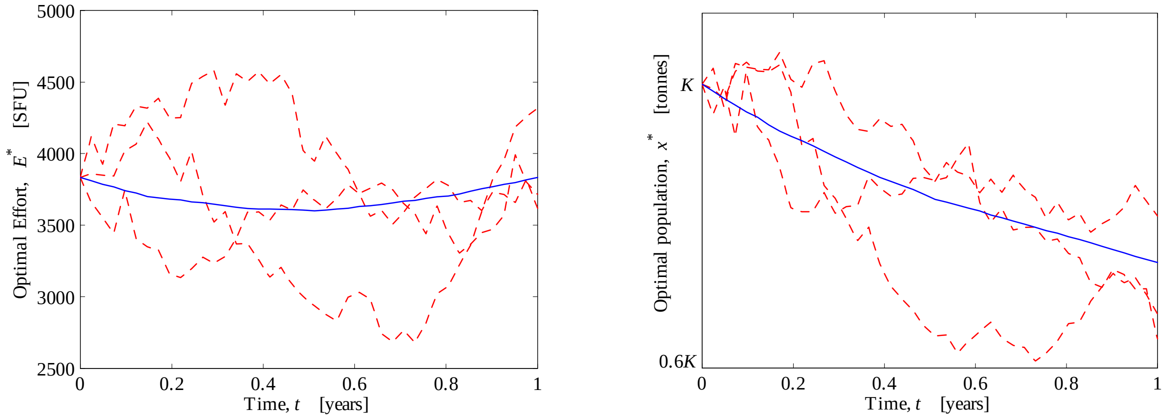


Figure 4.4: The optimal solution when both growth and price are random and $dW_1(t)$ and $dW_2(t)$ are uncorrelated, i.e., the correlation coefficient ρ is zero. Left: optimal effort solution; Right: optimal population growth.

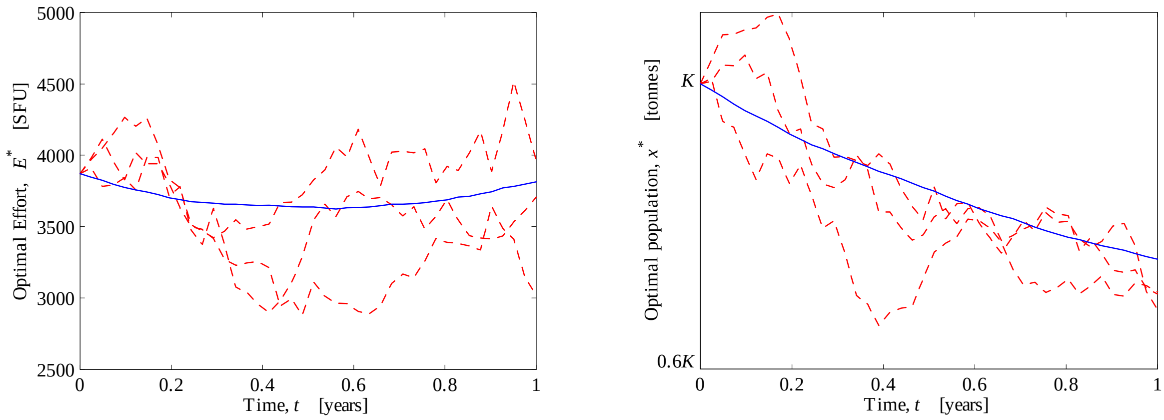


Figure 4.5: The optimal solution when both growth and price are random and $dW_1(t)$ and $dW_2(t)$ are correlated, i.e., the correlation coefficient ρ is non-zero; here $\rho = -0.5$. Left: optimal effort solution; Right: optimal population growth.

Note that we have fixed $\rho = -0.5$ to demonstrate a non-zero correlation between $dW_1(t)$ and $dW_2(t)$. We have assigned a negative sign to ρ because the stochastic fluctuations in growth and price are negatively correlated. For example, say the price increases as an effect of a random fluctuation; this makes harvesting more profitable and results in an increased harvest. The additional harvest leads to a drop in the population level. Thus, an unexpected rise in price results in an unexpected decline in stock. This indicates that there exists an inverse relationship between the random fluctuations in growth and price. As these random fluctuations for the growth and the price are represented by the Wiener increments $dW_1(t)$ and $dW_2(t)$ respectively, we therefore infer that $dW_1(t)$ and $dW_2(t)$ are negatively correlated. We notice that there is no significant difference between the optimal solutions corresponding to zero correlation and non-zero correlation, illustrated by Figures 4.4 and 4.5 respectively. As in the previous section, we now analyze the effect of growth and price variability on the average optimal solution where the average is always taken over 2000 realizations for the Wiener processes associated with stock growth and price dynamics.

Effect of variance: short-term solution

We first study the effect of a change in growth volatility and price volatility on the optimal solution associated with short-term harvesting. To accomplish this, we fix $T = 1$ year, the temporal step-size (dt) in the numerical scheme is $1/60th$ year. Figures 4.6 and 4.7 respectively illustrate the average optimal effort path and the corresponding average optimal population growth for different combinations of growth volatility σ_1 and price volatility σ_2.

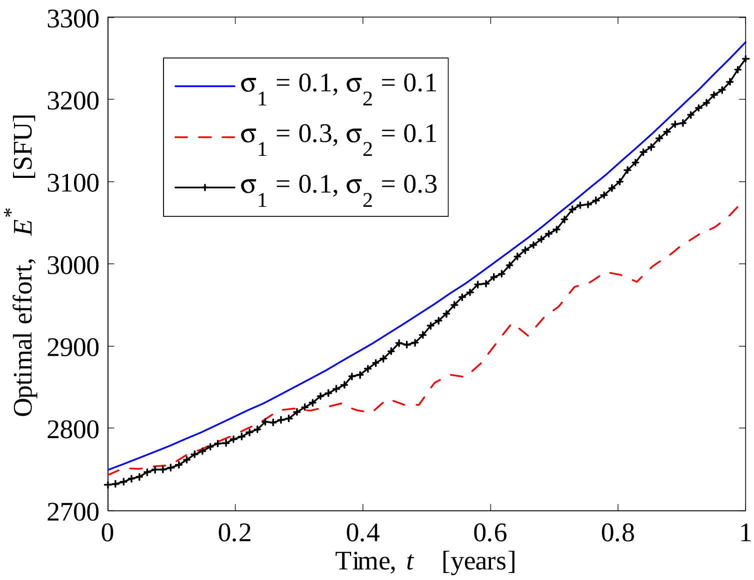


Figure 4.6: Effect of stochastic fluctuations in population and price on a short-term optimal effort solution for the case of random growth and random price. The increased variability results in decreased average optimal effort; the impact is more significant in the case of high growth variability.

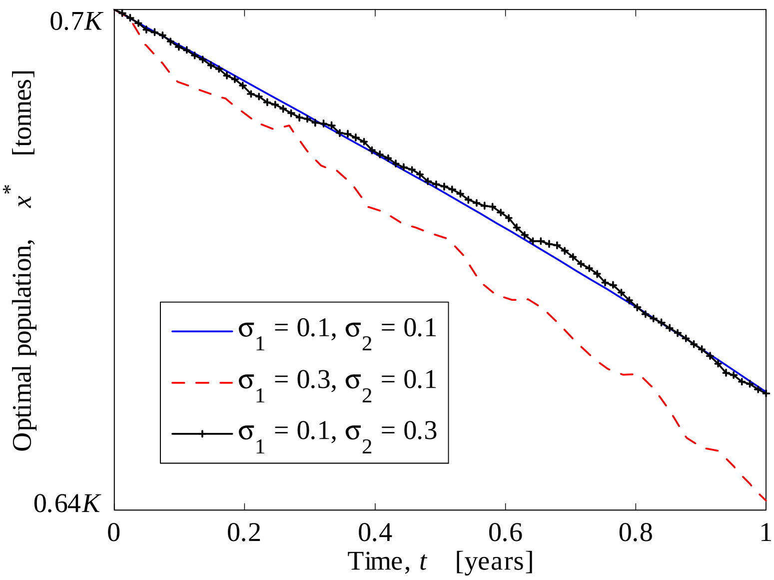


Figure 4.7: Effect of stochastic fluctuations in population and price on the mean optimal population path. On an average, a high growth volatility results in a pronounced decline in the optimal stock level whereas a high price volatility does not have a significant influence.

We notice that, on an average, the overall effect of increased variability is to shift the optimal effort and optimal population level downwards. However, a rise in σ_1 causes a bigger decline in the optimal effort solution than a rise in σ_2. Furthermore, for these parameter values, the increased price volatility has a negligible effect on the population growth whilst the increased growth volatility causes the stock level to drop substantially.

Effect of variance: long-term solution

Next we examine the influence of stochastic fluctuations in growth and price on the optimal solution for long-term harvesting; for this purpose, we fix $T = 20$ years and consider three different combinations of σ_1 and σ_2. Figure 4.8 presents the average optimal effort solution and the average optimal population growth. The qualitative behaviour of the mean optimal solution is similar to the mean optimal solution for the case of random growth and constant price, i.e., a rise in random effects lowers the steady state associated with the long-term optimal solution. However, as in the previous case, a rise in growth variability has a greater influence on the population growth than a rise in price variability.

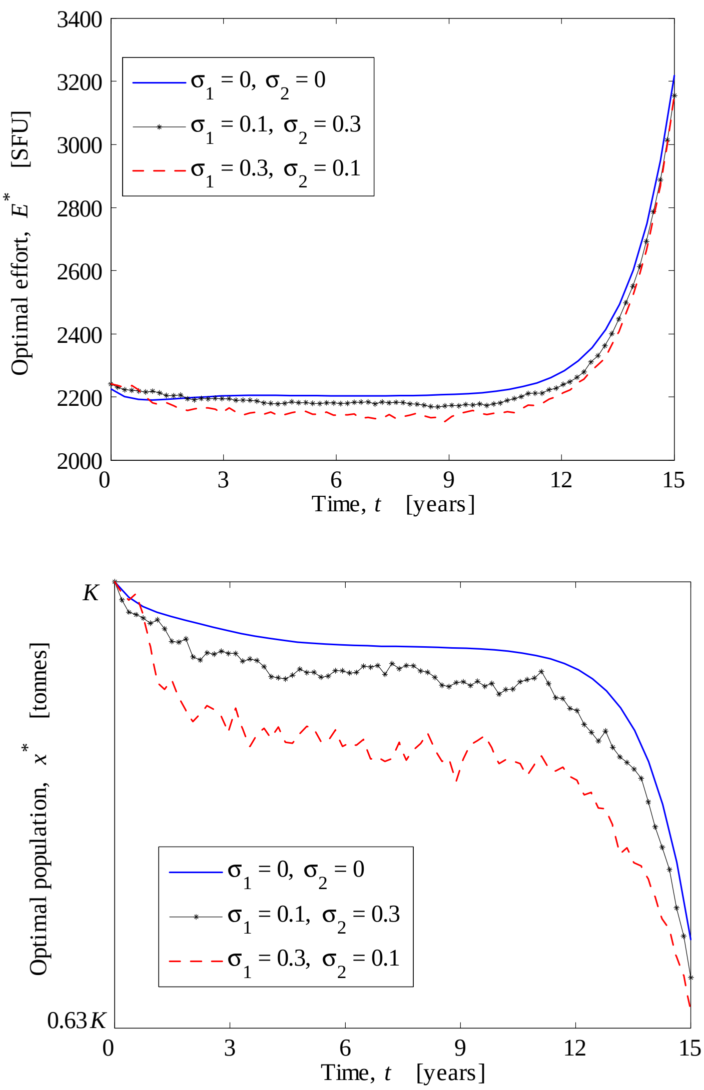


Figure 4.8: Effect of stochastic fluctuations in population and price on a long-term optimal solution for the case of random growth and random price. An increase in variability causes the average optimal effort as well as the average optimal population level to decline; similar to the short-term solution, the impact is more significant in the case of high growth volatility.

4.6 Sensitivity analysis

In Section 3.6 we studied the sensitivity of the net present value of total profit to the catchability q and the cost parameters c_1 and c_2. We now wish to perform the same analysis for the stochastic optimal harvesting problem with random growth and random price. The observations recorded here are based on the simulations of the Hamilton-Jacobi-Bellman equation (4.22); we consider a mean of 2000 realizations for the Wiener processes $dW_1(t)$ and $dW_2(t)$. As in the deterministic environment, we maintain the initial population level at K, the initial price at \$0.5/tonne, and fix $c_1 = \$0.01/\text{SFU}/\text{year}$ and $c_2 = \$0.01/\text{SFU}^2/\text{year}$ as the base case. The remaining parameter values are utilized from Table 4.1. We consider two scenarios: first where the catchability is low, demonstrated by fixing $q = 0.0001/\text{SFU}/\text{year}$; second where the catchability is high, demonstrated by fixing $q = 0.1/\text{SFU}/\text{year}$. In each scenario, we first record the expected net present value of total profit corresponding to the base case, then introduce variations in the values of and $c_1$and c_2 and record the corresponding expected net present value of total profit. The findings are summarized in Table 4.3 and discussed below.

Table 4.3: Sensitivity analysis of the expected total discounted profit with respect to the catchability and the cost parameters

q $[\text{SFU}^{-1}\ \text{year}^{-1}]$	c_1 $[\$\ \text{SFU}^{-1}\ \text{year}^{-1}]$	c_2 $[\$\ \text{SFU}^{-2}\ \text{year}^{-1}]$	$J^*(x(0) = K, 0)$ [\$]
0.0001	0.01	0.01	8.9364×10^4
	1	0.01	8.5946×10^4
	0.01	1	1.2310×10^3
0.1	0.01	0.01	2.5585×10^5
	1	0.01	2.5584×10^5
	0.01	1	2.5583×10^5

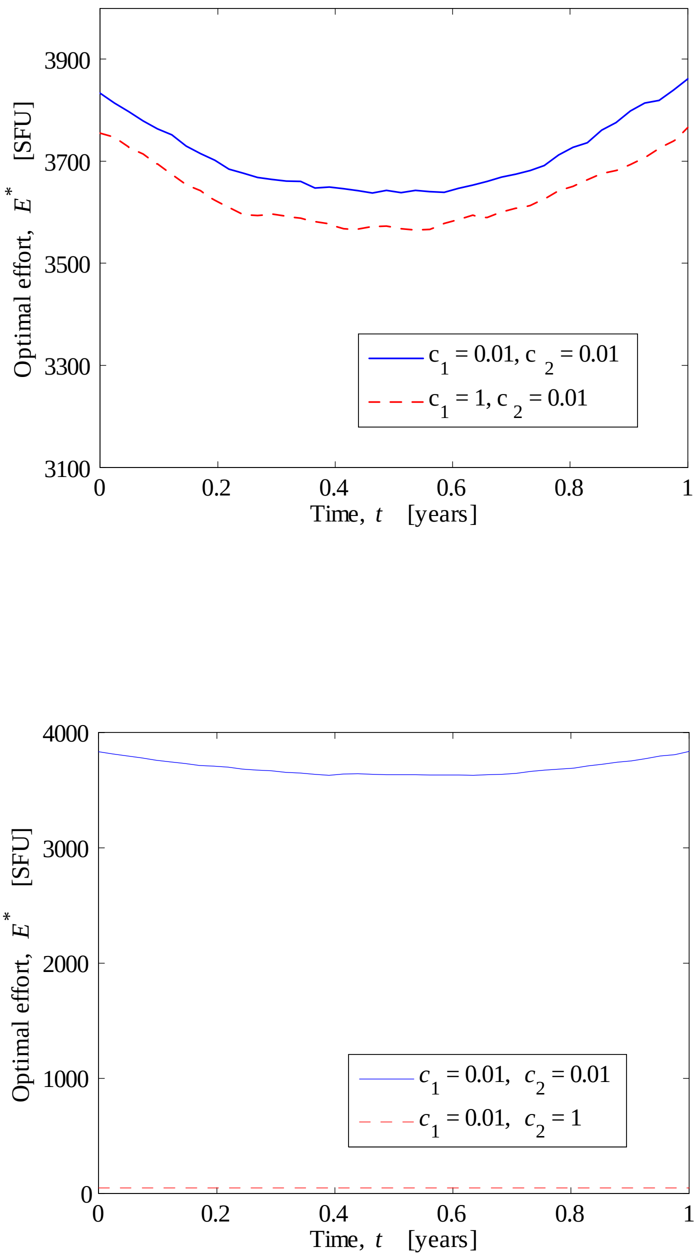


Figure 4.9: Effect of an increase in the cost parameters on the average optimal effort policy for a low catchability, demonstrated by fixing $q = 0.0001$/SFU/year. Top: c_2 is fixed while c_1 is allowed to vary; Bottom: c_1 is fixed while c_2 is allowed to vary. An increase in c_2 has a much greater impact on the optimal effort than an increase in c_1.

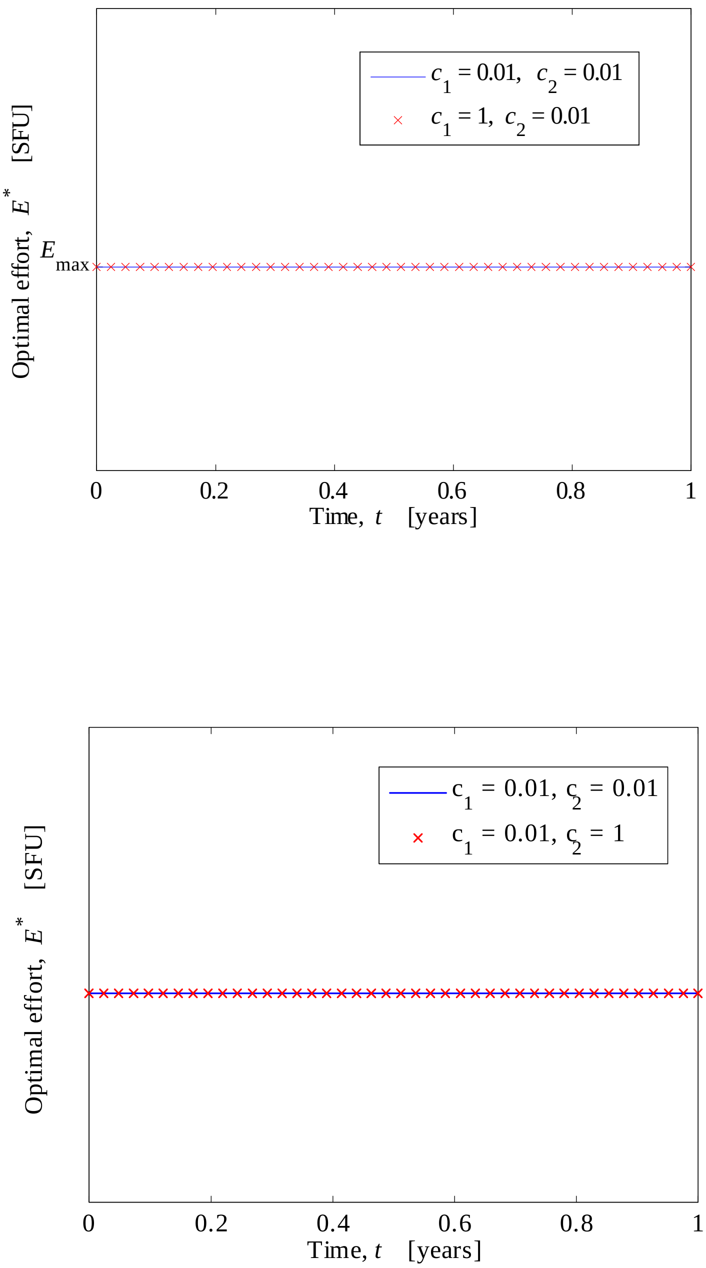


Figure 4.10: Effect of an increase in the cost parameters on the average optimal effort policy for a high catchability, demonstrated by fixing $q = 0.1/SFU/year$. Top: c_2 is fixed while c_1 is allowed to vary; Bottom: c_1 is fixed while c_2 is allowed to vary. The optimal effort stays at $E_{\max}$ in all the cases illustrated here.

For the lower value of q, a change in the linear cost-coefficient c_1 has a negligible effect on the expected net present value of the total profit J^*, whilst a similar change in the quadratic cost-coefficient c_2 has a pronounced effect; a significant drop is observed in the value of J^* following an increase in the value of c_2. The reason for this effect is illustrated by Figure 4.9 which shows the optimal solution under these circumstances. Evidently, a change in c_1 does not have a noticeable impact on the optimal solution, whereas a change in c_2 results in diminished (average) optimal effort and a consequent drop in the expected total discounted profit. On the other hand, it is optimal to harvest at full capacity when the catchability is high (see Figure 4.10). The optimal effort stays at $E_{\max}$ even after introducing a variation in the values for c_1 and c_2, consequently the expected total discounted profit stays more or less on the same level. Thus, neither of the cost coefficients has a considerable influence on the expected value of the total discounted profit for the case of high catchability. These observations are in accordance with the conclusions drawn in a deterministic environment (see Chapter 3).

4.7 Correlation between population and price

Next we focus on the correlation between fish price $p(t)$ and the population undergoing optimal harvest, $x^*(t)$. We assume that $dW_1(t)$ and $dW_2(t)$ are uncorrelated, and the fishery under consideration is relatively small so that the fish harvest has no effect on the world prices. Now the population dynamics, as given by Equation (4.1), include the effort $E(t)$. Furthermore, from Equation (4.23), the optimal effort $E^*(t)$ depends upon the price $p(t)$. This implies that the optimal population level is affected by the fish price through the optimal effort expended. The price dynamics given by Equation (4.3), however, are independent of the population level and the amount of effort exerted. By definition, the correlation between two random variables usually measures the strength of their dependency on each other. But here the stock level depends upon the price, whilst the price fluctuates randomly on its own. This indicates that the correlation between the population and the price is actually a measure of the dependency of the population on the price. The coefficient of correlation between the optimal population level and the price, denoted by $\rho_{x^*p}(t)$, can be calculated as

$$\rho_{x^*p}(t) = \frac{\mathcal{E}\left[x^*(t)p(t)\right] - \mathcal{E}\left[x^*(t)\right]\mathcal{E}\left[p(t)\right]}{\sqrt{Var\left[x^*(t)\right]}\sqrt{Var\left[p(t)\right]}} \tag{4.28}$$

As in Section 4.5, we use numerical methods to simulate Equation (4.22) and obtain 2000 realizations for $x^*(t)$ and $p(t)$ for $t \in [0,\ 1]$; the parameter values are taken from Table 4.2. Next, we calculate the expected value and the variance for both $x^*(t)$ and $p(t)$, and substitute them into Equation (4.28) to determine the correlation-coefficient $\rho_{x^*p}(t)$. Figure 4.11 illustrates the correlation between the population and the price during the harvest for various combinations of price and growth volatility. Some observations are discussed below.

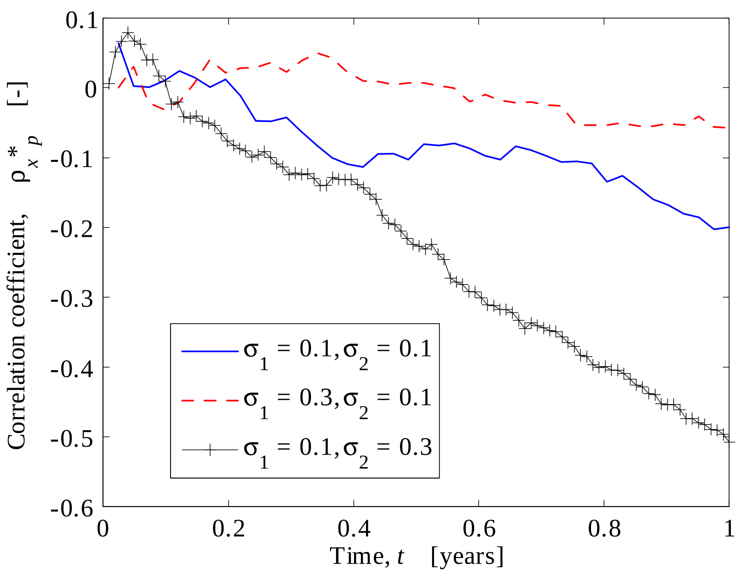


Figure 4.11: Correlation coefficient between fish price and the population undergoing optimal harvest for three different combinations of growth and price volatility. The correlation coefficient is seen to be mostly negative; it displays a considerable increase with a rise in price volatility whereas an increase in growth volatility does not produce a substantial effect.

The correlation-coefficient is mostly negative since an increase in price supports increased harvest which, in turn, brings the population level down. Thus, a rise in price has an adverse impact on stock level. It is further noted that the magnitude of the correlation-coefficient undergoes a high increase with an increase in price volatility, whereas an increase in growth volatility does not have a pronounced effect. This effect can be attributed to the assumption that stock level and effort have no influence on price. Since the price is independent of the population level, a change in growth volatility does not have a substantial effect on the correlation between stock and price. A high price-volatility, however, can introduce large variations in the stock growth through the effort expended, and these variations cause the magnitude of $\rho_{x^*p}(t)$ to increase considerably.

4.8 Summary

In this chapter we extended the deterministic model studied in Chapter 3 to its stochastic version by allowing the fish growth and the price per unit harvest to be stochastic processes. The optimal harvesting strategy was assumed to maximize the expected net present value of total profit earned during the harvesting period. Stochastic dynamic programming was employed to derive the Hamilton-Jacobi-Bellman partial differential equation describing the expected total discounted profit; the equation was solved numerically to determine the optimal policy.

We first restricted our study to the case of random growth and constant price. We discussed the mean optimal effort policy and the corresponding mean optimal population path for both short-term and long-term harvesting, and also investigated the effect of growth volatility on the mean optimal solution obtained in both cases. For short-term harvesting, we found that an increase in growth volatility caused the average optimal population level as well as the average optimal effort to decline. For long-term harvesting, on average, the optimal effort and the optimal population level stayed close to an equilibrium level for a while before the finiteness of the terminal time altered the optimal path. However, when close to the equilibrium level, the optimal solution was not completely stable and displayed fluctuations due to the presence of random effects. Furthermore, the equilibrium states associated with the optimal effort and the optimal population level shifted downwards as growth volatility increased.

Next, we studied the fully stochastic model where both stock growth and price per unit harvest were assumed to be random. Once again, we discussed the average optimal solution for short-term and long-term harvesting, and analyzed the sensitivity of the average optimal solution to various combinations of growth and price volatility. The observations were similar to those recorded for the case of random growth and constant price. Furthermore, as compared with price volatility, an increase in growth volatility resulted in a bigger drop in the average optimal solution.

Then we examined the sensitivity of the expected total discounted profit to the catchability coefficient q and the cost parameters c_1 and c_2. For a low value of q, the expected total discounted profit displayed a substantial drop with an increase in c_2, whereas an increase in c_1 did not have a significant influence on the value of the expected total discounted profit. For a high value of q, however, the expected total discounted profit displayed more or less the same value irrespective of the variations introduced in the values of c_1 and c_2.

Finally, we investigated the correlation between the stock level under optimal harvesting and the fish price. We noted that the correlation measured the dependency of the population level on price, and not *vice versa*, since the price was independent of the stock level. We found that the correlation coefficient was mostly negative. Furthermore, the magnitude of the correlation coefficient increased significantly with a rise in price volatility whilst a rise in growth volatility had a negligible effect.

Chapter 5

Harvesting as a real option

5.1 Introduction

Traditionally, the optimal harvesting strategy is defined in terms of fishing effort and is based on the *Expected Net Present Value* (ENPV) rule. The ENPV rule asserts that an investment project should be taken up only if the present value of the cash flows from the project is greater than or equal to the total costs associated with the project. For calculating the present value, future cash flows are discounted at a risk-adjusted discount rate reflecting the opportunity cost of capital; the opportunity cost is the expected rate of return that could be earned by undertaking another investment bearing a risk similar to the project under consideration. But, in practice, it is not always possible to correctly measure the opportunity cost of a project, and this undermines the validity of the ENPV rule as a decision-making tool for an investment opportunity.

The optimal policy obtained in Chapters 3 and 4 is based on the ENPV rule. Following this approach, the harvester calculates the expected flow of profit coming from harvesting and discounts it to the initial time in order to determine the expected net present value of the harvesting project; here, the discount rate is fixed arbitrarily. However, the above-mentioned calculations are based on the expected values of the stochastic variables underlying harvesting, and as the uncertainty is gradually resolved, these values may turn out to be different than expected. Therefore it might be beneficial for the harvester to change the initial decisions.

This option of altering the operating decisions during the life of the investment project is not considered in the ENPV rule; it treats the investment on a now-or-never basis and ignores the opportunity to wait before investing. Consequently, the managerial flexibility to revise later decisions is not accounted for. These issues have been discussed in great detail by a large number of researchers, who have put forward real options theory for evaluating capital investment projects.

Literature review

Real options are defined on capital investment projects exhibiting a possibility to alter the operational strategies, e.g. to defer, contract, shut down or abandon the investment during its life. Black & Scholes (1973) pioneered the option-pricing theory in finance. The books by Dixit & Pindyck (1994) and Trigeorgis (1996) provide a comprehensive description of real options along with their valuation techniques.

As pointed out in Dixit & Pindyck (1994), there are three main characteristics associated with a capital investment project which render evaluation using the ENPV rule misleading; these are: irreversibility; uncertainty; and an option to delay the investment decision. Most natural resource investments are irreversible because they are firm-specific. For instance, vessels and gear used for fish harvesting cannot be used in any other industry. There is also a high degree of uncertainty associated with output prices and stock growth. Furthermore, owning a harvesting opportunity is analogous to owning a financial call option. The owner of the call option can pay the strike price and exercise the option; (s)he has the right but not the obligation to do so. Similarly, the harvester can pay the associated costs (i.e. strike price) and exercise the harvesting option, thus acquiring the cash flows. But (s)he will harvest only if there is a profit to be earned, which implies that (s)he can delay harvesting. Hence it seems appropriate to apply financial techniques for valuing the harvesting option.

Real options theory incorporates two different valuation techniques; a detailed description with examples can be found in Insley & Wirjanto (2006). The first technique is *dynamic programming with a risk-adjusted (fixed) discount rate,* which implies that the rate of increase of total risk is constant with time (Trigeorgis, 1996), but this generally does not depict the true nature of the problem. Optimal tree harvesting was modeled as a real option by Insley (2002). The dynamic programming technique was employed and the harvesting decision was specified as an optimal stopping problem. The option value together with the optimal cutting time was determined numerically.

The second valuation technique in real options theory is *contingent claims analysis;* it involves building a risk-less portfolio consisting of the investment project under consideration along with other tradeable assets. Brennan & Schwartz (1985) used contingent claims analysis to derive the value of a mine with stochastic output prices. They first considered a cost function that was linear in mining rate, assuming that the mine had only two possible operating rates: zero when it was closed and the optimal rate when it was open. Switching from one operating rate to the other was assumed to incur costs. Following that, they discussed the case of convex (quadratic) costs, where the operating rate could vary continuously between zero and a fixed upper bound without incurring any expenses. They constructed a replicating portfolio with current value identical to the present value of the cash flow stream arising from the mining project; it was assumed that a futures market exists for the output commodity. Slade (2001) also used real options theory to evaluate mining investments.

Morck *et al.* (1989) used a real options approach to value forestry resources assuming stochastic inventories and prices. The forestry lease was valued as an option to harvest trees at the most profitable time. Assuming costs to be quadratic in harvest rate, they derived the partial differential equation governing the option-value and solved it numerically for the value of the forest and the cutting rate. Murillas & Chamorro (2006) developed a real options model to determine the value of the exploitation and the investment opportunity in a fishery with deterministic growth and stochastic price. They noted that a futures market does not exist for fish, therefore they considered a replicating portfolio consisting of existing assets spanning the risk in the output price. To accomplish this, they introduced a twin asset with a price that was perfectly correlated with the output price; the harvesting option was evaluated by building a portfolio consisting of the fishery and this twin asset.

None of the aforementioned models possessed an analytical solution but all could be solved by employing numerical techniques. A discussion on numerical methods and their application to real options problems is provided in Hull & White (1990), Geske & Shastri (1985) and Cortazar *et al.* (1998).

In this chapter we extend the model examined by Murillas & Chamorro (2006) and allow the fish stock to evolve stochastically. In order to overcome the restrictions imposed by the risk-adjusted discount rate approach of dynamic programming, we focus on a contingent claims approach. The chapter is organized as follows: In Section 5.2 the fish-harvesting problem is formulated using contingent claims analysis; this yields a partial differential equation determining the option value of harvesting. The partial differential equation is solved numerically, and comparisons with the optimal solution obtained by employing the ENPV rule are made in Section 5.3. Following that, a sensitivity analysis of the optimal solution with respect to various biological and economic parameters is presented. Section 5.4 provides a summary.

5.2 A real options model for harvesting

In this section we consider both population growth and fish price to be stochastic. As before, the growth dynamics for the fish stock is given by

$$dx(t) = \left\{ rx(t)\left(1 - \frac{x(t)}{K}\right) - qE(t)x(t) \right\} dt + \sigma_1 x(t) dW_1(t), \tag{5.1}$$

and the price dynamics follows

$$dp(t) = \mu_p p(t)dt + \sigma_2 p(t) dW_2(t),$$

where the notation has its usual interpretation (see Section 4.2) and we have again assumed that the fishery's harvest has no effect on world prices.

We normalize the population by defining $X(t) = \frac{x(t)}{K}$; then the growth equation (5.1) can be rewritten as

$$dX(t) = \left(rX(t)\left(1 - X(t)\right) - qE(t)X(t)\right) dt + \sigma_1 X(t) dW_1(t).$$

The *convenience yield* per unit of fish held is denoted by $Y(p(t))$. As suggested by Murillas & Chamorro (2006), it can be defined as the flow of net benefits provided by a stored unit of stock. In essence, convenience yield is like a dividend accrued to the owner of an asset, but not to the owner of a derivative contract on that asset. The harvester loses convenience yield if (s)he does not exercise the harvesting option; thus convenience yield acts like an additional opportunity cost for postponing the harvesting decision. We assume that the convenience yield is proportional to the fish price, i.e.,

$$Y(p(t)) = yp(t),$$

where y is a constant. This assumption has been adopted by a number of researchers, including Brennan & Schwartz (1985), Morck *et al.* (1989), and Murillas & Chamorro (2006), primarily for simplicity and tractability.

As noted by Murillas & Chamorro (2006), there does not exist a futures market for fish. Therefore we assume that the risk in the fish price is spanned by a combination of existing assets. This relies on a further assumption that the market is sufficiently complete, which means that we can find a tradeable asset whose stochastic changes exactly replicate the realizations for the Wiener process governing the fish price. We denote the price of this spanning asset by $S(t)$ and the expected return by μ_s (a constant). The spanning asset's price is assumed to evolve according to the following Itô-type stochastic differential equation

$$dS(t) = \mu_s S(t)dt + \sigma_s S(t)dW_2(t), \tag{5.2}$$

where σ_s (a constant) is the diffusion component; the risk for the fish price and the risk for the spanning asset are the same, both represented by $dW_2(t)$.

We assume that $\mu_s \geq \mu_p$ and present the following argument to justify our assumption. Given the opportunity, the harvester can either undertake harvesting or postpone it. If (s)he decides to wait, and in the meantime invests in the spanning asset, then the harvester's money grows at rate μ_s. On the other hand, if (s)he goes ahead with harvesting then the return from $p(t)$ must be discounted at rate μ_s, the reason being that the return from a project has to be discounted at a rate which mirrors its opportunity cost. In this case, the return from $S(t)$

would be considered as an opportunity cost of harvesting. Having $\mu_s < \mu_p$ implies a low discount rate, leading to the conclusion that the future is equally important. This in turn would mean that the harvester is better off waiting and, consequently, the harvesting option will never be exercised. Therefore our assumption, $\mu_s \geq \mu_p$, is reasonable.

We denote the returns from $p(t)$ and $S(t)$ by r_p and r_s respectively. The return from $p(t)$ is equal to the drift plus the return due to convenience yield. Hence we can write

$$r_p = \mu_p + y,$$

and the return from $S(t)$ is

$$r_s = \mu_s.$$

We can further assume that the diffusion component σ_s of the spanning asset is equal to the diffusion component σ_p of the fish price $p(t)$ (Murillas & Chamorro, 2006). If $\sigma_s \neq \sigma_2$, we can choose a riskfree asset and combine it with the spanning asset in such a proportion that the resulting asset has diffusion component equal to σ_2. Furthermore, using Capital Asset Pricing Model (described in textbooks like Brailsford *et al.*, 2006)

$$\mu_p + y = \mu_s. \tag{5.3}$$

We have assumed that $\mu_s \geq \mu_p$, therefore Equation (5.3) yields $y \geq 0$.

Following the intertemporal asset pricing model introduced by Merton (1973), the other assumptions underlying the capital market are:

- The assets are perfectly divisible, and trading in assets takes place continuously in time.
- There is no arbitrage.
- There are no transaction costs or taxes associated with the assets traded in the market.

As specified by Murillas & Chamorro (2006), we further assume that harvesting, once initiated, is perpetual and the effort can vary without costs between zero and an upper bound $E_{\max}$.

The case of zero correlation

We first consider the case when the two Wiener increments, $dW_1(t)$ and $dW_2(t)$, are uncorrelated so that $\mathcal{E}\left[dW_1(t)dW_2(t)\right] = 0$. We let $J(X(t), p(t), t)$ denote the option value of harvesting at time t; the change in the option value during the time interval dt is given by Itô's lemma (see Section 2.1) as

$$\begin{aligned} dJ \quad = \quad & \left[\frac{dJ}{dt} + \mu_p p(t)\frac{\partial J}{\partial p} + \left(rX(t)\left(1 - X(t)\right) - qE(t)X(t)\right)\frac{\partial J}{\partial X} \right. \\ & \left. + \frac{\sigma_2^2}{2}p(t)^2\frac{\partial^2 J}{\partial p^2} + \frac{\sigma_1^2}{2}X(t)^2\frac{\partial^2 J}{\partial X^2}\right] dt \\ & + \sigma_1 X(t)\frac{\partial J}{\partial X}dW_1(t) + \sigma_2 p(t)\frac{\partial J}{\partial p}dW_2(t). \end{aligned} \tag{5.4}$$

To evaluate the harvesting option, we construct a replicating portfolio by buying one unit of the harvesting option and selling short n units of the spanning asset $S(t)$. We assume that $dW_1(t)$ is a biological phenomenon independent of any market, which means that the risk depicted by $dW_1(t)$ cannot be hedged out. We hedge out the risk depicted by $dW_2(t)$ by using the replicating portfolio. For the hedging to be successful, the portfolio should be risk-free, in other words, the risk represented by $dW_2(t)$ should be eliminated. For this to happen, the component of risk in dJ corresponding to $dW_2(t)$, must cancel out the component of risk in ndS. Equating the coefficient of $dW_2(t)$ in Equation (5.4) with its counterpart in n times Equation (5.2) we obtain

$$\sigma_2 p(t)\frac{\partial J}{\partial p} = n\sigma_s S(t).$$

Since $\sigma_2 = \sigma_s$, the above relation reduces to

$$p(t)\frac{\partial J}{\partial p} = nS(t). \tag{5.5}$$

We calculate the total return from the portfolio as follows: We own one unit of the harvesting option in our portfolio and the return from the harvesting option, in time interval dt, is a sum of two components: the expected capital gain dJ, and the cash flow from harvest which is $\left(p(t)qKX(t) - c_1 - \frac{c_2}{2}E(t)\right)E(t)dt$. Additionally, we are selling n units of the spanning asset and these carry an expected return equal to ndS in time span dt. The expected return from the replicating portfolio is the difference between the expected return from the harvesting option and the expected return from n units of the spanning asset. The risk-free return from the portfolio over time interval dt is given by $\lambda(J - nS(t))dt$, where λ denotes the risk-free rate of interest. To avoid arbitrage, the risk-free return must be equal to the total expected return from the portfolio. Equivalently,

$$\lambda(J - nS(t))dt = \mathcal{E}\left[dJ - ndS\right] + \left(p(t)qKX(t) - c_1 - \frac{c_2}{2}E(t)\right)E(t)dt. \tag{5.6}$$

Substituting for the expected values in Equation (5.6) and dividing throughout by dt leads to

$$\begin{aligned}\lambda(J - nS(t)) &= \frac{dJ}{dt} + \left(rX(t)\left(1 - X(t)\right) - qE(t)X(t)\right)\frac{\partial J}{\partial X} + \mu_p p(t)\frac{\partial J}{\partial p} \\ &\quad + \frac{\sigma_1^2}{2}X(t)^2\frac{\partial^2 J}{\partial X^2} + \frac{\sigma_2^2}{2}p(t)^2\frac{\partial^2 J}{\partial p^2} \\ &\quad + \left(p(t)qKX(t) - c_1 - \frac{c_2}{2}E(t)\right)E(t) - \mu_s nS(t).\end{aligned}$$

Making use of Equations (5.3) and (5.5) gives

$$\begin{aligned}0 &= \frac{dJ}{dt} - \lambda J + \left(rX(t)\left(1 - X(t)\right) - qE(t)X(t)\right)\frac{\partial J}{\partial X} \\ &\quad + (\lambda - y)p(t)\frac{\partial J}{\partial p} + \frac{\sigma_1^2}{2}X(t)^2\frac{\partial^2 J}{\partial X^2} + \frac{\sigma_2^2}{2}p(t)^2\frac{\partial^2 J}{\partial p^2} \\ &\quad + \left(p(t)qKX(t) - c_1 - \frac{c_2}{2}E(t)\right)E(t).\end{aligned} \tag{5.7}$$

Equation (5.7) must be satisfied for each possible harvesting strategy. Therefore the value-maximizing harvesting policy can be determined by solving

$$\begin{aligned}0 &= \max_{E(t)}\left[\frac{dJ}{dt} - \lambda J + \left(rX(t)\left(1 - X(t)\right) - qE(t)X(t)\right)\frac{\partial J}{\partial X}\right. \\ &\quad + (\lambda - y)p(t)\frac{\partial J}{\partial p} + \frac{\sigma_1^2}{2}X(t)^2\frac{\partial^2 J}{\partial X^2} + \frac{\sigma_2^2}{2}p(t)^2\frac{\partial^2 J}{\partial p^2} \\ &\quad \left. + \left(p(t)qKX(t) - c_1 - \frac{c_2}{2}E(t)\right)E(t)\right],\end{aligned}$$

or alternatively,

$$\begin{aligned}-\frac{dJ}{dt} &= \max_{E(t)}\left[-\lambda J + \left(rX(t)\left(1 - X(t)\right) - qE(t)X(t)\right)\frac{\partial J}{\partial X}\right. \\ &\quad + (\lambda - y)p(t)\frac{\partial J}{\partial p} + \frac{\sigma_1^2}{2}X(t)^2\frac{\partial^2 J}{\partial X^2} + \frac{\sigma_2^2}{2}p(t)^2\frac{\partial^2 J}{\partial p^2} \\ &\quad \left. + \left(p(t)qKX(t) - c_1 - \frac{c_2}{2}E(t)\right)E(t)\right],\end{aligned} \tag{5.8}$$

where the optimal effort lies within $[0,\ E_{\max}]$. This is the partial differential equation, the solution to which is sought for the option value of harvesting corresponding to the case of zero correlation between $dW_1(t)$ and $dW_2(t)$.

The case of non-zero correlation

Next we assume that the two Wiener increments, $dW_1(t)$ and $dW_2(t)$, are correlated. We follow the same approach as described in Section 4.3 and suppose that the relation between $dW_1(t)$ and $dW_2(t)$ is given by

$$dW_1(t) = \rho dW_2(t) + \sqrt{1-\rho^2}dW_3(t) \tag{5.9}$$

where $-1 \leq \rho \leq 1$; where $dW_3(t)$ is a standard Wiener increment and $dW_2(t)$ and $dW_3(t)$ are uncorrelated. Recall that the correlation-coefficient between $dW_1(t)$ and $dW_2(t)$ is calculated as

$$\begin{aligned} Corr(dW_1(t), dW_2(t)) &= \frac{\mathcal{E}\left[dW_1(t)dW_2(t)\right] - \mathcal{E}\left[dW_1(t)\right]\mathcal{E}\left[dW_2(t)\right]}{\sqrt{Var(dW_1(t)}\sqrt{Var\left(dW_2(t)\right)}} \\ &= \frac{\rho dt}{dt} = \rho \end{aligned}$$

We further assume that the return from $p(t)$ is uncorrelated with the market portfolio. Then the Capital Asset Pricing Model asserts that the total return, r_p, from $p(t)$ is equal to the risk-free rate of return λ, which gives

$$\mu_p + y = \lambda. \tag{5.10}$$

This is the *risk-neutral valuation* technique followed by standard option pricing literature (e.g. Hull, 2005); here the market price of risk is assumed to be zero. In this case, the value of the harvesting option, represented by $J(X(t), p(t), t)$, evolves according to the stochastic differential equation

$$\begin{aligned} dJ &= \left[\frac{dJ}{dt} + \mu_p p(t)\frac{\partial J}{\partial p} + \left(rX(t)\left(1-X(t)\right) - qE(t)X(t)\right)\frac{\partial J}{\partial X}\right. \\ &\quad \left. + \frac{\sigma_2^2}{2}p(t)^2\frac{\partial^2 J}{\partial p^2} + \frac{\sigma_1^2}{2}X(t)^2\frac{\partial^2 J}{\partial X^2} + \rho\sigma_1 X(t)\sigma_2 p(t)\frac{\partial^2 J}{\partial X \partial p}\right]dt \\ &\quad + \sigma_1 X(t)\frac{\partial J}{\partial X}dW_1(t) + \sigma_2 p(t)\frac{\partial J}{\partial p}dW_2(t). \end{aligned} \tag{5.11}$$

Substituting for $dW_1(t)$ and μ_p from Equations (5.9) and (5.10) respectively, Equation (5.11) can be reformulated as

$$\begin{aligned} dJ &= \left[\frac{dJ}{dt} + (\lambda - y)p(t)\frac{\partial J}{\partial p} + \left(rX(t)\left(1 - X(t)\right) - qE(t)X(t)\right)\frac{\partial J}{\partial X}\right. \\ &\quad \left. + \frac{\sigma_2^2}{2}p(t)^2\frac{\partial^2 J}{\partial p^2} + \frac{\sigma_1^2}{2}X(t)^2\frac{\partial^2 J}{\partial X^2} + \rho\sigma_1 X(t)\sigma_2 p(t)\frac{\partial^2 J}{\partial X \partial p}\right] dt \\ &\quad + \sigma_1 X(t)\frac{\partial J}{\partial X}[\rho dW_2(t) + \sqrt{1-\rho^2}dW_3(t)] + \sigma_2 p(t)\frac{\partial J}{\partial p}dW_2(t). \end{aligned} \tag{5.12}$$

We again apply contingent claims analysis to find the value of the harvesting opportunity. This is achieved by constructing a replicating portfolio where we buy one unit of the harvesting option and sell short n units of the spanning asset $S(t)$. Here we assume that $dW_3(t)$ is a biological phenomenon, independent of any market, and hedge out the risk depicted by $dW_2(t)$. Equating the coefficients of $dW_2(t)$ in Equation (5.12) and n times Equation (5.2) yields

$$\rho\sigma_1 X(t)\frac{\partial J}{\partial X} + \sigma_2 p(t)\frac{\partial J}{\partial p} = n\sigma_s S(t). \tag{5.13}$$

Using $\sigma_2 = \sigma_s$, Equation (5.13) reduces to

$$\rho\frac{\sigma_1}{\sigma_2}X(t)\frac{\partial J}{\partial X} + p(t)\frac{\partial J}{\partial p} = nS(t). \tag{5.14}$$

To avoid arbitrage, we equate the total expected return from the portfolio to the risk-free return $\lambda(J - nS(t))dt$; this gives

$$\lambda(J - nS(t))dt = \mathcal{E}\left[dJ - ndS\right] + \left(p(t)qKX(t) - c_1 - \frac{c_2}{2}E(t)\right)E(t)dt. \tag{5.15}$$

Then, substituting for the expected values and using the relation (5.10) in Equation (5.15), we obtain

$$\begin{aligned} \lambda(J - nS(t)) &= \frac{dJ}{dt} + \left(rX(t)\left(1 - X(t)\right) - qE(t)X(t)\right)\frac{\partial J}{\partial X} + (\lambda - y)p(t)\frac{\partial J}{\partial p} \\ &\quad + \frac{\sigma_1^2}{2}X(t)^2\frac{\partial^2 J}{\partial X^2} + \frac{\sigma_2^2}{2}p(t)^2\frac{\partial^2 J}{\partial p^2} + \rho\sigma_1 X(t)\sigma_2 p(t)\frac{\partial^2 J}{\partial X\partial p} \\ &\quad + \left(p(t)qKX(t) - c_1 - \frac{c_2}{2}E(t)\right)E(t) - \mu_s nS(t). \end{aligned} \tag{5.16}$$

Using Equation (5.14), we then get

$$\begin{aligned}
\lambda J &= (\lambda-\mu_s)\left(\rho\frac{\sigma_1}{\sigma_2}X(t)\frac{\partial J}{\partial X}+p(t)\frac{\partial J}{\partial p}\right)+\frac{dJ}{dt} \\
&+(\lambda-y)p(t)\frac{\partial J}{\partial p}+\left(rX(t)\left(1-X(t)\right)-qE(t)X(t)\right)\frac{\partial J}{\partial X} \\
&+\frac{\sigma_1^2}{2}X(t)^2\frac{\partial^2 J}{\partial X^2}+\frac{\sigma_2^2}{2}p(t)^2\frac{\partial^2 J}{\partial p^2}+\rho\sigma_1 X(t)\sigma_2 p(t)\frac{\partial^2 J}{\partial X\partial p} \\
&+\left(p(t)qKX(t)-c_1-\frac{c_2}{2}E(t)\right)E(t). \qquad (5.17)
\end{aligned}$$

Substituting for μ_s from Equation (5.3) into Equation (5.17) leads to

$$\begin{aligned}
0 &= \frac{dJ}{dt}-\lambda J+\left(rX(t)\left(1-X(t)\right)-qE(t)X(t)\right)\frac{\partial J}{\partial X} \\
&+\left(\lambda-y\right)p(t)\frac{\partial J}{\partial p}+\frac{\sigma_1^2}{2}X(t)^2\frac{\partial^2 J}{\partial X^2}+\frac{\sigma_2^2}{2}p(t)^2\frac{\partial^2 J}{\partial p^2} \\
&+\left(p(t)qKX(t)-c_1-\frac{c_2}{2}E(t)\right)E(t)+\rho\sigma_1 X(t)\sigma_2 p(t)\frac{\partial^2 J}{\partial X\partial p}. \qquad (5.18)
\end{aligned}$$

The value of the harvesting option satisfies Equation (5.18) for each feasible harvesting strategy, therefore the problem of finding the optimal harvesting policy maximizing the option value can be stated as

$$\begin{aligned}
0 &= \max_{E(t)}\left[\frac{dJ}{dt}-\lambda J+\left(rX(t)\left(1-X(t)\right)-qE(t)X(t)\right)\frac{\partial J}{\partial X}\right. \\
&+(\lambda-y)p(t)\frac{\partial J}{\partial p}+\frac{\sigma_1^2}{2}X(t)^2\frac{\partial^2 J}{\partial X^2}+\frac{\sigma_2^2}{2}p(t)^2\frac{\partial^2 J}{\partial p^2} \\
&\left.+\left(p(t)qKX(t)-c_1-\frac{c_2}{2}E(t)\right)E(t)+\rho\sigma_1 X(t)\sigma_2 p(t)\frac{\partial^2 J}{\partial X\partial p}\right], \qquad (5.19)
\end{aligned}$$

where the effort is constrained according to $0 \leq E(t) \leq E_{\max}$.

Restructuring Equation (5.19) yields

$$\begin{aligned}
-\frac{dJ}{dt} &= \max_{E(t)}\left[-\lambda J+\left(rX(t)\left(1-X(t)\right)-qE(t)X(t)\right)\frac{\partial J}{\partial X}+(\lambda-y)p(t)\frac{\partial J}{\partial p}\right. \\
&+\frac{\sigma_1^2}{2}X(t)^2\frac{\partial^2 J}{\partial X^2}+\frac{\sigma_2^2}{2}p(t)^2\frac{\partial^2 J}{\partial p^2} \\
&\left.+\left(p(t)qKX(t)-c_1-\frac{c_2}{2}E(t)\right)E(t)+\rho\sigma_1 X(t)\sigma_2 p(t)\frac{\partial^2 J}{\partial X\partial p}\right], \qquad (5.20)
\end{aligned}$$

which is the partial differential equation determining the value of the harvesting opportunity in the case of non-zero correlation between $dW_1(t)$ and $dW_2(t)$.

Value-maximizing harvesting policy

Ignoring the term containing the correlation-coefficient ρ in Equation (5.20), we see that Equation (5.20) is the same as Equation (5.8), obtained for zero correlation. Hence the solution for the case of zero correlation can be deduced by simulating Equation (5.20) with $\rho = 0$. Therefore, we concentrate on Equation (5.20) for determining the value-maximizing harvesting strategy and the option value of harvesting. Denoting the control switching term in Equation (5.20) by D we have

$$D = \max_{E(t)} \left[\left(pqKX(t) - c_1 - \frac{c_2}{2} E(t) \right) E(t) - \frac{\partial J^*(X(t), t)}{\partial X} qE(t)X(t) \right].$$

The unconstrained optimal harvesting strategy, $E^*_{uc}(t)$, is obtained by setting the partial derivative $\frac{\partial D}{\partial E} = 0$ which gives

$$\begin{aligned} &(pqKX(t) - c_1 - c_2 E^*_{uc}(t)) - \frac{\partial J^* X((t), t)}{\partial X} qX(t) = 0 \\ &\Longrightarrow E^*_{uc}(t) = \left(p - \frac{\partial J^*(x(t), t)}{\partial X} \frac{1}{K} \right) \frac{qKX(t)}{c_2} - \frac{c_1}{c_2}. \end{aligned}$$

The constrained optimal effort is denoted by $E^*(t)$ and lies within $[0,\ E_{\max}]$. Using the optimal effort $E^*(t)$ in place of $E(t)$, Equation (5.20) transforms to

$$\begin{aligned} -\frac{dJ}{dt} \quad = \quad & \left(p(t)qKX(t) - c_1 - \frac{c_2}{2} E^*(t) \right) E^*(t) + (\lambda - y)p(t) \frac{\partial J}{\partial p} \\ & - \lambda J + (rX(t)(1 - X(t)) - qE^*(t)X(t)) \frac{\partial J}{\partial X} \\ & + \frac{\sigma_1^2}{2} X(t)^2 \frac{\partial^2 J}{\partial X^2} + \frac{\sigma_2^2}{2} p(t)^2 \frac{\partial^2 J}{\partial p^2} + \rho \sigma_1 X(t) \sigma_2 p(t) \frac{\partial^2 J}{\partial X \partial p} \end{aligned} \tag{5.21}$$

where

$$E^*(t) = \begin{cases} 0, & E^*_{uc}(t) < 0, \\ \left(p(t) - \frac{\partial J^*(X(t),t)}{\partial X(t)} \frac{1}{K} \right) \frac{qKX(t)}{c_2} - \frac{c_1}{c_2}, & 0 \le E^*_{uc}(t) \le E_{\max}, \\ E_{\max}, & E^*_{uc}(t) > E_{\max}. \end{cases} \tag{5.22}$$

Equation (5.21) is similar to the Hamilton-Jacobi-Bellman equation (4.22) describing the discounted flow of profit for stochastic growth and price; there are two differences as follows:

1. The risk-free rate λ in Equation (5.21) is replaced by the arbitrary discount rate δ in Equation (C.2).

2. The difference between the risk-free rate λ and the convenience yield y in Equation (5.21) is replaced by the price-drift, μ_p, in Equation(C.2) .

The boundary conditions associated with the problem can be specified as follows:

- If the price $p(t)$ is zero then the value of the fishery will also fall to zero, which gives

$$J(X, 0, t) = 0. \tag{5.23}$$

- If the stock falls to the minimum viable level $x_{\min}$ then there will be no harvesting. Consequently,

$$E^*(x_{\min}, p, t) = 0. \tag{5.24}$$

- At the final time T, the harvesting option cannot be delayed any further and therefore the option value is again zero. In other words,

$$J(X, p, 0) = 0. \tag{5.25}$$

Equations (5.21)-(5.25) represent a general model for the value of harvesting opportunity. This system has to be solved numerically due to the complexity and non-linearity involved; the numerical scheme is along the same lines as that for the Bellman equation (C.2).

5.3 Numerical results and discussion

Every solution illustrated in this section is a mean taken over 2000 realizations of the random variables included in stock and price evolution. Furthermore, J^* represents the value of the harvesting opportunity for contingent claims analysis, and the expected net present value of total profit for the ENPV rule. For the options approach, the time t marks the beginning of the options period. Therefore, $T - t$ is the time left before exercising the harvesting option. For the ENPV approach, t represents the time at which harvesting is initiated. The initial population level is always fixed at ninety percent of the carrying capacity, i.e. $x_0 = 0.9K$, unless otherwise stated. .

Comparison of ENPV and Options solution

We compare the optimal solution obtained using the ENPV rule and the real options approach for the parameter values listed in Table 5.1. Two scenarios are presented for the real options technique: one where the convenience yield is zero and the other where the convenience yield is non-zero. Figure 5.1 illustrates the difference between the expected net present value of total profit and the value of the harvesting option; Figure 5.2 presents the corresponding optimal effort paths. To examine the influence of correlation between $dW_1(t)$ and $dW_2(t)$, we plot separate graphs in each figure for $\rho = 0$ and $\rho \neq 0$. To illustrate the case of non-zero correlation, we fix $\rho = -0.5$. Following the explanation provided in Section 4.5, we have assigned a negative sign to the correlation-coefficient, ρ.

In Figure 5.1, we observe that the value of the harvesting option is greater than the discounted flow of profit in each case. This renders managerial flexibility important when making harvesting decisions and highlights the inefficiency of the arbitrarily-fixed discount rate introduced by the ENPV approach. Furthermore, the option value declines as convenience yield increases; this can be attributed to the dividend-like behaviour of convenience yield as follows: In substance, convenience yield corresponds to net benefits earned by holding additional units of the resource stock. As there is a possibility that the fish population might decline in the future, the harvester could earn extra profit by storing fish. Thus convenience yield is akin to a dividend paid to the owner of an asset and a rise in the value of a dividend has an inverse influence on the value of an option. The analogy here is that a dividend paying stock loses value (equal to the amount of expected dividend) each time a dividend is paid, therefore a high dividend would cause a bigger decrease in the stock value than a low dividend. The dividend is received by the owner, not by the option holder, but the option holder can see the stock losing value and, consequently, the option value declines.

Now we concentrate on the optimal effort policy. Figure 5.2 shows that, compared to the ENPV solution, the options approach yields lower optimal effort. This implies that optimal harvesting becomes more conservative when computed using contingent claims analysis. The optimal effort, however, increases with an increase in convenience yield. To explain this we will again consider the stock-dividend analogy. If a stock pays dividends at discrete time stages, the stock price is adjusted downwards each time a dividend is paid. On the other hand, if a stock pays a continuous dividend, the stock price has to be adjusted continuously in order to reflect the dividend paid. In either case, for a dividend paying stock, the stock price is highest in the beginning and then keeps falling; due to the dividend-type influence of convenience yield this behaviour can also be associated with the fish price. Thus with a high convenience yield the risk-adjusted prices are expected to fall in the future. Consequently, there is an incentive to opt for an increased initial harvest and cash in the benefits coming from a high convenience-yield.

We notice that the qualitative behaviour of the optimal solution for fishing effort and option value stays the same for $\rho = 0$ and $\rho \neq 0$. Hence, for these parameter values, correlation between $dW_1(t)$ and $dW_2(t)$ has no effect on the nature of the solution.

Table 5.1: Parameter values for the simulation of Hamilton-Jacobi-Bellman equation

Parameter	Description	Value	Unit
r	Intrinsic growth rate	0.71	year^{-1}
δ	Discount rate	0.12	year^{-1}
q	Catchability coefficient	0.0001	SFU^{-1} year^{-1}
K	Biological carrying capacity	10^6	tonnes
x_{min}	Minimum viable population level	$0.3K$	tonnes
p	Unit harvest price	0.5	\$ tonne^{-1}
T	Terminal time	5	year
c_1	Linear cost-coefficient	0.01	\$ SFU^{-1} year^{-1}
c_2	Quadratic cost-coefficient	0.01	\$ SFU^{-2} year^{-1}
μ	Price drift	0.02	year^{-1}
σ_1	Growth volatility	0.04	year$^{-1/2}$
σ_2	Price volatility	0.04	year$^{-1/2}$

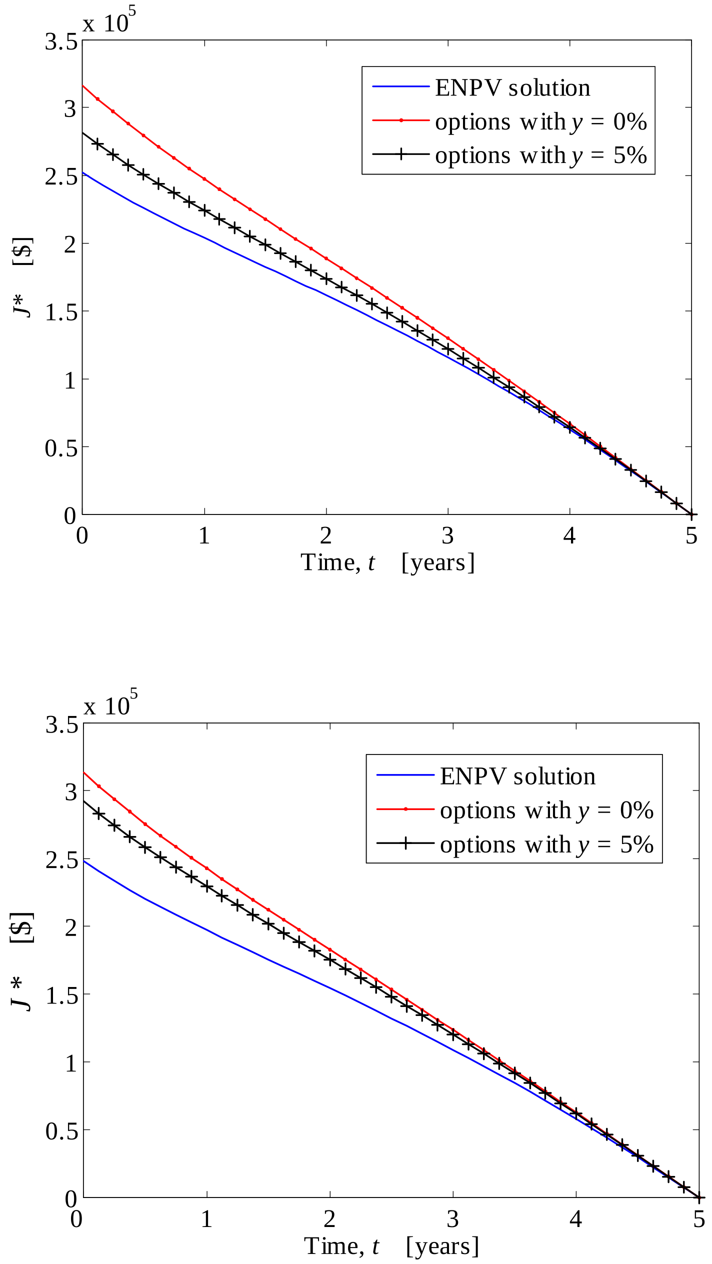


Figure 5.1: Comparison of the option value of harvesting with the expected net present value of total profit. Top: $\rho = 0$; Bottom: $\rho = -0.5$. In both cases, the value of the harvesting option exceeds the expected net present value of total profit.

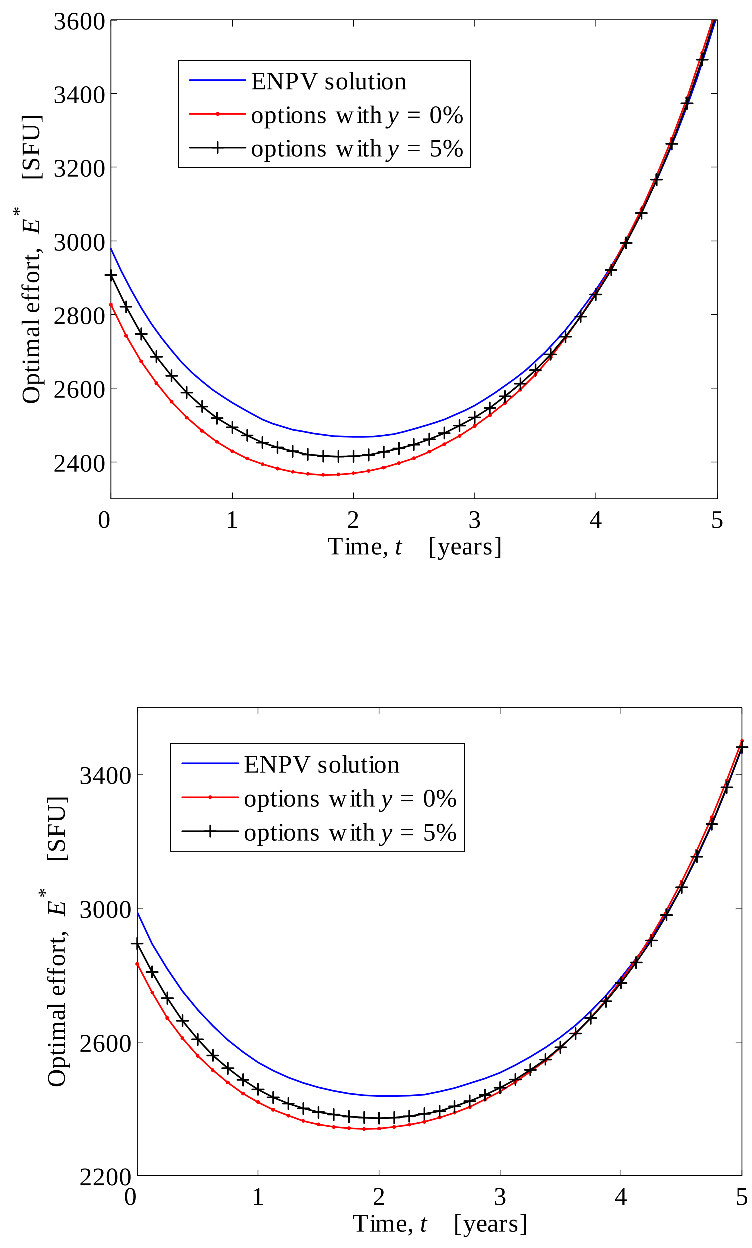


Figure 5.2: Comparison of the optimal effort solution obtained using the real options approach and the ENPV rule. Top: $\rho = 0$; Bottom: $\rho = -0.5$. In both cases, the policy yielded by real options indicates a lower effort level and therefore a more conservative harvest.

Sensitivity of the optimal solution to the initial population level

We now aim to study the change in the optimal solution due to a change in the initial population level, x_0. Figure 5.3 shows the value of harvesting opportunity and Figure 5.4 presents the optimal effort policy, both for three different initial population levels, where we have fixed $p_0 = \$1$/tonne and $T = 2$ years. The remaining parameter values are utilized from Table 5.1.

We see that both the optimal effort and the option value decline with a fall in x_0. This is reasonable as a low initial stock level implies that there are less fish available for harvesting, which in turn recommends harvesting at reduced levels to avoid over-exploitation of the fish. A diminished optimal effort is accompanied by a fall in the harvesting costs. However, this drop in the optimal effort and population levels also brings a drop in the revenue. The fall in revenue exceeds the fall in costs and, consequently, the value of the harvesting option decreases as x_0 decreases.

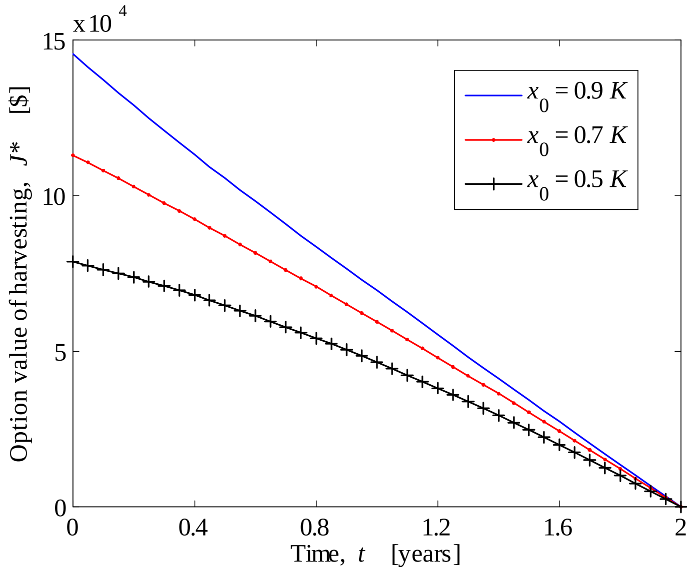


Figure 5.3: Sensitivity of the value of the harvesting option to the initial population level, x_0. The option value decreases as x_0 decreases.

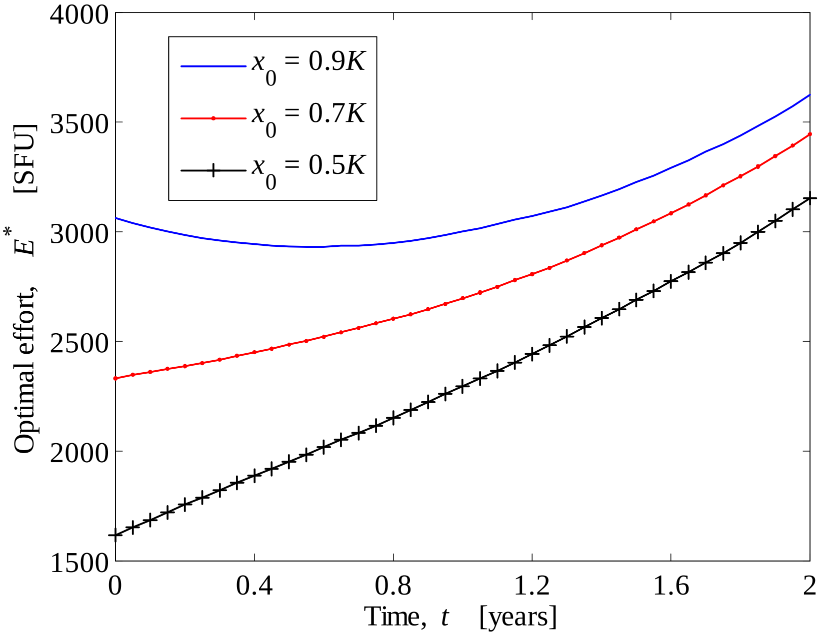


Figure 5.4: Sensitivity of the optimal effort policy to the initial population level, x_0. The optimal effort decreases as x_0 decreases.

Sensitivity of the optimal solution to the intrinsic growth rate of fish-stock

Next we examine the influence of the intrinsic rate of growth, r, on the optimal solution. Figure 5.5 shows the option value of harvesting and Figure 5.6 demonstrates the corresponding optimal effort path, both for three different values of r. Figure 5.7 illustrates the corresponding scenarios for the population growth under optimal harvesting. The parameter values are utilized from Table 5.1 with $T = 2$ years and $p_0 = \$1/\text{tonne}$.

We observe that a rise in r results in an increase in the optimal effort expended, as well as an increase in the value of the harvesting opportunity. This behaviour of the optimal solution can be accounted for by considering the fact that the biological growth of the resource stock is accelerated with a high value of r. Therefore, the population is harvested increasingly and the overall effect is a rise in the value of the harvesting option.

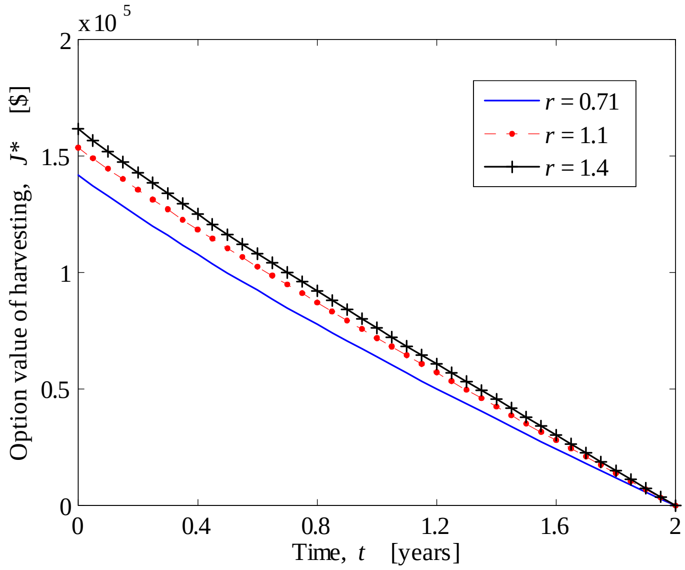


Figure 5.5: Sensitivity of the value of the harvesting option to the intrinsic rate of growth, r, of the resource stock. The option value is an increasing function of r.

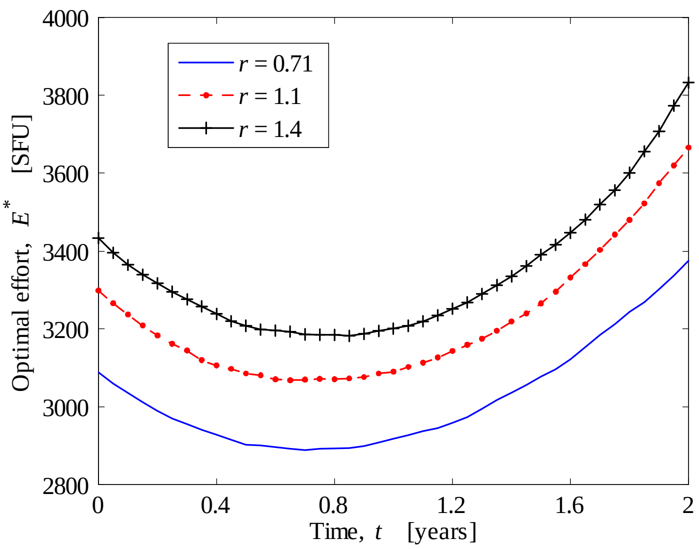


Figure 5.6: Sensitivity of the optimal effort solution to the intrinsic rate of growth, r, of the resource stock. The optimal effort increases with r.

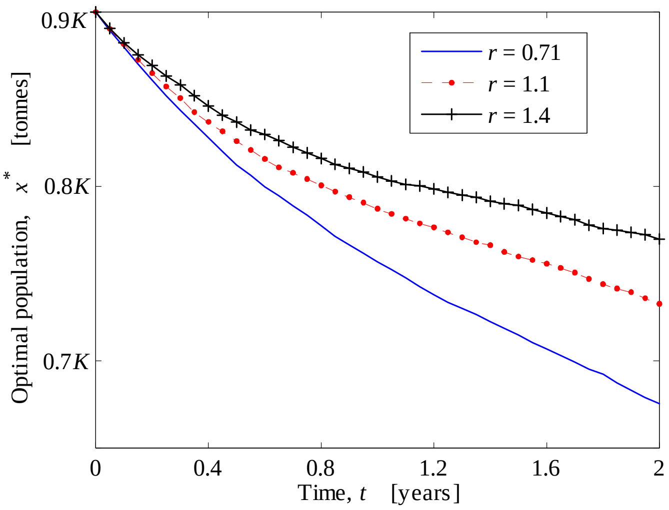
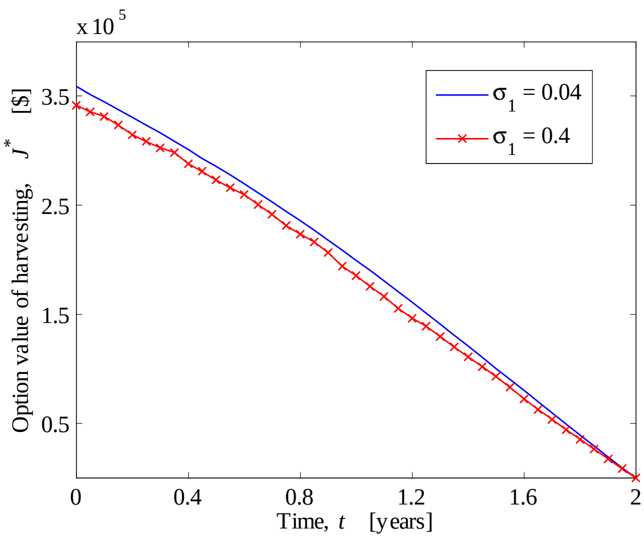


Figure 5.7: Population growth under optimal harvesting for three different values of r.

Sensitivity of the optimal solution to the growth volatility

Figure 5.8 presents the sensitivity of the option value of harvesting to a change in the growth volatility, σ_1. The corresponding effect on the optimal effort strategy is illustrated in Figure 5.9. We consider two scenarios for the initial stock size: (i) the initial population level is at half its carrying capacity, representing a low initial stock level; (ii) the initial population level is at ninety percent of its carrying capacity, representing a high initial stock level. We find that, in both cases, the option value decreases with an increase in the growth volatility. Furthermore, a rise in the magnitude of the growth volatility also leads to a considerable downward shift in the optimal effort path.

To understand these results we have to concentrate on the connection between the growth-variability and the stock level, demonstrated in Figure 5.10. With a high volatility, the population level can deviate significantly from its expected value. But there are biological and environmental constraints on the fish stock, and due to these limitations, the population level cannot rise above the carrying capacity K. Thus a stock-level higher than K cannot be maintained by the population, even if the stochastic fluctuations are large. However, with a high growth diffusion coefficient, the random fluctuations can result in the fish stock falling much below its predicted level. In other words, the upside potential of the resource stock is bounded by the carrying capacity whilst increased growth volatility raises the downside potential. Therefore the value of the harvesting opportunity declines as the growth volatility increases. However, for the parameter values considered for a high initial stock level, the fall in the option value is not substantial. Also, to avoid over-fishing, the optimal solution for effort suggests that harvesting be at reduced levels.



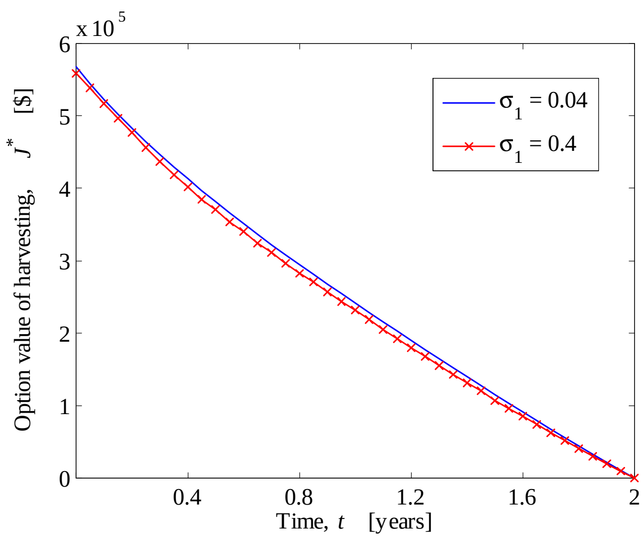
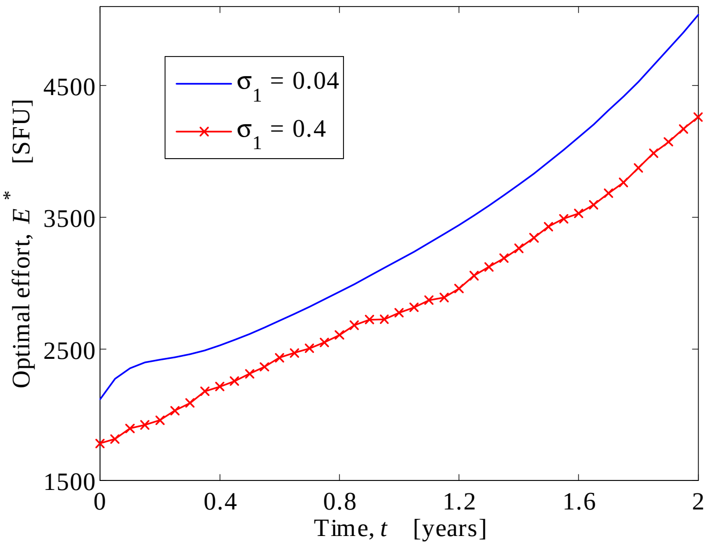


Figure 5.8: Sensitivity of the option value of harvesting to growth volatility. Top: $x_0 = 0.5K$*; Bottom:* $x_0 = 0.9K$*. In both cases, the option value decreases as growth volatility increases.*



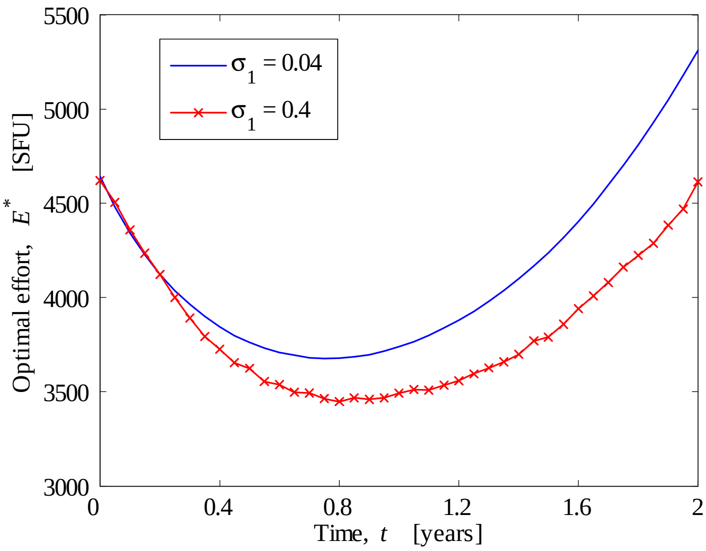


Figure 5.9: Sensitivity of the optimal effort policy to growth- volatility. Top: $x_0 = 0.5K$; Bottom: $x_0 = 0.9K$. In both cases, the optimal effort decreases as growth volatility increases.

Note that $x_0 = 0.5$ implies that the initial population level is very close to the minimum viable level. A high growth volatility in this case puts population at a greater risk of falling below the minimum vialble level. Consequently, the optimal population path in Figure 5.10 exhibits a high degree of fluctuations.

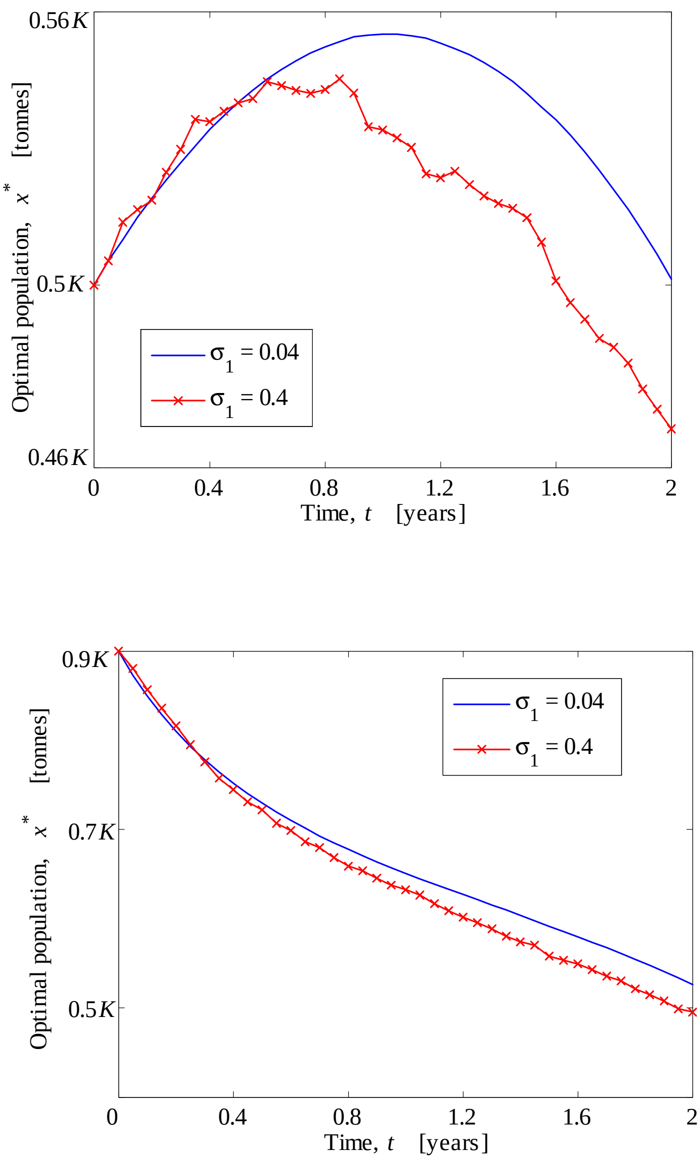


Figure 5.10: Sensitivity of the population dynamics to growth volatility. Top: $x_0 = 0.5K$; Bottom: $x_0 = 0.9K$. In both cases, the population growth is lowered with a rise in the growth volatility.

Sensitivity of the optimal solution to the price volatility

Figure 5.11 captures the effect of price variability on the value of the harvesting opportunity and Figure 5.12 displays the respective influence on the optimal effort strategy. We have again plotted two scenarios: one for $x_0 = 0.5K$ and the other for $x_0 = 0.9K$. We notice that, in both cases, the option value of harvesting increases with price volatility whereas the optimal effort path shifts downwards. This result is in accordance with standard option-pricing literature, for instance see Dixit & Pindyck (1994), where it is noted that an increased price volatility raises the option value.

To provide a general explanation for this observation, we consider a risky stock being traded in the market. To acquire a call option on this risky stock, a payment has to be made by the option holder. Let us suppose, for example, that the price volatility of the underlying stock has increased. This raises the upside as well as the downside potential for the stock price movement. If the stock price moves up and exceeds the strike price, the option holder can cash in the extra benefits by exercising the call option. On the other hand, if the stock price moves down, the option holder will not exercise the call option. Therefore the loss for the option holder is limited to the original cost of the option even if the stock price falls by a big amount, whereas a possible gain from a high price volatility is unbounded. Thus an increased price volatility amplifies the amount of money that the option-holder expects to earn, and this leads to a rise in the option value with a rise in the price volatility.

For the same reason, a high volatility of fish price implies a higher value of the harvesting option. Under unfavorable circumstances (non-positive profit), the harvester will not exercise the harvesting option, hence no costs will be incurred. Thus the downside potential here is limited to zero whilst the upside potential is again unlimited. Furthermore, a high option value indicates that there is an incentive in delaying the exercise of the option. Consequently, the optimal effort drops with an increase in fish price volatility.

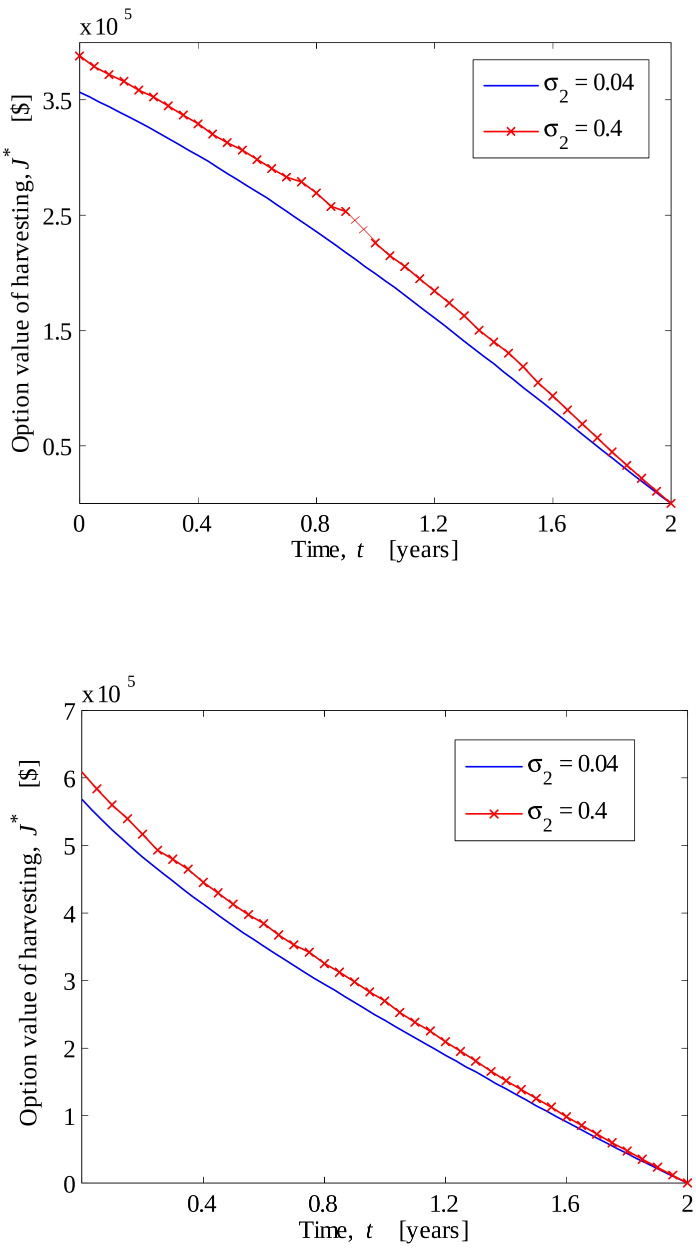


Figure 5.11: The effect of price volatility on the value of harvesting opportunity. Top: $x_0 = 0.5K$; Bottom: $x_0 = 0.9K$. In both the cases, the option-value increases as price volatility increases.

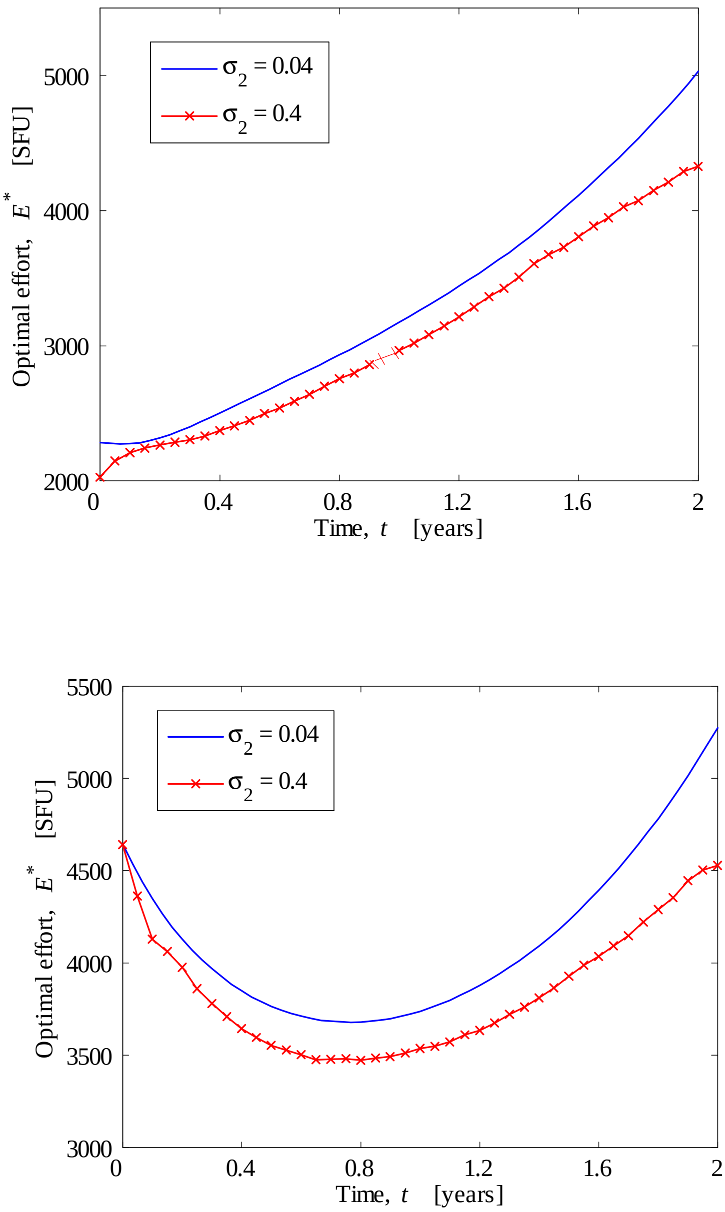


Figure 5.12: The effect of price volatility on the optimal effort strategy. Top: $x_0 = 0.5K$; Bottom: $x_0 = 0.9K$. In both cases, the optimal effort drops as price volatility rises.

5.4 Summary

In this chapter we presented an optimal harvesting model based on real options theory. We extended the basic model with deterministic growth and stochastic price, introduced by Murillas & Chamorro (2006), to include stochastic fluctuations in growth. We first restricted the analysis to the case of zero correlation between the Wiener increments underlying random growth and price effects. Following this, we included a non-zero correlation between the two Wiener increments. In each case, we obtained a partial differential equation governing the option value of harvesting; we solved these equations numerically.

We found that the real options method produced a higher value of the fishery (which is the value of the harvesting option) and a lower optimal effort compared to the ENPV approach. The correlation had no significant impact on the qualitative nature of the optimal solution. Furthermore, the convenience-yield produced a dividend-like effect on the option value and optimal effort policy.

Next we studied the rôle played by the intrinsic rate of growth in deciding the value-maximizing effort policy. We found that an increase in the growth rate triggered a higher biological growth as well as a higher optimal effort. The combined effect was to raise the option value of harvesting. Then we investigated the influence of the initial population level on the optimal solution. We observed that the optimal effort and the option value were increasing functions of the initial population level.

Finally, we performed a sensitivity analysis of the optimal solution with respect to the growth and the price variability. We found that the growth volatility had a pronounced effect on the optimal effort expended. The influence on the option value was less significant. Further, we noted that the price volatility had a positive effect on the value of harvesting opportunity. A rise in the magnitude of price volatility resulted in an increase in the option value, whereas the optimal effort declined.

Chapter 6

Elasticity

6.1 Introduction

A large proportion of the existing literature considers the demand for fish to be infinitely elastic (see, for example, Clark, 1975), the reason being that the harvested fish are supplied to a big market being catered for by a large number of fisheries. In Chapters 3-5 we followed the same route and only considered a small fishery such that the fishery's harvest had no effect on the evolution of market price. At most, we considered the stochastic fluctuations in stock growth and price to be correlated; we did not explicitly model the influence of fish harvest on price. In other words, the price dynamics did not explicitly contain any term corresponding to stock growth. The objective in this chapter is to demonstrate the effect of price elasticity of demand on the optimal harvesting strategy. This is achieved by assuming both stock growth and fish price to be random, and introducing a term that is dependent on stock size in the price dynamics. There is only a small amount of literature devoted to the role played by elasticity in fish harvesting. Some examples of such studies can be found in Gatto (1992), Briones (2006) and Danielsson (2002).

We first briefly explain some fundamental concepts underlying elasticity. A thorough explanation of all these economic concepts can be found in standard text-books, e.g. Jackson *et al.* (2007) and Swann & McEachern (2006). The *price elasticity of demand* is defined as a proportional change in the demand of a product due to a given change in its price. The coefficient of price elasticity of demand is denoted by E_d; it measures the sensitivity of the quantity demanded by consumers to a change in the price of a product, and is calculated as

$$E_d = \frac{\text{percentage change in quantity demanded}}{\text{percentage change in price}}.$$

In other words,

$$E_d = \frac{dq/q}{dp/p}, \tag{6.1}$$

where p denotes price and q denotes quantity demanded. If k denotes the slope of a demand-curve, relating a price that the market will pay to a specific quantity supplied to the market, then $k = dp/dq$ and Equation (6.1) becomes

$$E_d = \frac{1}{k}\frac{p}{q}. \tag{6.2}$$

Or,

$$k = \frac{1}{E_d}\frac{p}{q}. \tag{6.3}$$

The *law of economics* (Jackson, 2007) states that there is an inverse relationship between the price of a product and the quantity demanded of that product. As the price increases the quantity demanded declines, and *vice versa*, where all other things are held constant. Consequently, a market demand-curve is downward sloping which yields a negative slope k and, therefore, a negative coefficient of elasticity E_d. However, it is a convention in economics to represent a coefficient of elasticity in absolute value by dropping the negative sign in front of it.

A product is said to have *unit elasticity* when the percentage change in quantity demanded is equal to the percentage change in price, so that, $|E_d| = 1$ and $k = \frac{p}{q}$. This corresponds to the revenue-neutral case where the loss due to a fall in price is equally offset by the gain due to a rise in quantity demanded so that the total revenue remains unaffected. A *highly elastic* product has a coefficient of elasticity $|E_d| > 1$. A decline in price in this case is dominated by a rise in quantity demanded. Therefore the total revenue increases as price decreases, and *vice versa*. For a *perfectly elastic* (or *infinitely elastic*) product, the elasticity coefficient E_d is undefined and the demand-curve is parallel to the horizontal axis with slope $k = 0$. The demand for a good is called *relatively inelastic* when the quantity demanded does not change much with the price change. Such a product has a coefficient of elasticity $|E_d| < 1$. A fall in price for an inelastic good is accompanied by a relatively smaller increase in quantity demanded, and consequently, the total revenue moves in the same direction as price. Goods and services for which no substitutes exist are generally inelastic.

6.2 Model formulation with price elasticity of demand

We recall from Section 4.2 that the growth dynamics of resource stock follows:

$$dx(t) = \left\{ rx(t) \left(1 - \frac{x(t)}{K} \right) - qE(t)x(t) \right\} dt + \sigma_1 x(t) dW_1(t). \tag{6.4}$$

We aim to capture the effect of price elasticity in the model. To achieve this, we include an extra term in the price dynamics linking the fish growth with the price; the price changes are now specified as

$$dp(t) = \mu_p p(t)dt + \sigma_2 p(t)dW_2(t) + k(t)dx(t), \tag{6.5}$$

where $k(t)$ is the slope of a market demand-curve. When $k(t) = 0$, the problem reduces to the original case where the fishery is small and has no observable effect on the market price. Substituting $k = 0$ in Equation (6.2) yields the elasticity coefficient as ∞, which amounts to the demand being infinitely elastic. Therefore we can say that the problem discussed in Chapter 4 corresponds to infinite elasticity. Now, as $k(t)$ is the slope of a market demand-curve, we have

$$k(t) = \frac{dp(t)}{dx(t)}. \tag{6.6}$$

We assume that the coefficient of elasticity is a constant. Since a market demand-curve is downward sloping, i.e. it has a negative slope, $k(t)$ is negative. Consequently, the coefficient of elasticity is negative (see Equation (6.2)). We denote the magnitude of the elasticity coefficient by $m(> 0)$. Then, using Equation (6.1) and introducing the negative sign to capture the true effect of elasticity, we can write

$$\frac{dx(t)/x(t)}{dp(t)/p(t)} = -m,$$

which is equivalent to

$$\frac{dx(t)}{dp(t)} \frac{p(t)}{x(t)} = -m \tag{6.7}$$

Utilizing Equation (6.6) in Equation (6.7) we obtain

$$\begin{aligned}\frac{1}{k(t)}\frac{p(t)}{x(t)} &= -m \\ \Rightarrow k(t) &= -\frac{p(t)}{mx(t)}\end{aligned} \tag{6.8}$$

Using Equations (6.8) and (6.4) in Equation (6.5) yields

$$\begin{aligned}dp(t) &= \mu_p p(t)dt + \sigma_2 p(t)dW_2(t) \\ &\quad -\frac{p(t)}{mx(t)}\left\{rx(t)\left(1-\frac{x(t)}{K}\right) - qE(t)x(t)\right\}dt \\ &\quad -\frac{p(t)}{m}\sigma_1 dW_1(t).\end{aligned}$$

which leads to

$$dp(t) = \left\{\mu_p - \frac{r}{m}\left(1-\frac{x(t)}{K}\right) + \frac{qE(t)}{m}\right\}p(t)dt + \sigma_2 p(t)dW_2(t), \tag{6.9}$$

where we have ignored the contribution of stochastic fluctuations in stock growth (represented by $dW_1(t)$) towards the elasticity effect. The basic optimization problem faced by the harvester is still the same as the stochastic optimal control problem discussed in Chapter 4, where an optimal harvesting policy maximizing the expected present value of total flow of profit is sought, subject to random growth and price dynamics, boundary conditions and constraints on effort. The only difference now is in the price dynamics which have been modified to include the effect of price elasticity of demand. We proceed in the same manner as in Section (4.3) and obtain the Hamilton-Jacobi-Bellman equation for the discounted flow of profit as

$$\begin{aligned}-\frac{\partial J^*}{\partial t} = \max_{E(t)} \Bigg[&\left(p(t)qx(t) - c_1 - \frac{c_2}{2}E(t)\right)E(t) - \delta J^* \\ &+ \frac{\partial J^*}{\partial x}\left\{rx(t)\left(1-\frac{x(t)}{K}\right) - qE(t)x(t)\right\} \\ &+ \frac{\partial J^*}{\partial p}\left\{\mu_p - \frac{r}{m}\left(1-\frac{x(t)}{K}\right) + \frac{qE(t)}{m}\right\}p(t) \\ &+ \frac{1}{2}\frac{\partial^2 J^*}{\partial x^2}\sigma_1^2 x(t)^2 + \frac{1}{2}\frac{\partial^2 J^*}{\partial p^2}\sigma_2^2 p(t)^2\Bigg],\end{aligned} \tag{6.10}$$

where the optimal effort is given by

$$E^*(t) = \begin{cases} 0, & \left(p - \frac{\partial J^*}{\partial x}\right)\frac{qx(t)}{c_2} + \frac{qp(t)}{mc_2}\frac{\partial J^*}{\partial p} - \frac{c_1}{c_2} < 0 \\ \left(p - \frac{\partial J^*}{\partial x}\right)\frac{qx(t)}{c_2} + \frac{qp(t)}{mc_2}\frac{\partial J^*}{\partial p} - \frac{c_1}{c_2}, & 0 \le \left(p - \frac{\partial J^*}{\partial x}\right)\frac{qx(t)}{c_2} + \frac{qp(t)}{mc_2}\frac{\partial J^*}{\partial p} - \frac{c_1}{c_2} \le E_{\max}, \\ E_{\max}, & \left(p - \frac{\partial J^*}{\partial x}\right)\frac{qx(t)}{c_2} + \frac{qp(t)}{mc_2}\frac{\partial J^*}{\partial p} - \frac{c_1}{c_2} > E_{\max}. \end{cases} \tag{6.11}$$

As always, the Hamilton-Jacobi-Bellman equation (6.10) is solved numerically and the numerical procedure followed is same as in all the other cases. The numerical results are illustrated in the next section.

6.3 Numerical solution

We first examine the influence of a variation in the magnitude of the coefficient of elasticity on the optimal solution. Figure 6.1 presents the average optimal solution for three different values of the elasticity coefficient m, where the average is taken over 2000 realizations of the Wiener increments $dW_1(t)$ and $dW_2(t)$. The parameter values are listed in Table 6.1.

Table 6.1: Parameter values for the simulation of the Hamilton-Jacobi-Bellman equation

Parameter	Description	Value	Unit
r	Intrinsic growth rate	0.71	year^{-1}
δ	Discount rate	0.12	year^{-1}
μ	Price drift	0.02	year^{-1}
q	Catchability coefficient	0.0001	SFU^{-1} year^{-1}
K	Biological carrying capacity	10^6	tonnes
$x_{\min}$	Minimum viable population level	$0.4K$	tonnes
p_0	Initial price	0.5	\$ tonne^{-1}
c_1	Linear cost coefficient	0.01	\$ SFU^{-1} year^{-1}
c_2	Quadratic cost coefficient	0.01	\$ SFU^{-2} year^{-1}
σ_1	Growth volatility	0.1	year$^{-1/2}$
σ_2	Price volatility	0.1	year$^{-1/2}$

Compared to the solution for $m = 1$, the optimal effort is lower when $m > 1$ (demonstrated using $m = 1.2$) and higher when $m < 1$ (demonstrated using $m = 0.8$). We recall that $m = 1$ corresponds to the revenue neutral case (see Section 6.1). When $m > 1$, the total revenue moves in a direction opposite to the movement of price. This implies that $\frac{\partial J^*}{\partial p}$ is negative and this lowers the optimal effort given by Equation (6.11). On the other hand, when $m < 1$, the total revenue increases with an increase in price, and *vice versa*. Therefore, $\frac{\partial J^*}{\partial p}$ is positive and this leads to a rise in the optimal effort given by Equation (6.11). The consequence of the change in optimal effort on the corresponding population growth is visible in Figure 6.1. The optimal population level x^* undergoes an increase when $m = 1.2$ due to diminished optimal effort E^*, whereas, a decline is observed in x^* when $m = 0.8$ due to an increase in E^*.

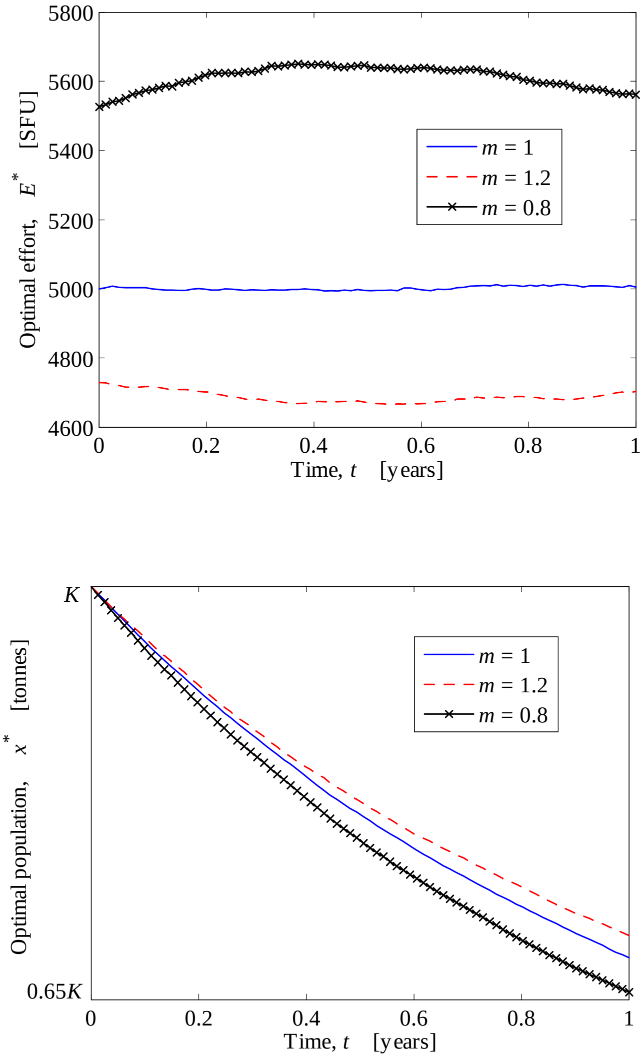
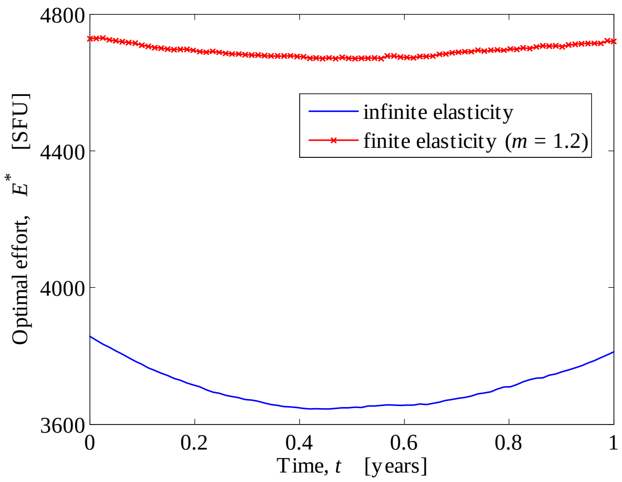


Figure 6.1: The optimal solution for three different magnitudes of the coefficient of elasticity, m. Above: optimal effort; Below: optimal stock level. As the magnitude of the elasticity coefficient decreases, the optimal effort increases whereas the optimal stock level drops.

Infinite elasticity vs. finite elasticity

We now compare the optimal solution associated with the case of infinite elasticity (which is the original problem discussed in Chapter 4) with the optimal solution corresponding to the case of finite elasticity; Figure 6.2 illustrates the comparison. We observe that the optimal effort is diminished when the demand is perfectly elastic. As the magnitude of the elasticity coefficient decreases, i.e. the demand becomes less elastic, the optimal effort increases causing the optimal stock level to fall.



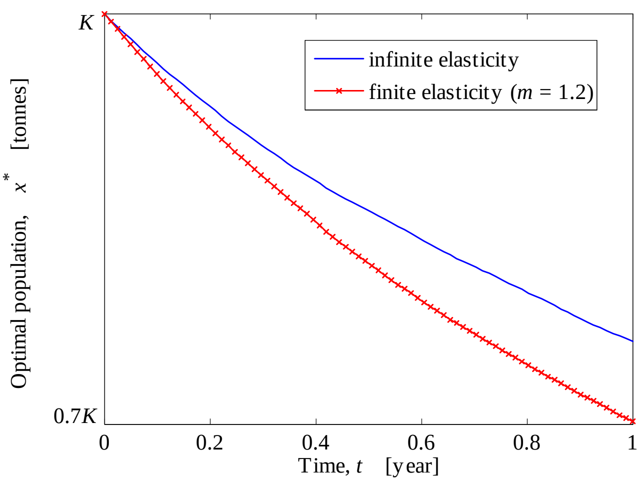


Figure 6.2: Comparison of the optimal solutions corresponding to infinite and finite elasticity. As demand becomes less elastic, the optimal effort rises while the optimal stock level declines.

6.4 Summary

In this chapter we extended the model studied in Chapter 4 to reflect the price elasticity of demand. In this context, we reformulated the evolution of price to include the product of stock growth with the slope of the demand curve. The fundamental constrained optimization problem and the initial and boundary conditions are exactly those as discussed in Chapter 4, the only change being the modified price dynamics. The Hamilton-Jacobi-Bellman equation associated with the problem was solved numerically. It was observed that, on average, a decrease in the magnitude of the elasticity coefficient resulted in a higher optimal effort and a subsequent drop in the optimal stock level. We also compared the optimal solution obtained for the original problem (infinite elasticity) with the optimal solution corresponding to a relatively less elastic demand. We noted that the optimal effort was lower while the optimal stock level was higher for the case of infinitely elastic demand.

We have confined our study to a constant coefficient of elasticity. Another approach would be to implement a linear demand curve with a constant slope k. This would imply that $dx(t)/dp(t)$ will be the same for all times t. The elasticity, however, would vary through time as $m(t) = k\frac{p(t)}{x(t)}$. It would be interesting to investigate this problem as a piece of future research.

Chapter 7

Summary and future directions

The main objective of this thesis was to determine optimal harvesting strategies for fisheries, in both deterministic and stochastic settings, by employing various optimization techniques and real options theory. We now present a summary of the work done.

7.1 Thesis summary

In Chapter 1 we discussed existing literature and introduced the Schaefer model, which laid the foundation for the fish population growth modelled in this study. In Chapter 2 we presented an outline of the optimization techniques used in this work. Specifically, we discussed the calculus of variations, optimal control theory, and dynamic programming in deterministic and stochastic frameworks. The applications to fish harvesting were also illustrated. In Chapter 3 we developed our model for a deterministic optimal harvesting problem with the following main assumptions:

(i) The population consists of a single species of fish.

(ii) The growth is only density dependent (hence the use of Schaefer model for representing growth dynamics).

(iii) The fishing effort is constrained to stay below a fixed maximum.

(iv) The harvesting costs are quadratic in fishing effort.

(v) There exists a minimum viable population level such that if the stock falls below this level then the population will become extinct.

The fish population was constrained to stay above a pre-specified minimum viable level throughout harvesting. The optimal harvesting strategy was assumed to maximize the net present value of total flow of profit. We determined the optimal solution corresponding to finite-horizon harvesting using three different optimization techniques: dynamic programming, optimal control theory (Hamiltonian method), and a variational method based on the calculus of variations. While dynamic programming was employed to solve the constrained problem, the constraints on fishing effort were dropped when using the other two approaches. A sensitivity analysis of the total discounted profit was carried out with respect to the catchability and cost parameters. The analysis highlighted the pronounced influence of the quadratic cost-coefficient on the total discounted profit for the case of low catchability; the linear cost-coefficient was seen to have a negligible effect. When catchability was high, the total discounted profit was observed to be unaffected by a change in the value of either of the cost-coefficients.

We also studied infinite-horizon harvesting where we found that the optimal effort and the optimal population approach a steady state in the long run. We determined the optimal steady states and compared them with the optimal solution corresponding to finite-horizon harvesting. It was noted that if the harvesting is carried on for a long (finite) period of time, the optimal effort and the optimal population stabilize at their respective steady states (associated with infinite-horizon harvesting) for a while. Due to the finiteness of the harvesting period, a deviation from the steady-state solution was observed in the last few time-stages.

In Chapter 4 we extended the optimal harvesting problem formulated in Chapter 3 to include random fluctuations in growth and price dynamics. The stochastic effects in the evolution of population and price were modelled using two different Wiener processes. We separately discussed the cases corresponding to zero correlation and a non-zero correlation between the two Wiener processes. The constraints on fishing effort and minimum viable population level remained as before. The optimal harvesting strategy, however, maximized the expected net present value (ENPV) of the total flow of profit. Stochastic dynamic programming was employed to determine the optimal solution. We first considered random growth and constant price and analyzed the effect of growth variability on the optimal harvesting policy. When harvesting was carried on over a short term, both optimal effort and optimal population level decreased with an increase in growth variability. For long-term harvesting, a rise in growth volatility lowered the steady state associated with the long-term optimal solution. Next we considered both growth and price to be random and examined the influence of growth and price volatility on the optimal solution. A sensitivity analysis with respect to the catchability and cost parameters was performed on the expected total discounted profit and the conclusions were found to be in agreement with those obtained in the deterministic environment. We also computed the correlation coefficient between the fish price and population undergoing optimal harvest. The coefficient of correlation was found to be mostly negative, and it displayed a high increase in magnitude following an increase in price variability.

Chapter 5 presented the above-mentioned stochastic optimal harvesting problem in a real-options framework. Instead of maximizing the expected net present value of total profit, this approach focused on determining the option value of harvesting. The optimal solution obtained using real-options theory was compared with the optimal solution determined using the ENPV approach in Chapter 4. It was found that the option value of harvesting exceeded the expected net present value of total profit while the optimal effort produced by the real options technique recommended harvesting at a reduced level. We also performed sensitivity analyses of the optimal effort path and the optimal population growth with respect to various parameters present in the model. Finally, Chapter 6 demonstrated the influence of price elasticity of demand on the optimal harvesting policy. The ENPV approach was employed to determine the optimal solution.

In each chapter, we obtained a partial differential equation which was always non-linear and highly complex. Therefore, the possibility of an analytical solution was ruled out and numerical methods were used to solve the partial differential equation and obtain the optimal solution. To approximate the spatial and temporal derivatives we can use a forward difference scheme, a backward difference scheme or an average of the two; this choice determines whether the finite-difference scheme is *explicit* or *implicit*. Each numerical scheme (explicit or implicit) has different stability criteria and truncation errors; further details can be found in Thomas (1995) and Dunn *et al.* (2006). The stability condition for an explicit method depends upon the temporal and spatial step-sizes. The HJB partial differential equation for deterministic case can be solved using an explicit finite-difference scheme. However, for random environment, the explicit scheme exhibits stability and convergence problems. Therefore, we use an implicit (Crank-Nicolson) finite difference method for solving the partial differential equations obtained in each chapter.

7.2 Future directions

This thesis is a comprehensive study of optimal harvesting policies for fisheries. Nevertheless, the problem studied here can be used as a basis for future research; some specific guidelines for further research are recorded below.

1. **Critical depensation**

Throughout this study we adopted variations of the Schaefer model, which is based on the logistic function, to represent biological growth. Recall from Section 1.1 that in the logistic models, the proportional growth rate $\frac{f(x)}{x}$ is a decreasing function of x, i.e. they are *pure compensation models*, whereas, if $\frac{f(x)}{x}$ is an increasing function of x for some values of x then *depensation* exists. In our study, the minimum viable population level was modelled as a constraint on the state. On the other hand, the minimum viable level can be included in the growth dynamics when modelling depensation. Consider the differential equation

$$\frac{dx}{dt} = rx\left(\frac{x}{K_0} - 1\right)\left(1 - \frac{x}{K}\right).$$

This equation possesses a critical threshold at $x = K_0$, which is the *minimum viable population level*. Solutions with initial conditions above K_0 approach K, whereas those starting below K_0 decay to zero. Since the net growth rate is negative at population levels lower than

K_0, this model exhibits *critical depensation* (Clark, 1990). Adding harvesting to the critical depensation model yields

$$\frac{dx}{dt} = rx\left(\frac{x}{K_0} - 1\right)\left(1 - \frac{x}{K}\right) - qEx.$$

There is a trivial equilibrium at $x^* = 0$ whilst the remaining two equilibria are the roots of the quadratic equation

$$r\left(\frac{x}{K_0} - 1\right)\left(1 - \frac{x}{K}\right) - qE = 0. \tag{7.1}$$

Some of the consequences of multiple equilibria in the context of fisheries are discussed in Larkin *et al.* (1964). Equation (7.1) undergoes bifurcations at the critical parameters E^*, since solutions of the system change discontinuously as E traces E^* (Kot, 2001). An interesting problem for future research would be the net profit maximization from harvesting with critical depensation in stock growth. This is especially challenging as the constraint on effort E must be chosen judiciously so that catastrophic effects on the population can be averted.

2. **Time-delays**

The logistic model does not explicitly consider recruitment, survival or lags in recruitment. In fisheries, our logistic model assumes a one-year time lag between changes in biomass and net production. A *Lagged Recruitment, Survival and Growth* (LRSG) model was analyzed by Hilborn & Mangel (1997). Another approach to represent net growth is to employ delay-difference models. These models are a little more comprehensive than simple production models because they explicitly contain the age-structured dynamics of population, including the lag between spawning and recruitment. A comprehensive reference in this area is Quin & Deriso (1999). Such models are important because they address the issue that ecosystems cannot respond simultaneously to recruitment and changes such as harvesting. A simple modification of the logistic growth equation to incorporate delays is due to Hutchinson & Wright (1948) and is given by the *delay-differential equation*

$$\frac{dx}{dt} = rx\left(1 - \frac{x(t-\tau)}{K}\right), \; x(s) = x_0(s), \quad -\tau \le s \le 0.$$

where the single time delay is represented by τ. Delay-differential equation models tend to exhibit destabilizing effects in the form of oscillations. In the context of harvesting, time delays can appear either in the growth function or in the harvesting term. Two recent works that consider such problems are Berezansky *et al.* (2004) and Cui & Li (2007). In both these works the time delay in the harvesting term appears in the population part. For example, in a delay model of a lobster fishery there is a delay in getting the information of the lobster population which yields the growth dynamics as

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - qE(t)x(t-\tau).$$

A future research problem would be to maximize the net profit from harvesting as formulated in this thesis, with the stock growth dynamics given by the above delay-differential equation.

3. **The theory of games**

Another area to explore is to determine optimal harvesting strategies using game-theory. We now briefly outline the concepts underlying game-theory and also provide a game-theoretic formulation of the optimal harvesting problem in the presence of two harvesters. The *theory of games* is designed to analyze strategic interactions among individuals; these individuals can be persons, firms, nations, or others. An elementary introduction to game theory can be found in Osborne (2004). The individuals in the theory of games are called *players* and have courses of actions available to them called *strategies.* Each player expects a return or *payoff* when following a particular action. The magnitude of this payoff depends on the other players' actions and payoffs. A satisfactory payoff to each player is termed a *solution* to the game. However, in many cases, such a solution does not exist. A two-person game is a *zero-sum* game if one player's gain is exactly equal to the other's loss, otherwise it is a *non-zero sum* game.

There are two broad classes of games: *cooperative* and *non-cooperative.* In a cooperative game the players can communicate with each other and have an incentive to cooperate. A solution to a cooperative game exists if no player is better off by unilaterally deviating from the cooperative solution. In a non-cooperative game, no communication exists between players.

Differential Games

A *differential game* is a mathematical model of a competitive situation which evolves continuously in time (Dockner *et al.*, 2000). The two main features of a differential game are: (i) the set of state variables characterizing the dynamical system, (ii) evolution of the state variables described by a set of differential equations. Due to their dynamic nature, differential games are well suited for modelling resource management problems. Consider a differential game played by N players over a time horizon $[0,\ T]$ where T could be infinite, in which case the time horizon becomes $[0,\ \infty)$. The n-dimensional state vector is denoted by x. When $N = 1$, the differential game reduces to an optimal control problem. The variable u_i, known as the control variable, denotes the action taken by player i and is, in general, an m-dimensional vector. If the game is non-zero sum, each player i seeks to maximize his total payoff over the planning horizon, discounted at the rate $\rho_i \geq 0$:

$$J_i = \phi_i(x(T), T) + \int_0^T e^{-\rho_i t} g_i(x,u,t)dt, \ldots, i = 1, 2, \ldots, N$$

subject to

$$\dot{x} = f(x,u,t).$$

where $u = (u_1, u_2, ..., u_N)$ is called a *strategy profile*. Most of the problems in economics fall into the category of non-cooperative games (Dockner *et al.*, 2000). Since we are investigating a bioeconomic model of a fishery, we confine our discussion to non-cooperative game theory. The simplest non-cooperative solution strategy is a *non-cooperative Nash equilibrium* (Nash, 1951). A strategy profile $(u_1^*, u_2^*, ..., u_i^*, ..., u_N^*)$ is said to be a Nash equilibrium if the following holds for $i = 1, 2, \ldots, N$:

$$J_i^*(u_1^*, u_2^*, \ldots u_i^*, \ldots, u_N^*) \geq J_i(u_1^*, u_2^*, \ldots u_i, \ldots, u_N^*) \quad \text{for each admissible } u_i.$$

i.e., unilateral deviation from a Nash strategy does not result in a better payoff. However, a Nash equilibrium may not be unique. Furthermore, for each player, we can define the Hamiltonian in the usual manner as

$$\mathcal{H}_i(x,u,\lambda,t) = e^{-\rho_i t} g_i(x,u,t) + \lambda^{\mathsf{T}} f(x,u,t).$$

Then the conditions for a Nash equilibrium can be specified as

$$\mathcal{H}_i^*(u_1^*, u_2^*, \ldots u_i^*, \ldots, u_N^*, \lambda, t) \geq \mathcal{H}_i(u_1^*, u_2^*, \ldots u_i, \ldots, u_N^*, \lambda, t). \tag{7.2}$$

In addition to Equations (7.2) we have the boundary conditions

$$\dot{x}_i = \frac{\partial \mathcal{H}_i^*}{\partial \lambda_i},$$

and costate equations

$$\dot{\lambda}_i = -\frac{\partial \mathcal{H}_i^*}{\partial x_i},$$

with transversality conditions

$$\lambda_i(T) = \frac{\partial \phi_i}{\partial x_i(T)}.$$

The solution to the game consists of the vector $u^* = [u_1^* \ldots u_N^*]^\mathsf{T}$ which provides the action of each player in an analytical form. A Nash equilibrium can be *open-loop* or *closed-loop*. An *open-loop strategy* is a strategy dependent on time t only, and is fixed from the outset. A *closed-loop strategy*, on the other hand, is conditioned both on time t and state vector $x(t)$. What occurs prior to time t is irrelevant, only the current state vector is of significance. For this reason, closed-loop strategies are also known as *Markovian strategies*.

Open-loop and closed-loop solutions of differential games

Consider the N-player differential game where the ith player wishes to choose his control u_i to maximize

$$J_i = \int_0^T g_i(x, t, u_1, \ldots, u_N)dt,$$

subject to the constraints

$$\dot{x} = f(x, t, u_1, \ldots, u_N), \quad x(0) = x_0,$$

where x is a state vector of dimension 1. We assume that there are no constraints either on the controls or the state variable.

Open-loop Nash solutions If the ith player is given the open-loop Nash controls $u_j^*(t)$, where $j \neq i$, for all his competitors then he can obtain a set of necessary conditions for his own open-loop Nash control using the established methods of optimal control theory, as follows: Define the Hamiltonian for the ith player as

$$\mathcal{H}_i(x, t, u_1, \ldots, u_N, \lambda_i) = g_i(x, t, u_1, \ldots, u_N) + \lambda_i f(x, t, u_1, \ldots, u_N).$$

Since $u_i^*(t)$ must maximize J_i when the other players use their Nash controls, the following first order necessary conditions must hold:

$$\begin{aligned}
\dot{x} &= f(x, t, u_1^*, \ldots, u_N^*), \quad ,x(0) = x_0, \\
\dot{\lambda}_i(t) &= -\frac{\partial \mathcal{H}_i(x, t, u_1^*, \ldots, u_N^*, \lambda_i)}{\partial x}, ...,\lambda_i(T) = 0, \\
& \frac{\partial \mathcal{H}_i(x, t, u_1^*, \ldots, u_N^*, \lambda_i)}{\partial u_i} = 0. \qquad (7.3) \\
& \frac{\partial^2 \mathcal{H}_i(x, t, u_1^*, \ldots, u_N^*, \lambda_i)}{\partial u_i^2} \text{is negative semi-definite on the optimal Nash path} \qquad (7.4)
\end{aligned}$$

Equations (7.3) for $i = 1, 2, \ldots, N$ provide a set of necessary conditions for the entire N-tuple of Nash open-loop strategies. These equations can be solved either numerically or analytically to obtain a control vector $u^*(t)$ as a function of $x, t, \lambda_1, \ldots, \lambda_N$. It must be noted that the ith player's Hamiltonian is maximized only with respect to his own control, i.e.

$$\frac{\partial \mathcal{H}_i(x, t, u_1^*, \ldots, u_N^*, \lambda_i)}{\partial u_j} \neq 0, i \neq j.$$

Closed-loop Nash solutions If the ith player is given the closed-loop strategies $u_j^*(x, t)$, where $j \neq i$, for all his competitors then he has to solve the system of following necessary conditions:

$$\begin{aligned}
\dot{x} &= f(x, t, u_1^*, \ldots, u_N^*), \ldots, x(0) = x_0, \\
\dot{\lambda}_i(t) &= -\frac{\partial \mathcal{H}_i(x, t, u_1^*, \ldots, u_N^*, \lambda_i)}{\partial x} - \sum_{j \neq i}^{N} \frac{\partial \mathcal{H}_i}{\partial u_j} \frac{\partial u_j}{\partial x}, \quad , \lambda_i(T) = 0, \\
& \frac{\partial \mathcal{H}_i(x, t, u_1^*, \ldots, u_N^*, \lambda_i)}{\partial u_i} = 0.
\end{aligned}$$

We see that for $N \geq 2$ the summation term $\sum_{j \neq i}^{N} \frac{\partial \mathcal{H}_i}{\partial u_j} \frac{\partial u_j}{\partial x}$ gives rise to a set of partial differential equations, generally extremely difficult to solve, unlike in the single player case ($N = 1$) where the summation term is absent.

Another approach dealing with closed-loop Nash controls is the dynamic programming technique. If $V_i(x_1, t_1)$ denotes the optimal profit for the ith player starting from state x_1 at time t_1 then

$$\begin{aligned} V_i(x_1, t_1) &= \max_{u_i} \left[g_i(x_1, t_1)\delta t + V_i(x_1 + \delta x_1, t_1 + \delta t) \right], \\ -\frac{\partial V_i}{\partial t} &= \max_{u_i} \mathcal{H}_i, \ldots, V(x(T), T) = 0. \end{aligned}$$

These are the generalized Hamilton-Jacobi-Bellman equations. The solutions to these are obtained by integrating backwards from the terminal boundary condition $V(x(T), T) = 0$.

Differential games in fisheries resource management

The problem of managing a fishery resource being shared by two different nations or firms was illustrated by Clark (1985), using the famous *Prisoner's Dilemma* (Luce, 1958) discussed extensively in game theory. To keep the problem simple, Clark ignored the fishing costs. The simplified problem was to maximize the discounted revenue

$$\int_0^\infty e^{-\rho t} p h(x) dt,$$

subject to

$$\dot{x} = f(x) - h(x),$$

where it was assumed that the two countries hold identical financial opportunities (same fish price p and discount rate ρ) and have the same harvesting power. We now present a brief review of the approach taken by Clark. Straightforward application of the Euler-Lagrange equation from Section 2.2 (Equation 2.21 with cost equal to zero) yields the optimal biomass

$$f'(x^*) = \rho.$$

The global economic return from the resource at $x = x^*$ is the discounted revenue $\frac{pf(x^*)}{\rho}$; this is the present value of the revenue earned by employing sustainable harvesting leading to conservation of the resource. Suppose that $x(0) = x^*$, and consider two possible strategies: "deplete the resource" and "conserve the resource". Since the available biomass is x^*, depleting the resource as rapidly as possible would yield an economic return of px^*. Note that the growth rate is assumed to be zero during the time taken for depleting the resource.

Thus if both countries adopt the depletion strategy, each would get a return of $\frac{px^*}{2}$. If, on the other hand, they both adopt the conservation strategy then each would get a return of $\frac{pf(x^*)}{2\rho}$. Hence the payoff matrix for the fishing game is obtained as

$$\begin{array}{cc} & \begin{array}{cc}\text{conserve} & \text{deplete}\end{array} \\ \begin{array}{c}\text{conserve} \\ \\ \text{deplete}\end{array} & \left(\begin{array}{cc} \left(\frac{pf(x^*)}{2\rho}, \frac{pf(x^*)}{2\rho}\right) & (0, px^*) \\ \\ (px^*, 0) & \left(\frac{px^*}{2}, \frac{px^*}{2}\right) \end{array}\right). \end{array}$$

If $x^* > \frac{f(x^*)}{\rho}$, the depletion strategy obviously works better for each country (regardless of the strategy that the other country adopts). This will motivate each country to deplete the resource, thus receiving a return of $\frac{px^*}{2}$. However, if there was one player involved, it would be adopting the conservation strategy only which means $\frac{pf(x^*)}{\rho} > px^*$. It follows then that $\frac{px^*}{2} < \frac{pf(x^*)}{2\rho}$, which in turn renders the mutual depletion strategy inferior to the mutual conservation strategy. The solution ("deplete the resource","deplete the resource"), referred to as the *competitive equilibrium*, is decidedly inferior to the solution ("conserve the resource","conserve the resource"), referred to as the *cooperative solution*. If $x^* < \frac{f(x^*)}{2\rho}$, the first country would be better off conserving if the other country followed the same strategy. This is however a highly unstable situation as one country might be tempted to deplete when the other one conserves, thus leaving the country following the conservation strategy with essentially nothing. If both countries choose to deplete, they again end up with the inferior return of $\frac{px^*}{2}$.

A simple differential game with two players was solved by Clark (1980). The problem was an extension of the problem presented in Section 2.2, from one harvester to two harvesters. The fish growth model was the Gordon-Schaefer model:

$$\begin{aligned} \dot{x} &= rx\left(1 - \frac{x}{K}\right) - q_1E_1x - q_2E_2x, \ldots, x(0) = x_0, \\ 0 &\leq E_i \leq E_i^{\max}, \ldots, i = 1, 2, \end{aligned}$$

where i stands for the ith player. The harvesting costs were assumed to be linear in fishing effort and the objective of each player was again to maximize his respective discounted profit

$$J_i = \int_0^\infty e^{-\rho_i t}(p_iq_ix - c_i)E_idt, \ldots, i = 1, 2.$$

The optimal biomass corresponding to the scenario where each player was operating alone were denoted by $x_i^*, i = 1, 2$. These could be determined by solving the quadratic Euler-Lagrange equations (see Section 2.2):

$$(-2rp_iq_i)(x_i^*)^2 + [c_ir_i + Kp_iq_i(r_i - \rho_i)]\, x_i^* + Kc_i\rho_i = 0.$$

Additionally, $x_i^\infty, i = 1, 2$, denoted the bionomic equilibria at which the economic rent vanished. These were given by $x_1^\infty = \frac{c_1}{p_1q_1}$ and $x_2^\infty = \frac{c_2}{p_2q_2}$ (see Section 1.1).

Assuming that $x_1^\infty < x_2^\infty$, i.e. the first player is more "efficient" than the second player, the non-cooperative Nash equilibrium was given by

$$E_1^*(x) = \begin{cases} E_1^{\max}, & x(t) > \min(x_1^*, x_2^\infty), \\ \frac{r(1-\frac{x}{K})}{q_1}, & x(t) = \min(x_1^*, x_2^\infty), \\ 0, & x(t) < \min(x_1^*, x_2^\infty); \end{cases}$$

$$E_2^*(x) = \begin{cases} E_2^{\max}, & x(t) > x_2^\infty, \\ 0, & x(t) \le x_2^\infty. \end{cases}$$

This implies that the first player, being the most efficient, started by harvesting enough stock, thereby reducing it to an appropriate level that eliminated the second player. That is, the second player was forced to exit the fishery. If the efficiency of the first player was much higher than the second player then the former could harvest x_1^*, tantamount to operating alone.

A differential game between two players, where one player makes the first move and the second player follows, was considered by Benchekroun & Long (2002). In this work, the first mover's catch rate was $h_1(t) = E_1(t)x(t)$ and the net benefit was $R_1(h_1(t))$. The net benefit $R_1(h_1(t))$ was independent of the other player's fishing effort, $E_2(t)$, because of the first move. The net benefit to the follower, however, was dependent on both $h_1(t)$ and $h_2(t)$ as well as $x(t)$, denoted by $R_2(h_1(t), h_2(t), x(t))$, where $h_2(t) = E_2(t)x(t)$. The rate of fish growth was given by

$$\dot{x} = f(x(t), E_1(t), E_2(t)).$$

The objective of the first mover was to maximize

$$J_1 = \int_0^\infty e^{-\rho t} R_1(h_1(t))dt,$$

and the objective of the follower was to maximize

$$J_2 = \int_0^\infty e^{-\rho t} R_2(h_1(t), h_2(t), x(t))dt.$$

It was noted that the first mover had to follow a more conservationist approach than he normally would have if he was operating alone. Deviating from that strategy would trigger an aggressive response from the second mover, leading to overexploitation of the resource. This scenario, where the action of the second player depends on the action of the first, admits a solution normally called a *Stackelberg equilibrium*. The action of the leader can be observed by the follower before taking an action. This problem is of Markovian nature with solutions $E_1^*(x)$ and $E_2^*(E_1^*(x), x)$.

Dockner *et al.* (1989) studied a two-player differential game, and provided both Nash and Stackelberg equilibrium solutions to it. They maximized the usual objective function

$$J_i = \int_0^\infty e^{-\rho_i t}(pq_i x - c_i)E_i dt, \qquad i = 1, 2,$$

with no constraints on the effort and assuming a Gompertz growth function for the stock, given by

$$\dot{x} = x(a - b\ln x) - q_1 E_1 x - q_2 E_2 x.$$

They concluded that the profits of both players were higher in the Stackelberg case than in the corresponding Nash case. Extinction of the fish population in finite time was ruled out due to the structure of the growth model.

Munro (1979) investigated the problem of optimal management of renewable resources jointly owned by two states using Nash's theory of two-person cooperative games (Nash, 1953). In this work Munro assumed that there was only one harvest function $h(t)$, to be allocated by means of harvest shares to both states: $\alpha h(t)$ to the first, and $(1 - \alpha)h(t)$ to the other. The objective functions of the two countries could then be expressed as

$$\begin{aligned} J_1 &= \int_0^\infty e^{-\rho_1 t}\alpha(p - c(x))h(t)dt, \\ J_2 &= \int_0^\infty e^{-\rho_2 t}(1-\alpha)(p - c(x))h(t)dt. \end{aligned}$$

Through an agreement, both states decided to maximize the weighted sum of the objective functionals

$$J = \beta J_1 + (1-\beta)J_2, \ldots, 0 \leq \beta \leq 1,$$

where β was a bargaining parameter weighing the preferences of each state: $\beta = 1$ indicated that the first state was dominant and $\beta = 0$ indicated that the second state was dominant. Munro considered three cases: (i) $\rho_1 \neq \rho_2$ with equal harvesting costs, (ii) $\rho_1 = \rho_2$ with variable harvesting costs, (iii) $\rho_1 = \rho_2$ with equal harvesting costs and variable fish prices. The first two cases turned out to be easy to tackle but the third one proved to be significantly more difficult, allowing for the possibility that the optimal control could fail to exist.

Nash's cooperative solution is not the only game theoretic model available. Another approach is based on the *Shapley value* (Shapely, 1953). In a survey paper related to fisheries management, Bjørndal *et al.* (2000) considered the Shapley value approach. According to Shapley, players form coalitions which define their contribution in the cooperative agreement and, as a consequence, their bargaining strengths. This is done by the definition of a characteristic function which assigns a particular value to each member of a coalition. The Shapley value is deemed to be a fair treatment to all players involved as it takes into account their relative strengths. A paper by White & Mace (1988) analyzed a four-player game between two harvesters and two processors based on the characteristic function approach.

Finally, the players may be receiving measurements of the state subject to noise. This situation gives rise to stochastic differential games where the state equation is generally of the form

$$\dot{x} = f(x) - \sum_{i=1}^{N} q_i E_i x + \sigma x dw,$$

and the objective, as usual, is to maximize the expectation of the present value of the discounted profit

$$J_i = \max_{E_i} E\left[\int_0^T e^{-\rho_i t}(pq_iE_ix - c_i(E_i))dt\right].$$

Examples of such games applied to fisheries are found in Jørgensen & Yeung (1996) and in Laukkanen (2003).

A formulation of a non-cooperative differential game with two players

We now follow a game-theoretic approach to formulate the optimal harvesting problem presented in Chapter 3. The differential game with two players is posed thus

$$J_i = \max_{u_i} \int_0^T e^{-\rho_i t}(p_iq_iE_ix - c_i(E_i))dt, \quad , i = 1, 2$$

subject to

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - q_1E_1x - q_2E_2x, \quad x(0) = x_0,$$

where $c_i(E_i)$ are the quadratic harvesting costs: $c_1(E_1) = a_1E_1 + \frac{b_1}{2}E_1^2$; $c_2(E_2) = a_2E_2 + \frac{b_2}{2}E_2^2$, and the index i refers to the ith player.

Open-loop Nash equilibrium We first present the case where fishing effort is only a function of time and is independent of the state. The Nash equilibrium obtained here would correspond to the open-loop solution. The Hamiltonians for the two players are formed as

$$\mathcal{H}_i = e^{-\rho_i t}(p_iq_iE_i(t)x - c_i(E_i(t))) + \lambda_i\left[rx\left(1 - \frac{x}{K}\right) - q_1E_1(t)x - q_2E_2(t)x\right], \quad i = 1, 2.$$

The necessary optimality conditions are:

$$\begin{aligned} \frac{\partial \mathcal{H}_i}{\partial E_i} &= e^{-\rho_i t}(p_iq_ix - c_i^{'}(E_i(t)) - \lambda_iq_ix = 0, \quad i = 1, 2, \\ \dot{\lambda}_i &= -e^{-\rho_i t}p_iq_iE_i(t) - r\lambda_i + 2r\lambda_i\frac{x}{K} + \lambda_i(q_1E_1(t) + q_2E_2(t)), \lambda_i(T) = 0, \quad i = 1, 2. \end{aligned}$$

Closed-loop Nash equilibrium We now consider the case where fishing effort is dependent on both current state and time. The two Hamiltonians as well as the stationarity conditions are the same as in the open-loop case, but E_1 and E_2 now are functions of the stock as well as time. The additional complication arises from the extra terms in the costate equations:

$$\begin{aligned}
\dot{\lambda}_i &= -e^{-\rho_i t}\left(p_i q_1 E_i(t,x(t)) + p_i q_i x \frac{\partial E_i(t,x(t))}{\partial x} - \frac{\partial c_i(E_i(t,x(t))}{\partial x}\right) \\
&\quad +\lambda_i \Big(-r + 2r\lambda_i \frac{x}{K} + q_1 E_1(t,x(t)) + q_2 E_2(t,x(t)) \\
&\qquad +q_1 x \frac{\partial E_1(t,x(t))}{\partial x} + q_2 x \frac{\partial E_2(t,x(t))}{\partial x}\Big), \quad i = 1,2, \\
\lambda_i(T) &= 0, \quad i = 1,2.
\end{aligned}$$

Some open problems for differential games in fish harvesting We have presented a brief open-loop and closed-loop formulation of the optimal harvesting problem in deterministic settings demonstrating the complexity of the two-harvester differential game. We identify below two major open problems in this area:

(i) The players can only access noise-corrupted measurements of the state, i.e., the amount of stock is not known precisely but its growth can be approximately modelled by a stochastic differential equation. This gives rise to stochastic differential games.

(ii) Some players adopt open-loop strategies while others adopt closed-loop strategies.

Further research is needed to provide more insights and deepen our understanding of optimal harvesting strategies for fisheries.

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Appendix A

Numerical solution for deterministic growth and constant price

We want to simulate the deterministic Hamilton Jacobi Bellman (HJB) partial differential equation given by

$$\begin{aligned} -\frac{\partial J^*(x(t),t)}{\partial t} &= \left(p(t)qx(t) - c_1 - \frac{c_2}{2}E(t)\right)E^*(t) - \delta J^*(x(t),t) \\ &\quad + \frac{\partial J^*(x(t),t)}{\partial x}\left\{rx(t)\left(1-\frac{x(t)}{K}\right) - qE^*(t)x(t)\right\}, \end{aligned} \tag{A.1}$$

where $E^*(t)$ is evaluated using

$$E^*(t) = \begin{cases} 0, & \left(p - \frac{\partial J^*}{\partial x}\right)\frac{qx(t)}{c_2} - \frac{c_1}{c_2} < 0 \\ \left(p - \frac{\partial J^*}{\partial x}\right)\frac{qx(t)}{c_2} - \frac{c_1}{c_2}, & 0 \le \left(p - \frac{\partial J^*}{\partial x}\right)\frac{qx(t)}{c_2} - \frac{c_1}{c_2} \le E_{\max}, \\ E_{\max}, & \left(p - \frac{\partial J^*}{\partial x}\right)\frac{qx(t)}{c_2} - \frac{c_1}{c_2} > E_{\max}. \end{cases} \tag{A.2}$$

subject to the boundary conditions $J^*(x,T) = 0$ and $E^*(x_{\min},t) = 0$.

We normalize the population with respect to the carrying capacity K and denote $X(t) = \frac{x(t)}{K}$. Using the normalized population, Equations (A.1) and (A.2) can be reformulated as:

$$\begin{aligned} -\frac{\partial J^*(X(t),t)}{\partial t} &= \left(pqKX(t) - c_1 - \frac{c_2}{2}E^*(t)\right)E^*(t) - \delta J^*(X(t),t) \\ &\quad + \frac{\partial J^*(X(t),t)}{\partial X(t)}\left\{rX(t)\left(1 - X(t)\right) - qE^*(t)X(t)\right\}, \end{aligned} \tag{A.3}$$

and

$$E^*(t) = \begin{cases} 0, & \left(p - \frac{1}{K}\frac{\partial J^*}{\partial X}\right)\frac{qKX(t)}{c_2} - \frac{c_1}{c_2} < 0 \\ \left(p - \frac{\partial J^*}{K\partial X}\right)\frac{qKX(t)}{c_2} - \frac{c_1}{c_2}, & 0 \leq \left(p - \frac{1}{K}\frac{\partial J^*}{\partial X}\right)\frac{qKX(t)}{c_2} - \frac{c_1}{c_2} \leq E_{\max}, \\ E_{\max}, & \left(p - \frac{1}{K}\frac{\partial J^*}{\partial X}\right)\frac{qKX(t)}{c_2} - \frac{c_1}{c_2} > E_{\max}. \end{cases} \tag{A.4}$$

We recall that harvesting is initiated at the initial time 0 and is terminated at the final time T (a fixed constant). The interval $[0,\, T]$ is partitioned as $0 = t_0, t_1, t_2, \ldots, t_N = T$ where each subinterval is of equal length, denoted by Δt. The normalized population $X(t)$ lies between $X_{\min} = x_{\min}/K$ and 1, and the interval $[X_{\min},\, 1]$ is partitioned as $X_{\min}, X_1, X_2, \ldots, X_M = 1$ where all the sub-partitions are of same length, denoted by ΔX.

We let $J^*_{i,n}$ and $E^*_{i,n}$ denote the values of J^* and E^* respectively at $X = X_i$ and $t = t_n$ where $0 \leq i \leq M$ and $0 \leq n \leq N$. The first order derivative of J^* with respect to the temporal variable t is approximated by:

$$\frac{\partial J^*}{\partial t} = \frac{J^*_{i,n+1} - J^*_{i,n}}{\triangle t} \quad \text{for } 0 \leq n \leq N-1.$$

The values of J^* and its spatial derivative are approximated as an average of the respective values at the nth and the $(n+1)th$ time step where $0 \leq n \leq N-1$. The first order spatial derivative is approximated by:

$$\begin{aligned} \frac{\partial J^*}{\partial X} &= \frac{1}{2}\left[\left(\frac{J^*_{i+1,n+1} - J^*_{i-1,n+1}}{2\triangle X}\right)\right. \\ &\quad \left. + \left(\frac{J^*_{i+1,n} - J^*_{i-1,n}}{2\triangle X}\right)\right] \quad \text{for } 0 \leq i \leq M-1, \\ \frac{\partial J^*}{\partial X} &= \frac{1}{2}\left[\left(\frac{3J^*_{M,n} - 4J^*_{M-1,n} + J^*_{M-2,n}}{2\triangle X}\right)\right. \\ &\quad \left. + \left(\frac{3J^*_{M,n+1} - 4J^*_{M-1,n+1} + J^*_{M-2,n+1}}{2\triangle X}\right)\right] \quad \text{for } i = M. \end{aligned}$$

The error in the temporal and the spatial derivatives above is of order $O(dt^2)$ and $O(dX^2)$ respectively.

These approximations are then substituted in Equations (A.3) and (A.4). The numerical solution is calculated by moving backwards in time since we have obtained the HJB equation by using dynamic programming which involves backward recursion while performing optimization. We know the J^* values at the terminal time $T = t_N$ which can be used to determine the J^* values at time t_{N-1} and so on. In other words, we feed the J^* values at each time t_{n+1} in the finite difference approximation of the HJB equation to evaluate the corresponding J^* values at time t_n. Discretizing the HJB equation for $1 \leq i \leq M-1$ we obtain

$$\begin{aligned}
-\frac{J^*_{i,n+1} - J^*_{i,n}}{\triangle t} &= \left[pqKX_i - c_1 - \frac{c_2}{2} E^*_{i,n+1}\right] E^*_{i,n+1} - \delta \left(\frac{J^*_{i,n}}{2} + \frac{J^*_{i,n+1}}{2} \right) \\
&+ \frac{1}{2} \left[\frac{J^*_{i+1,n} - J^*_{i-1,n}}{2\triangle X} \right] \left\{ rX_i \left(1 - X_i\right) - qE^*_{i,n+1} X_i \right\} \\
&+ \frac{1}{2} \left[\frac{J^*_{i+1,n+1} - J^*_{i-1,n+1}}{2\triangle X} \right] \left\{ rX_i \left(1 - X_i\right) - qE^*_{i,n+1} X_i \right\},
\end{aligned} \tag{A.5}$$

and

$$E^*_{i,n+1} = \left[p - \frac{J^*_{i+1,n+1} - J^*_{i-1,n+1}}{2\triangle XK} \right] \frac{qKX_i}{c_2} - \frac{c_1}{c_2}. \tag{A.6}$$

For $i = M$ we get the numerical scheme:

$$\begin{aligned}
-\frac{J^*_{M,n+1} - J^*_{M,n}}{\triangle t} &= \left[pqX_M - c_1 - \frac{c_2}{2} E^*_{M,n+1}\right] E^*_{M,n+1} - \delta \left(\frac{J^*_{M,n}}{2} + \frac{J^*_{M,n+1}}{2} \right) \\
&+ \frac{1}{2} \left[\frac{3J^*_{M,n} - 4J^*_{M-1,n} + J^*_{M-2,n}}{2\triangle X} \right] \left\{ rX_M \left(1 - X_M\right) - qE^*_{M,n+1} X_M \right\} \\
&+ \frac{1}{2} \left[\frac{3J^*_{M,n+1} - 4J^*_{M-1,n+1} + J^*_{M-2,n+1}}{2\triangle X} \right] \left\{ rX_M \left(1 - X_M\right) - qE^*_{M,n+1} X_M \right\}
\end{aligned} \tag{A.7}$$

and

$$E^*_{M,n+1} = \left[p - \frac{3J^*_{M,n+1} - 4J^*_{M-1,n+1} + J^*_{M-2,n+1}}{2\triangle XK} \right] \frac{qKX_M}{c_2} - \frac{c_1}{c_2}. \tag{A.8}$$

The optimal effort is forced to satisfy the constraints by performing the following check:

$$\begin{aligned}
&\text{if } \; E^*_{i,n+1} < 0 && \text{then put } \; E^*_{i,n+1} = 0, \\
&\text{if } \; E^*_{i,n+1} > E_{\max} && \text{then put } \; E^*_{i,n+1} = E_{\max}.
\end{aligned}$$

Rearranging terms in Equation (A.5) and collecting the coefficients gives

$$
\begin{aligned}
& \frac{\Delta t}{4\Delta X}\left\{rX_i\left(1-X_i\right)-qE^*_{i,n+1}X_i\right\}J^*_{i-1,n}+\left(1+\frac{\delta\Delta t}{2}\right)J^*_{i,n} \\
& -\frac{\Delta t}{4\Delta X}\left\{rx_i\left(1-X_i\right)-qE^*_{i,n+1}X_i\right\}J^*_{i+1,n} \\
= & -\frac{\Delta t}{4\Delta X}\left\{rX_i\left(1-X_i\right)-qE^*_{i,n+1}X_i\right\}J^*_{i-1,n+1}+\left(1-\frac{\delta\Delta t}{2}\right)J^*_{i,n+1} \\
& +\frac{\Delta t}{4\Delta X}\left\{rX_i\left(1-X_i\right)-qE^*_{i,n+1}X_i\right\}J^*_{i+1,n+1} \\
& +\left(pqKX_i-c_1-\frac{c_2}{2}E^*_{i,n+1}\right)E^*_{i,n+1}.
\end{aligned}
\tag{A.9}
$$

For $i=M$, rearranging terms in Equation (A.7) yields

$$
\begin{aligned}
& -\frac{\Delta t}{4\Delta X}\left\{rX_M\left(1-X_M\right)-qE^*_{M,n+1}X_M\right\}J^*_{M-2,n} \\
& +\frac{\Delta t}{\Delta X}\left\{rX_M\left(1-\frac{X_M}{K}\right)-qE^*_{M,n+1}X_M\right\}J^*_{M-1,n} \\
& +\left(1-\frac{3\Delta t}{4\Delta X}\left\{rX_M\left(1-X_M\right)-qE^*_{M,n+1}X_M\right\}+\frac{\delta\Delta t}{2}\right)J^*_{M,n} \\
= & \frac{\Delta t}{4\Delta X}\left\{rX_M\left(1-X_M\right)-qE^*_{M,n+1}X_M\right\}J^*_{M-2,n+1} \\
& -\frac{\Delta t}{\Delta X}\left\{rX_M\left(1-X_M\right)-qE^*_{M,n+1}X_M\right\}J^*_{M-1,n+1} \\
& +\left(1+\frac{3\Delta t}{4\Delta X}\left\{rX_M\left(1-X_M\right)-qE^*_{M,n+1}X_M\right\}-\frac{\delta\Delta t}{2}\right)J^*_{M,n+1} \\
& +\left(pqX_M-c_1-\frac{c_2}{2}E^*_{M,n+1}\right)E^*_{M,n+1}.
\end{aligned}
\tag{A.10}
$$

Hence we have obtained a system of M equations represented by Equations (A.9) and (A.10) and we can write this system in matrix form as follows:

$$
AJ^*_n = BJ^*_{n+1} + D \tag{A.11}
$$

where $J^*_n = \begin{bmatrix} J^*_{1,n} & J^*_{2,n} & \cdot & \cdot & J^*_{M,n} \end{bmatrix}^{\mathsf{T}}$.

We solve the system (A.11) for each $n = N-1, ..., 1, 0$ to find J^*_n. In this process the optimal effort is found implicitly from Equations (A.6) and (A.8). Finally we obtain two matrices, J^* and E^*, where each $J^*_{i,n}$ corresponds to the total discounted profit earned by initiating the harvest at time t_n with normalized initial population level $X\left(t_n\right) = X_i$, and each $E^*_{i,n}$ represents the required optimal effort when the population level is equal to X_i at time t_n. The optimal solution path starting from the given initial population level x_0 at the initial time t_0 can then be determined by interpolation.

Appendix B

Numerical solution for stochastic growth and fixed-price

The objective is to integrate numerically the Hamilton Jacobi Bellman partial differential equation for stochastic growth and fixed price, given by

$$
\begin{aligned}
-\frac{\partial J^*(x(t),t)}{\partial t} &= \left(pqx(t) - c_1 - \frac{c_2}{2}E^*(t)\right)E^*(t) - \delta J^*(x(t),t) \\
&+ \frac{\partial J^*(x(t),t)}{\partial x}\left\{rx(t)\left(1-\frac{x(t)}{K}\right) - qE^*(t)x(t)\right\} \\
&+ \frac{1}{2}\frac{\partial^2 J^*(x(t),t)}{\partial x^2}\sigma_1^2 x(t)^2, \qquad \text{(B.1)}
\end{aligned}
$$

where

$$
E^*(t) = \begin{cases} 0, & \left(p - \frac{\partial J^*}{\partial x}\right)\frac{qx(t)}{c_2} - \frac{c_1}{c_2} < 0 \\ \left(p - \frac{\partial J^*}{\partial x}\right)\frac{qx(t)}{c_2} - \frac{c_1}{c_2}, & 0 \le \left(p - \frac{\partial J^*}{\partial x}\right)\frac{qx(t)}{c_2} - \frac{c_1}{c_2} \le E_{\max}, \\ E_{\max}, & \left(p - \frac{\partial J^*}{\partial x}\right)\frac{qx(t)}{c_2} - \frac{c_1}{c_2} > E_{\max}. \end{cases} \qquad \text{(B.2)}
$$

subject to the boundary conditions $J^*(x,T) = 0$ and $E^*(x_{\min},t) = 0$.

Normalizing the population with respect to the carrying capacity K and denoting the normalized population level by $X(t)$ gives $X(t) = \frac{x(t)}{K}$. Using this, Equations (B.1) and (B.2) can be rewritten as:

$$\begin{aligned}-\frac{\partial J^*(X(t),t)}{\partial t} &= \left(pqKX(t) - c_1 - \frac{c}{2}E^*(t)\right)E^*(t) - \delta J^*(X(t),t)\\ &+ \frac{\partial J^*(X(t),t)}{\partial X}\left(rX(t)\left(1-X(t)\right) - qE^*(t)X(t)\right)\\ &+ \frac{1}{2}\frac{\partial^2 J^*(X(t),t)}{\partial X^2}\sigma_1^2 X^2, \end{aligned} \tag{B.3}$$

and

$$E^*(t) = \begin{cases} 0, & \left(p - \frac{1}{K}\frac{\partial J^*}{\partial X}\right)\frac{qKX(t)}{c_2} - \frac{c_1}{c_2} < 0 \\ \left(p - \frac{\partial J^*}{K\partial X}\right)\frac{qKX(t)}{c_2} - \frac{c_1}{c_2}, & 0 \le \left(p - \frac{1}{K}\frac{\partial J^*}{\partial X}\right)\frac{qKX(t)}{c_2} - \frac{c_1}{c_2} \le E_{\max}, \\ E_{\max}, & \left(p - \frac{1}{K}\frac{\partial J^*}{\partial X}\right)\frac{qKX(t)}{c_2} - \frac{c_1}{c_2} > E_{\max}. \end{cases} \tag{B.4}$$

As in the deterministic case, the harvesting is carried on from the initial time 0 to the fixed final time T and the interval $[0,\ T]$ is uniformly partitioned as $0 = t_0, t_1, t_2, \ldots, t_N = T$ where $t_{n+1} - t_n = \Delta t$ for $0 \le n \le N-1$. The spatial discretization is performed as $X_{\min} = X_0, X_1, X_2, \ldots, X_M = 1$ where the sub-partitions are of uniform length ΔX. Recall that we know the final boundary condition for the total discounted profit, so in order to determine the solution we need to perform numerical integration moving backwards in time. We let $J^*_{i,n}$ and $E^*_{i,n}$ denote the values of J^* and E^* respectively at $X = X_i$ and $t = t_n$ for $0 \le i \le M$ and $N \ge n \ge 0$. Using the Crank-Nicolson method, the spatial derivatives of $J^*$are approximated as an average of the corresponding values at the nth and the $(n+1)\,th$ time step for $0 \le n \le N-1$ as follows:

$$\begin{aligned}\frac{\partial J^*(X(t),p(t),t)}{\partial X} &= \frac{1}{2}\left[\left(\frac{J^*_{i+1,n+1} - J^*_{i-1,n+1}}{2\triangle X}\right)\right.\\ &\left. + \left(\frac{J^*_{i+1,n} - J^*_{i-1,n}}{2\triangle X}\right)\right] \quad \text{for } 1 \le i \le M-1,\\ \frac{\partial J^*(X(t),p(t),t)}{\partial X} &= \frac{1}{2}\left[\left(\frac{3J^*_{M,n} - 4J^*_{M-1,n} + J^*_{M-2,n}}{2\triangle X}\right)\right.\\ &\left. + \left(\frac{3J^*_{M,n+1} - 4J^*_{M-1,n+1} + J^*_{M-2,n+1}}{2\triangle X}\right)\right] \quad \text{for } i = M.\end{aligned}$$

The second order spatial derivatives of $J^*(X(t), p(t), t)$ are approximated by:

$$\begin{aligned}
\frac{\partial^2 J^*(X(t),p(t),t)}{\partial X^2} &= \frac{1}{2}\left[\left(\frac{J^*_{i+1,j,n+1} - 2J^*_{i,j,n+1} + J^*_{i-1,j,n+1}}{\triangle X^2}\right)\right. \\
&\quad \left. + \left(\frac{J^*_{i+1,j,n} - 2J^*_{i,j,n} + J^*_{i-1,j,n}}{\triangle X^2}\right)\right] \quad \text{for } 1 \le i \le M-1, \\
\frac{\partial^2 J^*(X(t),p(t),t)}{\partial X^2} &= \frac{1}{2}\left[\left(\frac{2J^*_{M,j,n+1} - 5J^*_{M-1,j,n+1} + 4J^*_{M-2,j,n+1} - J^*_{M-3,j,n+1}}{\triangle X^2}\right)\right. \\
&\quad \left. + \left(\frac{2J^*_{M,j,n} - 5J^*_{M-1,j,n} + 4J^*_{M-2,j,n} - J^*_{M-3,j,n}}{\triangle X^2}\right)\right] \quad \text{for } i = M.
\end{aligned}$$

The temporal derivative is approximated by:

$$\frac{\partial J}{\partial t} = \frac{J^*_{i,n+1} - J^*_{i,n}}{\triangle t} \qquad \text{for } 0 \le n \le N-1.$$

For $1 \le i \le M-1$, discretizing the HJB equation (B.3) gives

$$\begin{aligned}
-\frac{J^*_{i,n+1} - J^*_{i,n}}{\triangle t} &= \left(pqKX_i - c_1 - \frac{c_2}{2}E^*_{i,n+1}\right)E^*_{i,n+1} - \delta\left(\frac{J^*_{i,n}}{2} + \frac{J^*_{i,n+1}}{2}\right) \\
&\quad + \frac{1}{2}\left[\left(\frac{J^*_{i+1,n+1} - J^*_{i-1,n+1}}{2\triangle X}\right)\right. \\
&\quad \left. + \left(\frac{J^*_{i+1,n} - J^*_{i-1,n}}{2\triangle X}\right)\right]\left\{rX_i\left(1 - X_i\right) - qE^*_{i,n+1}X_i\right\} \\
&\quad + \frac{1}{2}\left[\left(\frac{J^*_{i+1,n+1} - 2J^*_{i,n+1} + J^*_{i-1,n+1}}{\triangle X^2}\right)\right. \\
&\quad \left. + \left(\frac{J^*_{i+1,n} - 2J^*_{i,n} + J^*_{i-1,n}}{\triangle X^2}\right)\right]\frac{\sigma_1^2 X_i^2}{2}
\end{aligned} \tag{B.5}$$

For $i = M$, we get the numerical scheme:

$$\begin{aligned}
-\frac{J^*_{M,n+1} - J^*_{M,n}}{\triangle t} &= \left(pqKX_M - c_1 - \frac{c_2}{2}E^*_{M,n+1}\right)E^*_{M,n+1} - \delta\left(\frac{J^*_{M,n}}{2} + \frac{J^*_{M,n+1}}{2}\right) \\
&\quad + \frac{1}{2}\left[\left(\frac{3J^*_{M,n+1} - 4J^*_{M-1,n+1} + J^*_{M-2,n+1}}{2\triangle X}\right)\right. \\
&\quad \left. + \left(\frac{3J^*_{M,n} - 4J^*_{M-1,n} + J^*_{M-2,n}}{2\triangle X}\right)\right]\left\{rX_M\left(1 - X_M\right) - qE^*_{M,n+1}X_M\right\} \\
&\quad + \frac{1}{2}\left[\left(\frac{2J^*_{M,n+1} - 5J^*_{M-1,n+1} + 4J^*_{M-2,n+1} - J^*_{M-3,n+1}}{\triangle X^2}\right)\right. \\
&\quad \left. + \left(\frac{2J^*_{M,n} - 5J^*_{M-1,n} + 4J^*_{M-2,n} - J^*_{M-3,n}}{\triangle X^2}\right)\right]\frac{\sigma_1^2 X_M^2}{2}.
\end{aligned} \tag{B.6}$$

The error in the temporal and the spatial derivatives approximated here is of order $O(dt^2)$ and $O(dX^2)$ respectively. The optimal effort is calculated from Equation (B.4) and is approximated as follows:

$$\begin{aligned}
E^*_{i,n+1} &= \left(p - \frac{J^*_{i+1,n+1} - J^*_{i-1,n+1}}{2\triangle X}\frac{1}{K}\right)\frac{qKX_i}{c_2} - \frac{c_1}{c_2} \quad \text{for } 1 \leq i \leq M-1, \\
E^*_{M,n+1} &= \left(p - \frac{3J^*_{M,n+1} - 4J^*_{M-1,n+1} + J^*_{M-2,n+1}}{2\triangle X}\frac{1}{K}\right)\frac{qKX_M}{c_2} - \frac{c_1}{c_2}.
\end{aligned}$$

We force the effort to fsatisfythe constraints by performing the following check:

$$\begin{aligned}
&\text{if } \; E^*_{i,n+1} < 0 && \text{then put } \; E^*_{i,n+1} = 0, \\
&\text{if } \; E^*_{i,n+1} > E_{\max} && \text{then put } \; E^*_{i,n+1} = E_{\max},
\end{aligned}$$

and substitute it in equations (B.5) and (B.6). Substituting and rearranging the terms in Equation (B.5) we get:

$$\begin{aligned}
&\left(\frac{\Delta t}{4\Delta X}\left\{rX_i\left(1-X_i\right) - qE^*_{i,n}X_i\right\} - \frac{\Delta t}{4}\frac{\sigma_1^2 X_i^2}{\triangle X^2}\right) J^*_{i-1,n} \\
&+ \left(1 + \frac{\delta\Delta t}{2} + \frac{\Delta t}{2}\frac{\sigma_1^2 X_i^2}{\triangle X^2}\right) J^*_{i,n} \\
&+ \left(-\frac{\Delta t}{4\Delta X}\left\{rX_i\left(1-X_i\right) - qE^*_{i,n+1}X_i\right\} - \frac{\Delta t}{4}\frac{\sigma_1^2 X_i^2}{\triangle X^2}\right) J^*_{i+1,n} \\
= \; &\left(pqKX_i - c_1 - \frac{c_2}{2}E^*_{i,n+1}\right) E^*_{i,n+1} \\
&+ \left(-\frac{\Delta t}{4\Delta X}\left\{rX_i\left(1-X_i\right) - qE^*_{i,n+1}X_i\right\} + \frac{\Delta t}{4}\frac{\sigma_1^2 X_i^2}{\triangle X^2}\right) J^*_{i-1,n+1} \\
&+ \left(1 - \frac{\delta\Delta t}{2} - \frac{\Delta t}{2}\frac{\sigma_1^2 X_i^2}{\triangle X^2}\right) J^*_{i,n+1} \\
&+ \left(\frac{\Delta t}{4\Delta X}\left\{rX_i\left(1-X_i\right) - qE^*_{i,n+1}X_i\right\} + \frac{\Delta t}{4}\frac{\sigma_1^2 X_i^2}{\triangle X^2}\right) J^*_{i+1,n+1}.
\end{aligned} \tag{B.7}$$

Rearranging the terms in Equation (B.6) we obtain:

$$\begin{aligned}
&\left(\frac{\Delta t}{4\Delta x^2}\sigma_1^2 X_M^2\right) J^*_{M-3,n} \\
&+\left(-\frac{\Delta t}{4\Delta X}\left\{rX_M\left(1-X_M\right)-qE^*_{M,n+1}X_M\right\}-\Delta t\frac{\sigma_1^2 X_M^2}{\triangle X^2}\right) J^*_{M-2,n} \\
&+\left(\frac{\Delta t}{\Delta X}\left\{rX_M\left(1-X_M\right)-qE^*_{M,n+1}X_M\right\}+\frac{5\Delta t}{4}\frac{\sigma_1^2 X_M^2}{\triangle X^2}\right) J^*_{M-1,n} \\
&+\left(1+\frac{\delta\Delta t}{2}-\frac{3\Delta t}{4\Delta X}\left\{rX_M\left(1-X_M\right)-qE^*_{M,n+1}X_M\right\}-\frac{\Delta t}{2}\frac{\sigma^2 X_M^2}{\triangle X^2}\right) J^*_{M,n} \\
= &\left(pqKX_M-c_1-\frac{c_2}{2}E^*_{M,n+1}\right)E^*_{M,n+1}+\left(\frac{-\Delta t}{4\Delta x^2}\sigma_1^2 X_M^2\right) J^*_{M-3,n+1} \\
&+\left(\frac{\Delta t}{4\Delta X}\left\{rX_M\left(1-X_M\right)-qE^*_{M,n+1}X_M\right\}+\Delta t\frac{\sigma_1^2 X_M^2}{\triangle X^2}\right) J^*_{M-2,n+1} \\
&+\left(-\frac{\Delta t}{\Delta X}\left\{rX_M\left(1-X_M\right)-qE^*_{M,n+1}X_M\right\}-\frac{5\Delta t}{4}\frac{\sigma_1^2 X_M^2}{\triangle X^2}\right) J^*_{M-1,n+1} \\
&+\left(1-\frac{\delta\Delta t}{2}+\frac{3\Delta t}{4\Delta X}\left\{rX_M\left(1-X_M\right)-qE^*_{M,n+1}X_M\right\}+\frac{\Delta t}{2}\frac{\sigma_1^2 X_M^2}{\triangle X^2}\right) J^*_{M,n+1}.
\end{aligned} \tag{B.8}$$

Equations (B.7) and (B.8) represent a system of M equations which can be written in the matrix form as:

$$AJ^*_n = BJ^*_{n+1} + D, \tag{B.9}$$

$$J^*_n = \begin{bmatrix} J^*_{1,n} & J^*_{2,n} & \cdot & \cdot & J^*_{M,n} \end{bmatrix}^{\top}.$$

The system (B.9), along with the boundary conditions, is solved for each $n = N-1, ..., 0$ using MATLAB. This procedure yields two matrices, J^* (the maximized total discounted profit) and E^* (the optimal effort), for each possible value of the initial population level. As in the deterministic case, the optimal harvesting policy for a particular initial population level can be determined by interpolation.

Appendix C

Numerical solution for stochastic growth and price

We wish to obtain a numerical solution of the Hamilton Jacobi Bellman equation corresponding to stochastic growth and stochastic price:

$$
\begin{aligned}
-\frac{\partial J^*}{\partial t} = \max_{E(t)} \Bigg[& \left(p(t)qx(t) - c_1 - \frac{c_2}{2}E(t)\right)E(t) - \delta J^* \\
& + \frac{\partial J^*}{\partial x(t)}\left\{rx(t)\left(1-\frac{x(t)}{K}\right) - qE(t)x(t)\right\} + \frac{\partial J^*}{\partial p(t)}\mu_p p(t) \\
& + \frac{1}{2}\frac{\partial^2 J^*}{\partial x(t)^2}\sigma_1^2 x(t)^2 + \frac{1}{2}\frac{\partial^2 J^*}{\partial p(t)^2}\sigma_2^2 p(t)^2 + \frac{\partial^2 J^*}{\partial x(t)\partial p(t)}\rho\sigma_1 x(t)\sigma_2 p(t)\Bigg],
\end{aligned} \tag{C.1}
$$

where the optimal effort is given by

$$
E^*(t) = \begin{cases}
0, & \left(p(t) - \frac{\partial J^*}{\partial x}\right)\frac{qx(t)}{c_2} - \frac{c_1}{c_2} < 0 \\
\left(p(t) - \frac{\partial J^*}{\partial x}\right)\frac{qx(t)}{c_2} - \frac{c_1}{c_2}, & 0 \le \left(p(t) - \frac{\partial J^*}{\partial x}\right)\frac{qx(t)}{c_2} - \frac{c_1}{c_2} \le E_{\max}, \\
E_{\max}, & \left(p(t) - \frac{\partial J^*}{\partial x}\right)\frac{qx(t)}{c_2} - \frac{c_1}{c_2} > E_{\max}.
\end{cases}
$$

We are solving the equation corresponding to a non-zero correlation between population and price. The solution for the case when fish stock and price are uncorrelated can be obtained by equating the correlation-coefficient ρ to zero.

Normalizing the population with respect to the carrying capacity K and denoting the normalized population by $X(t)$ we can rewrite the HJB equation as:

$$\begin{aligned}\frac{\partial J^*(X(t),p(t),t)}{\partial t} &= \Big[\Big(p(t)qKX(t) - c_1 - \frac{c_2}{2}E^*(t)\Big)E^*(t) - \delta J^*(X(t),p(t),t) \\ &+ \frac{\partial J^*(X(t),p(t),t)}{\partial X}\left(rX(t)\left(1-X(t)\right) - qE^*(t)KX(t)\right) \\ &+ \frac{\partial J^*(X(t),p(t),t)}{\partial p}\mu_p p(t) + \frac{\sigma_1^2 X(t)^2}{2}\frac{\partial^2 J^*(X(t),p(t),t)}{\partial X^2} \\ &+ \frac{\sigma_2^2 p(t)^2}{2}\frac{\partial^2 J^*(X(t),p(t),t)}{\partial p^2} + \frac{\partial^2 J^*}{\partial X \partial p}\rho\sigma_1 X(t)\sigma_2 p(t). \end{aligned} \tag{C.2}$$

where

$$E^*(t) = \begin{cases} 0, & \left(p(t) - \frac{1}{K}\frac{\partial J^*}{\partial X}\right)\frac{qKX(t)}{c_2} - \frac{c_1}{c_2} < 0 \\ \left(p(t) - \frac{1}{K}\frac{\partial J^*}{\partial X}\right)\frac{qKX(t)}{c_2} - \frac{c_1}{c_2}, & 0 \le \left(p(t) - \frac{1}{K}\frac{\partial J^*}{\partial X}\right)\frac{qKX(t)}{c_2} - \frac{c_1}{c_2} \le E_{\max}, \\ E_{\max}, & \left(p(t) - \frac{1}{K}\frac{\partial J^*}{\partial X}\right)\frac{qKX(t)}{c_2} - \frac{c_1}{c_2} > E_{\max}. \end{cases} \tag{C.3}$$

The boundary conditions associated with the problem are:

$$J^*(X,p,T) = 0,\ J^*(X,0,t) = 0 \text{ and } E^*(X_{\min},p,T) = 0, \tag{C.4}$$

where $X_{\min} = \frac{x_{\min}}{K}$ (normalized minimum viable population level).

We perform numerical integration using the Crank-Nicolson finite-difference method. As before, the initial time is assumed to be 0 and the final time is T (a fixed constant). We discretize over time by dividing the interval $[0,\ T]$ into N equal sub-intervals of size Δt as: $0, t_1, t_2, \ldots, t_N = T$. The normalized population lies between $[X_{\min},\ 1]$ and this interval is partitioned as $X_{\min}, X_1, X_2, \ldots, X_M = 1$ where all the sub-partitions are of uniform length ΔX. The price is assumed to lie between 0 and 1 and the interval is partitioned as $0, p_1, p_2, \ldots, p_L = 1$, again all the sub-partitions being of equal length Δp. We let $J^*_{i,j,n}$ and $E^*_{i,j,n}$ denote the values of J^* and E^*respectively at $X = X_i$, $p = p_j$ and $t = t_n$ where $0 \le i \le M$, $0 \le j \le L$ and $0 \le n \le N$. The values of J^* and its spatial derivatives are approximated as an average of the derivative approximation at the nth time step and the $(n+1)th$ time step.

For $0 \le n \le N-1$, the first order derivatives of $J^*(X(t),p(t),t)$ are approximated by:

$$\begin{aligned}
\frac{\partial J^*(X(t),p(t),t)}{\partial t} &= \frac{J^*_{i,j,n+1} - J^*_{i,j,n}}{\triangle t}, \\
\frac{\partial J^*(X(t),p(t),t)}{\partial X} &= \frac{1}{2}\left[\left(\frac{J^*_{i+1,j,n+1} - J^*_{i-1,j,n+1}}{2\triangle X}\right)\right. \\
&\quad \left. + \left(\frac{J^*_{i+1,j,n} - J^*_{i-1,j,n}}{2\triangle X}\right)\right], \\
\frac{\partial J^*(X(t),p(t),t)}{\partial p} &= \frac{1}{2}\left[\left(\frac{J^*_{i,j+1,n+1} - J^*_{i,j-1,n+1}}{2\triangle p}\right)\right. \\
&\quad \left. + \left(\frac{J^*_{i,j+1,n} - J^*_{i,j-1,n}}{2\triangle p}\right)\right], \qquad \text{(C.5)}
\end{aligned}$$

where $1 \le i \le M-1$ and $1 \le j \le L-1$.

For $i = M$,

$$\begin{aligned}
\frac{\partial J^*(X(t),p(t),t)}{\partial X} &= \frac{1}{2}\left[\left(\frac{3J^*_{M,j,n} - 4J^*_{M-1,j,n} + J^*_{M-2,j,n}}{2\triangle X}\right)\right. \\
&\quad \left. + \left(\frac{3J^*_{M,j,n+1} - 4J^*_{M-1,j,n+1} + J^*_{M-2,j,n+1}}{2\triangle X}\right)\right]. \qquad \text{(C.6)}
\end{aligned}$$

For $j = L$,

$$\begin{aligned}
\frac{\partial J^*(X(t),p(t),t)}{\partial p} &= \frac{1}{2}\left[\left(\frac{3J^*_{i,L,n+1} - 4J^*_{i,L-1,n+1} + J^*_{i,L-2,n+1}}{2\triangle p}\right)\right. \\
&\quad \left. + \left(\frac{3J^*_{i,L,n} - 4J^*_{i,L-1,n} + J^*_{i,L-2,n}}{2\triangle p}\right)\right]. \qquad \text{(C.7)}
\end{aligned}$$

The second order derivatives of $J^*(X(t),p(t),t)$ are approximated by:

$$\begin{aligned}
\frac{\partial^2 J^*(X(t),p(t),t)}{\partial X^2} &= \frac{1}{2}\left[\left(\frac{J^*_{i+1,j,n+1} - 2J^*_{i,j,n+1} + J^*_{i-1,j,n+1}}{\triangle X^2}\right)\right. \\
&\quad \left. + \left(\frac{J^*_{i+1,j,n} - 2J^*_{i,j,n} + J^*_{i-1,j,n}}{\triangle X^2}\right)\right], \\
\frac{\partial^2 J^*(X(t),p(t),t)}{\partial p^2} &= \frac{1}{2}\left[\left(\frac{J^*_{i,j+1,n+1} - 2J^*_{i,j,n+1} + J^*_{i,j-1,n+1}}{\triangle p^2}\right)\right. \\
&\quad \left. + \left(\frac{J^*_{i,j+1,n} - 2J^*_{i,j,n} + J^*_{i,j-1,n}}{\triangle p^2}\right)\right], \\
\frac{\partial^2 J^*(X(t),p(t),t)}{\partial X \partial p} &= \frac{1}{2}\left[\left(\frac{J^*_{i+1,j+1,n+1} - J_{i+1,j-1,n+1} - J^*_{i-1,j+1,n+1} + J^*_{i-1,j-1,n+1}}{4\Delta X \triangle p}\right)\right. \\
&\quad \left. + \left(\frac{J^*_{i+1,j+1,n} - J_{i+1,j-1,n} - J^*_{i-1,j+1,n} + J^*_{i-1,j-1,n}}{4\Delta X \triangle p}\right)\right]. \qquad \text{(C.8)}
\end{aligned}$$

where $0 \le n \le N-1$, $1 \le i \le M-1$ and $1 \le j \le L-1$.

For $i = M$, $1 \leq j \leq L-1$,

$$\begin{aligned}
\frac{\partial^2 J^*(X(t),p(t),t)}{\partial X^2} &= \frac{1}{2}\left[\left(\frac{2J^*_{M,j,n+1} - 5J^*_{M-1,j,n+1} + 4J^*_{M-2,j,n+1} - J^*_{M-3,j,n+1}}{\triangle X^2}\right)\right. \\
&\quad \left. + \left(\frac{2J^*_{M,j,n} - 5J^*_{M-1,j,n} + 4J^*_{M-2,j,n} - J^*_{M-3,j,n}}{\triangle X^2}\right)\right], \\
\frac{\partial^2 J^*(X(t),p(t),t)}{\partial p^2} &= \frac{1}{2}\left[\left(\frac{J^*_{M,j+1,n+1} - 2J^*_{M,j,n+1} + J^*_{M,j-1,n+1}}{\triangle p^2}\right)\right. \\
&\quad \left. + \left(\frac{J^*_{M,j+1,n} - 2J^*_{M,j,n} + J^*_{M,j-1,n}}{\triangle p^2}\right)\right], \\
\frac{\partial^2 J^*(X(t),p(t),t)}{\partial X \partial p} &= \frac{1}{2}\left[\left(\frac{5J^*_{M-1,j+1,n+1} - 5J^*_{M-1,j-1,n+1} - 8J^*_{M-2,j+1,n+1}}{4\triangle X \Delta p}\right.\right. \\
&\quad \left. + \frac{8J^*_{M-2,j-1,n+1} + 3J^*_{M-3,j+1,n+1} - 3J^*_{M-3,j-1,n+1}}{4\triangle X \Delta p}\right) \\
&\quad + \left(\frac{5J^*_{M-1,j+1,n} - 5J^*_{M-1,j-1,n} - 8J^*_{M-2,j+1,n}}{4\triangle X \Delta p}\right. . \\
&\quad \left.\left. + \frac{+8J^*_{M-2,j-1,n} + 3J^*_{M-3,j+1,n} - 3J^*_{M-3,j-1,n}}{4\triangle X \Delta p}\right)\right] \qquad \text{(C.9)}
\end{aligned}$$

For $j = L, 1 \leq i \leq M-1$,

$$\begin{aligned}
\frac{\partial^2 J^*(X(t),p(t),t)}{\partial X^2} &= \frac{1}{2}\left[\left(\frac{J^*_{i+1,L,n+1} - 2J^*_{i,L,n+1} + J^*_{i-1,L,n+1}}{\triangle X^2}\right)\right. \\
&\quad \left. + \left(\frac{J^*_{i+1,L,n} - 2J^*_{i,L,n} + J^*_{i-1,L,n}}{\triangle X^2}\right)\right], \\
\frac{\partial^2 J^*(X(t),p(t),t)}{\partial p^2} &= \frac{1}{2}\left[\left(\frac{2J^*_{i,L,n+1} - 5J^*_{i,L-1,n+1} + 4J^*_{i,L-2,n+1} - J^*_{i,L-3,n+1}}{\triangle p^2}\right)\right. \\
&\quad \left. + \left(\frac{2J^*_{i,L,n} - 5J^*_{i,L-1,n} + 4J^*_{i,L-2,n} - J^*_{i,L-3,n}}{\triangle p^2}\right)\right], \\
\frac{\partial^2 J^*(X(t),p(t),t)}{\partial X \partial p} &= \frac{1}{2}\left[\left(\frac{5J^*_{i+1,L-1,n+1} - 5J^*_{i-1,L-1,n+1} - 8J^*_{i+1,L-2,n+1}}{4\triangle X \Delta p}\right.\right. \\
&\quad \left. + \frac{8J^*_{i-1,L-2,n+1} + 3J^*_{i+1,L-3,n+1} - 3J^*_{i-1,L-3,n+1}}{4\triangle X \Delta p}\right) \\
&\quad + \left(\frac{5J^*_{i+1,L-1,n} - 5J^*_{i-1,L-1,n} - 8J^*_{i+1,L-2,n}}{4\triangle X \Delta p}\right. \\
&\quad \left.\left. + \frac{8J^*_{i-1,L-2,n} + 3J^*_{i+1,L-3,n} - 3J^*_{i-1,L-3,n}}{4\triangle X \Delta p}\right)\right]. \qquad \text{(C.10)}
\end{aligned}$$

For $i = M$, $j = L$,

$$\begin{aligned}
\frac{\partial^2 J^*(X(t),p(t),t)}{\partial X \partial p} &= \frac{1}{2}\Bigg[\Bigg(\frac{-11J^*_{M-1,L-2,n+1} + 8J^*_{M-1,L-1,n+1} - 11J^*_{M-2,L-1,n+1}}{2\triangle X \Delta p} \\
&\quad + \frac{14J^*_{M-2,L-2,n+1} + 3J^*_{M-1,L-3,n+1}}{2\triangle X \Delta p} \\
&\quad + \frac{-3J^*_{M-2,L-3,n+1} + 3J^*_{M-3,L-1,n+1} - 3J^*_{M-3,L-2,n+1}}{2\triangle X \Delta p}\Bigg) \\
&\quad + \Bigg(\frac{-11J^*_{M-1,L-2,n} + 8J^*_{M-1,L-1,n} - 11J^*_{M-2,L-1,n} +}{2\triangle X \Delta p} \\
&\quad + \frac{14J^*_{M-2,L-2,n} + 3J^*_{M-1,L-3,n}}{2\triangle X \Delta p} \\
&\quad + \frac{-3J^*_{M-2,L-3,n} + 3J^*_{M-3,L-1,n} - 3J^*_{M-3,L-2,n}}{2\triangle X \Delta p}\Bigg)\Bigg]. \qquad \text{(C.11)}
\end{aligned}$$

The optimal effort is constrained as follows:

$$\begin{aligned}
&\text{if } \; E^*_{i,n+1} < 0 && \text{then put } \; E^*_{i,n+1} = 0, \\
&\text{if } \; E^*_{i,n+1} > E_{\max} && \text{then put } \; E^*_{i,n+1} = E_{\max},
\end{aligned}$$

The error in the temporal and the spatial derivatives approximated here is of order $O(dt^2)$ and $O(dX^2)$ respectively.

The approximations (C.5)-(C.11) are substituted in Equations (C.2) and terms are rearranged.

For $1 \leq i \leq M-1$, $1 \leq j \leq L-1$ we obtain

$$
\begin{aligned}
& \left[\left\{rX_i\left(1-X_i\right)-qE_{i,j,n+1}X_i\right\}\frac{\Delta t}{4\Delta X}-\sigma_1^2X_i^2\frac{\Delta t}{4\Delta X^2}\right]J_{i-1,j,n} \\
& +\left[1+\frac{\delta\Delta t}{2}+\sigma_1^2X_i^2\frac{\Delta t}{2\Delta X^2}+\sigma_2^2p_j^2\frac{\Delta t}{2\Delta p^2}\right]J_{i,j,n} \\
& +\left[-\left\{rX_i\left(1-X_i\right)-qE_{i,j,n+1}X_i\right\}\frac{\Delta t}{4\Delta X}-\sigma_1^2X_i^2\frac{\Delta t}{4\Delta X^2}\right]J_{i+1,j,n} \\
& +\left[\mu_p p_j\frac{\Delta t}{4\Delta p}-\sigma_2^2p_j^2\frac{\Delta t}{4\Delta p^2}\right]J_{i,j-1,n} \\
& +\left[-\mu_p p_j\frac{\Delta t}{4\Delta p}-\sigma_2^2p_j^2\frac{\Delta t}{4\Delta p^2}\right]J_{i,j+1,n} \\
& +\left[-\sigma_1\sigma_2X_ip_j\rho\frac{\Delta t}{8\Delta X\Delta p}\right]J_{i-1,j-1,n} \\
& +\left[\sigma_1\sigma_2X_ip_j\rho\frac{\Delta t}{8\Delta X\Delta p}\right]J_{i-1,j+1,n} \\
& +\left[\sigma_1\sigma_2X_ip_j\rho\frac{\Delta t}{8\Delta X\Delta p}\right]J_{i+1,j-1,n} \\
& +\left[-\sigma_1\sigma_2X_ip_j\rho\frac{\Delta t}{8\Delta X\Delta p}\right]J_{i+1,j+1,n} \\
= & \left[pqKX_i-c_1-\frac{c_2}{2}E_{i,j,n+1}\right]E_{i,j,n+1} \\
& +\left[-\left\{rX_i\left(1-X_i\right)-qE_{i,j,n+1}X_i\right\}\frac{\Delta t}{4\Delta X}+\sigma_1^2X_i^2\frac{\Delta t}{4\Delta X^2}\right]J_{i-1,j,n+1} \\
& +\left[1-\frac{\delta\Delta t}{2}-\sigma_1^2X_i^2\frac{\Delta t}{2\Delta X^2}-\sigma_2^2p_j^2\frac{\Delta t}{2\Delta p^2}\right]J_{i,j,n+1} \\
& +\left[\left\{rX_i\left(1-X_i\right)-qE_{i,j,n+1}X_i\right\}\frac{\Delta t}{4\Delta X}+\sigma_1^2X_i^2\frac{\Delta t}{4\Delta X^2}\right]J_{i+1,j,n+1} \\
& +\left[-\mu_p p_j\frac{\Delta t}{4\Delta p}+\sigma_2^2p_j^2\frac{\Delta t}{4\Delta p^2}\right]J_{i,j-1,n+1} \\
& +\left[\mu_p p_j\frac{\Delta t}{4\Delta p}+\sigma_2^2p_j^2\frac{\Delta t}{4\Delta p^2}\right]J_{i,j+1,n+1} \\
& +\left[\sigma_1\sigma_2X_ip_j\rho\frac{\Delta t}{8\Delta X\Delta p}\right]J_{i-1,j-1,n+1} \\
& +\left[-\sigma_1\sigma_2X_ip_j\rho\frac{\Delta t}{8\Delta X\Delta p}\right]J_{i-1,j+1,n+1} \\
& +\left[-\sigma_1\sigma_2X_ip_j\rho\frac{\Delta t}{8\Delta X\Delta p}\right]J_{i+1,j-1,n+1} \\
& +\left[\sigma_1\sigma_2X_ip_j\rho\frac{\Delta t}{8\Delta X\Delta p}\right]J_{i+1,j+1,n+1}
\end{aligned}
\tag{C.12}
$$

For $i = M$, $1 \leq j \leq L-1$ we get

$$
\begin{aligned}
& \left[\sigma_1^2 X_M^2 \frac{\Delta t}{4\Delta X^2}\right] J_{M-3,j,n} \\
& \left[-\left\{rX_M\left(1-X_M\right) - qE_{M,j,n+1}X_M\right\} \frac{\Delta t}{4\Delta X} - \sigma_1^2 X_M^2 \frac{\Delta t}{\Delta X^2}\right] J_{M-2,j,n} \\
& + \left[\left\{rX_M\left(1-X_M\right) - qE_{M,j,n+1}X_M\right\} \frac{\Delta t}{\Delta X} + \sigma_1^2 X_M^2 \frac{5\Delta t}{4\Delta X^2}\right] J_{M-1,j,n} \\
& + \left[1 + \frac{\delta \Delta t}{2} - \left\{rX_M\left(1-X_M\right) - qE_{M,j,n+1}X_M\right\} \frac{3\Delta t}{4\Delta X}\right. \\
& \left. -\sigma_1^2 X_M^2 \frac{\Delta t}{2\Delta X^2} + \sigma_2^2 p_j^2 \frac{\Delta t}{2\Delta p^2}\right] J_{M,j,n} \\
& + \left[\mu_p p_j \frac{\Delta t}{4\Delta p} - \sigma_2^2 p_j^2 \frac{\Delta t}{4\Delta p^2}\right] J_{M,j-1,n} \\
& + \left[-\mu_p p_j \frac{\Delta t}{4\Delta p} - \sigma_2^2 p_j^2 \frac{\Delta t}{4\Delta p^2}\right] J_{M,j+1,n} \\
& + \left[-\sigma_1\sigma_2 X_M p_j \rho \frac{5\Delta t}{8\Delta X \Delta p}\right] J_{M-1,j+1,n} + \left[\sigma_1\sigma_2 X_M p_j \rho \frac{5\Delta t}{8\Delta X \Delta p}\right] J_{M-1,j-1,n} \\
& + \left[\sigma_1\sigma_2 X_M p_j \rho \frac{\Delta t}{\Delta X \Delta p}\right] J_{M-2,j+1,n} + \left[-\sigma_1\sigma_2 X_M p_j \rho \frac{\Delta t}{\Delta X \Delta p}\right] J_{M-2,j-1,n} \\
& + \left[-\sigma_1\sigma_2 X_M p_j \rho \frac{3\Delta t}{8\Delta X \Delta p}\right] J_{M-3,j+1,n} + \left[\sigma_1\sigma_2 X_M p_j \rho \frac{3\Delta t}{8\Delta X \Delta p}\right] J_{M-3,j-1,n} \\
= \; & \left[pqKX_M - c_1 - \frac{c_2}{2} E_{M,j,n+1}\right] E_{M,j,n+1} + \left[-\sigma_1^2 X_M^2 \frac{\Delta t}{4\Delta X^2}\right] J_{M-3,j,n+1} \\
& + \left[\left\{rX_M\left(1-X_M\right) - qE_{M,j,n+1}X_M\right\} \frac{\Delta t}{4\Delta X} + \sigma_1^2 X_M^2 \frac{\Delta t}{\Delta X^2}\right] J_{M-2,j,n+1} \\
& + \left[-\left\{rX_M\left(1-X_M\right) - qE_{M,j,n+1}X_M\right\} \frac{\Delta t}{\Delta X} - \sigma_1^2 X_M^2 \frac{5\Delta t}{4\Delta X^2}\right] J_{M-1,j,n+1} \\
& + \left[1 - \frac{\delta \Delta t}{2} + \left\{rX_M\left(1-X_M\right) - qE_{M,j,n+1}X_M\right\} \frac{3\Delta t}{4\Delta X}\right. \\
& \left. +\sigma_1^2 X_M^2 \frac{\Delta t}{2\Delta X^2} - \sigma_2^2 p_j^2 \frac{\Delta t}{2\Delta p^2}\right] J_{M,j,n+1} \\
& + \left[-\mu_p p_j \frac{\Delta t}{4\Delta p} + \sigma_2^2 p_j^2 \frac{\Delta t}{4\Delta p^2}\right] J_{M,j-1,n+1} \\
& + \left[\mu_p p_j \frac{\Delta t}{4\Delta p} + \sigma_2^2 p_j^2 \frac{\Delta t}{4\Delta p^2}\right] J_{M,j+1,n+1} \\
& + \left[\sigma_1\sigma_2 X_M p_j \rho \frac{5\Delta t}{8\Delta X \Delta p}\right] J_{M-1,j+1,n+1} + \left[-\sigma_1\sigma_2 X_M p_j \rho \frac{5\Delta t}{8\Delta X \Delta p}\right] J_{M-1,j-1,n+1} \\
& + \left[-\sigma_1\sigma_2 X_M p_j \rho \frac{\Delta t}{\Delta X \Delta p}\right] J_{M-2,j+1,n+1} + \left[\sigma_1\sigma_2 X_M p_j \rho \frac{\Delta t}{\Delta X \Delta p}\right] J_{M-2,j-1,n+1} \\
& + \left[\sigma_1\sigma_2 X_M p_j \rho \frac{3\Delta t}{8\Delta X \Delta p}\right] J_{M-3,j+1,n+1} + \left[-\sigma_1\sigma_2 X_M p_j \rho \frac{3\Delta t}{8\Delta X \Delta p}\right] J_{M-3,j-1,n+1}
\end{aligned} \tag{C.13}
$$

For $1 \leq i \leq M-1$, $j = L$ this gives

$$
\begin{aligned}
& \left[\{rX_i(1-X_i) - qE_{i,L,n+1}X_i\}\frac{\Delta t}{4\Delta X} - \sigma_1^2 X_i^2 \frac{\Delta t}{4\Delta X^2}\right] J_{i-1,L,n} \\
& + \left[1 + \frac{\delta \Delta t}{2} - \frac{3\Delta t}{4\Delta p}\mu_p p_L + \sigma_1^2 X_i^2 \frac{\Delta t}{2\Delta X^2} - \sigma_2^2 p_L^2 \frac{\Delta t}{2\Delta p^2}\right] J_{i,L,n} \\
& + \left[-\{rX_i(1-X_i) - qE_{i,L,n+1}X_i\}\frac{\Delta t}{4\Delta X} - \sigma_1^2 X_i^2 \frac{\Delta t}{4\Delta X^2}\right] J_{i+1,L,n} \\
& + \left[\sigma_2^2 p_L^2 \frac{\Delta t}{4\Delta p^2}\right] J_{i,L-3,n} \\
& + \left[-\mu_p p_L \frac{\Delta t}{4\Delta p} - \sigma_2^2 p_L^2 \frac{\Delta t}{\Delta p^2}\right] J_{i,L-2,n} \\
& + \left[\mu_p p_L \frac{\Delta t}{\Delta p} + \sigma_2^2 p_L^2 \frac{5\Delta t}{4\Delta p^2}\right] J_{i,L-1,n} \\
& + \left[-\sigma_1\sigma_2 X_i p_L \rho \frac{5\Delta t}{8\Delta X \Delta p}\right] J_{i+1,L-1,n} + \left[\sigma_1\sigma_2 X_i p_L \rho \frac{5\Delta t}{8\Delta X \Delta p}\right] J_{i-1,L-1,n} \\
& + \left[\sigma_1\sigma_2 X_i p_L \rho \frac{\Delta t}{\Delta X \Delta p}\right] J_{i+1,L-2,n} + \left[-\sigma_1\sigma_2 X_i p_L \rho \frac{\Delta t}{\Delta X \Delta p}\right] J_{i-1,L-2,n} \\
& + \left[-\sigma_1\sigma_2 X_i p_L \rho \frac{3\Delta t}{8\Delta X \Delta p}\right] J_{i+1,L-3,n} + \left[\sigma_1\sigma_2 X_i p_L \rho \frac{3\Delta t}{8\Delta X \Delta p}\right] J_{i-1,L-3,n} \\
= \; & \left[pqKX_i - c_1 - \frac{c_2}{2}E_{i,L,n+1}\right] E_{i,L,n+1} \\
& \left[-\{rX_i(1-X_i) - qE_{i,L,n+1}X_i\}\frac{\Delta t}{4\Delta X} + \sigma_1^2 X_i^2 \frac{\Delta t}{4\Delta X^2}\right] J_{i-1,L,n+1} \\
& + \left[1 - \frac{\delta \Delta t}{2} + \frac{3\Delta t}{4\Delta p}\mu_p p_L - \sigma_1^2 X_i^2 \frac{\Delta t}{2\Delta X^2} + \sigma_2^2 p_L^2 \frac{\Delta t}{2\Delta p^2}\right] J_{i,L,n+1} \\
& + \left[\{rX_i(1-X_i) - qE_{i,L,n+1}X_i\}\frac{\Delta t}{4\Delta X} + \sigma_1^2 X_i^2 \frac{\Delta t}{4\Delta X^2}\right] J_{i+1,L,n+1} \\
& + \left[-\sigma_2^2 p_L^2 \frac{\Delta t}{4\Delta p^2}\right] J_{i,L-3,n+1} \\
& + \left[\mu_p p_L \frac{\Delta t}{4\Delta p} + \sigma_2^2 p_L^2 \frac{\Delta t}{\Delta p^2}\right] J_{i,L-2,n+1} + \left[-\mu_p p_L \frac{\Delta t}{\Delta p} - \sigma_2^2 p_L^2 \frac{5\Delta t}{4\Delta p^2}\right] J_{i,L-1,n+1} \\
& + \left[\sigma_1\sigma_2 X_i p_L \rho \frac{5\Delta t}{8\Delta X \Delta p}\right] J_{i+1,L-1,n+1} + \left[-\sigma_1\sigma_2 X_i p_L \rho \frac{5\Delta t}{8\Delta X \Delta p}\right] J_{i-1,L-1,n+1} \\
& + \left[-\sigma_1\sigma_2 X_i p_L \rho \frac{\Delta t}{\Delta X \Delta p}\right] J_{i+1,L-2,n+1} + \left[\sigma_1\sigma_2 X_i p_L \rho \frac{\Delta t}{\Delta X \Delta p}\right] J_{i-1,L-2,n+1} \\
& + \left[\sigma_1\sigma_2 X_i p_L \rho \frac{3\Delta t}{8\Delta X \Delta p}\right] J_{i+1,L-3,n+1} + \left[-\sigma_1\sigma_2 X_i p_L \rho \frac{3\Delta t}{8\Delta X \Delta p}\right] J_{i-1,L-3,n+1}
\end{aligned}
\tag{C.14}
$$

And for $i = M$, $j = L$ we have

$$
\begin{aligned}
&\left[\sigma_1^2 X_M^2 \frac{\Delta t}{4\Delta X^2}\right] J_{M-3,L,n} \\
&+ \left[-\left\{rX_M\left(1-X_M\right) - qE_{M,L,n+1}X_M\right\}\frac{\Delta t}{4\Delta X} - \sigma_1^2 X_M^2 \frac{\Delta t}{\Delta X^2}\right] J_{M-2,L,n} \\
&+ \left[\left\{rX_M\left(1-X_M\right) - qE_{M,L,n+1}X_M\right\}\frac{\Delta t}{\Delta X} + \sigma_1^2 X_M^2 \frac{5\Delta t}{4\Delta X^2}\right] J_{M-1,L,n} \\
&+ \left[1 + \frac{\delta \Delta t}{2} - \left\{rX_M\left(1-X_M\right) - qE_{M,L,n+1}X_M\right\}\frac{3\Delta t}{4\Delta X}\right. \\
&\left. - \frac{3\Delta t}{4\Delta p}\mu_p p_L - \sigma_1^2 X_M^2 \frac{\Delta t}{2\Delta X^2} - \sigma_2^2 p_L^2 \frac{\Delta t}{2\Delta p^2}\right] J_{M,L,n} \\
&+ \left[\sigma_2^2 p_L^2 \frac{\Delta t}{4\Delta p^2}\right] J_{M,L-3,n} + \left[-\mu_p p_L \frac{\Delta t}{4\Delta p} - \sigma_2^2 p_L^2 \frac{\Delta t}{\Delta p^2}\right] J_{M,L-2,n} \\
&+ \left[\mu_p p_L \frac{\Delta t}{\Delta p} + \sigma_2^2 p_L^2 \frac{5\Delta t}{4\Delta p^2}\right] J_{M,L-1,n} \\
&+ \left[\sigma_1\sigma_2 X_M p_L \rho \frac{11\Delta t}{4\Delta X\Delta p}\right] J_{M-1,L-2,n} + \left[-\sigma_1\sigma_2 X_M p_L \rho \frac{7\Delta t}{2\Delta X\Delta p}\right] J_{M-2,L-2,n} \\
&+ \left[\sigma_1\sigma_2 X_M p_L \rho \frac{3\Delta t}{4\Delta X\Delta p}\right] J_{M-3,L-2,n} + \left[-\sigma_1\sigma_2 X_M p_L \rho \frac{2\Delta t}{\Delta X\Delta p}\right] J_{M-1,L-1,n} \\
&+ \left[\sigma_1\sigma_2 X_M p_L \rho \frac{11\Delta t}{4\Delta X\Delta p}\right] J_{M-2,L-1,n} + \left[-\sigma_1\sigma_2 X_M p_L \rho \frac{3\Delta t}{4\Delta X\Delta p}\right] J_{M-3,L-1,n} \\
&+ \left[-\sigma_1\sigma_2 X_M p_L \rho \frac{3\Delta t}{4\Delta X\Delta p}\right] J_{M-1,L-3,n} + \left[\sigma_1\sigma_2 X_M p_L \rho \frac{3\Delta t}{4\Delta X\Delta p}\right] J_{M-2,L-3,n} \\
= \quad &\left[pqKX_M - c_1 - \frac{c_2}{2}E_{M,L,n+1}\right] E_{M,L,n+1} + \left[-\sigma_1^2 X_M^2 \frac{\Delta t}{4\Delta X^2}\right] J_{M-3,L,n+1} \\
&+ \left[\left\{rX_M\left(1-X_M\right) - qE_{M,L,n+1}X_M\right\}\frac{\Delta t}{4\Delta X} + \sigma_1^2 X_M^2 \frac{\Delta t}{\Delta X^2}\right] J_{M-2,L,n+1} \\
&+ \left[-\left\{rX_M\left(1-X_M\right) - qE_{M,L,n+1}X_M\right\}\frac{\Delta t}{\Delta X} - \sigma_1^2 X_M^2 \frac{5\Delta t}{4\Delta X^2}\right] J_{M-1,L,n+1} \\
&+ \left[1 - \frac{\delta \Delta t}{2} + \left\{rX_M\left(1-X_M\right) - qE_{M,L,n+1}X_M\right\}\frac{3\Delta t}{4\Delta X}\right. \\
&\left. + \frac{3\Delta t}{4\Delta p}\mu_p p_L + \sigma_1^2 X_M^2 \frac{\Delta t}{2\Delta X^2} + \sigma_2^2 p_L^2 \frac{\Delta t}{2\Delta p^2}\right] J_{M,L,n+1} \\
&+ \left[-\sigma_2^2 p_L^2 \frac{\Delta t}{4\Delta p^2}\right] J_{M,L-3,n+1} + \left[\mu_p p_L \frac{\Delta t}{4\Delta p} + \sigma_2^2 p_L^2 \frac{\Delta t}{\Delta p^2}\right] J_{M,L-2,n+1} \\
&+ \left[-\mu_p p_L \frac{\Delta t}{\Delta p} - \sigma_2^2 p_L^2 \frac{5\Delta t}{4\Delta p^2}\right] J_{M,L-1,n+1} \\
&+ \left[-\sigma_1\sigma_2 X_M p_L \rho \frac{11\Delta t}{4\Delta X\Delta p}\right] J_{M-1,L-2,n+1} + \left[\sigma_1\sigma_2 X_M p_L \rho \frac{7\Delta t}{2\Delta X\Delta p}\right] J_{M-2,L-2,n+1} \\
&+ \left[-\sigma_1\sigma_2 X_M p_L \rho \frac{3\Delta t}{4\Delta X\Delta p}\right] J_{M-3,L-2,n+1} + \left[\sigma_1\sigma_2 X_M p_L \rho \frac{2\Delta t}{\Delta X\Delta p}\right] J_{M-1,L-1,n+1} \\
&+ \left[-\sigma_1\sigma_2 X_M p_L \rho \frac{11\Delta t}{4\Delta X\Delta p}\right] J_{M-2,L-1,n+1} + \left[\sigma_1\sigma_2 X_M p_L \rho \frac{3\Delta t}{4\Delta X\Delta p}\right] J_{M-3,L-1,n+1} \\
&+ \left[\sigma_1\sigma_2 X_M p_L \rho \frac{3\Delta t}{4\Delta X\Delta p}\right] J_{M-1,L-3,n+1} + \left[-\sigma_1\sigma_2 X_M p_L \rho \frac{3\Delta t}{4\Delta X\Delta p}\right] J_{M-2,L-3,n+1}
\end{aligned}
\tag{C.15}
$$

The numerical scheme thus obtained, to find the maximized total discounted profit and the optimal effort, is coded in MATLAB. At each stage, the optimal effort is calculated using the formula given by $E^*(t)$ and the solution is forced to satisfy the constraints. The fish stock is not allowed to fall below a minimum viable level denoted by $x_{\min}$.

Finally, we obtain two matrices J^* and E^* for all possible combinations of initial population level x_0 and initial price p_0. The optimal solution corresponding to a given combination of x_0 and p_0 is determined using two-dimensional interpolation.