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HOLOMORPHIC SOLUTIONS TO FUNCTIONAL DIFFERENTIAL
EQUATIONS

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Abstract

Functional differential equations with an entire functional argument g are examined and theory regarding the presence of holomorphic solutions to these equations presented. There are two main problems analysed, each related to the other. The first is the existence of local holomorphic solutions about a point fixed under g , and the second is the analytic continuation of such a solution throughout the complex plane. The local behaviour of g about its fixed points determines whether holomorphic solutions exist about such points, whilst the global behaviour of g under iteration determines the analytic continuation of these solutions. The dynamics of the functional argument g , therefore, is the driving force in both problems.

Both a local and global theory is developed for the existence of solutions, and for defining where such solutions are holomorphic. The case where g is a polynomial is considered in detail, although much of the theory applies equally well to the general case where g is entire.

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Introduction

Functional differential equations are ordinary differential equations in which one or more of the terms has a function of the independent variable as its argument. A simple such equation is the Pantograph equation

$$y'(x) + by(x) = cy(\alpha x) \tag{1.1}$$

where b , c , and α are constants such that $c \neq 0$. This equation, and its higher order analogues, has been studied extensively by many researchers, both on the real line and in the complex plane. A simple power series argument is sufficient to show that there exists a non-trivial solution holomorphic about $x = 0$ in the case where $|\alpha| \leq 1$, while no such solution exists in the case $|\alpha| > 1$ unless b and c satisfy some compatibility condition. The major difference between these two cases is that in the first case, points around the origin are moved closer to the origin under the functional argument, whereas in the second case they are pushed further away. This leads to the conjecture that the local behaviour of the functional argument g about the point $x = 0$, dictates whether local holomorphic solutions exist in a neighbourhood of 0 to the more general equation

$$y'(x) + by(x) = cy(g(x)), \tag{1.2}$$

where g is some function. In this thesis, we show that this conjecture holds true for such equations when g is entire. In fact, we will show that the functional argument g also heavily influences the global behaviour of any solutions, with equations generally admitting solutions with natural boundaries determined by g .

The work is divided into two main sections. Chapters 2 and 3 deal with the first and second order pantograph equations, where we look at the case where $|\alpha| > 1$ in more detail. Due to the lack of solutions holomorphic at $x = 0$, we look instead for solutions defined on the positive real axis, that satisfy boundary conditions at $x = 0$ and $x = \infty$. In Chapters 4, 5, and 6 we discuss the existence of holomorphic solutions about fixed points of g , and the analytic continuation of these solutions.

For the first order pantograph equation, Hall and Wake [11] examined the case where $c = b\alpha$, and found a Dirichlet series solution via Laplace transforms. Kim [19] showed further that there exists a unique solution in $L_1[0, \infty)$, for a small range of c . We extend this work in Chapter 2, by showing via analysis of the related Laplace problem that, in general, there exists an infinite number of solutions to such equations. Uniqueness can be obtained, however, if the decay rate of solutions is restricted. The solutions found exhibit an interesting symmetry in that they can be written in the form

$$y(x) = \sum_{n=0}^{\infty} c_n f(\alpha^n x) \quad (1.3)$$

for some function f , where the c_n 's are constant. Motivated by the fact that solutions of this form have been shown for some particular second order problems (cf. Kim [19]), we look to characterize first order equations that admit such solutions in Section 2.5. The Mellin transform is used here to convert the problem into a functional equation, and we show how the solutions to the Mellin problem can be used to find the form of the solution to the original problem.

Chapter 3 extends the work presented in Chapter 2 to second order pantograph equations. We now have the possibility of two linearly independent solutions, one for each root of the associated indicial equation that has positive real part. In fact, we find our two solutions are linearly independent only if the roots r_1 and r_2 are such that $r_1 \neq \alpha^k r_2$ for all integers k . If $r_1 = \alpha^k r_2$ for some integer k , then the second solution reduces to a multiple of the first. This is similar to the method of Frobenius for ordinary differential equations. B. van-Brunt *et. al.* [36] studied the associated Laplace equation in the case where only one root of the indicial equation has positive real part, and we extend this work to cover the case where both roots have positive real part. This leads to the discovery of a second linearly independent solution in the case where the roots differ by an integral power of α , and thus gives an analog to Frobenius' method for pantograph equations.

In Chapter 4 we look at the more general equation (1.2) where g is a non-linear polynomial. We seek solutions to (1.2) holomorphic about points that are fixed under g . Given suitable initial conditions, we obtain a unique holomorphic solution if the fixed point z_0 is attracting. The solution can then be analytically continued by iterating the functional argument g . This leads to an overview of complex dynamics in Section 4.3, where we present well-known results based on the work of Milnor [26]. We find that the Julia set $J(g)$ (or a subset of it) forms a natural boundary for our solution. Much of the material from this chapter has been published in the paper “Natural Boundaries for Solutions to a Certain Class of Functional Differential Equations” [22], with the exception of the generalisations in Section 4.5.

The existence of holomorphic solutions about repelling fixed points is discussed in the first four sections of Chapter 5, where we present work from the paper “An Eigenvalue problem for holomorphic solutions to a certain class of functional differential equations” [23]. As noted above, the pantograph equation admits non-trivial holomorphic solutions in the case where the fixed point is repelling only under certain arrangements of the coefficients b and c . We generalize this to polynomial functional arguments, and show, through a modest extension to the work of Oberg [28], that we can rewrite the equation as the eigenvalue problem

$$\mathcal{L}y(z) = \lambda y(z) \tag{1.4}$$

where \mathcal{L} is a compact operator. Thus, there is a discrete set of eigenvalues λ such that (1.4) has a holomorphic solution. Holomorphic solutions to the original equation, therefore, occur only as special cases. We illustrate this by finding the spectrum and associated eigenfunctions of the second order pantograph equation. Equations with non-linear polynomial functional arguments are then examined. Using the results on complex dynamics from Chapter 4, we show that generically, such equations have at most one non-zero eigenvalue. Finally, we exploit the results from compact operator theory (in particular the Fredholm Alternative) to establish an existence and uniqueness result for holomorphic solutions to non-homogeneous problems.

We return to the pantograph equation in Section 5.5, presenting work from the paper “A natural boundary for solutions to the second order pantograph equation” [25]. We show that, if $b > 0$ and $\alpha > 1$ is a positive integer, the Dirichlet series solution developed in Chapter 2 has the imaginary axis as a natural boundary. This

result can be generalized by using the properties of iterated functions observed in Chapter 4. We show that any solution singular at the origin, but holomorphic in an infinite strip S whose boundary contains the imaginary axis, must have the imaginary axis as a natural boundary. This is done by transforming the problem into one involving a set whose boundary is the unit circle. The functional argument of the transformed differential equation then has a multivalent inverse, which is used to show that any solutions must have a natural boundary on the unit circle. We thus establish the imaginary axis as a natural boundary for solutions to the original problem. We conclude Chapter 5 by showing how the analysis can be extended to higher order equations by considering the second order pantograph equation.

Chapter 6 presents results on holomorphic solutions about neutral fixed points from the paper “Holomorphic solutions to pantograph type equations with neutral arguments” [24]. We begin with a study of the pantograph equation in the case where $|\alpha| = 1$. As in the attracting case, entire solutions are obtained. The order of the solutions, however, depends on the arithmetic properties of α . A specific example is studied, and we show that if α is a root of unity, polynomial (order zero) solutions are obtained, whereas if α obeys a Siegel condition, the solution is of order ρ , where $\rho \in (0, 1)$ is bounded away from zero.

Section 6.2 introduces the different types of neutral fixed point, including Siegel points and Cremer points. An overview of the local behaviour about parabolic points is presented, as are the well-known results on the characterization of Siegel and Cremer points in terms of continued fraction expansions.

Holomorphic solutions about Siegel points (which are in the Fatou set) are then sought in Section 6.3. A careful study of the results in Chapter 4 on the existence and uniqueness of holomorphic solutions about attracting points, shows that the key requirement is that g maps small discs about the fixed point into themselves, i.e., for each δ sufficiently small, $g(D(z_0; \delta)) \subseteq D(z_0; \delta)$ where $D(z_0; \delta)$ is the disc with centre z_0 and radius δ . Attracting fixed points, however, obey the strict inequality $g(D(z_0; \delta)) \subset D(z_0; \delta)$. The allowance for equality leaves an opening for the Siegel case. We thus establish the existence of a unique holomorphic solution in the component of the Fatou set containing z_0 , called a Siegel disc. The boundary of the Siegel disc is then shown to form a natural boundary for the solution by applying the results on the Julia set from Chapter 4. We conclude Chapter 6 by showing that for equations with non-linear functional arguments, Siegel points are,

in general, the *only* type of neutral fixed point that admit holomorphic solutions.

The final chapter contains conclusions along with suggestions for future work in the field. In particular, we discuss the likelihood of holomorphic solutions being found within the attractive petals of parabolic points.

This thesis contains a number of images of the Julia sets of polynomials, which have been generated using a computer program written specifically for this purpose. The algorithm is detailed in Appendix A, along with notes on the techniques used to decrease computation time, and provide sharper images.

First Order Functional Differential Equations

In this chapter we consider equations of the form

$$y'(x) + p(x)y(x) = q(x)y(\alpha x),$$

where p and q are functions defined on the positive real axis, and α is a positive real constant. We seek solutions defined on the positive real axis. We start by considering the simplest such equation, where the functions p and q are constant.

2.1 The Pantograph Equation

Consider the functional differential equation

$$y'(x) + by(x) = cy(\alpha x), \tag{2.1}$$

where b and c are complex constants, with $c \neq 0$ and α is a positive constant. We note that if $\alpha = 1$, then (2.1) is a first order ordinary differential equation, which may be solved using the usual methods. We therefore assume that $\alpha \neq 1$. Equation (2.1) has been studied extensively by Derfel and Iserles [5], Kato and McLeod [18], and Iserles [16], among others. The equation arises in many applications ranging from current collection in an electric train [29] to cell growth [11]. Second and higher order analogues have been studied by Iserles [16], Kim [19] and Cooper *et*

al. [37], among others. Iserles gives further references to the pantograph equation and its applications.

We first seek a non-trivial solution to (2.1) holomorphic about the point $x = 0$. There are two separate cases to consider, depending on whether $\alpha < 1$ or $\alpha > 1$. Indeed, substituting a formal solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ into equation (2.1) yields the recurrence relation

$$a_{n+1} = \frac{c\alpha^n - b}{n+1} a_n,$$

for $n = 0, 1, \dots$. We thus obtain the formal solution

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{\prod_{k=0}^{n-1} (c\alpha^k - b)}{n!} x^n. \quad (2.2)$$

If $\alpha < 1$, then it is clear that the above series converges for all $x \in \mathbb{C}$ so that (2.1) has a solution holomorphic in \mathbb{C} given by (2.2) where the constant a_0 is determined by an initial condition at $x = 0$. If, however, $\alpha > 1$, then the series (2.2) diverges for all $x \neq 0$ unless one of the terms in the product is zero. Thus, if $c \neq \frac{b}{\alpha^k}$ for all non-negative integers k , then there is no solution holomorphic at $x = 0$. If $c = \frac{b}{\alpha^k}$ for some non-negative integer k , then the series terminates, and the solution is a polynomial. We therefore conclude that only certain equations of the form (2.1) yield a solution holomorphic at $x = 0$ if $\alpha > 1$. In fact, Kato and McLeod [18] showed that, though holomorphic solutions do not generally exist for the initial value problem, there are nevertheless an infinite number of solutions to (2.1) satisfying a specific initial condition $y(0) = y_0 \neq 0$. Thus the initial value problem is ill-posed.

For the remainder of this chapter, we consider a boundary value problem for the case where $\alpha > 1$. In particular, we are interested in solutions to (2.1) that satisfy the conditions

$$y(0) = y_0 \quad \text{and} \quad \lim_{x \rightarrow +\infty} y(x) = 0. \quad (2.3)$$

where y_0 is a constant, possibly zero. A.J. Hall and G.C. Wake [11] showed that a cell growth model based on a cell population structure with a single size variable x , where each parent cell divides evenly into α daughters all of the same size, results in an equation of the form (2.1) with $c = b\alpha$ where $b > 0$. They found a unique solution satisfying the boundary conditions (2.3) with $y_0 = 0$. In their model, y

represented the steady size distribution and so they required y to be a probability density function; therefore, they had the further requirement that

$$\int_0^{\infty} y(x) dx = 1.$$

Their solution was of the form of a Dirichlet series

$$y(x) = \sum_{n=0}^{\infty} a_n e^{-r\alpha^n x}, \quad (2.4)$$

where r and the a_n 's are constants. They found their unique solution by transforming the equation using the Laplace transform and solving the resulting equation explicitly. We will use this same technique in §2.4 when we investigate uniqueness criteria.

2.2 Dirichlet Series Solutions

Let y be a solution to (2.1) of the form (2.4), where $\alpha > 1$. Then

$$y'(x) = - \sum_{n=0}^{\infty} r\alpha^n a_n e^{-r\alpha^n x},$$

and

$$y(\alpha x) = \sum_{n=0}^{\infty} a_n e^{-r\alpha^{n+1}x} = \sum_{n=1}^{\infty} a_{n-1} e^{-r\alpha^n x}.$$

Substituting the series into (2.1) gives

$$\sum_{n=0}^{\infty} (b - r\alpha^n) a_n e^{-r\alpha^n x} = \sum_{n=1}^{\infty} c a_{n-1} e^{-r\alpha^n x}.$$

Now the set of functions

$$\{e^{-\alpha^n r x} : n \in \mathbb{Z}^+\},$$

is functionally independent on $[0, \infty)$; hence, we can equate the coefficients of like exponentials to obtain

$$(b - r)a_0 = 0,$$

and

$$(b - r\alpha^n)a_n = ca_{n-1}, \quad (2.5)$$

for $n = 1, 2, \dots$. For a non-trivial solution we require $a_k \neq 0$ for some k , hence $r = \frac{b}{\alpha^k}$ and $b \neq 0$. The recurrence relation (2.5) then gives

$$a_n = \frac{\left(\frac{c}{b}\right)^{n-k}}{\prod_{m=1}^{n-k} (1 - \alpha^m)} a_k,$$

for $n = k, k+1, \dots$, where we use the convention $\prod_{m=1}^0 (1 - \alpha^m) = 1$. Hence,

$$y(x) = a_k \sum_{n=k}^{\infty} \frac{\left(\frac{c}{b}\right)^{n-k}}{\prod_{m=1}^{n-k} (1 - \alpha^m)} e^{-b\alpha^{n-k}x},$$

which we can re-index to give the solution

$$y(x) = A \sum_{n=0}^{\infty} \frac{\left(\frac{c}{b}\right)^n}{\prod_{m=1}^n (1 - \alpha^m)} e^{-b\alpha^n x}, \quad (2.6)$$

where the constant A is arbitrary. Note that $r = \frac{b}{\alpha^k}$ for any non-negative integer k yields the same solution, so we will assume from now on that all such Dirichlet series have $a_0 \neq 0$.

As $\alpha > 1$, it is clear that the series converges uniformly for all x such that $\operatorname{Re}(bx) \geq 0$, and so we require $\operatorname{Re}(b) > 0$ in order to satisfy the boundary condition $y(\infty) = 0$. The constant A then must satisfy

$$A \sum_{n=0}^{\infty} \frac{\left(\frac{c}{b}\right)^n}{\prod_{m=1}^n (1 - \alpha^m)} = y_0.$$

Assuming the series above is non-zero, a constant A can be found so that $y(0) = y_0$. If the above series is zero, then we must have $y_0 = 0$ if a Dirichlet series solution is to exist. The constant A can then be chosen arbitrarily.

P. Fredrickson [10] used a Dirichlet series of the form

$$\phi(x, \beta) = \sum_{n=-\infty}^{\infty} a_n e^{\beta\alpha^n x}$$

involving an increased range of n to examine solutions to (2.1) subject to the

general initial condition $y(0) = k$. He showed that if $|b| \geq |c|$ and $\operatorname{Re}(b) < 0$, then there is only one Dirichlet series solution. If $|b| < |c|$ and $\beta \neq b\alpha^{-N}$ for a positive integer N , then the problem has a one-parameter family of Dirichlet series solutions in $L_p[0, \infty)$ where $\max(1, \frac{\log(\alpha)}{\log|\frac{c}{b}|}) < p \leq \infty$ if $\operatorname{Re}(\beta) < 0$. Another solution when $|b| < |c|$ suggested by Bowen (cf. L. Fox, D. Mayers, J. Ockendon and A. Taylor [9]) has the form

$$y(x) = A \sum_{n=0}^{\infty} \prod_{m=1}^n (\gamma\alpha^m - 1)^{-1} \left(\frac{c}{b}\right)^n e^{b\gamma\alpha^n x} + B \sum_{n=1}^{\infty} \prod_{m=0}^{n-1} (\gamma\alpha^{-m} - 1) \left(\frac{b}{c}\right)^n e^{b\gamma\alpha^{-n} x},$$

where A, B are real constants, $\gamma \neq \alpha^{-N}$ for any positive integer N and $\operatorname{Re}(b\gamma) < 0$.

T.S. Ferguson [7] showed an example of the case when $|b| = |c|$ and $b < 0$ in a paper on probability theory. He showed that there is a unique solution to the equation

$$y'(x) = y(\alpha x) - y(x),$$

satisfying the boundary conditions

$$y(0) = 0, \quad y(\infty) = 1,$$

such that the solution can be expressed as

$$y(x) = 1 - h \sum_{n=0}^{\infty} c_n e^{-\alpha^n x},$$

where $c_0 = 1, c_n = \prod_{m=1}^n (1 - \alpha^m)^{-1}$ and $h = (\sum_{n=0}^{\infty} c_n)^{-1}$.

For the case $b = 0$ and $c = 1$, G. R. Morris, A. Feldstein and E. W. Bowen [27] showed the existence of a solution of the form

$$y(x) = \sum_{n=-\infty}^{\infty} (-r_0)^n \alpha^{-\frac{1}{2}n(n-1)} e^{-r_0\alpha^{-n}x},$$

for $x \in \mathbb{R}$ where r_0 is a constant, provided $\operatorname{Re}(r_0\alpha^{-n}x) \geq 0$.

Finally, for the more general case when $b = 0$, Bowen (cf. L. Fox, D. Mayers,

J. Ockendon and A. Taylor [9]) suggested a solution of the form

$$y_\beta(x) = C_\beta \sum_{n=-\infty}^{\infty} (-c)^{-n} \alpha^{-\frac{1}{2}(n+\beta)^2} e^{-\alpha^{-n-\beta-\frac{1}{2}}x},$$

where $\beta \in \mathbb{R}$ and C_β is a constant.

2.3 Uniqueness

From the last section it is clear that there exist solutions to (2.1) satisfying (2.3) when $\alpha > 1$ and $\operatorname{Re} b > 0$. We now seek conditions under which these solutions are unique. We do this by adding the further restriction that $y \in L_1[0, \infty)$, so that

$$\int_0^\infty y(x) dx < \infty. \quad (2.7)$$

The dirichlet series solution (2.6) is uniformly convergent in the half-plane $\{z : \operatorname{Re}(bz) \geq 0\}$, and can thus be integrated term by term to show that it satisfies (2.7).

Integrating equation (2.1) from 0 to ∞ yields

$$y(0) = (b - \frac{c}{\alpha}) \int_0^\infty y(x) dx.$$

Hence, if $c \neq b\alpha$, we may replace the boundary conditions (2.3) with the normalizing conditions

$$\int_0^\infty y(x) dx = L_0, \quad \lim_{x \rightarrow +\infty} y(x) = 0, \quad (2.8)$$

where L_0 is some finite constant.

If $c = b\alpha$, it is clear that any solution in $L_1[0, \infty)$ must have $y(0) = 0$. As noted earlier, this is the case studied by Hall and Wake [11], who showed the existence of a unique solution when $L_0=1$. Kim [19] showed further that we can transform the problem into a Fredholm equation and thus obtain uniqueness for a range of values of c .

Multiplying both sides of (2.1) by the integrating factor e^{bx} gives

$$\frac{d}{dx} (e^{bx}y(x)) = ce^{bx}y(\alpha x),$$

and integrating the above equation from 0 to x yields

$$e^{bx}y(x) = y_0 + \frac{c}{\alpha} \int_0^{\alpha x} e^{b\frac{s}{\alpha}} y(s) ds;$$

therefore, y satisfies the Volterra equation

$$y(x) = y_0 e^{-bx} + \frac{c}{\alpha} \int_0^{\alpha x} e^{-b(x-\frac{s}{\alpha})} y(s) ds. \quad (2.9)$$

Let $f(x) = y_0 e^{-bx}$ and $\mathcal{K}y(x) = \int_0^{\alpha x} K(x, s)y(s) ds$ where

$$K(x, s) = \begin{cases} \frac{1}{\alpha} e^{-b(x-\frac{s}{\alpha})} & \text{if } 0 < s < \alpha x, \\ 0 & \text{if } s > \alpha x. \end{cases}$$

Then y satisfies

$$y(x) = f(x) + c\mathcal{K}y(x), \quad (2.10)$$

which is a Fredholm equation of the second kind. Let $\|f\| = \int_0^{\infty} |f(x)| dx$. It is known that if \mathcal{K} is a bounded operator with the property that

$$\|\mathcal{K}y_1 - \mathcal{K}y_2\| \leq M\|y_1 - y_2\|, \quad M < \infty,$$

then equation (2.10) has a unique solution $y \in L_1[0, \infty)$ for all $f \in L_1[0, \infty)$ and sufficiently small $|c|$. We can determine an appropriate range of c by the Contraction Mapping Theorem.

Theorem 2.1 *If $\operatorname{Re}(b) > 0$, $\alpha > 1$ and $|c| < \alpha \operatorname{Re}(b)$ then there exists a unique solution to (2.1) in $L_1[0, \infty)$ satisfying (2.8).*

Proof: For $y \in L_1[0, \infty)$, the condition $|c| < \alpha \operatorname{Re}(b)$ guarantees that y_0 is defined uniquely in terms of L_0 . Let \mathcal{T} be the operator defined by

$$\mathcal{T}y = y_0 e^{-bx} + \frac{c}{\alpha} \int_0^{\alpha x} e^{-b(x-\frac{s}{\alpha})} y(s) ds.$$

Then,

$$\|\mathcal{T}y\| \leq |y_0| \int_0^\infty e^{-\operatorname{Re}(b)x} dx + \frac{|c|}{\alpha} \int_0^\infty \int_0^{\alpha x} e^{-\operatorname{Re}(b)(x-\frac{s}{\alpha})} |y(s)| ds dx.$$

Exchanging the order of integration gives

$$\begin{aligned} \|\mathcal{T}y\| &\leq \frac{|y_0|}{\operatorname{Re}(b)} + \frac{|c|}{\alpha} \int_0^\infty \left(\int_{\frac{s}{\alpha}}^\infty e^{-\operatorname{Re}(b)(x-\frac{s}{\alpha})} dx \right) |y(s)| ds \\ &= \frac{|y_0|}{\operatorname{Re}(b)} + \frac{|c|}{\alpha \operatorname{Re}(b)} \int_0^\infty |y(s)| ds \\ &= \frac{|y_0|}{\operatorname{Re}(b)} + \frac{|c|}{\alpha \operatorname{Re}(b)} \|y\|. \end{aligned}$$

Thus, \mathcal{T} maps $L_1[0, \infty)$ into $L_1[0, \infty)$.

For $y_1, y_2 \in L_1[0, \infty)$,

$$\begin{aligned} \|\mathcal{T}y_1 - \mathcal{T}y_2\| &= \frac{|c|}{\alpha} \int_0^\infty \left| \int_0^{\alpha x} e^{-b(x-\frac{s}{\alpha})} (y_1(s) - y_2(s)) ds \right| dx \\ &\leq \frac{|c|}{\alpha} \int_0^\infty \left(\int_{\frac{s}{\alpha}}^\infty e^{-\operatorname{Re}(b)(x-\frac{s}{\alpha})} dx \right) |y_1(s) - y_2(s)| ds \\ &= \frac{|c|}{\alpha \operatorname{Re}(b)} \|y_1 - y_2\|. \end{aligned}$$

The result thus follows by the Contraction Mapping Theorem. \blacksquare

The Dirichlet series (2.6) is in $L_1[0, \infty)$, and is therefore the unique solution to (2.1) satisfying (2.8), if $|c| < \alpha \operatorname{Re}(b)$. We can extend the range of c further if we add an additional decay condition on y . Let $I_n[0, \infty)$ be the set of functions f defined on $[0, \infty)$ such that $x^k f \in L_1[0, \infty)$ for $k = 0, 1, \dots, n$, where n is a non-negative integer. On this space of functions we can impose the following normalizing condition

$$\int_0^\infty x^n y(x) dx = L_n, \quad \lim_{x \rightarrow +\infty} y(x) = 0, \quad (2.11)$$

where L_n is some finite constant. Note that L_n is the n -th moment of y . We can establish a relationship between the moments of y by the use of the **Mellin transform**

$$M(s) = \int_0^\infty x^{s-1} y(x) dx. \quad (2.12)$$

In general the Mellin transform is used with $s \in \mathbb{R}$; however we restrict the analysis

to $s \in \mathbb{Z}$ because we are concerned only with the moments of y . To avoid confusion, we use the notation of (2.11) (i.e. $L_s = M(s + 1)$). Use of the Mellin Transform will be further examined in §2.5 where we look at determining types of equations that exhibit solutions displaying certain symmetries.

Multiplying (2.1) by x^s and integrating between 0 and ∞ gives

$$\int_0^\infty x^s y'(x) dx + b \int_0^\infty x^s y(x) dx = c \int_0^\infty x^s y(\alpha x) dx.$$

Integrating the first term by parts and changing the variable of integration from x to αx in the last term yields

$$\lim_{x \rightarrow \infty} x^s y(x) - s \int_0^\infty x^{s-1} y(x) dx + b L_s = \frac{c}{\alpha^{s+1}} \int_0^\infty (\alpha x)^s y(\alpha x) d(\alpha x).$$

Now, assuming that the integrals converge (i.e. that the s -th moment of y exists), $x^s y(x) \rightarrow 0$ as $x \rightarrow \infty$, and thus

$$L_{s-1} = \frac{(b - \frac{c}{\alpha^{s+1}})}{s} L_s. \quad (2.13)$$

Clearly if L_n exists for a non-negative integer n , then L_0, L_1, \dots, L_{n-1} exist, and can be found recursively using (2.13). In particular,

$$L_0 = \frac{\prod_{m=0}^n (b - \frac{c}{\alpha^{m+1}})}{n!} L_n. \quad (2.14)$$

The Dirichlet series (2.6) is in $I_s[0, \infty)$ for all $s \geq 0$. Indeed,

$$\begin{aligned} \int_0^\infty x^s A \sum_{n=0}^\infty \frac{(\frac{c}{b})^n}{\prod_{m=1}^n (1 - \alpha^m)} e^{-b\alpha^n x} dx &= \frac{sA}{b} \sum_{n=0}^\infty \frac{(\frac{c}{b\alpha})^n}{\prod_{m=1}^n (1 - \alpha^m)} \int_0^\infty x^{s-1} e^{-b\alpha^n x} dx \\ &= \frac{As!}{b^s} \sum_{n=0}^\infty \frac{(\frac{c}{b\alpha^s})^n}{\prod_{m=1}^n (1 - \alpha^m)}. \end{aligned}$$

which converges for all s as $\alpha > 1$. We can extend Theorem 2.1 as follows.

Theorem 2.2 *Let $\operatorname{Re}(b) > 0$, $\alpha > 1$ and n be a non-negative integer. If $|c| < \alpha^{n+1} \operatorname{Re}(b)$, there exists a unique solution to (2.1) in $I_n[0, \infty)$ satisfying (2.11).*

Proof: As shown above, the Dirichlet series (2.6) satisfies the conditions of the

Theorem, so we need only show uniqueness. When $n = 0$, $I_n[0, \infty) = L_1[0, \infty)$, and the conditions (2.11) and (2.8) are equivalent so that Theorem 2.1 applies. Let $n = k$ be a positive integer greater than 1 and suppose the Theorem holds for $n = k - 1$. Suppose there exist two solutions y_1 and y_2 to (2.1) in $I_k[0, \infty)$ satisfying (2.11). Let $z_1(x) = \int_x^\infty y_1(t)dt$ and $z_2(x) = \int_x^\infty y_2(t)dt$. We show that $z_1 \equiv z_2$. Integrating (2.1) from x to ∞ gives

$$-y(x) + b \int_x^\infty y(t)dt = \frac{c}{\alpha} \int_{\alpha x}^\infty y(t)dt.$$

Making the substitution $z(x) = \int_x^\infty y(t)dt$ yields the equation

$$z'(x) + bz(x) = \frac{c}{\alpha} z(\alpha x). \quad (2.15)$$

Clearly z_1 and z_2 are solutions to (2.15), so it remains to show that $z_1, z_2 \in I_{k-1}[0, \infty)$ and that z_1, z_2 satisfy the same normalizing condition. For $i = 1, 2$ we have

$$\begin{aligned} \int_0^m |x^{k-1} z_i(x)| dx &\leq \int_0^m x^{k-1} \int_x^\infty |y_i(t)| dt dx \\ &= \frac{m^k}{k} \int_m^\infty |y_i(t)| dt + \frac{1}{k} \int_0^m x^k |y_i(x)| dx \\ &\leq \frac{1}{k} \int_m^\infty |t^k y_i(t)| dt + \frac{1}{k} \int_0^m |x^k y_i(x)| dx \\ &\leq \frac{1}{k} \int_0^\infty |x^k y_i(x)| dx \end{aligned}$$

The integral on the left is a monotonically increasing function of m that is bounded above, and therefore it converges to some finite number. Hence $z_1, z_2 \in I_{k-1}[0, \infty)$.

Also, for all $m > 0$,

$$\left| m^k \int_m^\infty y_i(t) dt \right| \leq \int_m^\infty |t^k y_i(t)| dt;$$

hence

$$\begin{aligned} \int_0^\infty x^{k-1} z_i(x) dx &= \left[\frac{x^k}{k} \int_x^\infty y_i(t) dt \right]_0^\infty + \frac{1}{k} \int_0^\infty x^k y_i(x) dx \\ &= \frac{1}{k} L_k. \end{aligned} \quad (2.16)$$

The Theorem holds for $n = k - 1$ by hypothesis, and so there exists a unique solution to (2.15) in $I_{k-1}[0, \infty)$ satisfying (2.16). Clearly z_1 and z_2 satisfy these

conditions and hence $z_1 \equiv z_2$. Thus $y_1 \equiv y_2$, and the Theorem is established by induction. ■

Note that the normalizing condition (2.11) may be replaced by the boundary conditions (2.3) as long as L_n is uniquely defined in terms of y_0 . Equation (2.14) shows that this occurs as long as $c \neq b\alpha^k$ for positive integers k . We may thus rewrite Theorem 2.2 as follows

Corollary 2.3 *Let n be a non-negative integer, $\operatorname{Re}(b) > 0$, $\alpha > 1$ and $c \neq b\alpha^k$ for $k = 1, 2, \dots, n$. If $|c| < \alpha^{n+1} \operatorname{Re}(b)$, there exists a unique solution to (2.1) in $I_n[0, \infty)$ satisfying (2.3).*

As shown above, the Dirichlet series solution (2.6) is in $I_n[0, \infty)$ for all positive n . Hence, it is the unique solution specified in Corollary 2.3 for any n . Therefore, any other solutions that exist must satisfy a maximal decay condition dependent on c , as they cannot be in $I_n[0, \infty)$ for

$$n > \frac{\ln\left(\frac{|c|}{\operatorname{Re}(b)}\right)}{\ln \alpha} - 1.$$

2.4 Laplace Analysis

In this section we use the Laplace transform to derive both the Dirichlet series solution presented earlier and more general solutions to the boundary value problem (2.1),(2.3). In particular, we seek solutions in $L_1[0, \infty)$.

Suppose $y \in L_1[0, \infty)$ is a solution to (2.1) satisfying (2.3). Then y is continuous on the interval $[0, \infty)$ and hence the Laplace transform

$$L(p) = \int_0^{\infty} e^{-px} y(x) dx,$$

is defined for all $p \geq 0$. Furthermore, $L(p)$ must be holomorphic in the right half plane $\Pi_0 = \{p \in \mathbb{C} : \operatorname{Re}(p) > 0\}$. Applying the Laplace transform to both sides of equation (2.1) and using (2.3) yields the functional equation

$$pL(p) + bL(p) = \frac{c}{\alpha} L\left(\frac{p}{\alpha}\right) + y_0, \quad (2.17)$$

and the condition that $y \in L_1[0, \infty)$ gives that

$$\lim_{p \rightarrow 0} L(p) < \infty. \quad (2.18)$$

We find solutions to equation (2.17) satisfying (2.18) by first constructing solutions holomorphic at $p = 0$ and then constructing the general solution. The reason for looking first at solutions holomorphic at the origin becomes apparent with the following result.

Definition 2.4 (Exponential Decay) *A function f defined on $[0, \infty)$ decays exponentially if there exists positive constants A and r such that*

$$|y(x)| \leq Ae^{-rx},$$

for all $x \in [0, \infty)$.

Theorem 2.5 *Let y be any function in $L_1[0, \infty)$ satisfying (2.3). Then y decays exponentially if and only if its Laplace transform is holomorphic at $p = 0$.*

Proof: Let y be a function defined on the positive real axis, and suppose that there are positive constants A and r such that

$$|y(x)| < Ae^{-rx},$$

for all $x \in [0, \infty)$. The Laplace transform L of y is given by

$$L(p) = \int_0^{\infty} e^{-px} y(x) dx.$$

For fixed x , the integrand is an entire function of p , thus expanding e^{-px} as a power series, and integrating term by term yields

$$\begin{aligned} \left| \int_0^{\infty} e^{-px} y(x) dx \right| &\leq \sum_{n=0}^{\infty} \frac{1}{n!} |p|^n \int_0^{\infty} |x^n y(x)| dx \\ &\leq \sum_{n=0}^{\infty} \frac{1}{n!} |p|^n \frac{An!}{r^{n+1}} \\ &= \frac{A}{r} \sum_{n=0}^{\infty} \left(\frac{|p|}{r} \right)^n. \end{aligned}$$

Thus, it is clear that L is holomorphic for $|p| < r$.

Suppose now that $L(p)$ is the Laplace transform of a function y and that L is holomorphic at $p = 0$. Then there exists an $r > 0$ such that L is holomorphic in the disc $D(0; r)$, hence the integral

$$\int_0^{\infty} e^{-px} y(x) dx,$$

converges for all $p \in D(0; r)$. In particular, it must converge for $p = -\frac{r}{2}$, and as y satisfies (2.3), it must decay exponentially. ■

Solutions to (2.1) that exhibit exponential decay (e.g. the Dirichlet series solution) are characterised by their Laplace transforms being holomorphic at the origin. We have the following existence and uniqueness result.

Lemma 2.6 *There exists a solution to (2.17) where $\alpha > 0$ unique among functions holomorphic at $p = 0$, provided $c \neq b\alpha^k$ for all positive integers k . If $c = b\alpha^k$ for some positive integer k then there exists a solution to (2.17) holomorphic at $p = 0$ only if $y_0 = 0$. This solution is unique up to an arbitrary multiplicative constant.*

Proof: Let $L(p) = \sum_{n=0}^{\infty} a_n p^n$ where the a_n 's are constants. Substituting into (2.17) and equating like powers of p gives

$$(b - \frac{c}{\alpha})a_0 = y_0,$$

and

$$(\frac{c}{\alpha^{n+1}} - b)a_n = a_{n-1}, \quad (2.19)$$

for $n = 1, 2, \dots$. Thus the a_n 's are uniquely determined if and only if $c \neq b\alpha^k$ for all $k = 1, 2, \dots$. If $c = b\alpha^k$ for some positive integer $k \geq 2$ then $a_{k-2} = \dots = a_0 = 0$ and y_0 must be 0. The constant a_{k-1} is then arbitrary. The recurrence relation (2.19) then holds for $n \geq k$, thus the a_n 's are determined up to an arbitrary constant multiple.

In either case, $\alpha > 1$ and (2.19) implies

$$\lim_{n \rightarrow \infty} \frac{a_n}{a_{n-1}} = \lim_{n \rightarrow \infty} (\frac{c}{\alpha^{n+1}} - b)^{-1} = -\frac{1}{b}.$$

The series thus has a non-zero radius of convergence; consequently, L is holomorphic at the origin as required. ■

Given that there exists a solution holomorphic at $p = 0$, we wish to continue it into a global solution, holomorphic in the right half plane Π_0 , so that it can be inverted to a solution of (2.1). We have the following continuation result.

Lemma 2.7 *The solution L to equation (2.17) that is holomorphic at the origin can be continued to produce a function meromorphic in \mathbb{C} . The only singularities that L can have are simple poles at $p = -b\alpha^n$ for some $n = 0, 1, 2, \dots$*

Proof: Evaluating equation (2.17) at $\frac{p}{\alpha}$ gives

$$L\left(\frac{p}{\alpha}\right) = \frac{\frac{c}{b\alpha}}{1 + \frac{p}{b\alpha}} L\left(\frac{p}{\alpha^2}\right) + \frac{y_0}{b} \frac{1}{1 + \frac{p}{b\alpha}},$$

which, upon substitution in (2.17) gives

$$L(p) = \frac{\left(\frac{c}{b\alpha}\right)^2}{\left(1 + \frac{p}{b}\right)\left(1 + \frac{p}{b\alpha}\right)} L\left(\frac{p}{\alpha^2}\right) + \frac{y_0}{b} \left\{ \frac{1}{\left(1 + \frac{p}{b}\right)\left(1 + \frac{p}{b\alpha}\right)} + \frac{1}{1 + \frac{p}{b\alpha}} \right\}.$$

Repeating the above steps n times yields

$$L(p) = \frac{\left(\frac{c}{b\alpha}\right)^{n+1} L\left(\frac{p}{\alpha^{n+1}}\right)}{\prod_{m=0}^n \left(1 + \frac{p}{b\alpha^m}\right)} + \frac{y_0}{b} \sum_{j=0}^n \frac{\left(\frac{c}{b\alpha}\right)^j}{\prod_{m=0}^j \left(1 + \frac{p}{b\alpha^m}\right)}. \quad (2.20)$$

We know there exists a $\delta > 0$ such that L is holomorphic in the disc $|p| < \delta$. Choose $p \in D(0, \delta)$. The right hand side of (2.20) is holomorphic in $D(0, \alpha^{n+1}\delta)$ except at the points $\{-b\alpha^m\}_{m=0}^n$ where it has, at worst, simple poles. Thus L may be continued to a meromorphic function in $D(0, \alpha^{n+1}\delta)$. Equation (2.20) holds for any n , and hence L must be meromorphic in \mathbb{C} . ■

Since L is meromorphic in \mathbb{C} , with possibly simple poles at $p = -b\alpha^m$ for $m = 0, 1, 2, \dots$, we must have

$$L(p) = \sum_{n=0}^{\infty} \frac{a_n}{p + b\alpha^n} + g(p),$$

where g is an entire function and the a_n 's are constants. In fact, we find that $g \equiv 0$, and the Mittag-Leffler expansion

$$L(p) = \sum_{n=0}^{\infty} \frac{a_n}{p + b\alpha^n}, \quad (2.21)$$

where the a_n 's are constants suffices. We show this as follows. Substituting the Mittag-Leffler expansion into equation (2.17) gives

$$\sum_{n=0}^{\infty} a_n = y_0, \quad (2.22)$$

and the recurrence relation

$$a_n = \frac{\frac{c}{b}}{1 - \alpha^n} a_{n-1},$$

for $n = 1, 2, \dots$. Thus,

$$a_n = a_0 \frac{\left(\frac{c}{b}\right)^n}{\prod_{m=1}^n (1 - \alpha^m)}. \quad (2.23)$$

Let

$$F(c) = \sum_{n=0}^{\infty} \frac{\left(\frac{c}{b}\right)^n}{\prod_{m=1}^n (1 - \alpha^m)}.$$

then (2.22) may be written as

$$a_0 F(c) = y_0.$$

Thus, assuming $F(c) \neq 0$, a_0 can be found for any initial value y_0 . The zeros of F , therefore, play the crucial role in determining whether a solution of this form exists. Setting $q = \frac{1}{\alpha}$ gives F in the form of a partition function

$$F(c) = \sum_{n=0}^{\infty} \frac{\left(-\frac{cq}{b}\right)^n q^{\frac{n(n-1)}{2}}}{\prod_{m=1}^n (1 - q^m)},$$

which can be converted to a product using the **Euler identity** [2]. Recall that, for $|t| < 1$, $|q| < 1$, the Euler identity is

$$1 + \sum_{n=1}^{\infty} \frac{t^n q^{\frac{n(n-1)}{2}}}{\prod_{m=1}^n (1 - q^m)} = \prod_{n=0}^{\infty} (1 + tq^n). \quad (2.24)$$

Therefore

$$F(c) = \prod_{n=0}^{\infty} \left(1 - \frac{cq^{n+1}}{b}\right) = \prod_{n=1}^{\infty} \left(1 - \frac{c}{b\alpha^n}\right), \quad (2.25)$$

provided that $|\frac{c}{b\alpha}| < 1$. However, even if $|\frac{c}{b\alpha}| \geq 1$ identity (2.25) is still valid by analytic continuation, since $F(c)$ is an entire function and the infinite product converges uniformly for all $c \in \mathbb{C}$.

It is clear, then, that if $c \neq b\alpha^k$ for all positive integers k , the Mittag-Leffler expansion (2.21) is a solution to (2.17) holomorphic at $p = 0$. In the case where $c = b\alpha^k$ for a positive integer k , we must have $y_0 = 0$, and a_0 can be chosen arbitrarily. In either case, Lemma 2.6 implies that it is the only such solution holomorphic at $p = 0$. Assuming $\text{Re}(b) > 0$, we may invert (2.21) term by term, giving the following result regarding solutions to (2.1).

Theorem 2.8 *If $\text{Re}(b) > 0$, $\alpha > 0$ and $c \neq b\alpha^k$ for all positive integers k , then there exists a unique solution to equation (2.1) satisfying (2.3) that decays exponentially. If $c = b\alpha^k$ for a positive integer k , then there exists a solution to (2.1) satisfying (2.3) that decays exponentially if and only if $y_0 = 0$. This solution is unique up to an arbitrary constant.*

We now drop the condition that L is holomorphic at $p = 0$ and instead require only that the solution be holomorphic in Π_0 so that the transform can be inverted. Clearly the Mittag-Leffler solution (2.21) is still valid, however there is now no guarantee of uniqueness. Let L denote a solution to (2.17) satisfying (2.18), L_h denote the Mittag-Leffler solution, and let $Z = L - L_h$. Then Z must satisfy

$$Z(p) = \frac{\frac{c}{b\alpha}}{1 + \frac{p}{b}} Z\left(\frac{p}{\alpha}\right), \quad (2.26)$$

and we must have

$$\lim_{p \rightarrow 0} Z(p) < \infty. \quad (2.27)$$

The condition (2.27) precludes a pole at $p = 0$, but Z may have a branch point at $p = 0$. With this knowledge, we can find the general solution of (2.26).

Let

$$Z(p) = W(p)U(p),$$

where

$$W(p) = \frac{p^\kappa}{\prod_{n=0}^{\infty} \left(1 + \frac{p}{b\alpha^n}\right)} \quad (2.28)$$

and

$$\kappa = \frac{\ln \left| \frac{c}{b\alpha} \right| + i\phi}{\ln(\alpha)}. \quad (2.29)$$

Here we are using $p^\kappa = e^{\kappa \text{Log}(p)}$, where $\text{Log}(z)$ is the principal branch of $\log(z)$ and $-\pi < \phi = \text{Arg}\left(\frac{c}{b\alpha}\right) < \pi$. Note that this precludes the case where $\frac{c}{b}$ is a negative real number. We can however choose a suitable branch of $\log(z)$ to take care of this scenario.

We have chosen $W(p)$ to be a particular solution to (2.26). Indeed,

$$W\left(\frac{p}{\alpha}\right) = \frac{1}{\alpha^\kappa} \frac{p^\kappa}{\prod_{n=1}^{\infty} \left(1 + \frac{p}{b\alpha^n}\right)} = \frac{1 + \frac{p}{b}}{\alpha^\kappa} W(p),$$

and $\alpha^\kappa = \frac{c}{b\alpha}$. Thus, substituting the expression for Z into equation (2.26) gives the relation

$$U(p) = U\left(\frac{p}{\alpha}\right).$$

It is helpful at this point to classify functions with this property.

Definition 2.9 (Log-Periodic Functions) *A function u such that*

$$u(z) = u(qz),$$

*for all $z \in \mathbb{C}$ where $q > 1$ is a positive real constant is said to be **log-periodic with period q** .*

We have the following result.

Theorem 2.10 *Let $u(z)$ be a log-periodic function, and suppose that*

$$\lim_{z \rightarrow 0} u(z) = c,$$

for some finite constant c . Then $u(z) \equiv c$ for all $z \in \mathbb{C}$.

Proof: Let $u(z)$ be a log-periodic function with period q . Then

$$u(z) = u\left(\frac{z}{q}\right),$$

for all $z \in \mathbb{C}$. Therefore

$$u(z) = u\left(\frac{z}{q^n}\right),$$

for all $n = 1, 2, \dots$, and so

$$u(z) = \lim_{n \rightarrow \infty} u\left(\frac{z}{q^n}\right) = u(0) = c,$$

for any z in C . Thus $u(z) \equiv c$ as required. ■

If $|c| \leq |b\alpha|$ then $\operatorname{Re}(\kappa) \leq 0$ so that $\lim_{p \rightarrow 0} W(p)$ does not exist. However, Z must satisfy (2.27), therefore, $\lim_{p \rightarrow 0} U(p) = 0$ and thus $U(p) \equiv 0$ by Theorem 2.10. If $|c| > |b\alpha|$ then $\operatorname{Re}(\kappa) > 0$ and so $\lim_{p \rightarrow 0} W(p) = 0$, and thus Z satisfies (2.27) as required. In summary, we have the following result on the solutions to the Laplace problem.

Theorem 2.11 *The general solution to (2.17) in Π_0 satisfying (2.18) is of the form*

$$L(p) = \sum_{n=0}^{\infty} \frac{a_n}{p + b\alpha^n}, \quad (2.30)$$

if $|c| \leq |b\alpha|$; and of the form

$$L(p) = \sum_{n=0}^{\infty} \frac{a_n}{p + b\alpha^n} + \frac{p^\kappa U(p)}{\prod_{n=0}^{\infty} \left(1 + \frac{p}{b\alpha^n}\right)}, \quad (2.31)$$

if $|c| > |b\alpha|$. The constants a_n and κ are defined by (2.22), (2.23), and (2.29), and U is an arbitrary log-periodic function of period α , holomorphic in the Π_0 .

Thus, in general, there is an infinite number of solutions defining the Laplace transform if $|c| > |b\alpha|$. However, if we define additional conditions at $p = 0$, we can obtain uniqueness for any value of c . We have the following result.

Theorem 2.12 *There exists a unique solution to (2.17) in Π_0 satisfying (2.18) such that*

$$\lim_{p \rightarrow 0} L^{(n)}(p), \quad (2.32)$$

exists, provided $|c| < |b\alpha^{n+1}|$, $\alpha > 1$ and $c \neq b\alpha^k$ for all $k = 2, 3, \dots, n$. If $c = b\alpha^k$ for some $k = 2, 3, \dots, n$ then the solution is unique up to an arbitrary constant.

Proof: The Theorem is true for $n = 0$ by Theorems 2.11 and 2.8, so assume $n > 0$.

Let

$$f(p) = p^\kappa U(p),$$

where U is a log-periodic function of period α , and

$$g(p) = \frac{1}{\prod_{n=0}^{\infty} \left(1 + \frac{p}{b\alpha^n}\right)}$$

so that $Z(p) = f(p)g(p)$. It is clear that (2.32) exists for the Mittag Leffler solution (2.30), so we just need to check that $\lim_{p \rightarrow 0} Z^{(n)}(p)$ exists for $p \in \Pi_0$. Now,

$$Z^{(n)}(p) = \sum_{k=0}^n \binom{n}{k} f^{(k)}(p) g^{(n-k)}(p), \quad (2.33)$$

and it is clear that g is holomorphic at $p = 0$ as the infinite product is an entire function, non-zero in Π_0 as $\operatorname{Re}(b) > 0$. We have

$$f^{(n)}(p) = \sum_{k=0}^n \prod_{j=0}^{k-1} (\kappa - j) p^{\kappa-k} U^{(n-k)}(p),$$

so that

$$\begin{aligned} f^{(n)}(\alpha p) &= \sum_{k=0}^n \alpha^{\kappa-k} \prod_{j=0}^{k-1} (\kappa - j) p^{\kappa-k} U^{(n-k)}(\alpha p) \\ &= \alpha^{\kappa-n} f^{(n)}(p), \end{aligned}$$

as $U^{(n-k)}(\alpha p) = \alpha^{k-n} U^{(n-k)}(p)$. We may thus write

$$f^{(n)}(p) = p^{\kappa-n} \hat{U}(p),$$

where \hat{U} is a log-periodic function of period α . From (2.29) we have

$$\operatorname{Re}(\kappa) = \frac{\ln \left| \frac{c}{b\alpha} \right|}{\ln(\alpha)} < \frac{\ln(\alpha^n)}{\ln(\alpha)} = n,$$

so for $\lim_{p \rightarrow 0} f^{(n)}(p)$ to exist we require $\lim_{p \rightarrow 0} \hat{U}(p) = 0$. Theorem 2.10 gives $\hat{U}(p) \equiv 0$ and so $Z(p) \equiv 0$. The only solution, therefore, is the Mittag-Leffler solution (2.30), which is holomorphic at the origin, so we apply Theorem 2.8 to obtain the uniqueness result. ■

Although we clearly have an infinite number of solutions to the Laplace equation, these need not correspond to transforms of solutions to the boundary problem. We still need to show that the functions defined in (2.31) are invertible. Let $H^2(\Pi_0)$ denote the Hardy space of functions g holomorphic in the right halfplane $\Pi_0 = \{z : \operatorname{Re}(z) > 0\}$ such that for any fixed $\mu > 0$ the integral

$$M_0^2(g) = \int_{-\infty}^{\infty} |g(\mu + i\tau)|^2 d\tau$$

is finite. We wish to show that (2.31) is in $H^2(\Pi_0)$ in order to show that inversion is valid. We first require the following lower bound for the growth of polynomials.

Lemma 2.13 *Let $f(z) = \sum_{j=0}^k f_j z^j$ be a polynomial of degree k , where $f_k \neq 0$. Then*

$$|f(z)| > \frac{|f_k|}{2} |z|^k,$$

for all $|z| > \frac{3}{|f_k|} \max\{|f_0|, |f_1|, \dots, |f_k|\}$.

Proof: Using the triangle inequality, we have

$$\left| \frac{f(z)}{z^k} \right| \geq \left| |f_k| - \sum_{j=0}^{k-1} \frac{|f_j|}{|z|^{k-j}} \right|,$$

and so for $|z| > \frac{3}{|f_k|} \max\{|f_0|, |f_1|, \dots, |f_k|\}$,

$$\begin{aligned} \left| \frac{f(z)}{z^k} \right| &> \left| |f_k| - \sum_{j=0}^{k-1} \frac{|f_j|}{3^{k-j}} \right| \\ &> |f_k| \left(1 - \sum_{j=1}^{\infty} 3^{-j} \right) \\ &= \frac{|f_k|}{2}. \end{aligned}$$

■

Theorem 2.14 *Let $\operatorname{Re}(b) > 0$, $\alpha > 1$, and let L be a solution to (2.17) of the form (2.31) where the log-periodic function U is bounded in Π_0 . Then $L \in H^2(\Pi_0)$.*

Proof: Let $L(p) = M(p) + U(p)W(p)$ where M is the Mittag-Leffler solution, W is as in (2.28), and U is log-periodic with period α and bounded in Π_0 (note that a

constant function U satisfies this condition.) The series for M is uniformly convergent in Π_0 , and each term is in $H^2(\Pi_0)$, hence $M \in H^2(\Pi_0)$. $U(p)$ is bounded in Π_0 , so all that remains is to show that $W \in H^2(\Pi_0)$.

Let N be a positive integer such that $N > \operatorname{Re}(\kappa)$. Then, for $p \in \Pi_0$,

$$|W(p)| = \frac{e^{\operatorname{Re}(\kappa) \operatorname{Log}(p)}}{\prod_{n=0}^N \left|1 + \frac{p}{b\alpha^n}\right| \prod_{n=N+1}^{\infty} \left|1 + \frac{p}{b\alpha^n}\right|} \leq \frac{e^{\operatorname{Re}(\kappa) \ln |p| - \operatorname{Im}(\kappa) \operatorname{Arg}(p)}}{\prod_{n=0}^N \left|1 + \frac{p}{b\alpha^n}\right|}.$$

Now, $\operatorname{Arg}(p) > -\pi/2$ so that

$$|W(p)| < \frac{|p|^{\operatorname{Re}(\kappa)} e^{\frac{|\operatorname{Im}(\kappa)|\pi}{2}}}{\prod_{n=0}^N \left|1 + \frac{p}{b\alpha^n}\right|},$$

and Lemma 2.13 gives

$$\prod_{n=0}^N \left|1 + \frac{p}{b\alpha^n}\right| > \frac{|p|^{N+1}}{2|b|^{N+1} \alpha^{\frac{N(N-1)}{2}}},$$

for $|p|$ sufficiently large. Hence,

$$\begin{aligned} |W(p)| &< \frac{|p|^{\operatorname{Re}(\kappa)} e^{\frac{|\operatorname{Im}(\kappa)|\pi}{2}} 2|b|^{N+1} \alpha^{\frac{N(N-1)}{2}}}{|p|^{N+1}} \\ &\leq \frac{A}{|p|} \end{aligned}$$

for some constant A and $|p|$ sufficiently large. Thus $W \in H^2(\Pi_0)$ and the theorem holds. ■

Note that the condition that U is bounded does not preclude non-constant functions.

Lemma 2.15 *Let u be of the form*

$$u(z) = \sum_{n=-\infty}^{\infty} w_n \exp\left(\frac{2n\pi i}{\ln(\alpha)} \operatorname{Log}(z)\right), \quad (2.34)$$

where the w_n 's are constants such that

$$\sum_{n=-\infty}^{\infty} |w_n| \exp\left(\frac{|n|\pi^2}{\ln(\alpha)}\right)$$

converges. Then u is log-periodic of period α , bounded in Π_0 .

Proof: Let u be of the form (2.34). Then

$$\begin{aligned} u(\alpha z) &= \sum_{n=-\infty}^{\infty} w_n \exp\left(\frac{2n\pi i}{\ln(\alpha)} \operatorname{Log}(\alpha z)\right) \\ &= \sum_{n=-\infty}^{\infty} w_n \exp\left(\frac{2n\pi i}{\ln(\alpha)} (\operatorname{Log}(z) + \ln(\alpha))\right) \\ &= u(z), \end{aligned}$$

so that u is log-periodic of period α . Now,

$$\begin{aligned} |u(z)| &\leq \sum_{n=-\infty}^{\infty} |w_n| \exp\left(\operatorname{Re}\left(\frac{2n\pi i}{\ln(\alpha)} \operatorname{Log}(z)\right)\right) \\ &= \sum_{n=-\infty}^{\infty} |w_n| \exp\left(\frac{-2n\pi}{\ln(\alpha)} \operatorname{Im}(\operatorname{Log}(z))\right) \\ &= \sum_{n=-\infty}^{\infty} |w_n| \exp\left(\frac{2n\pi}{\ln(\alpha)} \operatorname{Arg}(z)\right) \\ &\leq \sum_{n=-\infty}^{\infty} |w_n| \exp\left(\frac{|n|\pi^2}{\ln(\alpha)}\right). \end{aligned}$$

The series on the right converges, and thus u is bounded in Π_0 . ■

The significance of Theorem 2.14 is that we can invoke the Paley-Wiener theorem (cf. Hoffman [14]) to show that there is a unique $y \in L^2[0, \infty)$ such that

$$L(p) = \int_0^{\infty} y(x) e^{-px} dx.$$

Such an inverse y must be a solution to (2.1) satisfying (2.3).

Theorem 2.12 shows that the Dirichlet series solution (2.6) is the unique solution to (2.1) satisfying (2.11). The above result extends this by giving an infinite number of solutions in $L_2[0, \infty)$. One would expect that the condition that $\lim_{p \rightarrow 0^+} L(p)$ exists and is finite can be used to show that y is in $L_1[0, \infty)$, however this requires the use of a Tauberian result and has not yet been shown. Any inverse transforms of (2.31) that are in $L_1[0, \infty)$, however, must decay slower than $\frac{1}{x^{n+1}}$ as $x \rightarrow \infty$.

2.5 Equations with Variable Coefficients

Note that both the Dirichlet series solution (2.6), and the solutions to the Laplace equation (2.30) and (2.31), have increasing powers of α present. Indeed, the Dirichlet series solution takes the form

$$y(x) = \sum_{n=0}^{\infty} a_n y_0(\alpha^n x), \quad (2.35)$$

where the a_n 's are constants, and $y_0(x) = e^{-bx}$. In this case, y_0 is a solution to the homogeneous ordinary differential equation $y' + by = 0$. In this section we consider more general equations, and derive conditions on the types of equation that admit solutions of this form.

Consider the equation with variable coefficients

$$y'(x) + p(x)y(x) = q(x)y(\alpha x), \quad (2.36)$$

where p and q are functions holomorphic in the right half-plane. We wish to find conditions on p and q such that (2.36) admits solutions in $\operatorname{Re}(x) > 0$ of the form (2.35), where y_0 is a solution to the associated ordinary differential equation

$$y'(x) + p(x)y(x) = 0. \quad (2.37)$$

Substituting (2.35) into (2.36) gives

$$\sum_{n=0}^{\infty} a_n [\alpha^n y_0'(\alpha^n x) + p(x)y_0(\alpha^n x)] = \sum_{n=1}^{\infty} a_{n-1} q(x)y_0(\alpha^n x).$$

Now, y_0 is a solution to (2.37), so we have

$$y_0'(\alpha^n x) = -p(\alpha^n x)y_0(\alpha^n x);$$

hence,

$$\sum_{n=0}^{\infty} a_n [p(x) - \alpha^n p(\alpha^n x)]y_0(\alpha^n x) = \sum_{n=1}^{\infty} a_{n-1} q(x)y_0(\alpha^n x).$$

Equating the coefficients of $y_0(\alpha^n x)$ gives

$$a_n[p(x) - \alpha^n p(\alpha^n x)] = a_{n-1}q(x), \quad (2.38)$$

for $n > 0$. Hence

$$q(x) = \frac{a_n}{a_{n-1}}[p(x) - \alpha^n p(\alpha^n x)], \quad (2.39)$$

for $n = 1, 2, \dots$. Setting $n = 1$ we have

$$p(\alpha x) = \frac{1}{\alpha}[p(x) - \frac{a_0}{a_1}q(x)]. \quad (2.40)$$

Evaluating (2.39) at αx , using $n = 1$, gives

$$q(\alpha x) = \frac{a_1}{a_0}[p(\alpha x) - \alpha p(\alpha^2 x)].$$

Hence

$$\begin{aligned} q(\alpha x) &= \frac{a_1}{\alpha a_0}[p(x) - \frac{a_0}{a_1}q(x) - \alpha^2 p(\alpha^2 x)] && \text{by (2.40)} \\ &= -\frac{1}{\alpha}q(x) + \frac{a_1}{\alpha a_0}[p(x) - \alpha^2 p(\alpha^2 x)] \\ &= -\frac{1}{\alpha}q(x) + \frac{1}{\alpha} \frac{a_1}{a_0} \frac{a_1}{a_2} q(x) && \text{by (2.39)} \\ &= \frac{a_1^2 - a_0 a_2}{a_0 a_2 \alpha} q(x). \end{aligned}$$

Let $g(x) = x^{-\kappa} q(x)$ where

$$\kappa = \frac{\ln(\frac{a_1^2 - a_0 a_2}{a_0 a_2 \alpha})}{\ln(\alpha)}. \quad (2.41)$$

Then

$$g(\alpha x) = \alpha^{-\kappa} x^{-\kappa} q(\alpha x) = \frac{a_0 a_2 \alpha}{a_1^2 - a_0 a_2} x^{-\kappa} \frac{a_1^2 - a_0 a_2}{a_0 a_2 \alpha} q(x) = g(x).$$

Thus, q is of the form

$$q(x) = x^\kappa g(x) \quad (2.42)$$

where g is an arbitrary log-periodic function of period α . Using (2.39) with $n = 1$ we have

$$\frac{p(x) - p(\alpha x)}{q(x)} = \frac{p(x)}{q(x)} - \frac{a_1^2 - a_0 a_2}{a_0 a_2} \frac{p(\alpha x)}{q(\alpha x)} = \frac{a_0}{a_1}.$$

Setting $r(x) = \frac{p(x)}{q(x)}$ and using (2.41) gives

$$r(x) - \alpha^{-\kappa+1}r(\alpha x) = \frac{a_0}{a_1}.$$

Differentiating gives

$$r'(x) - \alpha^{-\kappa+2}r'(\alpha x) = 0$$

and thus r' is of the form

$$r'(x) = x^{\kappa-2}u(x)$$

where u is an arbitrary log-periodic function of period α . Thus it is clear r can be written as

$$r(x) = x^{\kappa-1}v(x) + C$$

where v is log-periodic, and C is a constant of integration. The form of p is therefore

$$p(x) = \frac{1}{x}w(x) + Cq(x)$$

for some log-periodic function w of period α . This leads us to the following result.

Theorem 2.16 *Let $q(x) = x^\kappa u_1(x)$ and $p(x) = \frac{1}{x}u_2(x) + Cq(x)$ where u_1 and u_2 are arbitrary log-periodic functions holomorphic in $\operatorname{Re}(x) > 0$ with period α , and C and κ are constants such that $C \neq 0$ and $\kappa > 0$. Let y_0 be a solution to (2.37), bounded on $[0, \infty)$ and suppose the set of functions $\{y_0(\alpha^n x)\}$ are functionally independent. Then the functional differential equation (2.36) has a solution of the form (2.35).*

Proof: Let y be of the form (2.35), where y_0 is as in the hypotheses. We need only verify that the constants a_n may be found so that the series converges and is a solution to (2.36). Substituting into equation (2.36) yields

$$\sum_{n=0}^{\infty} a_n [\alpha^n y_0'(\alpha^n x) + p(x)y_0(\alpha^n x)] = \sum_{n=1}^{\infty} a_{n-1} q(x)y_0(\alpha^n x).$$

As y_0 is a solution to (2.37), we have that $y_0'(\alpha^n x) = -p(\alpha^n x)y_0(\alpha^n x)$ giving

$$\sum_{n=0}^{\infty} a_n [p(x) - \alpha^n p(\alpha^n x)]y_0(\alpha^n x) = \sum_{n=1}^{\infty} a_{n-1} q(x)y_0(\alpha^n x).$$

Now,

$$p(x) - \alpha^n p(\alpha^n x) = C[1 - \alpha^{n(\kappa+1)}]q(x),$$

which gives

$$\sum_{n=0}^{\infty} a_n C[1 - \alpha^{n(\kappa+1)}]q(x)y_0(\alpha^n x) = \sum_{n=1}^{\infty} a_{n-1}q(x)y_0(\alpha^n x).$$

Hence we must have

$$a_n = \frac{a_{n-1}}{C(1 - \alpha^{(1+\kappa)n})},$$

so that

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{1}{C^n \prod_{m=0}^n (1 - \alpha^{(1+\kappa)m})} y_0(\alpha^n x). \quad (2.43)$$

As $\alpha > 1$, $\kappa > 0$ and y_0 is bounded on $[0, \infty)$, the series (2.43) converges for all $x \in [0, \infty)$. ■

In the special case where κ is an integer, and u_1 and u_2 are constant, we can transform the equation into one with constant coefficients. Consider the functional differential equation

$$y' + \left(\frac{A}{x} + Bx^k\right)y(x) = Cx^k y(\alpha x) \quad (2.44)$$

where A , B and C are complex constants, k is a positive integer, and $\alpha > 1$ is a constant, along with the conditions

$$\lim_{x \rightarrow 0^+} x^A y(x) = y_0 \quad \text{and} \quad \lim_{x \rightarrow \infty} x^A y(x) = 0. \quad (2.45)$$

where y_0 is some constant. Applying the transformations $z = \frac{x^{k+1}}{k+1}$ and $w(z) = x^A y(x)$ to (2.44) yields the equation

$$w'(z) + Bw(z) = \frac{C}{\alpha^A} w(\alpha^{k+1}z) \quad (2.46)$$

which is of the same form as (2.1). The boundary conditions (2.45) transform to the conditions

$$w(0) = y_0 \quad \text{and} \quad w(\infty) = 0. \quad (2.47)$$

Thus the results of the previous section can be readily extended for solutions to (2.44). For example we have the following existence and uniqueness result.

Theorem 2.17 *If $\operatorname{Re}(B) > 0$, $\alpha > 1$, and $|C| < \operatorname{Re}(B)\alpha^{\operatorname{Re}(A)+k+1}$ then there exists a unique solution to (2.44) satisfying (2.45) in the space of functions g such that $x^A g \in L_1[0, \infty)$.*

Proof: Let $y_1(x)$ and $y_2(x)$ be solutions to (2.44) satisfying (2.45) such that $x^A y_1, x^A y_2 \in L_1[0, \infty)$. Let $z = \frac{x^{k+1}}{k+1}$, $w_1(z) = x^A y_1(x)$ and $w_2(z) = x^A y_2(x)$. It is clear that w_1 and w_2 are solutions to (2.46) satisfying (2.47) and that $w_1, w_2 \in L_1[0, \infty)$. Theorem 2.1 thus implies that $w_1 \equiv w_2$ and hence $y_1 \equiv y_2$. ■

In order to better understand the symmetry in powers of α that arises, we look at the Mellin Transform of y .

2.5.1 The Mellin Transform

We begin with an example. Suppose we wish to find an exponentially decaying solution to (2.44). Multiplying (2.44) through by x^{s-1} and integrating from 0 to ∞ gives

$$\int_0^\infty x^{s-1} y'(x) dx + A \int_0^\infty x^{s-2} y(x) dx + B \int_0^\infty x^{s+k-1} y(x) dx = C \int_0^\infty x^{s+k-1} y(\alpha x) dx.$$

Integrating the first term by parts, and changing the variable of integration from x to αx in the last term yields, for $s \geq 1$,

$$\begin{aligned} \lim_{x \rightarrow \infty} x^{s-1} y(x) + (A - s + 1) \int_0^\infty x^{s-2} y(x) dx + B \int_0^\infty x^{s+k-1} y(x) dx \\ = \frac{C}{\alpha^{s+k}} \int_0^\infty (\alpha x)^{s+k-1} y(\alpha x) d(\alpha x), \end{aligned}$$

and, as we are looking for solutions that decay exponentially, $\lim_{x \rightarrow \infty} x^{s-1} y(x) = 0$. Using the Mellin transform as defined in (2.12), we therefore have

$$(s - A - 1)M(s - 1) = \left(B - \frac{C}{\alpha^{s+k}} \right) M(s + k), \tag{2.48}$$

for $s \geq 1$. This is a difference equation in s , for which a solution may be found as follows. Let $M(s) = F(s)Q(s)$ where F satisfies the difference equation

$$(s - A - 1)F(s - 1) = BF(s + k). \tag{2.49}$$

Then, assuming that $(s - A - 1)F(s - 1)$ is not identically zero, (2.48) reduces to

$$Q(s - 1) = \left(1 - \frac{C}{B\alpha^{s+k}}\right) Q(s + k). \quad (2.50)$$

Splitting up M in this way is useful, as F has been chosen so as to be independent of α . In fact, F corresponds to the Mellin transform of a solution to the homogeneous equation (2.37), as (2.49) is the transform of (2.37). We can therefore find F by solving the difference equation (2.49), or by finding the Mellin transform of the solution to (2.37). Either method yields

$$F(s) = \left(\frac{k+1}{B}\right)^{\frac{s-A}{k+1}} \Gamma\left(\frac{s-A}{k+1}\right).$$

Rewriting (2.50) using s in place of $s - 1$ gives

$$Q(s) = \left(1 - \frac{C}{B\alpha^{s+k+1}}\right) Q(s + k + 1),$$

thus

$$Q(s) = \left(1 - \frac{C}{B\alpha^{s+k+1}}\right) \left(1 - \frac{C}{B\alpha^{s+2(k+1)}}\right) Q(s + 2(k+1)),$$

and therefore a solution to (2.50) is

$$Q(s) = \prod_{n=1}^{\infty} \left(1 - \frac{C}{B\alpha^{s+n(k+1)}}\right).$$

The difference equation (2.48) gives information about M only at values of s separated by intervals of length $k + 1$, i.e., if we know M at a particular point s_0 , then equation (2.48) will yield information about the function only at the points $s_0 + n(k + 1)$ for integers n . We may therefore define M arbitrarily on any interval $[s_0, s_0 + k + 1)$, and can thus write the general solution to (2.48) as

$$M(s) = U(s) \left(\frac{k+1}{B}\right)^{\frac{s-A}{k+1}} \Gamma\left(\frac{s-A}{k+1}\right) \prod_{n=1}^{\infty} \left(1 - \frac{C}{B\alpha^{s+n(k+1)}}\right). \quad (2.51)$$

where U is an arbitrary periodic function of period $k + 1$. Note, however, that this solution to (2.48) may not correspond to the Mellin transform of a suitable solution to the original problem. Indeed, the addition of the arbitrary function U is not required when we consider that the inversion will only require knowledge of M

at the points $s = n(k + 1)$ for all positive integers n . This is equivalent to knowing every $k+1$ -th moment of some unknown function y and, using only this information, determining y completely. This is related to the well-known “Moment problem” (cf. Hausdorff [12], [13]). The moment problem consists of finding conditions on the sequence of numbers $\{\mu_k\}_{k=0}^{\infty}$ such that there exists a function $\alpha(t)$ of bounded variation in the interval $(0, 1)$ such that

$$\mu_k = \int_0^1 x^k d\alpha(t).$$

for all non-negative integers k . A variation to this problem, involving finding a function given information about its Laplace transform at the origin, is considered by D.V. Widder (cf. Widder [38] pp. 100-101). We exploit the relationship between the Mellin transform and derivatives of the Laplace transform at the origin to obtain the following result.

Theorem 2.18 *Let M be the Mellin transform of an unknown function y that satisfies*

$$|y(x)| < Ae^{-rx^k},$$

for constants $A > 0$, $r > 0$ and some positive integer k . If $M(nk)$ is known for all positive integers n , then y can be found completely.

Proof: From the definition of the Mellin transform, we have

$$M(nk) = \int_0^{\infty} x^{nk-1} y(x) dx.$$

Setting $x = f(w) = w^{1/k}$ gives

$$M(nk) = \int_0^{\infty} w^{n-1} y(f(w)) dw.$$

We therefore know all the moments of the function $Y(w) = y(f(w))$. Furthermore, as Y satisfies

$$|Y(w)| < Ae^{-rw},$$

for some constants $A > 0$, $r > 0$. Therefore,

$$\left| \int_0^{\infty} \frac{(-pw)^n}{n!} Y(w) dw \right| < A \int_0^{\infty} \frac{|p|^n w^n}{n!} e^{-rw} dw < \frac{A|p|^n}{r^{n+1}},$$

so that the series

$$\sum_{n=0}^{\infty} \frac{(-p)^n}{n!} M((n+1)k) = \sum_{n=0}^{\infty} \int_0^{\infty} \frac{(-pw)^n}{n!} Y(w) dw$$

converges absolutely for $|p| < r$. We may therefore exchange the order of summation and integration to obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-p)^n}{n!} M((n+1)k) &= \int_0^{\infty} \left(\sum_{n=0}^{\infty} \frac{(-pw)^n}{n!} \right) Y(w) dw \\ &= \int_0^{\infty} e^{-pw} Y(w) dw = L(p), \end{aligned}$$

where L is the Laplace transform of Y . The Laplace transform L of y is therefore defined uniquely, and as y decays exponentially, is holomorphic in Π_0 . We may therefore invert L to find y [32]. ■

We may therefore drop the arbitrary periodic function U in favour of a constant K . We thus have $M(s) = KF(s)Q(s)$ where α is only present in the Q term. The inverse of F is the solution y_0 to the homogeneous equation (2.37), whereas the inverse of M is a solution to the functional differential equation (2.36). The Q term, therefore, can be regarded as containing information about how y_0 should be modified to produce a solution to (2.36).

Let $M_0(s) = KF(s)$, $\beta = \alpha^{k+1}$ and let

$$M_n(s) = \left(1 - \frac{C}{B\alpha^s\beta^n} \right) M_{n-1}(s),$$

for $n = 1, 2, \dots$, so that $M(s) = \lim_{n \rightarrow \infty} M_n(s)$. We investigate the form of the solution y to (2.44) by examining the inverses of M_n in turn.

Clearly M_0 inverts to the solution y_0 of the homogeneous ODE (2.37). Define $y_n(x)$ as the inverse of $M_n(s)$ for $n > 0$. For any function h where the integrals are finite,

$$\int_0^{\infty} x^{s-1} h(\alpha x) dx = \frac{1}{\alpha^s} \int_0^{\infty} (\alpha x)^{s-1} h(\alpha x) d(\alpha x), \quad (2.52)$$

so that $\frac{1}{\alpha^s} M_{n-1}(s)$ inverts to $y_{n-1}(\alpha x)$; hence,

$$y_n(x) = y_{n-1}(x) - \frac{C}{B\beta^n} y_{n-1}(\alpha x). \quad (2.53)$$

Solving this recursion for the first few values of n yields

$$\begin{aligned} y_1(x) &= y_0(x) - \frac{C}{B} \left(\frac{1}{\beta} \right) y_0(\alpha x) \\ y_2(x) &= y_0(x) - \frac{C}{B} \left(\frac{1}{\beta} + \frac{1}{\beta^2} \right) y_0(\alpha x) + \left(\frac{C}{B} \right) \left(\frac{1}{\beta^3} \right) y_0(\alpha^2 x) \\ y_3(x) &= y_0(x) - \frac{C}{B} \left(\frac{1}{\beta} + \frac{1}{\beta^2} + \frac{1}{\beta^3} \right) y_0(\alpha x) + \left(\frac{C}{B} \right)^2 \left(\frac{1}{\beta^3} + \frac{1}{\beta^4} + \frac{1}{\beta^5} \right) y_0(\alpha^2 x) \\ &\quad - \left(\frac{C}{B} \right)^3 \left(\frac{1}{\beta^6} \right) y_0(\alpha^3 x). \end{aligned}$$

While it is not immediately obvious what the exact expression for y_n will be, it is clear that it will be a linear combination of the terms $y_0(\alpha^k x)$ for $k = 0, 1, \dots, n$. Thus, as expected, the inverse Mellin transform of M is a series in $y_0(\alpha^n x)$. We can find the series exactly by rewriting the product $Q(s)$ as a sum, and then inverting term by term.

Setting $c = \frac{C}{B\alpha^s}$ and β as before, we can write

$$Q(s) = \prod_{n=0}^{\infty} \left(1 - \frac{c}{\beta^n} \right) = \sum_{n=0}^{\infty} \frac{c^n}{\prod_{m=1}^n (1 - \beta^m)},$$

using the Euler Identity. Thus,

$$M(s) = K \sum_{n=0}^{\infty} a_n \frac{1}{\alpha^{ns}} F(s),$$

where

$$a_n = \frac{\left(\frac{C}{B} \right)^n}{\prod_{m=1}^n (1 - \beta^m)},$$

and K is an arbitrary constant. Replacing α in equation (2.52) with α^n and inverting term by term yields the inverse transform of M (and thus a solution to (2.44)) as

$$y(x) = K \sum_{n=0}^{\infty} \frac{\left(\frac{C}{B} \right)^n}{\prod_{m=1}^n (1 - \alpha^{(k+1)m})} y_0(\alpha^n x).$$

While the analysis above applies to one specific example, the principle holds for all functional differential equations. Assuming we can solve the associated Mellin equation, and that the solution found can be split into a product $Q(s)F(s)$ where $F(s)$ is independent of α , then Q will give an indication as to how to alter the inverse transform of F in order to obtain a solution to the original equation.

We end this section with a necessary condition for equation (2.36) to yield solutions of the form (2.35).

Theorem 2.19 *There exists a solution to (2.36) of the form (2.35), where y_0 is an exponentially decaying solution to the homogeneous equation (2.37), only if there exists a solution M to the associated Mellin equation that can be written in the form*

$$M(s) = F(s)Q(s),$$

where F is independent of α , and Q is holomorphic in α^{-s} .

Proof: Suppose y is a solution to (2.36) of the form (2.35), where y_0 is an exponentially decaying solution to (2.37). Then y must also decay exponentially, so that the Mellin transform M of y is defined for all $s \geq 1$. We therefore have

$$M(s) = \int_0^\infty x^{s-1} \sum_{n=0}^\infty a_n y_0(\alpha^n x) dx.$$

As y_0 decays exponentially, we may integrate the series term by term so that

$$\begin{aligned} M(s) &= \sum_{n=0}^\infty a_n \alpha^{-sn} \int_0^\infty (\alpha^n x)^{s-1} y_0(\alpha^n x) d(\alpha^n x) \\ &= \sum_{n=0}^\infty a_n \alpha^{-sn} F(s). \end{aligned}$$

where F is the Mellin transform of y_0 , which is necessarily independent of α . Thus, M has the form required. ■

Second Order Pantograph Equations

Consider the equation

$$y''(x) + ay'(x) + by(x) = \lambda y(\alpha x), \quad (3.1)$$

where a , b , λ , and α are real constants such that $\lambda \neq 0$ and $\alpha > 1$. As in the first order case, $\alpha > 1$ implies that the only solutions holomorphic at the origin are polynomials. This will be shown in detail in Chapter 5. For now, we concentrate on finding solutions defined on some halfplane, and for definiteness, we require solutions to be defined on the positive real axis.

3.1 Dirichlet Series Solutions

We expect solutions similar to those derived for the first order case, and therefore look for solutions of the form

$$y(x) = \sum_{n=0}^{\infty} c_n e^{-\alpha^n r x}, \quad (3.2)$$

where r and the coefficients c_n are to be determined. Substituting the above series into equation (3.1) leads to the recurrence relation

$$\frac{c_n}{c_{n-1}} = \frac{-\lambda}{\alpha^{2n} r^2 - a\alpha^n r + b}, \quad (3.3)$$

for $n = 1, 2, \dots$, and the indicial equation

$$r^2 - ar + b = 0. \quad (3.4)$$

A formal solution of (3.1) is thus given by

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^n (\alpha^{2m} r^2 - a\alpha^m r + b)} e^{-\alpha^n r x}, \quad (3.5)$$

where c_0 is an arbitrary constant, and r is a solution to (3.4). This series converges uniformly in the halfplane $\operatorname{Re}(rx) \geq 0$, and thus it is clear that the nature of the solution is determined by the roots of the indicial equation (3.4). This technique is analogous to the Frobenius method of finding solutions to ordinary differential equations, and as we shall see, there are a number of cases to consider.

The first complication encountered is the presence of two roots to the indicial equation (3.4). This allows the possibility of two linearly independent solutions to our functional differential equation. There are three cases to consider:

1. No solutions of the form (3.2) exist (no roots to (3.4) such that $\operatorname{Re}(r) > 0$).
2. One solution of the form (3.2) exists (one root to (3.4) such that $\operatorname{Re}(r) > 0$).
3. Two solutions of the form (3.2) exist (two roots to (3.4) such that $\operatorname{Re}(r) > 0$).

The first case leads to a divergent Dirichlet series and is consequently of little interest. The second case is satisfied if $a > 0$ and $b < 0$, and we can find similar results for existence and uniqueness (or lack thereof) as those found in Chapter 2 to the first order problem. Kim presents second order analogues to Theorem 2.1 and Theorem 2.2 ([19] pp. 48,49) and Laplace analysis similar to that done in Section 2.4 is covered by van-Brunt et. al. [36] where analogues to Lemma 2.6 and Theorem 2.11 are presented.

We briefly examine the third case, where the possibility of two Dirichlet series solutions exists. If we have 2 roots to the real quadratic (3.4), both with positive real part, then it is clear that both must be real and positive, or they are complex conjugates.

Suppose there exists distinct positive real roots r_1 and r_2 , where $r_1 > r_2$, to the indicial equation (3.4). Then two convergent Dirichlet series solutions $y_1(x) = \sum_{n=0}^{\infty} c_n e^{-\alpha^n r_1 x}$ and $y_2(x) = \sum_{n=0}^{\infty} d_n e^{-\alpha^n r_2 x}$ corresponding to r_1 and r_2 can be constructed. However, we must be careful with the treatment of y_2 when $r_1 = \alpha^n r_2$ for some $n \in N$ since

$$\alpha^{2n} r_2^2 - a\alpha^n r_2 + b = (\alpha^n r_2 - r_2)(\alpha^n r_2 - r_1),$$

and hence $\frac{d_n}{d_{n-1}}$ has a singularity. This is analogous to the case where roots differ by an integer in the Frobenius method. Note that we have no problem in the construction of y_1 as

$$\begin{aligned} \alpha^{2n} r_1^2 - a\alpha^n r_1 + b &= (\alpha^n r_1 - r_1)(\alpha^n r_1 - r_2) \\ &= (\alpha^n r_1 - r_1)(\alpha^{2n} r_2 - r_2) \\ &= r_1 r_2 (\alpha^n - 1)(\alpha^{2n} - 1) > 0. \end{aligned}$$

Indeed, as in the Frobenius case, y_2 is just a constant multiple of y_1 . Suppose that $r_1 = \alpha^k r_2$ for some $k \in N$. The recurrence relation for the d_n 's gives

$$(\alpha^{2k} r_2^2 - a\alpha^k r_2 + b)d_k = -\lambda d_{k-1},$$

and thus $d_{k-1} = 0$. We can then follow the recurrence relation back and find that $d_n = 0$ for $0 \leq n \leq k-1$. Thus,

$$\begin{aligned} y_2(x) &= \sum_{n=k}^{\infty} d_n e^{-\alpha^n r_2 x} \\ &= \sum_{n=k}^{\infty} d_n e^{-\alpha^{n-k} r_1 x}, \\ &= \sum_{n=0}^{\infty} d_{n+k} e^{-\alpha^n r_1 x}, \end{aligned}$$

and so the constants d_{n+k} must be identical up to a multiplicative constant to the constants c_n . Thus $y_2 = Ay_1$ and only one linearly independent solution exists. With the method of Frobenius, a second linearly independent solution can be found to the ordinary equation by multiplying the first solution by an appropriate logarithmic function. An analogous method for the pantograph equation will be

shown in the next section on Laplace analysis.

In the case where the roots differ by a non-integer power of α , we get linear independence. Indeed, for large x , the behaviour of the solutions is dominated by the first term in the series, i.e.,

$$\begin{aligned} y_1(x) &\sim c_0 e^{-r_1 x} \text{ and } y_2(x) \sim d_0 e^{-r_2 x} \\ y_1'(x) &\sim -c_0 r_1 e^{-r_1 x} \text{ and } y_2'(x) \sim -d_0 r_2 e^{-r_2 x} \end{aligned}$$

as $x \rightarrow \infty$. Therefore the Wronskian of y_1 and y_2 becomes

$$y_1(x)y_2'(x) - y_1'(x)y_2(x) \sim c_0 d_0 (r_1 - r_2) e^{-(r_1+r_2)x} \neq 0;$$

hence, $y_1(x)y_2'(x) - y_1'(x)y_2(x)$ is not identically zero for all x , and hence y_1 and y_2 are linearly independent. Thus, there exists an infinite number of solutions of the form $y(x) = y_1(x) + y_2(x)$ since c_0 and d_0 are arbitrary constants.

Lastly, we consider the case where r_1 and r_2 are complex conjugates. The solutions y_1 and y_2 are then complex valued functions. We let $r_1 = u + iv$ where $u = \frac{a}{2}$ and $v = \frac{\sqrt{4b-a^2}}{2}$, and thus $r_2 = u - iv$. The constants c_n and d_n are also complex, and we find that $d_n = A\bar{c}_n$ for some constant A . Indeed, the recurrence relation for c_n gives

$$\bar{c}_n = \frac{-\lambda \bar{c}_{n-1}}{\alpha^{2n} r_2^2 - a \alpha^n r_2 + b},$$

and so the d_n and \bar{c}_n follow the same recurrence relation, and are thus identical up to a multiplicative constant. We thus find that

$$\begin{aligned} y_2 &= A \sum_{n=0}^{\infty} \bar{c}_n e^{-\alpha^n \bar{r}_1 x} \\ &= A \sum_{n=0}^{\infty} c_n e^{-\alpha^n r_1 x} \\ &= A y_1(x). \end{aligned}$$

The real and imaginary parts of y_1 (or y_2) provide two linearly independent real

solutions. Let $c_n = c_{1n} + ic_{2n}$. These solutions can then be expressed as

$$\begin{aligned} w_1(x) &= \operatorname{Re}\left(\sum_{n=0}^{\infty} (c_{1n} + ic_{2n}) e^{-\alpha^n(u+iv)x}\right) \\ &= \sum_{n=0}^{\infty} (c_{1n} \cos(\alpha^n vx) + c_{2n} \sin(\alpha^n vx)) e^{-\alpha^n ux} \end{aligned}$$

and

$$\begin{aligned} w_2(x) &= \operatorname{Im}\left(\sum_{n=0}^{\infty} (c_{1n} + ic_{2n}) e^{-\alpha^n(u+iv)x}\right) \\ &= \sum_{n=0}^{\infty} (c_{2n} \cos(\alpha^n vx) - c_{1n} \sin(\alpha^n vx)) e^{-\alpha^n ux}. \end{aligned}$$

A study of the long term behaviour of these functions indicates that they are linearly independent.

3.2 Laplace Analysis

As with the first order case, transformation of the equation using the Laplace transform is useful in further tracking both the Dirichlet series solutions and other possible solution forms. Laplace analysis of equations of the form (3.1) has been studied by van-Brunt et. al. [36], although they consider only the case where there is a single root of the indicial equation (3.4) with $\operatorname{Re}(r) > 0$, and where solutions were subject to the initial conditions $y(0) = 0$ and $y'(0) = \frac{\lambda}{\alpha} - b$. We look at the case where both roots to the indicial equation (3.4) have $\operatorname{Re}(r) > 0$ and derive a further solution form in the special case where the roots of (3.4) differ by an integer power of α .

Consider equation (3.1) along with the initial conditions

$$y(0) = y_0, \quad y'(0) = y_1. \quad (3.6)$$

Applying the Laplace transform to both sides of equation (3.1), we get the functional relation

$$L(p) = \frac{\frac{\lambda}{\alpha} L\left(\frac{p}{\alpha}\right) + (p+a)y_0 + y_1}{(p+r_1)(p+r_2)}, \quad (3.7)$$

where r_1 and r_2 are the roots of the indicial equation (3.4). Equation (3.7) is an example of a class of equations known as ‘non-homogeneous q -difference equations’ with $q = \frac{1}{\alpha} < 1$. These equations have the form

$$f(p)Y(qp) + g(p)Y(p) + h(p) = 0.$$

Non-homogeneous q -difference equations have been studied by Adams [1] in addition to other researchers for the cases where the coefficients are holomorphic at $p = 0$. In particular, Adams shows that there are meromorphic solutions to equations such as (3.7) that are holomorphic at $p = 0$. Indeed, a simple power series substitution as was done with the first order problem in Chapter 2 is sufficient to show this. We have the following result.

Lemma 3.1 *There exists a unique holomorphic solution to (3.7) among functions holomorphic at $p = 0$, provided $\lambda \neq \alpha^n b$ for $n = 1, 2, \dots$*

Proof: Substituting a formal power series expression $L(p) = \sum_{n=0}^{\infty} c_n p^n$ into (3.7) and equating like powers gives

$$\begin{aligned} (b - \frac{\lambda}{\alpha})c_0 &= y_1 + ay_0, \\ (b - \frac{\lambda}{\alpha^2})c_1 &= y_0 - ac_0, \end{aligned}$$

and the recurrence relation

$$(b - \frac{\lambda}{\alpha^{n+1}})c_n = -ac_{n-1} - c_{n-2}, \quad (3.8)$$

for $n = 2, 3, \dots$. It is clear that as long as $\lambda \neq \frac{b}{\alpha^n}$ for $n = 1, 2, \dots$, these relations uniquely define the c_n 's, so we just need to check that the series converges. We do this by majorizing the series. Let $C_0 = |c_0|$ and $C_1 = |c_1|$ and define the rest of the C_n 's by the recurrence relation

$$\hat{b}C_n = |a|C_{n-1} + C_{n-2},$$

where $\hat{b} = \min_{n \in \mathbb{N}} \{|\lambda - \frac{b}{\alpha^n}|\}$. Clearly the series $F(p) = \sum_{n=0}^{\infty} C_n p^n$ majorizes our formal power series solution. It is also clear that $F(p)$ is the solution of the equation

$$\hat{b}F(p) - C_1 p - C_0 = |a|p(F(p) - C_0) + p^2 F(p),$$

which gives

$$F(p) = \frac{(C_1 - C_0|a|)p + C_0}{\hat{b} - |a|p - p^2}.$$

Since $\lambda \neq b/\alpha^n$ for all $n \in \mathbb{N}$, we have $\hat{b} > 0$; thus, F is holomorphic at $p = 0$. Now, the power series for F majorizes our formal power series, so that L must also be holomorphic at $p = 0$. ■

Note that in the case where $\lambda = b\alpha^n$ for some positive integer n , c_{n-1} is not uniquely defined, and can be chosen arbitrarily. This however forces a compatibility condition between the initial conditions y_0 and y_1 , and the constants a , b and α , that must be met. Assuming that this compatibility condition is met, the solution is still holomorphic at $p = 0$ and is unique up to the multiplicative constant c_{n-1} .

The solution holomorphic at the origin can be continued to a solution meromorphic in the complex plane as shown by the following Lemma.

Lemma 3.2 *The solution L to equation (3.7) that is holomorphic at the origin can be continued to a function meromorphic in \mathbb{C} . If L has any singularities, then they must be simple poles at $p = -\alpha^n r_1$ or $p = -\alpha^n r_2$ for some $n = 0, 1, 2, \dots$, unless $r_1 = \alpha^k r_2$ for some integer k . In this case L may have double poles at $p = -\alpha^n r_1$.*

Proof: Evaluating equation (3.7) at p/α gives

$$L\left(\frac{p}{\alpha}\right) = \frac{\frac{\lambda}{\alpha}}{\left(\frac{p}{\alpha} + r_1\right)\left(\frac{p}{\alpha} + r_2\right)} L\left(\frac{p}{\alpha^2}\right) + \frac{\left(\frac{p}{\alpha} + a\right)y_0 + y_1}{\left(\frac{p}{\alpha} + r_1\right)\left(\frac{p}{\alpha} + r_2\right)},$$

which, upon substitution in (3.7) gives

$$L(p) = \frac{\left(\frac{\lambda}{\alpha}\right)^2 L\left(\frac{p}{\alpha^2}\right)}{\prod_{m=0}^1 \left(\frac{p}{\alpha^m} + r_1\right)\left(\frac{p}{\alpha^m} + r_2\right)} + \sum_{j=0}^1 \frac{\left(\frac{\lambda}{\alpha}\right)^j \left[\left(\frac{p}{\alpha^j} + a\right)y_0 + y_1\right]}{\left(\prod_{m=0}^j \left(\frac{p}{\alpha^m} + r_1\right)\left(\frac{p}{\alpha^m} + r_2\right)\right)}.$$

Repeating the above steps n times yields

$$L(p) = \frac{\left(\frac{\lambda}{\alpha}\right)^{n+1} L\left(\frac{p}{\alpha^{n+1}}\right)}{\prod_{m=0}^n \left(\frac{p}{\alpha^m} + r_1\right)\left(\frac{p}{\alpha^m} + r_2\right)} + \sum_{j=0}^n \frac{\left(\frac{\lambda}{\alpha}\right)^j \left[\left(\frac{p}{\alpha^j} + a\right)y_0 + y_1\right]}{\prod_{m=0}^j \left(\frac{p}{\alpha^m} + r_1\right)\left(\frac{p}{\alpha^m} + r_2\right)}. \quad (3.9)$$

We know there exists a $\delta > 0$ such that L is holomorphic in the disc $|p| < \delta$. Choose

any complex number p and choose n so large that

$$\left| \frac{p}{\alpha^{n+1}} \right| < \delta.$$

If $p \neq -r_1\alpha^m$ and $p \neq -r_2\alpha^m$ for all $m = 0, 1, 2, \dots, n$, then (3.9) implies that L must be holomorphic at p . If $p = -\alpha^m r_1$ or $p = -\alpha^m r_2$ for some $m = 0, 1, 2, \dots, n$, then p can, at worst, be a pole of order 1, assuming $r_1 \neq \alpha^k r_2$ for all integers k . If $r_1 = \alpha^k r_2$ for an integer k then we obtain at worst poles of order 2. We may choose p to be any point in the complex plane, hence L must be meromorphic in \mathbb{C} . ■

Since L is meromorphic in \mathbb{C} , with possible poles at $p = \alpha^m r_1$ and $p = \alpha^m r_2$ for $m = 0, 1, 2, \dots$, we anticipate a solution in the form of a Mittag-Leffler expansion

$$L(p) = \sum_{n=0}^{\infty} \frac{c_n}{p + r_1 \alpha^n} + \sum_{n=0}^{\infty} \frac{d_n}{p + r_2 \alpha^n}, \quad (3.10)$$

in the case where $r_1 \neq \alpha^k r_2$ for all integers k , and of the form

$$L(p) = \sum_{n=0}^{\infty} \frac{c_n}{(p + r_1 \alpha^n)^2} + \sum_{n=0}^{\infty} \frac{d_n}{p + r_2 \alpha^n}, \quad (3.11)$$

when $r_1 = \alpha^k r_2$ for some integer k .

We may assume without loss of generality that $|r_1| \geq |r_2|$. Thus, there are 3 cases to consider:

1. $r_1 \neq \alpha^k r_2$ for all integers k ,
2. $r_1 = \alpha^k r_2$ for some positive integer k ,
3. $r_1 = r_2$.

We treat each of these cases in a separate section in order to simplify the notation.

3.2.1 The case where $r_1 \neq r_2\alpha^k$ for all integers k

Substituting the expression (3.10) into equation (3.7) and equating principal parts yields the recurrence relations

$$c_n = \frac{\lambda c_{n-1}}{(r_1 - r_1\alpha^n)(r_2 - r_1\alpha^n)},$$

and

$$d_n = \frac{\lambda d_{n-1}}{(r_1 - r_2\alpha^n)(r_2 - r_2\alpha^n)},$$

for $n = 1, 2, \dots$. The coefficients c_0 and d_0 must satisfy the relations

$$c_0 = \frac{(r_1 - a)y_0 + y_1}{r_1 - r_2} - \frac{\lambda}{r_1 - r_2} \sum_{n=0}^{\infty} \left\{ \frac{c_n}{r_1\alpha^{n+1} - r_1} + \frac{d_n}{r_2\alpha^{n+1} - r_1} \right\}, \quad (3.12)$$

and

$$d_0 = \frac{-(r_2 - a)y_0 - y_1}{r_1 - r_2} + \frac{\lambda}{r_1 - r_2} \sum_{n=0}^{\infty} \left\{ \frac{c_n}{r_1\alpha^{n+1} - r_2} + \frac{d_n}{r_2\alpha^{n+1} - r_2} \right\}. \quad (3.13)$$

It is clear from the recurrence relations for the c_n and d_n that the series will converge to a meromorphic function in \mathbb{C} with the correct poles, and so all that is left to do is verify that there exist a c_0 and a d_0 so that (3.10) is a solution to (3.7).

Adding equations (3.12) and (3.13) yields the relation

$$c_0 + d_0 = y_0 - \lambda \left\{ \sum_{n=1}^{\infty} \frac{c_{n-1}}{(r_1\alpha^n - r_1)(r_1\alpha^n - r_2)} + \sum_{n=1}^{\infty} \frac{d_{n-1}}{(r_2\alpha^n - r_1)(r_1\alpha^n - r_2)} \right\},$$

which when combined with the recurrence relations for c_n and d_n gives

$$\sum_{n=0}^{\infty} c_n + \sum_{n=0}^{\infty} d_n = y_0. \quad (3.14)$$

Similarly, subtracting (3.12) and (3.13) yields the further relation

$$(r_1 - r_2)(c_0 - d_0) = (r_1 + r_2 - 2a)y_0 + 2y_1 - \sum_{n=1}^{\infty} \{c_n(2r_1\alpha^n - a) + d_n(2r_2\alpha^n - a)\},$$

which can be simplified with the use of (3.14) and the fact that $r_1 + r_2 = a$ to the

expression

$$r_1 \sum_{n=0}^{\infty} \alpha^n c_n + r_1 \sum_{n=0}^{\infty} \alpha^n d_n = -y_1. \quad (3.15)$$

Let

$$F(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{\prod_{m=1}^n (r_1 - r_1 \alpha^m)(r_2 - r_1 \alpha^m)},$$

and

$$G(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{\prod_{m=1}^n (r_1 - r_2 \alpha^m)(r_2 - r_2 \alpha^m)}.$$

Then,

$$\sum_{n=0}^{\infty} c_n = c_0 F(\lambda), \quad \sum_{n=0}^{\infty} d_n = d_0 G(\lambda),$$

and

$$\sum_{n=0}^{\infty} \alpha^n c_n = c_0 F(\alpha\lambda), \quad \sum_{n=0}^{\infty} \alpha^n d_n = d_0 G(\alpha\lambda).$$

The relations (3.14) and (3.15) can therefore be reduced to the conditions

$$c_0 F(\lambda) + d_0 G(\lambda) = y_0, \quad (3.16)$$

and

$$c_0 r_1 F(\alpha\lambda) + d_0 r_2 G(\alpha\lambda) = -y_1. \quad (3.17)$$

For a fixed value of λ , equations (3.16) and (3.17) form a system of linear equations for the unknown constants c_0 and d_0 . The constants are determined uniquely provided

$$r_1 F(\alpha\lambda)G(\lambda) - r_2 F(\lambda)G(\alpha\lambda) \neq 0. \quad (3.18)$$

To check that (3.18) is satisfied for a given value of λ , we use the inherent recurrence in the terms of both F and G to establish functional identities that reduce (3.18) to something more tractable. Indeed, we see that

$$\begin{aligned} r_1^2 F(\alpha^2\lambda) + r_1(r_1 + r_2)F(\alpha\lambda) + r_1 r_2 F(\lambda) &= \sum_{n=0}^{\infty} \frac{(r_1 \alpha^n - r_1)(r_1 \alpha^n - r_2)\lambda^n}{\prod_{m=1}^n (r_1 - r_1 \alpha^m)(r_2 - r_1 \alpha^m)}, \\ &= \lambda \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{\prod_{m=1}^{n-1} (r_1 - r_1 \alpha^m)(r_2 - r_1 \alpha^m)}, \\ &= \lambda F(\lambda). \end{aligned}$$

so by replacing λ with $\frac{\lambda}{\alpha}$ and using the relationships $b = r_1 r_2$ and $a = r_1 + r_2$, we have the functional identity

$$r_1 F(\alpha\lambda) - aF(\lambda) = -\frac{(b - \frac{\lambda}{\alpha})}{r_1} F(\frac{\lambda}{\alpha}).$$

Similarly, we can establish the identity

$$r_2 G(\alpha\lambda) - aG(\lambda) = -\frac{(b - \frac{\lambda}{\alpha})}{r_2} G(\frac{\lambda}{\alpha}).$$

Multiplying the first identity by $G(\lambda)$ and subtracting the second identity multiplied by $F(\lambda)$ gives

$$r_1 F(\alpha\lambda)G(\lambda) - r_2 F(\lambda)G(\alpha\lambda) = (1 - \frac{\lambda}{\alpha b}) \left(r_1 F(\lambda)G(\frac{\lambda}{\alpha}) - r_2 F(\frac{\lambda}{\alpha})G(\lambda) \right).$$

Applying this relation n times yields

$$r_1 F(\alpha\lambda)G(\lambda) - r_2 F(\lambda)G(\alpha\lambda) = \left(\prod_{m=1}^n (1 - \frac{\lambda}{\alpha^m b}) \right) \left(r_1 F(\frac{\lambda}{\alpha^{n-1}})G(\frac{\lambda}{\alpha^n}) - r_2 F(\frac{\lambda}{\alpha^n})G(\frac{\lambda}{\alpha^{n-1}}) \right).$$

As $n \rightarrow \infty$ we have that $F(\frac{\lambda}{\alpha^n}) \rightarrow F(0) = 1$ and $G(\frac{\lambda}{\alpha^n}) \rightarrow G(0) = 1$; hence,

$$r_1 F(\alpha\lambda)G(\lambda) - r_2 F(\lambda)G(\alpha\lambda) = (r_1 - r_2) \prod_{m=1}^{\infty} (1 - \frac{\lambda}{\alpha^m b}). \quad (3.19)$$

Since $r_1 - r_2 \neq 0$, condition (3.18) is satisfied for all $\lambda \in \mathbb{C}$ unless $\lambda = \alpha^m b$ for some positive integer m .

In summary, provided that $\lambda \neq \alpha^m b$ for any positive integer m , Lemma 3.1 implies that the Mittag-Leffler expansion (3.10) is the only solution to equation (3.7) that is holomorphic at the origin. In the case where $\lambda = \alpha^m b$ for some positive integer m , the coefficients c_0 and d_0 are not determined uniquely, and furthermore the compatibility condition

$$r_1 F(\alpha\lambda)y_0 + F(\lambda)y_1 = 0 \quad (3.20)$$

must be met. Assuming that y_0 and y_1 satisfy (3.20), the solution is unique up to an arbitrary constant.

3.2.2 The case where $r_1 = r_2\alpha^k$ for some positive integer k

The second case is similar to case 1, although there is a coupling introduced between the constants c_n and d_n that makes life a little more difficult.

Substituting the expression (3.11) into equation (3.7) and equating principal parts yields the recurrence relations

$$c_n = \frac{\lambda\alpha c_{n-1}}{(r_1 - r_1\alpha^n)(r_2 - r_1\alpha^n)},$$

and

$$d_n = \begin{cases} \frac{\lambda d_{n-1}}{(r_1 - r_2\alpha^n)(r_2 - r_2\alpha^n)} & 0 < n < k, \\ \frac{\lambda d_{n-1} + (2r_2\alpha^n - a)c_{n-k}}{(r_1 - r_2\alpha^n)(r_2 - r_2\alpha^n)} & n > k, \end{cases}$$

where $c_0 = \frac{\lambda d_{k-1}}{r_2 - r_1}$. The coefficients d_0 and d_k must satisfy the relations

$$(r_1 - r_2)d_0 = \sum_{n=1}^{\infty} \frac{\lambda\alpha c_{n-1}}{(r_1\alpha^n - r_2)^2} + \sum_{n=1}^{\infty} \frac{\lambda d_{n-1}}{r_2\alpha^n - r_2} + r_1 y_0 + y_1, \quad (3.21)$$

and

$$(r_2 - r_1)d_k + c_0 = \sum_{n=1}^{\infty} \frac{\lambda\alpha c_{n-1}}{(r_1\alpha^n - r_1)^2} + \sum_{\substack{n=1 \\ n \neq k}}^{\infty} \frac{\lambda d_{n-1}}{r_2\alpha^n - r_1} + r_2 y_0 + y_1. \quad (3.22)$$

It is clear that the Mittag-Leffler expansion converges to a meromorphic function in \mathbb{C} with the correct poles, but we need to verify that there exist constants d_0 and d_k such that (3.11) is a solution to (3.7).

Subtracting equations (3.21) and (3.22), and using the recurrence relations for c_n and d_n gives

$$\sum_{n=0}^{\infty} d_n = y_0. \quad (3.23)$$

Similarly, adding (3.21) and (3.22) gives

$$2 \sum_{n=0}^{\infty} c_n - \sum_{n=0}^{\infty} (2r_2\alpha^n - a)d_n = ay_0 + 2y_1,$$

which can be simplified with the use of (3.23) to give

$$\sum_{n=0}^{\infty} c_n - r_2 \sum_{n=0}^{\infty} \alpha^n d_n = y_1. \quad (3.24)$$

Solving the recurrence relations for the c_n and d_n , we find

$$c_n = \frac{(\lambda\alpha)^n}{\prod_{m=1}^n (r_1 - r_1\alpha^m)(r_2 - r_1\alpha^m)} c_0,$$

and

$$d_n = \begin{cases} \frac{\lambda^n}{\prod_{m=1}^n (r_1 - r_2\alpha^m)(r_2 - r_2\alpha^m)} d_0 & n < k \\ \frac{\lambda^{n-k} \left(d_k + \sum_{m=1}^{n-k} \frac{\alpha^m (2r_1\alpha^m - a)}{(r_1 - r_1\alpha^m)(r_2 - r_1\alpha^m)} c_0 \right)}{\prod_{m=k+1}^n (r_1 - r_2\alpha^m)(r_2 - r_2\alpha^m)} & n > k. \end{cases}$$

Thus,

$$\sum_{n=0}^{\infty} d_n = d_k F(\lambda) + d_0 G(\lambda),$$

where

$$F(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{\prod_{m=1}^n (r_1 - r_1\alpha^m)(r_2 - r_1\alpha^m)},$$

and

$$G(\lambda) = \sum_{n=0}^{k-1} \frac{\lambda^n}{\prod_{m=1}^n (r_1 - r_2\alpha^m)(r_2 - r_2\alpha^m)} + \sum_{n=k+1}^{\infty} \frac{\lambda^n}{r_2 - r_1} \frac{\sum_{m=1}^{n-k} \frac{\alpha^m (2r_1\alpha^m - a)}{(r_1 - r_1\alpha^m)(r_2 - r_1\alpha^m)}}{\prod_{\substack{m=1 \\ m \neq k}}^n (r_1 - r_2\alpha^m)(r_2 - r_2\alpha^m)}.$$

Similarly, we find

$$\sum_{n=0}^{\infty} \alpha^n d_n = d_k \alpha^k F(\alpha\lambda) + d_0 G(\alpha\lambda),$$

and

$$\sum_{n=0}^{\infty} c_n = d_0 C_0(\lambda) F(\alpha\lambda),$$

where

$$C_0(\lambda) = \frac{\lambda^k}{(r_2 - r_1) \prod_{m=1}^k (r_1 - r_2 \alpha^m)(r_2 - r_2 \alpha^m)}.$$

We therefore reduce the relations (3.23) and (3.24) to

$$d_k F(\lambda) + d_0 G(\lambda) = y_0, \quad (3.25)$$

$$d_k r_1 F(\alpha \lambda) + d_0 (r_2 G(\alpha \lambda) - C_0(\lambda) F(\alpha \lambda)) = -y_1. \quad (3.26)$$

For a fixed value of λ , equations (3.25) and (3.26) form a system of linear equations for the unknown constants d_0 and d_k , where the constants are determined uniquely provided

$$r_1 F(\alpha \lambda) G(\lambda) - r_2 F(\lambda) G(\alpha \lambda) + C_0(\lambda) F(\lambda) F(\alpha \lambda) \neq 0. \quad (3.27)$$

In the same way as we did for the case where $r_1 \neq r_2 \alpha^k$, we use the inherent recurrence in the expressions for F and G to establish the functional identities

$$r_1 F(\alpha \lambda) - a F(\lambda) = -\frac{(b - \frac{\lambda}{\alpha})}{r_1} F(\frac{\lambda}{\alpha}),$$

and

$$r_2 G(\alpha \lambda) - C_0(\lambda) F(\alpha \lambda) - a G(\lambda) = -\frac{(b - \frac{\lambda}{\alpha})}{r_2} \left(G(\frac{\lambda}{\alpha}) + \frac{1}{r_1} C_0(\frac{\lambda}{\alpha}) F(\frac{\lambda}{\alpha}) \right).$$

Multiplying the first by $G(\lambda)$ and subtracting the second multiplied by $F(\lambda)$ yields

$$\begin{aligned} r_1 F(\alpha \lambda) G(\lambda) - r_2 F(\lambda) G(\alpha \lambda) + C_0(\lambda) F(\lambda) F(\alpha \lambda) = \\ \left(1 - \frac{\lambda}{\alpha b}\right) \left(r_1 F(\lambda) G(\frac{\lambda}{\alpha}) - r_2 F(\frac{\lambda}{\alpha}) G(\lambda) + C_0(\frac{\lambda}{\alpha}) F(\frac{\lambda}{\alpha}) F(\lambda) \right), \end{aligned} \quad (3.28)$$

and applying relation (3.28) n times gives

$$\begin{aligned} r_1 F(\alpha \lambda) G(\lambda) - r_2 F(\lambda) G(\alpha \lambda) + C_0(\lambda) F(\lambda) F(\alpha \lambda) = \\ \prod_{m=1}^n \left(1 - \frac{\lambda}{\alpha^m b}\right) \left(r_1 F(\frac{\lambda}{\alpha^{n-1}}) G(\frac{\lambda}{\alpha^n}) - r_2 F(\frac{\lambda}{\alpha^n}) G(\frac{\lambda}{\alpha^{n-1}}) + C_0(\frac{\lambda}{\alpha^n}) F(\frac{\lambda}{\alpha^n}) F(\frac{\lambda}{\alpha^{n-1}}) \right). \end{aligned}$$

Now, $G(\frac{\lambda}{\alpha^n}) \rightarrow G(0) = 1$, $F(\frac{\lambda}{\alpha^n}) \rightarrow F(0) = 1$, and $C_0(\frac{\lambda}{\alpha^n}) \rightarrow C_0(0) = 0$ as $n \rightarrow \infty$;

hence,

$$r_1 F(\alpha\lambda)G(\lambda) - r_2 F(\lambda)G(\alpha\lambda) + C_0(\lambda)F(\lambda)F(\alpha\lambda) = (r_1 - r_2) \prod_{m=1}^{\infty} \left(1 - \frac{\lambda}{\alpha^m b}\right). \quad (3.29)$$

Since $r_1 - r_2 \neq 0$, condition (3.27) is satisfied for all $\lambda \in \mathbb{C}$ unless $\lambda = \alpha^m b$ for some positive integer m .

Thus, provided that $\lambda \neq \alpha^m b$ for any positive integer m , the Mittag-Leffler expansion (3.11) is a valid solution to (3.7) that is holomorphic at $p = 0$. If $\lambda = \alpha^m b$ for some positive integer m , the coefficients d_0 and d_k are not determined uniquely, and furthermore the compatibility condition (3.20) must be met. Assuming that y_0 and y_1 satisfy (3.20), the solution is still valid, and will be unique up to an arbitrary constant.

3.2.3 The case where $r_1 = r_2$

In this last case, we have a repeated root to the indicial equation. Much of the analysis of the case where $r_1 = r_2 \alpha^k$ holds for this case as well, so we will present just the highlights. Once again, we may substitute the series (3.11) with r_2 replaced by r_1 into (3.7) and equate principal parts. The recurrence relations are

$$c_n = \frac{\lambda \alpha c_{n-1}}{(r_1 - r_1 \alpha^n)^2},$$

and

$$d_n = \frac{\lambda d_{n-1} + 2(r_1 \alpha^n - r_1) c_{n-1}}{(r_1 - r_1 \alpha^n)^2}.$$

The relations obtained for c_0 and d_0 yield the equations

$$\sum_{n=0}^{\infty} d_n = y_0, \quad (3.30)$$

and

$$\sum_{n=0}^{\infty} c_n - r_1 \sum_{n=0}^{\infty} \alpha^n d_n = y_1. \quad (3.31)$$

Solving the recurrence relations for c_n and d_n , equations (3.30),(3.31) can be recast

in the form

$$d_0 F(\lambda) + c_0 G(\lambda) = y_0, \quad (3.32)$$

$$d_0 r_1 F(\alpha\lambda) + c_0 (r_1 G(\alpha\lambda) - F(\alpha\lambda)) = -y_1. \quad (3.33)$$

where

$$F(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{\prod_{m=1}^n (r_1 - r_1 \alpha^m)^2},$$

and

$$G(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n \sum_{m=1}^n \frac{2\alpha^m}{(r_1 - r_1 \alpha^m)^2}}{\prod_{m=1}^n (r_1 - r_1 \alpha^m)^2}.$$

For a fixed value of λ , equations (3.32) and (3.33) form a system of linear equations for the unknown constants c_0 and d_0 . They determine the constants uniquely, provided

$$r_1 F(\alpha\lambda)G(\lambda) - r_1 F(\lambda)G(\alpha\lambda) + F(\lambda)F(\alpha\lambda) \neq 0. \quad (3.34)$$

Using the recurrence relations established so far, we obtain

$$\begin{aligned} & r_1 F(\alpha\lambda)G(\lambda) - r_1 F(\lambda)G(\alpha\lambda) + F(\lambda)F(\alpha\lambda) = \\ & \prod_{m=1}^n \left(1 - \frac{\lambda}{\alpha^m b}\right) \left(r_1 F\left(\frac{\lambda}{\alpha^{n-1}}\right)G\left(\frac{\lambda}{\alpha^n}\right) - r_1 F\left(\frac{\lambda}{\alpha^n}\right)G\left(\frac{\lambda}{\alpha^{n-1}}\right) + F\left(\frac{\lambda}{\alpha^n}\right)F\left(\frac{\lambda}{\alpha^{n-1}}\right) \right), \end{aligned}$$

for any $n > 0$. As $n \rightarrow \infty$, $F(\frac{\lambda}{\alpha^n}) \rightarrow F(0) = 1$ and $G(\frac{\lambda}{\alpha^n}) \rightarrow G(0) = 1$; hence,

$$r_1 F(\alpha\lambda)G(\lambda) - r_1 F(\lambda)G(\alpha\lambda) + F(\lambda)F(\alpha\lambda) = \prod_{m=1}^{\infty} \left(1 - \frac{\lambda}{\alpha^m b}\right). \quad (3.35)$$

Thus, the condition (3.34) is satisfied for all $\lambda \in \mathbb{C}$ unless $\lambda = \alpha^m b$ for some positive integer m .

In all 3 cases, the Mittag-Leffler solution is a valid solution to (3.7) that is holomorphic at $p = 0$. Lemma 3.1 implies that it is the only such solution if $\lambda \neq b\alpha^m$ for all positive integers m . It is clear that the inverse Laplace transform y of our Mittag-Leffler solution L decays exponentially, as it transforms to a series of exponentials. In fact we can show that y is the only solution to (3.1) satisfying (3.6) that decays

exponentially. Suppose we have two solutions y_1, y_2 to (3.1) satisfying (3.6) that decay exponentially. Then their Laplace transforms L_1, L_2 would be solutions to (3.7) holomorphic at $p = 0$. If $\lambda \neq b\alpha^m$ for all positive integers m , Lemma 3.1 implies that $L_1 \equiv L_2$; hence, $y_1 \equiv y_2$. We therefore have the following result.

Theorem 3.3 *If $\lambda \neq b\alpha^m$ for all positive integers m , there exists a unique solution y to (3.1) satisfying (3.6) that decays exponentially.*

Furthermore, we note that in the case where $\lambda = b\alpha^m$ for some non-negative integer m , if y_0 and y_1 satisfy (3.20), then we have an exponentially decaying solution to (3.1) unique up to a multiplicative constant. In fact, we know the form of the solution completely. We thus have the following analog of the Frobenius method for ordinary differential equations.

Theorem 3.4 (Frobenius Method for the Pantograph Equation) *Let r_1 and r_2 be roots of the indicial equation (3.4) such that $\operatorname{Re}(r_1) \geq \operatorname{Re}(r_2) > 0$. Then the solutions to (3.1) that decay exponentially take the form*

$$y(x) = y_1(x) + y_2(x),$$

where y_1 and y_2 are linearly independent solutions to (3.1). The solution y_1 is of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n e^{-r_1 \alpha^n x},$$

and the solution y_2 is one of the following forms.

1. If $r_2 \neq r_1 \alpha^m$ for all integers m ,

$$y_2(x) = \sum_{n=0}^{\infty} d_n e^{-r_2 \alpha^n x}$$

2. If $r_2 = r_1 \alpha^m$ for some non-negative integer m ,

$$y_2(x) = \sum_{n=0}^{\infty} c_n \alpha^n x e^{-r_1 \alpha^n x} + \sum_{n=0}^{\infty} d_n e^{-r_2 \alpha^n x}.$$

As in §2.4, the uniqueness of the solution to (3.7) depends crucially on the requirement that L be holomorphic at $p = 0$. We can drop this condition to obtain other solutions to (3.7) and therefore non-Dirichlet series solutions to (3.1). Indeed, all we require is that the solution to (3.7) be holomorphic in the right half plane $\Pi_0 = \{z : \operatorname{Re}(z) > 0\}$ so that the transform can be inverted.

Let $L(p) = L_h(p) + Z(p)$ where L_h is the Mittag-Leffler solution that we know is holomorphic at the origin. Then Z must satisfy

$$Z(p) = \frac{\frac{\lambda}{\alpha} Z\left(\frac{p}{\alpha}\right)}{(p + r_1)(p + r_2)}. \quad (3.36)$$

The function Z is therefore meromorphic in \mathbb{C} and using an argument similar to that used in Lemma 3.2, we see that ‘at worst’ Z has simple poles at $p = -r_1\alpha^k$ and $p = -r_2\alpha^k$ for non-negative integers k , unless $r_1 = r_2\alpha^m$ for some non-negative integer m where we may get double poles. In fact, (3.36) can be solved explicitly for Z .

Let

$$Z(p) = U(p)W(p), \quad (3.37)$$

where

$$W(p) = \frac{p^\kappa}{\prod_{m=0}^{\infty} \left(1 + \frac{p}{r_1\alpha^m}\right) \left(1 + \frac{p}{r_2\alpha^m}\right)},$$

and

$$\kappa = \frac{\ln \left| \frac{\lambda}{\alpha b} \right| + i\phi}{\ln \alpha}.$$

As in Chapter 2, we use the principle branch of $\log(z)$, and $-\pi < \phi = \arg\left(\frac{\lambda}{\alpha b}\right) < \pi$. Note that $W(p)$ so defined is a solution to (3.36) and, upon substituting (3.37) into equation (3.36), we see that

$$U(p) = U\left(\frac{p}{\alpha}\right),$$

so that U is log-periodic with period α . We have thus found all possible solutions to (3.7).

Theorem 3.5 *The general solution to (3.7) is of the form*

$$L(p) = \sum_{n=0}^{\infty} \frac{c_n}{p + r_1\alpha^n} + \sum_{n=0}^{\infty} \frac{d_n}{p + r_2\alpha^n} + \frac{p^\kappa U(p)}{\prod_{n=0}^{\infty} \left(1 + \frac{p}{r_1\alpha^n}\right) \left(1 + \frac{p}{r_2\alpha^n}\right)}, \quad (3.38)$$

if $r_1 \neq r_2\alpha^k$ for all integers k ; and of the form

$$L(p) = \sum_{n=0}^{\infty} \frac{c_n}{(p + r_1\alpha^n)^2} + \sum_{n=0}^{\infty} \frac{d_n}{p + r_2\alpha^n} + \frac{p^\kappa U(p)}{\prod_{n=0}^{\infty} (1 + \frac{p}{r_1\alpha^n})(1 + \frac{p}{r_2\alpha^n})} \quad (3.39)$$

if $r_1 = r_2\alpha^k$ for some non-negative integer k . Here, the constants c_n , d_n and κ are as defined earlier and U is a log-periodic function of period α .

We therefore have an infinite number of solutions to the Laplace equation (3.7), but this does not imply that each solution corresponds to a solution to (3.1) satisfying (3.6). We need to first show that the solutions of Theorem 3.5 can be inverted. As in the first order case, however, we can show that these solutions must be in the Hardy space $H^2(\Pi_0)$, and hence by the Paley-Weiner theorem (cf. Hoffman [14]) they are invertible to some function $y \in L_2[0, \infty)$ that is a solution to (3.1) satisfying (3.6). Thus, the second order pantograph equation has, in general, an infinite number of solutions defined on the positive real axis.

In order to obtain unique solutions to (3.1), we must specify a further decay condition on y such as requiring the solution be in $I_n[0, \infty)$. We have the following result.

Theorem 3.6 *If $|\lambda| < \alpha^{n+1}|b|$ and $\lambda \neq \alpha^k b$ for $k = 0, 1, \dots, n$, then there exists a unique solution to (3.1) in $I_n[0, \infty)$ satisfying (3.6).*

Proof: Let y be any solution in $I_n[0, \infty)$ to (3.1) satisfying (3.6), and let $L(p)$ be its Laplace transform. Then L is a solution to (3.7), and

$$L^{(n)}(p) = (-1)^n \int_0^\infty x^n y(x) e^{-px} dx$$

so that

$$\lim_{p \rightarrow 0} L^{(n)}(p)$$

is finite. By Theorem 3.38, L can be written in the form

$$L(p) = L_h(p) + Z(p).$$

where L_h and Z are as defined earlier. Now, $\lim_{p \rightarrow 0} L_h^{(n)}(p)$ is clearly finite, so that

$\lim_{p \rightarrow 0} Z^{(n)}(p)$ must be finite. Let

$$f(p) = p^\kappa U(p),$$

and

$$g(p) = \frac{1}{\prod_{n=0}^{\infty} (1 + \frac{p}{r_1 \alpha^n})(1 + \frac{p}{r_2 \alpha^n})},$$

so that $Z(p) = f(p)g(p)$. Clearly, $g(p)$ holomorphic at $p = 0$ as the infinite product is entire and non-zero for $|p| < r_2$, so that $\lim_{p \rightarrow 0} g^{(k)}(p)$ exists for all k . As was done in the proof of Theorem 2.12, we can show that $\lim_{p \rightarrow 0} f^{(k)}(p)$ exists for $k = 0, 1, \dots, n - 1$, and that we may write

$$f^{(n)}(p) = p^{\kappa-n} u(p),$$

where u is some log-periodic function of period α . Now, in order for $\lim_{p \rightarrow 0} Z^{(n)}(p)$ to be finite, we require $\lim_{p \rightarrow 0} f^{(n)}(p)$ to be finite, and as

$$\operatorname{Re}(\kappa) = \frac{\ln \left| \frac{\lambda}{\alpha b} \right|}{\ln \alpha} < n,$$

we therefore must have $\lim_{p \rightarrow 0} u(p) = 0$. Theorem 2.10 then implies that $u \equiv 0$, hence $Z \equiv 0$, and L consists solely of the Mittag-Leffler solution. Thus, L is holomorphic at $p = 0$, and hence must be unique by Theorem 3.1. ■

Clearly, the unique solution in Theorem 3.6 is the Dirichlet series solution. The additional information that Theorem 3.6 provides is that, for a given value of λ , the decay rate of any non-Dirichlet series solutions to (3.1) must be limited. Indeed, if $|\lambda| < \alpha^{n+1}|b|$, then any non-Dirichlet series solution must decay slower than $\frac{1}{x^{n+1}}$ as $x \rightarrow \infty$.

Natural Boundaries for Functional Differential Equations

In this chapter we consider functional differential equations of the form

$$y'(z) + ay(z) = by(g(z)), \quad (4.1)$$

where a and b are complex constants, and $g(z)$ is a non-constant entire function, along with an initial condition

$$y(z_0) = y_0. \quad (4.2)$$

Note that this problem is not well posed if $g(z_0) \neq z_0$, because the initial condition does not determine $y'(z_0)$ since $y(g(z_0))$ is unknown. We thus limit ourselves to the case where z_0 is fixed under g . Much of the material presented in this chapter (excluding §4.5) has been published in [22]. We start by considering some examples.

4.1 Examples

In this section we look at two simple examples that can be solved using power series expressions, and then look at a more complicated example in a somewhat ad-hoc manner in order to obtain an understanding of the requirements for the existence of holomorphic solutions.

4.1.1 Example 1

We start with the first order pantograph equation as discussed in Chapter 2. Consider the initial value problem

$$y'(z) + ay(z) = by(\alpha z), \quad (4.3)$$

$$y(0) = y_0, \quad (4.4)$$

where a , b and α are complex constants, and $y_0 \neq 0$. If a solution y is holomorphic at the origin, then y can be represented as a power series

$$y(z) = \sum_{n=0}^{\infty} c_n z^n \quad (4.5)$$

with a non-zero radius of convergence. Substituting this expression into equation (4.3) gives us

$$c_{n+1} = \frac{b\alpha^n - a}{n+1} c_n. \quad (4.6)$$

If $|\alpha| < 1$, then (4.6) shows that either the series terminates in the case where $b = a/\alpha^k$ for some non-negative integer k , or $\left| \frac{c_{n+1}}{c_n} \right| \rightarrow 0$ as $n \rightarrow \infty$, so that the series converges for all z . In either case, we obtain the entire solution

$$y(z) = y_0 \sum_{n=0}^{\infty} \frac{\prod_{m=0}^{n-1} (b\alpha^m - a)}{n!} z^n. \quad (4.7)$$

If $|\alpha| > 1$, then it is clear that the same formal power series solution would be obtained; however, the series would diverge unless the constants a and b were related such that one of the terms in the product is zero, thus terminating the series. We elucidate further on this case in Chapter 5.

The difference between these two cases is clear. The case where $|\alpha| < 1$ has a *retarded* functional argument. That is, all points are moved closer to the origin under the mapping g (i.e., $|g(z)| < |z|$). For the second case, however, the functional argument is *advanced*: all points are moved further from the origin under g (i.e., $|g(z)| > |z|$). This behaviour of the origin attracting, or repelling neighbouring points will be characterised in the next section, and is crucial to the underlying behaviour of solutions to functional differential equations.

A third case occurs when $|\alpha| = 1$. Here, the functional argument is *neutral*, neither advanced nor retarded. In this case, it is clear that the formal power series (4.7) we have established converges for all z . This, however, is a special case. The rotations αz where $|\alpha| = 1$ are the only entire functions for which $|g(z)| = |z|$ for all z . We consider the more general case for neutral functional arguments in Chapter 6.

The solution to the retarded version of the pantograph equation is entire, and one may wonder whether or not this is always the case. The functional argument αz , however, is special among entire functions in that it is the only function that has the origin fixed, and always remains retarded (i.e. $|g(z)| < |z|$ for all $z \in \mathbb{C}$). Liouville's theorem precludes any other entire function having this property other than multiples of αz . Indeed, let $f(z)$ be an entire function that fixes the origin, and is retarded for all z , (i.e. $|f(z)| < |z|$ for all z). Then $\frac{f(z)}{z}$ is an entire function, bounded above by 1. It is therefore a constant with modulus less than 1 by Liouville's theorem, and hence $f(z) = \beta z$ for some $|\beta| < 1$.

In contrast to the above example, we now present one whose solutions have a natural boundary.

4.1.2 Example 2

Consider the equation

$$y'(z) = y(z^2) \tag{4.8}$$

along with initial condition (4.4). It is clear that the functional argument $g(z) = z^2$ is retarded for all $|z| < 1$, but is advanced for all $|z| > 1$. We begin by seeking a solution of the form (4.5). Upon substitution into the differential equation, we find that $c_n = 0$ unless n is of the form $n = 2^k - 1$ for some non-negative integer k . The solution is

$$y(z) = y_0 \sum_{k=0}^{\infty} \frac{z^{2^k-1}}{\prod_{m=0}^k (2^m - 1)}, \tag{4.9}$$

and it is clear that the gaps between the non-zero terms of the series are growing of order 2^k as $k \rightarrow \infty$. Indeed, there exists a positive number ν such that the sequence of non-zero terms $\{n_k\} = \{2^k - 1\}$ satisfies $n_{k+1} > (1 + \nu)n_k$ for all $k \geq 1$. The *Hadamard gap theorem* [33] can thus be used to assert that the circle $|z| = 1$ is a natural boundary for (4.9).

It is evident that if we were to replace the functional argument z^2 in this example with any $g(z) = z^j$, where $j \geq 3$, then the natural boundary at $|z| = 1$ would persist. If we “unfold” these polynomials to more general j -th order polynomials that preserve the retarded character around the origin, then it is less clear, but certainly expected that a natural boundary will be present. The geometric form of this boundary, however, is not obvious. In this regard it is illuminating to unfold the z^2 in the above example, and approach it from a geometric perspective.

The natural boundary that occurred above coincided with the boundary of the region in which z^2 remained retarded. It is intuitive to think that this property will hold in general, however it quickly becomes clear that a better definition of the idea of a function being retarded is needed.

4.1.3 Example 3

Consider the example

$$y'(z) = y(z^2 + \epsilon z) \quad (4.10)$$

where ϵ is some complex number. Then, using our present definition for a retarded function, the functional argument $g(z) = z^2 + \epsilon z$ would be retarded in the set such that

$$|z^2 + \epsilon z| < |z|$$

which reduces to

$$D(-\epsilon; 1) = \{z \in \mathbb{C} : |z + \epsilon| < 1\}.$$

This set must include the origin, hence $|\epsilon| < 1$. Is this a good place to start looking for a possible natural boundary? Certainly, the function is retarded in this set; however, points that start in the set may have images outside of the set. Suppose that a solution y is holomorphic in some set Ω . Then y' is also holomorphic in Ω , and the differential equation then implies that y must then be holomorphic in $g(\Omega)$. It is clear, therefore, that the maximal set Ω for which the solution y is holomorphic must satisfy

$$g(\Omega) \subseteq \Omega. \quad (4.11)$$

The region $D(-\epsilon; 1)$ does not satisfy this property. For instance, consider the point on the boundary of $D(-\epsilon; 1)$ furthest from the origin, $\epsilon_0 = -\epsilon(\frac{1}{|\epsilon|} + 1)$. Then

$|g(\epsilon_0)| = |\epsilon_0|$ and $\arg g(\epsilon_0) = 2 \arg \epsilon_0$, hence $g(\epsilon_0)$ is outside of $D(-\epsilon; 1)$, except for the special case where ϵ is real and negative. By continuity, there exist points in $D(-\epsilon; 1)$ arbitrarily close to ϵ_0 which also get mapped outside of $D(-\epsilon, 1)$. Thus, $D(-\epsilon; 1)$ does not satisfy (4.11). For the special case where ϵ is real and negative, other suitable example points can be found, such as $-\epsilon + i$.

If we look instead at the largest disc centred at the origin contained in $D(-\epsilon; 1)$ we find that we have a region that satisfies (4.11). Indeed, let $\Omega_0 = D(0; 1 - |\epsilon|)$, and let $z \in \Omega_0$. Then

$$|g(z)| = |z||z + \epsilon| < (1 - |\epsilon|)(1 - |\epsilon| + |\epsilon|) < 1 - |\epsilon|,$$

and hence Ω_0 satisfies (4.11). This set, however, need not be maximal. We can extend this set by using the differential equation. Suppose y is holomorphic in Ω_0 . Choose any z such that $g(z) \in \Omega_0$. Then the right hand side of the differential equation is holomorphic at z , which implies the left hand side must also be holomorphic at z . Thus, we can expand our region of holomorphicity to the larger set $\Omega_1 = \{z : g(z) \in \Omega_0\}$. Repeating the procedure we find we can generate a sequence of regions, each larger than the previous one, that our solution must be holomorphic in. We can therefore build up an approximation to the boundary of the maximal region in which the solution to (4.10) is holomorphic by successively plotting the image of the circle $|z| = 1 - |\epsilon|$ under *inverses* of g .

The boundaries of the successive sets

$$\Omega_k = \{z : g(z) \in \Omega_{k-1}\}$$

for the function $g(z) = z^2 + 0.8z$ up to $k = 20$ are given in figure 4.1. Evidently the shape of the sets Ω_k increases in complexity as k increases, though the sequence of sets appears to be converging.

In figure 4.2 the same sets are plotted for the function $g(z) = z^2 + 0.9iz$, and exhibit a similar behaviour, although now the sets are disconnected. The initial set Ω_0 is the interior of the circle in the top centre of the figure. The second set Ω_1 extends Ω_0 slightly, and also adds the interior of the circle in the bottom centre. The next set, Ω_2 further expands these two circles and adds 2 new ones in the upper left and lower right parts of the figure. This continues, with each set consisting of more

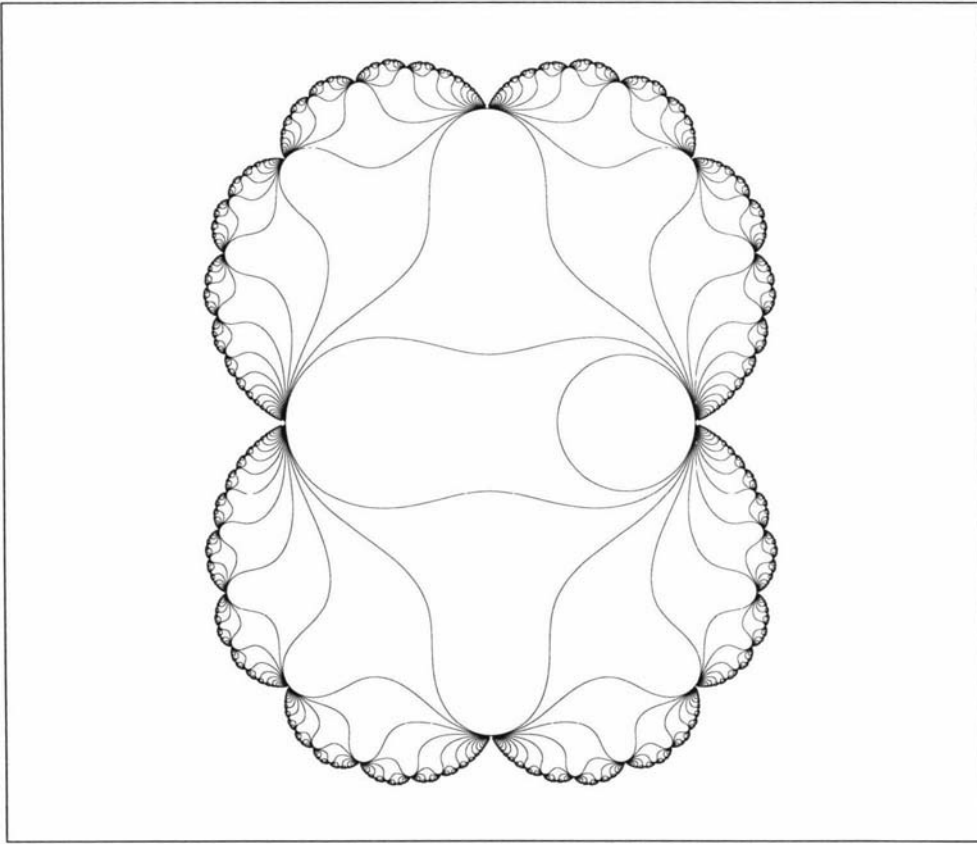


Figure 4.1: Plot of $\Omega_0, \dots, \Omega_{20}$ for $g(z) = z^2 + 0.8z$.

and more disconnected components, whilst also enlarging each of the components in turn. In fact, from $k = 7$ onwards, the sets “pull-in” and reconnect some of the disconnected elements, with Ω_7 containing the figure eight shaped region in the center of the plot which encloses the first two components of Ω_1 through Ω_6 . It appears as if the convergence observed in the previous case is also occurring in this case, although it is at a slower rate. In both of these plots, however, it is clear that the resulting sets have a very different shape than the simple circle in Example 2.

The preceding arguments are informal and gloss over many of the details; however, they suggest that the region in which any solution to a functional differential equation is holomorphic is governed by how points surrounding the initial point map under successive applications of g and its inverse. In the next section we introduce the definitions and machinery required to formally prove the above arguments.

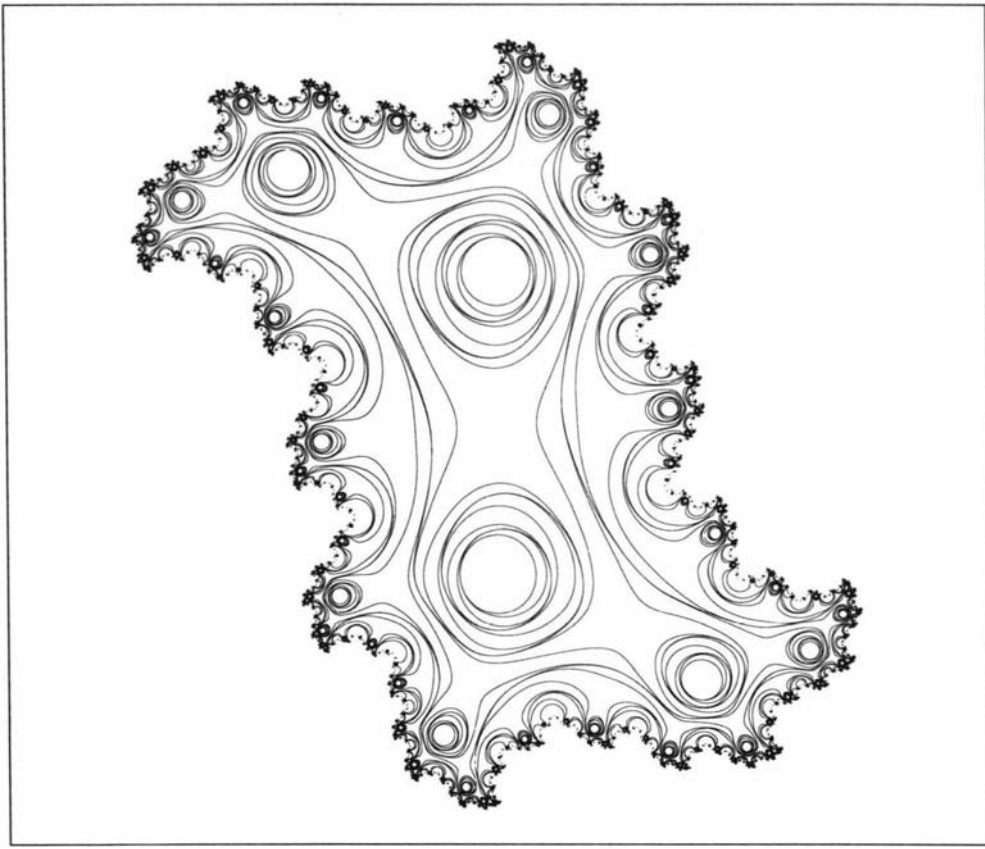


Figure 4.2: Plot of $\Omega_0, \dots, \Omega_{20}$ for $g(z) = z^2 + 0.9iz$.

4.2 Fixed and Periodic Points

The above discussion indicates that a precise definition of the region in which a function is retarded or advanced is required. Such a definition is linked to the behaviour of a function g about its fixed points.

Definition 4.1 (Fixed Point) *A fixed point of $g(z)$ is a point z_0 such that $g(z_0) = z_0$.*

Fixed points can be characterized into 3 types.

Definition 4.2 *A fixed point z_0 of a function g is*

1. **attracting** if $|g(z) - z_0| < |z - z_0|$ for all z in some neighbourhood of z_0 .
2. **repelling** if $|g(z) - z_0| > |z - z_0|$ for all z in some neighbourhood of z_0 .

If z_0 is neither attracting nor repelling, we say that z_0 is **neutral**.

The definition of the derivative of g at z_0 gives us the following method of classification, which is easier to use.

Theorem 4.3 *A fixed point z_0 of an analytic function $g(z)$ is*

1. **attracting** if and only if $|g'(z_0)| < 1$.
2. **repelling** if and only if $|g'(z_0)| > 1$.
3. **neutral** if and only if $|g'(z_0)| = 1$.

We observed in the last section that compositions of g^{-1} with itself are used to define the region in which a solution is likely to be holomorphic. Let

$$g^n(z)$$

denote the n -th composition of g with itself. If z_0 is a fixed point of g , then it will also be a fixed point of g^n for all n . It turns out that the fixed points of compositions of g are also useful.

Definition 4.4 (Periodic Point) *A fixed point z_0 of $g^n(z)$ is a **periodic point** of $g(z)$. The **period** of z_0 is the smallest n for which $g^n(z_0) = z_0$. The points $z_0, g(z_0), g^2(z_0), \dots, g^{n-1}(z_0)$ are all periodic points of g and are known as a **periodic orbit**.*

Periodic points of a function g can be classified as attracting, repelling or neutral in a similar way as was done for fixed points.

Theorem 4.5 *Let z_0 be a periodic point of the function $g(z)$ with period n . Then z_0 is*

1. **attracting** if and only if $|(g^n)'(z_0)| < 1$.
2. **repelling** if and only if $|(g^n)'(z_0)| > 1$.

3. neutral if and only if $|(g^n)'(z_0)| = 1$.

Definition 4.6 (Basin of Attraction) *Let z_0 be an attracting fixed point. The basin of attraction of z_0 for g is*

$$A_{z_0} = \left\{ z : \lim_{n \rightarrow \infty} g^n(z) = z_0 \right\}.$$

We can extend this definition to include periodic points by altering the definition to $A_{z_0} = \{z : \lim_{n \rightarrow \infty} g^{mn}(z) = z_0\}$, where m is the period of z_0 .

The set A_{z_0} need not be connected. The definition of an attracting fixed point, however, indicates that there is some neighbourhood U of z_0 that is in A_{z_0} , so that there is a non-empty connected open subset of A_{z_0} that contains z_0 . We define the **immediate basin of attraction** $A_{z_0}^0$ as the component of A_{z_0} containing the point z_0 . We are now in a position to show that there exist solutions to (4.1) holomorphic in $A_{z_0}^0$. Let $\mathcal{H}(\Omega)$ denote the set of functions holomorphic on the open set Ω .

Theorem 4.7 *Let z_0 be an attracting fixed point of the non-constant entire function g . Then there exists a solution to the initial-value problem (4.1),(4.2) that is unique among functions holomorphic at z_0 .*

Proof: The differential equation (4.1) can be written in the form

$$y'(z) = H(y(z), y(g(z))),$$

where $H(y(z), y(g(z))) = by(g(z)) - ay(z)$. We establish the existence of a local solution by showing that the associated integral equation

$$y(z) = \int_{z_0}^z H(y(\xi), y(g(\xi))) d\xi + y_0 \quad (4.12)$$

has a unique solution in $\mathcal{H}(\overline{D}(z_0, \delta))$ for some $\delta > 0$. We do this by showing that the integral operator is a contraction mapping for small δ .

Let $\mathcal{H}^\infty(\overline{D}(z_0; \delta))$ denote the Banach space of functions in $\mathcal{H}(\overline{D}(z_0; \delta))$ equipped

with the norm $\|\cdot\|$ defined by

$$\|y\| = \sup_{z \in \overline{D}(z_0; \delta)} |y(z)|,$$

for all $y \in \mathcal{H}^\infty(\overline{D}(z_0; \delta))$, and let \mathcal{T} be the operator defined by

$$\mathcal{T}y = \int_{z_0}^z H(y(\xi), y(g(\xi)))d\xi + y_0.$$

The function H is holomorphic in both variables, and since $g \in \mathcal{H}(\overline{D}(z_0, \delta))$ for any $\delta > 0$ we have that \mathcal{T} maps functions in $\mathcal{H}^\infty(\overline{D}(z_0; \delta))$ into $\mathcal{H}^\infty(\overline{D}(z_0; \delta))$. Now, for any $u, v \in \mathcal{H}^\infty(\overline{D}(z_0; \delta))$,

$$\begin{aligned} \|\mathcal{T}u - \mathcal{T}v\| &= \left\| \int_{z_0}^z H(u(\xi), u(g(\xi)))d\xi - \int_{z_0}^z H(v(\xi), v(g(\xi)))d\xi \right\| \\ &\leq |z - z_0| \|H(u(\xi), u(g(\xi))) - H(v(\xi), v(g(\xi)))\| \\ &= |z - z_0| \left(\sup_{z \in \overline{D}(z_0; \delta)} |b(u(g(z)) - v(g(z))) - a(u(z) - v(z))| \right) \\ &\leq \delta \left(|a| \sup_{z \in \overline{D}(z_0; \delta)} |u(z) - v(z)| + |b| \sup_{z \in \overline{D}(z_0; \delta)} |u(g(z)) - v(g(z))| \right). \end{aligned}$$

By hypothesis, z_0 is an attracting fixed point of g , and hence for δ sufficiently small

$$g(\overline{D}(z_0; \delta)) \subseteq \overline{D}(z_0; \delta). \quad (4.13)$$

Thus, for small $\delta > 0$,

$$\sup_{z \in \overline{D}(z_0; \delta)} |u(g(z)) - v(g(z))| \leq \sup_{z \in \overline{D}(z_0; \delta)} |u(z) - v(z)|,$$

and hence

$$\|\mathcal{T}u - \mathcal{T}v\| \leq \delta(|a| + |b|)\|u - v\|.$$

We may choose δ small enough so that $\delta(|a| + |b|) < 1$, and hence the operator \mathcal{T} is a contraction and we conclude that the integral equation (4.12) has a unique solution in $\mathcal{H}^\infty(\overline{D}(z_0; \delta))$. Therefore, the initial value problem (4.1),(4.2) has a unique solution in $\mathcal{H}(\overline{D}(z_0; \delta))$ for small δ . ■

We remark here that the above proof relies only on the condition that H is holomorphic in both its arguments, and that the fixed point z_0 is attracting. It can be readily extended to higher order functional differential equations by simply rewriting the higher order equation as a first order system, and then adjusting the Banach space accordingly. It also does not rely on the polynomial character of g . Indeed, all we require is that $g \in \mathcal{H}(\overline{D}(z_0; \delta))$. We address these issues among others in §4.5.

The local solution of Theorem 4.7 can be analytically continued using the functional differential equation (4.1) in a similar way as was done in the previous section.

Theorem 4.8 *Let z_0 be an attracting fixed point of the non-constant entire function g . Then there exists a unique solution to the initial value problem (4.1),(4.2) that is holomorphic in the set $A_{z_0}^0$.*

Proof: Theorem 4.7 shows that there is a $\delta > 0$ and a unique $y \in \mathcal{H}(\overline{D}(z_0; \delta))$ such that y is a solution to the initial value problem for all $z \in \overline{D}(z_0; \delta)$, and $\overline{D}(z_0; \delta) \subseteq A_{z_0}^0$. Suppose there exists a $w \in A_{z_0}^0$ at which y is not holomorphic. Since y is not holomorphic at w , the function $y' + ay$ is also not holomorphic at w , for if it was, y would be a solution to the equation $y' + ay = f$ where f is some holomorphic function, which yields a holomorphic solution for y . Thus, $y' + ay$ is not holomorphic at w , and equation (4.1) shows that y is not holomorphic at $g(w)$. The same argument can now be applied to $g(w)$ to show that y is not holomorphic at $g^2(w)$ also. A simple inductive argument shows that y is not holomorphic at $g^n(w)$ for all $n \in \mathbb{N}$. But $w \in A_{z_0}^0$, and therefore there is an $N > 0$ such that $|g^n(w) - z_0| < \delta$ for all $n > N$. Hence, there must be an $n \in \mathbb{N}$ such that $g^n(w) \in \overline{D}(z_0; \delta)$ and y is not holomorphic at $g^n(w)$. This contradicts the fact that $y \in \mathcal{H}(\overline{D}(z_0; \delta))$ and hence y must be holomorphic for all $w \in A_{z_0}^0$. Now, $A_{z_0}^0$ is connected, and contains $\overline{D}(z_0; \delta)$, therefore the local solution holomorphic at z_0 can be analytically continued to $A_{z_0}^0$. ■

Of interest here, is the arguments in the proof above apply equally well for any $w \in A_{z_0}$, not just those in the component $A_{z_0}^0$. This means that the local solution defines not only a unique function in $\mathcal{H}(A_{z_0}^0)$, but it also defines unique functions holomorphic in other components of A_{z_0} .

We will show that the boundary of the immediate basin of attraction forms a natural boundary for our solution. To accomplish this, we need a number of results from Complex Dynamics, specifically results on Julia sets.

4.3 Julia Sets

In this section we review some relevant facts regarding Julia sets. Many of the proofs in this section are based on those given by Milnor [26], where a more detailed account can be found.

We start with the definition of a normal family of functions.

Definition 4.9 *A collection \mathcal{F} of functions from a set $D \subseteq \mathbb{C}$ to a set $E \subseteq \mathbb{C}$ is normal if every infinite sequence of maps from \mathcal{F} contains a subsequence which converges locally uniformly to some limit function $g : D \rightarrow E$, or diverges uniformly to ∞ .*

We can now state the following definition for the Julia set.

Definition 4.10 *For a given function $g = g(z)$, we define the **Julia set** of g as*

$$J(g) = \{z : \{g^n\} \text{ is not a normal family in any neighbourhood of } z\}.$$

The above definition shows us that the Julia set must be closed, as any neighbourhood of any point on the boundary of the Julia set must include points in the Julia set, upon which $\{g^n\}$ is not a normal family. The complement, $\mathbb{C} \setminus J$ is called the **Fatou set**, or stable set. This definition of the Julia and Fatou sets is not convenient for determining the sets computationally for a particular function. We can, however, quickly determine some of the elements and properties of the Fatou and Julia sets as the following results show.

Theorem 4.11 (Milnor p. 39) *The Julia set and Fatou set of a function g are invariant under g .*

Proof: For any open set $U \in \mathbb{C}$, some sequence of iterates $\{g^{n_j}\}$ converges uniformly on compact subsets of U (or diverges uniformly to ∞) if and only if the corresponding sequence $\{g^{n_j+1}\}$ converges uniformly (or diverges uniformly to ∞) on compact subsets of the open set $g^{-1}(U)$. ■

This has two immediate consequences. The first is that the Julia set of a function must present a large amount of *self-similarity*. Indeed, if we have $g(z_1) = z_2$ for any $z_1 \in J(g)$ where $g'(z_1) \neq 0$, then there is a neighbourhood N_1 of z_1 that maps isomorphically under g to a neighbourhood N_2 of z_2 , and $N_1 \cap J(g)$ maps precisely onto $N_2 \cap J(g)$. Thus, any shape observed in the Julia set is generally seen infinitely many times. See figure 4.3 for example.

The second implication, is that we can use an inductive argument to conclude that the Julia set of g^k coincides with the Julia set of g for any positive integer k . This allows us to quickly find where the attracting and repelling periodic orbits must lie.

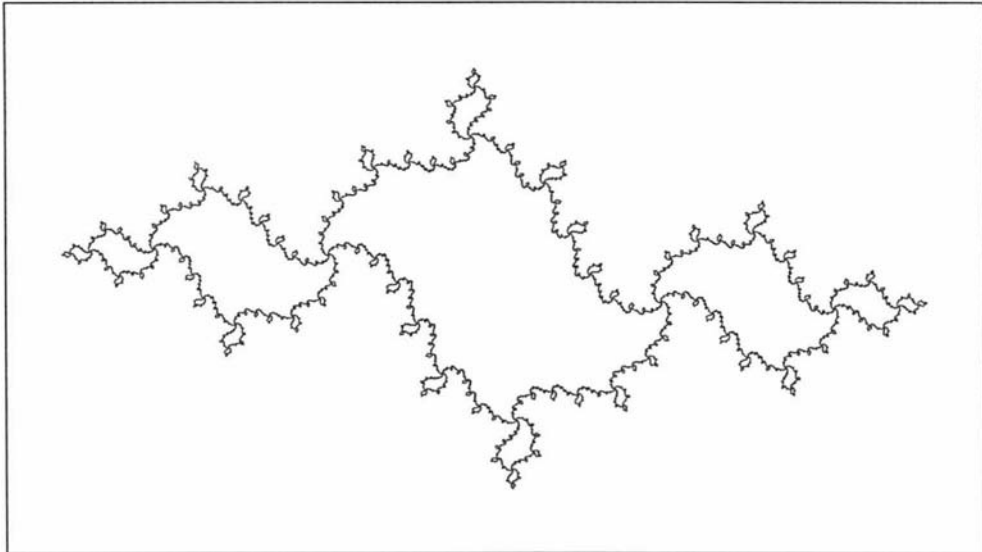


Figure 4.3: The Julia set for the function $g(z) = z^2 - 1 + 0.2i$.

Theorem 4.12 (Milnor p. 43) *Every attracting periodic orbit is contained in the Fatou set of g . In fact, the entire basin of attraction for any attracting orbit is in the Fatou set. Every repelling periodic point is contained in the Julia set.*

Proof: Consider a fixed point z_0 with $g'(z_0) = \alpha$. Then if $|\alpha| > 1$, the first derivative of g^k at z_0 is α^k which diverges as $k \rightarrow \infty$, hence no sequence of iterates of g can

converge uniformly near z_0 , and so z_0 must be in the Julia set. On the other hand, if $|\alpha| < 1$, we have an attracting fixed point, then there exists a constant c such that $|\alpha| < c < 1$ and $|g(z) - z_0| \leq c|z - z_0|$ for all z sufficiently close to z_0 . Thus, for any positive integer k , we have that $|g^k(z) - z_0| \leq c^k|z - z_0|$ for all z sufficiently close to z_0 ; hence, the successive iterates of g restricted to a small neighbourhood of z_0 converge uniformly to the constant function $h(z) = z_0$. This is extended readily to the basin of attraction of z_0 by noting that for any $z \in A_{z_0}$ there exists an integer $N > 0$ such that $|g^N(z) - z_0| < c|z - z_0|$ where c is as before. Thus, the sequence of iterates $\{g^{N+k}\}$ will converge uniformly to the constant function $h = z_0$ as before. These statements regarding fixed points generalize immediately to periodic points by making use of Theorem 4.11, since periodic points are simply fixed points of some iterate g^m . ■

The case of neutral periodic points is not so clear - indeed it turns out that some are in the Julia set, and some are in the Fatou set. The simplest case is as follows.

Definition 4.13 *Let z_0 be a neutral periodic point of period m , and let $\alpha = (g^m)'(z_0)$. If α is a root of unity, and no iterate of g is the identity map, then z_0 is called **parabolic**.*

Theorem 4.14 (Milnor p. 44) *Every parabolic periodic point belongs to the Julia set.*

Before we prove this result, let us first note that we may always consider any periodic point to be at the origin (or any other point that makes the analysis more convenient) by using a change of coordinates. Indeed, let z_0 be any periodic point of g , and consider the transformation $w = z - z_0$ and $h(w) = g(w + z_0) - z_0$. Then $h(0) = g(z_0) - z_0$ and $h'(0) = g'(z_0)$. Also,

$$\begin{aligned} h^2(0) &= h(h(0)) \\ &= h(g(z_0) - z_0) - z_0 \\ &= g((g(z_0) - z_0) + z_0) - z_0 \\ &= g^2(z_0) - z_0. \end{aligned}$$

and so $(h^2)'(0) = (g^2)'(z_0)$. A simple inductive argument then shows that $h^k(0) = g^k(z_0) - z_0$ and $(h^k)'(0) = (g^k)'(z_0)$ for all positive integers k . Thus, $h(w)$ will have a periodic point at the origin with the same properties as z_0 . We can perform whatever analysis is necessary with h , and then transform the results back to the original coordinate system.

Proof: Following the above argument, we may assume the periodic point is at the origin. Since the periodic point is parabolic, there is an iterate g^m that can be written using a Taylor series expansion as $g^m(z) = z + a_q z^q + a_{q+1} z^{q+1} + \dots$, where $q \geq 0$, and $a_q \neq 0$. It thus follows that any iterate g^{mk} corresponds to a power series of the form $z + ka_q z^q + \dots$. Thus, the q -th derivative of g^{mk} at 0 is equal to $q!ka_q$ which diverges to infinity as $k \rightarrow \infty$. Hence, no subsequence $\{g^{mk_j}\}$ can converge uniformly as $k_j \rightarrow \infty$. ■

Parabolic points are of course just one type of neutral point. Other types of neutral fixed points will be considered in Chapter 6, where we look for local holomorphic solutions about neutral fixed points.

The following result provides a characterisation of the Julia set in terms of exceptional points.

Definition 4.15 (Exceptional Points) *An exceptional point $w \in \mathbb{C}$ for a function f is a point such that $f(z) \neq w$ for all z in some open set $U \subseteq \mathbb{C}$.*

Theorem 4.16 (Montel) *Let $\{f_n\}$ be a family of functions in U where $U \subset \mathbb{C}$ is an open set. Suppose there exist $a, b \in \mathbb{C}$ such that $f_n(z) \neq a$ and $f_n(z) \neq b$ for all $n \in \mathbb{N}$ and all $z \in U$. Then $\{f_n\}$ is a normal family in U .*

A proof of this result can be found in [26] and is often used as a stepping stone to Picard's theorem [4].

Corollary 4.17 *Let $z \in J(g)$ and U be any neighbourhood of z . The set G defined as*

$$G = \bigcup_{n=1}^{\infty} g^n(U)$$

omits at most one point in \mathbb{C} .

Proof: Suppose G omits two points, a and b . Then $\{g^n\}$ is a family of functions in the open set U such that $g^n(z) \neq a$ and $g^n(z) \neq b$ for all $n \in \mathbb{N}$ and all $z \in U$. By Theorem 4.16, $\{g^n\}$ is a normal family, contradicting the hypothesis that $z \in J(g)$. ■

This result will be used later to show that the Julia set forms a natural boundary for our functional differential equation. In the special case where $g(z)$ is a polynomial, we can refine this result as follows.

Theorem 4.18 *Let $g(z)$ be a non-constant polynomial. Suppose that there is a point $w \in J(g)$ and a neighbourhood U of w such that*

$$\bigcup_{n=1}^{\infty} g^n(U) = \mathbb{C} \setminus \{a\}$$

for some $a \in \mathbb{C}$. Then $g(z) = a + b(z - a)^k$ for some $b \in \mathbb{C} \setminus \{0\}$ and some integer $k \geq 2$.

Proof: We first show that g must be of degree at least 2. Suppose g is linear. Then we may write $g(z) = c(z - d) + d$ for constants $c \neq 0$ and d . Now, $g^n(z) = c^n(z - d) + d$, and thus g has a single periodic point at $z = d$. This point must be repelling, as $J(g)$ is non-empty; hence, $|c| \geq 1$. Now, any neighbourhood U of d contains a disc $D(d; \epsilon)$ for some $\epsilon > 0$. But $g^n(D(d; \epsilon))$ is a disc of radius $|c|^n \epsilon$, thus

$$\bigcup_{n=1}^{\infty} g^n(U) = \mathbb{C}$$

for all neighbourhoods U of d . Thus g is must be at least degree 2.

By the Fundamental Theorem of Algebra, there exists a point $c \in \mathbb{C}$ such that $g(c) = a$. This pre-image of a cannot be in any of the sets $g^n(U)$ as a is not. The only possibility is that $c = a$, and thus a is a fixed point of z . Furthermore, a has no pre-images other than itself; therefore, the only solution to $g(z) - a = 0$ is a , hence g is of the form required. ■

Polynomials of this form (i.e. with exceptional values) will be referred to as cyclic polynomials.

Definition 4.19 (Cyclic Polynomials) *A polynomial of the form*

$$g(z) = a + b(z - a)^k,$$

where a and $b \neq 0$ are complex constants, and $k \geq 2$ is an integer is a **cyclic polynomial**.

We can characterize the fixed points of cyclic polynomials as follows.

Theorem 4.20 *A cyclic polynomial has only one attracting fixed point - the exceptional point. All other fixed points are repelling.*

Proof: Let g be a cyclic polynomial with exceptional point a , so that

$$g(z) = a + b(z - a)^k$$

for constants $b \neq 0$, $k \geq 2$. Clearly $z = a$ is an attracting fixed point of g , as $g(a) = a$ and $g'(a) = 0$. The other fixed points of g are given by the solutions z_1, \dots, z_{k-1} of

$$b(z - a)^{k-1} = 1.$$

These points must be repelling, as $g'(z) = kb(z - a)^{k-1}$, and hence $g'(z_m) = k$ for $m = 1, 2, \dots, k-1$. The only attracting fixed point of g is therefore the exceptional point a . ■

The above result shows that, for polynomials, the exceptional point must be in the Fatou set. For polynomials then, the iterated images of any neighbourhood of any point in the Julia set covers the Julia set. Indeed, the same result holds for general functions, although the exceptional point may be in the Julia set. In either case, we can establish the following result.

Theorem 4.21 (Milnor p. 47) *Let z_0 be any point in $J(g)$. Then the pre-images of z_0 are dense in $J(g)$, unless z_0 is the exceptional point.*

Proof: Choose any point $z_1 \in J(g)$. If z_0 is not the exceptional point, then there exists points z arbitrarily close to z_1 whose forward orbits contain z_0 by Corollary 4.17. Thus, we can find a pre-image of z_0 arbitrarily close to any point in $J(g)$, and so the preimages of z_0 are dense in $J(g)$. ■

This result gives an efficient method of computing pictures of the Julia set for polynomials - we simply start with any repelling point, and compute its pre-images until we have enough points to get a good plot of the Julia set. Note that it is most useful for quadratics, because the number of points generated grows as d^n , where d is the degree of the polynomial, and n is the length of the orbits we are generating. This growth of the number of points is the only real disadvantage with this technique, as many iterations may be required in order to get close to certain points in the Julia set.

The next theorem shows that the boundary of the immediate basin of attraction is contained in the Julia set.

Theorem 4.22 (Milnor p. 46) *If $A \in \mathbb{C}$ is the basin of attraction for some attracting periodic orbit, then the boundary $\partial A = \bar{A} \setminus A$ is equal to the entire Julia set.*

Proof: If N is any neighbourhood of a point of the Julia set, then (4.16) implies that some $g^n(N)$ intersects A , hence N itself intersects A by the definition of an attracting basin. Thus, $J \subset \bar{A}$. But J is disjoint from A , so it follows that $J \subset \partial A$. On the other hand, if N is any neighbourhood of a point of ∂A then any limit of iterates g^n evaluated on N must have a jump discontinuity between A and ∂A , since the points in A tend to the periodic orbit, whereas those on ∂A do not. Hence $\partial A \subset J$, and so $\partial A = J$. ■

Note here that we must make a distinction between ∂A and the union of the boundaries of the connected components of A . Often the latter set is smaller than the former. For example, the quadratic function $g(z) = z^2 + i$ has no attracting periodic points in \mathbb{C} , and thus no basins, yet its Julia set is non-empty (See figure 4.4).

The last two results in this section concern only polynomials, and are presented here without proof. These results show simply that, for polynomials, the Julia set is non-empty and that the number of attracting periodic points is finite and limited by the degree of the polynomial. The reader is referred to Milnor ([26] pp. 78-79, 136) for the details.

Theorem 4.23 *Every polynomial of degree at least two must have either a repelling fixed point, or a parabolic fixed point, or both.*

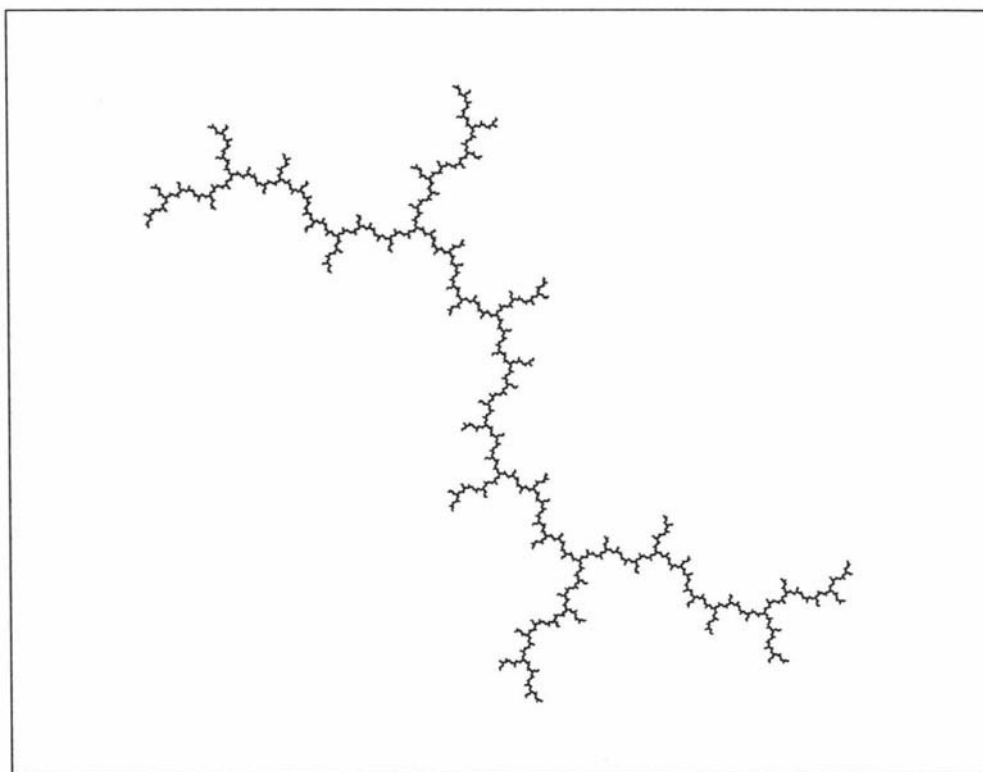


Figure 4.4: The Julia set for the function $g(z) = z^2 + i$, a *dendrite*.

Theorem 4.24 *For a polynomial of degree d , the number of attracting periodic points is at most $d - 1$.*

4.4 The Natural Boundary

We are now in a position to prove that the boundary of the immediate basin of attraction forms a natural boundary for the solution to our functional differential equation.

We first note that the immediate basin of attraction $A_{z_0}^0$ of our fixed point z_0 , has as its boundary a subset of the Julia set $J(g)$ by Theorem 4.22. Thus, if our solution is to be analytically continued outside of $A_{z_0}^0$, it must be holomorphic at some point $z \in J(g)$.

Lemma 4.25 *Let z_0 be an attracting fixed point of the non-constant entire function g , and let y be a solution to the differential equation (4.1) holomorphic at z_0 .*

Suppose that y is holomorphic at a point $w \in J(g)$. Then y is holomorphic in $\mathbb{C} \setminus \{g_0\}$ where g_0 is the exceptional point of g .

Proof: If g is linear, with attracting point z_0 , it may be written in the form $g(z) = \alpha(z - z_0) + z_0$, where $|\alpha| < 1$. Thus, $g^n(z) - z_0 = \alpha^n(z - z_0)$, and hence $A_{z_0} = \mathbb{C}$. Theorem 4.12 thus gives $J(g) = \emptyset$, so we need only prove the result in the case where g is non-linear.

Suppose that y is holomorphic at $w \in J(g)$. Then there is a neighbourhood U of w such that $y \in \mathcal{H}(U)$. Now, y is holomorphic in U and hence the function $y' + ay$ is also holomorphic in U . Since y is a solution to equation (4.1), y must be holomorphic in $g(U)$, i.e. $y \in \mathcal{H}(g(U))$. We apply the same argument to $g(U)$ to deduce that $y \in \mathcal{H}(g^2(U))$, and by induction $y \in \mathcal{H}(g^n(U))$ for all $n \in \mathbb{N}$. Therefore, $y \in \mathcal{H}(G)$, where

$$G = \bigcup_{n=1}^{\infty} g^n(U).$$

Corollary 4.17 then gives that y is either entire, or is holomorphic at all but the exceptional point of g . ■

In the case of polynomials, we can eliminate the possibility of a singularity at the exceptional point, and can then establish the existence of the natural boundary.

Corollary 4.26 *Let z_0 be an attracting fixed point of the non-constant polynomial g , and let y be a solution to the differential equation (4.1) holomorphic at z_0 . Suppose that y is holomorphic at a point $w \in J(g)$. Then y is an entire function.*

Proof: Let G be as in the proof of Lemma 4.25 and let w be any point in $J(g)$. As g is a polynomial we can apply Theorem 4.18, so the set G is either the entire complex plane, or $G = \mathbb{C} - \{a\}$ for some constant a , and g is a cyclic polynomial. If g is a cyclic polynomial, then it has a as its sole attracting fixed point by Theorem 4.20. By hypothesis z_0 is an attracting fixed point of g , thus $a = z_0$, and so y is holomorphic at a . Hence, y is entire. ■

Theorem 4.27 *Let $y_0 \neq 0$, $a \neq b$, $b \neq 0$, and let z_0 be an attracting fixed point for the non-linear polynomial g . Then the solution to equation (4.1) satisfying (4.2) has $\partial A_{z_0}^0$ as a natural boundary.*

Proof: Theorem 4.8 shows that the solution y to the initial-value problem is holomorphic in $A_{z_0}^0$. We are therefore required to show that y is singular at every point on the boundary $\partial A_{z_0}^0$.

Suppose that y is holomorphic at some point $w \in \partial A_{z_0}^0$. Since $\partial A_{z_0}^0 \subseteq J(g)$, Corollary 4.26 implies that y must be entire. We show by examining the growth of y that this is impossible.

Let $C_R = \partial D(0; R)$ and let $M(R) = \sup_{z \in D(0; R)} |y(z)|$. Since y is entire, the maximum modulus principle implies that

$$M(R) = \sup_{z \in C_R} |y(z)|$$

for all $R > 0$. The conditions $a \neq b$ and $y_0 \neq 0$ preclude y from being a constant, and in fact y cannot be linear, since equation (4.1) would imply that g is linear, contrary to our hypothesis. Thus, $M(R_1) < M(R_2)$ for any $R_1 < R_2$.

Let $z \in C_R$. The Cauchy integral formula implies that for any non-negative integer k ,

$$y^{(k)}(z) = \frac{k!}{2\pi i} \int_{\gamma} \frac{y(\xi)}{(\xi - z)^{k+1}} d\xi,$$

where γ is any simple closed contour orientated anticlockwise that encloses z . Thus, for any $\delta > 0$, we may use the contour of the form $\gamma_{\delta} = \{\xi : |\xi - z| = \delta\}$ which yields the inequality

$$|y^{(k)}(z)| \leq \frac{k!}{\delta^k} \sup_{\xi \in \gamma_{\delta}} |y(\xi)|.$$

Now, γ_{δ} is contained entirely in $C_{R+\delta}$, thus the maximum modulus principle implies that

$$|y^{(k)}(z)| \leq \frac{k!}{\delta^k} M(R + \delta). \quad (4.14)$$

For any $z \in C_R$, inequality (4.14) with $\delta = 1$ and the differential equation (4.1) yield

$$|y(g(z))| \leq \left(1 + \frac{|a|}{|b|}\right) M(R + \delta). \quad (4.15)$$

Now since g is a polynomial with degree at least 2, there is an $R_1 > 0$ and an $\alpha > 0$ such that

$$|g(z)| > \alpha R^2 \quad (4.16)$$

for all $z \in C_R$ with $R > R_1$. For such a choice of R , the curve $\Gamma = \{g(z) : z \in C_R\}$ is

closed, and encloses the curve $C_{\alpha R^2}$. The maximum modulus principle thus implies that there is a $z_1 \in C_R$ such that

$$|y(g(z_1))| > M(\alpha R^2). \quad (4.17)$$

Combining this with inequality (4.15) yields

$$M(\alpha R^2) < \left(1 + \frac{|a|}{|b|}\right) M(R + \delta). \quad (4.18)$$

We now show that this gives a contradiction. Let

$$h(z) = \frac{y(z) - y(0)}{z}. \quad (4.19)$$

Since y is entire and non-constant, h must be entire, and so $M_h(R) = \sup_{z \in C_R} |h(z)|$ where $M_h(R) = \sup_{z \in D(0;R)} |h(z)|$ by the maximum modulus principle. Taking the modulus of (4.19) and using the triangle inequality yields the relations

$$\begin{aligned} M_h(R) &\leq \frac{M(R) + |y(0)|}{R} \\ M(R) &\leq R M_h(R) + |y(0)|. \end{aligned}$$

Combining these with inequality (4.18) yields

$$\begin{aligned} M_h(\alpha R^2) &\leq \frac{M(\alpha R^2) + |y(0)|}{\alpha R^2} \\ &\leq \frac{\left(1 + \frac{|a|}{|b|}\right) M(R + \delta) + |y(0)|}{\alpha R^2} \\ &\leq \frac{\left(1 + \frac{|a|}{|b|}\right) (R + \delta) M_h(R + \delta) + \left(2 + \frac{|a|}{|b|}\right) |y(0)|}{\alpha R^2} \\ &\leq \left\{ \frac{\left(1 + \frac{|a|}{|b|}\right) (R + \delta)}{\alpha R^2} + \frac{\left(2 + \frac{|a|}{|b|}\right) |y(0)|}{\alpha R^2 M_h(R + \delta)} \right\} M_h(R + \delta). \end{aligned}$$

For R sufficiently large, we have

$$\frac{\left(1 + \frac{|a|}{|b|}\right) (R + \delta)}{\alpha R^2} + \frac{\left(2 + \frac{|a|}{|b|}\right) |y(0)|}{\alpha R^2 M_h(R + \delta)} < 1$$

and

$$\alpha R^2 > R + \delta.$$

For such R , we therefore have that

$$M_h(\alpha R^2) < M_h(R + \delta).$$

But y is not a linear function, and hence h cannot be a constant, so we have a contradiction to the maximum modulus principle. We conclude that y is not an entire function, and hence that y is not holomorphic at w . Since our choice of $w \in \partial A_{z_0}^0$ was arbitrary, we see that y is not holomorphic at any point of $\partial A_{z_0}^0$, and hence that $\partial A_{z_0}^0$ is a natural boundary. ■

An interesting feature of solutions to (4.1) is that the natural boundary does not depend on the choice of the constants a and b (other than requiring $a \neq b$): it is determined entirely by the functional argument g . In the following section, we see how this independence from the differential operator allows some simple generalisations to be made. Note also that if $a = b$, then the initial value problem admits the constant solution $y = y_0$ regardless of g . By Theorem 4.7 this solution is unique for the attracting fixed point y_0 ; hence, there is no natural boundary in this case.

4.5 Generalisations

In the last sections, we proved a number of results regarding the solutions to simple first order functional differential equations with constant coefficients, and a polynomial functional argument. We show in this section how some of these results may be extended to more general equations.

4.5.1 Higher order equations

Let us first investigate how we may extend our results on first order equations to n -th order equations with constant coefficients.

Consider the functional differential equation

$$Ly = by(g(z)) \quad (4.20)$$

where $b \neq 0$ is a constant, $g(z)$ is a non-constant polynomial, and L is the linear differential operator

$$Ly = y^{(n)}(z) + a_{n-1}y^{(n-1)}(z) + \dots + a_1y'(z) + a_0y(z), \quad (4.21)$$

where a_0, a_1, \dots, a_{n-1} are complex constants. We seek solutions to (4.20) satisfying the initial conditions

$$y(z_0) = y_0, y'(z_0) = y'_0, \dots, y^{(n-1)}(z_0) = y_0^{(n-1)}, \quad (4.22)$$

where z_0 is a fixed point of g .

We start by modifying the proof to Theorem 4.7 to obtain a local analyticity result for these higher order equations.

Let $y_1 = y, y_2 = y', \dots, y_n = y^{(n-1)}$. The differential equation (4.20) can then be written as

$$\mathbf{y}'(z) = \mathbf{H}(\mathbf{y}(z), \mathbf{y}(g(z))), \quad (4.23)$$

where $\mathbf{y} = (y_1, y_2, \dots, y_n)^T$ and

$$\mathbf{H}(\mathbf{y}(z), \mathbf{y}(g(z))) = \begin{bmatrix} y_2(z) \\ y_3(z) \\ \vdots \\ y_n(z) \\ by_1(g(z)) - \sum_{k=1}^n a_{k-1}y_k(z) \end{bmatrix}.$$

The associated integral equation is then

$$\mathbf{y}(z) = \int_{z_0}^z \mathbf{H}(\mathbf{y}(\xi), \mathbf{y}(g(\xi)))d\xi + \mathbf{y}_0, \quad (4.24)$$

where $\mathbf{y}_0 = (y_0, y'_0, \dots, y_0^{(n-1)})^T$.

Let $\mathcal{H}^\infty(\overline{D}(z_0; \delta))$ denote the Banach space of functions in the n -fold cartesian product $\mathcal{H}(\overline{D}(z_0; \delta)) \times \mathcal{H}(\overline{D}(z_0; \delta)) \times \dots \times \mathcal{H}(\overline{D}(z_0; \delta))$ equipped with the norm $\|\cdot\|$.

defined by

$$\|\mathbf{y}\| = \sum_{k=1}^n \sup_{z \in \overline{D}(z_0; \delta)} |y_k(z)|,$$

for all $\mathbf{y} = (y_1, \dots, y_n)^T \in \mathcal{H}^\infty(\overline{D}(z_0; \delta))$, and let \mathcal{T} be the operator defined by

$$\mathcal{T}\mathbf{y} = \int_{z_0}^z \mathbf{H}(\mathbf{y}(\xi), \mathbf{y}(g(\xi))) d\xi + \mathbf{y}_0.$$

For any $\delta > 0$ the function \mathbf{H} is holomorphic in y_1, \dots, y_n and $y_1(g)$, and since $g \in \mathcal{H}(D(z_0, \delta))$ we have that \mathcal{T} maps functions in $\mathcal{H}^\infty(\overline{D}(z_0; \delta))$ into $\mathcal{H}^\infty(\overline{D}(z_0; \delta))$.

Now, for any $\mathbf{u}, \mathbf{v} \in \mathcal{H}^\infty(\overline{D}(z_0; \delta))$,

$$\begin{aligned} \|\mathcal{T}\mathbf{u} - \mathcal{T}\mathbf{v}\| &= \left\| \int_{z_0}^z \mathbf{H}(\mathbf{u}(\xi), \mathbf{u}(g(\xi))) d\xi - \int_{z_0}^z \mathbf{H}(\mathbf{v}(\xi), \mathbf{v}(g(\xi))) d\xi \right\| \\ &\leq |z - z_0| \|\mathbf{H}(\mathbf{u}(\xi), \mathbf{u}(g(\xi))) - \mathbf{H}(\mathbf{v}(\xi), \mathbf{v}(g(\xi)))\| \\ &= |z - z_0| \left(\sum_{k=2}^n \sup_{z \in \overline{D}(z_0; \delta)} |u_k(z) - v_k(z)| \right. \\ &\quad \left. + \sup_{z \in \overline{D}(z_0; \delta)} |b(u_1(g(z)) - v_1(g(z))) - \sum_{k=1}^n a_{k-1}(u_k(z) - v_k(z))| \right) \\ &\leq \delta \left(\sum_{k=2}^n (1 + |a_{k-1}|) \sup_{z \in \overline{D}(z_0; \delta)} |u_k(z) - v_k(z)| \right. \\ &\quad \left. + |a_0| \sup_{z \in \overline{D}(z_0; \delta)} |u_1(z) - v_1(z)| + |b| \sup_{z \in \overline{D}(z_0; \delta)} |u(g(z)) - v(g(z))| \right). \end{aligned}$$

If z_0 is an attracting fixed point of g , then for sufficiently small δ

$$g(\overline{D}(z_0; \delta)) \subseteq \overline{D}(z_0; \delta).$$

Thus, for small $\delta > 0$, the maximum modulus principle implies that

$$\sup_{z \in \overline{D}(z_0; \delta)} |u_1(g(z)) - v_1(g(z))| \leq \sup_{z \in \overline{D}(z_0; \delta)} |u_1(z) - v_1(z)|.$$

Setting $M = \max\{1 + |a_1|, 1 + |a_2|, \dots, 1 + |a_{n-1}|, |a_0| + |b|\}$ gives, for $\delta > 0$ sufficiently small and $z \in \overline{D}(z_0; \delta)$,

$$\|\mathcal{T}\mathbf{u} - \mathcal{T}\mathbf{v}\| \leq \delta M \|\mathbf{u} - \mathbf{v}\|.$$

For δ sufficiently small, $\delta M < 1$, and hence the operator \mathcal{T} is a contraction and we

conclude that the integral equation (4.24) has a unique solution in $\mathcal{H}^\infty(\overline{D}(z_0; \delta))$. We can thus generalize Theorem 4.7 to the following.

Theorem 4.28 *Let z_0 be any attracting fixed point of the non-constant entire function g . Then there exists a local solution to (4.20) satisfying (4.22), unique among functions holomorphic at z_0 .*

Given that the n -th order differential operator L in (4.21) maps a holomorphic function y to a holomorphic function, the proof for Theorem 4.8 can be readily extended.

Theorem 4.29 *Let z_0 be an attracting fixed point of the non-constant entire function g . Then there exists a unique solution to the initial value problem (4.20), (4.22) that is holomorphic in the set $A_{z_0}^0$.*

In order to extend Theorem 4.27 to handle higher order operators, we need only show that we can establish an equivalent inequality to (4.15). Indeed, for $z \in C_R$, the Cauchy integral formula and maximum modulus principle yield

$$|Ly| \leq \left(\frac{n!}{\delta^n} + \frac{|a_{n-1}|(n-1)!}{\delta^{n-1}} + \dots + \frac{|a_1|}{\delta} + |a_0| \right) M(R + \delta);$$

hence,

$$|y(g(z))| \leq \frac{1}{|b|} \left(\frac{n!}{\delta^n} + \frac{|a_{n-1}|(n-1)!}{\delta^{n-1}} + \dots + \frac{|a_1|}{\delta} + |a_0| \right) M(R + \delta).$$

For R sufficiently large, there exists a δ_0 such that

$$\frac{1}{|b|} \left(\frac{n!}{\delta_0^n} + \frac{|a_{n-1}|(n-1)!}{\delta_0^{n-1}} + \dots + \frac{|a_1|}{\delta_0} + |a_0| \right) \leq 1 + \frac{|a_0|}{b}.$$

Replacing inequality (4.15) with

$$|y(g(z))| \leq \left(1 + \frac{|a_0|}{|b|} \right) M(R + \delta_0)$$

in the proof of Theorem 4.27, and all references to a and δ with a_0 and δ_0 respectively, the result follows *mutatis mutandis*.

Theorem 4.30 *Let $y_0 \neq 0$, $a_0 \neq b$, and let z_0 be an attracting fixed point for the non-linear polynomial g . Then the solution to the equation (4.20) satisfying (4.22) has $\partial A_{z_0}^0$ as a natural boundary.*

4.5.2 Entire functional arguments

We now look to extend Theorem 4.27 and 4.30 to the case where g is a non-constant entire function. The main issue is that g may have an exceptional point at which y may not be holomorphic.

In the polynomial case, if g had an exceptional point, it was the only attracting point of g . However, no such claims can be made in the non-polynomial case. For example, the entire function $f(z) = \lambda ze^z$ has $z = 0$ as an exceptional point (for $z = 0$ is the only solution to $f(z) = 0$), yet $f'(0) = \lambda$. Thus the exceptional point need not be in the Fatou set, and no conclusion can be immediately drawn on whether y is holomorphic there.

We can conclude that if y is a solution to (4.20), g is an entire function, and y is holomorphic at some point on the Julia set $J(g)$, then y must be either an entire function, or holomorphic at all points in \mathbb{C} with at most one exception. In fact, we can alter the proof for Theorem 4.27 and Theorem 4.30 to eliminate the possibility of non-constant entire solutions.

Theorem 4.31 *Let y be any solution to (4.20) where g is a non-linear entire function with a fixed point z_0 . If y is holomorphic at some point $w \in J(g)$, then one of the following must hold:*

1. y is a constant.
2. y is holomorphic in $\mathbb{C} \setminus \{g_0\}$ and is singular at g_0 , where g_0 is the exceptional point of g .

Proof: Let y be a solution of (4.20) holomorphic at some point $w \in J(g)$. Lemma 4.25 shows that either y is entire, or is holomorphic throughout $\mathbb{C} \setminus \{g_0\}$ where g_0 is the exceptional point of g . All we must show, therefore, is that y cannot be a non-constant entire function.

Assume that y is a non-constant entire function. As was done in Theorem 4.27, we can establish the inequality

$$|y(g(z))| \leq \left(1 + \frac{a_0}{b}\right) M(R + \delta_0),$$

for all $z \in C_R$ for R sufficiently large. We are required to establish a lower bound on $|y(g(z))|$ for some point $z \in C_R$ to replace (4.17). In order to establish this lower bound, we need to be able to have $g(0) = 0$.

Suppose that $g(0) \neq 0$, i.e. the origin is not a fixed point of g . Let $w = z - z_0$, $h(w) = g(w + z_0) - z_0$, and $f(w) = y(w + z_0)$. Then (4.20) transforms to

$$f^{(n)}(w) + a_{n-1}f^{(n-1)}(w) + \dots + a_1f'(w) + a_0f(w) = bf(h(w)), \quad (4.25)$$

and $h(0) = 0$. Thus f is a non-constant entire solution to (4.25). We may therefore assume that $g(0) = 0$.

For entire functions f and g such that $g(0) = 0$, Pólya [30] showed that there is a constant c , $0 < c < 1$ such that

$$M_{f \circ g}(R) \geq M_f(cM_g(\frac{R}{2})), \quad (4.26)$$

for all $R > 0$. Now, g is non-linear, so there exists a constant $\beta > 0$ such that $M_g(R) > \beta R^2$ for all R sufficiently large by Liouville's Theorem. Hence, for R large, there exists a $z_1 \in C_R$ such that

$$|y(g(z_1))| \geq M_y(c\frac{\beta}{4}R^2).$$

Setting $\alpha = c\frac{\beta}{4}$ we have established the inequality (4.17), and the rest of the proof of Theorem 4.27 then shows a contradiction to the maximum modulus principle. Thus, y must be a constant function. ■

The first case in Theorem 4.31 can be ruled out by placing the same conditions on y_0 , a and b as were used in Theorem 4.27. Assuming g has no exceptional point (or the exceptional point is in the immediate basin of attraction), we can therefore conclude that y cannot be analytically continued across the Julia set. Thus, $\partial A_{z_0}^0$ must be a natural boundary. We sum up the results as follows.

Theorem 4.32 *Let $y_0 \neq 0$ and $a \neq b$, and let y be any solution to the initial value problem (4.20), (4.22) where z_0 is an attracting fixed point of the non-linear entire function $g(z)$. Then y is holomorphic in the entire complex plane with the exception of a single point, or y is holomorphic within the immediate basin of attraction $A_{z_0}^0$, and has $\partial A_{z_0}^0$ as a natural boundary.*

If g has an exceptional point that is outside of the basin of attraction, then we can not conclude that y can be analytically continued across the Julia set - the Julia set may still form a natural boundary for y . The likelihood of (4.20) having a solution that is holomorphic at some point in the Julia set (in particular at a repelling fixed point) will be further examined in Chapter 5.

4.5.3 Equations with polynomial coefficients

In this last section, we examine the possibility of replacing the constants a_k , $k = 0, \dots, n-1$ and b in the functional differential equation (4.20) with functions holomorphic at the initial point z_0 .

Consider the equation

$$y^{(n)}(z) + a_{n-1}(z)y^{(n-1)}(z) + \dots + a_1(z)y'(z) + a_0(z)y(z) = b(z)y(g(z)), \quad (4.27)$$

where the functions $a_k(z)$, $k = 0, \dots, n-1$, $b(z)$ and $g(z)$ are entire.

Let $\mathbf{H}(y(z), y(g(z)))$ be defined as in the proof of Theorem 4.28, but with the a_k 's and b being replaced by functions of z , and let \mathcal{T} be as before. Then,

$$\begin{aligned} \|\mathcal{T}\mathbf{u} - \mathcal{T}\mathbf{v}\| &\leq |z - z_0| \left(\sum_{k=2}^n \sup_{z \in \overline{D}(z_0; \delta)} |u_k(z) - v_k(z)| \right. \\ &\quad \left. + \sup_{z \in \overline{D}(z_0; \delta)} |b(z)(u_1(g(z)) - v_1(g(z))) - \sum_{k=1}^n a_{k-1}(z)(u_k(z) - v_k(z))| \right) \\ &\leq \delta \left(\sum_{k=2}^n (1 + A_{k-1}(\delta)) \sup_{z \in \overline{D}(z_0; \delta)} |u_k(z) - v_k(z)| \right. \\ &\quad \left. + A_0(\delta) \sup_{z \in \overline{D}(z_0; \delta)} |u_1(z) - v_1(z)| + B(\delta) \sup_{z \in \overline{D}(z_0; \delta)} |u(g(z)) - v(g(z))| \right) \end{aligned}$$

where $B(\delta) = \sup_{z \in \overline{D}(z_0; \delta)} |b(z)|$ and $A_k(\delta) = \sup_{z \in \overline{D}(z_0; \delta)} |a_k(z)|$ for $k = 0, \dots, n-1$. Hence, for δ sufficiently small and $z \in \overline{D}(z_0; \delta)$,

$$\|\mathcal{T}\mathbf{u} - \mathcal{T}\mathbf{v}\| \leq \delta M \|\mathbf{u} - \mathbf{v}\|,$$

where $M = \max\{1 + A_1(\delta), 1 + A_2(\delta), \dots, 1 + A_{n-1}(\delta), A(\delta) + B(\delta)\}$. The functions a_k and b are all holomorphic in $\overline{D}(z_0; \delta)$, hence A_k and B are monotone increasing functions of δ by the maximum modulus principle. We can thus find a δ sufficiently small such that $\delta M < 1$ as before.

Note also that the proof of Theorem 4.8 can be easily extended to this case as it relies only on the fact that y is holomorphic at some point w if and only if the differential operator is holomorphic at w . This certainly holds for the differential operator $Ly = y^n(z) + a_{n-1}(z)y^{(n-1)}(z) + \dots + a_1(z)y'(z) + a_0(z)y(z)$. We thus have the following result.

Theorem 4.33 *Let z_0 be an attracting fixed point of the non-constant entire function g . Then there exists a unique solution to (4.27) satisfying (4.22) that is holomorphic in $A_{z_0}^0$.*

There are two key difficulties that we face when attempting to prove that $\partial A_{z_0}^0$ must be a natural boundary for our solution. The first problem is that the proof of Lemma 4.26 breaks down when we hit the zeros of $b(z)$. We can however work around this problem as follows. Suppose y is the solution from Theorem 4.33 and that y is holomorphic at some point $w \in J(g)$. Then there is a neighbourhood U of w such that $y \in \mathcal{H}(U)$. Let w_1 be any point in $g(U)$. There then exists a w_0 in U such that $w_1 = g(w_0)$. As y is holomorphic at w_0 , the linear differential operator

$$Ly = y^{(n)}(z) + a_{n-1}(z)y^{(n-1)}(z) + \dots + a_1(z)y'(z) + a_0(z)y(z) \quad (4.28)$$

is also holomorphic at w_0 . Now, y is a solution to (4.27); hence, either y is holomorphic at w_1 , or w_0 is a zero of $b(z)$ and y has a pole at w_1 . Suppose that y has a pole at w_1 . Then by following the same argument, y must have singularities (they need not be poles; indeed, they can not possibly all be poles) at all preimages of w_1 , with the possible exceptions of those preimages where b has a zero that matches the singularity of y . The solution y must also be singular at all the images of w_1 , and

all of their respective preimages as well (once again, with the possible exception of points where b has a matching zero). For any point $\hat{w} \in J(g)$, there exists points arbitrarily close that are preimages of w_1 by Theorem 4.21, hence \hat{w} is a limit point of singularities, and thus must be a singularity of y . Hence y must be singular at every point in the Julia set, which gives us a contradiction to our hypothesis that y was holomorphic at w . Therefore y is holomorphic at w_1 , and as w_1 is an arbitrary point in $g(U)$, $y \in \mathcal{H}(g(U))$.

Repeating the above arguments with $g(U)$ in the place of U shows that $y \in \mathcal{H}(g^2(U))$, and an inductive argument then shows that $y \in \mathcal{H}(G)$, where

$$G = \bigcup_{n=0}^{\infty} g^n(U).$$

We have thus established the following results.

Lemma 4.34 *Let z_0 be an attracting fixed point of the non-constant entire function g , and let y be the solution to the differential equation (4.27) satisfying (4.22). Suppose that y is holomorphic at a point $w \in J(g)$. Then y is holomorphic in $\mathbb{C} \setminus g_0$ where g_0 is the exceptional point of g .*

Corollary 4.35 *Let z_0 be an attracting fixed point of the polynomial g , and let y be the solution to the differential equation (4.27) satisfying (4.22). Suppose that y is holomorphic at a point $w \in J(g)$. Then y is an entire function.*

The second problem that arises when trying to obtain a natural boundary for the solution to an equation with variable coefficients is that the growth of the differential operator is dependent on the growth of the coefficients $a_k(z)$. This growth must then be offset with the growth of $y(g(z))$ and $b(z)$. In general this can be difficult to track, but if we restrict ourselves to polynomials for the coefficients a_k and b and the functional argument g we can make progress. We start with a lemma that concerns the growth of entire functions.

Lemma 4.36 *Let f be an entire function, δ, γ, c , positive constants, and G, N positive integers. If f satisfies the inequality*

$$M_f(\gamma R^G) \leq cR^N M_f(R + \delta) \tag{4.29}$$

for all R sufficiently large, then f is a polynomial of degree at most $\lceil \frac{N}{G-1} \rceil$.

Proof: Let $f_0 = f$, $c_0 = c$, and

$$f_1(z) = \frac{f_0(z) - f_0(0)}{z}.$$

Then f_1 is also an entire function, and by the triangle inequality we have the relations

$$\begin{aligned} M_{f_1}(R) &\leq \frac{M_{f_0}(R) + |f_0(0)|}{R} \\ M_{f_0}(R) &\leq RM_{f_1}(R) + |f_0(0)|. \end{aligned}$$

Combining these with inequality (4.29) yields

$$\begin{aligned} M_{f_1}(\gamma R^G) &\leq \frac{M_{f_0}(\gamma R^G) + |f_0(0)|}{\gamma R^G} \\ &\leq \frac{c_0 R^N M_{f_0}(R + \delta) + |f_0(0)|}{\gamma R^G} \\ &\leq \frac{c_0 R^N (R + \delta) M_{f_1}(R + \delta) + (c_0 R^N + 1) |f_0(0)|}{\gamma R^G} \\ &\leq \left\{ \frac{c_0 R^N (R + \delta)}{\gamma R^G} + \frac{(1 + c_0 R^N) |f_0(0)|}{\gamma R^G M_{f_1}(R + \delta)} \right\} M_{f_1}(R + \delta). \end{aligned}$$

For R sufficiently large, there is a $c_1 > 0$ such that

$$\frac{c_0 R^N (R + \delta)}{\gamma R^G} + \frac{(1 + c_0 R^N) |f_0(0)|}{\gamma R^G M_{f_1}(R + \delta)} < c_1 R^{N-(G-1)},$$

thus

$$M_{f_1}(\gamma R^G) < c_1 R^{N-(G-1)} M_{f_1}(R + \delta).$$

Repeating the process inductively using

$$f_k(z) = \frac{f_{k-1}(z) - f_{k-1}(0)}{z}$$

yields further constants c_k such that for R sufficiently large,

$$M_{f_k}(\gamma R^G) < c_k R^{N-k(G-1)} M_{f_k}(R + \delta).$$

Eventually, we reach the value of k where $N - k(G - 1) < 0$. For this value of k ,

and sufficiently large R , we have the inequalities

$$M_{f_k}(\gamma R^G) < M_{f_k}(R + \delta),$$

and

$$\gamma R^G > R + \delta.$$

Hence, by the maximum modulus theorem, f_k must be a constant function (possibly identically zero), and therefore f must be a polynomial of at most degree k . ■

Theorem 4.37 *Let z_0 be an attracting fixed point for the non-linear polynomial g . Then the solution to the equation (4.27) satisfying (4.22) either has $\partial A_{z_0}^0$ as a natural boundary, or is a polynomial.*

Proof: Let y be the solution to (4.27) satisfying (4.22), and let L be the differential operator defined in (4.28). Then the Cauchy integral formula, and the maximum modulus theorem give

$$\sup_{z \in C_R} |Ly| \leq \left(\frac{n!}{\delta^n} + M_{a_{n-1}}(R) \frac{(n-1)!}{\delta^{n-1}} + \dots + M_{a_1}(R) \frac{1}{\delta} + M_{a_0}(R) \right) M_y(R + \delta) \quad (4.30)$$

where $0 < \delta < R$. Let the degree of the polynomial $a_k(z)$ be A_k for $k = 0, 1, \dots, n-1$, and the degree of $b(z)$ be B . Then there exists positive constants R_1 , β , and α_k such that

$$M_{a_k}(R) \leq \alpha_k R^{A_k} \quad (4.31)$$

for $k = 0, 1, \dots, n-1$ and

$$M_b(R) \geq \beta R^B \quad (4.32)$$

for all $R > R_1$ by Liouville's Theorem. Now, a similar argument as used in the proof of Theorem 4.27, shows that there exists an $R_2 > 0$ and $\gamma > 0$ such that

$$|g(z)| > \gamma R^G$$

for all $z \in C_R$ with $R > R_2$, where G is the degree of $g(z)$. Thus, for such R , the curve $\Gamma = \{g(z) : z \in C_R\}$ completely encloses the curve $C_{\gamma R^G}$, and the maximum modulus principle implies that there is a $z_1 \in C_R$ such that

$$|y(g(z_1))| > M_y(\gamma R^G).$$

Combining this with inequalities (4.30), (4.31) and (4.32) yields

$$\begin{aligned} M_y(\gamma R^G) &< \frac{1}{\beta R^B} \left(\frac{n!}{\delta^n} + \alpha_{n-1} R^{A_{n-1}} \frac{(n-1)!}{\delta^{n-1}} + \dots + \alpha_1 R^{A_1} \frac{1}{\delta} + \alpha_0 R^{A_0} \right) M_y(R + \delta) \\ &< \frac{1}{\beta} \left(\frac{n!}{\delta^n} + \alpha_{n-1} \frac{(n-1)!}{\delta^{n-1}} + \dots + \alpha_1 \frac{1}{\delta} + \alpha_0 \right) R^N M_y(R + \delta) \end{aligned}$$

where $N = \max\{A_0, A_1, \dots, A_{n-1}, B\} - B$. For R sufficiently large, there exists a δ_0 such that

$$\frac{1}{\beta} \left(\frac{n!}{\delta_0^n} + \alpha_{n-1} \frac{(n-1)!}{\delta_0^{n-1}} + \dots + \alpha_1 \frac{1}{\delta_0} + \alpha_0 \right) < 1 + \frac{\alpha_0}{\beta};$$

thus,

$$M_y(\gamma R^G) < \left(1 + \frac{\alpha_0}{\beta} \right) R^N M_y(R + \delta_0),$$

and so by Lemma 4.36, y must be a polynomial of degree at most $\lceil \frac{N}{G-1} \rceil$.

If y is not a polynomial, then we have a contradiction and thus y is not holomorphic at any point on $\partial A_{z_0}^0$, and hence $\partial A_{z_0}^0$ is a natural boundary for y . ■

Note that, although polynomial coefficients allow for polynomial solutions, these solutions need not exist. Indeed, both the degrees of the polynomials a_k , b and g and their coefficients put constraints on the solution in addition to the initial conditions (4.22). For example, consider the functional differential equation

$$y'(z) + (a_1 z + a_0)y(z) = (b_1 z + b_0)y(z^2), \quad (4.33)$$

along with the initial condition

$$y(0) = c_0,$$

where the $a_k, b_k, k = 0, 1$ and c_0 are complex constants. From the proof of Theorem 4.37 we note that any polynomial solution to (4.33) must have degree at most 1. Let $y(z) = c_1 z + c_0$ where c_1 is a constant. Then

$$c_1 + (a_1 z + a_0)(c_1 z + c_0) = (b_1 z + b_0)(c_1 z^2 + c_0),$$

and thus we have the relationships

$$b_1c_1 = 0, \quad (4.34)$$

$$a_1c_1 - b_0c_1 = 0, \quad (4.35)$$

$$a_0c_1 + a_1c_0 - b_1c_0 = 0, \quad (4.36)$$

$$a_0c_0 - b_0c_0 + c_1 = 0. \quad (4.37)$$

Suppose that $c_1 = 0$. Then from (4.36) and (4.37) we obtain that $b_0 = a_0$ and $b_1 = a_1$ in order to get a non-trivial constant solution. If $c_1 \neq 0$, then (4.34) and (4.35) yield that $b_0 = a_1$ and $b_1 = 0$, and upon substituting these into (4.36) and (4.37) we have the further requirement that either $a_0 = -1$, or

$$a_1 = \frac{a_0^2}{a_0 + 1}.$$

Thus, relatively few examples of (4.33) actually yield entire solutions - the rest all have natural boundaries on $D(0; 1)$ by Theorem 4.37.

Holomorphic Solutions about Repelling Fixed Points

In this chapter we consider solutions to functional differential equations holomorphic about a repelling fixed point. Consider the functional differential equation

$$y''(z) + p(z)y'(z) + q(z)y(z) = \lambda r(z)y(g(z)) \quad (5.1)$$

where p, q, r and g are functions holomorphic at z_0 , a repelling fixed point of g , λ is a constant, and $r(z_0) \neq 0$. We wish to find non-trivial holomorphic solutions to (5.1) about the point z_0 . The paper [23] contains much of what is presented in this chapter, excluding §5.5.

From our experience with the first order pantograph equation in Chapter 2, we would expect that this more general equation will only admit holomorphic solutions for specific configurations of the functions p, q and r and the constant λ . Most of the literature concerning functional differential equations deals with the case where g is linear ([11], [19], [5], [18], [37]) where little mention is made of the case where the solution is holomorphic at a repelling point. A notable exception is Oberg [28] who studied the first order equation

$$y'(z) = \lambda q(z)y(g(z)), \quad (5.2)$$

where q and g are holomorphic at z_0 , and z_0 is a repelling fixed point of g . Oberg's analysis shows that equation (5.2) has non-trivial solutions holomorphic at z_0 only

for certain values of λ . He also showed that the values of λ that yielded a holomorphic solution formed a discrete set.

The key to Oberg's results is that equation (5.2) can be replaced by one that involves a compact operator. Once the reformulation is done, we are left with an eigenvalue problem for analyticity, where the operator is compact. We can therefore use the formidable array of results from the spectral theory of compact operators to gain more understanding into the likelihood of holomorphic solutions. Oberg limits his use of the spectral theory to the discrete nature of the spectrum, and the finite dimensional nature of the corresponding eigenspaces. His analysis, however, can be readily adapted to more general equations such as equation (5.1). The requirement of analyticity allows use of the Cauchy Integral formula, and this coupled with the repelling nature of the fixed point provides the mechanism whereby the problem can be reposed as an eigenvalue problem with a compact operator.

5.1 The Eigenvalue Problem for Holomorphic Solutions

We start by transforming (5.1) to an eigenvalue problem. Let \mathcal{L} denote the operator defined by

$$\mathcal{L}y(z) = a_2(z)y''(z) + a_1(z)y'(z) + a_0(z)y(z) \quad (5.3)$$

where a_0 , a_1 , and a_2 are functions holomorphic at z_0 with $a_2(z_0) \neq 0$. Setting

$$a_2(z) = \frac{1}{r(z)}, \quad a_1(z) = \frac{p(z)}{r(z)}, \quad \text{and} \quad a_0(z) = \frac{q(z)}{r(z)}.$$

we have that a_0 , a_1 and a_2 are holomorphic at z_0 , and $a_2(z_0) \neq 0$, so we may write (5.1) in the form

$$\mathcal{L}y(z) = \lambda y(g(z)). \quad (5.4)$$

We thus consider the problem of determining the values of $\lambda \in \mathbb{C}$ such that equation (5.4) has a non-trivial holomorphic solution at z_0 . The set of all such λ , i.e. the spectrum, will be denoted by S .

The character of S depends crucially on the nature of the fixed point. From Theorem 4.33 we see that if z_0 is attracting, then $S = \mathbb{C}$. However, in the repelling case,

we will first show that the spectrum must be discrete, before using the results of Chapter 4 to conclude that in many cases, the spectrum consists of just one point. We start by altering the analysis of Oberg to show that the spectrum is discrete, and that $\lambda = 0$ is the only possible limit point of S .

Since z_0 is a repelling fixed point, $g'(z_0) \neq 0$, and g has a unique inverse function h holomorphic at $g(z_0) = z_0$. Moreover, since $|g'(z_0)| > 1$ we have $|h'(z_0)| < 1$; therefore, z_0 is an attracting fixed point for h . Let \mathcal{L}_h denote the operator defined by

$$\mathcal{L}_h y(z) = a_2(h(z))y''(h(z)) + a_1(h(z))y'(h(z)) + a_0(h(z))y(h(z)).$$

Equation (5.4) can be replaced by the equation

$$\mathcal{L}_h y(z) = \lambda y(z), \tag{5.5}$$

and, since y is holomorphic at z_0 if and only if $y \circ h$ is holomorphic at z_0 , the problem of finding holomorphic solutions to equation (5.4) at z_0 is equivalent to that of finding holomorphic solutions to equation (5.5) at z_0 . This seemingly innocuous step has the effect of transforming the original problem into an eigenvalue problem with a compact operator.

Theorem 5.1 *Let $\mathcal{H}^\infty(D(z_0; \rho))$ denote the Banach space of functions holomorphic in $D(z_0; \rho)$ and continuous in $\overline{D}(z_0; \rho)$ equipped with the norm*

$$\|f\|_\infty = \sup_{|z|=\rho} |f(z)|.$$

There is a $\rho > 0$ such that \mathcal{L}_h is a compact operator from $\mathcal{H}^\infty(D(z_0; \rho))$ into $\mathcal{H}^\infty(D(z_0; \rho))$.

Proof: Since a_0, a_1, a_2 are holomorphic at z_0 , and $|h'(z_0)| < 1$, there are numbers $R > \rho > 0$ such that $a_0, a_1, a_2, h \in \mathcal{H}(\overline{D}(z_0; R))$ and

$$\overline{W} \subset D(z_0; \rho),$$

where \overline{W} denotes the closure of the set $W = \{h(z) : z \in D(z_0; R)\}$. For such a choice of R and ρ , it is evident that \mathcal{L}_h maps $\mathcal{H}^\infty(D(z_0; \rho))$ into $\mathcal{H}^\infty(D(z_0; \rho))$.

We use the Arzela-Ascoli Theorem [21] to show that \mathcal{L}_h is compact. Let $F = \{f_n\}$ be a bounded family of functions in $\mathcal{H}^\infty(D(z_0; \rho))$. Then there exists a number M such that $\|f_n\|_\infty \leq M$ for all n . Moreover, there exists a number Λ such that $\|a_k\|_\infty \leq \Lambda$, for $k = 0, 1, 2$. Now, for any $f_n \in F$ and $z \in \overline{D}(z_0; \rho)$ we have that $h(z) = w \in \overline{W}$ and

$$\begin{aligned} |\mathcal{L}_h f_n(z)| &\leq \sum_{k=0}^2 |a_k(w)| |f_n^{(k)}(w)| \\ &< \Lambda \sum_{k=0}^2 \left| \frac{k!}{2\pi i} \int_{|\xi|=R} \frac{f_n(\xi)}{(\xi-w)^{(k+1)}} d\xi \right| \\ &< \Lambda R M \left\{ \frac{2}{(R-\rho)^3} + \frac{1}{(R-\rho)^2} + \frac{1}{R-\rho} \right\}; \end{aligned}$$

hence, $\{\mathcal{L}_h f_n\}$ is uniformly bounded in $\mathcal{H}^\infty(D(z_0; \rho))$.

We now show that L_h is equicontinuous in $\mathcal{H}^\infty(D(z_0; \rho))$. Let $z_1, z_2 \in \overline{D}(z_0; R)$. Then for any $f_n \in F$ we have $w_1, w_2 \in \overline{W}$ and

$$\begin{aligned} &|a_2(w_1)f_n''(w_1) - a_2(w_2)f_n''(w_2)| \\ &= \frac{1}{\pi} \left| a_2(w_1) \int_{|\xi|=R} \frac{f_n(\xi)}{(\xi-w_1)^3} d\xi - a_2(w_2) \int_{|\xi|=R} \frac{f_n(\xi)}{(\xi-w_2)^3} d\xi \right| \\ &= \frac{1}{\pi} \left| a_2(w_1) \int_{|\xi|=R} f_n(\xi) \left(\frac{1}{(\xi-w_1)^3} - \frac{1}{(\xi-w_2)^3} \right) d\xi \right. \\ &\quad \left. + (a_2(w_1) - a_2(w_2)) \int_{|\xi|=R} \frac{f_n(\xi)}{(\xi-w_2)^3} d\xi \right| \\ &= \frac{1}{\pi} \left| a_2(w_1) \int_{|\xi|=R} f_n(\xi) \left(\frac{(w_1-w_2)((\xi-w_1)^2 + (\xi-w_1)(\xi-w_2) + (\xi-w_2)^2)}{(\xi-w_1)^3(\xi-w_2)^3} \right) d\xi \right. \\ &\quad \left. + (a_2(w_1) - a_2(w_2)) \int_{|\xi|=R} \frac{f_n(\xi)}{(\xi-w_2)^3} d\xi \right| \\ &\leq \frac{2RM}{(R-\rho)^3} \left\{ \frac{3R^2\Lambda}{(R-\rho)^3} |w_1 - w_2| + |a_2(w_1) - a_2(w_2)| \right\}. \end{aligned}$$

Since $a_m \in H(D(z_0; \rho))$ for $m = 0, 1, 2$, there are numbers γ_m such that

$$|a_m(w_1) - a_m(w_2)| \leq \gamma_m |w_1 - w_2|$$

for $m = 0, 1, 2$. We thus have

$$|a_2(w_1)f_n''(w_1) - a_2(w_2)f_n''(w_2)| \leq K_2|w_1 - w_2|,$$

where

$$K_2 = \frac{2RM}{(R - \rho)^3} \left\{ \frac{3R^2\Lambda}{(R - \rho)^3} + \gamma_2 \right\}.$$

In a similar manner, we can find numbers K_0 and K_1 such that

$$|a_1(w_1)f_n''(w_1) - a_1(w_2)f_n''(w_2)| \leq K_1|w_1 - w_2|$$

and

$$|a_0(w_1)f_n''(w_1) - a_0(w_2)f_n''(w_2)| \leq K_0|w_1 - w_2|.$$

Indeed, the constants K_m can be found from

$$K_m = \frac{m!RM}{(R - \rho)^m} \left\{ \frac{mR^{m-1}\Lambda}{(R - \rho)^m} + \gamma_m \right\},$$

for $m = 0, 1, 2$.

Since $h \in H(D(z_0; \rho))$ there is a number γ_h such that

$$|w_1 - w_2| = |h(z_1) - h(z_2)| \leq \gamma_h|z_1 - z_2|.$$

Thus, there is a constant $K = \gamma_h \max\{K_0, K_1, K_2\}$ such that for all $z_1, z_2 \in \overline{D}(z_0; R)$

$$|\mathcal{L}_h y(z_1) - \mathcal{L}_h y(z_2)| \leq K|z_1 - z_2|,$$

and hence \mathcal{L}_h is equicontinuous. The Arzeli-Ascoli Theorem therefore implies that \mathcal{L}_h is compact. ■

A basic property of a compact operator is that its spectrum is discrete; moreover, the associated eigenspace must be finite dimensional. Theorem 5.1 thus immediately yields the following result.

Corollary 5.2 *The set S has no limit point except perhaps at $\lambda = 0$. The eigenspace associated with any $\lambda \in S$ is finite dimensional.*

5.2 Spectrum of the Pantograph Equation

In Chapter 2, we found that in the repelling case of the first order pantograph equation, the only solutions holomorphic at the origin were polynomials and they arose when $\lambda = b/\alpha^n$. In the first order case then,

$$S = \left\{ \frac{b}{\alpha^n} \right\}_{n=0}^{\infty}$$

which has a limit point at $\lambda = 0$ as allowed by Corollary 5.2.

In this section, we study the spectrum associated with the second order pantograph equation and establish that the spectrum is the same as for the first order result, although to do so requires a more complicated method. Consider the functional differential equation

$$y''(z) + ay'(z) + by(z) = \lambda y(\alpha z) \quad (5.6)$$

where a, b, λ and α are constants with $|\alpha| > 1$. Substituting a formal power series $y(z) = \sum_{n=0}^{\infty} c_n z^n$ into (5.6) yields the recurrence relation

$$c_{n+2} = \frac{(\lambda \alpha^n - b)c_n - a c_{n+1}(n+1)}{(n+2)(n+1)}.$$

Unlike the first order case, it is not immediately clear from the above recurrence relation that the power series would diverge for $z \neq 0$. The coefficients c_n look as if they would grow large, though it is hard to draw anything conclusive from the above expression due to the difference present in the numerator. We instead attack the problem using some of the techniques from Chapter 4.

The Julia set of the functional argument αz when $|\alpha| > 1$ consists entirely of the origin by Theorem 4.21, as the only point fixed by αz is $z = 0$, and this point is repelling and has no other pre-images. Thus, by the arguments used in the proof of Lemma 4.26, any solution to (5.6) holomorphic at the origin must be entire. We start by proving the following lemma.

Lemma 5.3 *Let f be an entire function, and let $M(R) = \sup_{|z|=R} |f(z)|$ for all $R > 0$. Suppose there exists constants $\beta > 1$ and $\delta > 0$, and a non-negative integer n such that*

$$M(\beta R) \leq \beta^n M(R + \delta), \quad (5.7)$$

for all R sufficiently large. Then f is a polynomial of degree $d \leq n$.

Proof: Pòlya & Szegö [31] show that, for any fixed number μ , $0 < \mu < 1$, if f is a transcendental entire function then

$$\lim_{R \rightarrow \infty} \frac{M(\mu R)}{M(R)} = 0. \quad (5.8)$$

Given inequality (5.7), there is a number γ , $1 < \gamma < \beta$ such that

$$M(\beta R) \leq \beta^n M(\gamma R),$$

for all R sufficiently large. Note that we can choose γ arbitrarily close to 1 by increasing R , as we require only that $\gamma > 1 + \delta/R$, and δ does not depend on R . Therefore

$$\frac{M(\mu \rho)}{M(\rho)} \geq \frac{1}{\beta^n} > 0, \quad (5.9)$$

for all ρ large, where $1/\beta < \mu = \gamma/\beta < 1$. Hence, relation (5.8) cannot be satisfied, and therefore f is a polynomial. Suppose that f has degree d . Then there exists constants η_1, η_2 such that $\eta_1 R^d < M(R) < \eta_2 R^d$ for R large by Liouville's Theorem, where the ratio $\frac{\eta_1}{\eta_2}$ tends to 1 as R increases. The relation (5.9) then yields

$$\frac{\eta_2 \mu^d \rho^d}{\eta_1 \rho^d} \geq \frac{1}{\beta^n}.$$

Hence,

$$\frac{\eta_2}{\eta_1} \gamma^d \geq \frac{1}{\beta^{n-d}},$$

and as $\beta > 1$ and both $\frac{\eta_2}{\eta_1}$ and γ can be made arbitrarily close to 1, it follows that $d \leq n$. ■

We are now in a position to prove that any solution holomorphic at the origin must be a polynomial.

Theorem 5.4 *Suppose that $\lambda \neq 0$ and y is a non-trivial solution to (5.6) that is holomorphic at $z = 0$. Then $\lambda \in \overline{D}(0; |b|)$ and y is a polynomial.*

Proof: As y is holomorphic at a point on the Julia set, it is entire. Using the same

argument as in the proof for Theorem 4.27, we can quickly establish the inequality

$$M(|\alpha|R) \leq \frac{1}{|\lambda|} \left(\frac{2}{\delta^2} + \frac{|a|}{\delta} + |b| \right) M(R + \delta), \quad (5.10)$$

for constants δ, R such that $0 < \delta < R$.

Suppose that $|\lambda| > |b|$. Then for δ and R sufficiently large we have

$$\frac{1}{|\lambda|} \left(\frac{2}{\delta^2} + \frac{|a|}{\delta} + |b| \right) < 1.$$

Fix δ and choose R so large that $|\alpha|R > R + \delta$. Then the above inequality implies

$$M(|\alpha|R) < M(R + \delta),$$

which contradicts the maximum modulus principle. Hence, $|\lambda| \leq |b|$.

Suppose that $\lambda \in \overline{D}(0; |b|)$. Then there is a non-negative integer n such that

$$\frac{|b|}{|\alpha|^{n+1}} < |\lambda| \leq \frac{|b|}{|\alpha|^n}. \quad (5.11)$$

Inequalities 5.10 and 5.11 imply

$$M(|\alpha|R) \leq \frac{\alpha^{n+1}}{|b|} \left(\frac{2}{\delta^2} + \frac{|a|}{\delta} + |b| \right) M(R + \delta),$$

and hence for δ and R sufficiently large

$$M(|\alpha|R) \leq |\alpha|^{n+1} M(R + \delta).$$

Lemma 5.3 thus implies that y is a polynomial of degree $d \leq n + 1$. ■

Note that if $b = 0$, the above analysis shows that the set of non-zero eigenvalues is empty. For $b \neq 0$ we have the following.

Corollary 5.5 *Let $b \neq 0$. The spectrum associated with (5.6) is $S = \{b/\alpha^k\}_{k=0}^{\infty}$. For any non-negative integer k , the eigenfunction associated with $\lambda_k = b/\alpha^k$ is a polynomial of degree k that is determined uniquely up to a multiplicative constant.*

Proof: Theorem 5.4 shows that any non-trivial solution to equation (5.6) holomorphic at $z = 0$ must be a polynomial. Therefore, there exists a non-negative integer n , and $n + 1$ constants c_n, \dots, c_0 with $c_n \neq 0$ such that

$$y(z) = c_n z^n + c_{n-1} z^{n-1} + \dots + c_1 z + c_0. \quad (5.12)$$

Substituting this expression into the functional differential equation (5.6) and equating coefficients of z^n gives

$$(b - \lambda \alpha^n) c_n = 0, \quad (5.13)$$

and hence

$$\lambda = \frac{b}{\alpha^n}.$$

We thus conclude that if $\lambda \in S$ then $\lambda = b/\alpha^n$ for some non-negative integer n .

Suppose that $\lambda = b/\alpha^n$ for some non-negative integer n . Equation (5.13) shows us that c_n can be chosen arbitrarily, and equating coefficients of z^{n-1} gives

$$c_{n-1} = \frac{an}{b(1/\alpha - 1)} c_n,$$

and the general recurrence relation

$$(k + 2)(k + 1)c_{k+2} + a(k + 1)c_{k+1} + b(1 - \alpha^{k-n})c_k = 0,$$

for $k = 0, 1, \dots, n - 2$. Given a value of c_n , we can thus determine the other coefficients uniquely. Hence, for each non-negative integer n the choice $\lambda = b/\alpha^n$ corresponds to a polynomial solution to (5.6), and therefore $\lambda \in S$. Moreover, the polynomial solution is determined uniquely up to the choice of the multiplicative constant c_n . ■

It is clear that the arguments used to show that the second order pantograph equation has spectrum $S = \{b/\alpha^n\}_{n=0}^{\infty}$ will apply equally well to higher order pantograph equations. All pantograph equations, therefore, have no solutions holomorphic at $z = 0$ except the polynomial solutions that arise when $\lambda \in S$.

5.3 Non-linear Polynomial Arguments

Having studied the spectrum of the pantograph equation, it seems natural to investigate the existence of solutions about repelling fixed points of equations with more general functional arguments. Consider the functional differential equation

$$y''(z) + ay'(z) + by(z) = \lambda y(g(z)), \quad (5.14)$$

where a and b are complex constants, and g is a polynomial of degree $d \geq 2$. One might expect that the discrete spectrum is much like that of the pantograph equation, albeit more complicated to determine, but it turns out that, in general, the spectrum is simpler. In fact we show that there is precisely one non-zero eigenvalue unless g is a cyclic polynomial.

If we drop the condition in Lemma 4.26 that y be holomorphic at an attracting point z_0 , replacing it with the condition that g be non-cyclic, we obtain the following.

Corollary 5.6 *Let g be a non-cyclic polynomial, and let z_0 be a repelling fixed point of g . If y is a solution to (5.14) that is holomorphic at z_0 then y is an entire function.*

Theorem 4.31, however, completely eliminates the possibility of non-constant entire solutions, so any solution to (5.14) holomorphic at a repelling fixed point of g must be a constant. Substituting a constant solution into our equation quickly yields the spectrum as follows.

Theorem 5.7 *Let g be a non-cyclic polynomial of degree at least 2, and let z_0 be a repelling fixed point for g . There exists a solution to (5.14) holomorphic at z_0 and not identically 0, if and only if $\lambda = b$. In this case, y is a constant function.*

Theorem 5.7 shows that if g is non-cyclic, and if $b \neq 0$ then the only non-zero eigenvalue is $\lambda = b$. If $b = 0$, then the set of non-zero eigenvalues is empty.

If g is cyclic, the character of the spectrum can be somewhat different. Consider, for example, the equation

$$y''(z) = \lambda y(z^n), \quad (5.15)$$

where $n \geq 2$. The functional argument z^n is cyclic, and has a repelling fixed point at $z = 1$. It also has an exceptional point at $z = 0$. If a solution were to be holomorphic at $z = 1$, then, it must either be a constant, or be holomorphic in the entire plane excluding the origin by Theorem 4.31. The only constant solution available is the trivial solution, so any eigenfunctions, if they exist, will have an isolated singularity at $z = 0$. Let us first try a function of the form

$$y_0(z) = C_0 \frac{1}{z^m}, \quad (5.16)$$

where C_0 is a constant. Substituting (5.16) into (5.15) we find that m must satisfy

$$m = \frac{2}{n-1}, \quad (5.17)$$

and $\lambda = m(m+1)$. Thus we have at least one non-zero eigenvalue and its corresponding eigenfunction which is holomorphic throughout the plane except at the origin. Unlike the case where g is non-cyclic, however, we can find other solutions holomorphic at $z = 1$ for different choices of $\lambda \neq 0$. In fact, (5.15) can be transformed into a pantograph equation. Let

$$y(z) = \frac{1}{z^m} Y(z),$$

where m satisfies (5.17). Then (5.15) transforms to

$$z^2 Y''(z) - 2mz Y'(z) + m(m+1)Y(z) = \lambda Y(z^n).$$

Let $z = e^w$ and $Y(z) = P(w)$. Then the above equation transforms to

$$P''(w) - (2m+1)P'(w) + m(m+1)P(w) = \lambda P(nw), \quad (5.18)$$

which is a second order pantograph equation of the form (5.6) with $\alpha = n > 1$. The repelling fixed point $z_0 = 1$ for z^n is mapped to the repelling fixed point $w_0 = 0$ for nw , and we thus know that (5.18) has eigenvalues of the form

$$\lambda = \frac{m(m+1)}{n^k},$$

for non-negative integers k . We can thus deduce that (5.15) also has these eigen-

values, and that the corresponding eigenfunctions are of the form

$$y(z) = \frac{1}{z^m} L_k(z),$$

where $L_k(z)$ is a polynomial in $\log z$ of degree k .

5.4 Non-homogeneous Equations

In this section, we address the existence of holomorphic solutions about repelling fixed points to non-homogeneous functional differential equations. Consider the equation

$$\mathcal{L}y(z) - \lambda y(g(z)) = f(z), \quad (5.19)$$

where \mathcal{L} is defined as in §5.1, f and g are given functions holomorphic at z_0 , and z_0 is a repelling fixed point of g .

It was shown in §5.1 that the homogeneous equation (5.4) could be replaced by (5.5) which involves a compact operator. Evidently, (5.19) can be replaced by

$$\mathcal{L}_h y(z) - \lambda y(z) = F(z), \quad (5.20)$$

where

$$F(z) = f(h(z)).$$

The value of this reformulation is that extensions of the Fredholm type theorems are available for compact operators (cf. [15], [20]). For example, the following result is an immediate consequence of the compactness of \mathcal{L}_h .

Theorem 5.8 (Existence and Uniqueness) *Let z_0 be a repelling fixed point for g and suppose $\lambda \neq 0$. Then (5.19) has a solution y that is holomorphic at z_0 for any given function f holomorphic at z_0 if and only if the only solution to (5.4) that is holomorphic at z_0 is the trivial solution. In this case the solution to (5.19) is unique.*

Indeed, if λ is not in the spectrum of \mathcal{L}_h , then the solution y can be represented by

the Neumann series

$$y = -\frac{1}{\lambda} \left(F + \frac{1}{\lambda} \mathcal{L}_h F + \frac{1}{\lambda^2} \mathcal{L}_h^2 F + \dots \right), \quad (5.21)$$

where, for $n \geq 2$,

$$\mathcal{L}_h^n F = \mathcal{L}_h (\mathcal{L}_h^{n-1} F).$$

Theorem 5.8 contrasts sharply with the analogous problem for the case where z_0 is an attracting fixed point. In this case, it can be shown through a simple alteration of the proof of Theorem 4.28 that holomorphic solutions to (5.19) exist for any $\lambda \in \mathbb{C}$ and any choice of f that is holomorphic at z_0 . Given λ and f , however, there is no uniqueness because it is always possible to prescribe the initial values $y(z_0)$ and $y'(z_0)$. It is thus remarkable that the repelling fixed point case yields a unique solution without any initial data.

In §5.2 we determined the spectrum for the second order pantograph equation. Since this spectrum is known explicitly, we can use the above theorem to generate a result specific to the pantograph equation.

Corollary 5.9 *Let λ and b be non-zero numbers such that $\lambda \neq b/\alpha^n$ for any non-negative integer n . For any given function f holomorphic at $z = 0$ there exists a unique solution y to the equation*

$$y''(z) + ay'(z) + by(z) - \lambda y(\alpha z) = f(z),$$

that is holomorphic at $z = 0$. Here, a and α are complex constants such that $|\alpha| > 1$.

As a simple example, consider the equation

$$y''(z) - y(z) + \alpha y(\alpha z) = z, \quad (5.22)$$

which corresponds to the choice of $\lambda = -\alpha$ and $f(z) = z$. This value of λ is not in the spectrum, and hence the above corollary asserts that there is a unique solution to (5.22) holomorphic at 0. We may find the solution either by inspection or by use of the Neumann series. Using the latter approach by way of illustration, we have

$F(z) = z/\alpha$ and

$$\mathcal{L}_h F = F''(z/\alpha) - F'(z/\alpha).$$

We can quickly generate the needed terms for the Neumann series as

$$\begin{aligned}\mathcal{L}_h F &= -\frac{z}{\alpha^2}, \\ \mathcal{L}_h^2 F &= \frac{z}{\alpha^3},\end{aligned}$$

and in general

$$\mathcal{L}_h^n F = (-1)^n \frac{z}{\alpha^{n+1}}.$$

Hence

$$\begin{aligned}y(z) &= \frac{1}{\alpha} \left(F(z) - \frac{1}{\alpha} \mathcal{L}_h F(z) + \frac{1}{\alpha^2} \mathcal{L}_h^2 F(z) + \dots \right) \\ &= \frac{1}{\alpha} \left(\frac{z}{\alpha} + \frac{z}{\alpha^3} + \frac{z}{\alpha^5} + \dots \right) \\ &= \frac{z}{\alpha^2 - 1}.\end{aligned}$$

In contrast, consider the equation

$$y''(z) - y(z) + y(\alpha z) = 1, \tag{5.23}$$

which corresponds to the choice $\lambda = -1$ and $f(z) = 1$. This value of λ is in the spectrum and the corresponding eigenfunction is a constant. Suppose that there is a holomorphic solution y to equation (5.23). Then y can be represented as a power series

$$y(z) = \sum_{n=0}^{\infty} c_n z^n,$$

with non-zero radius of convergence. Substituting the power series into equation (5.23) gives

$$c_2 = \frac{1}{2},$$

and, for all $k \geq 2$,

$$c_{2k} = \frac{\prod_{m=1}^{k-1} (1 - \alpha^{2m})}{(2k)!}.$$

Consequently,

$$\lim_{n \rightarrow \infty} \left| \frac{c_{2n+2} z^{2n+2}}{c_{2n} z^{2n}} \right| = \frac{|1 - \alpha^{2n}|}{(2n+2)(2n+1)} |z|^2 \rightarrow \infty$$

for all $z \neq 0$ and therefore the power series must diverge. There is no solution to (5.23) holomorphic at $z = 0$.

Before we end our discussion of non-homogeneous functional differential equations, we note that Theorem 5.8 is part of a more general result for compact operators, the Fredholm Alternative, that addresses the existence problem for solutions to the non-homogeneous equation when λ is an eigenvalue. A version of the Fredholm Alternative is available for compact operators in general Banach spaces (cf. Kreyszig [20]), but for our case the result would involve a Banach space adjoint and the dual space of \mathcal{H}^∞ . It is more tractable to first recast the problem in some Hilbert space. We are interested primarily in local existence results as opposed to the behaviour of solutions on some disc boundary, and with this in mind we can recast the problem to one that involves a compact operator on the Hardy space $\mathcal{H}^2(D(z_0; \rho))$ for a suitable choice of $\rho > 0$. Recall that $\mathcal{H}^2(D(z_0; \rho))$ is the set of all functions $f \in H(D(z_0; \rho))$ such that

$$\|f\|_2 = \left\{ \int_0^{2\pi} |f(\rho e^{i\phi})|^2 d\phi \right\}^{1/2} < \infty,$$

and that $\mathcal{H}^2(D(z_0; \rho))$ is a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{|z|=\rho} f(z) \bar{g}(z) \frac{1}{z} dz,$$

for $f, g \in \mathcal{H}^2(D(z_0; \rho))$. The dual of $\mathcal{H}^2(D(z_0; \rho))$ is $\mathcal{H}^2(D(z_0; \rho))$. Let $\mathcal{T}_\lambda : \mathcal{H}^2(D(z_0; \rho)) \rightarrow \mathcal{H}^2(D(z_0; \rho))$ be the operator defined by

$$\mathcal{T}_\lambda y(z) = \mathcal{L}_h y(z) - \lambda y(z),$$

and let \mathcal{T}_λ^* denote the (Hilbert space) adjoint of \mathcal{T}_λ . Since $\mathcal{L}_h : \mathcal{H}^2(D(z_0; \rho)) \rightarrow \mathcal{H}^2(D(z_0; \rho))$ is compact we have the following version of the Fredholm Alternative for functional differential equations of the form (5.19)

Theorem 5.10 (Fredholm Alternative) *Suppose that $\lambda \neq 0$. Then one of the two following alternatives holds:*

1. The homogeneous equation $\mathcal{T}_\lambda y = 0$ has only the trivial solution. In which case the non-homogeneous equation (5.19) has exactly one solution for each $f \in \mathcal{H}^2(D(z_0; \rho))$.
2. The homogeneous equation $\mathcal{T}_\lambda y = 0$ has a non-trivial solution. In which case, for $f \in \mathcal{H}^2(D(z_0; \rho))$, the non-homogeneous equation (5.19) has a solution if and only if $\langle F, K \rangle = 0$ for every solution K to the adjoint equation $\mathcal{T}_\lambda^* K = 0$.

5.5 A Natural Boundary for the Pantograph Equation

As a final application of the theory we developed in Chapter 4, we return to the pantograph equation, and show that if a solution is not holomorphic at the origin, but is holomorphic in some strip whose boundary includes a line through the origin, then that line forms a natural boundary for the solution. Note that the line in question does not correspond to the Julia set of the functional argument, which is $\{0\}$, and thus the techniques used in Chapter 4 can not be applied directly. We can, however, use the properties of iterated functions that we have thus far observed to obtain a natural boundary nevertheless. The material below may also be found in [25]. We begin with the first order case as an example.

Recall that the first order pantograph equation studied in Chapter 2,

$$y'(z) + by(z) = \lambda y(\alpha z) \quad (5.24)$$

where b, λ and α are real constants with $b > 0$ and $\alpha > 1$, has Dirichlet series solutions of the form

$$y(z) = \sum_{n=0}^{\infty} a_n e^{-b\alpha^n z}, \quad (5.25)$$

where the a_n 's are constant. We showed in Section 2.2 that solutions of this form are holomorphic in the right half plane $\Pi_0 = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ and, as they are clearly not polynomials, must have a singularity at $z = 0$.

However, we also saw in Section 2.4 that (5.24) exhibited non-uniqueness if $|\lambda|$ is large enough, and so this indicates that other non-Dirichlet series solutions are present. These too must be singular at $z = 0$.

In the special case where $b = 0$, Fredrickson [10] showed that the associated Dirichlet series solution to equation (5.24) defines a function that has the imaginary axis as a natural boundary. His method of proof, however, is crucially dependent on b being zero, which led him to comment “The hypothesis $b = 0$ in this theorem seems quite unnatural (although it is essential in the proof). Are *any* of the solutions we have generated continuable beyond the half plane given?”

5.5.1 The case where α is a positive integer

We start by considering the case where α is a positive integer. We showed in Chapter 2, that equation (5.24) has non-trivial solutions of the form (5.25) where $a_0 \neq 0$ and

$$\frac{a_n}{a_{n-1}} = \frac{\lambda}{b(1 - \alpha^n)} \quad (5.26)$$

for $n \geq 1$.

Consider the transformation defined by

$$w = e^{-bz}. \quad (5.27)$$

This transformation maps the half plane Π_0 to the unit disc $D(0; 1)$, and the imaginary axis to the circle $\partial D(0; 1)$. Under this transformation, the Dirichlet series (5.25) becomes a series of the form

$$h(w) = \sum_{n=0}^{\infty} a_n w^{\alpha^n}. \quad (5.28)$$

Now, α is an integer, so (5.28) is a power series in w . Taking the modulus of a ratio of consecutive terms, and using (5.26) gives

$$\left| \frac{a_n w^{\alpha^n}}{a_{n-1} w^{\alpha^{n-1}}} \right| = \frac{|\lambda|}{b(\alpha^n - 1)} |w|^{\alpha^{n-1}(\alpha-1)},$$

which converges to 0 as $n \rightarrow \infty$ if $|w| < 1$, yet diverges if $|w| > 1$. Consequently, the power series defines a function h holomorphic in $D(0; 1)$ that has at least one singularity on $\partial D(0; 1)$.

Note that the power series (5.28) has large gaps between the non-zero terms. Indeed,

if we write h in the standard form

$$h(w) = \sum_{j=0}^{\infty} c_j w^j,$$

we have that $c_j = 0$ except when j belongs to the sequence $\{j_k\} = \{\alpha^k\}$. Since $\alpha > 1$, there exists a number ν such that $j_{k+1} > (1 + \nu)j_k$ for all $k \geq 1$. We can thus appeal to the *Hadamard gap theorem* [33] to assert that $\partial D(0; 1)$ is a natural boundary for h .

If we assume that the Dirichlet series solution y can be analytically continued across the imaginary axis, then it follows that h must be analytically continued across $\partial D(0; 1)$, which gives a contradiction. We have thus shown the following result.

Theorem 5.11 *Let $\alpha > 1$ be an integer, and suppose that y is a non-trivial Dirichlet series solution of the form (5.25) with coefficients satisfying (5.26). Then the imaginary axis is a natural boundary for y .*

Note that the conditions on α and y are somewhat restrictive and are certainly “unnatural” to use Fredrickson’s term. Unfortunately, when α is not an integer, the transformed series (5.28) is not a power series, and so we cannot appeal to Hadamard’s gap theorem. Furthermore, there is no obvious way in which we could extend the above analysis to non-Dirichlet series solutions. An additional complication is the existence of polynomial solutions to (5.24) for certain configurations of the constants b, c and α which clearly do not have the imaginary axis as a natural boundary. In order to rule the polynomial solutions out, we show in the following section that the condition that the solutions are singular at $z = 0$ is all that is needed to imply the existence of a natural boundary.

5.5.2 The general first order equation

Whether or not a function f has the imaginary axis as a natural boundary is only meaningful if f is holomorphic in some open set $\Omega \subset \mathbb{C}$ the boundary of which includes the imaginary axis. Let Υ be the set of functions f such that:

1. there exists an open set $\Omega \subseteq \Pi_0$ such that $\partial\Pi_0 \subseteq \partial\Omega$ and $f \in \mathcal{H}(\Omega)$;
2. f has a singularity at $z = 0$; and
3. f satisfies equation (5.24) for all $z \in \Omega$.

Our goal is to show that the imaginary axis is a natural boundary for any function $y \in \Upsilon$. In order to do this, we use the transformation (5.27) that arose naturally from the Dirichlet series solutions to transform the problem to one concerning the unit circle $\partial D(0; 1)$. This yields a new functional differential equation whose functional argument has a multivalent inverse. We can exploit the multivalent nature of this inverse to show that the unit circle forms a natural boundary for the transformed problem. A simple argument by contradiction then shows that the original problem has the imaginary axis as a natural boundary.

Suppose that $y \in \Upsilon$. Then there is a set Ω such that $\partial\Pi_0 \subseteq \partial\Omega$ and $f \in \mathcal{H}(\Omega)$. Let $\Psi = \{w \in \mathbb{C} : w = e^{-bz}, z \in \Omega\}$. Under the transformation (5.27), if $h(w) = y(z)$, then $y'(z) = -bwh'(w)$ and $y(\alpha z) = h(w^\alpha)$; thus, (5.24) transforms to the Cauchy-Euler type functional differential equation

$$h'(w) - \frac{1}{w}h(w) = \frac{-\lambda}{bw}h(w^\alpha). \quad (5.29)$$

Let

$$\hat{h}(w) = y\left(-\frac{1}{b}\log(w)\right) \quad (5.30)$$

for $w \in \Psi$. Without loss of generality we can assume that Ω does not contain the point at infinity, so that Ψ does not contain $w = 0$. For any choice of $w \in \Psi$, then, there is a branch of $\log(w)$ such that \hat{h} is holomorphic at w . Moreover, \hat{h} is a solution to (5.29) for any $y \in \Upsilon$. We show that $\partial D(0; 1)$ is a natural boundary for \hat{h} , i.e., there is no branch of $\log(w)$ such that \hat{h} can be analytically continued to values in $\partial D(0; 1)$.

In the case where α is a positive integer, then the functional argument $g(w) = w^\alpha$ in equation (5.29) is an entire function, so we can apply the results of Chapter 4 to obtain the natural boundary. Indeed, \hat{h} is a solution to (5.29) and has a singularity at $w = 1$, as y is singular at $z = 0$. Using the arguments used to prove Theorem 4.8, we conclude that every pre-image of $w = 1$ is also a singularity for \hat{h} , and

seeing as $1 \in J(g)$, these singularities are dense in the Julia set $J(g)$ by Theorem 4.21. Now, $J(g) = \partial D(0; 1)$ and hence $\partial D(0; 1)$ is a natural boundary for \hat{h} .

The above analysis is only applicable in the case where α is an integer, as otherwise the functional argument g is not an entire function, and so we cannot apply the results of Chapter 4. In this more general case, however, we can use a similar argument by considering the pre-images of 1. We start by showing that the preimages of 1 under the the functional argument w^α are dense on $\partial D(0; 1)$.

Lemma 5.12 *For all non-negative integers k, n let*

$$\phi_{n,k} = \frac{k}{\alpha^n}, \quad (5.31)$$

where $\alpha > 1$. The set Φ defined by

$$\Phi = \{\phi_{n,k} : k \leq \alpha^n\},$$

is dense on the interval $[0, 1]$.

Proof: Choose any two distinct points $\rho_0, \rho_1 \in [0, 1]$. Without loss of generality we may assume that $\rho_0 < \rho_1$. Since $\alpha > 1$ we have that $\alpha^{-n} \rightarrow 0$ as $n \rightarrow \infty$, and hence there is a positive integer N such that

$$\frac{1}{\alpha^N} < \rho_1 - \rho_0.$$

Now the points $0, \alpha^{-N}, 2\alpha^{-N}, \dots, [\alpha^N]\alpha^{-N}$ are equally spaced on the interval $[0, 1]$, with a maximum distance α^{-N} between consecutive points. Hence, there exists a $k \in \mathbb{Z}^*$, $k \leq \alpha^N$ such that

$$\rho_0 < \frac{k}{\alpha^N} < \rho_1.$$

We may choose ρ_0 and ρ_1 to be anywhere in the interval $[0, 1]$, and arbitrarily close together, hence the set Φ is dense on $[0, 1]$. ■

Now, let $w_{n,k} = e^{2\phi_{n,k}\pi i}$, where $\phi_{n,k}$ are as in (5.31), and let $\Gamma = \{w_{n,k}\}$. Then $w_{0,k} = 1$ for all k , and we have the relation

$$(w_{n+1,k})^\alpha = w_{n,k}$$

for all k . By induction, each element of Γ is a pre-image of 1 under the functional argument $g(w) = w^\alpha$. Lemma 5.12 shows that the set Φ is dense on $[0, 1]$ and therefore Γ must be dense on $\partial D(0; 1)$. As all points in Γ are pre-images of 1, the pre-images of 1 are dense on $\partial D(0; 1)$. We can now prove the following result.

Lemma 5.13 *Suppose that h is a solution to (5.29) holomorphic in some set $\tilde{\Psi} \subseteq D(0; 1)$ such that $\partial D(0; 1) \subseteq \partial \tilde{\Psi}$. Suppose further that h has a singularity at $w = 1$. Then $\partial D(0; 1)$ is a natural boundary for h .*

Proof: By hypothesis, h is singular at $w = 1$ and, as $w_{0,k} = 1$ for all k , it is clear that h has a singularity at the points $w_{0,k}$ for all k . Suppose that h has singularities at all the points $w_{n,k}$ for all k , where n is some non-negative integer. Then $h' + \frac{1}{w}h$ also has singularities at these points. Since

$$(w_{n+1,k})^\alpha = w_{n,k}$$

for all n, k , and since h is a solution to (5.29), we conclude that h is also singular at $w_{n+1,k}$ for all k . Induction then shows that h is singular at all points in Γ , and as the points in Γ are dense on $\partial D(0; 1)$, we have established the natural boundary. ■

We are now in a position to prove the main result.

Theorem 5.14 *If $y \in \Upsilon$, then the imaginary axis is a natural boundary for y .*

Proof: Suppose that $y \in \Upsilon$, and that there exists an analytic continuation Y of y across the imaginary axis. Therefore there is a point z_0 with $\operatorname{Re}(z_0) = 0$ such that Y is holomorphic at z_0 . Let h be a function defined by equation (5.30). Then h is a solution to (5.29) and for some branch of $\log w$ the function

$$H(w) = Y\left(-\frac{1}{b} \log w\right)$$

is holomorphic in some open set Δ containing the point $w_0 = e^{-bz_0} \in \partial D(0; 1)$. Since $y \in \Upsilon$, property (1) is satisfied, and hence there is an open set $\tilde{\Psi} \subseteq \Psi \subseteq D(0; 1)$ such that $g \in \mathcal{H}(\tilde{\Psi})$ and $\partial D(0; 1) \subseteq \partial \tilde{\Psi}$. Since $H(w) = h(w)$ for all $w \in \tilde{\Psi}$

and $\tilde{\Psi} \cap \Delta \neq 0$ we have that H must be an analytic continuation of h across $\partial D(0; 1)$.

Now, h must have a singularity at $w = 1$, for otherwise $y(z) = h(w)$ would imply that y is not singular at $z = 0$. But Lemma 5.13 implies that $\partial D(0; 1)$ is a natural boundary for h , and this contradicts the existence of the analytic continuation H . We thus conclude that for $y \in \Upsilon$ there does not exist an analytic continuation across the imaginary axis, and thus the imaginary axis is a natural boundary. ■

This result is readily extended to second and higher order pantograph type equations, as suitable transformations of the form (5.27) always result in a Cauchy-Euler type equation of the form (5.29), and much of the above analysis can be applied with only minor changes needed. We briefly discuss the second order case in order to show the necessary modifications.

5.5.3 The second order pantograph equation

Consider the equation

$$y''(z) + ay'(z) + by(z) = \lambda y(\alpha z), \quad (5.32)$$

where a, b, λ and α are real constants such that $\lambda \neq 0$ and $\alpha > 1$. In this case, the natural boundary will depend on the roots to the indicial equation

$$r^2 - ar + b = 0. \quad (5.33)$$

As was shown in Chapter 3, there are two linearly independent Dirichlet series solutions to (5.32) corresponding to the two solutions r_1 and r_2 to the indicial equation (5.33). These solutions are holomorphic in the halfplane $\text{Re}(r_j x) > 0$ for $j = 1, 2$. It is clear then, that if r_1 and r_2 are not real and positive, the natural boundary will no longer be the imaginary axis, but rather the line $\text{Re}(r_j x) = 0$. i.e. Each solution may have a different natural boundary, depending on which root of the indicial equation that solution corresponds to. We modify our definition of the set Υ as follows.

Let r be a root of the indicial equation (5.33), $\Pi(r) = \{z \in \mathbb{C} : \text{Re}(rz) > 0\}$, and define the set $\Upsilon(r)$ as the set of functions f such that:

1. there exists an open set $\Omega \subseteq \Pi(r)$ such that $\partial\Pi(r) \subseteq \partial\Omega$ and $f \in \mathcal{H}(\Omega)$;
2. f has a singularity at $z = 0$; and
3. f satisfies equation (5.32) for all $z \in \Omega$.

The transformation

$$w = e^{-rz} \tag{5.34}$$

then takes the set $\Pi(r)$ to the unit disc $D(0; 1)$, mapping the boundary $\operatorname{Re}(rz) = 0$ to $\partial D(0; 1)$ and the repelling point $z = 0$ to $w = 1$. Setting $h(w) = y(z)$, we can transform equation (5.32) to

$$h''(w) + \frac{r-a}{rw}h'(w) + \frac{b}{r^2w^2}h(w) = \frac{c}{r^2w^2}h(w^\alpha). \tag{5.35}$$

For any $y \in \Upsilon(r)$, we can construct the function

$$\hat{h}(w) = y\left(-\frac{1}{r} \log z\right)$$

which will be (for a suitable branch of $\log z$) a solution to (5.35) holomorphic in the set $\Psi = \{w \in \mathbb{C} : w = e^{-rz}, z \in \Omega\}$, where $\partial D(0; 1) \subseteq \partial\Psi$. Furthermore, \hat{h} must be singular at $w = 1$.

The differential operator does not affect the proof of Lemma 5.13 so we can conclude that $\partial D(0; 1)$ forms a natural boundary for \hat{h} . The same arguments as used in Theorem 5.14 then give us the following result.

Theorem 5.15 *Let r be a root of the indicial equation (5.33), and let $y \in \Upsilon(r)$. Then the line $\operatorname{Re}(rx) = 0$ is a natural boundary for y .*

Thus, a solution to a pantograph equation corresponding to a particular root of the associated indicial equation will have a line through the origin as a natural boundary, unless the solution is holomorphic at the origin. If a particular solution corresponds to more than one root of the indicial equation (such as a sum of two linearly independent solutions), then that solution will be holomorphic in the intersection of the halfplanes associated with each root. The boundary of the resultant wedge will then be the natural boundary. As an example, consider the equation

$$y''(z) + 2y'(z) + 2y(z) = 3y(2z).$$

The associated indicial equation is $r^2 - 2r + 2 = 0$ which has roots $r_1 = 1 + i$ and $r_2 = 1 - i$. A Dirichlet series solution is given by

$$y(z) = y_1(z) + y_2(z),$$

where

$$y_1(z) = \sum_{n=0}^{\infty} c_n e^{-2^n r_1 z},$$

and

$$y_2(z) = \sum_{n=0}^{\infty} d_n e^{-2^n r_2 z},$$

for constants c_n and d_n . The function y_1 is holomorphic in the halfplane $\operatorname{Re}(r_1 z) \geq 0$ and is singular at $z = 0$, so by Theorem 5.15 has a natural boundary on the line $\{z : \operatorname{Re}(z) = \operatorname{Im}(z)\}$. Similarly, y_2 is holomorphic in the halfplane $\operatorname{Re}(r_2 z) \geq 0$ and thus has a natural boundary on the line $\{z : \operatorname{Re}(z) = -\operatorname{Im}(z)\}$. Therefore, y must be holomorphic only where these two halfplanes intersect, i.e. in the wedge

$$\Lambda = \left\{z : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}\right\}.$$

The boundary of this wedge will be a natural boundary of y .

Finally, we note that we cannot conclude that *every* solution to the pantograph equation that is singular at $z = 0$ must have a natural boundary along a line passing through the origin. For that we would be required to show that all solutions are holomorphic in some strip that includes a line through the origin as part of its boundary. However, the analysis used to discover the non-Dirichlet series solutions in §3.2, for instance, showed only that they must be in $L_2[0, \infty)$, and be singular at $z = 0$. An interesting open problem, then, is to show that these general solutions must also have natural boundaries.

Holomorphic Solutions about Neutral Fixed Points

In Chapter 4 we discussed the existence of solutions to functional differential equations about attracting fixed points and found that there is always a holomorphic solution available. In Chapter 5 we looked at the corresponding situation for repelling fixed points and found that the existence of holomorphic solutions is equivalent to an eigenvalue problem involving a compact operator – only certain equations yield holomorphic solutions. In this chapter we consider the case where the fixed point is neutral, presenting material from [24].

Recall that a neutral fixed point z_0 of a function g is a point such that $g(z_0) = z_0$ and $|g'(z_0)| = 1$. Consider the functional differential equation

$$y'(z) + by(z) = \lambda y(g(z)), \quad (6.1)$$

where b and λ are complex constants, and g is an entire function. We seek solutions to (6.1) holomorphic about a neutral fixed point z_0 of g .

There are different types of neutral points depending on whether the point is in the Julia set. Parabolic points, for instance, were discussed in §4.3 and were found to be in the Julia set. Theorem 4.31 then shows that solutions holomorphic at a parabolic fixed point are either constant functions, or are holomorphic everywhere except the exceptional point of g . Constant functions require a specific balancing in the differential operator, and if g has no exceptional point, then holomorphic

solutions to (6.1) do not exist at parabolic fixed points. We will show, however, that other types of neutral fixed points, admit local holomorphic solutions regardless of the differential operator involved, and that it is the arithmetic properties of $g'(z_0)$ that determine the type of neutral point.

6.1 The Linear Case

We start by examining the case where g is a linear function of the form $g(z) = \alpha z$ where $|\alpha| = 1$. This is the pantograph equation with neutral argument. We study non-trivial solutions to (6.1) that are holomorphic at $z = 0$.

Suppose that y is a non-trivial solution to (6.1) that is holomorphic at $z = 0$. Then y can be represented as a power series of the form

$$y(z) = \sum_{n=0}^{\infty} c_n z^n$$

that has a non-zero radius of convergence. Substituting this power series into equation (6.1) and equating coefficients of z^n gives

$$c_n = \frac{c_0}{n!} \prod_{k=1}^n (\lambda \alpha^{k-1} - b) \quad (6.2)$$

for $n \geq 1$. As $|\alpha| = 1$, we have

$$|c_n| \leq |c_0| \frac{(|\lambda| + |b|)^n}{n!}, \quad (6.3)$$

so that the solution is entire. We also have

$$|c_n| \geq |c_0| \frac{||\lambda| - |b||^n}{n!}, \quad (6.4)$$

and thus have both upper and lower bounds on the coefficients c_n with which we can find the order of the entire function.

The order ρ of an entire function represented by a power series is given by

$$\rho = \limsup_{n \rightarrow \infty} \frac{n \ln n}{\ln \left(\frac{1}{|c_n|} \right)}. \quad (6.5)$$

If $|\lambda| \neq |b|$, then inequality (6.3) gives

$$\ln \left(\frac{1}{|c_n|} \right) \geq \ln(n!) - \ln |c_0| - n \ln (|\lambda| + |b|),$$

and so

$$\begin{aligned} \rho &\leq \lim_{n \rightarrow \infty} \frac{n \ln n}{\ln(n!) - \ln |c_0| - n \ln (|\lambda| + |b|)} \\ &= 1; \end{aligned}$$

hence, the maximum order for the entire solution is 1. A similar calculation using the lower bound (6.4) gives, if $|b| \neq |\lambda|$,

$$\begin{aligned} \rho &\geq \lim_{n \rightarrow \infty} \frac{n \ln n}{\ln(n!) - \ln |c_0| - n \ln (|\lambda| - |b|)} \\ &= 1, \end{aligned}$$

thus the order of the entire solution is 1. If $|\lambda| = |b|$ then we have a somewhat different scenario. We still have the upper bound (6.3) and so the solution is entire and is at most of order 1, however the lower bound is now zero, so we cannot determine the actual order of the solution. If $\lambda = b/\alpha^k$ for some non-negative integer k , then it is clear that the solution is a polynomial, and thus order 0. The interesting case occurs when $|\lambda| = |b|$ but $\lambda \neq b/\alpha^k$ for all non-negative integers k . In this case, the order of the entire solution depends on the arithmetic properties of α , in particular on whether or not α is a root of unity. To illustrate this comment, consider the example

$$y'(z) + y(z) = \alpha y(\alpha z). \quad (6.6)$$

For this equation, the coefficients c_n are given by

$$c_n = \frac{c_0}{n!} \prod_{k=1}^n (\alpha^k - 1).$$

If α is a root of unity, then $\alpha^j = 1$ for some positive integer j , and consequently the entire solution y must be a polynomial. Suppose that α is not a root of unity. Evidently y cannot be a polynomial, but the order of y depends on “how close” α approximates a root of unity. From the discussion above, the order of the solution

y is no greater than 1. Suppose that α is such that

$$-\ln |\alpha^n - 1| = O(\ln n), \quad \text{as } n \rightarrow \infty. \quad (6.7)$$

This condition corresponds to Siegel's condition for a stable neutral fixed point ([34], p. 608) which we will discuss further in the next section. In this case, there is an $L > 0$ such that

$$-\ln |\alpha^n - 1| < L \ln n,$$

for n sufficiently large; consequently,

$$\ln \left(\frac{1}{|c_n|} \right) \leq \ln(n!) + nL \ln n - \ln c_0,$$

for n large. Therefore,

$$\begin{aligned} \rho &\geq \lim_{n \rightarrow \infty} \frac{n \ln n}{\ln(n!) + nL \ln n - \ln |c_0|} \\ &= \frac{1}{1 + L}. \end{aligned}$$

For such α , the order ρ of y must satisfy

$$\frac{1}{1 + L} \leq \rho \leq 1.$$

One suspects that there may be other conditions (Cremer type relations) for non roots of unity that would limit the order of the function from above. At any rate, the order of the entire function for this example is linked to the arithmetic properties of α .

6.2 Types Of Neutral Points

There are different types of neutral points, some of which are in the Fatou set, and others of which are in the Julia set. We know, for instance, that all parabolic neutral points are in the Julia set. Here we introduce the other types of neutral fixed point. Let z_0 be a neutral fixed point of g , and set $\alpha = g'(z_0)$ so that $|\alpha| = 1$. We call α the *multiplier* of z_0 . We start with the case where α is a root of unity (i.e. it is a rational rotation).

Suppose $\alpha^n = 1$ for some positive integer n . If $g^n(z) \neq z$ then we have a parabolic fixed point, which we know is in the Julia set. Figure 6.1 shows the Julia set of the function $g(z) = z^2 + e^{2\pi\xi i}z$ with $\xi = 2/5$, which has a neutral periodic point at $z = 0$ of period 5, which is the centre of the 5 “petals” in the top left of the image.

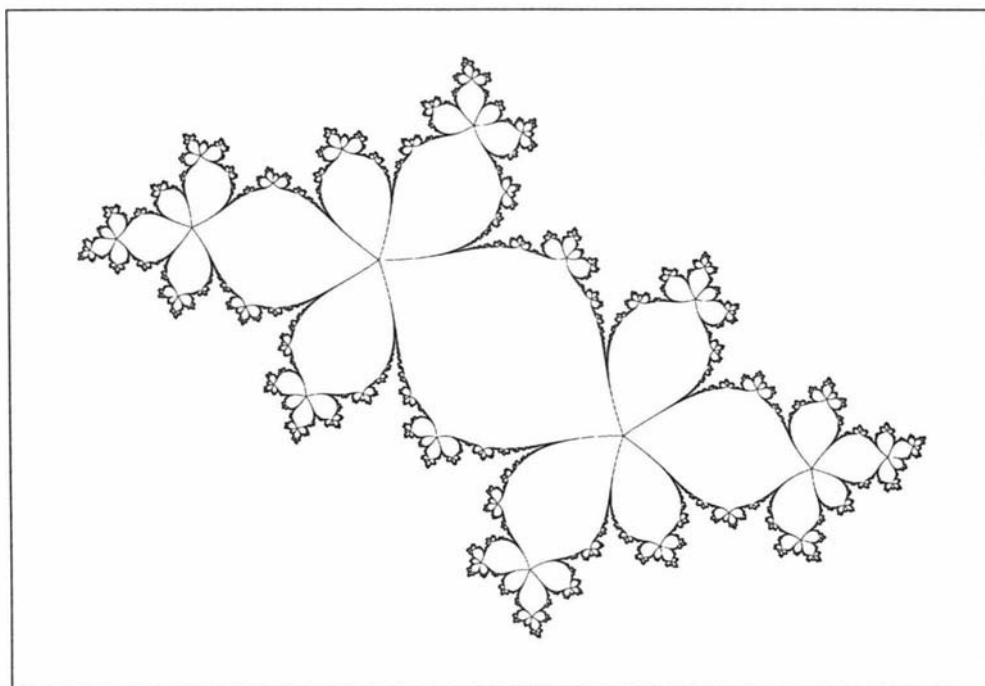


Figure 6.1: The Julia set for $g(z) = z^2 + e^{2\pi\xi i}z$ with $\xi = 2/5$.

The behaviour of points close to a parabolic fixed point z_0 gives us a little insight as to why they fall in the Julia set. If z_0 is a parabolic fixed point of g , then it can be shown that in any neighbourhood of z_0 there exist both points that are attracted to z_0 under g , and points that are repelled from z_0 under g . Specifically, we have the following well-known result (cf. Milnor [26] p. 105).

Theorem 6.1 (Leau-Fatou Flower Theorem) *Let z_0 be a parabolic fixed point of g such that $g'(z_0) = \alpha$, where $\alpha^n = 1$ for some positive integer n , where n is chosen to be as small as possible. Then there exist n attracting petals P_1, P_2, \dots, P_n such that any orbit under g^n containing points in P_k will converge to z_0 from within P_k . Similarly, there exist n repelling petals P'_1, P'_2, \dots, P'_n such that any orbit under the inverse of g^n containing points in P'_k will converge to z_0 from within P'_k . The union of these $2n$ petals along with the point z_0 form a neighbourhood of z_0 , and*

furthermore, they are alternately arranged cyclically at z_0 such that any repelling petal intersects only the two attractive petals on either side.

Figure 6.1 clearly shows the attractive petals about the parabolic point. The repelling petals overlap the attractive petals and are each centered on one of the “spokes” of the Julia set emanating from the parabolic point. As there are attractive petals about z_0 , we may define the **basin of attraction** of z_0 as the set of all points that eventually map into an attractive petal of z_0 . Note, however, that we cannot generate local holomorphic solutions within this attractive basin, as the presence of repelling petals (and thus repelling periodic points arbitrarily close to z_0) forces any local holomorphic solution to be continuable throughout the plane, from which we obtain only constant solutions, unless g has an exceptional point. The presence of repelling points means there is no neighbourhood U of z_0 such that $g(U) \subseteq U$, which as we will see, is the crucial property that allows (and guarantees) holomorphic solutions at z_0 .

If $\alpha^n = 1$ for some positive integer n and $g^n(z) = z$, we no longer have a parabolic point. However, we find that we have already dealt with this case in the previous section - g must be a rotation. We start by showing that g is linear.

Assume g is non-linear. As g is entire, there exists constants $\beta > 0$ and $R > 0$ such that

$$|g(z)| > \beta|z|^2,$$

for all $|z| > R$ by Liouville's Theorem. For R sufficiently large, then

$$|g^2(z)| > \beta^3|z|^4,$$

$$|g^3(z)| > \beta^7|z|^8,$$

and in general,

$$|g^n(z)| > \beta^{2^n-1}|z|^{2^n},$$

for all $|z| > R$. But $g^n(z) = z$ for all z , so we have a contradiction. Thus, g must be linear.

Let $g(z) = az + b$ where a and b are complex constants. Substituting this expression

into $g^n(z) = z$ and equating the coefficients of z^1 and z^0 gives

$$\begin{aligned} a^n - 1 &= 0 \\ b(a^{n-1} + a^{n-2} + \dots + a^1 + 1) &= 0; \end{aligned}$$

hence, a must be a root of unity. If $a = 1$, then $b = 0$ and $g(z) = z$, which is a trivial rotation. If $a \neq 1$ then b can be any complex number. Setting

$$z_0 = \frac{b}{1-a},$$

then gives g in the form

$$g = a(z - z_0) + z_0,$$

which is a rotation about the point z_0 through an angle of $\arg(a)$. We can (if necessary) change the problem so that $z_0 = 0$ by using a simple translation. Thus, the functional differential equation is of the form analysed in the previous section, and will always have a solution holomorphic about z_0 . In fact the solution will be entire.

We now consider the case where α is not a root of unity. In this case we can characterize our neutral points into a further two categories: Siegel points, and Cremer points.

Definition 6.2 (Siegel and Cremer Points) *Let z_0 be a neutral fixed point of g such that $\alpha = g'(z_0)$ is an irrational rotation. Suppose that there exists a simply connected neighbourhood U of z_0 such that $g(U) \subseteq U$. Then z_0 is a Siegel point. If no such neighbourhood exists, z_0 is a Cremer point.*

It is clear from Definition 4.10 of the Julia set that Siegel points are in the Fatou set, whereas Cremer points are in the Julia set. We would therefore expect holomorphic solutions about Siegel points to exist as they do about attracting fixed points. Likewise, we would expect the local theory developed for repelling points to be applicable in some way to Cremer fixed points. Figure 6.2 contains the Julia set for the quadratic $g(z) = z^2 + e^{2\pi\xi i}$ where ξ is the Golden Ratio $\xi = \frac{1+\sqrt{5}}{2}$. The Siegel point at $z = 0$ is in the centre of the large white region in the lower left. The large white region containing the Siegel point is the region mapped into itself by

g. There is no image of a Cremer point available to the best of my knowledge (See Appendix 1).

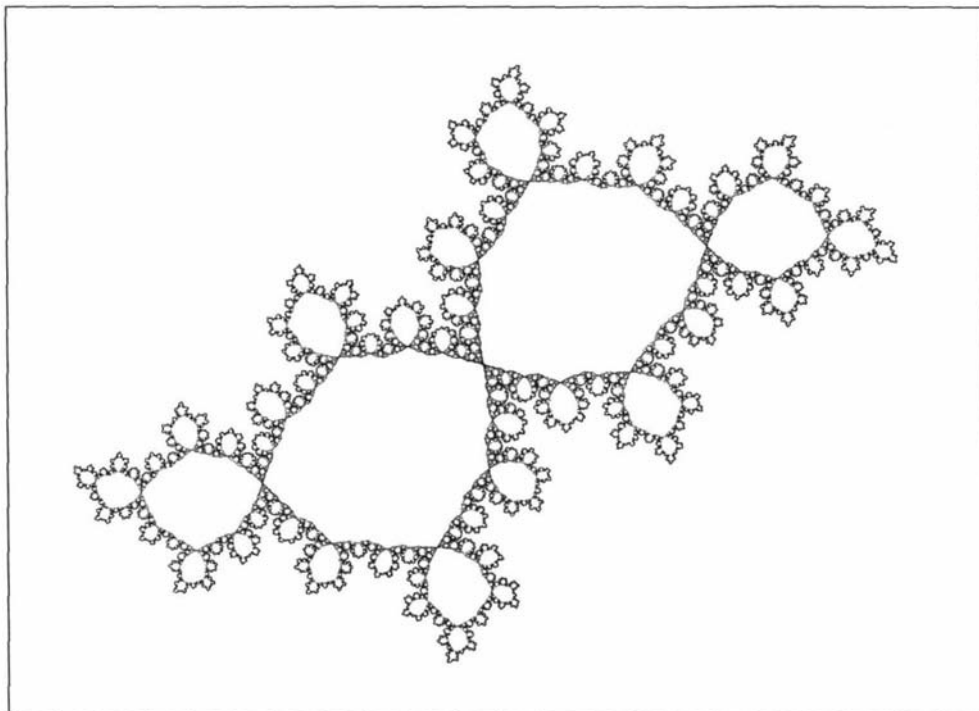


Figure 6.2: The Julia set for $g(z) = z^2 + e^{2\pi\xi i} z$ with $\xi = \frac{1+\sqrt{5}}{2}$.

Before we examine the theory regarding holomorphic solutions to functional differential equations about Siegel and Cremer points, we first note some results characterizing such points. We begin with some definitions.

Definition 6.3 (Diophantine Numbers) *An irrational number ξ is said to be Diophantine of order κ where $\kappa > 0$ if there exists an $\epsilon > 0$ so that*

$$\left| \xi - \frac{p}{q} \right| > \frac{\epsilon}{q^\kappa} \quad \text{for every rational number } \frac{p}{q}.$$

Definition 6.4 (Continued Fractions) *For any irrational number $\xi \in (0, 1)$ we define the continued fraction expansion of ξ as*

$$\xi = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$$

where the a_n are uniquely defined positive integers. The a_n may be computed inductively by the formula

$$a_{n+1} = \left\lfloor \frac{1}{\xi_n} \right\rfloor, \quad \xi_{n+1} = \frac{1}{\xi_n} - a_{n+1}, \quad \text{with } a_1 = \left\lfloor \frac{1}{\xi} \right\rfloor, \quad \xi_1 = \frac{1}{\xi} - a_1.$$

The rational number

$$\frac{p_n}{q_n} = \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_{n-1}}}}$$

is called the n -th convergent to ξ .

The first characterisation we provide is due to Siegel [34].

Theorem 6.5 (Siegel's Theorem) *If the angle ξ is Diophantine of any order, then any neutral fixed point with multiplier $\alpha = e^{2\pi\xi i}$ is a Siegel point.*

A much sharper version of Theorem 6.5 that makes use of the denominators of the convergents to ξ was proved in 1972 by Bryuno.

Theorem 6.6 (Theorem of Bryuno) *Let q_n be the denominator of the n -th convergent to ξ . If*

$$\sum_{n=0}^{\infty} \frac{\log(q_{n+1})}{q_n} < \infty, \tag{6.8}$$

then any neutral fixed point with multiplier $\alpha = e^{2\pi\xi i}$ is a Siegel point.

Yoccoz, in 1987 showed that this is the best possible result, by completely analysing the quadratic case.

Theorem 6.7 (Theorem of Yoccoz) *Conversely, if the sum (6.8) diverges, then the quadratic map $f(z) = z^2 + \alpha z$ with $\alpha = e^{2\pi\xi i}$ has a Cremer point at $z = 0$.*

We end this section with a sufficient condition for Cremer points.

Theorem 6.8 (Cremer's Theorem) *Suppose g is a polynomial of degree d with a neutral fixed point z_0 with multiplier α . If the d^q -th root of $\frac{1}{|\alpha^q - 1|}$ is unbounded as $q \rightarrow \infty$, then z_0 is a Cremer point.*

6.3 Holomorphic Solutions About Siegel Points

In this section, we address the task of finding a holomorphic solution to (6.1) about a Siegel point z_0 . The theory developed for attracting fixed points can be applied, with suitable alterations, to the Siegel point case.

Note that the proof of the local existence and uniqueness result, Theorem 4.7, uses the condition that z_0 is an attracting fixed point only to deduce that

$$g(\overline{D}(z_0; \delta)) \subseteq \overline{D}(z_0; \delta), \quad (6.9)$$

for all $\delta > 0$ sufficiently small. In fact, attracting fixed points obey the strict inequality $g(\overline{D}(z_0; \delta)) \subset \overline{D}(z_0; \delta)$ by definition. The allowance for equality leaves an opening that enables a similar result to be obtained for the Siegel case. We also note that the discussion preceding Theorem 4.33 indicates that we can replace the constants a and b in (6.1) with functions $a(z)$, $b(z)$ holomorphic at z_0 and still obtain a unique holomorphic solution. We can thus relax the conditions of Theorem 4.7 as follows.

Corollary 6.9 *Let p , q and g be functions holomorphic at a fixed point z_0 of g , and suppose that there is a $\delta_1 > 0$ such that g satisfies (6.9) for all $\delta < \delta_1$. Then there exists a solution to*

$$y'(z) + p(z)y(z) = q(z)y(g(z)), \quad (6.10)$$

satisfying the initial condition (4.2) unique among functions holomorphic at z_0 .

Corollary 6.9 alone is not enough to show that there exists a holomorphic solution to (6.1) about a Siegel point, as the neighbourhood U in the definition of a Siegel point need not contain a disc that satisfies property (6.9). We can, however, get around this difficulty by transforming equation (6.1) to one in which the functional argument satisfies (6.9).

Let z_0 be a Siegel point of g , and let U be a simply connected open neighbourhood of z_0 such that $g(U) \subseteq U$. Let ϕ be a conformal map of the unit disc $D(0; 1)$ onto U which takes the origin to z_0 . Let $G = \phi^{-1} \circ g \circ \phi$ be the map of $D(0; 1)$ into itself induced by g . Now, $G(0) = 0$, and so by Schwarz' lemma, either $|G(w)| < |w|$ for

all $w \in D(0; 1)$ or else $G(w) = \beta w$ where $|\beta| = 1$. Clearly $G'(0) = g'(0)$ and so we must have $G(w) = \alpha w$, so that G satisfies relation (6.9). We use the map ϕ to recast equation (6.1) into one where the functional argument is G .

Let $z = \phi(w)$, and $Y(w) = y(z)$. Then (6.1) is transformed to

$$Y'(w) + b\phi'(w)Y(w) - \lambda\phi'(w)Y(\alpha w) = 0. \quad (6.11)$$

We now show that this transformed equation has a unique holomorphic solution that can be analytically continued throughout the unit disc.

Lemma 6.10 *For any $Y_0 \in \mathbb{C}$, there exists a unique solution $Y \in \mathcal{H}(D(0; 1))$ to equation (6.11) that satisfies $Y(0) = Y_0$.*

Proof: Since $\phi \in \mathcal{H}(D(0; 1))$ and αw satisfies (6.9) for any $\delta > 0$, the conditions of Corollary 6.9 are satisfied. Hence, for any $Y_0 \in \mathbb{C}$ there is an $\epsilon > 0$ and a function $Y \in \mathcal{H}(D(0; \epsilon))$ such that Y is a solution to equation (6.11) and satisfies $Y(0) = Y_0$. Moreover, Y is unique.

Corollary 6.9 guarantees a unique solution to the initial-value problem, but the result is local in character and it remains to show that the holomorphic solution can be analytically continued throughout the disc $D(0; 1)$. If equation (6.11) were an ordinary differential equation we could appeal to well-known results concerning the analytic continuation of solutions to discs of radius at least that of the radius of convergence of ϕ (cf. [3]). Nonetheless, we can mimic the proof of this result for the neutral case and employ the method of majorants to deduce that $Y \in \mathcal{H}(D(0; 1))$. Specifically, we have that Y can be represented in the form

$$Y(w) = \sum_{n=0}^{\infty} C_n w^n,$$

for $w \in D(0; \epsilon)$, and that $\phi'(w)$ can be represented in the form

$$\phi'(w) = \sum_{n=0}^{\infty} k_n w^n,$$

for $w \in D(0; 1)$. Substituting these power series into equation (6.11) gives

$$C_{n+1} = \frac{1}{n+1} \sum_{j=0}^n k_{n-j} C_j (\lambda \alpha^j - b).$$

For any r , $0 < r < 1$ there is a number Λ_r such that

$$|k_n| \leq \Lambda_r r^{-n},$$

for all n . Let $w \in D(0; r)$, $q = |\lambda| + |b|$,

$$K_n = q \Lambda_r r^{-n},$$

and define the function H as

$$\begin{aligned} H(w) &= \sum_{n=0}^{\infty} K_n w^n \\ &= q \Lambda_r \sum_{n=0}^{\infty} \left(\frac{w}{r}\right)^n \\ &= \frac{q \Lambda_r}{1 - \frac{w}{r}}. \end{aligned}$$

By construction, the differential equation

$$P'(w) + H(w)P(w) = 0 \tag{6.12}$$

majorizes equation (6.11) under the condition

$$P(0) = |C_0|, \tag{6.13}$$

so that if P is represented by the power series

$$P(w) = \sum_{n=0}^{\infty} b_n w^n,$$

then $b_n > |C_n|$ for all n . Now, equation (6.12) with the initial condition (6.13) has the unique holomorphic solution

$$P(w) = \left(1 - \frac{w}{r}\right)^{-r q \Lambda_r} |C_0|.$$

Since $P \in \mathcal{H}(D(0; r))$ and r can be chosen arbitrarily close to 1, we conclude that $Y \in \mathcal{H}(D(0; 1))$. ■

The set U is simply a neighbourhood of z_0 that is mapped into itself under g . We now find the largest such set for a particular z_0 . Let Ω be the component of the Fatou set of g containing the Siegel point z_0 . It is clear then that $g(\Omega) \subseteq \Omega$, for by Theorem 4.11 the Fatou set $F(g)$ is invariant under g , and thus components of $F(g)$ must map to components of $F(g)$ under g . The point z_0 is fixed under g , and since g is an entire function, g must map the set Ω into itself. Thus, we can set $U = \Omega$, and therefore Ω is conformally equivalent to the unit disc $D(0; 1)$ under ϕ . The set Ω is called the **Siegel disc** associated with the Siegel point z_0 . We then have the following result.

Theorem 6.11 *Let z_0 be a Siegel point for g , and let Ω be the Siegel disc associated with z_0 . Then there exists a unique solution to equation (6.1) satisfying the initial condition (4.2) holomorphic in Ω .*

Proof: We have a unique solution Y to (6.11) satisfying $Y(0) = y_0$, where y_0 is as in (4.2), holomorphic in $D(0; 1)$ by Lemma 6.10. Since ϕ defined above was a conformal map, $y = Y \circ \phi^{-1}$ will be a function holomorphic in Ω which satisfies (6.1) and has $y(z_0) = y_0$. Thus, there exists a solution to (6.1) satisfying (4.2) holomorphic in $\mathcal{H}(\Omega)$. ■

The local theory about Siegel points is identical to that about attracting fixed points - there exists a local holomorphic solution which can be analytically continued up to a component of the Fatou set. In the attracting case, the boundary of this component formed a natural boundary for our solution. This was due to the boundary of Fatou components being in the Julia set, which must also hold for Siegel discs. We can thus claim a natural boundary in the majority of cases as follows.

Theorem 6.12 *Let $y_0 \neq 0$, $b \neq \lambda$, and let y be any solution to the initial value problem (6.1) satisfying (4.2) where z_0 is a Siegel point for the non-linear entire function g . Then one of the following must hold.*

1. y is holomorphic in $\mathbb{C} \setminus \{g_0\}$ where g_0 is the exceptional point of g .
2. y is holomorphic in the Siegel disc Ω containing z_0 , and has $\partial\Omega$ as a natural boundary.

Proof: We have a solution y to (6.1) satisfying (4.2) which is holomorphic in the Siegel disc Ω by Theorem 6.11. If there is no point $w \in \partial\Omega$ at which y is holomorphic, then $\partial\Omega$ forms a natural boundary for y , and so y satisfies the second option. Suppose there is a $w \in \partial\Omega$ such that y is holomorphic at w . Then Theorem 4.31 can be applied to show that either y is a constant, or $y \in \mathcal{H}(\mathbb{C} \setminus \{g_0\})$. The conditions $y_0 \neq 0$ and $b \neq \lambda$, however, rule out constant solutions; therefore, y satisfies the first option. ■

If the class of equations is restricted to those where g is a polynomial, then only the second option in Theorem 6.12 is available. The key result is a Corollary of Theorem 4.31, which holds for all neutral fixed points, not just Siegel points, of non-linear polynomials.

Corollary 6.13 *Let y be a solution to (6.1) where $g(z)$ is a non-linear polynomial with a neutral fixed point z_0 . If y is holomorphic at some point $w \in J(g)$, then y is a constant.*

Proof: Cyclic polynomials have no neutral fixed points by Theorem 4.20, thus g must be non-cyclic. It therefore has no exceptional point. Theorem 4.31 can then be applied to show that y is a constant. ■

For non-linear polynomial functional arguments with Siegel points, non-constant solutions have a natural boundary.

Corollary 6.14 *Let $y_0 \neq 0$, $b \neq \lambda$, and let z_0 be a Siegel point for the non-linear polynomial g . Then the solution y to equation (6.1) satisfying (4.2) is holomorphic in Ω , and has $\partial\Omega$ as a natural boundary.*

In fact, once we rule out constant solutions, we can then rule out the existence of holomorphic solutions about Cremer points and parabolic points altogether.

Corollary 6.15 *Let $y_0 \neq 0$, $b \neq \lambda$, and let z_0 be a neutral fixed point of the non-linear polynomial g . Then there is a solution to (6.1) satisfying (4.2) holomorphic at z_0 if and only if z_0 is a Siegel point.*

Proof: The neutral fixed point z_0 must be either a parabolic point, a Siegel point, or a Cremer point. Clearly if z_0 is a Siegel point, the Corollary holds. If z_0 is a parabolic point or Cremer point, then $z_0 \in J(g)$, and so any solution holomorphic at z_0 is a constant by Corollary 6.13. The conditions $y_0 \neq 0$ and $b \neq \lambda$ preclude constant solutions; consequently, no solutions holomorphic at z_0 can exist unless z_0 is a Siegel point. ■

Conclusion

Functional differential equations with an entire functional argument have been examined and theory regarding the presence of holomorphic solutions to these equations presented. The main results come in Chapters 4 through 6 where two separate problems are analysed, each one related to the other. The first is the existence of local holomorphic solutions, and the second is the analytic continuation of such a local solution throughout the complex plane.

Both problems depend predominantly on the behaviour of g , the functional argument. The local behaviour of g about its fixed points determines whether a holomorphic solution exists about that point. Similarly, the analytic continuation of such a solution occurs through iteration of g . The dynamics of g therefore is the driving force in both problems.

If the initial fixed point is attracting, then a unique local holomorphic solution exists given appropriate initial conditions, and the dynamics of g dictate that the solution can be analytically continued into the immediate basin of attraction of the fixed point. A similar result can be found if the initial point is a Siegel point. In this case, the solution can be analytically continued throughout the Siegel disc. In both these cases, however, the boundary of this region of analyticity is part of the Julia set $J(g)$, which is the closure of the set of repelling periodic points of g . Therefore, in order for solutions to be analytically continued beyond these sets, they must be holomorphic about some repelling periodic point of g .

Repelling fixed points are more difficult to analyse at the local level. Transforming the problem into one involving the inverse of g and considering it as an eigenvalue problem, however, shows that any solutions that are locally holomorphic are rare,

corresponding to eigenfunctions of a compact operator. Furthermore, if g is an entire map, then we can analytically continue any such solutions throughout the entire complex plane, with the possible exception of a single point. Thus, either these solutions are entire, or have a singularity at the exceptional point. Entire solutions, however, are at the mercy of the maximum modulus principle, and from this we can conclude that their existence is even less likely, as the form of equation severely limits the growth rate (often down to constant functions).

In the general case, ruling out the occurrence of solutions with a single singularity in the complex plane is difficult. If we restrict the functional argument g to be a polynomial, however, we can make more progress. Indeed, the only polynomials that have exceptional points are the cyclic polynomials. We can thus conclude that, functional differential equations with non-cyclic polynomial arguments, have no holomorphic solutions about repelling fixed points. Furthermore, any polynomial with a neutral fixed point cannot be cyclic, hence no functional differential equation with a polynomial functional argument can have non-entire solutions that are holomorphic about a Cremer or parabolic fixed point. These results, once fed back in to the problem regarding the analytic continuation of solutions about attracting fixed points, imply that the boundary of the immediate basin of attraction (or Siegel disc in the case of a Siegel point) forms a natural boundary for the solution.

Additionally, in Chapters 2 and 3 we considered solutions away from the fixed point, by examining the simplest functional differential equation with a repelling fixed point, the advanced pantograph equation. Through studying both the equation on the real line and its Laplace transformed analogue, we have shown that the well known Dirichlet series solutions are the only ones that decay exponentially along the positive real axis. Furthermore, we have established the existence of an infinite number of other solutions in $L_2[0, \infty)$. A decay condition is therefore necessary to obtain uniqueness, the required condition being dependent on the coefficients of the particular equation being analysed.

7.1 Future Work

Of interest is how the Laplace and Mellin transforms are useful in gaining insight into a given equation - particularly pantograph type equations. The Laplace transform often allows an advanced equation to be rewritten as a retarded equation in the Laplace variable, whereas the Mellin transform reduces pantograph type equations with polynomial coefficients to difference equations. Both lead to simpler equations in many circumstances. A useful further research project would be the analysing of the Mellin transformed equation and its application in determining possible solution forms for other pantograph type equations with polynomial coefficients. The symmetries present in the Dirichlet series solutions in Chapter 2, for instance, may be applicable to a broader class of equations, and other symmetry forms may also be found.

Another potential area for new results is in the uniqueness problem for the pantograph equation. The uniqueness results of Chapter 2 (Theorems 2.2, 2.8, and 2.12), limit available solutions to the Dirichlet series solution. However, the Laplace analysis of §2.4 indicates that if we relax the conditions of these theorems, there are an infinite number of solutions available, though they must decay slowly along the positive real axis. There is an outstanding question as to whether such solutions are in $L_1[0, \infty)$. The condition that $\lim_{p \rightarrow 0} L(p)$ exists and is finite suggests that this may be true, but has not been shown. This leads to questions regarding the natural space for solutions to this problem. For instance, it is clear that the Dirichlet series solutions are in the Hardy space $H^2(\Pi_0)$. A suitable choice of the solution space may be enough to guarantee a unique solution.

Further work on the neutral case also looks promising. In particular, parabolic fixed points are known to be in the Julia set (and thus do not generally admit local holomorphic solutions), but it is likely that holomorphic solutions may be found in a set Ω whose boundary contains the parabolic point. Indeed, Theorem 6.1 implies there exists a small wedge Λ whose tip is the parabolic point such that $g^n(\Lambda) \subseteq g(\Lambda)$ for some positive integer n . With this knowledge, it may be possible to establish the existence of a solution within such a wedge via a contraction mapping. Such a solution could then be analytically continuable throughout the attracting petal containing the wedge in the same manner as was done for attracting fixed points.

Lastly, we note that there is a relatively simple extension to be made using periodic

rather than fixed points for the initial point z_0 . Indeed, if z_0 was periodic of period 2, then we would need only know $y(z_0)$ and $y(g(z_0))$ in order to determine all the derivatives of y at z_0 and $g(z_0)$ using the differential equation directly. Thus, one can find local holomorphic solutions about attractive periodic points. Much of the theory of Chapter 4 would then be readily extendible to the case where z_0 was periodic.

Notes on Computer Graphics

A.1 Introduction

This section presents a basic overview of the techniques used to produce the images featured in this thesis. The methods presented apply only to polynomials. We divide the area we wish to investigate into a grid of pixels, and for each pixel, we choose a point inside (usually the midpoint of the pixel). We then study the forward images of this point to determine whether it is in the Julia set or the Fatou set, colouring the pixel accordingly. For instance, if the forward images grow without bound in magnitude, then it cannot be in the Julia set as the Julia set for polynomials is bounded, and forward images of points in the Julia set are in the Julia set. Similarly, if the forward images tend towards some fixed orbit, then the point must have been in a basin of attraction, so once again is not in the Julia set.

There are two problems with this approach. Firstly, the chance that the midpoint of a particular pixel is in the Julia set is remote. The Julia set is a closed set and often consists of very thin sections (The Julia set for $g = z^2$, for instance, is the circle $|z| = 1$) and thus it can pass through a pixel while not containing the midpoint. Thus, pixels would be characterized as being in the Fatou set, when they contain points in the Julia set. This problem is tackled by using a *distance estimator* to estimate how far a given point is away from any point on the Julia set. We detail such an estimate in section A.2.

The second problem is that we often require a great number (i.e. millions) of

iterations in order to determine if a point is in the Fatou set – particularly when we are dealing with neutral fixed points. Consider, for instance, a point within a distance of 10^{-3} of the parabolic fixed point $z = 0$ of the function $g(z) = z + z^3$. In one iteration, we see that it will have moved no more than 10^{-9} , as $|g(z) - z| = |z|^3$. It will thus take at least 10^6 iterations for the image of the point to move more than 10^{-3} . This problem is overcome by identifying known regions of the Fatou set, so that we may terminate the iteration whenever an orbit falls into one of the regions. This is detailed in section A.3.

Finally, we note that we have not presented a figure of a Cremer point, and there is no way to generate one using a computer as far as I am aware. The problem is, of course, that a Cremer point z_0 must have $g'(z_0) = e^{2\pi\xi i}$ where ξ is an irrational number that is “close” to a rational number. This is close to impossible to generate using a computer, as all the usual computer representations of numbers are floating point, and therefore rationals. Thus, it is extremely difficult to create a number using a computer that is “close enough” to a rational, yet isn’t actually rational.

A.2 Distance Estimator

Use of a distance estimator is crucial as the Julia set often contains very thin sections or filaments that may pass through many pixels while missing their midpoints. Distance estimators are built by identifying a conformal isomorphism between a region U of the complex plane (or the Riemann sphere if we are considering the attractive basin at infinity) and the unit disc, and then using this isomorphism to determine an appropriate distance estimator. For instance, Milnor ([26], Appendix H) details a distance estimator for an attractive basin U of a superattracting fixed point at the origin. (i.e., one in which $g(z) = az^n + b_1z^{n+1} + \dots$ for some $n > 1$). It is shown that the distance from the boundary of U to a point z is given (up to a factor of 2) by

$$\frac{|\sinh(G(z))|}{\|G'(z)\|},$$

where, for any orbit $z_0 \mapsto z_1 \mapsto \dots$ in U ,

$$G(z_0) = \lim_{k \rightarrow \infty} \frac{\log |z_k|}{n^k},$$

and

$$\|G'(z_0)\| = \lim_{k \rightarrow \infty} \frac{|dz_k/dz_0|}{n^k |z_k|}.$$

In both cases, the successive terms can be computed in an inductive manner. If z_0 is very close to the boundary of U , then G will be small, and hence the distance estimator is very close to the ratio $|G(z_0)|/\|G'(z_0)\|$ which saves the computation of the sinh. It also has the additional benefit that the ratio of successive terms for $G(z_0)$ and $\|G'(z_0)\|$ may be used as an approximate for the distance estimator, allowing additional simplification of the formula which reduces computation time. Indeed, we have

$$\frac{|G(z_0)|}{\|G'(z_0)\|} \approx \frac{|z_k| \log |z_k|}{|dz_k/dz_0|}$$

Therefore, we need only retain $|z_k|$ and $|dz_k/dz_0|$ from the previous iteration. At the next iteration, these can then be updated by

$$\begin{aligned} \left| \frac{dz_{k+1}}{dz_0} \right| &= |g'(z_k)| \left| \frac{dz_k}{dz_0} \right|, \\ |z_{k+1}| &= |g(z_k)|. \end{aligned}$$

Thus, the distance estimator requires only the additional computation of the derivative of g at the previous orbit point.

Similar distance estimates can be found for other regions of the Fatou set, although, for polynomials, we find that just using a distance estimator on the attractive basin at infinity is usually enough to give a sharp picture of $J(g)$.

A.3 Identifying Regions of the Fatou Set

For polynomials, the orbits of points in the complex plane can behave in the following ways:

1. Orbits tend to the point at infinity.
2. Orbits tend to some attracting periodic orbit.
3. Orbits map into an attractive petal of a parabolic point.
4. Orbits map into a Siegel disc.

5. Orbits remain in the Julia set.

Note that the last case occurs if and only if one of the previous cases doesn't occur. We examine each of the first 4 cases above in turn, but we first note that we must be able to find the attractive and neutral periodic points of a polynomial g before we can identify such regions.

A.3.1 Finding attractive and neutral periodic points

Given a polynomial g , we must find the periodic points of g that are not in $J(g)$. Theorem 4.24 gives a limit on the number of attracting periodic points that g may have, though we do not know the period of such points. Thus, we would require to find the roots of g^k for $k = 0, 1, \dots$ until we had the required number of points. Note that this is an expensive operation, as the degree of g^k is d^k where d is the degree of g . Note also that Theorem 4.24 does not give how many attracting periodic points g has, it gives only an upper bound on the number of such points. Additionally, we have no information about how many neutral periodic points g may have. Fortunately, this is remedied by examining the orbits of the critical points of g , which we will refer to as *critical orbits*. We make use of the following result, presented without proof. The reader is referred to Milnor ([26], pp. 78,146) for the details.

Theorem A.1 *Every attractive basin of a polynomial g contains some point in a critical orbit of g . Similarly, every Siegel disc, or cycle of Siegel discs contains points from a critical orbit in its boundary.*

We find the critical points of g (by using a numerical polynomial solver if necessary), and then examine each of their orbits. From the orbits we can identify the periodicity of any periodic points, and their approximate location. We may then examine the appropriate iterate of g , and find the fixed point required, along with its type by using a numerical polynomial solver.

Once we have identified the attractive and neutral fixed points of g , we can now identify regions of $F(g)$ as detailed in the following sections.

A.3.2 Orbits tend to the point at infinity

Such orbits can be identified readily, as $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$. Indeed, all we need to find is the *escape radius* r such that $|g(z)| > |z|$ for all $|z| > r$. We may use Lemma 2.13 to determine r for a given polynomial. If any of the iterates of a point z has modulus greater than r , we know immediately that z must be in the attractive basin of infinity. We then use the distance estimator from section A.2 to check that our pixel does not contain points in $J(g)$ away from the midpoint.

A.3.3 Orbits tend to some attracting periodic cycle

Identifying points within an attractive basin is accomplished by establishing a neighbourhood U of the periodic points such that g maps points in U to points in U . Any orbits that end up in the set U , then, must be in the basin of attraction of the periodic cycle. We may establish such a neighbourhood U as follows. Suppose that we have an attractive periodic point z_0 of a polynomial g . Then some iterate g^k of g may be written in the form

$$g^k(z) = z_0 + g_1(z - z_0) + (z - z_0)^2 \sum_{j=2}^n g_j (z - z_0)^{j-2},$$

where n is the degree of g^k . Thus

$$|g^k(z) - z_0| \leq |z - z_0| \left(|g_1| + |z - z_0| \sum_{j=2}^n |g_j| |z - z_0|^{j-2} \right),$$

and so for any $z \in \overline{D}(z_0; 1)$ we have

$$|g^k(z) - z_0| \leq |z - z_0| \left(|g_1| + |z - z_0| \sum_{j=2}^n |g_j| \right).$$

Now $|g_1| < 1$ as z_0 is an attractive fixed point of g^k , so that

$$r = \frac{1 - |g_1|}{\sum_{j=2}^n |g_j|}$$

is positive. Choosing $\delta < \min\{r, 1\}$, then gives

$$|g(z) - z_0| \leq |z - z_0|,$$

for all $z \in U = \overline{D}(z_0, \delta)$.

A.3.4 Orbits map into an attractive petal of a parabolic periodic point

This case is similar to the previous case, although now the convergence is much slower, and we need to be able to differentiate between the different attractive petals of the parabolic periodic point z_0 . The Julia set will necessarily be very thin near parabolic points, so some pixels may contain points from two or more different attractive petals. We can determine such pixels by examining the orbits of the corners of each pixel, and examining the angle by which each orbit approaches z_0 . Points that have corners whose orbits approach z_0 from different directions must also contain points in the Julia set, whereas points with corners whose orbits all approach z_0 from the same direction must be in an attractive petal. Note that we may still require a great number of iterations to get sufficiently close to a parabolic point so as to be able to reliably compute the “approaching angle”. Fortunately, modern computers are fast enough to perform the necessary computations in a reasonable time frame. For instance, figure 6.1 consists of approximately 6 million pixels and took around 12 hours to render on a 2GHz Athlon processor. Note that almost all that time was spent on points in a small neighbourhood of the parabolic point at $z = 0$. Relatively few iterations are required for orbits to map into the disc $D(0;0.1)$, for instance, but it takes many more iterations from that stage to distinguish each attractive petal.

A.3.5 Orbits map into a Siegel disc

In this last case, the problem that arises is that points in the Siegel disc do not move closer to the Siegel point z_0 under iteration, as g is conformally equivalent to an irrational rotation. We are therefore required to know the Siegel disc in its entirety. Fortunately, this is a simple task, as not only does the boundary of the

Siegel disc U contain points from a critical orbit by Theorem A.1, but these points are in fact dense on the boundary ∂U (cf. Milnor [26], pp 128). We can therefore compute ∂U to the required accuracy using the orbits of the appropriate critical point. It is then a trivial matter to determine if a particular orbit lies within this boundary or not.

A.4 The Algorithm

The algorithm used, therefore, is quite simple. The input data consists of a polynomial g , a region R of the plane on which we wish to draw the Julia set $J(g)$, and the resolution (number of pixels) we require.

We begin by finding the critical points of g and, by examining their orbits, we obtain the periodicity and approximate locations of the attractive and neutral periodic points of g . These points are then found precisely using a numerical solver on the appropriate iterate of g . Known regions F_1, F_2, \dots of the Fatou set can then be identified as detailed in the previous section.

The region R is divided up into a pixel grid so as to give the required resolution, and each pixel's midpoint is iterated under g . At each iteration (or after a given number of iterations), we test to see whether the current point z in the orbit is contained within one of the regions F_k . If $z \in F_k$ for some k , we optionally use the distance estimator to check that the initial point was sufficiently far from the Julia set, before assigning the pixel an appropriate colour. If $z \notin F_k$ for all k , we loop back and perform more iterations. Note that a maximum number of iterations must be specified, as we may have midpoints that are in the Julia set, whose orbits will never lie in one of the terminating sets F_k . If the maximum number of iterations is exceeded for a particular pixel, we issue a warning notice, apply the distance estimator for the attractive basin at infinity, and colour the pixel accordingly. Once we have coloured all our pixels, the image of the Julia set is displayed, and optionally saved to a permanent file.

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