Modelling Repeated Epidemics with General Infection Kernels.

This thesis is presented in partial fulfilment of the requirement for the Degree of Masters of Information Science in Mathematics At Massey University, Albany, New Zealand

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Abstract

This thesis is on mathematical modelling in epidemiology, exploring the generic characteristics of diseases in two different population structures.

Integral equations are used, to model the epidemics in each generation (of the epidemic). Difference equations are then used to model the change in the populations between epidemics. Initially, single dimension populations are modelled, where the entire population is considered to be one class. Then the population is split into two classes and a similar analysis is performed, with critical differences noted between the two structures. An analytical approach is taken, with numerical examples.

The work in this thesis is not specific to one disease, the main focus is to develop a stepped process between generations of the epidemic and analyse the behaviour.

Acknowledgements

I would like to thank my supervisor, Associate Professor Mick Roberts, for his patience and persistence with me throughout the year, and without whom this topic would have been lost on me!

I would also like to thank my family and friends, for without their support full time study would not have been possible.

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Chapter 1 Introduction.

Since 1760, when Daniel Bernoulli developed a mathematical model of the impact of vaccination against smallpox[1], there has been an increasing demand for models of various infections. Mathematical models help us achieve a better understanding of the dynamics of infections, which may then allow us to efficiently implement control techniques.

Infections are continually changing, whether it is change due to drug resistance or just mutation within the infectious agent. Our population dynamics are also changing, which has a large effect on the transmission and probability of infection – and so the mathematical models must also change.

This thesis will look at a simple model for an infection, with six generic examples given throughout the analysis. An integral equation technique is used to model the epidemic and then a discrete mapping system is used to model the dynamics of the population between successive epidemics. We first need to describe the assumptions and terminology used when modelling a disease.

1.1 The Model

Consider a population that can be split into three classes (in relation to an epidemic): those who are susceptible to the infection, the infectious people, and those who are removed from the epidemic (through immunity or

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[1] Dietz & Heesterbeek (2002)

death), i.e. no one member of the population may be infected twice. In the following analysis, we assume that the total population is constant.

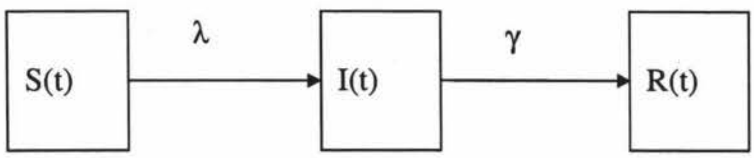

Figure 1.1 SIR Model – the population is divided into three compartments in relation to the infection: susceptibles (*S*), infectives (*I*)and removed (*R*).

The simplest model is depicted in Figure 1.1. The three compartments can represent population density or population size – as the total population size is assumed to be constant it makes no difference. Susceptibles becomes infected at a rate λ resulting from contact with infectives. Contact here is very loosely defined, as the amount of contact needed to become infected will be depend on the infection being modelled. Infectives then become part of the removed compartment at a constant rate γ. As the population size is constant, we know that the change in the population will be zero, .i.e.

$$\frac{dS}{dt}+\frac{dI}{dt}+\frac{dR}{dt}=0 \tag{1.1}$$

The differential equations to describe this model are:

$$\begin{aligned}\frac{dS}{dt}&=-\beta\chi\frac{SI}{N}\\ \frac{dI}{dt}&=\beta\chi\frac{SI}{N}-\gamma I\\ \frac{dR}{dt}&=\gamma I\end{aligned} \tag{1.2}$$

Where χ is the rate at which susceptibles contact other members of the population and β is the probability of a susceptible being infected given contact with an infected member of the population. We have assumed that the population size is constant, that is: $S(t)+I(t)+R(t)=N$, so it is easy to see that one of the above equations is redundant.

The rate λ is the force of the infection, that is, the rate that susceptibles become infected[2]. We have

$$\lambda = \beta\chi\frac{I}{N} \tag{1.3}$$

We can solve equations (1.2) to find a relation between the susceptibles and the infectives in the population[3].

A differential equations approach has been used for numerous mathematical models, and there is a large amount of information available for the analysis of such systems. However, with constant contact parameters, the time spent in each compartment is exponentially distributed among the members of the population – this does not fit actual results. So we turn to a slightly different way of constructing a model with the use of integral equations.

Using integral equations to model an infection is more intuitive than a differential equations approach, and will match the actual data more closely. However, the down side is, there is not a lot of information

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[2] Anderson & May (1991)

[3] See Roberts & Heesterbeek (2000).

published relating to the analysis of such systems. This thesis is based on the integral equation approach that will now be defined[4].

1.2 Probability of Infection

The probability of a susceptible being infected depends on their contact with an infective and the probability of infection given this contact, which depends on the time since the infective was itself infected. If we let $p(\tau)$ be the probability of contact and infection, and $\chi(\tau)$ be the contact rate with an infective, where τ is the time since the infective was initially infected, we let

$$A(\tau) = p(\tau)\chi(\tau) \tag{1.4}$$

So the function $A(\tau)$ represents the probability of contact and infection with an infective at infection time τ (the time since infection took place).

Throughout the following work, six different functions $A(\tau)$ will be used to illustrate the model, where $\tau \geq 0$.

1.2.1 Distribution 1

$$A(\tau) = \begin{cases} 0, & \tau < T_1 \\ a, & T_1 \leq \tau \leq T_2 \\ 0 & \tau > T_2 \end{cases}$$

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[4]Please refer to Diekmann & Heesterbeek (2000) for further elaboration on the integral equation approach.

When τ is less than some specified time T_1 or greater than a second specified time T_2 there is no chance of a susceptible being infected when contacting an infective. When τ lies between the two specified times, there is a constant probability of infection when a susceptible comes in contact with an infective. The period between time zero and T_1 can be seen as a latency period in the infection.

1.2.2 Distribution 2

$$A(\tau)=\begin{cases} a, & 0\le\tau\le T_1 \\ 0, & \tau>T_1 \end{cases}$$

This is similar to distribution one, but now there is a constant probability of contact and infection with an infective from time zero to time T_1. At any other time there is no chance of infection.

1.2.3 Distribution 3

$$A(\tau)=ae^{-c\tau},\ \tau\ge 0$$

Where a and c are positive constants. For this distribution, the probability of contact and infection decreases in a negative exponential in the time since infection. Note that this is the same as for the differential equations model, as a member of the population will spend an exponential amount of time within a compartment.

1.2.4 Distribution 4

$$A(\tau)=a\tau e^{-c\tau},\ \tau\ge 0$$

Again, a and c are positive constants. Here, the probability of contact and infection has a similar shape to the gamma distribution (see Figure 1.2 for further clarification).

1.2.5 Distribution 5

$$A(\tau) = ae^{-c(\tau - T_1)^2}, \ \tau \geq 0$$

As expected, a, c and T_1 are positive constants. The probability of contact and infection takes the shape of a shifted normal distribution curve, but we further truncate this, as we are dealing only on a positive time scale.

1.2.6 Distribution 6

$$A(\tau) = \begin{cases} \dfrac{a}{T_2 - T_1}(\tau - T_1), & T_1 \leq \tau \leq T_2 \\ a, & T_2 < \tau < T_3 \\ \dfrac{-a}{T_4 - T_3}(\tau - T_4), & T_3 \leq \tau \leq T_4 \\ 0, & otherwise \end{cases}$$

The probability of contact and infection takes the form of a trapezium. From $\tau = 0$ to T_1 there is a latency period, and the from T_1 to T_2 the probability of contact and infection increases linearly, to reach its maximum at T_2. This maximum lasts until T_3 when it starts to decrease linearly to zero at T_4. This is one of the most flexible distributions and was recently used by Roberts[5] to model SARS.

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[5] Refer Roberts (in prep.)

Example plots are given below to further elaborate on the above explanations.

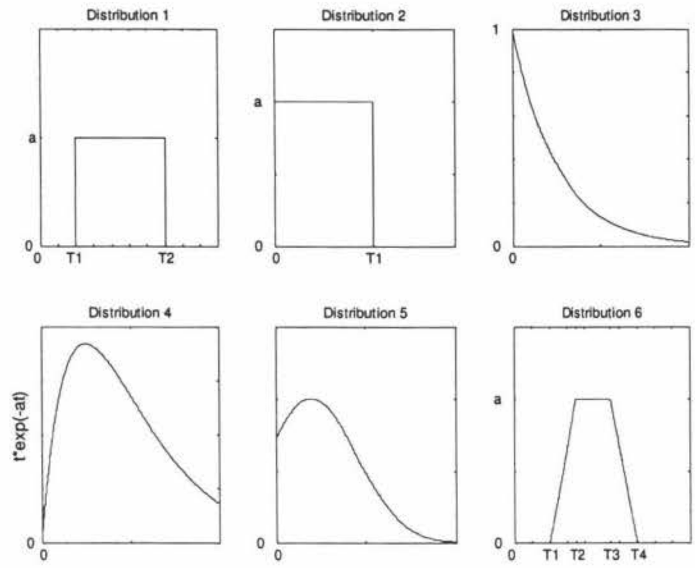

Figure 1.2 Contact Rate/Probability Distributions, time on the x axis

1.3 The Basic Reproduction Ratio

To see if an infection will persist within a population, we consider the basic reproduction ratio, represented by R_0, of the epidemic. We define the basic reproduction ratio as follows:

> The basic reproduction ratio is the number of secondary cases that arise from a primary case in a susceptible population (Diekmann & Heesterbeek 2000).

So the critical value of R_0 is one. If $R_0 < 1$ then the epidemic will not persist in the population, and the number of infectives will decrease. If $R_0 > 1$ then the epidemic will continue through the population, and the number of infectives will increase while the number of susceptibles will decrease. We can see that R_0 will depend on the population size, the contact rates and the probability of infection, hence:

$$R_0 = S(0)\int_0^{\infty} A(\tau)\,\mathrm{d}\tau \qquad (1.5)$$

1.4 The Incidence of Infection.

The incidence of infection $i(t)$ is the number of new cases per unit time. So we see that it will be equal to the change in the susceptible population (as we have ignored changes in the susceptible population due to other causes).

At time *t*, the number of new cases of the infection depends on the contacts between susceptibles and infectives – those who were infected themselves before time *t*. So we have:

$$i(t) = i_0\delta(t) + S(t)\int_0^{t} A(\tau)i(t-\tau)\,\mathrm{d}\tau \qquad (1.6)$$

where the $i_0\delta(t)$ accounts for the initial introduction of the infection into the population[6].

We may also rewrite this in terms of the change in the susceptible population:

$$-\frac{\mathrm{d}S(t)}{\mathrm{d}t} = i_0\delta(t) - S(t)\int_0^{t} A(\tau)\frac{\mathrm{d}S(t-\tau)}{\mathrm{d}t}\,\mathrm{d}\tau \qquad (1.7)$$

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[6] $\delta(t)$ is the Dirac's delta function

1.5 The Initial Growth Rate

The number of infectives can be modelled by an exponential during the initial phases of infection. So we let

$$i(t) \approx ke^{rt} \tag{1.8}$$

for some positive constant *r*, which we call the initial growth rate of the infection. As we said above, the change in infectives is proportional to the incidence of infection. We can then state:

$$ke^{rt} = i_0\delta(t) + S(t)k\int_0^\infty A(\tau)e^{r(t-\tau)}\mathrm{d}\tau \tag{1.9}$$

As we are examining the initial growth of the infection, we do not need to include the initial introduction of the infection into our population; hence we can omit the $i_0\delta(t)$ term. We also set the size of the susceptible population equal to its initial value[7], $S(t) \equiv S(0)$. So we solve:

$$1 = S(0)\int_0^\infty A(\tau)e^{-r\tau}\mathrm{d}\tau \tag{1.10}$$

It is shown in Diekmann and Heesterbeek (2000), that there is a unique real *r* that solves equation (1.10). Note that equation (1.10) is similar to our equation for the basic reproduction ratio (equation (1.5)) The correlation between the two lead to two important facts: $r > 0$ if and only if the basic reproduction ratio is greater than one, and $r < 0$ if and only if the basic

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[7] Note: $S(t) \gg i_0$, and so we let $S(0^+) = S(0^-)$. i_0 will usually be assumed to be equal to one, i.e. there will be one initial case to introduce the infection into the susceptible population.

reproduction ratio is less than one. That is, we only have initial growth of the infection if we have an epidemic.

1.6 Overview

The purpose of the following exercises is to determine R_0 (the basic reproduction ratio), r (the initial growth rate) and the final size equation of an epidemic, given a function $A(\tau)$ that characterises the epidemic. The susceptible population will first be viewed as one class, and will then be split into two classes with intra-class mixing introduced. All the calculations will be based on the following relation for the incidence of the epidemic

$$i(t)=i_0\delta(t)+S(t)\int_0^t A(\tau)i(t-\tau)\mathrm{d}\tau \tag{1.11}$$

for our six functions $A(\tau)$ and for constant and non-constant $S(t)$.

Two methods will be used to calculate the final size of the infection. For a small epidemic, $R_0<1$, we assume that $S(t)$ is constant and equal to the initial susceptible population (as there will be no major epidemic, so the change in the population due to the infection is slower that any other change in the population). For a larger epidemic, $R_0>1$, we can not assume that the susceptible population is constant, so we use a direct method applied to equation (1.7) to calculate the final size of the epidemic.

Using the methods outlines above, we then construct a repeated epidemic process, where we consider epidemics on a discrete generation basis. Initially we assume that the entire population is susceptible, and we let an epidemic occur. We then calculate the final number of susceptibles and let a portion of them continue on to the next epidemic generation. New susceptibles are introduced into the population to maintain a constant population. We then let another epidemic occur, calculate the final number of susceptibles from this second generation and let a proportion continue and introduce new members into the population. This is repeated, with either an epidemic occurring each generation or the infection not persisting within the population. Numerical calculations are given for this, and then a full analytic proof into the nature of the solution is given.

We then repeat out analysis of the six functions $A(\tau)$ when the population is split into two subclasses. The basic reproduction ratio will be calculated for four different mixing schemes between classes. The same methods can be used as for the one dimensional case, with slight alterations to the equations. The final size equations are calculated, and a brief introduction into applying a repeated epidemic process is given.

MATLAB has been used to generate the numerical examples with this thesis, and Maple was used for some of the analytical work.

Chapter 2 The One Dimensional Model: Epidemic Properties

The six distributions mentioned previously will now be considered in detail. For each distribution the basic reproduction ratio, initial growth rate and the final size equations for small and large epidemics will be calculated.

The basic reproduction ratio is the number of secondary cases that arise from one primary case in a fully susceptible population. So we may calculate this value by multiplying the number of susceptibles in the population by the chance of being infected given contact with an infective, that is:

$$R_0 = S(0)\int_0^{\infty} A(\tau)\,\mathrm{d}\tau \tag{2.1}$$

where τ is the time since infection. As stated previously, R_0 acts as a threshold to our infection: if $R_0 < 1$ then the infection will not persist within the population, but if $R_0 > 1$ then an epidemic will occur. R_0 is dependent on the distribution being considered, so we will calculate it for each distribution, and then provide an example at the end of the chapter.

The initial growth rate of the infection can be calculated in two ways by using normal integration techniques and by using the method of Laplace transforms. For the direct method we use the equation for the incidence of infection

$$i(t) = i_0\delta(t) + S(t)\int_0^{t} A(\tau)i(t-\tau)\,\mathrm{d}\tau \tag{2.2}$$

as explained in chapter one. If we assume that the infection increases approximately exponentially at the beginning of the epidemic, we let

$$i(t) = ke^{rt} \tag{2.3}$$

where r is the initial growth rate and k is a constant. So equation (2.2) becomes:

$$ke^{rt} = i_0\delta(t) + S(t)k\int_0^t A(\tau)e^{r(t-\tau)}\mathrm{d}\tau \tag{2.4}$$

As we are looking at the initial growth rate of the infection, we do not include the initial cases that introduced the infection to our population, hence we can exclude the $i_0\delta(t)$ term. Initially, we have said that the entire population is susceptible, so we can approximate $S(t) \equiv S(0)$. Equation (2.4) becomes:

$$\begin{aligned} ke^{rt} &= S(0)k\int_0^t A(\tau)e^{r(t-\tau)}\mathrm{d}\tau \\ 1 &= S(0)\int_0^t A(\tau)e^{-r\tau}\mathrm{d}\tau \end{aligned} \tag{2.5}$$

Even though this is the initial growth rate, we let $t \to \infty$ in the integral, as we want to include all the contacts between susceptibles and infectives (similar to the basic reproduction ratio). We can easily calculate the above integral and solve (either analytically or numerically) for r. To use the method of Laplace transforms we start again with the equation for the incidence of infection (2.2) and approximate $S(t) \equiv S(0)$, as the entire population is susceptible to the infection initially. So we have:

$$i(t) = i_0\delta(t) + S(0)\int_0^t A(\tau)i(t-\tau)\mathrm{d}\tau \tag{2.6}$$

We can then take the Laplace transform of this, using the convolution product[8]:

$$\begin{aligned}\bar{i}(s) &= i_0 + S(0)\bar{A}(s)\bar{i}(s) \\ &= \frac{i_0}{1 - S(0)\bar{A}(s)}\end{aligned} \tag{2.7}$$

(note that the over-bar represents the Laplace transform). If we assume that function has only simple poles at $s = s_n$, then the inverse Laplace transform is the same as calculating the sum of the residues at $s=s_n$ of equation (2.7) multiplied by $e^{s_n t}$. Hence, the growth rate is determined by the dominant term of this series, and is the value of s_n with greatest real part.. So the initial growth rate of the infection is the value of s that makes the denominator of equation (2.7) equal to zero. If more than one solution to the equation exists, the initial growth rate is taken to be the real part of the value of s with the largest real part. Both methods yield the same solution and both methods will be used to calculate the initial growth rate for our six distributions.

The final size of an epidemic is the number of susceptibles before the epidemic less the number of susceptibles after the epidemic:

$$S(0) - S(\infty) \tag{2.8}$$

To calculate $S(\infty)$ for a small epidemic (where we assume that the number of susceptibles is constant, $S(t) \equiv S(0)$), we use the method of Laplace

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[8] See Borrelli and Coleman (1998), pp326-330 for details

transforms on the equation for the incidence of infection (2.2). We know that

$$\int_0^\infty i(t)\mathrm{d}t = \lim_{s\to 0}\int_0^\infty i(t)e^{-st}\mathrm{d}t = \lim_{s\to 0}\bar{i}(s) \tag{2.9}$$

and

$$i(t) = -\frac{\mathrm{d}S(t)}{\mathrm{d}t} \tag{2.10}$$

Combining the above two equations:

$$-\int_0^\infty \frac{\mathrm{d}S(t)}{\mathrm{d}t}\mathrm{d}t = \lim_{s\to 0}\bar{i}(s) \tag{2.11}$$

Calculating the integral gives us the final size equation:

$$S(0) - S(\infty) = \lim_{s\to 0}\bar{i}(s) \tag{2.12}$$

Recall from equation (2.7) and the definition of Laplace transforms:

$$S(0) - S(\infty) = \lim_{s\to 0}\frac{i_0}{1 - S(0)\int_0^\infty A(\tau)e^{-s\tau}\mathrm{d}\tau} \tag{2.13}$$

Using equation (2.1) we have:

$$S(0) - S(\infty) = \frac{i_0}{1 - R_0} \tag{2.14}$$

Although this is independent of the distribution, we shall use this method to calculate the final size of the epidemic for each of the six distributions.

To calculate the final size equation for large epidemics, we can not assume that the susceptible population remains constant. We use the equation for the incidence of infection (2.2) and the fact $i(t) = -\frac{dS(t)}{dt}$, which yields

$$-\frac{dS(t)}{dt} = i_0\delta(t) + S(t)\int_0^t A(\tau)\left[-\frac{dS(t-\tau)}{dt}\right]d\tau \tag{2.15}$$

As we are calculating the final size of the epidemic, we can let t>0, and we do not need to include the initial introduction of the infection in the population.

$$\frac{dS(t)}{dt} = S(t)\int_0^t A(\tau)\left[\frac{dS(t-\tau)}{dt}\right]d\tau \tag{2.16}$$

If we now integrate with respect to time:

$$\int_0^\infty \frac{1}{S(t)}\frac{dS(t)}{dt}dt = \int_0^\infty \int_0^t A(\tau)\left[\frac{dS(t-\tau)}{dt}\right]d\tau dt \tag{2.17}$$

Changing the order of integration on the right hand side gives:

$$\begin{aligned}\int_{S(0)}^{S(\infty)} \frac{1}{S(t)}dS(t) &= \int_0^\infty \int_\tau^\infty A(\tau)\left[\frac{dS(t-\tau)}{dt}\right]dt d\tau \\ \log\left(\frac{S(\infty)}{S(0)}\right) &= \left(S(\infty)-S(0)\right)\int_0^\infty A(\tau)d\tau \\ &= \left(\frac{S(\infty)}{S(0)}-1\right)R_0\end{aligned} \tag{2.18}$$

Figure 2.1 Sample final size curves, intersection of lines gives S(∞) where S(0)=1000, and R_0=3. shows the two curves $y=\frac{S(\infty)}{S(0)}$ and $y=e^{\left(\frac{S(\infty)}{S(0)}-1\right)R_0}$ plotted against the same axis. The two curves intercept when $S(\infty)=S(0)$ (corresponding to no epidemic) and at a second point where $S(\infty)<S(0)$ (corresponding to an epidemic). Given values for $S(0)$ and R_0 we can solve equation (2.18) numerically for $S(\infty)$ and calculate the final size equation (2.8).

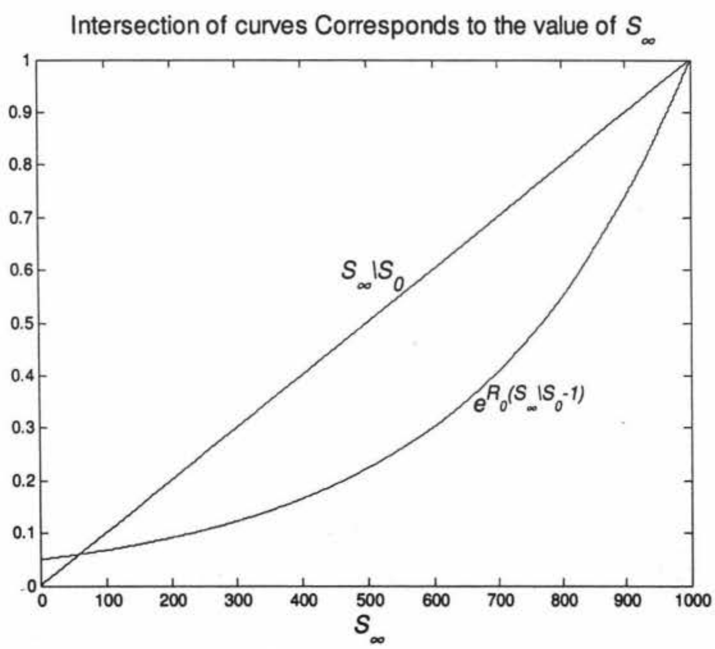

Figure 2.1 Sample final size curves, intersection of lines gives S(∞) where S(0)=1000, and R_0=3.

From equation (2.18) we can see that the final size equation for a large epidemic only depends on the basic reproduction ratio of the infection (which we have assumed to be known) and the initial susceptible population size (which we also assume to be known). So we will not include this calculation in our analysis of the six distributions.

2.1 Distribution One

$$A(\tau)=\begin{cases}0, & \tau<T_1\\ a, & T_1\le\tau\le T_2\\ 0, & \tau\ge T_2\end{cases} \tag{2.19}$$

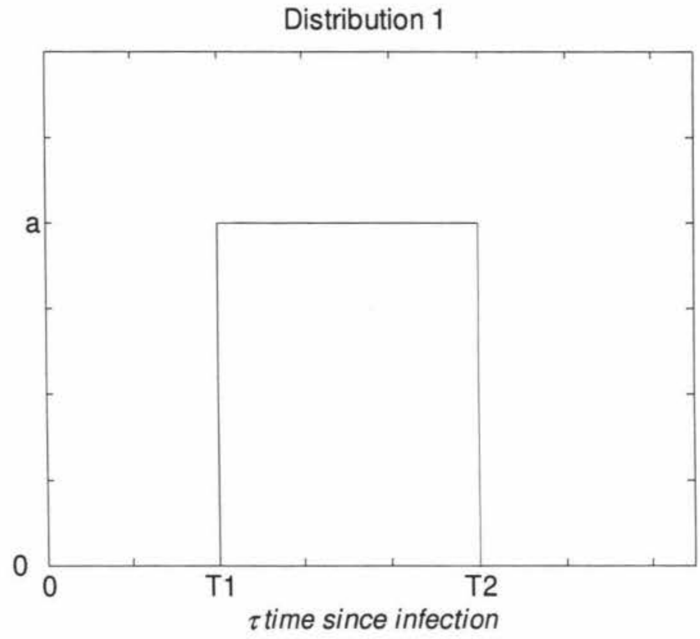

Figure 2.2 Distribution 1

2.1.1 The Basic Reproduction Ratio

The basic reproduction ratio will be:

$$\begin{aligned}R_0&=S(0)\int_0^{\infty}A(\tau)\mathrm{d}\tau\\ &=S(0)\left(\int_0^{T_1}A(\tau)\mathrm{d}\tau+\int_{T_1}^{T_2}A(\tau)\mathrm{d}\tau+\int_{T_2}^{\infty}A(\tau)\mathrm{d}\tau\right)\\ &=S(0)a(T_2-T_1)\end{aligned} \tag{2.20}$$

That is, the basic reproduction ratio is given by in the initial number of susceptibles multiplied by the probability of infection and the time over which infection can occur.

2.1.2 The Initial Growth Rate

We will first compute the initial growth rate using the direct method as outlined above. From equation (2.5)

$$\begin{aligned} 1 &= S(0)a\int_{T_1}^{T_2} e^{-r\tau}\mathrm{d}\tau \\ &= S(0)e^{rt}a\left[\frac{e^{-rT_1} - e^{rT_2}}{r}\right] \end{aligned}$$

Hence

$$r = S(0)a\left(e^{-rT_1} - e^{rT_2}\right) \tag{2.21}$$

We may solve this numerically using MATLAB. A graphical example of this is shown below. The straight line is the left hand side of the above equation ($y=r$) and the curves are the right hand side of the above equation for two different basic reproduction ratios.

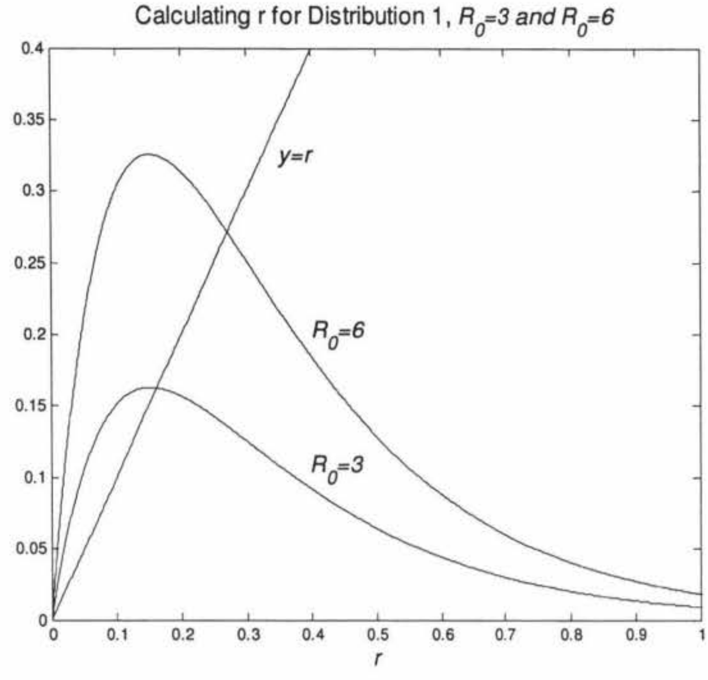

Figure 2.3 Example of calculating r for distribution 1 given R_0=3 and R_0=6. Other parameter values are listed in Examples section

We may also calculate the initial growth rate using the method of Laplace transforms, recall equation (2.7):

$$\begin{aligned}\bar{i}(s) &= i_0 + S(0)\bar{A}(s)\bar{i}(s) \\ &= \frac{i_0}{1 - S(0)\bar{A}(s)}\end{aligned} \tag{2.22}$$

The Laplace transform of $A(\tau)$ is:

$$\begin{aligned}\bar{A}(s) &= \int_0^\infty e^{-s\tau} A(\tau)\,\mathrm{d}\tau \\ &= a\int_{T_1}^{T_2} e^{-s\tau}\,\mathrm{d}\tau \\ &= \frac{-a}{s}\left(e^{-sT_2} - e^{-sT_1}\right)\end{aligned} \tag{2.23}$$

So we have:

$$\bar{i}(s)=\frac{i_0}{1+S(0)\frac{a}{s}\left(e^{-sT_2}-e^{-sT_1}\right)} \tag{2.24}$$

To find the initial growth rate, we need to find the value of s for which the denominator is zero in the above equation. That is:

$$\begin{aligned}1+S(0)\frac{a}{s}\left(e^{-sT_2}-e^{-sT_1}\right)&=0\\ S(0)a\left(e^{-sT_1}-e^{-sT_2}\right)&=s\end{aligned} \tag{2.25}$$

This is the same as equation (2.21), that we calculated using the direct method.

2.1.3 The Final Size Equation – Small Epidemic

Using the method of Laplace transforms, from equation (2.7), where the Laplace transform of $A(\tau)$ is calculated above, we have:

$$\begin{aligned}\bar{i}(s)&=\frac{i_0}{1+S(0)\frac{a}{s}\left(e^{-sT_2}-e^{-sT_1}\right)}\\ &=\frac{si_0}{s+S(0)a\left(e^{-sT_2}-e^{-sT_1}\right)}\end{aligned}$$

Substituting this into equation (2.11) and using L'Hôpital's Rule:

$$\begin{aligned} S(0)-S(\infty) &= \lim_{s\to 0}\frac{i_0}{1+S(0)a\left(-T_2e^{-sT_2}+T_1e^{-sT_1}\right)} \\ &= \frac{i_0}{1+S(0)a(T_1-T_2)} \\ &= \frac{i_0}{1-R_0} \end{aligned} \tag{2.26}$$

2.2 Distribution Two

$$A(\tau)=\begin{cases} 0, & 0>\tau \text{ or } \tau>\mathrm{T}_1 \\ a, & 0\le\tau\le T_1 \end{cases} \tag{2.27}$$

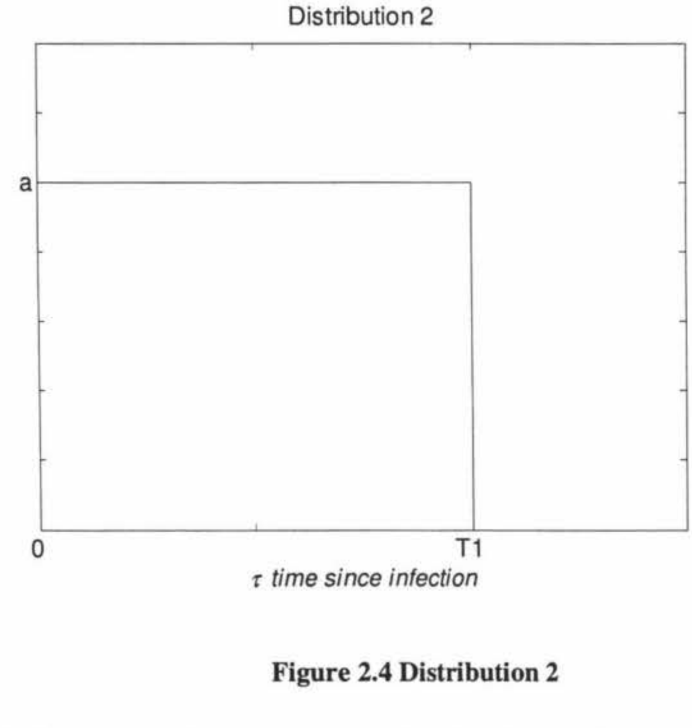

Figure 2.4 Distribution 2

This is the same as the previous distribution, but we have replaced T_2 with T_1 and let T_1 be zero.

2.2.1 The Basic Reproduction Ratio

We calculate the basic reproduction ratio by using equation (2.1)

$$R_0 = S(0)\int_0^\infty A(\tau)\mathrm{d}\tau = S(0)aT_1 \quad (2.28)$$

2.2.2 The Initial Growth Rate

To calculate r using the direct method, we use equation (2.5), which gives:

$$r = S(0)a\left(1-e^{-rT_1}\right) \quad (2.29)$$

The solution to the above equation can be calculated numerically. The plot below shows an example given values of R_0 and $S(0)$. The intersection of the line and curve is the value of r that solves the above equation, note that two values for the basic reproduction ratio are shown below.

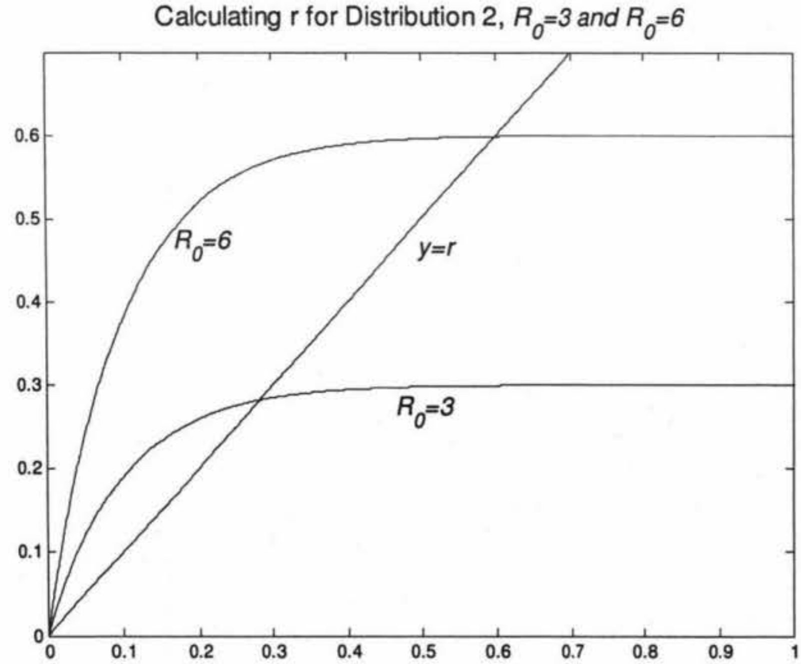

Figure 2.5 Example of calculating r for Distribution 2, given R_0=3 and R_0=6. Parameter values listed in Examples section.

Again, we can also calculate the initial growth rate using Laplace transform method. The Laplace transform of $A(\tau)$ is easily computed, and yields

$$\bar{A}(s)=\frac{-a}{s}\left(e^{-sT_1}-1\right) \tag{2.30}$$

Substituting this into equation (2.7) produces:

$$\bar{i}(s)=\frac{i_0}{1+S(0)\frac{a}{s}\left(e^{-sT_1}-1\right)} \tag{2.31}$$

The initial growth rate is found by solving when the denominator of the above equation is zero:

$$\begin{aligned}0&=1+S(0)\frac{a}{s}\left(e^{-sT_1}-1\right)\\ s&=S(0)a\left(1-e^{-sT_1}\right)\end{aligned} \tag{2.32}$$

Again this is the same as equation (2.29) that we found using the direct method.

2.2.3 The Final Size Equation – Small Epidemics

Using Laplace transforms to find the final size equation, from equation (2.31) we have :

$$\bar{i}(s)=\frac{si_0}{s+S(0)a\left(e^{-sT_1}-1\right)} \tag{2.33}$$

Substituting this into equation (2.9) and using L'Hôpital's rule yields:

$$S(0) - S(\infty) = \frac{i_0}{1 - R_0} \tag{2.34}$$

2.3 Distribution Three

$$A(\tau) = ae^{-c\tau},\ \tau \geq 0 \tag{2.35}$$

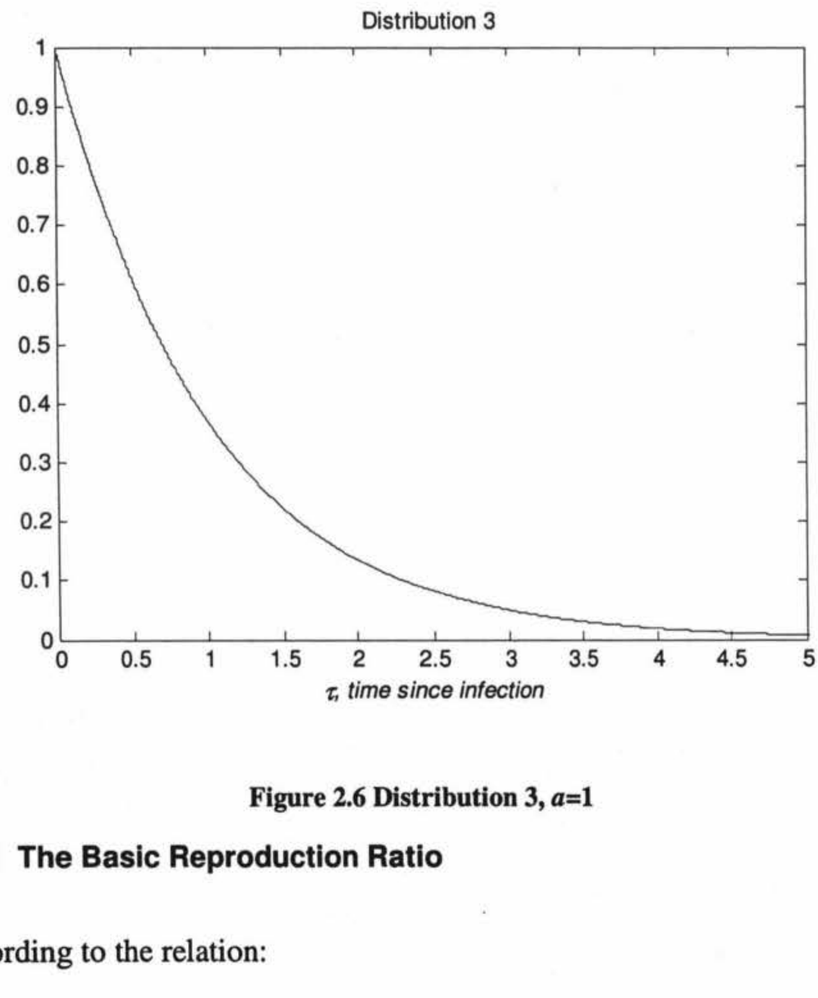

Figure 2.6 Distribution 3, *a*=1

2.3.1 The Basic Reproduction Ratio

According to the relation:

$$R_0 = S(0) \int_0^\infty A(\tau) \mathrm{d}\tau \tag{2.36}$$

We gain

$$\begin{aligned} R_0 &= S(0)a\int_0^\infty e^{-c\tau}\mathrm{d}\tau \\ &= \frac{aS(0)}{c} \end{aligned} \tag{2.37}$$

2.3.2 The Initial Growth Rate

Calculating the initial growth rate using the direct method we have:

$$\begin{aligned} 1 &= S(0)a\int_0^t e^{-\tau(r+c)}\mathrm{d}\tau \\ &= S(0)a\left[\frac{e^{-\tau(r+c)}}{-(r+c)}\right]_{\tau=0}^{\tau=t} \\ &= \frac{aS(0)}{(r+c)}\left[1-e^{-t(r+c)}\right] \\ r &= aS(0)\left[1-e^{-t(r+a)}\right]-c \end{aligned} \tag{2.38}$$

We then let $t \to \infty$:

$$r = aS(0) - c \tag{2.39}$$

This is simple to solve as $S(0)$, a and c are known.

We may also using the method of Laplace transforms to calculate the initial growth rate when $S(t) \equiv S(0)$. The Laplace transform of the equation for the incidence of infection is:

$$\begin{aligned} \bar{i}(s) &= i_0 + S(0)\bar{A}(s)\bar{i}(s) \\ &= \frac{i_0}{1-S(0)\bar{A}(s)} \end{aligned} \tag{2.40}$$

Where

$$\bar{A}(s)=\frac{a}{s+c} \tag{2.41}$$

So we gain

$$\bar{i}(s)=\frac{i_0}{1-\frac{a}{(s+c)}S(0)} \tag{2.42}$$

Solving when the denominator of the above equation is zero:

$$s=aS(0)-c \tag{2.43}$$

This is the same as equation (2.39) that we calculated previously.

2.3.3 The Final Size Equation – Small Epidemics

We use Laplace transforms on our equation for the incidence of infection. From equation(2.42):

$$\bar{i}(s)=\frac{i_0(s+c)}{s+c-aS(0)} \tag{2.44}$$

Using equation (2.9) we have

$$S(0)-S(\infty)=\frac{i_0}{1-R_0} \tag{2.45}$$

2.4 Distribution Four

$$A(\tau)=a\tau e^{-c\tau},\ \tau\geq 0 \tag{2.46}$$

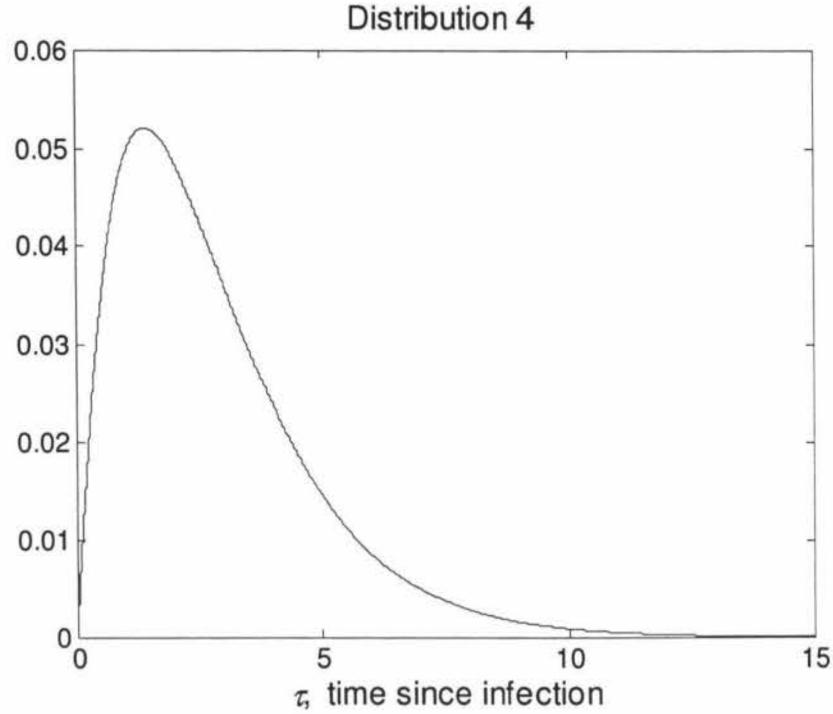

Figure 2.7 Distribution 4, a = 0.1000 and c = 0.7071.

2.4.1 The Basic Reproduction Ratio

Using equation (2.1) we have:

$$\begin{aligned} R_0 &= S(0)\int_0^\infty A(\tau)\,\mathrm{d}\tau \\ &= \frac{aS(0)}{c^2} \end{aligned} \tag{2.47}$$

by using integration by parts.

2.4.2 The Initial Growth Rate

Using equation (2.5) to calculate the initial growth rate:

$$\begin{aligned}1&=S(0)a\int_0^t \tau e^{-\tau(r+c)}\mathrm{d}\tau\\&=\frac{S(0)a}{(c+r)^2}\left[1-\left(t(c+r)+1\right)e^{-(c+r)t}\right]\end{aligned} \quad (2.48)$$

We then let $t\to\infty$ and the above equation simplifies to

$$\begin{aligned}1&=\frac{aS(0)}{(c+r)^2}\\0&=r^2+2cr+c^2-aS(0)\\r&=-c\pm\sqrt{aS(0)}\end{aligned} \quad (2.49)$$

We have only taken the positive root, as we want to find the growth rate! Again, this may be easily calculated, as *c*, *a* and *S*(0) are known. When $c^2 > aS(0)$ implies that we have a negative value for the initial growth rate, and corresponds to having a basic reproduction ratio less than one (that is, no epidemic).

We may also using the method of Laplace transforms to calculate the initial growth rate when $S(t)\equiv S(0)$. The Laplace transform of the equation for the incidence of infection is:

$$\begin{aligned}\bar{i}(s)&=i_0+S(0)\bar{A}(s)\bar{i}(s)\\&=\frac{i_0}{1-S(0)\bar{A}(s)}\end{aligned} \quad (2.50)$$

Where

$$\bar{A}(s)=\frac{a}{(s+c)^2} \quad (2.51)$$

So we get

$$\bar{i}(s)=\frac{i_0}{1-\frac{a}{(s+c)^2}S(0)} \tag{2.52}$$

The initial growth rate of the infection is found by solving when the denominator of the above equation is zero:

$$\begin{aligned}(s+c)^2&=aS(0)\\ s&=-c\pm\sqrt{aS(0)}\end{aligned} \tag{2.53}$$

Where we only consider the positive root.

2.4.3 The Final Size Equation – Small Epidemics

We use Laplace transforms on our equation for the incidence of infection, to gain equation (2.50). Thus

$$\bar{i}(s)=\frac{i_0(s+c)^2}{(s+c)^2-aS(0)} \tag{2.54}$$

Using equation (2.9) we have

$$S(0)-S(\infty)=\frac{i_0}{1-R_0} \tag{2.55}$$

2.5 Distribution Five

$$A(\tau)=ae^{-c(\tau-T_1)^2},\ \tau\geq 0 \tag{2.56}$$

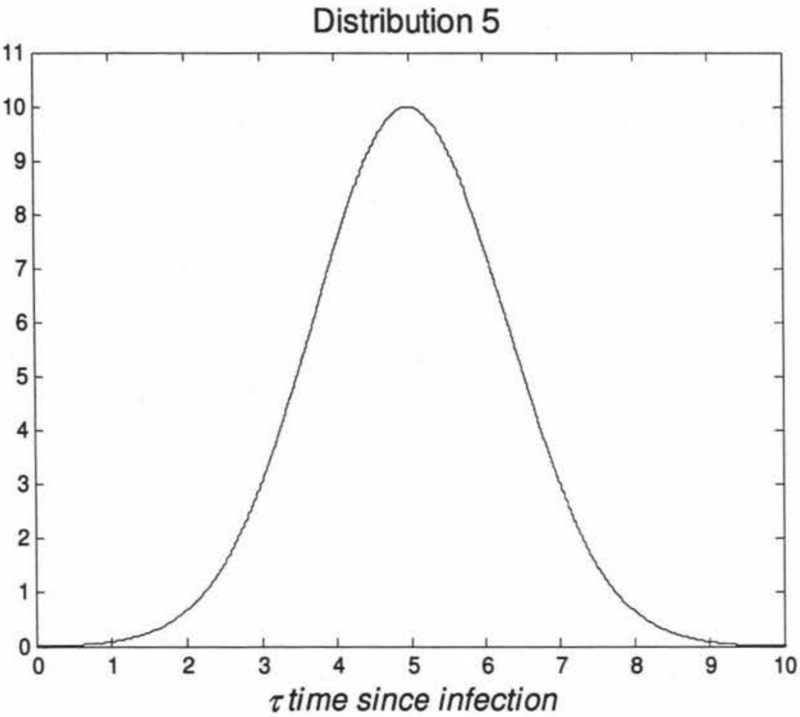

Figure 2.8 Distribution 5, T_1=5, $S(0)$=1000, a=10 and c=0.3

2.5.1 The Basic Reproduction Ratio

We find the basic reproduction ratio in the normal way:

$$\begin{aligned} R_0 &= S(0)\int_0^\infty A(\tau)\,\mathrm{d}\tau \\ &= S(0)a\int_0^\infty e^{-c(\tau-T_1)^2}\,\mathrm{d}\tau \end{aligned} \tag{2.57}$$

To calculate this, we make the substitution

$$\begin{aligned} u &= \sqrt{c}\,(\tau - T_1) \\ \frac{\mathrm{d}u}{\mathrm{d}\tau} &= \sqrt{c} \end{aligned} \tag{2.58}$$

This gives us:

$$\begin{aligned} R_0 &= \frac{aS(0)}{\sqrt{c}} \int_{-\sqrt{c}T_1}^{\infty} e^{-u^2} \mathrm{d}u \\ &= \frac{aS(0)}{\sqrt{c}} \left[\frac{\sqrt{\pi}}{2} + \int_0^{\sqrt{c}T_1} e^{-u^2} \mathrm{d}u \right] \\ &= \frac{aS(0)}{\sqrt{c}} \frac{\sqrt{\pi}}{2} \left[1 + \operatorname{erf}\left(\sqrt{c}T_1 \right) \right] \end{aligned} \tag{2.59}$$

Where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, \mathrm{d}t$$

This has to be calculated numerically, given constant values of a, c and T_1.

2.5.2 The Initial Growth Rate

To calculate the initial growth rate we utilise equation (2.5):

$$ke^{rt} = i_0\delta(t) + S(t)k \int_0^t A(\tau) e^{r(t-\tau)} \mathrm{d}\tau \tag{2.60}$$

As explained previously, this becomes:

$$\begin{aligned} 1 &= S(0) \int_0^t A(\tau) e^{-r\tau} \mathrm{d}\tau \\ &= aS(0) \int_0^t e^{-c(\tau - T_1)^2} e^{-r\tau} \mathrm{d}\tau \\ &= aS(0) e^{-cT_1^2} \int_0^t e^{-c\tau^2 + \tau(2cT_1 - r)} \mathrm{d}\tau \\ &= aS(0) e^{-cT_1^2 + c\left(T_1 - \frac{r}{2c}\right)^2} \int_0^t e^{-c\left(\tau^2 - \tau\left(2T_1 - \frac{r}{c}\right) + \left(T_1 - \frac{r}{2c}\right)^2\right)} \mathrm{d}\tau \\ &= aS(0) e^{-T_1 r + \frac{r^2}{4c}} \int_0^t e^{-c\left(\tau - T_1 + \frac{r}{2c}\right)^2} \mathrm{d}\tau \end{aligned} \tag{2.61}$$

Again, we use substitution to calculate the above integral.

$$u=\sqrt{c}\left(\tau-T_1+\frac{r}{2c}\right)$$
$$\frac{\mathrm{d}u}{\mathrm{d}\tau}=\sqrt{c} \tag{2.62}$$

If we substitute this, and let $t\to\infty$ we have

$$\begin{aligned}1&=\frac{aS(0)}{\sqrt{c}}e^{-T_1r+\frac{r^2}{4c}}\int_{-\sqrt{c}\left(T_1-\frac{r}{2c}\right)}^{\infty}e^{-u^2}\mathrm{d}\tau\\&=\frac{aS(0)}{\sqrt{c}}e^{-T_1r+\frac{r^2}{4c}}\left[\frac{\sqrt{\pi}}{2}+\int_0^{\sqrt{c}\left(T_1-\frac{r}{2c}\right)}e^{-u^2}\mathrm{d}\tau\right]\\&=\frac{aS(0)\sqrt{\pi}}{2\sqrt{c}}e^{-T_1r+\frac{r^2}{4c}}\left(1+\mathrm{erf}\left(\sqrt{c}\left(T_1-\frac{r}{2c}\right)\right)\right)\end{aligned} \tag{2.63}$$

This can be solved numerically using MATLAB, when T_1, a, $c>0$.

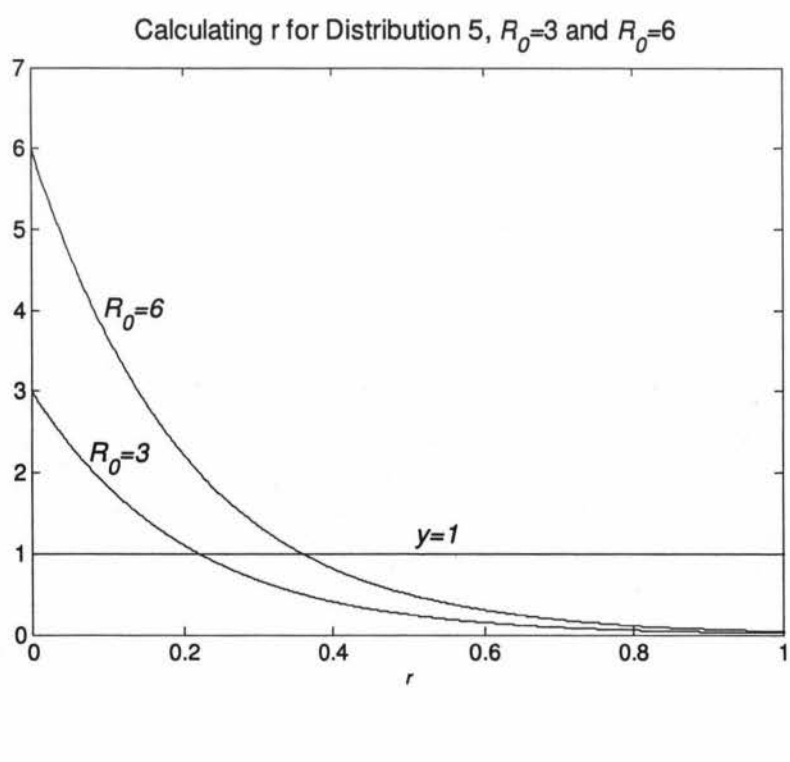

Figure 2.9 Example of calculating r for distribution 5 given R_0=3 and R_0=6, and $S(0)$=1000. Please see table in Examples for other parameter values.

We may also using the method of Laplace transforms to calculate the initial growth rate when $S(t) \equiv S(0)$. The Laplace transform of the equation for the incidence of infection is:

$$\begin{aligned} \bar{i}(s) &= i_0 + S(0)\bar{A}(s)\bar{i}(s) \\ &= \frac{i_0}{1 - S(0)\bar{A}(s)} \end{aligned} \tag{2.64}$$

Where

$$\bar{A}(s) = \frac{a\sqrt{\pi}}{2\sqrt{c}} e^{-T_1 s + \frac{s^2}{4c^2}} \left(1 + \operatorname{erf}\left(\sqrt{c}\left(T_1 - \frac{s}{2c}\right)\right)\right) \tag{2.65}$$

So we gain

$$\bar{i}(s) = \frac{i_0}{1 - \frac{a\sqrt{\pi}}{2\sqrt{c}} e^{-T_1 s + \frac{s^2}{4c^2}} \left(1 + \operatorname{erf}\left(\sqrt{c}\left(T_1 - \frac{s}{2c}\right)\right)\right) S(0)} \tag{2.66}$$

The initial growth rate of the infection is found by solving when the denominator of the above equation is zero:

$$1 = \frac{a\sqrt{\pi}}{2\sqrt{c}} e^{-T_1 s + \frac{s^2}{4c^2}} \left(1 + \operatorname{erf}\left(\sqrt{c}\left(T_1 - \frac{s}{2c}\right)\right)\right) \tag{2.67}$$

The value of s that satisfies the above equation is the initial growth rate for the infection. Note that if more than one solution exists, we take the real part of the value of s with the largest real part.

2.5.3 The Final Size Equation – Small Epidemics

Using equation (2.65) in equation (2.7) we have:

$$\lim_{s\to 0}\bar{i}(s)=\lim_{s\to 0}\frac{i_0}{1-S(0)\frac{a\sqrt{\pi}}{2\sqrt{c}}e^{-T_1 s+\frac{s^2}{4c^2}}\left(1+\operatorname{erf}\left(\sqrt{c}\left(T_1-\frac{s}{2c}\right)\right)\right)}$$

$$=\lim_{s\to 0}\frac{2i_0\sqrt{c}}{2\sqrt{c}-aS(0)\sqrt{\pi}e^{-T_1 s+\frac{s^2}{4c^2}}\left(1+\operatorname{erf}\left(\sqrt{c}\left(T_1-\frac{s}{2c}\right)\right)\right)} \quad (2.68)$$

$$=\frac{2\sqrt{c}i_0}{2\sqrt{c}-aS(0)\sqrt{\pi}\left(1+\operatorname{erf}\left(\sqrt{c}T_1\right)\right)}$$

$$=\frac{i_0}{1-R_0}$$

From equation (2.9) we have

$$S(\infty)-S(0)=\frac{i_0}{1-R_0} \quad (2.69)$$

2.6 Distribution Six

$$A(\tau)=\begin{cases}\frac{a}{T_2-T_1}(\tau-T_1), & T_1\le\tau\le T_2\\ a, & T_2<\tau<T_3\\ \frac{-a}{T_4-T_3}(\tau-T_4), & T_3\le\tau\le T_4\\ 0. & otherwise\end{cases} \quad (2.70)$$

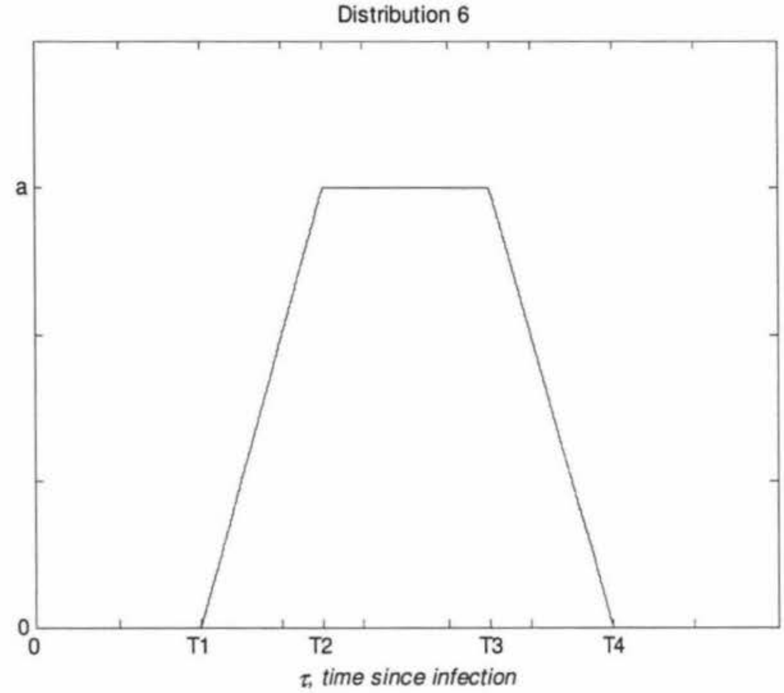

Figure 2.10 Distribution 6

2.6.1 The Basic Reproduction Ratio

Again using the relation:

$$R_0 = S(0)\int_0^{\infty} A(\tau)\,\mathrm{d}\tau \tag{2.71}$$

This is just *S(0)* multiplied by the area of the trapezium, so we have:

$$R_0 = \frac{S(0)a}{2}\left(-T_1 - T_2 + T_3 + T_4\right) \tag{2.72}$$

2.6.2 The Initial Growth Rate

To find the initial growth rate we use equation (2.5):

$$
\begin{aligned}
1 &= S(0)\int_0^t A(\tau)e^{-r\tau}\mathrm{d}\tau \\
&= S(0)\left[\left\{\frac{a}{T_2-T_1}\int_{T_1}^{T_2}(\tau-T_1)+a\int_{T_2}^{T_3}-\frac{a}{T_4-T_3}\int_{T_3}^{T_4}(\tau-T_4)\right\}e^{-r\tau}\mathrm{d}\tau\right] \\
&= \frac{-aS(0)}{r^2(T_2-T_1)(T_4-T_3)}\left(-e^{-T_1 r}(T_4-T_3)+e^{-T_2 r}(T_4-T_3)+e^{-T_3 r}(T_2-T_1)-e^{-T_4 r}(T_2-T_1)\right) \\
r^2 &= \frac{aS(0)}{(T_2-T_1)(T_4-T_3)}\left((T_4-T_3)\left(e^{-T_1 r}-e^{-T_2 r}\right)+(T_2-T_1)\left(e^{-T_4 r}-e^{-T_3 r}\right)\right)
\end{aligned}
\tag{2.73}
$$

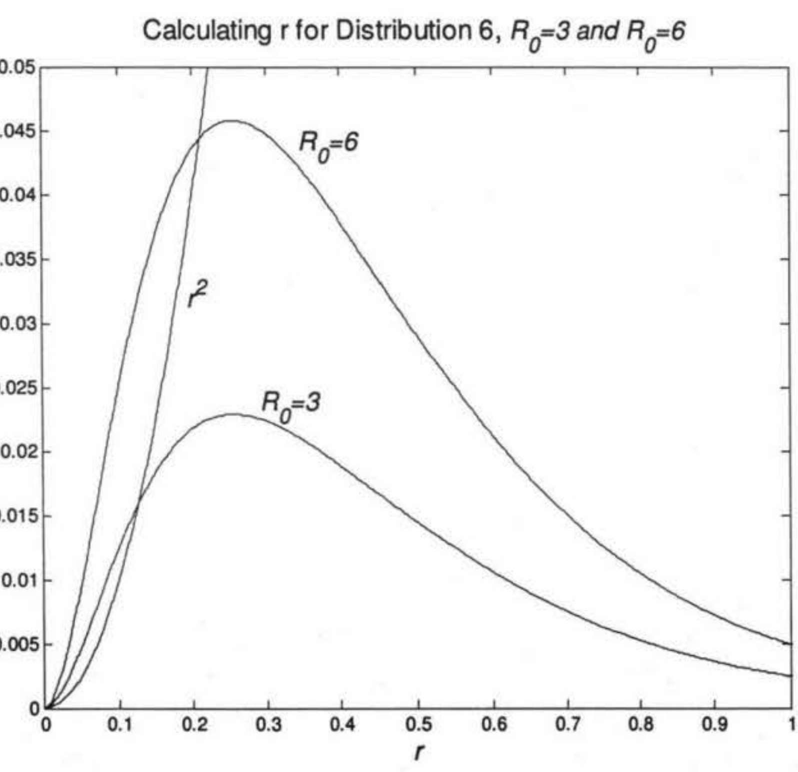

Figure 2.11 Example of calculating r for Distribution 6, given R_0=3 and 6. Parameter values are listed in Examples section to follow.

As we are considering the initial growth rate, we need only find the largest real positive value r that satisfies equation (2.73).

We may also using the method of Laplace transforms to calculate the initial growth rate when $S(t)\equiv S(0)$. The Laplace transform of the equation for the incidence of infection is:

$$\begin{aligned}\bar{i}(s)&=i_0+S(0)\bar{A}(s)\bar{i}(s)\\&=\frac{i_0}{1-S(0)\bar{A}(s)}\end{aligned}\tag{2.74}$$

Where

$$\begin{aligned}\bar{A}(s)&=\int_0^{\infty}e^{-s\tau}A(\tau)\mathrm{d}\tau\\&=\left[\left\{\frac{a}{T_2-T_1}\int_{T_1}^{T_2}(\tau-T_1)+a\int_{T_2}^{T_3}-\frac{a}{T_4-T_3}\int_{T_3}^{T_4}(\tau-T_4)\right\}e^{-s\tau}\mathrm{d}\tau\right]\\&=\frac{a}{s^2(T_2-T_1)(T_4-T_3)}\left((T_4-T_3)\left(e^{-T_1s}-e^{-T_2s}\right)+\left(e^{-T_4s}-e^{-T_3s}\right)(T_2-T_1)\right)\end{aligned}\tag{2.75}$$

So we gain

$$\bar{i}(s)=\frac{i_0}{1-\frac{a}{s^2(T_2-T_1)(T_4-T_3)}\left((T_4-T_3)\left(e^{-T_1s}-e^{-T_2s}\right)+\left(e^{-T_4s}-e^{-T_3s}\right)(T_2-T_1)\right)S(0)}\tag{2.76}$$

The initial growth rate of the infection is found by solving when the denominator of the above equation is zero:

$$s^2=\frac{a}{(T_2-T_1)(T_4-T_3)}\left((T_4-T_3)\left(e^{-T_1s}-e^{-T_2s}\right)+\left(e^{-T_4s}-e^{-T_3s}\right)(T_2-T_1)\right)\tag{2.77}$$

The value of s that satisfies the above equation is the initial growth rate for the infection; this can be found numerically using MATLAB.

2.6.3 The Final Size Equation – Small Epidemics

For a small epidemic, we are assuming that *S(t)* is constant, that is *S(t)=S(0)*. We use Laplace transforms on our equation for the incidence of infection, to gain equation (2.74). Where the Laplace transform of $A(\tau)$ is:

$$\bar{A}(s)=\frac{a}{s^2(T_2-T_1)(T_4-T_3)}\left((T_4-T_3)\left(e^{-T_1s}-e^{-T_2s}\right)+\left(e^{-T_4s}-e^{-T_3s}\right)(T_2-T_1)\right) \tag{2.78}$$

as shown previously. So we gain

$$\begin{aligned}\bar{i}(s)&=\frac{i_0}{1-S(0)\frac{a}{s^2(T_2-T_1)(T_4-T_3)}\left((T_4-T_3)\left(e^{-T_1s}-e^{-T_2s}\right)+\left(e^{-T_4s}-e^{-T_3s}\right)(T_2-T_1)\right)}\\&=\frac{i_0s^2(T_2-T_1)(T_4-T_3)}{s^2(T_2-T_1)(T_4-T_3)-S(0)a\left((T_4-T_3)\left(e^{-T_1s}-e^{-T_2s}\right)+\left(e^{-T_4s}-e^{-T_3s}\right)(T_2-T_1)\right)}\end{aligned} \tag{2.79}$$

To find the final size equation we use:

$$\int_0^\infty i(t)\,\mathrm{d}t=\lim_{s\to 0}\bar{i}(s) \tag{2.80}$$

To calculate the limit, we using L'Hôpital's Rule twice:

$$
\begin{aligned}
\lim_{s\to 0}\bar{i}(s) &= \lim_{s\to 0}\frac{i_0 s^2(T_2-T_1)(T_4-T_3)}{s^2(T_2-T_1)(T_4-T_3)-S(0)a\begin{pmatrix}(T_4-T_3)\left(e^{-T_1 s}-e^{-T_2 s}\right)\\ +\left(e^{-T_4 s}-e^{-T_3 s}\right)(T_2-T_1)\end{pmatrix}} \\
&= \lim_{s\to 0}\frac{2i_0 s(T_2-T_1)(T_4-T_3)}{2s(T_2-T_1)(T_4-T_3)-S(0)a\begin{pmatrix}(T_4-T_3)\left(-T_1e^{-T_1 s}+T_2e^{-T_2 s}\right)\\ +\left(-T_4e^{-T_4 s}+T_3e^{-T_3 s}\right)(T_2-T_1)\end{pmatrix}} \\
&= \lim_{s\to 0}\frac{2i_0(T_2-T_1)(T_4-T_3)}{2(T_2-T_1)(T_4-T_3)-S(0)a\begin{pmatrix}(T_4-T_3)\left(T_1^2e^{-T_1 s}-T_2^2e^{-T_2 s}\right)\\ +\left(T_4^2e^{-T_4 s}-T_3^2e^{-T_3 s}\right)(T_2-T_1)\end{pmatrix}} \\
&= \frac{2i_0(T_2-T_1)(T_4-T_3)}{2(T_2-T_1)(T_4-T_3)-S(0)a\left((T_4-T_3)\left(T_1^2-T_2^2\right)+\left(T_4^2-T_3^2\right)(T_2-T_1)\right)} \\
&= \frac{2i_0(T_2-T_1)(T_4-T_3)}{2(T_2-T_1)(T_4-T_3)-S(0)a\begin{pmatrix}(T_4-T_3)(T_1+T_2)(T_1-T_2)\\ +(T_4+T_3)(T_4-T_3)(T_2-T_1)\end{pmatrix}} \\
&= \frac{2i_0(T_2-T_1)(T_4-T_3)}{(T_2-T_1)(T_4-T_3)\left[2-S(0)a\left(-(T_1+T_2)+(T_4+T_3)\right)\right]} \\
&= \frac{2i_0}{\left[2-S(0)a\left(-(T_1+T_2)+(T_4+T_3)\right)\right]} \\
&= \frac{i_0}{1-R_0}
\end{aligned}
\tag{2.81}
$$

Therefore:

$$
S(0)-S(\infty)=\frac{i_0}{1-R_0} \tag{2.82}
$$

2.7 Examples

Below is a table showing the initial growth rates for the six distributions, when R_0=3 then R_0=6 and $S(0)$=1000. As $S(\infty)$ is not dependent on the distribution, only on the values of R_0 and $S(0)$ we will know that it will be the same for every distribution.

When R_0=*3*, solving the nonlinear equation gives us $S(\infty) = 59.52$.

Distribution		*Parameters*	*r*
1	Shifted Rectangle	T_1=4, T_2=10,	0.1626
2	Rectangle at zero	T_1=10, a=0.0003	0.2821
3	Negative exponential	a=0.003, c=1	2
4	Gamma Function	a=0.00005, c=0.4082	0.2989
5	Shifted normal	a=0.01, c=34.9, T_1=5	0.2198
6	Trapezium	T_1=4, T_2=7, T_3=11, T_4=14, a=0.00042	0.1263

Table 2.1 Initial growth rates for the six distributions when R_0=3

When R_0=*6* solving the nonlinear equation gives us $S(\infty) = 2.5165$.

Distribution		*Parameters*	*r*
1	Shifted rectangle	T_1=4, T_2=10, a=0.001	0.2423
2	Rectangle at zero	T_1=10, a=0.0006	0.5985
3	Negative exponential	a=0.006, c=1	5
4	Gamma Function	a=0.0005, c=0.2887	0.4184
5	Shifted normal	a=0.01, c=8.699, T_1=5	0.3594
6	Trapezium	T_1=4, T_2=7, T_3=11, T_4=14, a=0.00086	0.2108

Table 2.2 Initial growth rates for the six distributions when R_0=6

Chapter 3 Cobwebs – The One Dimensional Model

When an infection is introduced into a population, an epidemic may or may not occur depending on the number of susceptibles within the population. If an epidemic does occur, the next question is will there be another epidemic in the following year or will it develop into some form of cycle where there is an epidemic every n^{th} year?

We can develop a discrete map for the susceptible population from epidemic generation to epidemic generation.[9] From our work in the previous chapter we can calculate the size of the susceptible population after an epidemic has occurred, when the basic reproduction ratio is given.

Initially the entire population is susceptible to the infection. We then let an epidemic occur, and calculate $S(\infty)$ - the number of susceptibles in the population when the epidemic is finished. We let a portion of these susceptibles stay in the population, and introduce new susceptibles to keep the population size constant. By calculating the new basic reproduction ratio for this population, we see whether another epidemic occurs. At the end of each epidemic we assume there are no infectives from the previous epidemic present.

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[9] This idea initially came from Andreason (2003), although in his paper new infection is introduced in each generation.

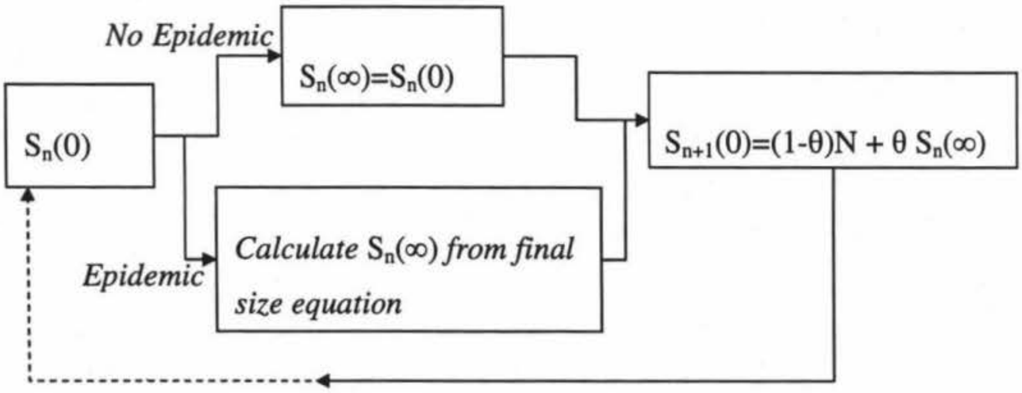

Figure 3.1 Schematic for discrete map from epidemic generation to generation

An example of when this method can be applied is with children at a school. Suppose all the children are susceptible to an infection, and the reproduction ratio of a certain infection is greater than one – then an epidemic occurs within the school. At the end of the school year, a proportion of the children will leave the school and new students will attend. We then let another epidemic occur. A 'reshuffling' of susceptibles will occur at the end of every epidemic, and then another epidemic may or may not occur depending on the new basic reproduction ratio.

3.1 The Method

As we have seen in the previous chapter, for every distribution considered, the basic reproduction ratio is:

$$R_0 = S(0)\int_0^\infty A(\tau)\,\mathrm{d}\tau \tag{3.1}$$

The final size equation is given by:

$$\log\left(\frac{S(\infty)}{S(0)}\right) = R_0\left(\frac{S(\infty)}{S(0)} - 1\right) \tag{3.2}$$

where $S(\infty)$ is the number of susceptibles left in the population after the epidemic, and $S(0)$ is the initial number of susceptibles. After each epidemic, we assume that a proportion θ of the final number of susceptibles continue into the next epidemic.

Let N be the population size, which we assume remains constant. Initially we have $S_{0,0} = N$, that is, the susceptible population before the first epidemic is just the entire population. The proportion of susceptibles in the entire population is then $x_0 = \frac{S_{0,0}}{N} = 1$, and the basic reproduction ratio is $R_0 = N\int_0^\infty A(\tau)\,\mathrm{d}\tau$. If $R_0 < 1$ there is no epidemic, and the proportion of susceptibles in the population at the next generation will become:

$$x_1 = 1 - \theta + \theta x_0 \tag{3.3}$$

and therefore, $x_1 = 1$ .However, if $R_0 > 1$ then there will be an epidemic and we must solve the final size equation.

$$\log\left(\frac{z}{x_0}\right) = R_0\left(\frac{z}{x_0} - 1\right) \tag{3.4}$$

where $z = \frac{S_{\infty,0}}{N}$. In this case, we then replace x_0 with z in equation (3.3).

We can then repeat this process for subsequent epidemics. So in general we have:

$$\begin{aligned} x_n &= 1 - \theta + \theta z_{n-1} \\ R_n &= x_n R_0 \end{aligned} \tag{3.5}$$

If $R_n > 1$ we have an epidemic, so we solve:

$$\log\left(\frac{z_n}{x_n}\right) = x_n R_0 \left(\frac{z_n}{x_n} - 1\right) \tag{3.6}$$

for z_n. This has a unique solution[10] $z_n \neq x_n$. If $R_n < 1$ then there is no epidemic, and we set $z_n = x_n$.

The map is

$$x_{n+1} = f(x_n)$$

Where

$$f(x_n) = \begin{cases} 1-\theta+\theta z_n, & \text{if } x_n > \dfrac{1}{R_0}, \text{ i.e. epidemeic} \\ 1-\theta+\theta x_n, & \text{otherwise, i.e. no epidemic} \end{cases} \tag{3.7}$$

and z_n solves equation (3.6).

Some plots of this map are shown below, with a cobweb added to show how the solutions converge to a steady state. The cobweb plot shows the convergence of repeated iterations of the piecewise continuous function given in equation (3.7), when the entire population is initially susceptible.

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[10] See Diekmann & Heesterbeek (2000) for a complete proof

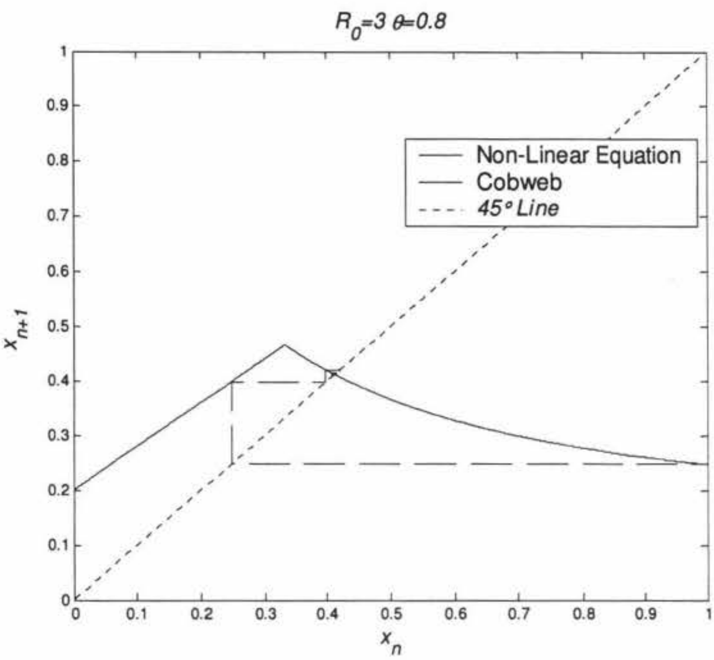

Figure 3.2 Cobweb plot, R_0=3, θ=0.8, S_0=N=1000, x_0=1

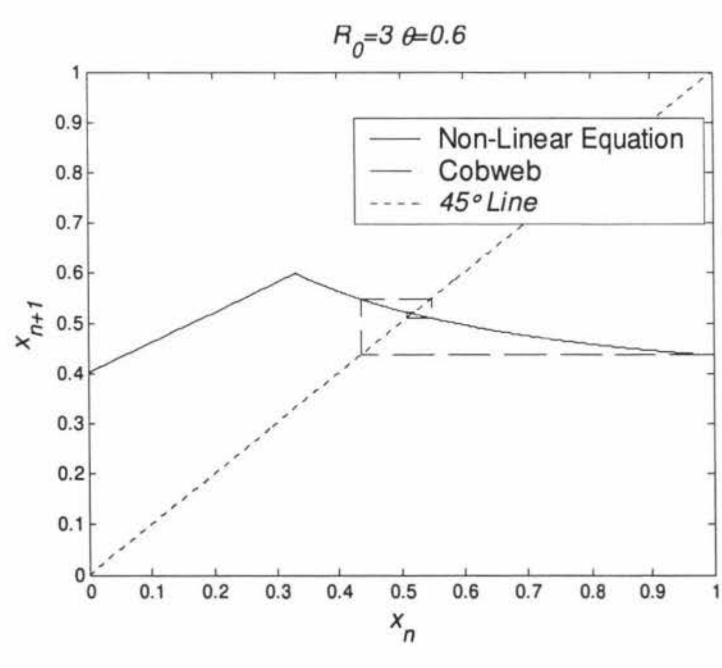

Figure 3.3 Cobweb plot, R_0=3, θ=0.6, S_0=N=1000, x_0=1

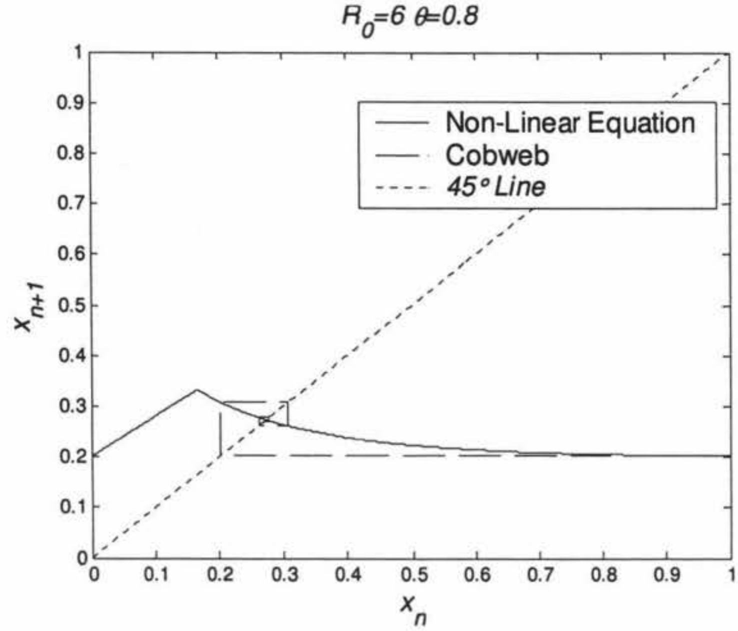

Figure 3.4 Cobweb plot, R_0=6, θ=0.8, S_0=N=1000, x_0=1

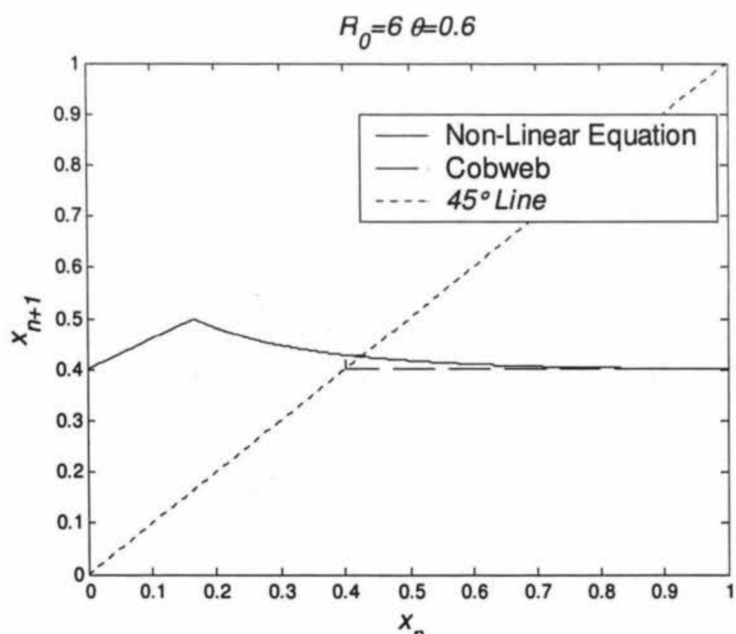

Figure 3.5 Cobweb plot, R_0=6, θ=0.6, S_0=N=1000, x_0=1

3.2 The Stability of the Fixed Point

In Figure 3.2-Figure 3.5, note that there is a straight line part corresponding to no epidemics. We know that there is no epidemic if the basic reproduction ratio is less than one for that generation, that is

$$R_n = x_n R_0 < 1 \tag{3.8}$$

So we see that the straight line spans $0 < x_n < \frac{1}{R_0}$.

If we let the straight line part of the cobweb be $g(x) = 1 - \theta + \theta x$ and the curved part be the solution to

$$\log\left(\frac{x_{n+1} - (1-\theta)}{\theta x_n}\right) = x_n R_0 \left(\frac{x_{n+1} - (1-\theta)}{\theta x_n} - 1\right)$$

defined by the function $x_{n+1} = f(x_n)$, we have:

$$\log\left(\frac{f(x_n) - (1-\theta)}{\theta x_n}\right) = R_0 \left(\frac{f(x_n) - (1-\theta)}{\theta} - x_n\right) \tag{3.9}$$

Note that *g(x)* and the line *y=x* only intersect at *x*=0 and *x*=1, so the steady state must lie at some value $\frac{1}{R_0} < X < 1$, where X is our steady state.

To find the stability of the fixed point (although we can gather from the cobweb plots that it is stable), we find the derivative of *f(x)*.

$$\frac{\theta x}{f(x) - (1-\theta)} \left(\frac{f'(x)}{\theta x} - \frac{f(x) - (1-\theta)}{\theta x^2}\right) = \frac{R_0}{\theta}\left(f'(x) - \theta\right) \tag{3.10}$$

Note that at the fixed point we have *f(X)=X*, so we can simplify the above:

$$\frac{\theta X}{X-(1-\theta)}\left(\frac{f'(X)X-X+(1-\theta)}{\theta X^2}\right)=\frac{R_0}{\theta}\left(f'(X)-\theta\right)$$

$$f'(X)\left(\frac{1}{X-(1-\theta)}-\frac{R_0}{\theta}\right)=\frac{1}{X}-R_0$$

$$f'(X)\left(\frac{\theta-R_0\left(X-(1-\theta)\right)}{\theta\left(X-(1-\theta)\right)}\right)=\frac{1-R_0X}{X}$$

$$f'(X)=\frac{\theta(1-R_0X)\left(X-(1-\theta)\right)}{X\left(\theta-R_0\left(X-(1-\theta)\right)\right)} \qquad (3.11)$$

We can also use (3.9) to find an expression for R_0 at the steady state.

$$\theta\log\left(\frac{X-(1-\theta)}{\theta X}\right)=R_0\left(X-(1-\theta)-\theta X\right)$$

$$\theta\log\left(\frac{X-(1-\theta)}{\theta X}\right)=-R_0(1-\theta)(1-X)$$

$$R_0=\frac{\theta}{(1-\theta)(1-X)}\log\left(\frac{\theta X}{X-(1-\theta)}\right) \qquad (3.12)$$

Recall that R_0 and θ are usually known and we are solving for the fixed point X. We know that the fixed point lies within the range $\left(\frac{1}{R_0},1\right)$, so by using equation (3.12) we can calculate the basic reproduction ratio at every fixed point, and then use this information to calculate equation (3.11) – a back-to-front way compared to how we have been working! We have used this method to produce Figure 3.6. The first plot (in Figure 3.6) shows the value of R_0 at varying fixed points X and selected values of θ. The line

$X = \frac{1}{R_0}$ has been added just to emphasize the fact that fixed point lies in the range $\left(\frac{1}{R_0}, 1\right)$. The second plot shows *f'(X)* at varying fixed points *X* and different values of θ.

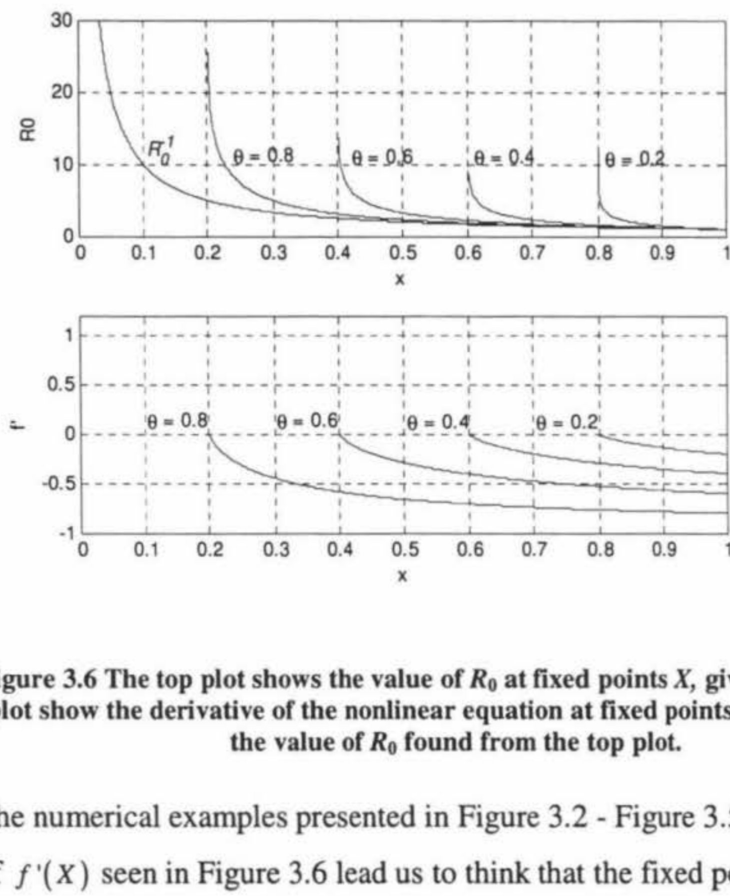

Figure 3.6 The top plot shows the value of R_0 at fixed points *X*, given θ. The second plot show the derivative of the nonlinear equation at fixed points *X*, given θ using the value of R_0 found from the top plot.

The numerical examples presented in Figure 3.2 - Figure 3.5 and the values of $f'(X)$ seen in Figure 3.6 lead us to think that the fixed point will always be stable. We shall now establish this with an analytical proof.

If we let *X*=1 in equation (3.11) we gain:

$$f'(1) = \frac{\theta(1-R_0)(1-(1-\theta))}{(\theta - R_0(1-(1-\theta)))} = \frac{\theta^2(1-R_0)}{\theta - \theta R_0} = \theta \qquad (3.13)$$

However, from the bottom plot in Figure 3.6, we can see that $f'(1) < 0$. We have neglected the fact that R_0 is also dependent on X. The top plot in Figure 3.6 suggests that R_0=1 at X=1. To calculate R_0 when X=1, we let $X=1-\varepsilon$ in equation (3.12):

$$\begin{aligned} R_0 &= \frac{\theta}{(1-\theta)(1-(1-\varepsilon))}\log\left(\frac{\theta(1-\varepsilon)}{1-\varepsilon-(1-\theta)}\right) \\ &= \frac{\theta}{\varepsilon(1-\theta)}\log\left(\frac{\theta(1-\varepsilon)}{\theta-\varepsilon}\right) \\ &= \frac{\theta}{\varepsilon(1-\theta)}\left[\log\theta+\log(1-\varepsilon)-\log(\theta-\varepsilon)\right] \end{aligned} \tag{3.14}$$

Expanding in powers of ε we gain:

$$R_0 = 1+\frac{\varepsilon}{2\theta}(1+\theta)+O\left(\varepsilon^2\right) \tag{3.15}$$

Claim 1: $f'(1) = -\theta$.

Proof: Let $X = 1-\varepsilon$, then we already have $R_0 \approx 1+\frac{\varepsilon}{2\theta}(1+\theta)$. Note that

$$R_0 X = 1-\varepsilon+\frac{\varepsilon(1+\theta)}{2\theta}+O\left(\varepsilon^2\right)$$

Substituting this into equation (3.11):

$$f'(1-\varepsilon)=\frac{\theta\left(\varepsilon-\frac{\varepsilon(1+\theta)}{2\theta}\right)(\theta-\varepsilon)}{(1-\varepsilon)\left(\varepsilon-\frac{\varepsilon(1+\theta)}{2}\right)}$$
$$=\frac{(2\varepsilon\theta-\varepsilon(1+\theta))(\theta-\varepsilon)}{(1-\varepsilon)(2\varepsilon-\varepsilon-\varepsilon\theta)}$$
$$=\frac{-(1-\theta)(\theta-\varepsilon)}{(1-\varepsilon)(1-\theta)}$$
$$=\frac{-(\theta-\varepsilon)}{1-\varepsilon}$$

By then letting $\varepsilon\to 0$, we gain $f'(1)=-\theta$

□

By letting $X=\frac{1}{R_0}$ in equation (3.11), we have $f'\left(\frac{1}{R_0}\right)\equiv 0$, as there is a zero in the numerator.

To show that the fixed point is stable, we must show $|f'(X)|<1$ within our range of consideration, $\frac{1}{R_0}<X<1$, where $R_0>1$ as we have an epidemic. We first show that $f'(X)$ is continuous within this range.

Claim 2: $f'(X)$ is continuous over $X\in\left(\frac{1}{R_0},1\right)$.

Proof: From equation (3.11), we can see that $f'(X)$ has a singularity when

$$X\left(\theta-R_0\left(X-(1-\theta)\right)\right)=0 \qquad (3.16)$$

i.e. when $X=0$ or

$$\theta - R_0\left(X - (1-\theta)\right) = 0 \tag{3.17}$$

As $X = 0$ is not in our range, we need only consider the solution to equation (3.17):

$$X = \frac{\theta}{R_0} + 1 - \theta \tag{3.18}$$

At this point we know equation (3.12) holds, therefore:

$$\begin{aligned} R_0 &= \frac{\theta}{(1-\theta)\left(1 - \frac{\theta}{R_0} - 1 + \theta\right)} \log\left(\frac{\theta\left(\frac{\theta}{R_0} + 1 - \theta\right)}{\frac{\theta}{R_0} + 1 - \theta - 1 + \theta}\right) \\ R_0 &= \frac{1}{(1-\theta)\left(1 - \frac{1}{R_0}\right)} \log\left(R_0\left(\frac{\theta}{R_0} + 1 - \theta\right)\right) \end{aligned} \tag{3.19}$$

$$R_0(1-\theta) - (1-\theta) = \log\left(\theta + R_0(1-\theta)\right) \tag{3.20}$$

Let $\psi = R_0(1-\theta) + \theta$ then the above equation becomes

$$\psi - 1 = \log\psi \tag{3.21}$$

Now let $p = \psi - 1$ and $q = \log\psi$, for $\psi > 0$. Figure 3.7 shows p and q plotted together.

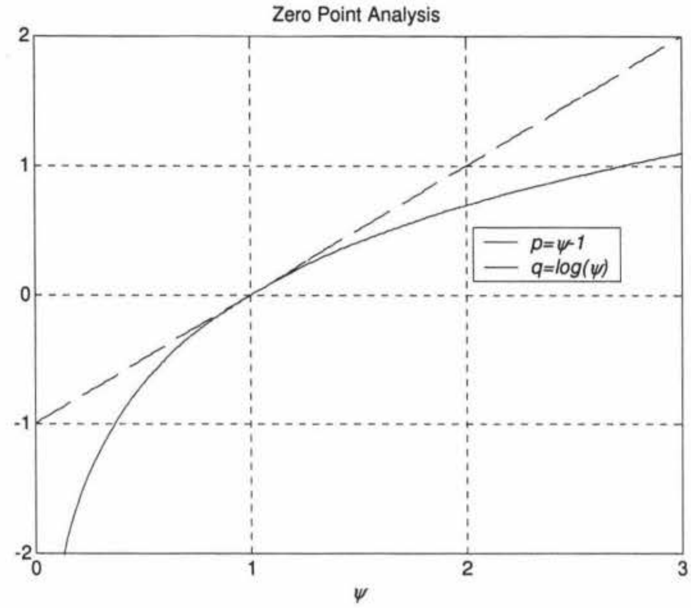

Figure 3.7 Plot to depict where the zero point of *f'(X)* lies

The values of p and q are equal at the point $\psi = 1$. Note that as $p' = 1$ and $q' = \frac{1}{\psi}$, it is easy to see that the lines are tangent when $\psi = 1$. From their definitions, we know that $p > q$ everywhere else. The tangent point, $\psi = 1$, gives:

$$\begin{aligned} R_0(1-\theta) + \theta = 1 \\ R_0 = \frac{1-\theta}{1-\theta} \equiv 1 \end{aligned} \tag{3.22}$$

As stated previously, the fixed point lies in the range, $\frac{1}{R_0} < X < 1$, so the singularity is not in our range of consideration.

□

From the above proof we have gained the new fact:

$$X > 1-\theta+\frac{\theta}{R_0} \tag{3.23}$$

We have now seen that $f'(X)$ is continuous over $X \in \left(\frac{1}{R_0},1\right)$,

$f'\left(\frac{1}{R_0}\right)=0$ and $f'(1)=-\theta$. So to prove that $|f'(X)|<1$ (that is, the fixed point is stable[11]), we only need to show that $f'(X)$ has no turning points over $X \in \left(\frac{1}{R_0},1\right)$.

Claim 3: $f'(X)$ has no critical points over $X \in \left(\frac{1}{R_0},1\right)$.

Proof: We have

$$f'(X)=\frac{\theta(1-R_0X)\left(X-(1-\theta)\right)}{X\left(\theta-R_0\left(X-(1-\theta)\right)\right)}$$

With some calculation it can be shown that:

$$\begin{aligned}\frac{\mathrm{d}f'(X)}{\mathrm{d}X}&=\frac{-\theta(-1+\theta)\left(\theta-R_0\theta+R_0-2R_0X+R_0X^2\right)}{X^2\left(-\theta+R_0X-R_0+R_0\theta\right)^2}\\&=\frac{\theta(1-\theta)\left(\theta(1-R_0)+R_0X(X-2)+R_0\right)}{X^2\left(\theta(R_0-1)+R_0(X-1)\right)^2}\end{aligned} \tag{3.24}$$

A critical point must satisfy $\frac{\mathrm{d}f'(X)}{\mathrm{d}X}=0$, i.e.

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[11] See Strogatz (1994) pp 349-350 for a proof of stability analysis.

$$\begin{aligned}\theta(1-R_0)+R_0X(X-2)+R_0=0\\ R_0X^2-2R_0X+R_0+\theta(1-R_0)=0\end{aligned} \tag{3.25}$$

Solving this quadratic in X yields

$$\begin{aligned}X&=\frac{2R_0\pm\sqrt{4R_0^2-4R_0\left(R_0+\theta(1-R_0)\right)}}{2R_0}\\ &=\frac{R_0\pm\sqrt{R_0\theta(R_0-1)}}{R_0}\\ &=1\pm\sqrt{\frac{\theta(R_0-1)}{R_0}}\end{aligned} \tag{3.26}$$

Recall that we are only considering $X\in\left(\frac{1}{R_0},1\right)$, so we look at

$$X=1-\sqrt{\frac{\theta(R_0-1)}{R_0}} \tag{3.27}$$

We have two possibilities:

If $\frac{\theta(R_0-1)}{R_0}\geq 1$, then from equation (3.27) X would be negative (or zero) – which is not in our range of consideration.

If $\frac{\theta(R_0-1)}{R_0}<1$ then :

$$\begin{aligned}X&=1-\sqrt{\frac{\theta(R_0-1)}{R_0}}\\ &<1-\frac{\theta(R_0-1)}{R_0}\end{aligned} \tag{3.28}$$

Which contradicts equations (3.23).

Therefore $f'(X)$ has no critical points over $X \in \left(\frac{1}{R_0}, 1\right)$.

□

We have shown $f'(1) = -\theta, f'\left(\frac{1}{R_0}\right) = 0$, the singularity $X = 1 - \theta + \frac{R_0}{\theta}$ in equation (3.11) does not lie in the region we are considering, $\frac{1}{R_0} < X < 1$ and that *f'(X)* does not have any turning points within this range. So $0 \geq f'(X) \geq -\theta > -1$, thus the fixed point *X* that satisfies equation (3.11) is stable.

This means that when the infection is introduced into a fully susceptible population with the basic reproduction ratio greater than one, and the population size remains constant after each epidemic, then there will be an epidemic each and every year.

Chapter 4 The Two Dimensional Model – Epidemic Properties

We will now consider the population being split into two distinct classes: for example, people in the susceptible population who are under 30 and those who are above, or a male female spilt. We will look at the same six distributions as discussed earlier, and calculate the basic reproduction ratio and the initial growth rate for each distribution.

To calculate the basic reproduction ratio we look at the spectral radius (largest eigenvalue in absolute value) of the next generation matrix[12] K:

$$K = S(t)C\int_0^\infty A(\tau)\mathrm{d}\tau \tag{4.1}$$

where

$$S(t) = \begin{pmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{pmatrix} \tag{4.2}$$

$$C = \begin{pmatrix} C_{11} & C_{21} \\ C_{21} & C_{22} \end{pmatrix} \tag{4.3}$$

and

$$A(\tau) = \begin{pmatrix} A_1(\tau) & 0 \\ 0 & A_2(\tau) \end{pmatrix} \tag{4.4}$$

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[12] Refer to Diekmann and Heesterbeek (2000), pp74-76 for a detailed discussion of this

$S_j(t)$ (j=1,2) are the susceptible populations for each class and C is the mixing matrix, i.e. C_{ij} describes the contacts between classes i and j. For all our examples, we shall assume that $A_1(\tau) = A_2(\tau) = A(\tau)$. Recall from our definition for the one dimensional problems, that $A(\tau) = \chi(\tau)\, p(\tau)$, where $\chi(\tau)$ is the contact rate with infectives and $p(\tau)$ is the probability that a contact with an infective will lead to infection. The matrix C then describes the mixing between the classes, and the contact rate is included in the function $A(\tau)$.

Previously we have always assumed that the whole population is initially susceptible, as the definition of the basic reproduction ratio is the number of secondary cases that arise from a primary case in a susceptible population. So R_0 becomes:

$$R_0 = \rho(K) = \rho\left(S(0)\int_0^\infty A(\tau)\mathrm{d}\tau\right) \tag{4.5}$$

Where $\rho(K)$ is the spectral radius (dominant eigenvalue) of K. To calculate R_0 we solve the equation:

$$\det(K - \lambda I) = 0 \tag{4.6}$$

That is:

$$\begin{aligned} &\lambda^2 - \lambda\,\mathrm{tr}(K) + \det(K) = 0 \\ &2\lambda = \mathrm{tr}(K) \pm \sqrt{\mathrm{tr}(K)^2 - 4\det(K)} \end{aligned} \tag{4.7}$$

As the susceptible population is now split into two classes, we may consider different types of interaction between these classes. In particular we shall calculate the basic reproduction ratio for four special cases of interaction: preferential mixing, sexual mixing, proportionate mixing and reduced proportionate mixing. For each of these special cases, we only need to change the mixing matrix C.

Preferential Mixing

Preferential mixing occurs when there is only mixing within each class and no mixing between classes; for example people in class one will only mix with other people in class one. This can be seen in our model by letting $C_{12}=C_{21}=0$ in our mixing matrix. Note that this system becomes decoupled at this stage, and we are just solving two one dimensional problems.

Sexual Mixing

Sexual mixing is when there is only mixing between the two classes, and none within each class. We can achieve this by letting $C_{11}=C_{22}=0$ in our model. This type of interaction can be seen in host parasite models. In Roberts & Heesterbeek (2003), a model is given for dengue fever where the infection can only be passed from human to mosquito, and vice versa, but there is no transmission from mosquito to mosquito or from human to human.

Proportionate Mixing

When the mixing rate between classes is proportional to the product of the mixing rates within each class, we have proportionate mixing. In our model, this refers to the case when $C_{12}=C_{21}=\sqrt{C_{11}C_{22}}$.

Reduced Proportionate Mixing

This is similar to proportionate mixing, but we now weight the inter-class mixing by a parameter ε, which is a constant less than one. In our model this equates to $C_{12} = C_{21} = \varepsilon\sqrt{C_{11}C_{22}}$. An example of this mixing structure can be seen in Roberts & Tobias (2000), where a mixing matrix is constructed using reduced proportionate mixing to relate four distinct classes within the susceptible population.

The equation for the incidence of infection is now:

$$\underline{i}(t) = \underline{i}_0\delta(t) + S(t)C\int_0^t A(\tau)\underline{i}(t-\tau)\mathrm{d}\tau \qquad (4.8)$$

Where

$$\underline{i}(t) = \begin{pmatrix} i_1(t) \\ i_2(t) \end{pmatrix} \qquad (4.9)$$

$i_j(t)$ (j=1,2) is the incidence of infection in each of the two classes. We can use the method of Laplace transforms in the same manner as we did for the one dimensional problem to calculate the initial growth rate for the infection. As we are now dealing with matrices we have:

$$\overline{\underline{i}(s)} = \left[I - \overline{A(s)} \begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12} \\ S_2(0)C_{21} & S_2(0)C_{22} \end{pmatrix} \right]^{-1} \underline{i}_0 \qquad (4.10)$$

The initial growth rate is the value of s such that the above matrix is not invertible, i.e. the determinant is equal to zero. So we solve:

$$\det(W) = 0 \qquad (4.11)$$

where

$$W = \left(I - \bar{A}(s) \begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12} \\ S_2(0)C_{21} & S_2(0)C_{22} \end{pmatrix} \right) \tag{4.12}$$

We have already calculated the Laplace transform for our six distributions in Chapter 2, so to calculate the initial growth rate for each distribution we need only solve equation (4.11).

We shall now calculate the basic reproduction ratio (and the four special cases for the mixing matrix) and the initial growth rate for our six distributions. At the end of each distribution there are numerical examples for all the cases given.

4.1 Distribution One

$$A(\tau) = \begin{cases} 0 & \tau < T_1 \\ a & T_1 \le \tau \le T_2 \\ 0 & T_2 < \tau \end{cases} \tag{4.13}$$

To find R_0 we calculate the spectral radius of the next generation matrix, K.

$$\begin{aligned} K &= \begin{pmatrix} S_1(0) & 0 \\ 0 & S_2(0) \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \int_0^\infty A(\tau) d\tau \\ &= \begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12} \\ S_2(0)C_{21} & S_2(0)C_{22} \end{pmatrix} a(T_2 - T_1) \end{aligned} \tag{4.14}$$

From equation (4.7) we need to solve:

$$2\lambda = \operatorname{tr}(K) \pm \sqrt{\operatorname{tr}(K)^2 - 4\det(K)} \tag{4.15}$$

That is:

$$2\lambda = a(T_2 - T_1)\left(S_1(0)C_{11} + S_2(0)C_{22}\right) \pm \sqrt{\begin{aligned}&a^2(T_2 - T_1)^2\left(S_1(0)C_{11} + S_2(0)C_{22}\right)^2\\ &-4a^2(T_2 - T_1)^2 S_1(0)S_2(0)(C_{11}C_{22} - C_{12}C_{21})\end{aligned}} \tag{4.16}$$

Simplifying the above equation yields:

$$2\lambda = a(T_2 - T_1)\left[\begin{aligned}&\left(S_1(0)C_{11} + S_2(0)C_{22}\right) \pm\\ &\sqrt{\left(S_1(0)C_{11} - S_2(0)C_{22}\right)^2 + 4S_1(0)S_2(0)C_{12}C_{21}}\end{aligned}\right] \tag{4.17}$$

Thus, the basic reproduction ratio is:

$$R_0 = \max\left(\frac{a}{2}(T_2 - T_1)\left(\begin{aligned}&\left(S_1(0)C_{11} + S_2(0)C_{22}\right)\\ &\pm\sqrt{(S_1(0)C_{11} - S_2(0)C_{22})^2 + 4S_1(0)S_2(0)C_{12}C_{21}}\end{aligned}\right)\right) \tag{4.18}$$

As we are taking the maximum value, we will always take the positive root that will be determined by the relative magnitudes of $S_1(0)C_{11}$ and $S_2(0)C_{22}$. See the preferential mixing case below for further clarification.

4.1.1 Special Cases

For the four special cases listed below, we only need to change the values of C_{ij} in equation (4.18).

Preferential Mixing

This is when there is only mixing within each class: $C_{21} = C_{12} = 0$

$$\begin{aligned}R_{pref} &= \frac{a}{2}(T_2 - T_1)\max\left(\left(S_2(0)C_{22} + S_1(0)C_{11}\right) \pm \sqrt{(S_2(0)C_{22} - S_1(0)C_{11})^2}\right)\\ &= a(T_2 - T_1)\max\left(S_1(0)C_{11}, S_2(0)C_{22}\right)\\ &= \max(R_{pref1}, R_{pref2})\end{aligned} \tag{4.19}$$

Where

$$\begin{aligned}R_{pref1} &= a(T_2 - T_1)S_1(0)C_{11}\\ R_{pref2} &= a(T_2 - T_1)S_2(0)C_{22}\end{aligned} \tag{4.20}$$

Sexual Mixing

We now let $C_{11} = C_{22} = 0$ in equation (4.18), which gives us:

$$R_{sex} = a(T_2 - T_1)\sqrt{S_1(0)S_2(0)C_{21}C_{12}} \tag{4.21}$$

Proportionate Mixing

If we let the mixing between the two different classes be proportionate to the mixing between two same classes, i.e. $C_{12} = C_{21} = \sqrt{C_{11}C_{22}}$, in equation (4.18), we have:

$$\begin{aligned}R_{prop} &= \frac{a}{2}(T_2 - T_1)\max\left(\left(S_2(0)C_{22} + S_1(0)C_{11}\right) \pm \sqrt{\begin{matrix}(S_2(0)C_{22} - S_1(0)C_{11})^2\\ +4S_1(0)S_2(0)C_{11}C_{22}\end{matrix}}\right)\\ &= \frac{a}{2}(T_2 - T_1)\left(\left(S_2(0)C_{22} + S_1(0)C_{11}\right) + \sqrt{(S_2(0)C_{22} + S_1(0)C_{11})^2}\right)\\ &= a(T_2 - T_1)\left(S_2(0)C_{22} + S_1(0)C_{11}\right)\\ &= R_{pref1} + R_{pref1}\end{aligned} \tag{4.22}$$

Reduced Proportionate Mixing

This is similar to the above, but we introduce a constant ε, such that $C_{12} = C_{21} = \varepsilon\sqrt{C_{11}C_{22}}$. Substituting this into equation (4.18), gives:

$$
\begin{aligned}
R_{prop} &= \frac{a}{2}(T_2 - T_1)\max\left((S_2(0)C_{22} + S_1(0)C_{11}) \pm \sqrt{\begin{matrix}(S_2(0)C_{22} - S_1(0)C_{11})^2 \\ +4\varepsilon^2 S_1(0)S_2(0)C_{11}C_{22}\end{matrix}} \right) \\
&= \frac{a}{2}(T_2 - T_1)\left((S_2(0)C_{22} + S_1(0)C_{11}) + \sqrt{(S_2(0)C_{22} - S_1(0)C_{11})^2 + 4\varepsilon^2 S_1(0)S_2(0)C_{11}C_{22}} \right) \\
&= \frac{R_{prop}}{2} + \frac{a}{2}(T_2 - T_1)\sqrt{(S_2(0)C_{22} - S_1(0)C_{11})^2 + 4\varepsilon^2 S_1(0)S_2(0)C_{11}C_{22}} \\
&= \frac{R_{pref1} + R_{pref2}}{2} + \frac{1}{2}\sqrt{\left(R_{pref1} - R_{pref2}\right)^2 + 4\varepsilon^2 R_{pref1}R_{pref2}} \\
&= \frac{1}{2}\left(R_{prop} + \sqrt{\left(R_{pref1} - R_{pref2}\right)^2 + 4\varepsilon^2 R_{pref1}R_{pref2}} \right)
\end{aligned}
\qquad (4.23)
$$

4.1.2 Initial Growth Rate

As we have seen before, the Laplace transform of $A(\tau)$ is:

$$\overline{A(s)} = \frac{a}{s}\left(e^{-sT_1} - e^{-sT_2}\right) \qquad (4.24)$$

Substituting this into equation (4.10):

$$\overline{\underline{i}(s)} = \left[I - \overline{A(s)}\begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12} \\ S_2(0)C_{21} & S_2(0)C_{22} \end{pmatrix} \right]^{-1} \underline{i}_0 \qquad (4.25)$$

We then find when the matrix is not invertible:

$$\det\left[I-\frac{a}{s}\left(e^{-sT_1}-e^{-sT_2}\right)\begin{pmatrix}S_1(0)C_{11} & S_1(0)C_{12}\\ S_2(0)C_{21} & S_2(0)C_{22}\end{pmatrix}\right]=0 \tag{4.26}$$

We solve the following for *s*:

$$\begin{aligned}\left(1-\frac{a}{s}\left(e^{-sT_1}-e^{-sT_2}\right)S_1(0)C_{11}\right)\left(1-\frac{a}{s}\left(e^{-sT_1}-e^{-sT_2}\right)S_2(0)C_{22}\right)\\ -\frac{a^2}{s^2}\left(e^{-sT_1}-e^{-sT_2}\right)^2 S_1(0)S_2(0)C_{12}C_{21}=0\\ 1-\frac{a}{s}\left(e^{-sT_1}-e^{-sT_2}\right)S_1(0)C_{11}-\frac{a}{s}\left(e^{-sT_1}-e^{-sT_2}\right)S_2(0)C_{22}\\ +\frac{a^2}{s^2}\left(e^{-sT_1}-e^{-sT_2}\right)^2 S_1(0)S_2(0)C_{12}C_{21}=0\end{aligned} \tag{4.27}$$

As we are solving for *s*, which will be the initial growth rate of the infection, we require $s \neq 0$. So we may multiply equation (4.27) throughout by s^2.

$$s^2-as\left(e^{-sT_1}-e^{-sT_2}\right)\left(S_1(0)C_{11}+S_2(0)C_{22}\right)+a^2\left(e^{-sT_1}-e^{-sT_2}\right)^2 S_1(0)S_2(0)C_{12}C_{21}=0 \tag{4.28}$$

This can be solved numerically using MATLAB.

4.1.3 Examples

The table below shows some numerical examples of the basic reproduction ratio and the initial growth rate for the four special cases given above, and one general case. In each example, the basic reproduction ratio is approximately 3 – this is not an exact value as we chose convenient values for parameters that gave approximately the same R_0 in each case.

	a	$S_1(0)$	$S_2(0)$	C_{11}	C_{12}	C_{21}	C_{22}	R_0	r
General Case	0.15	8	9	0.2	0.15	0.1	0.25	3.1646	0.1319
Preferential Mixing	0.3	7	9	0.2	0	0	0.15	2.94	0.1190
Sexual Mixing	0.25	5	10	0	0.25	0.3	0	3.3889	0.1400
Proportionate Mixing	0.15	7	9	0.2	$\sqrt{C_{11}C_{22}}$	$\sqrt{C_{11}C_{22}}$	0.15	2.8875	0.1211
Reduced Proportionate Mixing (ε=0.25)	0.2	0.6	11	0.25	$\varepsilon\sqrt{C_{11}C_{22}}$	$\varepsilon\sqrt{C_{11}C_{22}}$	0.2	3.3927	0.1402

Table 4.1. Numerically calculated values of the basic reproduction ratio and the initial growth rate for the four special mixing cases and a general case for distribution 1. $R_0 \approx 3$, T_1=5.5 and T_2=12.5 in each case.

4.2 Distribution Two

$$A(\tau) = \begin{cases} a & 0 \le \tau \le T_1 \\ 0 & otherwise \end{cases} \qquad (4.29)$$

As this is similar to distribution 1, we can just let $T_1=0$ and then replace T_2 with T_1 in our above calculations to gain:

$$R_0 = \frac{a}{2}T_1 \max\left(\left(S_1(0)C_{11} + S_2(0)C_{22}\right) \pm \sqrt{(S_1(0)C_{11} - S_2(0)C_{22})^2 + 4S_1(0)S_2(0)C_{12}C_{21}}\right)$$

(4.30)

4.2.1 Special Cases

We shall now calculate the basic reproduction ratio when each of our four mixing methods are applied.

Preferential Mixing

We let $C_{12} = C_{21} = 0$ in equation (4.30), to gain:

$$\begin{aligned} R_{pref} &= \frac{a}{2}T_1 \max\left(\left(S_2(0)C_{22} + S_1(0)C_{11} \right) \pm \sqrt{\left(S_2(0)C_{22} - S_1(0)C_{11} \right)^2} \right) \\ &= aT_1 \max\left(S_2(0)C_{22}, S_1(0)C_{11} \right) \\ &= \max\left(R_{pref\,2}, R_{pref\,1} \right) \end{aligned} \tag{4.31}$$

Where

$$\begin{aligned} R_{pref\,1} &= aT_1 S_1(0)C_{11} \\ R_{pref\,2} &= aT_1 S_2(0)C_{22} \end{aligned} \tag{4.32}$$

Sexual Mixing

If there is only mixing between different classes: $C_{11} = C_{22} = 0$, equation (4.30) becomes:

$$R_{sex} = aT_1\sqrt{S_1(0)S_2(0)C_{12}C_{21}} \tag{4.33}$$

Proportionate and Reduced Proportionate Mixing

If we let $C_{12} = C_{21} = \varepsilon\sqrt{C_{11}C_{22}}$ in equation (4.30)

$$R_{red}=\frac{a}{2}T_1\max\left(\left(S_2(0)C_{22}+S_1(0)C_{11}\right)\pm\sqrt{(S_2(0)C_{22}-S_1(0)C_{11})^2+4\varepsilon^2S_1(0)S_2(0)C_{11}C_{22}}\right)$$
$$=\frac{a}{2}T_1\left(\left(S_2(0)C_{22}+S_1(0)C_{11}\right)+\sqrt{\left(S_2(0)C_{22}-S_1(0)C_{11}\right)^2+4\varepsilon^2S_1(0)S_2(0)C_{11}C_{22}}\right)$$

(4.34)

For proportionate mixing we let $\varepsilon=0$ in the above equation:

$$R_{prop}=\frac{a}{2}T_1\left(\left(S_2(0)C_{22}+S_1(0)C_{11}\right)+\sqrt{\left(S_2(0)C_{22}-S_1(0)C_{11}\right)^2+4S_1(0)S_2(0)C_{11}C_{22}}\right)$$
$$=aT_1\left(S_2(0)C_{22}+S_1(0)C_{11}\right)$$
$$=R_{pref\,2}+R_{pref\,1}$$

(4.35)

We can then simplify the basic reproduction ratio for the reduced proportionate mixing:

$$R_{red}=\frac{a}{2}T_1\max\left(\left(S_2(0)C_{22}+S_1(0)C_{11}\right)\pm\sqrt{(S_2(0)C_{22}-S_1(0)C_{11})^2+4\varepsilon^2S_1(0)S_2(0)C_{11}C_{22}}\right)$$
$$=\frac{a}{2}T_1\left(\left(S_2(0)C_{22}+S_1(0)C_{11}\right)+\sqrt{\left(S_2(0)C_{22}-S_1(0)C_{11}\right)^2+4\varepsilon^2S_1(0)S_2(0)C_{11}C_{22}}\right)$$
$$=\frac{R_{prop}}{2}+\frac{1}{2}\sqrt{a^2T_1^2\left(S_2(0)C_{22}-S_1(0)C_{11}\right)^2+4\varepsilon^2a^2T_1^2S_1(0)S_2(0)C_{11}C_{22}}$$
$$=\frac{R_{pref\,2}+R_{pref\,1}}{2}+\frac{1}{2}\sqrt{\left(R_{pref\,2}-R_{pref\,1}\right)^2+4\varepsilon^2R_{pref\,2}R_{pref\,1}}$$
$$=\frac{1}{2}\left(R_{prop}+\sqrt{\left(R_{pref\,2}-R_{pref\,1}\right)^2+4\varepsilon^2R_{pref\,2}R_{pref\,1}}\right)$$

(4.36)

4.2.2 Initial Growth Rate

To calculate the initial growth rate we use equation (4.10)

$$\overline{\underline{i}(s)} = \left[I - \overline{A(s)} \begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12} \\ S_2(0)C_{21} & S_2(0)C_{22} \end{pmatrix} \right]^{-1} \underline{i}_0$$

Where the Laplace transform of $A(\tau)$ has been previously calculated:

$$\overline{A(s)} = \frac{a}{s}\left(1 - e^{-sT_1}\right) \tag{4.37}$$

So we have

$$\overline{\underline{i}(s)} = \left[I - \frac{a}{s}\left(1 - e^{-sT_1}\right) \begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12} \\ S_2(0)C_{21} & S_2(0)C_{22} \end{pmatrix} \right]^{-1} \underline{i}_0$$

We then solve for s such that the above matrix is not invertible:

$$\det\left[I - \frac{a}{s}\left(1 - e^{-sT_1}\right) \begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12} \\ S_2(0)C_{21} & S_2(0)C_{22} \end{pmatrix} \right] = 0$$

$$\left(1 - \frac{a}{s}\left(1 - e^{-sT_1}\right) S_1(0)C_{11}\right)\left(1 - \frac{a}{s}\left(1 - e^{-sT_1}\right) S_2(0)C_{22}\right) - \frac{a^2}{s^2}\left(1 - e^{-sT_1}\right)^2 S_1(0)S_2(0)C_{12}C_{21} = 0$$

$$1 + \left(\frac{a}{s}\left(1 - e^{-sT_1}\right)\right)^2 S_1(0)S_2(0)(C_{11}C_{22} - C_{12}C_{21}) - \frac{a}{s}\left(1 - e^{-sT_1}\right)\left(S_1(0)C_{11} + S_2(0)C_{22}\right) = 0$$

$$s^2 + \left(a\left(1 - e^{-sT_1}\right)\right)^2 S_1(0)S_2(0)(C_{11}C_{22} - C_{12}C_{21}) - as\left(1 - e^{-sT_1}\right)\left(S_1(0)C_{11} + S_2(0)C_{22}\right) = 0 \tag{4.38}$$

This can be solved for varying values of C_{ij} using MATLAB.

4.2.3 Examples

Using MATLAB to solve the equations given above, the following numerical results have been calculated for the four special mixing cases and one general case. T_1=5.5 and R_0 is approximately three for each case.

	a	$S_1(0)$	$S_2(0)$	C_{11}	C_{12}	C_{21}	C_{22}	R_0	r
General Case	0.15	7	9	0.2	0.3	0.15	0.25	2.938	0.5001
Preferential Mixing	0.25	7	9	0.15	0	0	0.25	3.094	0.5324
Sexual Mixing	0.2	7	9	0	0.4	0.3	0	3.025	0.5181
Proportionate Mixing	0.15	7	8	0.2	$\sqrt{C_{11}C_{22}}$	$\sqrt{C_{11}C_{22}}$	0.25	2.805	0.4720
Reduced Proportionate Mixing (ε=0.1)	0.15	8	10	0.35	$\varepsilon\sqrt{C_{11}C_{22}}$	$\varepsilon\sqrt{C_{11}C_{22}}$	0.3	2.6454	0.4377

Table 4.2. Numerically calculated values of the basic reproduction ratio and the initial growth rate for the four special mixing cases for distribution 2. $R_0 \approx 3$ and T_1=5.5 in each case.

4.3 Distribution Three

$$A(\tau) = ae^{-c\tau},\ \tau \geq 0 \tag{4.39}$$

where a and c are positive constants.

The next generation matrix is found by computing $K = S(0)C\int_0^\infty A(\tau)\mathrm{d}\tau$

So we have:

$$K=\begin{pmatrix} S_1(0) & 0 \\ 0 & S_2(0)\end{pmatrix}\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22}\end{pmatrix}\int_0^\infty A(\tau)\mathrm{d}\tau$$
$$=a\begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12} \\ S_2(0)C_{21} & S_2(0)C_{22}\end{pmatrix}\int_0^\infty e^{-c\tau}\mathrm{d}\tau \qquad (4.40)$$
$$=\begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12} \\ S_2(0)C_{21} & S_2(0)C_{22}\end{pmatrix}\frac{a}{c}$$

The basic reproduction ratio is the largest eigenvalue (in absolute value) of the above matrix. Using equation (4.7) we have:

$$\lambda=\frac{a}{2c}\left(S_1(0)C_{11}+S_2(0)C_{22}\right)\pm\sqrt{\frac{a^2}{c^2}\left(S_1(0)C_{11}+S_2(0)C_{22}\right)^2-\frac{4a^2}{c^2}S_1(0)S_2(0)\left(C_{11}C_{22}-C_{12}C_{21}\right)}$$
$$=\frac{a}{2c}\left(\left(S_1(0)C_{11}+S_2(0)C_{22}\right)\pm\sqrt{\left(S_1(0)C_{11}-S_2(0)C_{22}\right)^2+4S_1(0)S_2(0)C_{12}C_{21}}\right) \qquad (4.41)$$

Then, the basic reproduction ratio is the largest of these two eigenvalues:

$$R_0=\frac{a}{2c}\max\left(\left(S_1(0)C_{11}+S_2(0)C_{22}\right)\pm\sqrt{\left(S_1(0)C_{11}-S_2(0)C_{22}\right)^2+4S_1(0)S_2(0)C_{12}C_{21}}\right) \qquad (4.42)$$

4.3.1 Special Cases

We can now calculate our four special mixing cases.

Preferential Mixing

By letting $C_{12}=C_{21}=0$ in equation (4.42) we gain:

$$\begin{aligned} R_{pref} &= \frac{a}{c}\max\left(S_1(0)C_{11}, S_2(0)C_{22}\right) \\ &= \max\left(R_{pref1}, R_{pref2}\right) \end{aligned} \tag{4.43}$$

Where

$$\begin{aligned} R_{pref1} &= \frac{a}{c}S_1(0)C_{11} \\ R_{pref2} &= \frac{a}{c}S_2(0)C_{22} \end{aligned} \tag{4.44}$$

Sexual Mixing

By letting $C_{11} = C_{22} = 0$ in equation (4.42), it simplifies to:

$$R_{sex} = \frac{a}{c}\sqrt{S_1(0)S_2(0)C_{12}C_{21}}$$

Proportionate and Reduced Proportionate Mixing

If we let $C_{12} = C_{21} = \varepsilon\sqrt{C_{11}C_{22}}$ in equation (4.42) we gain:

$$R_{red} = \frac{a}{2c}\max\left(\left(S_1(0)C_{11} + S_2(0)C_{22}\right) \pm \sqrt{\left(S_1(0)C_{11} - S_2(0)C_{22}\right)^2 + 4\varepsilon^2 S_1(0)S_2(0)C_{11}C_{22}}\right) \tag{4.45}$$

If we let $\varepsilon = 1$, we have the basic reproduction ratio for proportionate mixing:

$$\begin{aligned} R_{prop} &= \frac{a}{c}\left(S_1(0)C_{11} + S_2(0)C_{22}\right) \\ &= R_{pref1} + R_{pref2} \end{aligned} \tag{4.46}$$

We may then simplify the basic reproduction ratio for reduced proportionate mixing:

$$
\begin{aligned}
R_{red} &= \frac{a}{2c}\max\left(\left(S_1(0)C_{11}+S_2(0)C_{22}\right)\pm\sqrt{\left(S_1(0)C_{11}-S_2(0)C_{22}\right)^2+4\varepsilon^2 S_1(0)S_2(0)C_{11}C_{22}}\right)\\
&= \frac{a}{2c}\left(\left(S_1(0)C_{11}+S_2(0)C_{22}\right)+\sqrt{\left(S_1(0)C_{11}-S_2(0)C_{22}\right)^2+4\varepsilon^2 S_1(0)S_2(0)C_{11}C_{22}}\right)\\
&= \frac{R_{pref1}+R_{pref2}}{2}+\frac{1}{2}\sqrt{\frac{a^2}{c^2}\left(S_1(0)C_{11}-S_2(0)C_{22}\right)^2+4\frac{a^2}{c^2}\varepsilon^2 S_1(0)S_2(0)C_{11}C_{22}}\\
&= \frac{R_{pref1}+R_{pref2}}{2}+\frac{1}{2}\sqrt{\left(R_{pref1}-R_{pref2}\right)^2+4\varepsilon^2 R_{pref1}R_{pref2}}\\
&= \frac{R_{prop}}{2}+\frac{1}{2}\sqrt{\left(R_{pref1}-R_{pref2}\right)^2+4\varepsilon^2 R_{pref1}R_{pref2}}
\end{aligned}
\tag{4.47}
$$

4.3.2 Initial Growth Rate

Using the method given previously, we calculate:

$$
\overline{\underline{i}(s)} = \left[I-\overline{A(s)}\begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12}\\ S_2(0)C_{21} & S_2(0)C_{22}\end{pmatrix}\right]^{-1}\underline{i}_0 \tag{4.48}
$$

where the Laplace transform of $A(\tau)$ has already been calculated:

$$
\begin{aligned}
\overline{A(s)} &= \int_0^\infty e^{-s\tau}A(\tau)d\tau\\
&= \frac{a}{s+c}
\end{aligned}
\tag{4.49}
$$

The initial growth rate is the value of s for which the above matrix is not invertible. That is when:

$$\det\left[I-\frac{a}{s+c}\begin{pmatrix}S_1(0)C_{11} & S_1(0)C_{12}\\ S_2(0)C_{21} & S_2(0)C_{22}\end{pmatrix}\right]=0$$

$$\left(1-\frac{aS_1(0)C_{11}}{s+c}\right)\left(1-\frac{aS_2(0)C_{22}}{s+c}\right)-\frac{a^2S_1(0)S_2(0)C_{12}C_{21}}{(s+c)^2}=$$

$$1-a\frac{S_1(0)C_{11}+S_2(0)C_{22}}{s+c}+a^2\frac{S_1(0)S_2(0)C_{11}C_{22}}{(s+c)^2}-a^2\frac{S_1(0)S_2(0)C_{12}C_{21}}{(s+c)^2}=$$

$$(s+c)^2-a(s+c)\left(S_1(0)C_{11}+S_2(0)C_{22}\right)+a^2S_1(0)S_2(0)(C_{11}C_{22}-C_{12}C_{21})=$$

The above may be solved explicitly:

$$\begin{aligned}s+c&=\frac{a\left(S_1(0)C_{11}+S_2(0)C_{22}\right)\pm\sqrt{\begin{matrix}a^2\left(S_1(0)C_{11}+S_2(0)C_{22}\right)^2\\ -4a^2S_1(0)S_2(0)(C_{11}C_{22}-C_{12}C_{21})\end{matrix}}}{2}\\ &=a\frac{\left(S_1(0)C_{11}+S_2(0)C_{22}\right)\pm\sqrt{\left(S_1(0)C_{11}-S_2(0)C_{22}\right)^2+4S_1(0)S_2(0)C_{12}C_{21}}}{2}\end{aligned}$$

(4.50)

4.3.3 Examples

Below we have some numerical examples, calculated using MATLAB, for the basic reproduction ratio and the initial growth rate for our four special cases and one general case. We have a=1, S_{10}=7 and S_{20}=10 in each case.

	c	C_{11}	C_{12}	C_{21}	C_{22}	R_0	r
General Case	1.35	0.15	0.2	0.25	0.3	3.063	2.785
Preferential Mixing	0.5	0.1	0	0	0.15	3	1
Sexual Mixing	0.68	0	0.3	0.2	0	3.014	1.370
Proportionate Mixing	1.35	0.3	$\sqrt{C_{11}C_{22}}$	$\sqrt{C_{11}C_{22}}$	0.2	3.037	2.750
Reduced Proportionate Mixing (ε=0.14)	1.2	0.3	$\varepsilon\sqrt{C_{11}C_{22}}$	$\varepsilon\sqrt{C_{11}C_{22}}$	0.35	2.997	2.396

Table 4.3. Numerically calculated values of the basic reproduction ratio and the initial growth rate for the four special mixing cases and one general case for distribution 3. $R_0 \approx 3$ $S_1(t)=7$, $S_2(t)=10$ and $a=1$ in each case.

4.4 Distribution Four

$$A(\tau) = a\tau e^{-c\tau}, \ \tau \geq 0 \tag{4.51}$$

where all the constants are positive.

To find R_0 we find the next generation matrix: $K = S(0)C\int_0^\infty A(\tau)d\tau$

$$K=\begin{pmatrix} S_1(0) & 0 \\ 0 & S_2(0) \end{pmatrix}\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}\int_0^\infty A(\tau)\mathrm{d}\tau$$
$$=\begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12} \\ S_2(0)C_{21} & S_2(0)C_{22} \end{pmatrix} a\int_0^\infty \tau e^{-c\tau}\mathrm{d}\tau \qquad (4.52)$$
$$=\begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12} \\ S_2(0)C_{21} & S_2(0)C_{22} \end{pmatrix}\frac{a}{c^2}$$

The basic reproduction ratio is the largest eigenvalue of the above matrix, so we calculate:

$$2\lambda = a^2\frac{S_1(0)C_{11}+S_2(0)C_{22}}{c^2} \pm \sqrt{\frac{a^4\left(S_1(0)C_{11}+S_2(0)C_{22}\right)^2}{c^4} - \frac{4a^4}{c^4}S_1(0)S_2(0)(C_{11}C_{22}-C_{21}C_{12})}$$
$$= a^2\frac{S_1(0)C_{11}+S_2(0)C_{22}}{c^2} \pm \frac{a^2}{c^2}\sqrt{\left(S_1(0)C_{11}+S_2(0)C_{22}\right)^2 - 4S_1(0)S_2(0)(C_{11}C_{22}-C_{21}C_{12})}$$
$$= a^2\frac{S_1(0)C_{11}+S_2(0)C_{22}}{c^2} \pm \frac{a^2}{c^2}\sqrt{\left(S_1(0)C_{11}-S_2(0)C_{22}\right)^2 + 4S_1(0)S_2(0)C_{12}C_{21}}$$
$$(4.53)$$

Then, the basic reproduction ratio is:

$$R_0=\frac{a^2}{2c^2}\max\left(S_1(0)C_{11}+S_2(0)C_{22} \pm \sqrt{\left(S_1(0)C_{11}-S_2(0)C_{22}\right)^2 + 4S_1(0)S_2(0)C_{12}C_{21}}\right)$$
$$(4.54)$$

4.4.1 Special Cases

We shall now apply our four special cases for mixing, and calculate the basic reproduction ratio for each case.

Preferential Mixing

By letting $C_{12} = C_{21} = 0$ in equation (4.54) we gain the basic reproduction ratio when preferential mixing is present:

$$\begin{aligned} R_{pref} &= \frac{a^2}{2c^2}\max\left(\left(S_1(0)C_{11} + S_2(0)C_{22}\right) \pm \sqrt{\left(S_1(0)C_{11} - S_2(0)C_{22}\right)^2}\right) \\ &= \frac{a^2}{c^2}\max\left(S_1(0)C_{11}, S_2(0)C_{22}\right) \\ &= \max\left(R_{pref1}, R_{pref2}\right) \end{aligned} \tag{4.55}$$

Where

$$\begin{aligned} R_{pref1} &= \frac{a^2}{c^2}S_1(0)C_{11} \\ R_{pref2} &= \frac{a^2}{c^2}S_2(0)C_{22} \end{aligned} \tag{4.56}$$

Sexual Mixing

We let $C_{11} = C_{22} = 0$ in equation (4.54):

$$R_{sex} = \frac{a^2}{c^2}\sqrt{S_1(0)S_2(0)C_{12}C_{21}}$$

Proportionate and Reduced Proportionate Mixing

If we let $C_{12} = C_{21} = \varepsilon\sqrt{C_{11}C_{22}}$ in equation (4.54) we can calculate the basic reproduction ratios for both the proportionate and reduced proportionate mixing:

$$R_{red} = \max\left(\frac{a^2}{2c^2}\left(S_1(0)C_{11} + S_2(0)C_{22} \pm \sqrt{\left(S_1(0)C_{11} - S_2(0)C_{22}\right)^2 + 4\varepsilon^2 S_1(0)S_2(0)C_{11}C_{22}}\right)\right) \tag{4.57}$$

By letting $\varepsilon = 1$ we obtain the basic reproduction ratio for proportionate mixing:

$$\begin{aligned} R_{prop} &= \frac{a^2}{2c^2}\max\left(S_1(0)C_{11} + S_2(0)C_{22} \pm \sqrt{\left(S_1(0)C_{11} - S_2(0)C_{22}\right)^2 + 4S_1(0)S_2(0)C_{11}C_{22}}\right) \\ &= \frac{a^2}{c^2}\left(S_1(0)C_{11} + S_2(0)C_{22}\right) \\ &= R_{pref1} + R_{pref2} \end{aligned} \tag{4.58}$$

We can then use this to simplify (4.57), so the basic reproduction ratio for reduced proportionate mixing is:

$$\begin{aligned} R_{red} &= \frac{R_{pref1} + R_{pref2}}{2} + \frac{1}{2}\sqrt{\left(R_{pref1} - R_{pref2}\right)^2 + 4\varepsilon^2 R_{pref1}R_{pref2}} \\ &= \frac{R_{prop}}{2} + \frac{1}{2}\sqrt{\left(R_{pref1} - R_{pref2}\right)^2 + 4\varepsilon^2 R_{pref1}R_{pref2}} \end{aligned} \tag{4.59}$$

4.4.2 Initial Growth Rate

To calculate the initial growth rate of the infection, we use equation (4.10), where we have previously calculated the Laplace transform of our distribution. So we find when the following matrix is not invertible, and solve for s.

$$\overline{\underline{i}(s)} = \left[I - \frac{a}{(s+c)^2}\begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12} \\ S_2(0)C_{21} & S_2(0)C_{22} \end{pmatrix}\right]^{-1} \underline{i}_0 \tag{4.60}$$

So we want the determinate of the above matrix to be zero:

$$\det\left[I-\frac{a}{(s+c)^2}\begin{pmatrix}S_1(0)C_{11} & S_1(0)C_{12}\\ S_2(0)C_{21} & S_2(0)C_{22}\end{pmatrix}\right]=0$$

$$\left(1-\frac{aS_1(0)C_{11}}{(s+c)^2}\right)\left(1-\frac{aS_2(0)C_{22}}{(s+c)^2}\right)-\frac{a^2S_1(0)S_2(0)C_{12}C_{21}}{(s+c)^4}=$$

$$1-a\frac{S_1(0)C_{11}+S_2(0)C_{22}}{(s+c)^2}+\frac{a^2S_1(0)S_2(0)C_{11}C_{22}}{(s+c)^4}-\frac{a^2S_1(0)S_2(0)C_{12}C_{21}}{(s+c)^4}=$$

$$(s+c)^4-a(s+c)^2(S_1(0)C_{11}+S_2(0)C_{22})+a^2S_1(0)S_2(0)(C_{11}C_{22}-C_{12}C_{21})=$$

(4.61)

The above may be solved explicitly

$$s+c=\left[\frac{a(S_1(0)C_{11}+S_2(0)C_{22})\pm\sqrt{a^2(S_1(0)C_{11}-S_2(0)C_{22})^2+4a^2S_1(0)S_2(0)C_{12}C_{21}}}{2}\right]^{\frac{-1}{2}}$$

(4.62)

4.4.3 Examples

Using MATLAB, the basic reproduction ratios and initial growth rate for the four special mixing cases and one general case are given below. In each case a=1.

	a	$S_1(0)$	$S_2(0)$	C_{11}	C_{12}	C_{21}	C_{22}	R_0	r
General Case-	1.2	7	10	0.15	0.2	0.28	0.3	2.9389	0.8572
Preferential Mixing	0.7	6	7	0.24	0	0	0.15	2.9388	0.5000
Sexual Mixing	0.8	15	10	0	0.12	0.2	0	2.9646	0.5774
Proportionate Mixing	1.18	7	10	0.3	$\sqrt{C_{11}C_{22}}$	$\sqrt{C_{11}C_{22}}$	0.2	2.9446	0.8448
Reduced Proportionate Mixing (ε=0.14)	1.11	7	10	0.3	$\varepsilon\sqrt{C_{11}C_{22}}$	$\varepsilon\sqrt{C_{11}C_{22}}$	0.35	2.9188	0.7864

Table 4.4. Numerically calculated values of the basic reproduction ratio and the initial growth rate for the four special mixing cases and one general case for distribution 4. $R_0 \approx 3$.amd a=1 in each case

4.5 Distribution Five

$$A(\tau) = ae^{-c(\tau - T_1)^2}, \ \tau \geq 0$$

Where all the constants are positive.

The next generation matrix is given by:

$$\begin{aligned} K &= \begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12} \\ S_2(0)C_{21} & S_2(0)C_{22} \end{pmatrix} a\int_0^\infty e^{-c(\tau - T_1)^2} \mathrm{d}\tau \\ &= \frac{a\sqrt{\pi}}{2\sqrt{c}}\left(1 + \operatorname{erf}\left(\sqrt{c}T_1\right)\right)\begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12} \\ S_2(0)C_{21} & S_2(0)C_{22} \end{pmatrix} \end{aligned} \quad (4.63)$$

Where

$$\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}t$$

We can then calculate R_0 (the largest eigenvalue of K).

$$\lambda^2-\mathrm{tr}(K)\lambda+\det(K)=0$$
$$2\lambda=\mathrm{tr}(K)\pm\sqrt{\mathrm{tr}(K)^2-4\det(K)}$$

That is,

$$2\lambda=\frac{a\sqrt{\pi}}{2\sqrt{c}}\left(1+\mathrm{erf}(\sqrt{c}T_1)\right)\left((S_1(0)C_{11}+S_2(0)C_{22})\pm\sqrt{\begin{matrix}(S_1(0)C_{11}+S_2(0)C_{22})^2\\-4S_1(0)S_2(0)(C_{11}C_{22}-C_{12}C_{21})\end{matrix}}\right)$$
$$\lambda=\frac{a\sqrt{\pi}}{4\sqrt{c}}\left(1+\mathrm{erf}(\sqrt{c}T_1)\right)\left((S_1(0)C_{11}+S_2(0)C_{22})\pm\sqrt{(S_1(0)C_{11}-S_2(0)C_{22})^2+4S_1(0)S_2(0)C_{12}C_{21}}\right)$$

(4.64)

So R_0 is:

$$R_0=\max\left[\frac{a\sqrt{\pi}}{4\sqrt{c}}\left(1+\mathrm{erf}(\sqrt{c}T_1)\right)\left(\begin{matrix}(S_1(0)C_{11}+S_2(0)C_{22})\\\pm\sqrt{(S_1(0)C_{11}-S_2(0)C_{22})^2+4S_1(0)S_2(0)C_{12}C_{21}}\end{matrix}\right)\right]$$

(4.65)

4.5.1 Special Cases

Using equation (4.65) we can now calculate our four special cases for mixing.

Preferential Mixing

Letting $C_{12}=C_{21}=0$ in equation (4.65) we gain:

$$R_{pref} = \frac{a\sqrt{\pi}}{4\sqrt{c}}\left(1+\mathrm{erf}(\sqrt{c}T_1)\right)\max\left[\left(S_1(0)C_{11}+S_2(0)C_{22}\right)\pm\sqrt{\left(S_1(0)C_{11}-S_2(0)C_{22}\right)^2}\right]$$
$$= \frac{a\sqrt{\pi}}{2\sqrt{c}}\left(1+\mathrm{erf}(\sqrt{c}T_1)\right)\max\left[S_1(0)C_{11}, S_2(0)C_{22}\right] \quad (4.66)$$
$$= \max\left(R_{pref1}, R_{pref2}\right)$$

Where

$$R_{pref1} = \frac{a\sqrt{\pi}}{2\sqrt{c}}\left(1+\mathrm{erf}(\sqrt{c}T_1)\right)S_1(0)C_{11}$$
$$R_{pref2} = \frac{a\sqrt{\pi}}{2\sqrt{c}}\left(1+\mathrm{erf}(\sqrt{c}T_1)\right)S_2(0)C_{22} \quad (4.67)$$

Sexual Mixing

To calculate the basic reproduction ratio when sexual mixing occurs, we set $C_{11} = C_{22} = 0$ in equation (4.65):

$$R_{sex} = \frac{a\sqrt{\pi}}{2\sqrt{c}}\left(1+\mathrm{erf}(\sqrt{c}T_1)\right)\left(\sqrt{S_1(0)S_2(0)C_{12}C_{21}}\right) \quad (4.68)$$

Proportionate and Reduced Proportionate Mixing

We now $C_{12} = C_{21} = \varepsilon\sqrt{C_{11}C_{22}}$ in equation (4.65)

$$R_{red} = \frac{a\sqrt{\pi}}{4\sqrt{c}}\left(1+\mathrm{erf}(\sqrt{c}T_1)\right)\max\left[\begin{array}{l}\left(S_1(0)C_{11}+S_2(0)C_{22}\right)\\ \pm\sqrt{\left(S_1(0)C_{11}-S_2(0)C_{22}\right)^2+4\varepsilon^2 S_1(0)S_2(0)C_{11}C_{22}}\end{array}\right]$$
$$(4.69)$$

By letting $\varepsilon = 1$ in the above equation, we obtain the basic reproduction ratio for proportionate mixing:

$$\begin{aligned} R_{prop} &= \frac{a\sqrt{\pi}}{4\sqrt{c}}\left(1+\operatorname{erf}(\sqrt{c}T_1)\right)\max\left[\left(S_1(0)C_{11}+S_2(0)C_{22}\right)\pm\left(S_1(0)C_{11}+S_2(0)C_{22}\right)\right] \\ &= \frac{a\sqrt{\pi}}{2\sqrt{c}}\left(1+\operatorname{erf}(\sqrt{c}T_1)\right)\left(S_1(0)C_{11}+S_2(0)C_{22}\right) \\ &= R_{pref1}+R_{pref2} \end{aligned} \tag{4.70}$$

We can then use this to simplify equation (4.69), so the basic reproduction ratio when reduced proportionate mixing is applied is:

$$\begin{aligned} R_{red} &= \frac{R_{prop}}{2}+\frac{a\sqrt{\pi}}{4\sqrt{c}}\left(1+\operatorname{erf}(\sqrt{c}T_1)\right)\sqrt{\left(S_1(0)C_{11}-S_2(0)C_{22}\right)^2+4\varepsilon^2 S_1(0)S_2(0)C_{11}C_{22}} \\ &= \frac{R_{prop}}{2}+\frac{1}{2}\sqrt{\left(R_{pref1}-R_{pref2}\right)^2+4\varepsilon^2 R_{pref1}R_{pref2}} \end{aligned} \tag{4.71}$$

4.5.2 Initial Growth Rate

To calculate the initial growth rate, r, we consider

$$\overline{\underline{i}(s)} = \left[I-\overline{A(s)}\begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12} \\ S_2(0)C_{21} & S_2(0)C_{22}\end{pmatrix}\right]^{-1}\underline{i}_0$$

where the Laplace transform of the distribution is as calculated before. We find the value of s such that the above matrix is not invertible, i.e. when its determinant is zero.

$$\begin{aligned} \left(1-\overline{A(s)}S_1(0)C_{11}\right)\left(1-\overline{A(s)}S_2(0)C_{22}\right)-\overline{A(s)}^2 S_1(0)S_2(0)C_{12}C_{21} = 0 \\ 1-\overline{A(s)}\left(S_1(0)C_{11}+S_2(0)C_{22}\right)+\overline{A(s)}^2 S_1(0)S_2(0)\left(C_{11}C_{22}-C_{12}C_{21}\right) = \end{aligned} \tag{4.72}$$

Where

$$\bar{A}(s)=\frac{a\sqrt{\pi}}{2\sqrt{c}}e^{-T_1 s+\frac{s^2}{4c^2}}\left(1+\operatorname{erf}\left(\sqrt{c}\left(T_1-\frac{s}{2c}\right)\right)\right) \qquad (4.73)$$

We can solve equation (4.72) numerically using MATLAB.

4.5.3 Examples

Below are some numerical calculations of the basic reproduction ratio and initial growth rate for the four special cases and one general case. In each calculation, a=1.

	c	T_1	$S_1(0)$	$S_2(0)$	C_{11}	C_{12}	C_{21}	C_{22}	R_0	r
General Case	2.4	5.5	7	8	0.15	0.2	0.28	0.1	3.0894	0.2059
Preferential Mixing	1.11	4	6	9	0.15	0	0	0.2	3.0282	0.2815
Sexual Mixing	1.2	5.5	15	10	0	0.12	0.2	0	3.0700	0.2055
Proportionate Mixing	1.4	5.5	7	10	0.15	$\sqrt{C_{11}C_{22}}$	$\sqrt{C_{11}C_{22}}$	0.1	3.0709	0.2054
Reduced Proportionate Mixing (ε=0.14)	0.9	5.5	7	10	0.2	$\varepsilon\sqrt{C_{11}C_{22}}$	$\varepsilon\sqrt{C_{11}C_{22}}$	0.15	3.0995	0.2079

Table 4.5. Numerically calculated values of the basic reproduction ratio and the initial growth rate for the four special mixing cases and one general case for distribution 5. $R_0 \approx 3$ and a=1 in each case.

4.6 Distribution Six

$$A(\tau)=\begin{cases}\dfrac{a}{T_2-T_1}(\tau-T_1), & T_1\le\tau\le T_2\\ a, & T_2<\tau<T_3\\ \dfrac{-a}{T_4-T_3}(\tau-T_4), & T_3\le\tau\le T_4\\ 0, & otherwise\end{cases} \tag{4.74}$$

To calculate R_0 we find the next generation matrix K.

$$\begin{aligned}K&=\begin{pmatrix}S_1(0)C_{11} & S_1(0)C_{12}\\ S_2(0)C_{21} & S_2(0)C_{22}\end{pmatrix}\int_0^{\infty}A(\tau)\,d\tau\\ &=\frac{a}{2}(T_4+T_3-T_2-T_1)\begin{pmatrix}S_1(0)C_{11} & S_1(0)C_{12}\\ S_2(0)C_{21} & S_2(0)C_{22}\end{pmatrix}\end{aligned} \tag{4.75}$$

R_0 is then the largest (in absolute value) eigenvalue of K.

$$\begin{aligned}&\lambda^2-\lambda\mathrm{tr}(K)+\det(K)=0\\ &2\lambda=\mathrm{tr}(K)\pm\sqrt{\mathrm{tr}(K)^2-4\det(K)}\end{aligned}$$

$$R_0=\frac{a}{4}(T_4+T_3-T_2-T_1)\max\left(S_1(0)C_{11}+S_2(0)C_{22}\pm\sqrt{\begin{aligned}&\left(S_1(0)C_{11}-S_2(0)C_{22}\right)^2\\ &+4S_1(0)S_2(0)C_{12}C_{21}\end{aligned}}\right) \tag{4.76}$$

4.6.1 Special Cases

Using equation (4.76) we shall now investigate the special cases for mixing.

Preferential Mixing

Letting $C_{12} = C_{21} = 0$ in equation (4.76)

$$\begin{aligned} R_{pref} &= \frac{a}{4}(T_4 + T_3 - T_2 - T_1)\max\left(S_1(0)C_{11} + S_2(0)C_{22} \pm \sqrt{\left(S_1(0)C_{11} - S_2(0)C_{22}\right)^2} \right) \\ &= \frac{a}{2}(T_4 + T_3 - T_2 - T_1)\max\left(S_{1,0}C_{11}, S_{2,0}C_{22}\right) \\ &= \max(R_{pref1}, R_{pref2}) \end{aligned}$$

(4.77)

where

$$\begin{aligned} R_{pref1} &= \frac{a}{2}(T_4 + T_3 - T_2 - T_1)S_1(0)C_{11} \\ R_{pref2} &= \frac{a}{2}(T_4 + T_3 - T_2 - T_1)S_2(0)C_{22} \end{aligned} \quad (4.78)$$

Sexual Mixing

Substitute $C_{11} = C_{22} = 0$ in equation (4.76) to gain:

$$R_{sex} = \frac{a}{4}(T_4 + T_3 - T_2 - T_1)\sqrt{S_1(0)S_2(0)C_{12}C_{21}} \quad (4.79)$$

Proportionate and Reduced Proportionate Mixing

If we let $C_{12} = C_{21} = \varepsilon\sqrt{C_{11}C_{22}}$ in equation (4.76) we have:

$$R_{red} = \frac{a}{4}(T_4 + T_3 - T_2 - T_1)\left(\begin{array}{l} S_1(0)C_{11} + S_2(0)C_{22} \\ +\sqrt{\left(S_1(0)C_{11} - S_2(0)C_{22}\right)^2 + 4\varepsilon^2 S_1(0)S_2(0)C_{11}C_{22}} \end{array} \right)$$

(4.80)

By letting $\varepsilon=1$ in the above relation, we gain the basic reproduction ratio when there is proportionate mixing:

$$\begin{aligned} R_{prop} &= \frac{a}{2}(T_4+T_3-T_2-T_1)\left(S_1(0)C_{11}+S_2(0)C_{22}\right) \\ &= R_{pref1}+R_{pref2} \end{aligned} \tag{4.81}$$

We may then rewrite equation (4.80), to gain the basic reproduction ratio for reduced proportionate mixing:

$$\begin{aligned} R_{red} &= \frac{a}{4}(T_4+T_3-T_2-T_1)\left(S_1(0)C_{11}+S_2(0)C_{22}+\sqrt{\begin{matrix}\left(S_1(0)C_{11}-S_2(0)C_{22}\right)^2 \\ +4\varepsilon^2 S_1(0)S_2(0)C_{11}C_{22}\end{matrix}}\right) \\ &= \frac{1}{2}\left(\left(R_{pref1}+R_{pref2}\right)+\sqrt{\left(R_{pref1}-R_{pref2}\right)^2+4\varepsilon^2 R_{pref1}R_{pref2}}\right) \\ &= \frac{1}{2}\left(R_{prop}+\sqrt{\left(R_{pref1}-R_{pref2}\right)^2+4\varepsilon^2 R_{pref1}R_{pref2}}\right) \end{aligned} \tag{4.82}$$

4.6.2 Initial Growth Rate

To calculate the initial growth rate we use equation (4.10). The Laplace transform of $A(\tau)$ has been calculated previously, and is:

$$\overline{A}(s) = \frac{a}{s^2(T_2-T_1)(T_4-T_3)}\left(\left(e^{-T_1 s}-e^{-T_2 s}\right)(T_4-T_3)+\left(e^{-T_4 s}-e^{-T_3 s}\right)(T_2-T_1)\right) \tag{4.83}$$

We need to find when the following matrix is not invertible:

$$\overline{\underline{i}(s)} = \left[I-\overline{A(s)}\begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12} \\ S_2(0)C_{21} & S_2(0)C_{22}\end{pmatrix}\right]^{-1}\underline{i}_0$$

That is when the determinant is zero:

$$\begin{aligned}&\left(1-\overline{A(s)}S_1(0)C_{11}\right)\left(1-\overline{A(s)}S_2(0)C_{22}\right)-\overline{A(s)}^2S_1(0)S_2(0)C_{12}C_{21}=0\\&1-\overline{A(s)}\left(S_1(0)C_{11}+S_2(0)C_{22}\right)+\overline{A(s)}^2S_1(0)S_2(0)\left(C_{11}C_{22}-C_{12}C_{21}\right)=0\end{aligned}\tag{4.84}$$

The value of s that satisfies the above equation can be calculated numerically using MATLAB.

4.6.3 Examples

By letting T_1=4, T_2=7, T_3=11, and T_4=14 we can calculate the basic reproduction ratio and initial growth rates for our four special cases and one general case, using MALTAB.

	p	$S_1(0)$	$S_2(0)$	C_{11}	C_{12}	C_{21}	C_{22}	R_0	r
General Case	0.15	8	9	0.25	0.1	0.15	0.2	3.0912	0.1299
Preferential Mixing	0.25	7	9	0.15	0	0	0.2	3.15	0.1321
Sexual Mixing	0.2	7	9	0	0.25	0.3	0	3.0432	0.1280
Proportionate Mixing	0.15	7	10	0.2	$\sqrt{C_{11}C_{22}}$	$\sqrt{C_{11}C_{22}}$	0.15	3.0450	0.1281
Reduced Proportionate Mixing (ε=0.15)	0.2	8	9	0.25	$\varepsilon\sqrt{C_{11}C_{22}}$	$\varepsilon\sqrt{C_{11}C_{22}}$	0.2	3.0823	0.1296

Table 4.6. Numerically calculated values of the basic reproduction ratio and the initial growth rate for the four special mixing cases and one general case for distribution 6. $R_0\approx 3$, T_1=4, T_2=7, T_3=11 and T_4=14 in each case.

4.7 The General Case

As we have seen in the workings for the above six distributions, the resulting equations are similar, so we can refine our method. To calculate the basic reproduction ratio, we look at the eigenvalues of the next generation matrix, and then take the largest value. That is:

$$R_0 = \max\left(\frac{1}{2}\mathrm{tr}(K) \pm \frac{1}{2}\sqrt{\mathrm{tr}(K)^2 - 4\det(K)}\right) \tag{4.85}$$

Where

$$K = \begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12} \\ S_2(0)C_{21} & S_2(0)C_{22} \end{pmatrix} \int_0^\infty A(\tau)\mathrm{d}\tau \tag{4.86}$$

Putting the above two equations together gives us:

$$\begin{aligned} R_0 &= \max\left(\frac{1}{2}\left(S_1(0)C_{11} + S_2(0)C_{22}\right)\int_0^\infty A(\tau)\mathrm{d}\tau \pm \frac{1}{2}\sqrt{\begin{matrix}\left(\left(S_1(0)C_{11} + S_2(0)C_{22}\right)\int_0^\infty A(\tau)\mathrm{d}\tau\right)^2 \\ -4S_1(0)S_2(0)\begin{pmatrix} C_{11}C_{22} \\ -C_{12}C_{21}\end{pmatrix}\left(\int_0^\infty A(\tau)\mathrm{d}\tau\right)^2\end{matrix}}\right) \\ &= \frac{1}{2}\max\left(\left(S_1(0)C_{11} + S_2(0)C_{22}\right) \pm \sqrt{\begin{matrix}\left(S_1(0)C_{11} + S_2(0)C_{22}\right)^2 \\ -4S_1(0)S_2(0)\left(C_{11}C_{22} - C_{12}C_{21}\right)\end{matrix}}\right)\int_0^\infty A(\tau)\mathrm{d}\tau \\ &= \frac{1}{2}\max\left(\left(S_1(0)C_{11} + S_2(0)C_{22}\right) \pm \sqrt{\begin{matrix}\left(S_1(0)C_{11} - S_2(0)C_{22}\right)^2 \\ +4S_1(0)S_2(0)C_{12}C_{21}\end{matrix}}\right)\int_0^\infty A(\tau)\mathrm{d}\tau \end{aligned} \tag{4.87}$$

4.7.1 Special Cases

We can now look at our four special cases for mixing by varying the C_{ik} parameters in equation (4.87).

Preferential Mixing

Let $C_{12} = C_{21} = 0$

$$\begin{aligned} R_{pref} &= \frac{1}{2}\max\left(\left(S_1(0)C_{11} + S_2(0)C_{22}\right) \pm \left(S_1(0)C_{11} - S_2(0)C_{22}\right)\right) \int_0^\infty A(\tau)\mathrm{d}\tau \\ &= \max\left(S_{1,0}C_{11}, S_{2,0}C_{22}\right) \int_0^\infty A(\tau)\mathrm{d}\tau \\ &= \max\left(R_{pref1}, R_{pref2}\right) \end{aligned} \tag{4.88}$$

Where

$$\begin{aligned} R_{pref1} &= S_1(0)C_{11} \int_0^\infty A(\tau)\mathrm{d}\tau \\ R_{pref2} &= S_2(0)C_{22} \int_0^\infty A(\tau)\mathrm{d}\tau \end{aligned} \tag{4.89}$$

Sexual Mixing

Let $C_{11} = C_{22} = 0$

$$R_{sex} = \left(\sqrt{S_1(0)S_2(0)C_{12}C_{21}}\right) \int_0^\infty A(\tau)\mathrm{d}\tau \tag{4.90}$$

Proportionate Mixing

Let $C_{12} = C_{21} = \sqrt{C_{11}C_{22}}$

$$R_{prop} = \left(S_1(0)C_{11} + S_2(0)C_{22}\right)\int_0^\infty A(\tau)\,\mathrm{d}\tau \\ = R_{pref1} + R_{pref2} \tag{4.91}$$

Reduced Proportionate Mixing

Let $C_{12} = C_{21} = \varepsilon\sqrt{C_{11}C_{22}}$

$$R_{red} = \frac{1}{2}\left(\left(S_1(0)C_{11} + S_2(0)C_{22}\right) + \sqrt{\left(S_1(0)C_{11} - S_2(0)C_{22}\right)^2 + 4\varepsilon^2 S_1(0)S_2(0)C_{11}C_{22}}\right)\int_0^\infty A(\tau)\,\mathrm{d}\tau \\ = \frac{1}{2}R_{prop} + \frac{1}{2}\sqrt{\left(R_{pref1} - R_{pref2}\right)^2 + 4\varepsilon^2 R_{pref1}R_{pref2}} \tag{4.92}$$

There is a clear link between the basic reproduction ratios for preferential mixing, proportionate mixing and reduced proportionate mixing.

To find the initial growth rate, we solve for following equation for s ($\neq 0$):

$$1 - \overline{A(s)}\left(S_1(0)C_{11} + S_2(0)C_{22}\right) + \overline{A(s)}^2 S_1(0)S_2(0)\left(C_{11}C_{22} - C_{12}C_{21}\right) = 0 \tag{4.93}$$

We have calculated the basic reproduction ratio and final size equations for the general case and four special cases for each of our six distributions and for a generic distribution, where all the equations need to be solve numerically. In each case, we have assumed that $A(\tau)$ is the same for each of the two classes. However, this will not be realistic, for example for sexually transmitted diseases, you would expect the transmission characteristics to be different for males and females.

4.8 Different $A(\tau)$ *For Each Class*

We shall now assume that the function $A(\tau)$ is different for each population class. Using the same methods as previously, our next generation matrix is

$$K = S(0)C\int_0^\infty A(\tau)\mathrm{d}\tau \tag{4.94}$$

Where

$$S(t) = \begin{pmatrix} S_1(0) & 0 \\ 0 & S_2(0) \end{pmatrix} \tag{4.95}$$

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \tag{4.96}$$

and now

$$A(\tau) = \begin{pmatrix} A_1(\tau) & 0 \\ 0 & A_2(\tau) \end{pmatrix} \tag{4.97}$$

Substituting these into equation (4.94)

$$\begin{aligned} K &= \begin{pmatrix} S_1(0)C_{11} & S_1(0)C_{12} \\ S_2(0)C_{21} & S_2(0)C_{22} \end{pmatrix} \begin{pmatrix} \int_0^\infty A_1(\tau)\mathrm{d}\tau & 0 \\ 0 & \int_0^\infty A_2(\tau)\mathrm{d}\tau \end{pmatrix} \\ &= \begin{pmatrix} S_1(0)C_{11}\int_0^\infty A_1(\tau)\mathrm{d}\tau & S_1(0)C_{12}\int_0^\infty A_2(\tau)\mathrm{d}\tau \\ S_2(0)C_{21}\int_0^\infty A_1(\tau)\mathrm{d}\tau & S_2(0)C_{22}\int_0^\infty A_2(\tau)\mathrm{d}\tau \end{pmatrix} \end{aligned} \tag{4.98}$$

The basic reproduction ratio is the largest eigenvalue of the next generation matrix.

$$R_0 = \max\left(\frac{1}{2}\mathrm{tr}(K) \pm \frac{1}{2}\sqrt{\mathrm{tr}(K)^2 - 4\det(K)}\right) \tag{4.99}$$

That is:

$$R_0 = \frac{1}{2}\max\left(\begin{array}{l}\left(S_1(0)C_{11}\int_0^\infty A_1(\tau)\mathrm{d}\tau + S_2(0)C_{22}\int_0^\infty A_2(\tau)\mathrm{d}\tau\right) \\ \pm\sqrt{\begin{array}{l}\left(S_1(0)C_{11}\int_0^\infty A_1(\tau)\mathrm{d}\tau - S_2(0)C_{22}\int_0^\infty A_2(\tau)\mathrm{d}\tau\right)^2 \\ +4S_1(0)S_2(0)C_{12}C_{21}\int_0^\infty A_1(\tau)\mathrm{d}\tau\int_0^\infty A_2(\tau)\mathrm{d}\tau\end{array}}\end{array}\right) \tag{4.100}$$

We can then calculate the four special mixing cases in the same way as before, and we come to very similar results.

Preferential Mixing

$$\begin{aligned} R_{pref} &= \max\left(S_1(0)C_{11}\int_0^\infty A_1(\tau)\mathrm{d}\tau, S_2(0)C_{22}\int_0^\infty A_2(\tau)\mathrm{d}\tau\right) \\ &= \max\left(R_{pref\,1}, R_{pref\,2}\right) \end{aligned} \tag{4.101}$$

where

$$\begin{aligned} R_{pref\,1} &= S_1(0)C_{11}\int_0^\infty A_1(\tau)\mathrm{d}\tau \\ R_{pref\,2} &= S_2(0)C_{22}\int_0^\infty A_2(\tau)\mathrm{d}\tau \end{aligned} \tag{4.102}$$

Sexual Mixing

$$R_{sex} = \sqrt{S_1(0)S_2(0)C_{12}C_{21}\int_0^\infty A_1(\tau)\mathrm{d}\tau\int_0^\infty A_2(\tau)\mathrm{d}\tau} \tag{4.103}$$

Proportionate Mixing

$$\begin{aligned} R_{prop} &= S_1(0)C_{11}\int_0^\infty A_1(\tau)\mathrm{d}\tau + S_2(0)C_{22}\int_0^\infty A_2(\tau)\mathrm{d}\tau \\ &= R_{pref1} + R_{pref2} \end{aligned} \tag{4.104}$$

Reduced Proportionate Mixing

$$\begin{aligned} R_{red} &= \frac{1}{2}\left(\begin{aligned} &\left(S_1(0)C_{11}\int_0^\infty A_1(\tau)\mathrm{d}\tau + S_2(0)C_{22}\int_0^\infty A_2(\tau)\mathrm{d}\tau\right) \\ &+\sqrt{\begin{aligned}&\left(S_1(0)C_{11}\int_0^\infty A_1(\tau)\mathrm{d}\tau - S_2(0)C_{22}\int_0^\infty A_2(\tau)\mathrm{d}\tau\right)^2 \\ &+4\varepsilon^2 S_1(0)S_2(0)C_{11}C_{22}\int_0^\infty A_1(\tau)\mathrm{d}\tau\int_0^\infty A_2(\tau)\mathrm{d}\tau\end{aligned}} \end{aligned} \right) \\ &= \frac{R_{prop}}{2} + \frac{1}{2}\sqrt{\left(R_{pref1} - R_{pref2}\right)^2 + 4\varepsilon^2 R_{pref1}R_{pref2}} \end{aligned} \tag{4.105}$$

Therefore, when each class has the same or different contact/probability of infection functions $A(\tau)$, the relations between preferential, sexual, proportionate and reduced proportionate mixing are the same as when both classes had the same function.

The same analysis can be extended to include more subclasses of the population, as would be the case in more realistic examples when the population can naturally be split into multiple categories.

Chapter 5 The Final Size Equation – Two Dimensional Model

For the one dimensional model, we calculated the final size equation for a small epidemic and a large epidemic. Hence, we calculate the final size equation for a large epidemic in the two dimensional model. We need to calculate the final size in anticipation of solving the two dimensional problem with repeated epidemics.

The equation for the incidence of infection is:

$$\begin{pmatrix} i_1(t) \\ i_2(t) \end{pmatrix} = \delta(t)\begin{pmatrix} i_1(0) \\ i_2(0) \end{pmatrix} + \begin{pmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{pmatrix}\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}\int_0^t A(\tau)\begin{pmatrix} i_1(t-\tau) \\ i_2(t-\tau) \end{pmatrix}\mathrm{d}\tau \tag{5.1}$$

Where

$$A(\tau) = \begin{pmatrix} A_1(\tau) & 0 \\ 0 & A_2(\tau) \end{pmatrix} \tag{5.2}$$

As we are calculating the final size, we can ignore the initial cases of the infection in our population, i.e. the $\delta(t)$ term. We can also utilise the fact

$$-\frac{\mathrm{d}S(t)}{\mathrm{d}t} = \underline{i}(t) \tag{5.3}$$

Thus we gain:

$$\begin{pmatrix} \frac{\mathrm{d}S_1(t)}{\mathrm{d}t} & 0 \\ 0 & \frac{\mathrm{d}S_2(t)}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \int_0^t \begin{pmatrix} \mathrm{A}_1(\tau)\frac{\mathrm{d}S_1(t-\tau)}{\mathrm{d}t} & 0 \\ 0 & \mathrm{A}_2(\tau)\frac{\mathrm{d}S_2(t-\tau)}{\mathrm{d}t} \end{pmatrix} \mathrm{d}\tau \tag{5.4}$$

We can easily multiply by the inverse of the matrix $\begin{pmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{pmatrix}$:

$$\begin{aligned} &\begin{pmatrix} \frac{1}{S_1(t)} & 0 \\ 0 & \frac{1}{S_2(t)} \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}S_1(t)}{\mathrm{d}t} & 0 \\ 0 & \frac{\mathrm{d}S_2(t)}{\mathrm{d}t} \end{pmatrix} \\ &= \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \int_0^t \begin{pmatrix} \mathrm{A}_1(\tau)\frac{\mathrm{d}S_1(t-\tau)}{\mathrm{d}t} & 0 \\ 0 & \mathrm{A}_2(\tau)\frac{\mathrm{d}S_2(t-\tau)}{\mathrm{d}t} \end{pmatrix} \mathrm{d}\tau \end{aligned} \tag{5.5}$$

Performing usual matrix algebra yields:

$$\begin{aligned} &\begin{pmatrix} \frac{1}{S_1(t)}\frac{\mathrm{d}S_1(t)}{\mathrm{d}t} & 0 \\ 0 & \frac{1}{S_2(t)}\frac{\mathrm{d}S_2(t)}{\mathrm{d}t} \end{pmatrix} \\ &= \begin{pmatrix} C_{11}\int_0^t A_1(\tau)\frac{\mathrm{d}S_1(t-\tau)}{\mathrm{d}t}\mathrm{d}\tau & C_{12}\int_0^t A_2(\tau)\frac{\mathrm{d}S_2(t-\tau)}{\mathrm{d}t}\mathrm{d}\tau \\ C_{21}\int_0^t A_1(\tau)\frac{\mathrm{d}S_1(t-\tau)}{\mathrm{d}t}\mathrm{d}\tau & C_{22}\int_0^t A_2(\tau)\frac{\mathrm{d}S_2(t-\tau)}{\mathrm{d}t}\mathrm{d}\tau \end{pmatrix} \end{aligned} \tag{5.6}$$

We can then integrate equation (5.6) with respect to t from $0 \rightarrow \infty$

$$\begin{pmatrix} \int_0^\infty \frac{1}{S_1(t)}\frac{\mathrm{d}S_1(t)}{\mathrm{d}t}\mathrm{d}t & 0 \\ 0 & \int_0^\infty \frac{1}{S_2(t)}\frac{\mathrm{d}S_2(t)}{\mathrm{d}t}\mathrm{d}\tau \end{pmatrix} = \begin{pmatrix} C_{11}\int_0^\infty \int_0^t A_1(\tau)\frac{\mathrm{d}S_1(t-\tau)}{\mathrm{d}t}\mathrm{d}\tau\mathrm{d}t & C_{12}\int_0^\infty \int_0^t A_2(\tau)\frac{\mathrm{d}S_2(t-\tau)}{\mathrm{d}t}\mathrm{d}\tau\mathrm{d}t \\ C_{21}\int_0^\infty \int_0^t A_1(\tau)\frac{\mathrm{d}S_1(t-\tau)}{\mathrm{d}t}\mathrm{d}\tau\mathrm{d}t & C_{22}\int_0^\infty \int_0^t A_2(\tau)\frac{\mathrm{d}S_2(t-\tau)}{\mathrm{d}t}\mathrm{d}\tau\mathrm{d}t \end{pmatrix} \tag{5.7}$$

Changing the order of integration of the right hand sides gives:

$$\begin{pmatrix} \log\left(\frac{S_1(\infty)}{S_1(0)}\right) & 0 \\ 0 & \log\left(\frac{S_2(\infty)}{S_2(0)}\right) \end{pmatrix} = \begin{pmatrix} C_{11}\int_0^\infty \int_\tau^\infty A_1(\tau)\frac{\mathrm{d}S_1(t-\tau)}{\mathrm{d}t}\mathrm{d}t\mathrm{d}\tau & C_{12}\int_0^\infty \int_\tau^\infty A_2(\tau)\frac{\mathrm{d}S_2(t-\tau)}{\mathrm{d}t}\mathrm{d}t\mathrm{d}\tau \\ C_{21}\int_0^\infty \int_\tau^\infty A_1(\tau)\frac{\mathrm{d}S_1(t-\tau)}{\mathrm{d}t}\mathrm{d}t\mathrm{d}\tau & C_{22}\int_0^\infty \int_\tau^\infty A_2(\tau)\frac{\mathrm{d}S_2(t-\tau)}{\mathrm{d}t}\mathrm{d}t\mathrm{d}\tau \end{pmatrix} \tag{5.8}$$

$$\begin{pmatrix} \log\left(\frac{S_1(\infty)}{S_1(0)}\right) & 0 \\ 0 & \log\left(\frac{S_2(\infty)}{S_2(0)}\right) \end{pmatrix} = \begin{pmatrix} C_{11}\left(S_1(\infty)-S_1(0)\right)\int_0^\infty A_1(\tau)\mathrm{d}\tau & C_{12}\left(S_2(\infty)-S_2(0)\right)\int_0^\infty A_2(\tau)\mathrm{d}\tau \\ C_{21}\left(S_1(\infty)-S_1(0)\right)\int_0^\infty A_1(\tau)\mathrm{d}\tau & C_{22}\left(S_2(\infty)-S_2(0)\right)\int_0^\infty A_2(\tau)\mathrm{d}\tau \end{pmatrix} \tag{5.9}$$

We can write this as a system:

$$\begin{pmatrix} \log\left(\dfrac{S_1(\infty)}{S_1(0)}\right) \\ \log\left(\dfrac{S_2(\infty)}{S_2(0)}\right) \end{pmatrix} = \begin{pmatrix} C_{11}\int_0^\infty A_1(\tau)\mathrm{d}\tau & C_{12}\int_0^\infty A_2(\tau)\mathrm{d}\tau \\ C_{21}\int_0^\infty A_1(\tau)\mathrm{d}\tau & C_{22}\int_0^\infty A_2(\tau)\mathrm{d}\tau \end{pmatrix} \begin{pmatrix} S_1(\infty)-S_1(0) \\ S_2(\infty)-S_2(0) \end{pmatrix} \quad (5.10)$$

So we look for solutions of the following two equations:

$$\begin{aligned} \log\left(\frac{S_1(\infty)}{S_1(0)}\right) &= C_{11}\left(S_1(\infty)-S_1(0)\right)\int_0^\infty A_1(\tau)\mathrm{d}\tau + C_{12}\left(S_2(\infty)-S_2(0)\right)\int_0^\infty A_2(\tau)\mathrm{d}\tau \\ \log\left(\frac{S_2(\infty)}{S_2(0)}\right) &= C_{21}\left(S_1(\infty)-S_1(0)\right)\int_0^\infty A_1(\tau)\mathrm{d}\tau + C_{22}\left(S_2(\infty)-S_2(0)\right)\int_0^\infty A_2(\tau)\mathrm{d}\tau \end{aligned} \quad (5.11)$$

The above equations need to be solved simultaneously and numerically. We shall only consider the four special cases for mixing given in the previous chapter.

5.1 Special Cases

We shall now calculate the final size equation for the four special cases of mixing: preferential, sexual, proportionate and reduced proportionate mixing. For each case, equation (5.11) will be used and the values of C_{ij} altered appropriately.

Preferential Mixing

Preferential mixing is when there is only mixing within each class, so we let $C_{12} = C_{21} = 0$. So equation (5.11) becomes:

$$\begin{aligned} \log\left(\frac{S_1(\infty)}{S_1(0)}\right) &= C_{11}\left(S_1(\infty)-S_1(0)\right)\int_0^\infty A_1(\tau)\,\mathrm{d}\tau \\ \log\left(\frac{S_2(\infty)}{S_2(0)}\right) &= C_{22}\left(S_2(\infty)-S_2(0)\right)\int_0^\infty A_2(\tau)\,\mathrm{d}\tau \end{aligned} \tag{5.12}$$

Using the definitions for the previous chapter, this is:

$$\begin{aligned} \log\left(\frac{S_1(\infty)}{S_1(0)}\right) &= \left(\frac{S_1(\infty)}{S_1(0)}-1\right)R_{pref1} \\ \log\left(\frac{S_2(\infty)}{S_2(0)}\right) &= \left(\frac{S_2(\infty)}{S_2(0)}-1\right)R_{pref2} \end{aligned} \tag{5.13}$$

The two equations have become decoupled, and we are just solving two one dimensional equations similar to those in chapter 2.

Sexual Mixing

Sexual mixing refers to there only being mixing between the classes, but not within a class, so we have $C_{11} = C_{22} = 0$. Substituting this into equation (5.11):

$$\begin{aligned} \log\left(\frac{S_1(\infty)}{S_1(0)}\right) &= C_{12}\left(S_2(\infty)-S_2(0)\right)\int_0^\infty A_2(\tau)\,\mathrm{d}\tau \\ \log\left(\frac{S_2(\infty)}{S_2(0)}\right) &= C_{21}\left(S_1(\infty)-S_1(0)\right)\int_0^\infty A_1(\tau)\,\mathrm{d}\tau \end{aligned} \tag{5.14}$$

More analysis will be performed on the above equations in the following section.

Proportionate and Reduced Proportionate Mixing

We now let the mixing between classes to proportional to the mixing within classes, $C_{12} = C_{21} = \varepsilon\sqrt{C_{11}C_{22}}$ (for proportionate mixing, we let $\varepsilon = 1$). Substituting into equation (5.11), we gain:

$$\log\left(\frac{S_1(\infty)}{S_1(0)}\right) = C_{11}\left(S_1(\infty) - S_1(0)\right)\int_0^\infty A_1(\tau)\,\mathrm{d}\tau + \varepsilon\sqrt{C_{11}C_{22}}\left(S_2(\infty) - S_2(0)\right)\int_0^\infty A_2(\tau)\,\mathrm{d}\tau$$
$$\log\left(\frac{S_2(\infty)}{S_2(0)}\right) = \varepsilon\sqrt{C_{11}C_{22}}\left(S_1(\infty) - S_1(0)\right)\int_0^\infty A_1(\tau)\,\mathrm{d}\tau + C_{22}\left(S_2(\infty) - S_2(0)\right)\int_0^\infty A_2(\tau)\,\mathrm{d}\tau$$

(5.15)

A numerical method is required to solve this.

5.2 Two Dimensional Cobwebs for Sexual Mixing

We have already produced a discrete map for the one dimensional case, and we would like to apply the same process to the two dimensional case. From the calculations above, we see that solving the final size equations in two dimensions is harder than for our one dimensional case.

From equation (5.12), the final size equations for preferential mixing are decoupled of the two population classes, so we can use the cobwebbing method for the one dimension case. For proportionate and reduced proportionate mixing, the final size equation must be solved simultaneously and numerically, so repeated epidemics for this mixing method will not be considered here. When sexual mixing is used, we see from equation (5.14) that we may continue our analysis further.

We need to solve the following two equations simultaneously to calculate the final size of each population class.

$$\log\left(\frac{S_1(\infty)}{S_1(0)}\right)=C_{12}\left(S_2(\infty)-S_2(0)\right)\int_0^\infty A_2(\tau)\,\mathrm{d}\tau \qquad (5.16)$$

$$\log\left(\frac{S_2(\infty)}{S_2(0)}\right)=C_{21}\left(S_1(\infty)-S_1(0)\right)\int_0^\infty A_1(\tau)\,\mathrm{d}\tau \qquad (5.17)$$

Rearranging equation (5.16) for $S_1(\infty)$:

$$S_1(\infty)=S_1(0)e^{C_{12}\left(S_2(\infty)-S_2(0)\right)\int_0^\infty A_2(\tau)\mathrm{d}\tau} \qquad (5.18)$$

Then substituting equation (5.18) into equation (5.17):

$$\log\left(\frac{S_2(\infty)}{S_2(0)}\right)=C_{21}\left(S_1(0)e^{C_{12}\left(S_2(\infty)-S_2(0)\right)\int_0^\infty A_2(\tau)\mathrm{d}\tau}-S_1(0)\right)\int_0^\infty A_1(\tau)\,\mathrm{d}\tau \qquad (5.19)$$

which we may solve numerically.

To implement the repeated epidemic scheme as we did for the one dimensional population, we now need to solve two equations ((5.18) and (5.19)) for each infection generation.

Chapter 6 Conclusion

Systems of differential equation have been used to model many diseases, and there is a vast amount of information available on the analysis of such systems. However, when constant parameters are used to model the dynamics between the susceptible, infective and recovered populations, each member spends an exponential amount of time within each compartment. This kind of assumption does not often model the data closely, so an integral equation method has been developed.

We have shown methods to calculate the basic reproduction ratio, initial growth rate of an infection and the final size of an infection. When calculating the final size, we have used two methods: one for a small epidemic, when the basic reproduction ratio is less than one and we approximate the susceptible population by the initial number of susceptibles in the population; and a method for a large epidemic, when the basic reproduction ratio is greater than one, and we can not make the assumption that the susceptible population is constant.

By using integral equations to model the dynamics of the susceptible population we are able to generate a discrete map that looks at the effect of an epidemic on the susceptible population from one epidemic generation to the next. This was shown to converge to a stable solution where an epidemic occurs each year.

Looking at the cobweb maps (Figure 3.2 Figure 3.5), as their shape is similar to the tent map we would think that there should be some interesting dynamics present. However, it has been shown that the gradient of the line that corresponds to epidemics is always less than one, and so we always have a stable solution, where an epidemic occurs each year.

We have also applied the integral equation method to a susceptible population that was split into two subpopulations. Various mixing schemes between the two classes were analysed and the basic reproduction ratio was calculated for each case. It can be noted from Chapter 4.7 , that the basic reproduction ratios for reduce proportionate and proportionate mixing depend on the basic reproduction ratio for preferential mixing. The sexual mixing case does not hold any relation to the other mixing methods.

When calculating the final size equation for the model when the population had been divided into two distinct classes, numerical methods must be used. When preferential mixing is applied the problem becomes disjoint, and we solved two one dimensional problems. When sexual mixing is applied a system of equations was derived and the problem is left open to further analysis. Preliminary attempts to generate cobweb plots for the sexual mixing case showed this to be a difficult numerical problem, due to the double exponential nature of the equation.

Further development to the model may include splitting the population into more subclasses, and to implement numerical methods to solve the final size equations, which would assist in our understanding of repeated epidemics of infectious diseases.

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