Monitoring the Mean with Locally Weighted Averages for Skewed Processes

by

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Declaration

I hereby declare that this dissertation is my own work carried out under the supervision of Dr Jonathan Godfrey and Dr K Govindaraju. I have not submitted this work in whole or in part for any other degree or qualification in this or any other university. To the best of my knowledge, this work contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

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Abstract

Averaging functions are used in many research areas such as decision making, image processing, pattern recognition and statistics. The basic averaging function, arithmetic mean, is most widely used in statistical quality control to monitor a particular quality characteristic. However, other averaging functions such as weighted averages can be used in control charting to improve the probability of detection in process level shifts when a process distribution deviates from the normality assumption. This study focused on applying locally weighted averages as the control statistic in quality control charts to detect the process mean of a right-skewed process.

Six weights were defined: Max-weight – based on the maximum distance; PDF-weight – based on the probability density function of the process; CoPDF-weight – based on the complement of the probability density function of the process; CDF-weight – based on the cumulative probability density function of the process; CoCDF-weight – based on the complement of the cumulative density function of the process; and Haz-weight – based on the hazard function of the process. Weighted average control charts; $\tilde{X}_{max}$, $\tilde{X}_{pdf}$, $\tilde{X}_{1-pdf}$, $\tilde{X}_{cdf}$, $\tilde{X}_{1-cdf}$, and $\tilde{X}_{haz}$ were proposed to monitor the process mean using the weighted averages based on Max-weight, PDF-weight, CoPDF-weight, CDF-weight, CoCDF-weight, and Haz-weight, respectively as the control statistic. First, the behaviour of these control statistics was explored for symmetric distributions using the standard normal distribution. Second, the performance of these control charts was compared to Shewhart $\bar{X}$ control chart for right-skewed distributions using the average run length (ARL) and the standard deviation of the run length ($SDRL$). Exponential and three gamma distributions were considered to illustrate positively skewed distributions in this study. Monte-Carlo simulations were used in evaluating the ARLs and $SDRL$s and control limits for Phase II applications. Then Phase I control limits were established for all the distributions considered using bootstrapping.

When the process is symmetric, $\bar{X}$ control chart was suitable for monitoring the process mean as expected. On the other hand, $\tilde{X}_{cdf}$ and $\tilde{X}_{1-cdf}$ control charts were able to detect the variance of symmetric distributions. The importance of these results is that the weighted average control charts and the $\bar{X}$ control chart can be plotted in the same graph facilitating to simultaneously detect the mean and the variance, this is discussed as joint monitoring in the literature.

Weighted average control charts cannot monitor the process mean when the underlying distribution of the quality characteristic is identified as exponential. However, when the quality characteristic follows a gamma distribution, weighted averages outperformed the Shewhart $\bar{X}$ control chart in a variety of situations. Therefore, the locally weighted averages proposed in this study are useful in monitoring the process mean for gamma-distributed data and variance of symmetric distributions.

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Glossary

ARL	Average Run Length
$SDRL$	Standard Deviation of Run Length
SPC	Statistical Process Control
IN	in-control
OC	out-of-control
LCL	Lower Control Limit
CL	Center Line
UCL	Upper Control Limit
x_{ij}	j^{th} observation of the i^{th} sample
n	Sample size
m	Number of samples
$\bar{X}$	Sample average/ Unweighted average
$\tilde{X}_{max}$	Maximum distance based weighted average
$\tilde{X}_{pdf}$	Probability density function based weighted average
$\tilde{X}_{1-pdf}$	Complement of probability density function based weighted average
$\tilde{X}_{haz}$	Hazard function based weighted average
$\tilde{X}_{cdf}$	Cumulative function based weighted average
$\tilde{X}_{1-cdf}$	Complement of cumulative function based weighted average

Chapter 1 - Introduction

1.1 Background of the Study

Statistical quality control refers to applying statistical techniques to enhance product quality. In statistical quality control, a sample is drawn from an ongoing manufacturing process and examined for the interested quality characteristic because no manufacturing process produces all items of acceptable quality. Every production process has a specific stable pattern of variation. This natural variation is known as chance variation, and they are practically impossible to eliminate. The source of other variations can often be identified and therefore eliminated, and these are known as assignable variations. When the process runs without this assignable variation, it is said to be stable or in-control. The main purpose of statistical quality control is to identify whether the process is in-control or not. The determination of process stability is done via a control chart. Shewhart first introduced the $\bar{X}$ control chart (Montgomery, 2020), and plenty of research has been conducted in constructing control charts for various situations. The distribution of the quality characteristic is assumed to be normal in Shewhart $\bar{X}$ control chart. However, in a real-life scenario, the validity of this assumption is questionable since there are situations where the process cannot be modelled by a normal distribution. For example, the false alarm (signal indicating an out-of-control) rate will increase in a Shewhart $\bar{X}$ chart as the skewness of the quality characteristic increases (Yourstone & Zimmer, 1992).

The average (arithmetic mean) of the sample is used for examining the quality characteristic in the Shewhart $\bar{X}$ control chart. The arithmetic mean gives equal weight to all observations in the sample. Therefore, it is inappropriate when the sample contains extreme values or individual observations with differing levels of importance. Thus, the arithmetic average is not a good representation of a skewed process. As a solution, each observation can be given a weight based on its relative importance, resulting in a weighted average. Hence a weighted average can be a better representation of a skewed process. Although the use of locally weighted averaging functions in constructing a control chart is not studied previously, this research proposes their use for determining a weighted average as a control statistic in implementing a control chart for skewed processes. This implementation has to be done in two Phases. Phase I establishes the control limits, and Phase II uses these control limits in monitoring the process mean.

1.2 Research Objectives

The use of a weighted average as a control statistic in implementing a control chart for skewed processes was studied by identifying the following objectives as the focus of the research.

1 Identify locally weighted averages that can be used to monitor a positively skewed process.
2 Explore the behaviour of the weighted averages in symmetric and positively skewed processes.
3 Compare and contrast the performance of weighted average control charts with the Shewhart $\bar{X}$ chart.
4 Implementation of Phase I control chart for weighted averages.
5 Identify other possible uses of weighted average control charts.

1.3 Literature Review

1.3.1 Averaging Functions

Averaging functions extract and aggregate information from a sample. The output of an averaging function is a single value that represents the information contained in several input values. The output value lies between the minimum and maximum inputs. Beliakov et al. (2016) discussed averaging functions and their properties. Some commonly used averaging functions are listed below.

1.3.1.1 Arithmetic Mean

The most widely used averaging function is the arithmetic mean. For example, in statistical quality control charts, the arithmetic mean is used as the control statistic to monitor the process mean. The arithmetic mean is defined as, $\sum_{i=1}^{n} x_i/n$, where x_i 's are the observations, and n is the sample size.

1.3.1.2 Weighted Average

When the inputs are given weights according to their relative contribution to the total value, it provides a weighted average of the input data. The weight has two properties.

- The weight should be a value between zero and 1 ($w_i \in [0,1]$)
- The summation of the weights should equal one ($\sum w_i = 1$)

A weighted average is defined as, $\sum_{i=1}^{n} w_i\, x_i$, where x_i 's are the observations, w_i is the weight associated with x_i and n is the sample size.

1.3.1.3 Geometric and Harmonic Means

The geometric mean is defined as $(\prod_{i=1}^{n} x_i)^{1/n}$. The harmonic mean is defined as $n\left(\sum_{i=1}^{n} {}^{1}/_{x_i}\right)^{-1}$. Here x_i 's are the observations, and n is the sample size. They are widely used when discussing the rates.

1.3.1.4 Ordered Weighted Average (OWA)

The ordered weighted average (OWA) is an averaging function introduced by Yager (1988). It differs from the weighted arithmetic mean in terms of weights, where OWA considers the size of the input. OWA is defined as, $\sum_{i=1}^{n} w_i\, x_{(i)}$, where w_i is the weight associated with x_i, n is the sample size, $x_{(i)}$ is the i^{th} ordered observation in the sample, $w_i \geq 0$, and $\sum w_i = 1$. The weight can be chosen using the methods based on data, a measure of dispersion and weight generating functions etc. Xu and Da (2002) introduced the ordered weighted geometric (OWG) operator based on the geometric mean similar to the OWA operator. OWA has been used in various areas such as database systems, neural networks, decision making, mathematical programming and image processing. For more information, refer; Torra and Godo (2002), Chiclana et al. (2002), Bustince et al. (2011), and Farias et al. (2018).

1.3.1.5 Density Based Weighted Average

Angelov and Yager (2013) proposed a weighted average based on the relative density of the data sample and is defined as $\sum_{i=1}^{n} w_i(X)\, x_i$, where $w_i(X)$ is the density based weight. Sadiq and Tesfamariam (2007) used the probability density functions, which extended the method of creating OWA weights using the normal distribution function. The use of probability in density based weighting and OWA is discussed by Merigó (2010). A probabilistic-weighted-average (PWA) and probabilistic-ordered-weighted-averaging (POWA) is proposed by Merigo (2012a) and Merigo (2012b), respectively. Su et al. (2016) considered commonly used probability distributions for determining the weighting vector in weighted averages.

Averaging is commonly utilized in a variety of applications. For example, instead of arithmetic means, averaging functions were used to compute the empirical risk by Shibzukhov (2017), Paternain et al. (2015) considered averaging functions to develop an image reduction algorithm, Lin and Jiang (2014) used weighted averages in decision making, etc. For more information, see Allen (1988), Lecomte (2014), Pogromsky and Matveev (2015), and Wilkin et al. (2014). In addition, identification of the stability of the mean of a process in statistical quality control is done using averages. However, the use of locally weighted averages in constructing control charts is minimal. Therefore, this study is focused on applying locally weighted averages to control charts.

1.3.2 Control Charts for the Process Mean

Control charts are frequently used to identify the status of a process in statistical process control (SPC). These statuses are in-control (IC) and out-of-control (OC) of a process measured using two control limits: Upper control limit (UCL) and Lower control limit (LCL). In general, the Shewhart $\bar{X}$ control chart is used in process monitoring by assuming the underlying process follows a normal distribution. However, process data violate the assumption of normality in many situations.

The effect of non-normality in constructing $\bar{X}$ control chart has been studied by Burrows (1962), Schilling and Nelson (1976), Balakrishnan and Kocherlakota (1986), Yourstone and Zimmer (1992), Spedding and Rawlings (1994), Pyzde (1995), and Amhemad (2010). Tukey's λ-family of symmetric distributions was used to discuss the impact of a process departure from normality by Chan et al. (1988). These studies showed that there is an extensive influence on the performance of $\bar{X}$ control chart when the process is skewed. Various techniques have been discussed to deal with the problem which arose from the non-normality of the underlying distribution.

The first method is data transformation. The data is transformed to follow a normal distribution, and by using these transformed data, the control chart is then constructed. Chou et al. (1998) used a simulation study to show that the mean-squared error of the standard deviation estimate is increased when the process deviates from the normal assumption. As a solution, they discussed the suitability of the Johnson transformation of the data. The difficulty in this approach is determining a suitable technique and interpreting the results since the control chart is drawn in a different scale of measurements.

A second approach is to increase the sample size. Then by applying the central limit theorem, the distribution of the sample mean is approximated to a normal distribution. However, increasing the sample size is not always easy and can be expensive.

A third option is to use asymmetric control limits to monitor the process mean of a skewed process. Chen and Kuo (2007) compared the symmetric and asymmetric control limits for $\bar{X}$ chart for skewed distributions. Gamma distribution was used to illustrate positively skewed distribution and Johnson unbounded distribution demonstrates the negatively skewed distribution. They found that when the process is positively skewed, the symmetric control limits show a lower average run length (ARL) than the asymmetric control limits for large shifts in the mean. In contrast, symmetric control limits show a lower ARL than the asymmetric control limits for small shifts in the mean when the process is negatively skewed. Thus, choosing the control limits for monitoring a quality characteristic of a skewed process would be based on the direction of the skewness and the shift of the process.

Another approach is to assume that the exact distribution of the data is known and to derive the control limits. Kantam et al. (2006) determined control charts and the corresponding control limits for the process mean and process range, where the underlying distribution is log-logistic. Control charts for the process mean, where the data comes from an exponential-gamma distribution, were discussed by Rao and Kumar (2015), while Adewara et al. (2020) examined the data from a Gompertz distribution. The limitation of this technique is that the identification of the distribution needs to be accurate. Otherwise, the decisions made by examining the control chart may be faulty.

Another way of handling skewed data is to design control charts based on heuristic methods. The prevailing heuristic methods are the weighted variance (WV), the scaled weighted variance (SWV), the weighted standard deviation (WSD) and the skewness correction (SC) proposed by Bai and Choi (1995), Castagliola (2000), Chang and Bai (2001) and Chan and Cui (2003), respectively.

Bai and Choi (1995) suggested the weighted variance control chart (WV- $\bar{X}$) by splitting the skewed distribution into two parts from the mean. The standard deviation is calculated for the two parts separately. The upper control limit produced by this approach is larger than the lower limit if the data is positively skewed, while it is shorter if the data is negatively skewed. For symmetric distributions, the limits are the same as the usual $\bar{X}$ chart. Another method of obtaining the WV control limits was proposed by Choobineh and Ballard (1987) by splitting the skewed distribution into two parts from the mean of the process, and two symmetrical probability density functions are defined for each piece. The upper control limit (UCL) and lower control limit (LCL) are instituted using these probability density functions. Scaled weighted variance control chart (SWV-$\bar{X}$) proposed by Castagliola (2000) was an enhancement of the weighted variance control chart (WV- $\bar{X}$). Even though the data is split into two functions in WV method, they are not probability density functions, and the replaced probability density functions are bell-shaped but not normally distributed. Therefore, a scaled factor is used to derive the control limits. The SWV$-\bar{X}$ control chart is better in identifying downward shifts than the WV-$\bar{X}$ chart and the $\bar{X}$ chart.

Chang and Bai (2001) introduced the weighted standard deviation (WSD) control chart. First, the standard deviation of the skewed process is decomposed into two parts at its mean to create two symmetric distributions. Then the standard deviations of these symmetric distributions have been used in constructing the control limits. Finally, the sum of these two standard deviations should be equal to the standard deviation of the original distribution.

The skewness correction (SC) method proposed by Chan and Cui (2003) revised the Shewhart 3σ control limits by adding a skewness adjustment factor to the control limits. The skewness correction method was extended by Mehmood et al. (2020) for unknown parameters and unknown skewed probability distributions. The skewness correction factor defined by Mehmood depends on the skewness, dispersion, sample size and the number of estimation samples. Pongpullponsak et al. (2007) compare the efficiency of the control charts for skewed distributions. They examined the WV control chart, SWV control chart, empirical quantiles method, and extreme-value theory for skewed data. The average run length (ARL) was used to determine the performance of the control charts. A univariate control chart was introduced utilising the skewness correction method, denoted as SC$-\bar{X}_S$ by Zain et al. (2020). They demonstrated the SC$-\bar{X}_S$ control chart performs well for the non-normal data by using a simulation study and a health care example. Although SC control charts perform better than the other heuristic control charts in detecting out-of-control signals for skewed distributions, the WV$-\bar{X}$ charts are easier to implement than the SC control charts.

The synthetic control chart outperforms the Shewhart $\bar{X}$ control chart when the underlying distribution is normal. Khoo et al. (2008) presented a new synthetic control chart for monitoring shifts in the process mean of skewed populations using the WV technique. Positively skewed distributions; weibull, lognormal and gamma are used with the normal distribution to compare the performance of this new control chart with the existing control charts using the Monte Carlo simulations. Although the WSD$-\bar{X}$ control chart shows less ARL than the WV$-\bar{X}$ control chart, the synthetic WSD$-\bar{X}$ control chart gives high ARL than the synthetic WV$-\bar{X}$ control chart. It was extended to a synthetic scaled weighted variance (synthetic SWV) control chart by Castagliola and Khoo (2009) and demonstrated synthetic SWV$-\bar{X}$ control chart was superior to the synthetic WV$-\bar{X}$ control chart for detecting the downward shifts in the mean and the results were robust with the degree of the skewness.

Lin and Chou (2007) discussed the performance of variable parameter (VP) $\bar{X}$ charts when the data is not normally distributed. The gamma and t distributions were considered to illustrate that the VP$-\bar{X}$ control chart was more effective in detecting small mean shifts regardless of the nature of the distribution.

1.3.3 Application of Averaging Functions in Statistical Quality Control Charts

Using averaging functions in control charts have been found in the literature. For example, a distance-based weight was used in constructing a control chart when the data contains outliers by Kao (2016). He used the dispersion of data represented by the distance between the data and the mean, where the shorter distances were assigned a smaller weight. In contrast, outliers with a considerable distance were given more significant weight. A weighted average control chart and an inter-quartile control chart monitor the mean and the range, respectively. The average and range of a subgroup were replaced from a trimmed mean and the trimmed mean of the subgroup ranges, respectively, to implement a new control chart named the trimmed $\bar{X}$ control chart by Langenberg and Iglewicz (1986).

The control limits were found by using an averaging function instead of the subgroup mean used in $\bar{X}$ control chart. Schoonhoven and Does (2010) studied non-normality in $\bar{X}$ control chart and established new control limits based on a pooled sample standard deviation and Gini's mean sample differences. Further, Karagöz (2018) proposed a modified Shewhart (MS), a modified weighted variance (MWV), and a modified skewness correction (MSC) method to construct control limits by replacing the overall mean by a trimmed mean.

The extensive literature review found that locally weighted averages were largely underused and were not assessed properly in the context of constructing control charts. Therefore, this study focused on identifying a locally weighted average as the control statistic to implement a control chart for positively skewed underlying distribution.

1.4 Layout of the Chapters

Chapter 1: Introduction

Chapter 1 includes a background of the study followed by the objectives. A comprehensive review of the use of control charts for process mean and averaging functions is presented, and then use of averaging functions in control charts for the skewed process is discussed.

Chapter 2: Methodology

Chapter 2 defines the locally weighted averages that can be used as control statistic and describes the Phase I and Phase II applications of statistical control charts and the tools used to implement the control charts: Monte-Carlo simulation and Bootstrapping. In addition, the methodologies for finding the control limits for Phase I application and obtaining the average run lengths (ARL) and standard deviation of run lengths ($SDRL$) are detailed.

Chapter 3: Results

The results obtained in Phase II monitoring (simulation results of identifying the ARL and $SDRL$ of control charts constructed) with appropriate descriptive analysis are included in Chapter 3.

Chapter 4: Discussion

A detailed discussion on the results in Chapter 3 and the construction of Phase I control charts are discussed in Chapter 4. Moreover, the distribution and the behaviour of the weighted averages are analysed. The detection of a shift in variance of a normal distribution is also included.

Chapter 5: Conclusions and Future Study

The major findings of the study will be summarised in Chapter 5. In addition, a summary of research objectives, achievements, and limitations are included with the future research directions.

Chapter 2 - Methodology

2.1 Introduction

This chapter includes the definitions of proposed weighted averages and the probability distributions used in this study, including the methodologies for comparing each weighted average control chart against the state of the art, Shewhart $\bar{X}$ control chart and for constructing control limits. Further, the Monte-Carlo simulation and the bootstrapping methodologies are discussed.

2.2 Proposed Control Statistic

Six weighted averages are proposed as new control statistic to detect the shifts in mean and variance from the in-control distribution where the underlying distribution is symmetric or positively skewed.

Let X_{ij} denote the j^{th} observation of the i^{th} sample of a particular quality characteristic where $i = 1,2,\dots,m$, $j = 1,2,\dots,n$, m is the number of samples, and n is the sample size. The unweighted average is defined as,

$$\bar{X} = \frac{\sum_i^m \sum_j^n X_{ij}}{n \times m}$$

Unweighted average is used as the default control statistic in Shewhart $\bar{X}$ control chart.

The weights proposed in this study explored the possible scenario of assigning a value to the data according to its relative importance. Six possibilities were considered as follows.

2.2.1 Maximum Distance Based Weighted (Max-weight) Average

The weight based on the maximum distance is referred to as *Max-weight* and denoted as W_{max}. The Max-weight was defined using the distance from the maximum value in the sample to the observation as,

$$W_{j(max)} = \frac{|X_{i(n)} - X_{ij}|}{\sum_{\forall j} |X_{i(n)} - X_{ij}|},$$

where $X_{i(n)}$ is the maximum observation in the i^{th} sample. A weighted average $\tilde{X}_{max}$ is defined as

$$\tilde{X}_{max} = \sum_{j=1}^{n} W_{j(max)} \times X_{ij}.$$

$\tilde{X}_{max}$ is proposed as a control statistic in this study, and the control chart designed using the statistic $\tilde{X}_{max}$ is denoted as the $\tilde{X}_{max}$ control chart.

2.2.2 Density Based Weighted (PDF-weight) Average

The probability density function of the underlying distribution was considered to define a novel weight referred to as *PDF-weight* and denoted as W_{pdf}. The probability density at the observation X_{ij} denoted by $f(X_{ij})$ is observed and W_{pdf} is defined as

$$W_{j(pdf)} = \frac{f(X_{ij})}{\sum \forall j \, f(X_{ij})}.$$

A weighted average $\tilde{X}_{pdf}$ is defined as

$$\tilde{X}_{pdf} = \sum_{j=1}^{n} W_{j(pdf)} \times X_{ij}.$$

$\tilde{X}_{pdf}$ is proposed as a control statistic in this study, and the control chart designed using the statistic $\tilde{X}_{pdf}$ is denoted as the $\tilde{X}_{pdf}$ control chart.

2.2.3 Complement of Density Based Weighted (CoPDF-weight) Average

The complement of the probability density function of the underlying distribution was considered to define a weight referred to as *CoPDF-weight* and denoted as W_{1-pdf}. The complement of the probability density at the observation X_{ij} is observed and W_{1-pdf} and is defined as,

$$W_{j(1-pdf)} = \frac{1 - f(X_{ij})}{\sum \forall j \left(1 - f\left(X_{ij}\right)\right)}.$$

A weighted average $\tilde{X}_{1-pdf}$ is defined as

$$\tilde{X}_{1-pdf} = \sum_{j=1}^{n} W_{j(1-pdf)} \times X_{ij}.$$

$\tilde{X}_{1-pdf}$ is proposed as a control statistic in this study, and the control chart designed using the statistic $\tilde{X}_{1-pdf}$ is denoted as the $\tilde{X}_{1-pdf}$ control chart.

2.2.4 Hazard Function Based Weighted (Haz-weight) Average

The weight based on the hazard function (h_{ij}) is referred to as *Haz-weight* and denoted as W_{haz}. The hazard function is defined as

$$h_{ij} = \frac{f(X_{ij})}{1 - P(X \le X_{ij})},$$

and the Haz-weight is defined as

$$W_{j(haz)} = \frac{h_{ij}}{\sum_{\forall j} h_{ij}}.$$

A weighted average $\tilde{X}_{haz}$ is defined as

$$\tilde{X}_{haz} = \sum_{j=1}^{n} W_{j(haz)} \times X_{ij}.$$

$\tilde{X}_{haz}$ is proposed as a control statistic in this study, and the control chart designed using the statistic $\tilde{X}_{haz}$ is denoted as the $\tilde{X}_{haz}$ control chart.

2.2.5 Cumulative Function Based Weighted (CDF-weight) Average

The *CDF-weight* is defined using the cumulative probability of the underlying distribution and denoted as W_{cdf} defined as

$$W_{j(cdf)} = \frac{P(X \le X_{ij})}{\sum_{\forall j} P(X \le X_{ij})}.$$

A weighted average $\tilde{X}_{cdf}$ is defined as

$$\tilde{X}_{cdf} = \sum_{j=1}^{n} W_{j(cdf)} \times X_{ij}.$$

$\tilde{X}_{cdf}$ is proposed as a control statistic in this study, and the control chart designed using the statistic $\tilde{X}_{cdf}$ is denoted as the $\tilde{X}_{cdf}$ control chart.

2.2.6 Complement of Cumulative Function Based Weighted (CoCDF-weight) Average

The weight based on the complement of the cumulative probability function is referred to as *CoCDF-weight* and denoted as $W_{(1-cdf)}$ and is defined as

$$W_{j(1-cdf)} = \frac{1 - P(X \le X_{ij})}{\sum_{\forall j} \left(1 - P(X \le X_{ij})\right)}$$

A weighted average $\tilde{X}_{(1-cdf)}$ is defined as

$$\tilde{X}_{(1-cdf)} = \sum_{j=1}^{n} W_{j(1-cdf)} \times X_{ij}.$$

$\tilde{X}_{(1-cdf)}$ is proposed as a control statistic in this study, and the control chart designed using the statistic $\tilde{X}_{(1-cdf)}$ is denoted as $\tilde{X}_{(1-cdf)}$ control chart.

2.3 Probability Distributions

2.3.1 Normal Distribution

The normal distribution with mean μ and standard deviation σ denoted $N(\mu, \sigma^2)$ is defined as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \text{ ; for } -\infty < x < \infty, -\infty < \mu < \infty \text{ and } \sigma > 0.$$

When $\mu = 0$ and $\sigma^2 = 1$, the normal distribution is referred to as a standard normal distribution. The sample average $\bar{X}$ of a normal random variable follows a normal distribution with the mean μ and standard deviation $\sigma/\sqrt{n}$, where n is the sample size.

2.3.2 Exponential Distribution

An exponential random variable X with rate λ is denoted as $X \sim \exp(\lambda)$. The probability density function is given by,

$$f(x) = \lambda e^{-\lambda x} \text{ ; } x \geq 0.$$

The mean and the variance of the exponential distribution is $1/\lambda$ and $1/\lambda^2$, respectively. For random samples derived from an exponential distribution, the sample average $\bar{X}$ follows a gamma distribution with shape parameter n and the scale parameter $1/n\lambda$ where n is the sample size (Larsen & Marx, 2005).

2.3.3 Gamma Distribution

The gamma distribution with shape parameter α and scale parameter β is defined as,

$$f(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} \text{ ; } x > 0 \text{ and } \alpha, \beta > 0$$

and denoted as $Gam(\alpha, \beta)$. The mean and the variance of the gamma distribution is $\alpha\beta$ and $\alpha\beta^2$, respectively. When $\alpha = 1$, the gamma distribution becomes an exponential distribution with mean $1/\beta$. When the sampling distribution of X follows a gamma distribution, then $\bar{X}$ also follows a gamma distribution with shape and scale parameters $n\alpha$ and β/n, respectively (Larsen & Marx, 2005).

2.4 Phase I and Phase II Control Chart

In practice, the structure of the statistical process control consists of two phases, Phase I is known as the retrospective phase, and Phase II is known as the prospective or monitoring phase. The aim of Phase I is to understand the process and determine the stability of the process. In Phase I, the data is evaluated to verify no assignable causes of variations and determine the process control limits. The control limits specified in Phase I are trial control limits, and these control limits are modified until there are no assignable causes of variations present. After these iterative steps, the unknown parameters are estimated. The Phase II control limits can be determined using the estimations in Phase I and used in monitoring the process. When the observed control statistic lies within the control limits, the process is said to be in control, and if the control statistic goes beyond the control limits, the process is said to be out of control. Phase I is crucial because the process monitoring (Phase II) efficacy depends on Phase I estimates. (Montgomery, 2020)

2.5 Average Run Length (ARL)

Average run length (ARL) is the average number of samples before an out-of-control signal is exhibited. The ARL is defined as $1/P$, where P is the probability of a point exceeding the control limits. The in control ARL, denoted by ARL_0 is longer and the out-of-control ARL (ARL_1) is shorter. When a process follows a normal distribution, it gives $P = 0.0027$ for an in control $\bar{X}$ chart. Therefore, the ARL_0 for a $\bar{X}$ chart with three-sigma limits is 370. ARL can be used to evaluate the performance of control charts (Aroian & Levene, 1950). Control charts have the same false alarm rate when their in-control ARL are equal. ARL_1 of two control charts with the same ARL_0 can be directly compared and the control chart with less ARL_1 value outperforms the other control chart.

2.6 Monte Carlo Simulation

Monte Carlo simulation is a mathematical technique that is used in modelling by generating random samples. A random sample is generated from the probability distribution to obtain the parameters of interest. Then the simulation is conducted using those parameters. Kalos and Whitlock (2009) summarized the steps in Monte Carlo simulation as,

1. Define possible inputs and identify the statistical probability distribution.
2. Generate possible inputs through random sampling from the probability distribution over the domain.
3. Perform simulation with these input parameters.
4. Aggregate and analyze the output results statistically.

Monte-Carlo simulation assumes that the distribution is known, and therefore, it was used to find the run length distribution of the control statistic in Phase II in this study.

2.7 Bootstrapping

Efron and Lepage (1992) discussed the bootstrap method. The main concept of bootstrapping is generating samples from the original data with replacement. It is a powerful method of making inferences about a statistic when the distribution of the statistic is unknown. The steps in bootstrapping are,

1. Draw a random sample of size n.
2. Calculate the statistic of interest.
3. Generate B random samples of size n from the original data, known as the bootstrap sample.
4. Construct the sampling distribution of the statistic of interest from the bootstrap sample.

In practice, the distribution of the data is unknown. Therefore, the distribution of the observations is identified, and parameters are estimated. However, the distributions of the weighted averages are also unknown. Hence, bootstrapping is used in constructing the control limits for Phase I in this study.

2.8 Comparison of Weighted Average Control Charts with Shewhart $\bar{X}$ Control Chart in Phase II

The performance of the Shewhart $\bar{X}$ chart and the weighed average control charts ($\tilde{X}$ charts) were compared in terms of their ARLs. A control chart is superior when it has the lowest ARL_1 because it detects process changes quickly. ARLs of the weighted average control charts and the $\bar{X}$ control chart were estimated using the Monte Carlo simulation approach. The simulation consists of two stages. In the first simulation stage, the in-control control limits were found for known parameters, and the ARL for the control charts were obtained by simulating the run length distribution in the second stage. The steps of each stage are described below.

Stage I: Determined the control limits

Step 1: Generated a random sample of size n (5 and 10) from the in-control process.

Step2: Calculated the unweighted average $\bar{X}$ and weighted averages, $\tilde{X}_{max}$, $\tilde{X}_{pdf}$, $\tilde{X}_{1-pdf}$, $\tilde{X}_{cdf}$, $\tilde{X}_{(1-cdf)}$ and $\tilde{X}_{haz}$.

Step 3: For each sample size, Steps 1 and 2 were repeated for $N = 1,000,000$ times to obtain the empirical distribution of the control statistic ($\bar{X}$ and $\tilde{X}$s).

Step 4: Found the α^{th} and $(1-\alpha)^{th}$ percentiles of the empirical distributions of $\bar{X}$ and $\tilde{X}$s to obtain the control limits of the control statistic ($\alpha = 0.01$).

Stage 2: Obtained the ARL and $SDRL$

Step 1: Generated a random sample of size n (5 and 10) from the particular distribution and obtained the control statistics; $\bar{X}$ and $\tilde{X}$s.

Step 2: Repeated Step 1 until a false alarm occurred and counted the number of occurrences for the control statistic, which refers to the run length of the control statistic.

Step 3: Repeated Step 2 for N (20000) times to obtain the run length distributions.

Step 4: Calculated the mean and the standard deviation of the run-length distributions, which refers to ARL_0 and $SDRL_0$ of the control chart.

Step 5: Changed the parameter values and repeated Steps 1 to 3 and calculated the mean and standard deviation of the run-length distributions, which refers to ARL_1 and $SDRL_1$ for several out-of-control states.

The standard normal distribution was considered to examine the behaviour of the weighted averages when the underlying distribution is symmetric. The Average run length (ARL) and the standard deviation of the run length ($SDRL$) were found for all weighted averages and the unweighted average for several shifts. Shifts occurred by an increase in the mean while keeping variance in-control, an increase in the variance while keeping mean in-control, then increasing the mean and the variance simultaneously and finally increasing the variance while the mean decreased was considered.

Exponential and gamma distributions were used to discuss the performance of the weighted averages when the underlying distribution is positively skewed. The standard exponential distribution, $exp(1)$, was considered in this study. The weighted averages, $\tilde{X}_{max}$,$\tilde{X}_{pdf}$,$\tilde{X}_{(1-pdf)}$,$\tilde{X}_{cdf}$ and $\tilde{X}_{(1-cdf)}$ were discussed as the control statistic and $\tilde{X}_{haz}$ was not considered because the hazard function is a constant (λ). So $\tilde{X}_{haz}$ is the same as $\bar{X}$. The ARL and the $SDRL$ were determined for shifts occurring in the mean in order to discuss the performance of the weighted average control charts.

The shape of the gamma distribution is based on the selection of the parameters. This study considered three gamma distributions: a shape parameter less than one, a shape parameter greater than one and less than the scale parameter and a shape parameter greater than one and the scale parameter. Figure 2.1 shows the shapes of the densities used in this study.

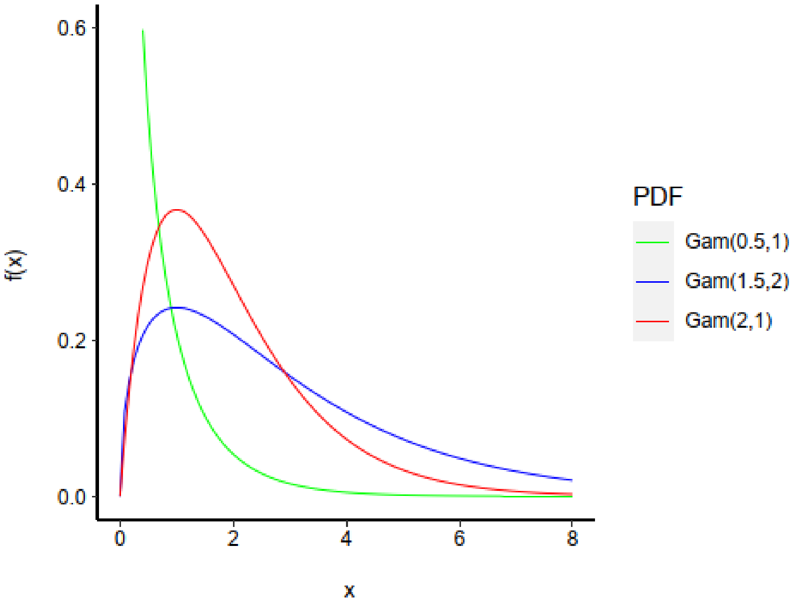

Figure 2.1: Gamma density plots

The mean of the gamma distribution depends on both parameters. A change in the mean can occur in three ways: a change in any one parameter or both parameters, which may cause an out-of-control state in the process. Four ways considered in this study were:

1. The mean was shifted by changing only the shape parameter.
2. The mean was shifted by changing only the scale parameter.
3. The mean was shifted by changing both parameters (both parameters either were increased or decreased, one parameter was increased while the other parameter was decreased)
4. The mean was in-control, but the parameters were shifted (which shifts the variance of the process from in-control to out-of-control).

A shift in the mean affects the ARL of the control chart. The performance of the weighted average control charts will be discussed in terms of the ARL of the particular control chart. The in-control ARL was used as 100 $(\alpha = 0.01)$ in the simulations of this study. The control limits for the different control statistic were found using the steps highlighted in Stage 1 to have the ARL_0 approximately 100. Then, the ARL of the control charts were found using the steps stated in Stage 2 for several shifts in the parameters, making the in-control mean into out-of-control. The ARL_1 from each weighted average control chart was compared with the ARL_1 of Shewhart $\bar{X}$ control chart to determine the performance of the weighted average control charts.

2.9 Implementation of Control Charts in Phase I

Control limits established for Phase I is repetitive. First, data was randomly generated. Then the distribution of the observations was identified, and the parameters were estimated. However, the distribution of the weighted averages is often unknown. Therefore, the bootstrap technique was used in this study to find the control limits in Phase I. The steps for constructing control limits were as follows:

Step 1: Considered m (25) subgroups of size n $(5 \text{ or } 10)$ generated from the known distribution.

Step 2: Obtained a random sample of size n with replacement from the data generated in Step 1.

Step 3: Calculated the unweighted and weighted averages from the bootstrap sample drawn in Step 2.

Step 4: Repeated Steps 2 and 3 for B $(100{,}000)$ number of bootstrap samples to obtain the empirical distribution of the averages.

Step 5: Found the α^{th}, 0.5^{th} and $(1-\alpha)^{th}$ percentiles of the empirical distributions as the control limits $(\alpha = 0.0027)$.

Using the bootstrap control limits, the $\bar{X}$ control chart and weighted average control charts were drawn to identify whether the process is in control or not. If the process was identified as out of control, the control limits need to be revised. Otherwise, the control limits were used in Phase II for monitoring the process.

2.10 Summary

The weighted averages proposed in section 2.2 were considered as the control statistic to construct the control charts in this study. The performance of the weighted average control charts was compared to the Shewhart $\bar{X}$ control chart using ARL. The ARLs obtained, and relative descriptive analysis are presented in Chapter 3.

Chapter 3 - Results

3.1 Introduction

This chapter presents the tables of ARL and $SDRL$ for the control charts discussed in this study for different situations with their relative descriptive analyses. Also, the performance of the proposed weighted average control charts is compared to the existing Shewhart $\bar{X}$ control chart. Standard normal, exponential and gamma distributions were used as the underlying distributions, and the parameters are assumed to be known. Since the ARL and $SDRL$ are approximately in the same order, the conclusions made on the performance of the control charts based on ARL and $SDRL$ are also the same.

3.2 Symmetric Distributions

The standard normal distribution was considered for illustrating the performance of weighted average control charts when the underlying distribution is symmetric. Figure 3.1 shows the $N(0,1)$ density curve with the mean of the control statistic calculated from a randomly generated empirical distribution for subgroup size ten. The mean of the control statistic based on CDF-weight and Haz-weight were positioned to the right, while Max-weight and CoCDF-weight were positioned to the left of the unweighted average. The weighted average based on the PDF-weight overlapped with the unweighted average for symmetric distributions.

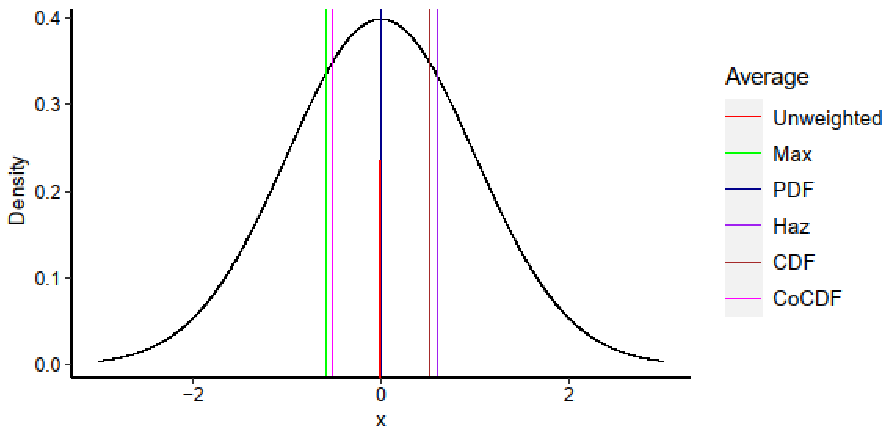

Figure 3.1: N(0,1) density curve with the means of control statistics

The Shewhart $\bar{X}$ control chart gave less ARL_1 compared to the weighted average control charts discussed in this study in detecting the mean shifts. The corresponding ARL and $SDRL$ are given in Table 3.1. The upward and downward mean shifts reacted similarly because of the symmetry. Figure 3.2 shows the in-control and out-of-control probability density functions for upward shifts of the mean. The density functions shifted to the right as the mean increased from its in-control value and shifted to the left for the mean decreased from its in-control value.

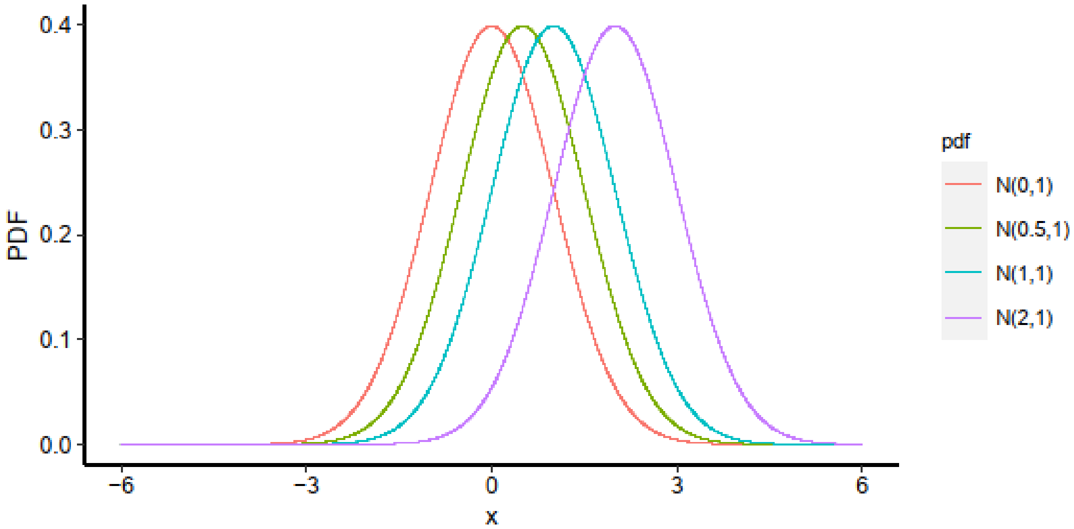

Figure 3.2: In-control and out-of-control pdfs for upward mean shift, variance in-control - N(0,1)

The ARL and $SDRL$ for variance increases, while the mean is in-control, are given in Table 3.2. The weighted average control charts $\tilde{X}_{max}$, $\tilde{X}_{haz}$, $\tilde{X}_{cdf}$ and $\tilde{X}_{(1-cdf)}$ showed less ARL_1 compared to the $\bar{X}$ control chart in detecting an increase in the variance when the mean is in its in-control state. The smallest ARL_1 was given by $\tilde{X}_{(1-cdf)}$ and $\tilde{X}_{cdf}$ control charts. Figure 3.3 demonstrates the in-control and out-of-control pdf for variance increase, and it shows that the pdfs were flattened as the variance increases.

Table 3.1: ARL and $SDRL$ for upward mean shift, variance in-control - N(0,1)

Sample size	Mean	Standard Deviation	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{haz}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart	
			ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**0.00**	**1**	**100.67**	**100.43**	**99.64**	**98.57**	**100.45**	**100.02**	**100.62**	**100.16**	**100.18**	**99.86**	**100.02**	**98.9**
	0.20	1	56.60	56.48	58.20	57.61	66.33	65.85	88.12	87.57	78.19	77.83	60.98	60.29
	0.50	1	13.88	13.39	16.60	16.06	20.62	20.1	35.36	34.86	26.20	25.63	19.67	19.24
	1.00	1	2.73	2.18	3.49	2.95	4.60	4.08	7.03	6.48	4.86	4.33	4.61	4.06
	1.50	1	1.28	0.60	1.53	0.9	1.96	1.37	2.23	1.67	1.72	1.12	2.00	1.42
n=10	**0.00**	**1**	**100.68**	**100.40**	**98.08**	**97.48**	**99.7**	**99.08**	**100.17**	**99.66**	**100.27**	**99.28**	**100.05**	**99.21**
	0.20	1	37.17	36.70	41.32	40.82	47.90	47.56	70.18	69.43	57.87	57.22	48.38	47.71
	0.50	1	6.26	5.73	7.90	7.42	10.17	9.67	18.03	17.5	12.39	11.90	11.27	10.76
	1.00	1	1.39	0.73	1.68	1.07	2.13	1.55	2.85	2.31	2.08	1.49	2.56	2.00
	1.50	1	1.02	0.13	1.06	0.26	1.20	0.49	1.2	0.5	1.09	0.31	1.38	0.72

Table 3.2: ARL and $SDRL$ for variance increase, mean in-control - N(0,1)

Sample size	Mean	Standard Deviation	Variance	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{haz}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart	
				ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	0.00	**1.0**	**1.00**	**100.67**	**100.43**	**99.64**	**98.57**	**100.45**	**100.02**	**100.62**	**100.16**	**100.18**	**99.86**	**100.02**	**98.9**
	0.00	**1.2**	**1.44**	31.46	31.04	29.04	28.45	50.01	49.6	26.21	25.77	26.08	25.67	26.31	25.67
	0.00	**1.5**	**2.25**	11.71	11.16	8.72	8.21	24.42	24.03	7.37	6.85	7.22	6.70	7.27	6.76
	0.00	**2.0**	**4.00**	5.06	4.55	3.32	2.78	11.46	10.90	2.77	2.21	2.71	2.16	2.73	2.17
n=10	0.00	**1.0**	**1.00**	**100.68**	**100.40**	**98.08**	**97.48**	**99.7**	**99.08**	**100.17**	**99.66**	**100.27**	**99.28**	**100.05**	**99.21**
	0.00	**1.2**	**1.44**	31.23	30.80	32.02	31.53	51.71	51.5	20.15	19.67	21.00	20.55	20.81	20.32
	0.00	**1.5**	**2.25**	11.68	11.18	10.57	9.96	27.33	26.79	4.48	3.97	4.69	4.19	4.70	4.17
	0.00	**2.0**	**4.00**	5.04	4.52	4.12	3.57	14.00	13.49	1.76	1.16	1.79	1.19	1.79	1.19

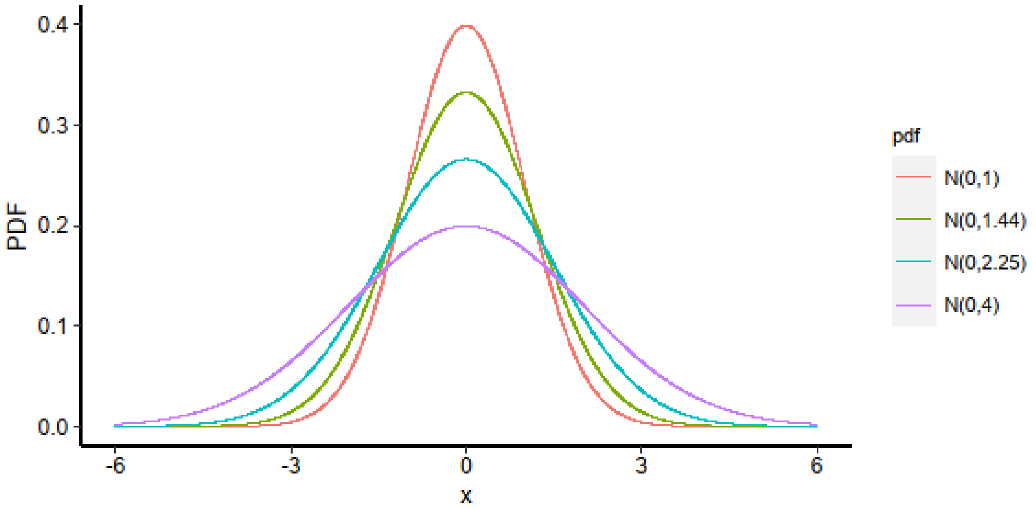

Figure 3.3: In-control and out-of-control pdfs for variance increases, mean in-control - N(0,1)

Table 3.3 provides the ARL and $SDRL$ for simultaneous mean and variance increase. The $\tilde{X}_{cdf}$ chart showed less ARL_1 than the proposed weighted average control charts in this study and the $\bar{X}$ control chart for detecting the shifts that increase the mean and the variance. When the increase in the variance is small compared to the increase in the mean, the $\bar{X}$ control chart provided a lower ARL than the $\tilde{X}_{cdf}$ control chart. However, when the sample size increases, both $\tilde{X}_{cdf}$ and $\bar{X}$ charts led to approximately equal ARL_1. Figure 3.4 shows the in-control and the out-of-control pdf discussed in Table 3.3. As the mean and the variance increased, the density curve shifted to the right and flattened.

ARL and $SDRL$ for the mean shift downward while the variance increases are given in Table 3.4. The $\tilde{X}_{(1-cdf)}$ control chart showed the lowest ARL_1 among the discussed weighted average control charts and the $\bar{X}$ control chart in detecting the mean shifts downward with increased variance. When the mean shift is large and the increase in the variance is comparatively low, the ARL_1 of $\bar{X}$ control chart was lower than the ARL_1 of the $\tilde{X}_{(1-cdf)}$ control chart. However, with increasing sample size, the $\tilde{X}_{(1-cdf)}$ control chart showed less ARL_1. Further, $\tilde{X}_{max}$ and $\tilde{X}_{haz}$ control charts gave lower ARL_1 than the $\bar{X}$ control chart for small downward and upward mean shifts, respectively. Figure 3.5 shows the in-control and the out-of-control pdf for the mean decreases and variance increases, and it demonstrates that the density curve moves to the left and is flattened.

Table 3.3: ARL and $SDRL$ for upward mean shift, variance increases - N(0,1)

Sample size	Mean	Standard Deviation	Variance	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{haz}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart	
				ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**0.0**	**1.0**	**1.00**	**100.67**	**100.43**	**99.64**	**98.57**	**100.45**	**100.02**	**100.62**	**100.16**	**100.18**	**99.86**	**100.02**	**98.9**
	0.2	**1.2**	**1.44**	22.97	22.45	29.93	29.49	40.39	39.99	19.38	18.86	18.36	17.91	28.33	27.87
		1.5	**2.25**	10.03	9.50	12.26	11.69	22.48	22.09	5.82	5.3	5.66	5.14	8.81	8.30
		2.0	**4.00**	4.82	4.28	4.78	4.24	11.16	10.64	2.46	1.88	2.41	1.84	3.08	2.54
	0.5	**1.2**	**1.44**	8.89	8.38	14.29	13.72	19.01	18.49	10.31	9.82	9.03	8.53	17.80	17.28
		1.5	**2.25**	5.83	5.27	10.31	9.81	15.42	15.02	4.00	3.45	3.80	3.27	9.57	9.09
		2.0	**4.00**	3.77	3.24	5.31	4.81	9.83	9.28	2.06	1.48	2.01	1.43	3.58	3.03
	1.0	**1.2**	**1.44**	2.58	2.02	4.01	3.49	5.78	5.25	3.75	3.2	3.14	2.59	5.89	5.39
		1.5	**2.25**	2.44	1.87	4.53	3.99	6.97	6.44	2.28	1.7	2.12	1.54	6.45	5.95
		2.0	**4.00**	2.27	1.70	4.30	3.77	6.84	6.35	1.57	0.95	1.53	0.90	3.98	3.44
	1.5	**1.2**	**1.44**	1.35	0.69	1.81	1.2	2.59	2.03	1.81	1.21	1.57	0.95	2.65	2.09
		1.5	**2.25**	1.44	0.80	2.23	1.67	3.59	3.06	1.49	0.86	1.42	0.77	3.57	3.03
		2.0	**4.00**	1.53	0.90	2.77	2.22	4.51	4.00	1.28	0.6	1.26	0.57	3.57	3.03
n=10	**0.00**	**1.0**	**1.00**	**100.68**	**100.40**	**98.08**	**97.48**	**99.7**	**99.08**	**100.17**	**99.66**	**100.27**	**99.28**	**100.05**	**99.21**
	0.2	**1.2**	**1.44**	17.58	17.06	27.42	26.87	34.78	34.3	11.89	11.36	11.27	10.79	28.80	28.35
		1.5	**2.25**	8.78	8.27	11.8	11.2	23.01	22.45	3.33	2.78	3.36	2.82	6.54	6.04
		2.0	**4.00**	4.54	4.01	4.25	3.7	13.26	12.71	1.56	0.93	1.58	0.95	2.06	1.48
	0.5	**1.2**	**1.44**	4.89	4.36	8.83	8.23	11.6	11.07	5.22	4.69	4.55	4.01	16.95	16.58
		1.5	**2.25**	3.88	3.34	8.69	8.14	12.44	11.95	2.2	1.63	2.15	1.57	9.24	8.74
		2.0	**4.00**	3.03	2.47	5.17	4.63	10.65	10.13	1.35	0.68	1.35	0.68	2.57	2.00
	1.0	**1.2**	**1.44**	1.45	0.81	2.10	1.52	3.02	2.47	1.85	1.25	1.62	1.01	4.33	3.80
		1.5	**2.25**	1.53	0.90	2.77	2.22	4.5	3.98	1.36	0.7	1.31	0.64	6.88	6.35
		2.0	**4.00**	1.62	1.00	3.54	2.98	6.24	5.7	1.13	-0.38	1.12	0.37	3.47	2.92
	1.5	**1.2**	**1.44**	1.04	0.20	1.17	0.44	1.53	0.91	1.14	0.4	1.08	0.29	1.97	1.38
		1.5	**2.25**	1.08	0.29	1.41	0.77	2.3	1.72	1.08	0.29	1.06	0.25	3.37	2.82
		2.0	**4.00**	1.16	0.43	1.94	1.34	3.78	3.25	1.04	0.20	1.04	0.19	3.67	3.12

Table 3.4: ARL and $SDRL$ for downward mean shift, variance increases - N(0,1)

Sample size	Mean	Standard Deviation	Variance	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{haz}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart	
				ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**0.0**	**1.0**	**1.00**	**100.67**	**100.43**	**99.64**	**98.57**	**100.45**	**100.02**	**100.62**	**100.16**	**100.18**	**99.86**	**100.02**	**98.9**
	-0.2	**1.2**	**1.44**	22.78	22.30	22.39	21.89	40.06	36.91	27.41	26.97	28.27	27.88	18.35	17.82
		1.5	**2.25**	10.07	9.61	7.92	7.41	22.29	21.79	8.75	8.28	8.75	8.17	5.67	5.13
		2.0	**4.00**	4.80	4.27	3.45	2.91	11.13	10.63	3.13	2.57	3.07	2.53	2.43	1.86
	-0.5	**1.2**	**1.44**	8.84	8.32	10.27	9.74	18.88	18.37	17.23	16.65	17.91	17.40	9.06	8.55
		1.5	**2.25**	5.80	5.27	4.82	4.27	15.37	14.88	9.56	9.09	9.56	9.01	3.83	3.29
		2.0	**4.00**	3.78	3.25	2.65	2.08	9.83	9.31	3.67	3.14	3.57	3.04	2.01	1.43
	-1.0	**1.2**	**1.44**	2.58	2.02	3.42	2.87	5.77	5.2	5.74	5.17	5.93	5.37	3.15	2.61
		1.5	**2.25**	2.44	1.88	2.38	1.82	6.97	6.46	6.43	5.89	6.48	5.95	2.12	1.54
		2.0	**4.00**	2.29	1.71	1.78	1.18	6.83	6.28	4.14	3.64	3.98	3.46	1.54	0.91
	-1.5	**1.2**	**1.44**	1.35	0.69	1.67	1.06	2.6	2.04	2.59	2.03	2.66	2.11	1.57	0.94
		1.5	**2.25**	1.44	0.79	1.49	0.86	3.57	3.03	3.52	2.98	3.56	3.03	1.41	0.76
		2.0	**4.00**	1.54	0.91	1.36	0.7	4.49	3.96	3.67	3.14	3.57	3.02	1.26	0.57
n=10	**0.0**	**1.0**	**1.00**	**100.68**	**100.40**	**98.08**	**97.48**	**99.7**	**99.08**	**100.17**	**99.66**	**100.27**	**99.28**	**100.05**	**99.21**
	-0.2	**1.2**	**1.44**	17.83	17.32	14.96	14.46	35.23	34.62	25.68	25.09	29.04	28.71	11.25	10.74
		1.5	**2.25**	8.83	8.34	5.42	4.9	23.16	22.74	6.04	5.52	6.55	6.04	3.35	2.80
		2.0	**4.00**	4.55	4.01	2.59	2.02	13.35	12.87	2.01	1.43	2.06	1.48	1.57	0.95
	-0.5	**1.2**	**1.44**	4.94	4.40	5.12	4.56	11.81	11.27	15.38	14.92	16.95	16.42	4.52	3.97
		1.5	**2.25**	3.91	3.35	2.84	2.28	12.6	12.12	8.13	7.58	9.25	8.72	2.15	1.57
		2.0	**4.00**	3.05	2.49	1.85	1.25	10.75	10.21	2.44	1.92-	2.56	2.01	1.34	0.68
	-1.0	**1.2**	**1.44**	1.45	0.81	1.65	1.04	3.05	2.5	4.05	3.53	4.33	3.80	1.62	1.00
		1.5	**2.25**	1.54	0.91	1.42	0.77	4.57	4.05	6.24	5.72	6.94	6.43	1.31	0.64
		2.0	**4.00**	1.62	1.01	1.27	0.59	6.32	5.77	3.28	2.72	3.47	2.93	1.12	0.37
	-1.5	**1.2**	**1.44**	1.04	0.20	1.08	0.3	1.55	0.92	1.87	1.27	1.97	1.37	1.08	0.29
		1.5	**2.25**	1.08	0.30	1.08	0.29	2.31	1.74	3.15	2.61	3.37	2.82	1.06	0.25
		2.0	**4.00**	1.16	0.43	1.07	0.28	3.83	3.29	3.51	2.98	3.67	3.15	1.03	0.19

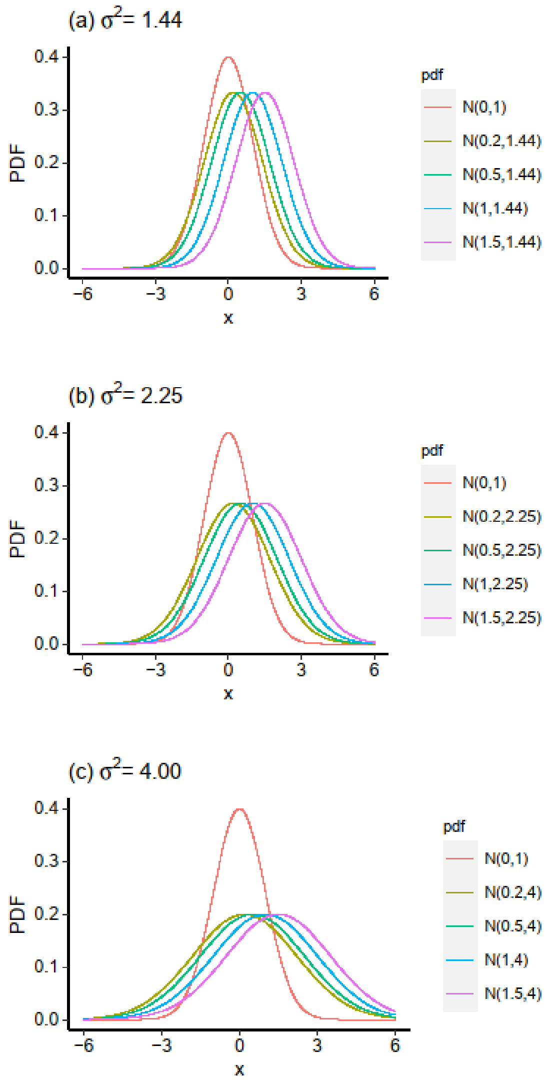

Figure 3.4: In-control and out-of-control pdfs for upward mean shift, variance increase - N(0,1)

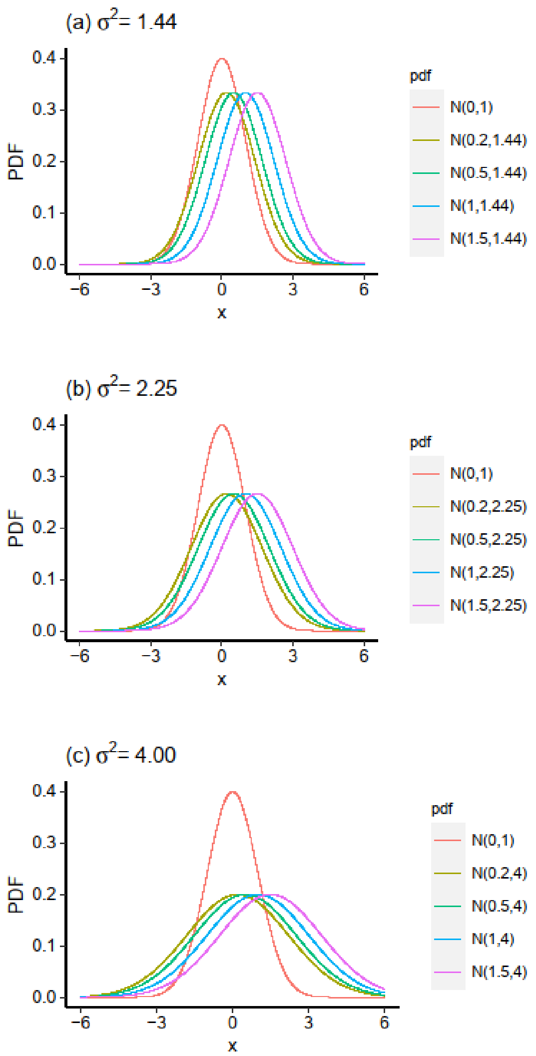

Figure 3.5: In-control and out-of-control pdfs for downward mean shift, variance increase - N(0,1)

3.3 Positively Skewed Distributions

The exponential and gamma distributions were used to examine the behaviour of the weighted average control charts when the underlying distribution is positively skewed.

3.3.1 Exponential Distribution

The rate of the exponential distribution was considered as one in this study. Exponential distribution occurs when the shape parameter equals one in gamma distribution. The probability density function was drawn with the means of the control statistics, calculated from an empirical distribution generated randomly for subgroup size ten is shown in Figure 3.6. The weighted averages based on the CoPDF-weight and CDF-weight are the same and positioned to the right of the unweighted average. In contrast, the weighted averages based on PDF-weight and the CoCDF-weight were similar and positioned to the left of the simple average. Also, the weighted average based on Max-weight was left to the unweighted average. Since the hazard function of the exponential distribution is a constant (λ), the weighted average $\tilde{X}_{haz}$ was the same as the unweighted average $\bar{X}$. Hence, the control statistic defined based on Haz-weight was not considered for the exponential distribution.

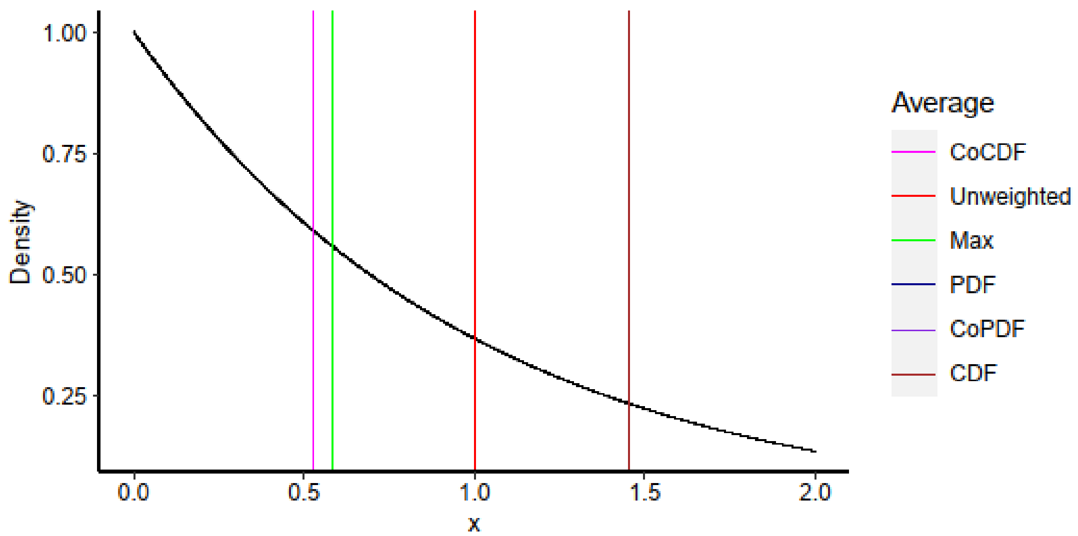

Figure 3.6: exp(1) density curve with means of control statistics

Tables 3.5 and 3.6 give the ARL and $SDRL$ for mean shifts upward and downward from the in-control value, respectively. The $\bar{X}$ control chart gave the lowest ARL_1 compared to all the weighted average control charts in detecting mean shifts. Figure 3.7 shows the in-control and out-of-control pdf for the increasing and decreasing mean changes. When the mean increases, the density curve was stretched to the right, while the density curve was shrunk to the left when the mean decreases from the in-control value.

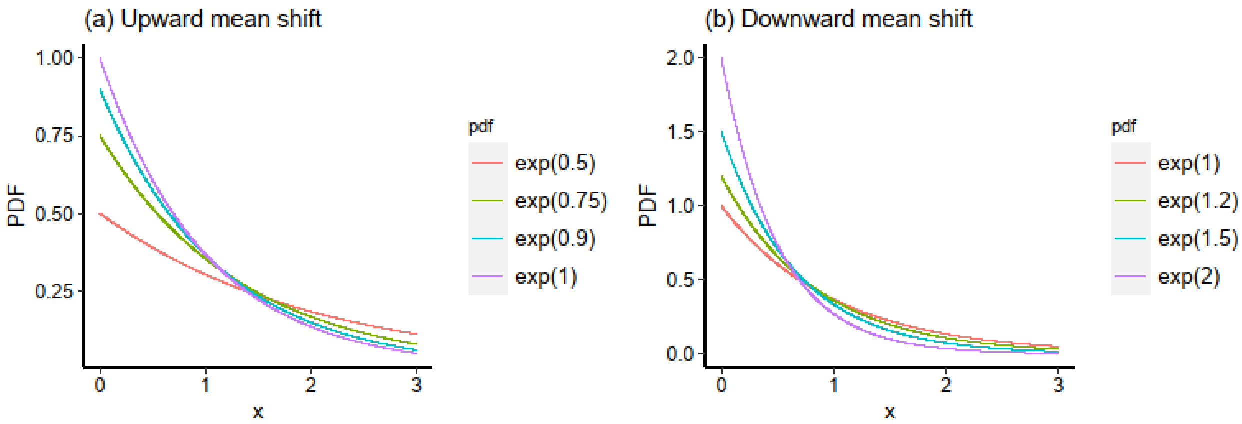

Figure 3.7: In-control and out-of-control pdfs for mean shift - exp(1)

3.3.2 Gamma Distribution

Gam(0.5,1), Gam(1.5,2) and Gam(2,1) were considered as the in-control distributions to illustrate the shape parameter less than one, the shape parameter greater than one and less than the scale parameter and shape parameter greater than one and scale parameter, respectively. In addition, mean shifts that change the shape and scale parameters in various ways will be discussed for two sample sizes, 5 and 10 in the following sections.

3.3.2.1 Shape Parameter Less than One – Gam(0.5,1)

Figure 3.8 shows the probability density function of Gam(0.5,1) with the means of the control statistics, calculated from a randomly generated empirical distribution of subgroup size ten. As illustrated in Figure 3.8, the mean of control statistic based on the CDF-weight was shifted to the right of the unweighted average. In contrast, all the other weighted averages were to the left of the unweighted average.

Table 3.5: ARL and $SDRL$ for upward mean shift - exp(1)

Sample size	rate	Standard Deviation	Variance	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{1-pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart	
				ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**1**	**1.00**	**1.00**	**98.98**	**98.70**	**99.45**	**98.83**	**99.02**	**98.57**	**98.59**	**98.26**	**98.59**	**98.26**	**99.02**	**98.57**
	0.90	1.11	1.23	65.09	64.84	72.28	71.90	78.47	78.14	74.32	74.12	74.32	74.12	78.47	78.14
	0.75	1.33	1.78	23.15	22.57	30.73	30.07	43.28	42.58	32.15	31.56	32.15	31.56	43.28	42.58
	0.50	2.00	4.00	4.04	3.52	5.90	5.38	12.82	12.27	5.65	5.15	5.65	5.15	12.82	12.27
	0.25	4.00	16.00	1.26	0.57	1.54	0.90	3.76	3.22	1.42	0.77	1.42	0.77	3.76	3.22
n=10	**1**	**1.00**	**1.00**	**101.1**	**100.42**	**100.33**	**60.79**	**100.48**	**99.49**	**98.35**	**97.68**	**98.35**	**97.68**	**100.48**	**99.49**
	0.90	1.11	1.23	57.19	56.45	61.31	16.92	77.13	76.68	64.68	63.85	64.68	63.85	77.13	76.68
	0.75	1.33	1.78	14.32	13.77	17.48	2.13	36.93	36.29	19.5	18.92	19.5	18.92	36.93	36.29
	0.50	2.00	4.00	2.19	1.62	2.69	0.27	9.66	9.18	2.82	2.27	2.82	2.27	9.66	9.18
	0.25	4.00	16.00	1.03	0.18	1.07	60.79	3.2	2.64	1.06	0.26	1.06	0.26	3.2	2.64

Table 3.6: ARL and $SDRL$ for downward mean shift - exp(1)

Sample size	rate	Standard Deviation	Variance	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{1-pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart	
				ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**1**	**1.00**	**1.00**	**98.98**	**98.70**	**99.45**	**98.83**	**99.02**	**98.27**	**98.59**	**98.26**	**98.59**	**98.26**	**99.02**	**98.27**
	1.20	0.83	0.69	87.49	86.83	92.60	92.50	98.69	98.24	84.17	83.65	84.17	83.65	98.69	98.24
	1.50	0.67	0.44	39.90	39.62	48.55	48.07	56.00	55.33	41.14	40.86	41.14	40.86	56.00	55.33
	2.00	0.50	0.25	14.56	13.98	19.53	19.04	22.11	21.53	15.47	14.90	15.47	14.9	22.11	21.53
	2.50	0.40	0.16	7.26	6.73	10.38	9.86	11.06	10.52	7.86	7.31	7.86	7.31	11.06	10.52
n=10	**1**	**1.00**	**1.00**	**101.1**	**100.42**	**100.33**	**100.16**	**100.48**	**99.49**	**98.35**	**97.68**	**98.35**	**97.68**	**100.48**	**99.49**
	1.20	0.83	0.69	60.66	60.1	64.85	64.16	80.74	79.7	61.34	60.6	61.34	60.6	80.74	79.7
	1.50	0.67	0.44	17.39	16.86	20.47	19.94	32.03	31.71	19.2	18.73	19.2	18.73	32.03	31.71
	2.00	0.50	0.25	4.65	4.14	5.77	5.25	9.25	8.69	5.46	4.93	5.46	4.93	9.25	8.69
	2.50	0.40	0.16	2.23	1.65	2.75	2.2	4.08	3.55	2.65	2.1	2.65	2.1	4.08	3.55

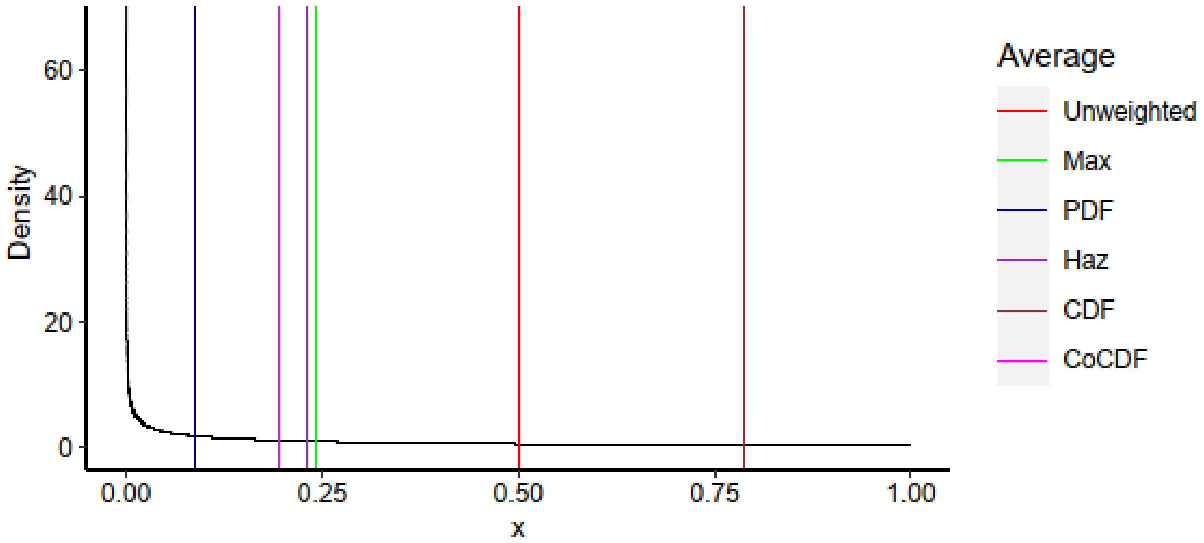

Figure 3.8: Gam(0.5,1) density curve with the means of control statistics

Tables 3.7 and 3.8 summaries the ARL and $SDRL$ for the mean increase and decreases occurred by a shift in either shape or scale parameter, respectively. The weighted average control charts $\tilde{X}_{max}$, $\tilde{X}_{pdf}$, $\tilde{X}_{haz}$ and $\tilde{X}_{(1-cdf)}$ showed lower ARL_1 compared to $\bar{X}$ chart in detecting a shift in the mean, which occurs due to a change in the shape parameter. The $\tilde{X}_{haz}$ control chart indicated the lowest ARL_1 compared to other control charts. In contrast, the $\bar{X}$ chart was better for detecting mean shifts from a change in the scale parameter. Also, the ARL_1 turned out to be shorter with an increase in the sample size. Further, $\tilde{X}_{cdf}$ control chart was insensitive for a slight decrease in the shape parameter, and all the control charts were insensitive in detecting a slight decrease in the scale parameter when the sample size is five.

Figures 3.9 and 3.10 summaries the in-control and out-of-control pdf for the mean shifts occurred by a single parameter. When the scale increases, the density curve was expanded to the right, whereas the scale decreases, the density curve was shrunk to the left. A decrease in the shape parameter could move the density curve to the left, and an increase in the shape parameter could flattened the density curve and moved to the right.

Table 3.7: ARL and $SDRL$ for upward mean shift - change in one parameter - Gam(0.5,1)

Sample size	Shape	Scale	Mean	Variance	Skewness	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart		$\tilde{X}_{haz}$ Chart	
						ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**0.5**	**1**	**0.5**	**0.5**	**2.8**	**100.52**	**99.62**	**100.81**	**100.27**	**99.47**	**99.13**	**100.04**	**99.99**	**99.29**	**98.63**	**98.95**	**98.55**
	0.70	1.00	0.7	0.7	2.4	51.60	51.17	37.09	36.73	38.51	37.89	125.45	125.94	37.78	37.18	36.76	36.44
	1.00	1.00	1.0	1.0	2.0	12.43	11.92	7.56	7.06	8.31	7.84	54.61	53.74	8.04	7.61	7.49	6.98
	1.50	1.00	1.5	1.5	1.6	2.99	2.44	1.97	1.38	2.32	1.75	11.93	11.36	2.26	1.69	1.98	1.39
	2.00	1.00	2.0	2.0	1.4	1.50	0.86	1.19	0.48	1.38	0.73	3.93	3.40	1.35	0.68	1.20	0.49
	0.50	**1.20**	0.6	0.7	2.8	52.06	51.67	60.89	60.53	70.43	69.51	61.12	61.08	74.39	74.14	54.95	55.13
	0.50	**1.50**	0.8	1.1	2.8	19.84	19.31	27.43	26.98	45.10	44.38	25.78	25.07	46.79	46.11	23.99	23.62
	0.50	**2.00**	1.0	2.0	2.8	7.28	6.76	10.82	10.29	25.58	25.30	9.56	9.04	25.86	25.55	9.36	8.84
	0.50	**2.50**	1.3	3.1	2.8	4.10	3.59	6.10	5.57	17.52	16.93	5.28	4.77	17.36	16.87	5.42	4.85
n=10	**0.5**	**1**	**0.5**	**0.5**	**2.8**	**99.69**	**98.97**	**99.92**	**99.52**	**99.38**	**98.4**	**99.85**	**99.49**	**99.36**	**98.45**	**98.38**	**97.25**
	0.70	1.00	0.7	0.7	2.4	30.18	29.64	20.45	19.88	19.01	18.37	91.42	89.93	18.09	17.43	17.18	16.59
	1.00	1.00	1.0	1.0	2.0	5.12	4.59	3.22	2.66	3.20	2.63	24.45	24.18	2.94	2.38	2.70	2.14
	1.50	1.00	1.5	1.5	1.6	1.39	0.74	1.15	0.41	1.23	0.53	3.97	3.45	1.16	0.43	1.09	0.32
	2.00	1.00	2.0	2.0	1.4	1.03	0.18	1.01	0.08	1.03	0.19	1.53	0.89	1.01	0.12	1.00	0.05
	0.50	**1.20**	0.6	0.7	2.8	42.06	41.55	47.93	47.24	69.96	69.06	49.72	49.22	72.31	71.85	47.25	47.17
	0.50	**1.50**	0.8	1.1	2.8	12.46	11.96	15.67	15.17	43.24	42.39	15.88	15.60	40.73	39.67	17.24	16.56
	0.50	**2.00**	1.0	2.0	2.8	4.01	3.47	5.15	4.63	24.50	24.12	5.16	4.66	20.77	20.28	6.05	5.49
	0.50	**2.50**	1.3	3.1	2.8	2.30	1.73	2.86	2.29	16.80	16.20	2.76	2.21	13.64	13.11	3.44	2.90

Table 3.8: ARL and $SDRL$ for downward mean shift - change in one parameter - Gam(0.5,1)

Sample size	Shape	Scale	Mean	Variance	Skewness	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart		$\tilde{X}_{haz}$ Chart	
						ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**0.5**	**1.00**	**0.50**	**0.50**	**2.83**	**100.52**	**99.62**	**100.81**	**100.27**	**99.47**	**99.13**	**100.04**	**99.99**	**99.29**	**98.63**	**98.95**	**98.55**
	0.40	1.00	0.40	0.40	3.16	48.75	48.22	42.82	42.47	41.22	40.57	49.54	48.81	48.78	47.40	37.96	37.51
	0.30	1.00	0.30	0.30	3.65	15.99	15.41	11.78	11.26	10.22	9.70	19.34	18.85	14.75	14.29	8.87	8.42
	0.25	1.00	0.25	0.25	4.00	9.17	8.64	6.36	5.83	5.21	4.64	11.59	10.96	8.25	7.69	4.57	4.01
	0.50	**0.80**	0.40	0.32	2.83	109.10	108.57	112.97	112.90	129.79	130.31	104.21	103.33	104.49	104.13	127.87	127.96
	0.50	**0.50**	0.25	0.13	2.83	41.23	40.96	52.51	51.89	124.75	122.39	41.91	41.20	48.16	48.00	81.32	80.14
	0.50	**0.25**	0.13	0.03	2.83	9.64	9.10	14.73	14.22	72.36	72.51	9.85	9.32	11.04	10.53	34.55	33.88
n=10	**0.5**	**1.00**	**0.50**	**0.50**	**2.83**	**99.69**	**98.97**	**99.92**	**99.52**	**99.38**	**98.4**	**99.85**	**99.49**	**99.36**	**98.45**	**98.38**	**97.25**
	0.40	1.00	0.40	0.40	3.16	39.27	38.82	32.81	32.29	36.72	35.78	45.40	45.02	31.79	31.03	33.16	32.60
	0.30	1.00	0.30	0.30	3.65	10.58	10.07	7.47	6.95	7.79	7.33	16.58	16.02	6.95	6.41	6.71	6.22
	0.25	1.00	0.25	0.25	4.00	5.80	5.29	3.98	3.45	3.73	3.23	10.10	9.53	3.61	3.08	3.26	2.71
	0.50	**0.80**	0.40	0.32	2.83	79.51	78.18	81.93	81.03	130.28	128.52	76.59	76.40	88.79	87.57	129.80	129.84
	0.50	**0.50**	0.25	0.13	2.83	14.85	14.32	17.34	16.83	127.51	127.59	15.49	15.10	23.82	23.30	78.66	77.89
	0.50	**0.25**	0.13	0.03	2.83	2.33	1.74	2.87	2.33	73.91	74.42	2.55	1.99	3.64	3.11	30.93	30.56

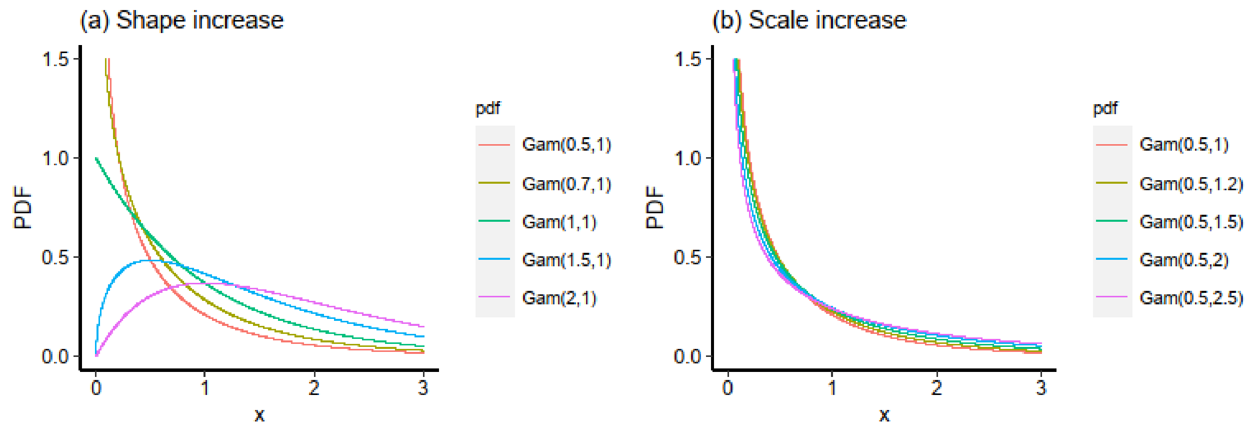

Figure 3.9: In-control and out-of-control pdfs for upward mean shift - change in one parameter - Gam(0.5,1)

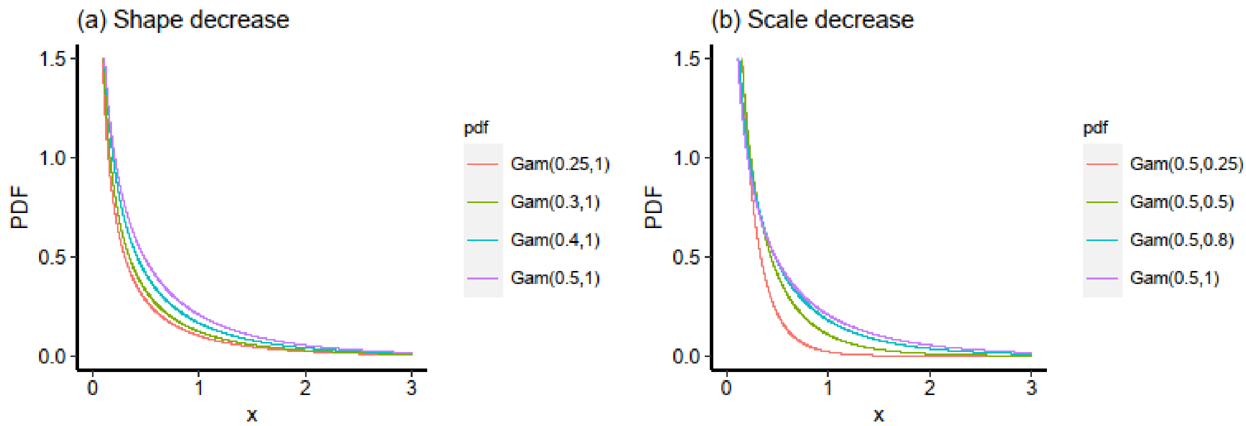

Figure 3.10: In- control and out-of-control pdfs for downward mean shift - change in one parameter – Gam(0.5,1)

Table 3.9 presented the ARL and $SDRL$ for mean shifts when both parameters were shifted simultaneously to the same direction. The $\tilde{X}_{max}$ and $\tilde{X}_{haz}$ control charts provided lower ARL_1 than the $\bar{X}$ control chart for both upward and downward mean shifts. All the other control charts had longer ARL_1 in comparison to the $\bar{X}$ control chart. Figure 3.11 shows the in-control and out-of-control pdfs for the mean shifts from their in-control states when both parameters were shifted to the same direction. The density curve was flattened and expanded to the right when both parameters were increased. When both parameters decrease, the density curve was shrunk to the left.

Table 3.9: ARL and $SDRL$ for mean shifts - change in both parameters - same direction - Gam(0.5,1)

Sample size	Shape	Scale	Mean	Variance	Skewness	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart		$\tilde{X}_{haz}$ Chart	
						ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
Mean increase																	
n=5	**0.5**	**1.00**	**0.50**	**0.50**	**2.83**	**100.52**	**99.62**	**100.81**	**100.27**	**99.47**	**99.13**	**100.04**	**99.99**	**99.29**	**98.63**	**98.95**	**98.55**
	0.70	**1.20**	0.84	1.01	2.39	19.33	18.75	16.96	16.54	22.83	22.20	45.23	44.39	22.21	21.53	15.57	15.07
	1.00	**1.20**	1.20	1.44	2.00	5.70	5.19	4.28	3.76	5.78	5.23	18.39	17.60	5.60	5.04	4.02	3.49
	1.00	**1.50**	1.50	2.25	2.00	2.91	2.35	2.59	2.02	4.06	3.48	6.69	6.08	3.92	3.36	2.34	1.79
	1.50	**1.20**	1.80	2.16	1.63	1.89	1.29	1.50	0.87	1.93	1.34	4.96	4.40	1.87	1.28	1.45	0.81
n=10	**0.5**	**1.00**	**0.50**	**0.50**	**2.83**	**99.69**	**98.97**	**99.92**	**99.52**	**99.38**	**98.4**	**99.85**	**99.49**	**99.36**	**98.45**	**98.38**	**97.25**
	0.70	**1.20**	0.84	1.01	0.70	9.78	9.30	8.07	7.56	12.02	11.58	25.40	24.85	10.63	10.24	7.25	6.75
	1.00	**1.20**	1.20	1.44	1.00	2.51	1.95	1.95	1.35	2.52	1.96	7.53	7.01	2.24	1.67	1.73	1.13
	1.00	**1.50**	1.50	2.25	1.00	1.50	0.86	1.35	0.68	2.08	1.46	2.87	2.30	1.81	1.21	1.27	0.59
	1.50	**1.20**	1.80	2.16	1.50	1.12	0.37	1.05	0.22	1.17	0.45	1.95	1.36	1.01	0.33	1.03	0.16
Mean decrease																	
n=5	**0.5**	**1.00**	**0.50**	**0.50**	**2.83**	**100.52**	**99.62**	**100.81**	**100.27**	**99.47**	**99.13**	**100.04**	**99.99**	**99.29**	**98.63**	**98.95**	**98.55**
	0.40	**0.80**	0.32	0.26	3.16	35.63	35.11	32.26	31.83	38.52	38.13	39.83	39.65	35.88	35.09	31.61	31.29
	0.40	**0.50**	0.20	0.10	3.16	15.53	15.03	16.22	15.74	29.98	29.38	17.95	17.37	16.61	16.08	19.90	19.10
	0.25	**0.80**	0.20	0.16	4.00	7.16	6.61	5.26	4.73	4.85	4.35	9.65	9.04	6.57	6.12	4.08	3.57
	0.25	**0.50**	0.13	0.06	4.00	4.30	3.75	3.55	3.02	4.13	3.62	5.75	5.25	4.08	3.57	3.21	2.67
	0.25	**0.25**	0.06	0.02	4.00	2.23	1.65	2.13	1.56	3.17	2.60	2.84	2.26	2.18	1.59	2.33	1.76
n=10	**0.5**	**1.00**	**0.50**	**0.50**	**2.83**	**99.69**	**98.97**	**99.92**	**99.52**	**99.38**	**98.4**	**99.85**	**99.49**	**99.36**	**98.45**	**98.38**	**97.25**
	0.40	**0.80**	0.32	0.26	3.16	20.90	20.44	18.14	17.65	34.02	33.01	27.49	27.16	19.62	19.06	26.90	26.55
	0.40	**0.50**	0.20	0.10	3.16	5.82	5.29	5.69	5.15	26.32	25.46	7.60	7.05	6.86	6.34	16.35	15.72
	0.25	**0.80**	0.20	0.16	4.00	3.95	3.41	2.92	2.37	3.48	2.90	7.00	6.49	2.84	2.26	2.89	2.36
	0.25	**0.50**	0.13	0.06	4.00	2.02	1.44	1.71	1.10	2.96	2.43	3.23	2.71	1.79	1.20	2.25	1.67
	0.25	**0.25**	0.06	0.02	4.00	1.14	0.40	1.11	0.35	2.32	1.77	1.48	0.84	1.15	0.41	1.65	1.04

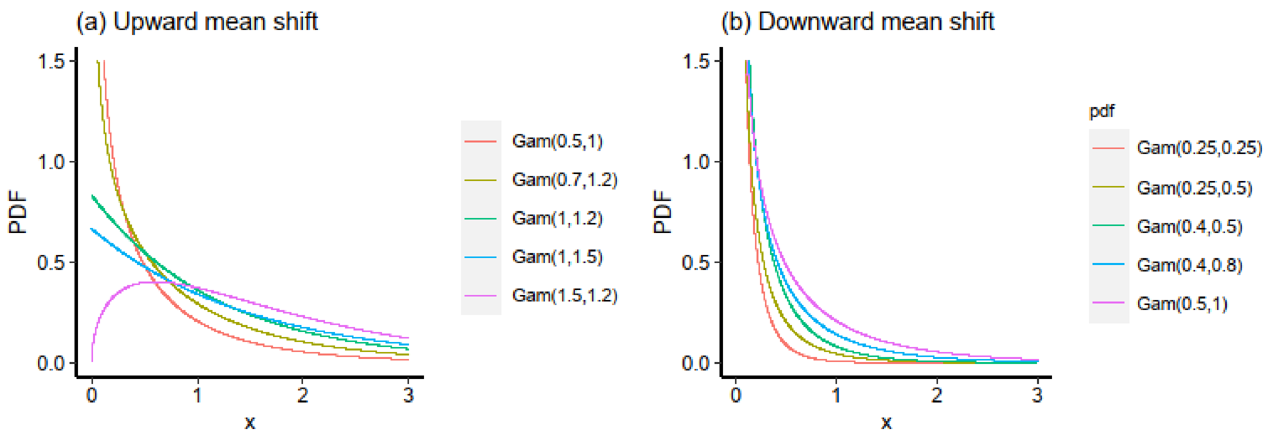

Figure 3.11: In-control and out-of-control pdfs - change in both parameters - same direction - Gam(0.5,1)

The mean of the process was varied by changing the parameters in the opposite direction. ARL and $SDRL$ for upward mean shifts are given in Tables 3.10. The weighted average control charts $\tilde{X}_{max}$, $\tilde{X}_{pdf}$, $\tilde{X}_{(1-cdf)}$ and $\tilde{X}_{haz}$ gave lower ARL_1 than the $\bar{X}$ control chart while $\tilde{X}_{cdf}$ gave higher ARL_1 than the $\bar{X}$ control chart for the shifts occurred by an increase in the shape parameter and decrease in the scale parameter. Although the $\tilde{X}_{pdf}$ and $\tilde{X}_{(1-cdf)}$ charts showed lower ARL_1 than ARL_0 for both sample sizes, the $\bar{X}$ and $\tilde{X}_{cdf}$ control charts were insensitive to small changes in the mean. Further, $\tilde{X}_{max}$ and $\tilde{X}_{haz}$ control charts were insensitive to small shifts when the sample size is five and gave smaller ARL_1 compared to ARL_0 for subgroup size 10. In contrast, $\tilde{X}_{cdf}$ control chart gave lower ARL_1 compared to all the other control charts for shifts occurred by a decrease in the shape parameter and increase in the scale parameter. Moreover, $\tilde{X}_{haz}$ control chart also resulted less ARL_1 than the $\bar{X}$ control chart for a significant increase in the mean.

Figure 3.12 shows the in-control and out-of-control pdf considered for the mean increased by shifting the parameters opposite. The density curve was flattened and moved to the right when the shape increases and scale decreases, while the density curve was condensed to the left when the shape decreases and scale increases.

Table 3.10: ARL and $SDRL$ for upward mean shift - change in both parameters - opposite direction - Gam(0.5,1)

Sample size	Shape	Scale	Mean	Variance	Skewness	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart		$\tilde{X}_{haz}$ Chart	
						ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**0.5**	**1.00**	**0.50**	**0.50**	**2.83**	**100.52**	**99.62**	**100.81**	**100.27**	**99.47**	**99.13**	**100.04**	**99.99**	**99.29**	**98.63**	**98.95**	**98.55**
	0.70	**0.80**	0.56	0.45	2.39	223.79	223.37	116.90	116.40	82.44	82.55	421.90	423.10	81.97	81.47	136.67	136.71
	1.00	**0.80**	0.80	0.64	2.00	46.48	45.82	18.68	18.16	15.03	14.40	290.86	292.89	14.84	14.40	21.21	20.69
	2.00	**0.50**	1.00	0.50	1.41	33.51	32.87	5.13	4.61	3.24	2.71	1530.52	1533.92	3.22	2.67	8.13	7.53
	1.50	**0.80**	1.20	0.96	1.63	7.22	6.68	3.30	2.76	3.19	2.66	54.62	54.38	3.11	2.58	3.61	3.04
	2.00	**0.80**	1.60	1.28	1.41	2.50	1.93	1.50	0.87	1.61	0.99	12.94	12.45	1.57	0.95	1.58	0.96
	0.40	**1.50**	0.60	0.90	3.16	30.17	29.54	39.76	39.45	39.06	38.46	27.77	27.26	56.22	55.83	32.20	32.14
	0.25	**2.50**	0.63	1.56	4.00	10.85	10.35	11.80	11.30	6.45	5.90	8.21	7.69	15.60	15.08	6.93	6.39
	0.40	**2.00**	0.80	1.60	3.16	12.01	11.57	20.71	19.89	31.95	31.40	11.95	11.75	43.30	42.67	17.27	16.64
	0.40	**2.50**	1.00	2.50	3.16	6.37	5.87	11.56	11.05	25.92	25.50	6.82	6.36	32.70	32.43	10.07	9.45
n=10	**0.5**	**1.00**	**0.50**	**0.50**	**2.83**	**99.69**	**98.97**	**99.92**	**99.52**	**99.38**	**98.4**	**99.85**	**99.49**	**99.36**	**98.45**	**98.38**	**97.25**
	0.70	**0.80**	0.56	0.45	2.39	189.13	188.36	94.03	93.41	39.35	38.41	424.71	423.86	41.55	41.03	72.64	72.51
	1.00	**0.80**	0.80	0.64	2.00	20.11	19.69	8.42	7.87	4.84	4.31	181.85	178.36	4.68	4.14	6.51	5.99
	2.00	**0.50**	1.00	0.50	1.41	7.83	7.30	2.01	1.42	1.19	0.47	526.59	532.16	1.18	0.45	1.73	1.13
	1.50	**0.80**	1.20	0.96	1.63	2.54	1.98	1.52	0.90	1.36	0.70	17.89	17.36	1.29	0.61	1.36	0.70
	2.00	**0.80**	1.60	1.28	1.41	1.20	0.49	1.04	0.20	1.05	0.22	3.61	3.10	1.02	0.16	1.02	0.15
	0.40	**1.50**	0.60	0.90	3.16	25.73	25.12	36.22	35.78	38.41	37.61	21.19	20.76	54.13	53.80	31.48	31.03
	0.25	**2.50**	0.63	1.56	4.00	9.94	9.40	13.18	12.69	4.71	4.13	5.77	5.25	8.64	8.09	5.44	4.97
	0.40	**2.00**	0.80	1.60	3.16	7.72	7.17	12.60	12.12	34.61	33.72	7.24	6.69	49.16	48.61	15.63	15.06
	0.40	**2.50**	1.00	2.50	3.16	3.82	3.30	5.93	5.43	29.87	29.40	3.80	3.22	36.92	36.51	8.53	7.96

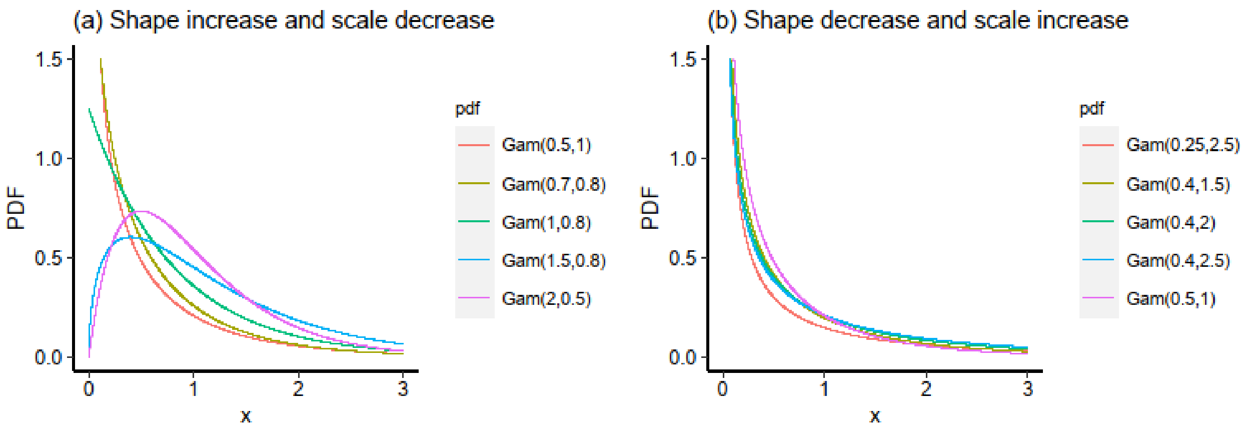

Figure 3.12: In-control and out-of-control pdfs for upward mean shift - change in both parameters - opposite direction - Gam(0.5,1)

The ARL and $SDRL$ for downward mean shifts are given in Table 3.11. The $\tilde{X}_{max}$, $\tilde{X}_{pdf}$ and $\tilde{X}_{haz}$ control charts gave small ARL_1 compared to the $\bar{X}$ control chart for the downward mean shift occurred by shape parameter decreases and scale parameter increases. In addition, $\tilde{X}_{cdf}$ control chart gave the lower ARL_1 for small mean shifts while $\tilde{X}_{(1-cdf)}$ showed lower ARL_1 for significant mean shifts when compared to $\bar{X}$ control chart. The lowest ARL_1 was given by $\tilde{X}_{haz}$ control chart. $\tilde{X}_{haz}$ and $\tilde{X}_{pdf}$ control charts were insensitive in detecting downward mean shifts occurred by increasing the shape parameter and decreasing the scale parameter. However, ARL_1 was smaller than ARL_0 for $\tilde{X}_{cdf}$ control chart when the subgroup size is ten and $\bar{X}$, $\tilde{X}_{max}$ and $\tilde{X}_{(1-cdf)}$ control charts for significant mean shifts. However, the $\tilde{X}_{cdf}$ control chart showed smaller ARL_1 than $\bar{X}$ control chart.

Figure 3.13 shows the in-control and out-of-control pdf for the downward mean shifts when the parameters were shifted in the opposite direction. The density curve was moved to the right when the mean was decreased by shape increases and scale decreases, and vice versa.

Table 3.11: ARL and $SDRL$ for mean downward mean shift - change in both parameters - opposite direction - Gam (0.5,1)

Sample size	Shape	Scale	Mean	Variance	Skewness	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart		$\tilde{X}_{haz}$ Chart	
						ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**0.5**	**1.00**	**0.50**	**0.50**	**2.83**	**100.52**	**99.62**	**100.81**	**100.27**	**99.47**	**99.13**	**100.04**	**99.99**	**99.29**	**98.63**	**98.95**	**98.55**
	0.40	**1.20**	0.48	0.58	3.16	48.90	48.36	47.83	47.38	41.36	40.71	44.69	43.99	56.22	55.74	39.45	39.12
	0.25	**1.50**	0.38	0.56	4.00	13.23	12.74	8.92	8.36	5.76	5.22	13.74	13.26	11.70	11.09	5.72	5.22
	0.25	**1.20**	0.30	0.36	4.00	11.15	10.48	7.47	6.93	5.47	4.91	13.37	12.74	9.61	9.01	5.07	4.50
	0.70	**0.50**	0.35	0.18	2.39	360.60	360.02	628.28	628.46	670.76	665.71	264.44	261.53	344.95	344.76	1432.00	1445.53
	0.70	**0.25**	0.18	0.04	2.39	43.83	43.83	112.13	111.63	1174.02	1173.38	33.37	32.77	60.46	60.17	485.03	483.73
n=10	**0.5**	**1.00**	**0.50**	**0.50**	**2.83**	**99.69**	**98.97**	**99.92**	**99.52**	**99.38**	**98.4**	**99.85**	**99.49**	**99.36**	**98.45**	**98.38**	**97.25**
	0.40	**1.20**	0.48	0.58	3.16	48.72	48.02	46.56	45.98	38.47	37.49	43.56	43.05	43.95	43.17	36.50	35.72
	0.25	**1.50**	0.38	0.56	4.00	11.77	11.34	7.50	6.99	4.21	3.67	14.06	13.60	5.54	5.00	4.11	3.53
	0.25	**1.20**	0.30	0.36	4.00	8.08	7.58	5.24	4.74	3.92	3.37	13.08	12.54	4.38	3.88	3.61	3.05
	0.70	**0.50**	0.35	0.18	2.39	149.84	149.13	288.66	287.50	353.92	353.56	84.94	84.85	297.05	294.95	1701.46	1699.98
	0.70	**0.25**	0.18	0.04	2.39	6.88	6.33	13.49	12.88	1577.07	1574.88	5.11	4.54	21.16	20.45	572.56	564.16

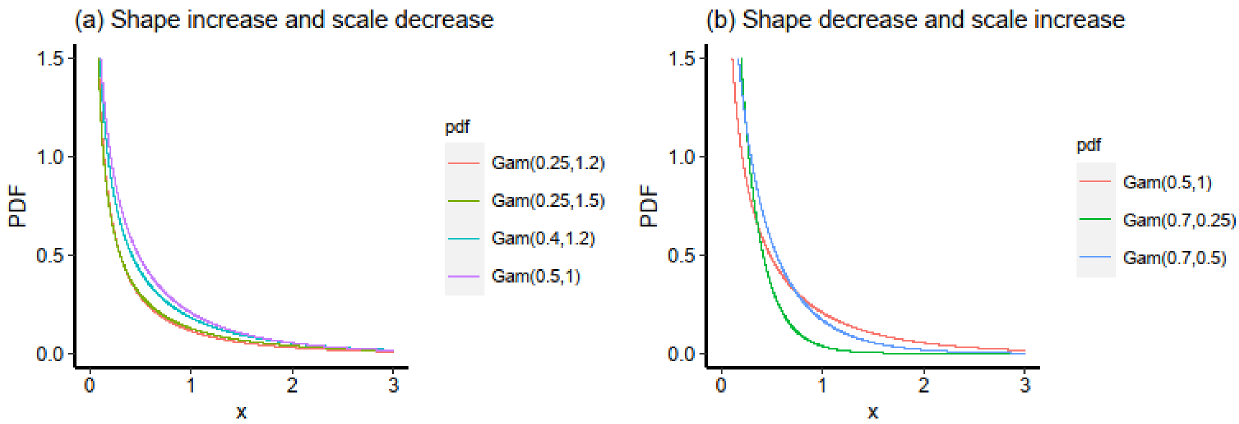

Figure 3.13: In-control and out-of-control pdfs for downward mean shift - change in both parameters - opposite direction - Gam(0.5,1)

A shift in parameters may not change the mean from its in-control state. However, the variance and the skewness may change. The ARL and $SDRL$ for these kinds of shifts are given in Table 3.12. All the control charts were insensitive to the changes in scale parameter decreases and shape parameter increases. However, the $\tilde{X}_{(1-cdf)}$ control chart resulted smaller ARL_1 values than the ARL_0 value for sample size 10. On the other hand, $\tilde{X}_{cdf}$ and $\tilde{X}_{max}$ control charts showed higher ARL_1 than $\bar{X}$ control chart and the $\tilde{X}_{pdf}$,$\tilde{X}_{cdf}$ and $\tilde{X}_{haz}$ control charts showed less ARL_1 compared to the $\bar{X}$ control chart when the scale increases and the shape decrease without shifting the mean. Further, the smallest ARL_1 was given by the $\tilde{X}_{haz}$ control chart. The corresponding in-control and out-of-control pdfs are shown in Figure 3.14. As the shape increase and scale decreases, the density curve was moved to the right and vice versa.

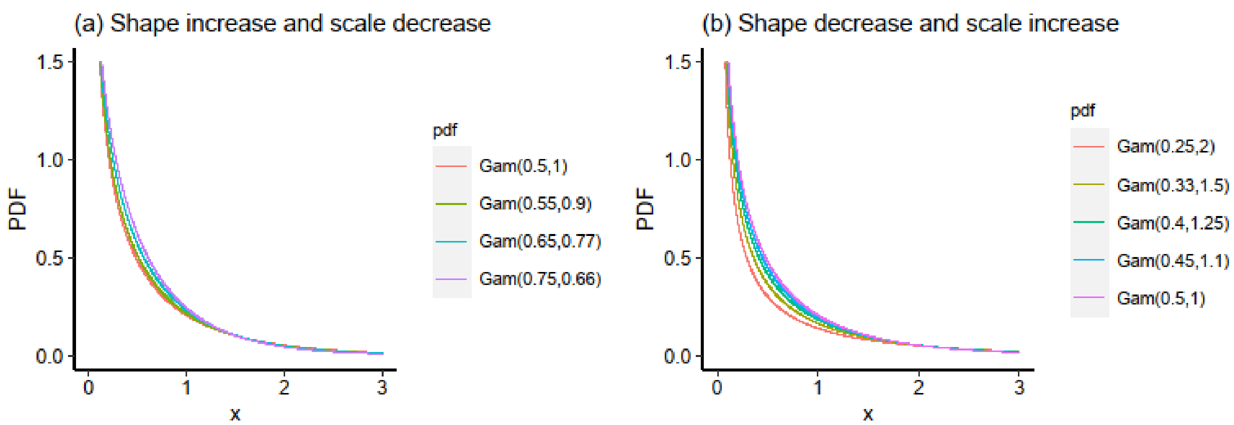

Figure 3.14: In-control and out-of-control pdfs for shifts in both parameters, mean in-control - Gam(0.5,1)

Table 3.12: ARL and $SDRL$ for parameters shift, mean in-control - Gam(0.5,1)

Sample size	Shape	Scale	Mean	Variance	Skewness	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart		$\tilde{X}_{haz}$ Chart	
						ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**0.5**	**1.00**	**0.50**	**0.50**	**2.83**	**100.52**	**99.62**	**100.81**	**100.27**	**99.47**	**99.13**	**100.04**	**99.99**	**99.29**	**98.63**	**98.95**	**98.55**
	0.55	0.90	0.50	0.45	2.70	151.11	151.05	137.63	135.31	124.48	122.95	158.05	156.48	117.81	116.94	148.56	147.58
	0.65	0.77	0.50	0.39	2.48	301.48	301.65	194.81	194.98	134.46	134.06	361.72	359.69	127.44	126.81	240.48	241.35
	0.75	0.66	0.50	0.33	2.31	655.41	655.01	260.40	258.94	129.75	129.44	809.50	818.75	130.06	130.13	387.40	394.04
	0.45	1.10	0.50	0.54	2.98	69.74	69.44	71.34	70.05	68.88	67.72	66.35	65.63	78.49	77.48	64.40	63.97
	0.40	1.25	0.50	0.63	3.16	46.66	46.27	47.83	47.17	41.14	40.52	42.57	42.08	57.10	56.20	39.45	39.12
	0.33	1.50	0.50	0.74	3.48	26.90	26.48	24.72	24.11	17.34	16.79	22.86	22.45	31.17	30.99	17.04	16.62
	0.25	2.00	0.50	1.00	4.00	13.55	12.93	11.02	10.56	6.21	5.64	11.16	10.66	14.03	13.58	6.54	6.02
n=10	**0.5**	**1.00**	**0.50**	**0.50**	**2.83**	**99.69**	**98.97**	**99.92**	**99.52**	**99.38**	**98.4**	**99.85**	**99.49**	**99.36**	**98.45**	**98.38**	**97.25**
	0.55	0.90	0.50	0.45	2.70	149.44	149.66	137.76	137.61	108.14	107.24	154.17	153.87	109.66	110.17	136.71	136.64
	0.65	0.77	0.50	0.39	2.48	302.85	303.67	197.15	198.53	108.14	73.47	323.55	324.97	83.78	82.33	170.66	171.01
	0.75	0.66	0.50	0.33	2.31	649.84	646.14	279.79	279.34	52.06	51.19	590.57	593.34	62.83	62.13	211.83	209.74
	0.45	1.10	0.50	0.54	2.98	70.28	70.52	70.70	70.17	70.37	69.28	65.41	64.45	74.62	74.79	65.50	64.81
	0.40	1.25	0.50	0.63	3.16	46.43	45.91	46.98	45.99	38.78	37.69	39.91	39.71	46.39	45.75	36.82	36.10
	0.33	1.50	0.50	0.74	3.48	27.15	26.45	24.91	24.38	14.25	13.70	20.71	20.14	19.71	19.08	14.52	13.86
	0.25	2.00	0.50	1.00	4.00	13.96	13.62	11.44	10.75	2.47	3.90	9.37	8.78	7.25	6.73	4.84	4.30

3.3.2.2 Shape Parameter Greater than One and Less than Scale Parameter – Gam(1.5,2)

Figure 3.15 shows the Gam(1.5,2) probability density function with the means of the control statistics, calculated from an empirical distribution randomly generated for subgroup size 10. The averages based on PDF-weight, Max-weight, and CoCDF-weight were to the left of the unweighted average, while the averages based on Haz-weight and CDF-weight were right to the unweighted average.

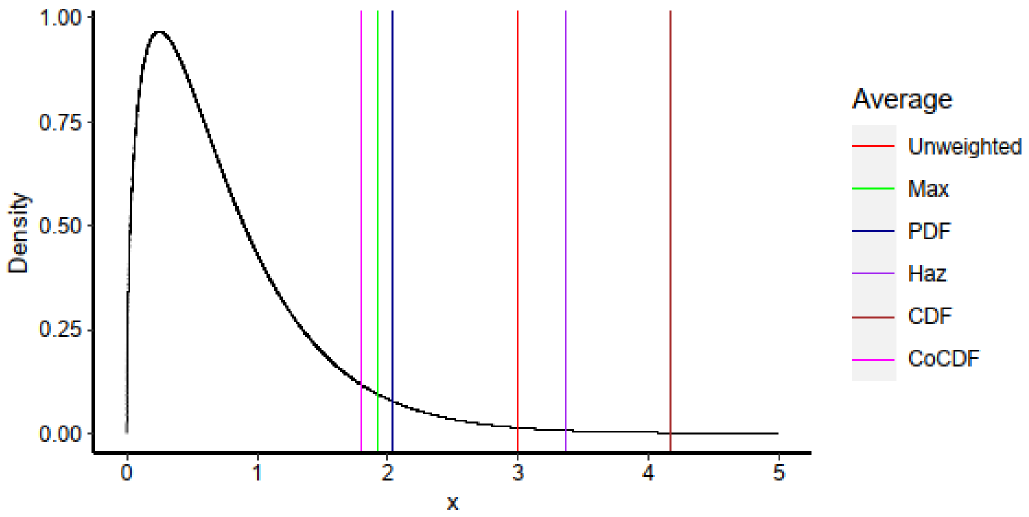

Figure 3.15: Gam(1.5,2) density curve with means of control statistics

The ARL and $SDRL$ for upward and downward mean shifts by changing either shape or scale parameters are given in Tables 3.13 and 3.14, respectively. The $\tilde{X}_{max}$ control chart gave smaller ARL_1 compared to $\bar{X}$ chart when the mean shift was occurred by a change of the shape parameter. Further, $\tilde{X}_{pdf}$ and $\tilde{X}_{(1-cdf)}$ charts gave lower ARL_1 values in comparison with $\bar{X}$ chart when the shape parameter was decreased by a small amount. In contrast, the $\bar{X}$ chart showed the lowest ARL_1 for the mean shifts which occurred by a change of the scale parameter. However, when the sample size is 5, $\tilde{X}_{haz}$ showed lower ARL_1 values than the $\bar{X}$ chart, for a significant change in the scale parameter, which increases the process mean. Figures 3.16 and 3.17 show the in-control and out-of-control pdfs for mean shift occurred only by one parameter. The density curve was flattened and was moved to the right when the shape or scale parameter increases. The density curve was shrunk to the left when either shape or scale parameter decreases.

Table 3.13: ARL and $SDRL$ for upward mean shift - change in one parameter - Gam(1.5,2)

Sample size	Shape	Scale	Mean	Variance	Skewness	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart		$\tilde{X}_{haz}$ Chart	
						ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**1.5**	**2.00**	**3.00**	**6.00**	**1.63**	**98.54**	**97.38**	**99.94**	**101.13**	**101.54**	**101.24**	**100.27**	**99.67**	**100.69**	**101.28**	**98.36**	**97.58**
	1.70	2.00	3.40	6.80	1.53	75.31	73.76	71.26	70.57	74.62	74.51	120.46	120.25	75.05	74.92	85.96	85.78
	2.00	2.00	4.00	8.00	1.41	28.44	27.73	24.66	24.15	27.71	27.27	79.11	78.41	27.93	27.52	36.85	36.62
	2.50	2.00	5.00	10.00	1.26	7.32	6.78	6.21	5.73	7.62	7.07	27.20	26.76	7.69	7.14	10.06	9.62
	3.00	2.00	6.00	12.00	1.15	3.00	2.43	2.63	2.04	3.36	2.86	10.27	9.77	3.39	2.85	3.98	3.46
	1.50	**2.20**	3.30	7.26	1.63	63.83	64.04	71.61	71.50	78.17	79.33	77.27	76.74	77.55	77.99	65.76	65.3
	1.50	**2.50**	3.75	9.38	1.63	27.20	26.29	35.31	34.82	46.27	45.69	39.75	39.55	45.95	45.58	28.5	27.84
	1.50	**2.75**	4.13	11.34	1.63	14.67	14.10	20.56	19.97	30.13	29.57	14.06	13.45	21.38	20.63	9.57	8.97
	1.50	**3.00**	4.50	13.50	1.63	8.99	8.52	12.91	12.43	21.19	20.76	6.75	6.15	12.16	11.67	4.71	4.21
n=10	**1.5**	**2.00**	**3.00**	**6.00**	**1.63**	***100.13***	***100.29***	***100.94***	***100.46***	***103.03***	***102.16***	***100.77***	***100.72***	***101.31***	***101.02***	***100.50***	***100.59***
	1.70	2.00	3.40	6.80	1.53	56.30	55.53	50.75	49.98	54.68	54.15	101.08	99.86	53.99	53.25	67.81	66.88
	2.00	2.00	4.00	8.00	1.41	14.02	13.40	11.47	10.96	13.55	13.18	45.13	45.44	13.66	13.14	19.35	18.82
	2.50	2.00	5.00	10.00	1.26	3.00	2.43	2.49	1.95	3.14	2.57	10.66	10.11	3.19	2.63	4.07	3.56
	3.00	2.00	6.00	12.00	1.15	1.42	0.77	1.29	0.61	1.55	0.92	3.51	2.96	1.59	0.96	1.72	1.11
	1.50	**2.20**	3.30	7.26	1.63	54.24	53.56	59.62	58.37	73.52	72.72	65.83	64.74	74.34	73.3	56.18	55.16
	1.50	**2.50**	3.75	9.38	1.63	16.83	16.25	20.53	19.86	36.06	35.96	24.74	23.97	38.24	37.96	17.64	17.07
	1.50	**2.75**	4.13	11.34	1.63	8.07	7.58	10.14	9.77	21.57	20.85	6.94	6.51	15.78	15.18	4.94	4.46
	1.50	**3.00**	4.50	13.50	1.63	4.70	4.15	6.00	5.46	14.24	13.72	3.2	2.66	8.83	8.26	2.42	1.86

Table 3.14: ARL and $SDRL$ for downward mean shift - change in one parameter - Gam(1.5,2)

Sample size	Shape	Scale	Mean	Variance	Skewness	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart		$\tilde{X}_{haz}$ Chart	
						ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**1.50**	**2.00**	**3.00**	**6.00**	**1.63**	**98.54**	**97.38**	**99.94**	**101.13**	**101.54**	**101.24**	**100.27**	**99.67**	**100.69**	**101.28**	**98.36**	**97.58**
	1.30	2.00	2.60	5.20	1.75	52.67	52.21	49.86	49.00	59.18	58.53	56.52	55.97	53.56	52.99	56.63	56.32
	1.00	2.00	2.00	4.00	2.00	11.93	11.43	9.82	9.38	14.21	13.88	17.47	16.86	10.94	10.49	14.61	14.28
	0.75	2.00	1.50	3.00	2.31	4.08	3.54	3.21	2.63	5.07	4.48	6.96	6.42	3.6	3.08	5.53	5.06
	0.50	2.00	1.00	2.00	2.83	1.88	1.29	1.51	0.86	2.40	1.84	3.22	2.70	1.61	0.98	2.6	2.05
	1.50	**1.80**	2.70	4.86	1.63	94.96	95.18	97.88	97.61	99.71	100.64	92.45	91.59	104.59	105.56	94.76	93.65
	1.50	**1.50**	2.25	3.38	1.63	44.09	43.58	53.14	52.99	55.95	55.37	46.12	45.38	63.87	62.89	45.15	44.75
	1.50	**1.25**	1.88	2.34	1.63	18.93	18.48	24.64	24.04	25.87	25.39	20.64	20.19	30.83	30.23	19.33	19.01
	1.50	**1.00**	1.50	1.50	1.63	7.56	7.01	10.55	10.13	10.18	9.58	8.55	8.02	12.48	11.86	7.78	7.24
n=10	**1.50**	**2.00**	**3.00**	**6.00**	**1.63**	**100.13**	**100.29**	**100.94**	**100.46**	**103.03**	**102.16**	**100.77**	**100.72**	**101.31**	**101.02**	**100.50**	**100.59**
	1.30	2.00	2.60	5.20	1.75	39.49	39.07	35.76	35.30	42.48	41.61	50.34	49.95	37.56	36.48	45.04	45.13
	1.00	2.00	2.00	4.00	2.00	6.37	5.87	5.14	4.57	7.15	6.60	12.75	12.29	5.34	4.78	8.79	8.3
	0.75	2.00	1.50	3.00	2.31	2.18	1.60	1.79	1.19	2.48	1.93	5.09	4.57	1.8	1.21	3.2	2.69
	0.50	2.00	1.00	2.00	2.83	1.21	0.52	1.10	0.32	1.36	0.71	2.54	1.98	1.09	0.32	1.7	1.1
	1.50	**1.80**	2.70	4.86	1.63	74.98	74.17	79.33	78.40	86.14	85.55	75.49	74.65	91.67	90.65	75.43	73.97
	1.50	**1.50**	2.25	3.38	1.63	20.91	20.42	24.87	24.86	32.18	31.66	23.59	22.91	38.83	38.28	21.59	21.26
	1.50	**1.25**	1.88	2.34	1.63	6.69	6.15	8.51	7.97	11.01	10.44	8.22	7.59	14.12	13.69	7.08	6.52
	1.50	**1.00**	1.50	1.50	1.63	2.47	1.88	3.13	2.59	3.79	3.24	3.12	2.56	4.82	4.33	2.54	1.97

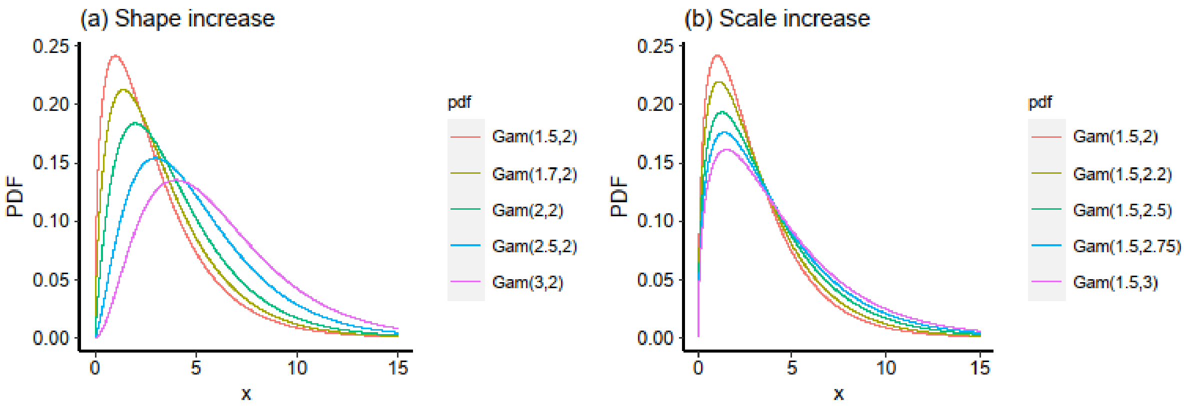

Figure 3.16: In-control and out-of-control pdfs for upward mean shift - change in one parameter - Gam (1.5,2)

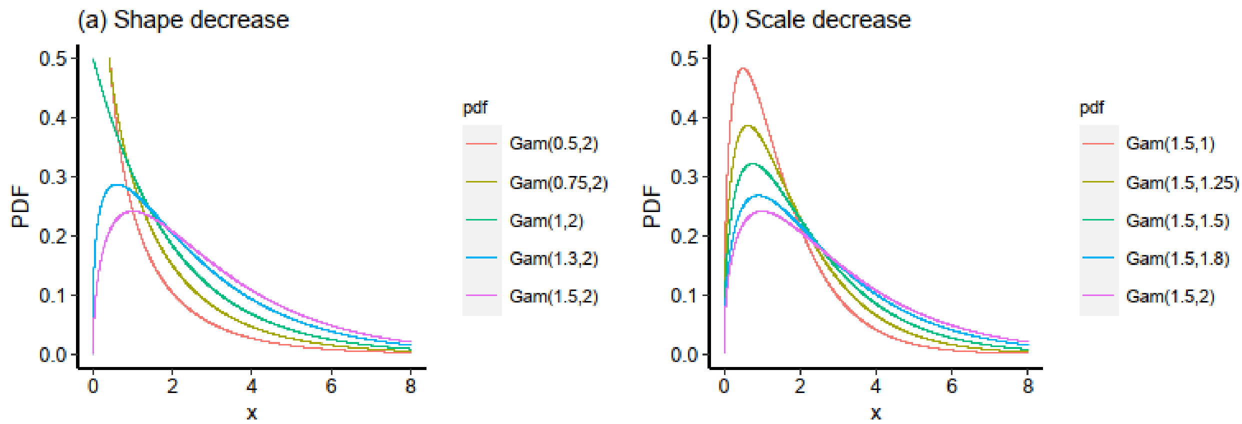

Figure 3.17: In-control and out-of-control pdf for downward mean shift - change in one parameter - Gam (1.5,2)

Table 3.15 summarises the ARL and $SDRL$ for mean shifts that simultaneously increase /decrease both parameters. The $\bar{X}$ and $\tilde{X}_{max}$ control charts gave the lowest ARL_1 for mean increases and decreases, respectively. $\tilde{X}_{pdf}$, $\tilde{X}_{cdf}$, $\tilde{X}_{(1-cdf)}$ and $\tilde{X}_{haz}$ control charts showed larger ARL_1 compared to $\bar{X}$ control chart. In-control and out-of-control pdf are shown in Figure 3.18. The density curve was flattened and was expanded to the right when the mean increases. The peak of the curve rose when the mean decreases.

Table 3.15: ARL and $SDRL$ for mean shift - change in both parameters - same direction - Gam(1.5,2)

Sample size	Shape	Scale	Mean	Variance	Skewness	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart		$\tilde{X}_{haz}$ Chart	
						ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
Mean increase																	
n=5	**1.50**	**2.00**	**3.00**	**6.00**	**1.63**	**98.54**	**97.38**	**99.94**	**101.13**	**101.54**	**101.24**	**100.27**	**99.67**	**100.69**	**101.28**	**98.36**	**97.58**
	1.70	2.20	3.74	8.23	1.53	35.06	34.52	38.10	38.00	44.74	43.90	65.07	63.61	45.09	44.12	40.55	39.71
	1.70	2.50	4.25	10.63	1.53	14.19	13.50	16.91	16.38	23.82	23.40	28.03	27.83	24.18	23.69	16.15	15.57
	2.00	2.50	5.00	12.50	1.41	6.42	6.00	6.88	6.37	10.12	9.64	15.26	14.68	10.27	9.74	7.72	7.21
	2.50	2.50	6.25	15.63	1.26	2.54	1.98	2.57	2.01	3.70	3.21	6.02	5.5	3.75	3.28	3.05	2.49
	2.50	3.00	7.50	22.50	1.26	1.56	0.93	1.64	1.03	2.53	1.98	2.65	2.12	2.55	1.97	1.74	1.14
n=10	**1.50**	**2.00**	**3.00**	**6.00**	**1.63**	**100.13**	**100.29**	**100.94**	**100.46**	**103.03**	**102.16**	**100.77**	**100.72**	**101.31**	**101.02**	**100.50**	**100.59**
	1.70	2.20	3.74	8.23	1.53	20.82	20.22	20.93	20.31	28.47	27.74	42.16	42.09	29.28	28.5	25.14	24.2
	1.70	2.50	4.25	10.63	1.41	7.22	6.67	7.83	7.35	13.97	13.48	14.1	13.51	14.81	14.44	8.49	7.98
	2.00	2.50	5.00	12.50	1.26	2.90	2.37	2.92	2.34	4.86	4.35	6.49	5.94	5.06	4.51	3.51	2.95
	2.50	2.50	6.25	15.63	1.26	1.34	0.68	1.31	0.63	1.85	1.26	2.41	1.84	1.93	1.32	1.51	0.86
	2.50	3.00	7.50	22.50	1.53	1.07	1.34	1.07	0.28	1.47	0.82	1.34	0.68	1.53	0.9	1.12	0.36
Mean decrease																	
n=5	**1.50**	**2.00**	**3.00**	**6.00**	**1.63**	**98.54**	**97.38**	**99.94**	**101.13**	**101.54**	**101.24**	**100.27**	**99.67**	**100.69**	**101.28**	**98.36**	**97.58**
	1.30	1.80	2.34	4.21	1.75	35.66	35.00	35.25	34.50	42.48	42.38	41.73	40.78	40.56	39.85	39.14	38.9
	1.00	1.80	1.80	3.24	2.00	8.56	8.13	7.48	6.96	10.76	10.14	13.04	12.57	8.65	8.18	10.64	10.11
	1.00	1.50	1.50	2.25	2.00	5.05	4.49	4.78	4.25	6.50	6.00	7.78	7.3	5.63	5.09	6.29	5.71
	0.50	1.80	0.90	1.62	2.83	1.67	1.05	1.39	0.72	2.14	1.56	2.79	2.25	1.49	0.84	2.27	1.7
n=10	**1.50**	**2.00**	**3.00**	**6.00**	**1.63**	**100.13**	**100.29**	**100.94**	**100.46**	**103.03**	**102.16**	**100.77**	**100.72**	**101.31**	**101.02**	**100.50**	**100.59**
	1.30	1.80	2.34	4.21	1.75	20.24	20.10	19.59	19.06	25.50	25.09	29.15	28.71	23.7	23	23.76	23.42
	1.00	1.80	1.80	3.24	2.00	4.16	3.60	3.58	3.05	5.00	4.46	8.24	7.72	4.01	3.51	5.64	5.07
	1.00	1.50	1.50	2.25	2.00	2.29	1.71	2.13	1.56	2.88	2.34	4.25	3.7	2.51	1.95	2.94	2.38
	0.50	1.80	0.90	1.62	2.83	1.13	0.39	1.06	0.25	1.26	0.57	2.13	1.56	1.06	0.26	1.49	0.85

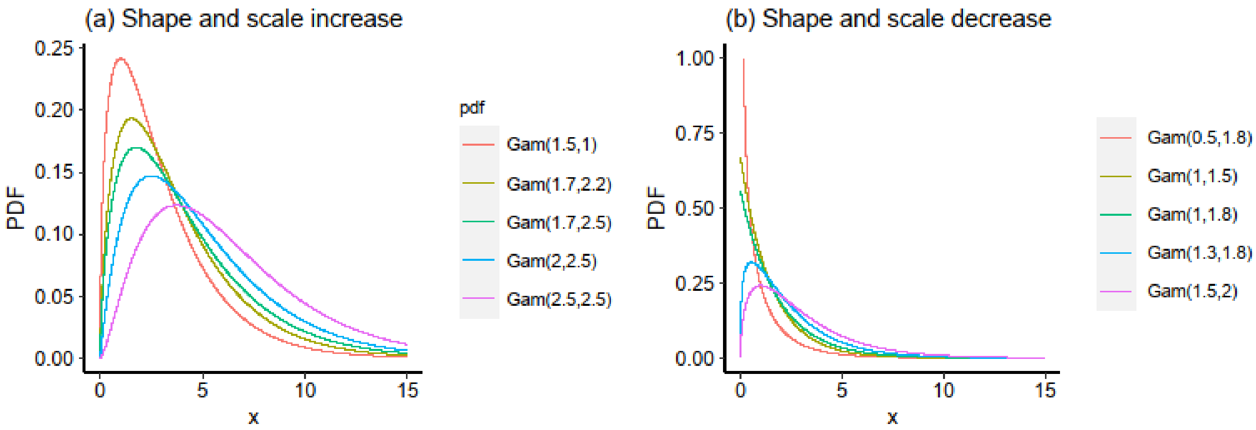

Figure 3.18: In-control and out-of-control pdfs - change in both parameters - same direction - Gam(1.5,2)

Tables 3.16 provide the ARL and $SDRL$ for the mean shift upward by shifting both parameters in opposite directions. All the control charts were insensitive when there is a slight increase in the shape parameter and a slight decrease in the scale parameter. However, for significant mean shifts, the weighted average control charts $\tilde{X}_{max}$, $\tilde{X}_{pdf}$ and $\tilde{X}_{(1-cdf)}$ gave less ARL_1 compared to $\bar{X}$ control chart. $\tilde{X}_{cdf}$ and $\tilde{X}_{haz}$ control charts gave small ARL_1 compared to $\bar{X}$ control chart for the mean increase occurred by a shape parameter decreases and scale parameter increase. The in-control and out-of-control pdfs are shown in Figure 3.19. The density curve was expanded to the right when the shape parameter increases and the scale parameter decreases, while it was shrunk to the left when the shape parameter decreases and the scale parameter increases.

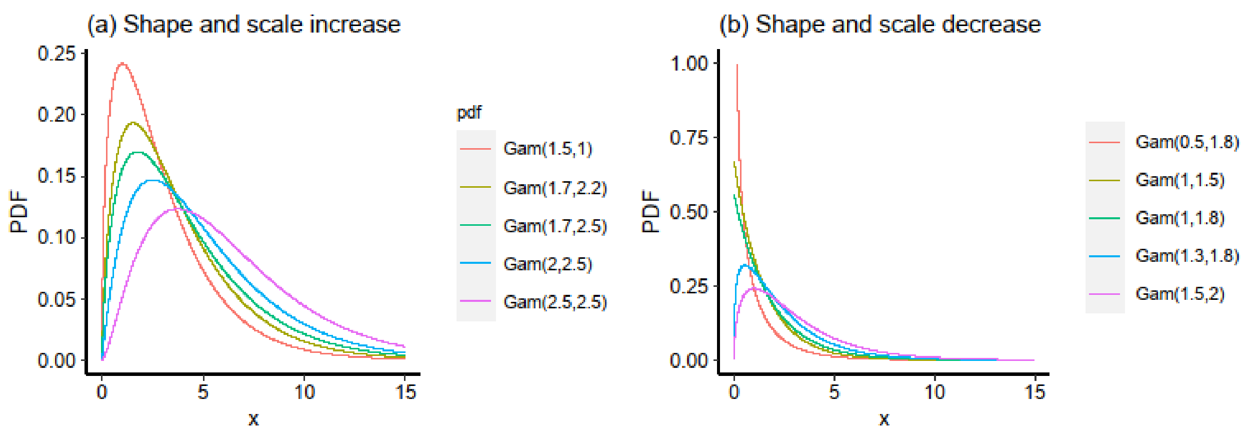

Figure 3.19: In-control and out-of-control pdfs for upward mean shift - change in both parameters - opposite direction - Gam(1.5,2)

Table 3.16: ARL and $SDRL$ for upward mean shift - change both parameters - opposite directions - Gam(1.5,2)

Sample size	Shape	Scale	Mean	Variance	Skewness	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart		$\tilde{X}_{haz}$ Chart	
						ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**1.50**	**2.00**	**3.00**	**6.00**	**1.63**	**98.54**	**97.38**	**99.94**	**101.13**	**101.54**	**101.24**	**100.27**	**99.67**	**100.69**	**101.28**	**98.36**	**97.58**
	1.70	**1.80**	3.06	5.51	1.53	154.42	154.51	135.18	134.70	122.55	122.25	170.12	169.66	127.66	127.22	160.65	159.36
	2.00	**1.80**	3.60	6.48	1.41	72.17	71.56	53.84	52.97	52.19	52.05	188.10	187.26	52.14	51.50	96.68	96.46
	2.50	**1.80**	4.50	8.10	1.26	15.46	14.85	11.22	10.66	12.26	11.78	70.73	70.15	12.28	11.82	22.64	22.36
	1.30	**2.50**	3.25	8.13	1.75	46.03	45.38	56.66	56.35	66.91	66.75	45.76	45.56	59.69	59.02	43.67	43.68
	1.00	**3.50**	3.50	12.25	2.00	18.34	17.82	27.17	26.78	37.61	37.50	15.08	14.65	26.91	26.46	15.38	14.95
	1.30	**3.00**	3.90	11.70	1.75	16.68	16.28	26.62	26.10	40.08	38.92	19.85	19.14	37.74	37.40	16.05	15.33
n=10	**1.50**	**2.00**	**3.00**	**6.00**	**1.63**	**100.13**	**100.29**	**100.94**	**100.46**	**103.03**	**102.16**	**100.77**	**100.72**	**101.31**	**101.02**	**100.50**	**100.59**
	1.70	**1.80**	3.06	5.51	1.53	152.58	151.15	129.32	128.51	108.48	107.09	167.01	165.48	108.47	107.69	160.61	158.88
	2.00	**1.80**	3.60	6.48	1.41	42.94	42.03	29.16	28.64	26.87	26.18	146.72	146.65	26.1	25.68	63.24	62.98
	2.50	**1.80**	4.50	8.10	1.26	6.13	5.64	4.31	3.78	4.69	4.18	31.24	30.57	4.67	4.15	9.39	8.87
	1.30	**2.50**	3.25	8.13	1.75	42.24	41.56	53.20	52.41	70.37	70.32	37.48	37.17	61.83	61.15	37.81	37.42
	1.00	**3.50**	3.50	12.25	2.00	15.06	14.53	25.21	24.75	38.07	37.73	11.59	11.08	23.33	22.84	11.4	10.96
	1.30	**3.00**	3.90	11.70	1.75	10.54	10.01	15.62	15.18	37.09	36.61	9.93	9.47	39.07	38.77	9.93	9.42

Tables 3.17 provides the ARL and $SDRL$ for the mean shift downward by shifting both parameters in opposite directions. $\tilde{X}_{cdf}$ and $\tilde{X}_{haz}$ control charts gave lower ARL_1 than $\bar{X}$ control chart for mean decreases occurred by shape parameter increases while scale parameter decreases for subgroup size 10. All the control charts were insensitive to a slight increase in the scale parameter for sample size five. In contrast, $\tilde{X}_{max}$, $\tilde{X}_{pdf}$ and $\tilde{X}_{(1-cdf)}$ control charts gave lower ARL_1 comparing to the $\bar{X}$ control chart for mean decreased by a decrease in the shape parameter and an increase in the scale parameter. The in-control and out-of-control pdfs are shown in Figure 3.20. The density curve was expanded to the right when the shape parameter decreases and the scale parameter increases, while the density curve was moved to the left when the shape parameter increases, and the scale parameter decreases.

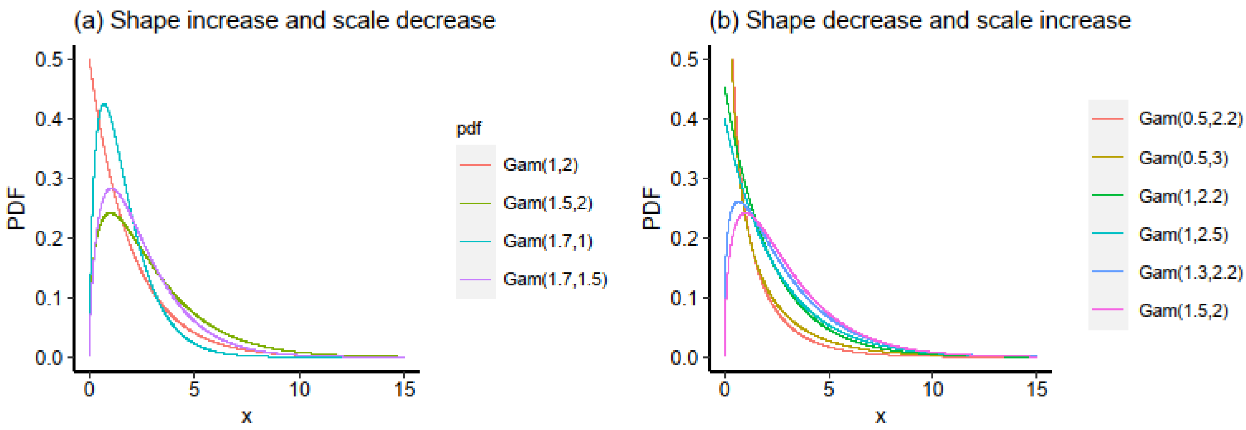

Figure 3.20: In-control and out-of-control pdfs for downward mean shift - change in both parameters - opposite direction - Gam(1.5,2)

Table 3.18 summaries the ARL and $SDRL$ values for some shifts in the parameters, which does not change the mean from its in-control value. All the control charts were insensitive to the shifts that occurred by increasing the shape parameter and decreasing the scale parameter. The $\tilde{X}_{cdf}$ and $\tilde{X}_{haz}$ control charts gave small ARL_1 than the $\bar{X}$ control chart where the shape parameter decreases, and the scale parameter increases. The corresponding pdf of in-control and out-of-control states are shown in Figure 3.21. The density curve was shrunk to the left when the shape parameter increases, and the scale parameter decreases. In contrast, the density curve was expanded to the right when the shape parameter decreases, and the scale parameter increases.

Table 3.17: ARL and $SDRL$ for downward mean shift - change in both parameters - opposite direction - Gam(1.5,2)

Sample size	Shape	Scale	Mean	Variance	Skewness	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart		$\tilde{X}_{haz}$ Chart	
						ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**1.50**	**2.00**	**3.00**	**6.00**	**1.63**	**98.54**	**97.38**	**99.94**	**101.13**	**101.54**	**101.24**	**100.27**	**99.67**	**100.69**	**101.28**	**98.36**	**97.58**
	1.70	1.50	2.55	3.83	1.53	121.59	121.29	150.81	149.37	121.55	120.99	104.30	103.14	168.72	168.53	112.47	112.39
	2.00	1.00	2.00	2.00	1.41	50.90	50.09	102.17	101.71	50.49	49.99	38.07	37.28	135.16	136.18	42.7	41.97
	1.70	1.00	1.70	1.70	1.53	15.09	14.63	24.39	23.82	15.03	14.80	14.76	14.17	30.21	29.80	14.44	13.97
	1.30	2.20	2.86	6.29	1.75	62.47	62.40	61.35	60.05	62.51	61.77	61.30	60.58	61.65	61.33	62.71	62.49
	1.00	2.50	2.50	6.25	2.00	23.31	22.66	18.34	17.94	23.36	22.89	27.73	26.97	18.33	17.79	27.15	26.39
	1.00	2.20	2.20	4.84	2.00	16.08	15.54	12.70	12.23	16.23	15.63	22.35	21.96	13.82	13.13	19.8	19.31
	0.50	3.00	1.50	4.50	2.83	3.24	2.71	2.19	1.60	3.19	2.65	5.55	5.08	2.23	1.68	4.69	4.14
	0.50	2.20	1.10	2.42	2.83	2.11	1.56	1.62	0.99	2.11	1.55	3.69	3.21	1.73	1.13	2.98	2.45
n=10	**1.50**	**2.00**	**3.00**	**6.00**	**1.63**	**100.13**	**100.29**	**100.94**	**100.46**	**103.03**	**102.16**	**100.77**	**100.72**	**101.31**	**101.02**	**100.50**	**100.59**
	1.70	1.50	2.55	3.83	1.53	72.88	73.42	98.56	97.47	73.23	73.95	59.01	58.68	148.43	149.07	64.66	63.86
	2.00	1.00	2.00	2.00	1.41	14.54	14.22	28.76	28.27	14.80	14.25	10.38	9.80	73.58	72.88	11.66	11.1
	1.70	1.00	1.70	1.70	1.53	4.42	3.91	6.61	6.11	4.37	3.87	4.69	4.13	12.29	11.9	4.24	3.69
	1.30	2.20	2.86	6.29	1.75	60.27	59.82	57.21	56.42	60.51	60.00	61.85	61.62	52.79	52.08	63.19	62.38
	1.00	2.50	2.50	6.25	2.00	17.61	17.13	12.96	12.39	17.55	16.99	27.63	27.08	10.23	9.65	23.79	23.27
	1.00	2.20	2.20	4.84	2.00	9.61	9.15	7.53	7.07	9.64	9.16	18.96	18.66	7.02	6.48	13.68	13.35
	0.50	3.00	1.50	4.50	2.83	2.05	1.49	1.50	0.86	2.05	1.47	5.39	4.93	1.32	0.66	3.58	3.08
	0.50	2.20	1.10	2.42	2.83	1.33	0.66	1.15	0.42	1.33	0.66	3.00	2.44	1.13	0.39	1.97	1.37

Table 3.18: ARL and $SDRL$ for parameters shift, mean in-control - Gam(1.5,2)

Sample size	Shape	Scale	Mean	Variance	Skewness	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart		$\tilde{X}_{haz}$ Chart	
						ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**1.50**	**2.00**	**3.00**	**6.00**	**1.63**	**98.54**	**97.38**	**99.94**	**101.13**	**101.54**	**101.24**	**100.27**	**99.67**	**100.69**	**101.28**	**98.36**	**97.58**
	1.65	1.82	3.00	5.47	1.56	144.61	143.14	133.52	132.89	123.11	123.30	150.66	149.72	127.90	125.96	147.76	146.37
	2.00	1.50	3.00	4.50	1.41	339.26	336.29	227.42	227.97	169.50	168.27	353.66	351.20	179.06	179.20	357.95	356.85
	2.50	1.20	3.00	3.60	1.26	1133.87	1133.43	366.72	365.95	213.44	212.35	982.44	980.10	214.13	213.21	1210.57	1209.96
	1.40	2.14	3.00	6.41	1.69	76.56	75.76	80.19	79.64	86.21	84.63	77.51	77.26	80.83	79.47	76.43	75.36
	1.30	2.31	3.00	6.94	1.75	60.38	60.03	62.76	61.90	71.18	70.97	58.22	57.31	63.14	63.14	59.15	58.65
	1.00	3.00	3.00	9.00	2.00	27.66	27.28	27.12	26.09	35.78	35.42	23.99	23.57	24.71	24.03	25.82	25.30
	0.80	3.75	3.00	11.25	2.24	16.02	15.58	14.04	13.57	20.26	19.86	13.10	12.64	11.80	11.30	14.67	14.09
n=10	**1.50**	**2.00**	**3.00**	**6.00**	**1.63**	**100.13**	**100.29**	**100.94**	**100.46**	**103.03**	**102.16**	**100.77**	**100.72**	**101.31**	**101.02**	**100.50**	**100.59**
	1.65	1.82	3.00	5.47	1.56	146.59	146.59	133.39	132.33	117.35	116.63	146.76	145.99	119.28	118.54	146.08	145.4
	2.00	1.50	3.00	4.50	1.41	346.20	345.07	224.58	223.60	122.42	121.12	272.95	270.55	111.12	109.6	335.72	336.41
	2.50	1.20	3.00	3.60	1.26	1162.52	1157.85	334.74	336.55	98.00	98.01	483.24	477.95	78.09	77.93	1009.25	1002.51
	1.40	2.14	3.00	6.41	1.69	78.68	78.77	80.02	79.58	86.42	84.95	76.31	75.51	80.15	78.59	77.67	76.75
	1.30	2.31	3.00	6.94	1.75	60.84	60.59	63.05	62.51	69.95	69.40	57.12	56.61	58.54	57.88	60.16	59.63
	1.00	3.00	3.00	9.00	2.00	28.29	27.69	26.25	25.95	28.67	28.21	20.41	19.80	17.13	16.82	25.27	24.63
	0.80	3.75	3.00	11.25	2.24	16.50	16.22	13.71	13.25	14.33	13.91	10.18	9.66	7.01	6.56	13.56	13.12

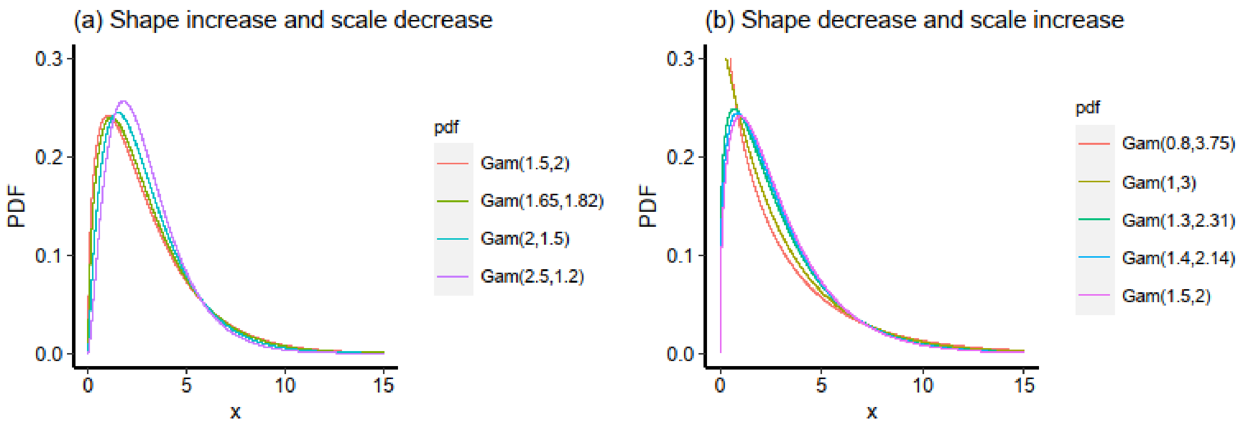

Figure 3.21: In-control and out-of-control pdfs for both parameters shift, mean in-control - Gam(1.5,2)

3.3.2.3 Shape Parameter Greater than One and the Scale Parameter – Gam(2,1)

Figure 3.22 shows the Gam(2,1) probability density curve with the means of the control statistics, obtained from an empirical distribution generated randomly for sample size ten. The weighted averages based on PDF-weight, Max-weight and CoCDF-weight were found to the left of the unweighted average, while the weighted averages based on Haz-weight and CDF-weight were found to the right of the unweighted average.

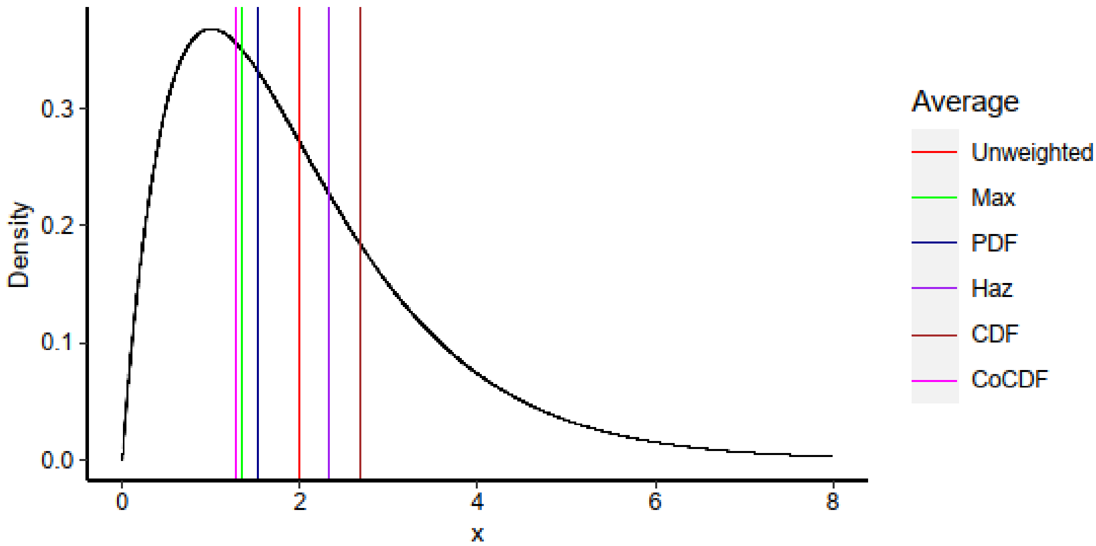

Figure 3.22: Gam(2,1) density curve with means of control statistics

Tables 3.19 and 3.20 provide the ARL and $SDRL$ for upward and downward mean shift by shifting the shape or scale parameter. Compared to $\bar{X}$ control chart, the $\tilde{X}_{max}$ control chart showed lower ARL_1 when the mean change occurred by a shift in the shape parameter. The $\bar{X}$ chart gave the lowest ARL_1 for the mean changes occurred by a shift in the scale parameter. The in-control and out-of-control pdf for mean increase and decrease is shown in Figures 3.23 and 3.24, respectively. The pdf was expanded to the right when the mean increases. Conversely, the pdf was shrunk to the left when the mean decreases.

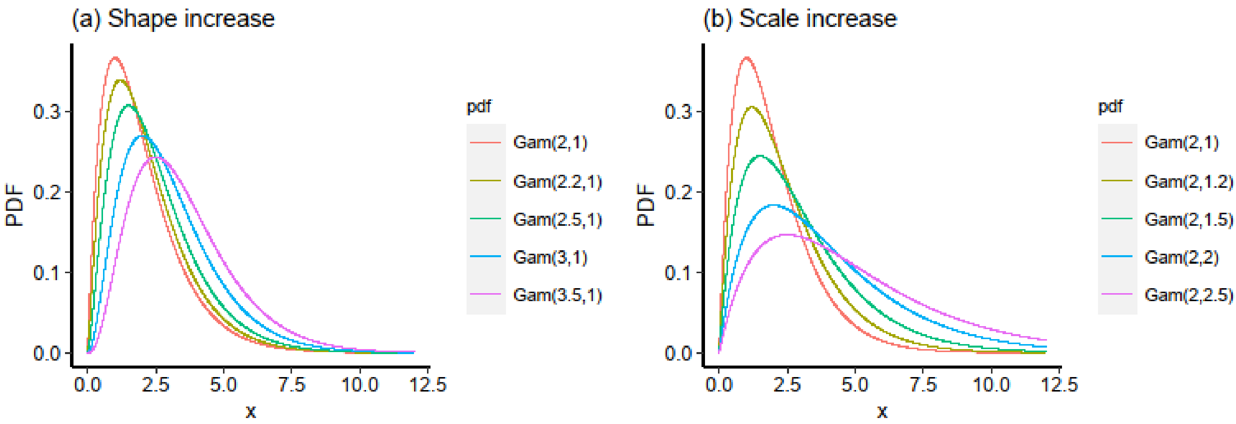

Figure 3.23: In-control and out-of-control pdfs for upward mean shift - change in one parameter - Gam(2,1)

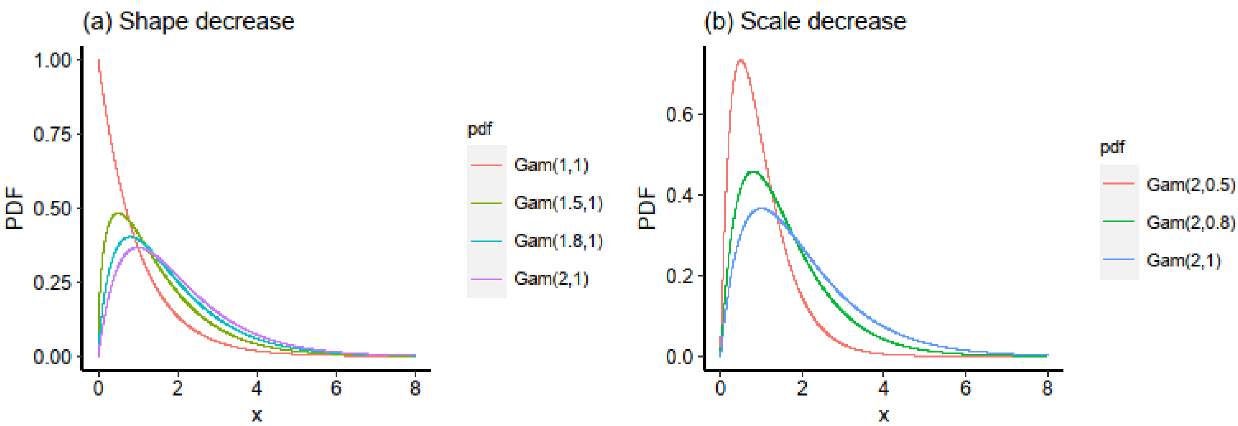

Figure 3.24: In-control and out-of-control pdfs for downward mean shift - change in one parameter - Gam(2,1)

Table 3.19: ARL and $SDRL$ for upward mean shift - change in one parameter - Gam(2,1)

Sample size	Shape	Scale	Mean	Variance	Skewness	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart		$\tilde{X}_{haz}$ Chart	
						ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**2.00**	**1.00**	**2.00**	**2.00**	**1.41**	**99.75**	**100.11**	**99.13**	**99.28**	**99.82**	**100.11**	**101.22**	**100.17**	**99.02**	**99.10**	**100.33**	**100.29**
	2.20	1.00	2.20	2.20	1.35	79.22	78.73	77.01	76.01	79.05	78.65	116.28	114.72	79.85	79.32	94.35	92.65
	2.50	1.00	2.50	2.50	1.26	33.74	33.15	31.37	30.82	34.76	34.57	85.04	84.46	35.31	34.52	49.47	48.76
	3.00	1.00	3.00	3.00	1.15	9.59	9.12	8.61	8.07	10.38	9.86	32.84	31.93	10.44	10.00	15.30	14.84
	3.50	1.00	3.50	3.50	1.07	3.85	3.32	3.53	2.97	4.48	3.94	13.16	12.50	4.50	4.01	6.13	5.54
	2.00	1.20	2.40	2.88	1.41	30.81	30.23	38.47	38.28	48.40	48.08	44.47	44.34	47.24	46.53	34.41	34.07
	2.00	1.50	3.00	4.50	1.41	6.90	6.31	9.88	9.37	16.00	15.38	11.05	10.48	16.11	15.55	7.87	7.26
	2.00	2.00	4.00	8.00	1.41	2.19	1.63	3.04	2.52	5.89	5.35	3.08	2.52	5.92	5.35	2.39	1.83
	2.00	2.50	5.00	12.50	1.41	1.39	0.73	1.77	1.17	3.52	2.95	1.72	1.12	3.58	3.03	1.48	0.84
n=10	**2.00**	**1.00**	**2.00**	**2.00**	**1.41**	**99.73**	**98.32**	**99.61**	**98.73**	**100.00**	**99.64**	**99.96**	**99.93**	**100.77**	**100.72**	**101.21**	**100.47**
	2.20	1.00	2.20	2.20	1.35	62.11	61.80	57.28	56.70	63.49	62.81	99.28	98.47	62.92	62.64	77.87	76.76
	2.50	1.00	2.50	2.50	1.26	17.56	16.99	15.33	14.75	18.82	18.28	50.39	49.96	18.83	18.33	27.66	27.04
	3.00	1.00	3.00	3.00	1.15	3.91	3.38	3.37	2.80	4.40	3.90	13.56	12.92	4.53	4.02	6.43	5.90
	3.50	1.00	3.50	3.50	1.07	1.69	1.09	1.56	0.93	1.99	1.41	4.64	4.14	2.07	1.48	2.44	1.86
	2.00	1.20	2.40	2.88	1.41	**19.27**	18.72	23.54	22.79	37.00	36.31	28.42	27.87	39.60	38.73	21.83	21.13
	2.00	1.50	3.00	4.50	1.41	**3.45**	2.94	4.46	3.97	9.85	9.39	5.29	4.80	11.52	10.93	3.93	3.40
	2.00	2.00	4.00	8.00	1.41	**1.31**	0.63	1.51	0.88	3.61	3.07	1.60	0.98	4.21	3.66	1.38	0.74
	2.00	2.50	5.00	12.50	1.41	**1.05**	0.23	1.12	0.36	2.35	1.77	1.13	0.39	2.66	2.11	1.07	0.28

Table 3.20: ARL and $SDRL$ for downward mean shift - change in one parameter - Gam(2,1)

Sample size	Shape	Scale	Mean	Variance	Skewness	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart		$\tilde{X}_{haz}$ Chart	
						ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**2.00**	**1.00**	**2.00**	**2.00**	**1.41**	**99.75**	**100.11**	**99.13**	**99.28**	**99.82**	**100.11**	**101.22**	**100.17**	**99.02**	**99.10**	**100.33**	**100.29**
	1.80	1.00	1.80	1.80	1.49	60.86	60.57	59.88	59.82	70.25	70.40	64.93	63.84	64.33	63.18	65.47	64.42
	1.50	1.00	1.50	1.50	1.63	17.88	17.29	15.88	15.43	23.40	23.13	24.79	23.97	18.11	17.69	22.85	22.20
	1.00	1.00	1.00	1.00	2.00	3.17	2.65	2.62	2.04	4.67	4.18	5.72	5.21	2.93	2.38	4.76	4.22
	2.00	0.80	1.60	1.28	1.41	47.65	46.77	56.54	56.36	59.50	59.48	50.15	49.68	70.44	70.08	49.01	48.61
	2.00	0.50	1.00	0.50	1.41	4.64	4.05	6.86	6.35	6.18	5.67	5.57	5.16	8.10	7.49	5.03	4.55
n=10	**2.00**	**1.00**	**2.00**	**2.00**	**1.41**	**99.73**	**98.32**	**99.61**	**98.73**	**100.00**	**99.64**	**99.96**	**99.93**	**100.77**	**100.72**	**101.21**	**100.47**
	1.80	1.00	1.80	1.80	1.49	47.75	47.25	**45.50**	45.04	53.62	52.75	58.17	57.68	48.69	48.48	55.47	55.16
	1.50	1.00	1.50	1.50	1.63	9.75	9.33	**8.42**	7.95	12.33	11.92	18.40	17.65	9.39	9.04	14.24	13.74
	1.00	1.00	1.00	1.00	2.00	1.70	1.08	**1.49**	0.86	2.23	1.65	3.92	3.37	1.54	0.91	2.81	2.22
	2.00	0.80	1.60	1.28	1.41	24.26	23.78	28.61	27.94	33.82	33.45	28.06	27.80	43.72	43.00	25.59	25.19
	2.00	0.50	1.00	0.50	1.41	1.69	1.07	2.11	1.52	2.31	1.74	2.23	1.67	3.03	2.49	1.87	1.28

The ARL and $SDRL$ for mean shifts that occurred by simultaneously increasing /decreasing both parameters are given in Table 3.21. The $\bar{X}$ control chart gave the smallest ARL_1 compared with the weighted average control charts when the mean was shifted by changing both parameters to the same direction. The corresponding pdf is shown in Figure 3.25. The pdf was flattened and was moved to the right when the mean increases, and the density curve's height rises when the mean decreases.

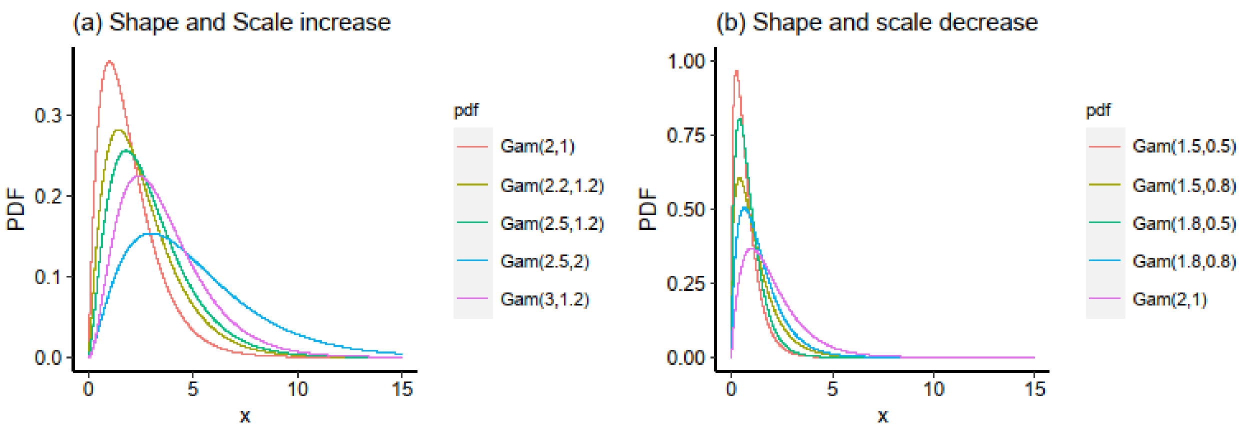

Figure 3.25: In-control and out-of-control pdfs – change in both parameters - same direction - Gam(2,1)

Tables 3.22 and 3.23 give the ARL and $SDRL$ for mean increases and decreases from the in-control state by shifting both parameters in opposite directions, respectively. According to Table 3.22, the $\tilde{X}_{max}$, $\tilde{X}_{pdf}$ and $\tilde{X}_{(1-cdf)}$ control charts showed lower ARL_1 than the $\bar{X}$ control chart detects an increase in the mean when the shape parameter increases and the scale parameter decreases from their in-control values. $\bar{X}$ control chart gave the lowest ARL_1, when the shape parameter decreased, the scale parameter increased from the in-control value. The corresponding pdf is shown in Figure 3.2. The density curve was expanded to the right when the parameters shift in opposite direction, where the mean was increased.

Table 3.21: ARL and $SDRL$ for mean shifts - change in both parameters - same direction - Gam(2,1)

Sample size	Shape	Scale	Mean	Variance	Skewness	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart		$\tilde{X}_{haz}$ Chart	
						ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
Mean increase																	
n=5	**2.00**	**1.00**	**2.00**	**2.00**	**1.41**	**99.75**	**100.11**	**99.13**	**99.28**	**99.82**	**100.11**	**101.22**	**100.17**	**99.02**	**99.10**	**100.33**	**100.29**
	2.20	1.20	2.64	3.17	1.35	17.11	16.51	20.75	20.42	27.52	27.22	33.24	32.47	17.37	16.77	21.53	20.98
	2.50	1.20	3.00	3.60	1.26	8.15	7.60	9.10	8.53	12.62	12.22	18.98	18.36	8.42	7.92	11.09	10.47
	3.00	1.20	3.60	4.32	1.15	3.21	2.65	3.42	2.84	4.84	4.29	8.02	7.48	3.13	2.58	4.46	3.91
	2.50	2.00	5.00	10.00	1.26	1.34	0.68	1.57	0.94	2.68	2.10	1.78	1.19	1.13	0.39	1.46	0.82
n=10	**2.00**	**1.00**	**2.00**	**2.00**	**1.41**	**99.73**	**98.32**	**99.61**	**98.73**	**100.00**	**99.64**	**99.96**	**99.93**	**100.77**	**100.72**	**101.21**	**100.47**
	2.20	1.20	2.64	3.17	1.35	8.81	8.28	10.05	9.64	16.50	15.83	17.37	16.77	17.60	16.89	11.38	10.83
	2.50	1.20	3.00	3.60	1.26	3.65	3.11	3.81	3.27	6.30	5.75	8.42	7.92	6.73	6.17	5.09	4.58
	3.00	1.20	3.60	4.32	1.15	1.57	0.95	1.57	0.94	2.29	1.72	3.13	2.58	2.45	1.89	2.02	1.43
	2.50	2.00	5.00	10.00	1.26	1.04	0.20	1.06	0.25	1.70	1.10	1.13	0.39	1.86	1.26	1.06	0.25
Mean decrease																	
n=5	**2.00**	**1.00**	**2.00**	**2.00**	**1.41**	**99.75**	**100.11**	**99.13**	**99.28**	**99.82**	**100.11**	**101.22**	**100.17**	**99.02**	**99.10**	**100.33**	**100.29**
	1.80	0.80	1.44	1.15	1.49	21.15	20.64	23.68	23.09	27.98	27.48	25.71	25.33	29.54	29.12	23.95	23.35
	1.50	0.80	1.20	0.96	1.63	7.30	6.83	7.37	6.79	10.18	9.58	10.75	10.31	8.91	8.41	9.37	8.90
	1.80	0.50	0.90	0.45	1.49	3.02	2.50	4.01	3.45	4.01	3.51	3.98	3.46	4.67	4.17	3.50	2.91
	1.50	0.50	0.75	0.38	1.63	1.86	1.28	2.14	1.57	2.40	1.82	2.62	2.06	2.37	1.78	2.24	1.68
n=10	**2.00**	**1.00**	**2.00**	**2.00**	**1.41**	**99.73**	**98.32**	**99.61**	**98.73**	**100.00**	**99.64**	**99.96**	**99.93**	**100.77**	**100.72**	**101.21**	**100.47**
	1.80	0.80	1.44	1.15	1.49	9.46	8.92	10.28	9.72	13.48	12.99	17.37	16.77	14.80	14.27	11.49	11.06
	1.50	0.80	1.20	0.96	1.63	3.14	2.60	3.12	2.61	4.40	3.85	8.42	7.92	3.90	3.40	4.34	3.83
	1.80	0.50	0.90	0.45	1.49	1.33	0.67	1.51	0.87	1.67	1.07	3.13	2.58	1.90	1.30	1.50	0.87
	1.50	0.50	0.75	0.38	1.63	1.09	0.31	1.12	0.38	1.24	0.54	1.13	0.39	1.24	0.55	1.21	0.51

Table 3.22: ARL and $SDRL$ for upward mean shift - change in both parameters - opposite directions – Gam(2,1)

Sample size	Shape	Scale	Mean	Variance	Skewness	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart		$\tilde{X}_{haz}$ Chart	
						ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**2.00**	**1.00**	**2.00**	**2.00**	**1.41**	**99.75**	**100.11**	**99.13**	**99.28**	**99.82**	**100.11**	**101.22**	**100.17**	**99.02**	**99.10**	**100.33**	**100.29**
	2.50	**0.90**	2.25	2.03	1.26	95.79	94.50	73.90	74.10	71.91	71.25	196.36	194.27	72.23	71.73	136.92	136.74
	3.00	**0.80**	2.40	1.92	1.15	78.91	77.57	46.83	46.11	42.80	42.91	321.45	324.01	42.15	41.93	153.03	152.06
	3.50	**0.80**	2.80	2.24	1.07	20.30	19.58	13.16	12.74	13.06	12.49	124.36	123.83	13.21	12.75	43.50	43.56
	1.80	**1.20**	2.16	2.59	1.49	49.89	49.45	59.24	58.25	70.78	69.83	53.52	53.12	63.90	63.86	49.67	48.86
	1.50	**1.50**	2.25	3.38	1.63	25.47	24.97	36.06	35.67	48.52	48.65	23.12	22.51	36.88	36.35	21.91	21.25
	1.80	**1.50**	2.70	4.05	1.49	11.54	10.87	17.97	17.40	27.60	27.20	15.60	14.98	27.22	26.72	12.08	11.55
	1.50	**2.00**	3.00	6.00	1.63	5.71	5.19	10.64	10.12	20.29	20.00	6.45	6.03	18.79	18.45	5.48	4.97
n=10	**2.00**	**1.00**	**2.00**	**2.00**	**1.41**	**99.73**	**98.32**	**99.61**	**98.73**	**100.00**	**99.64**	**99.96**	**99.93**	**100.77**	**100.72**	**101.21**	**100.47**
	2.50	**0.90**	2.25	2.03	1.26	65.52	65.00	47.71	47.24	45.60	45.07	169.14	168.58	42.75	42.29	107.07	107.27
	3.00	**0.80**	2.40	1.92	1.15	40.48	39.61	22.72	22.40	19.29	18.64	242.72	244.56	17.17	16.58	95.40	95.78
	3.50	**0.80**	2.80	2.24	1.07	7.66	7.11	4.93	4.46	4.88	4.36	55.96	55.55	4.63	4.05	17.85	17.46
	1.80	**1.20**	2.16	2.59	1.49	43.61	42.90	53.89	53.25	70.27	68.96	42.88	42.52	65.96	65.59	40.36	39.72
	1.50	**1.50**	2.25	3.38	1.63	20.53	42.90	32.43	32.19	51.64	51.37	15.47	15.13	37.05	36.14	15.68	15.12
	1.80	**1.50**	2.70	4.05	1.49	6.42	42.90	9.25	8.82	21.12	20.84	7.94	7.45	24.36	23.96	6.46	5.99
	1.50	**2.00**	3.00	6.00	1.63	3.25	42.90	5.30	4.81	17.39	17.14	3.34	2.77	20.61	20.45	2.97	2.39

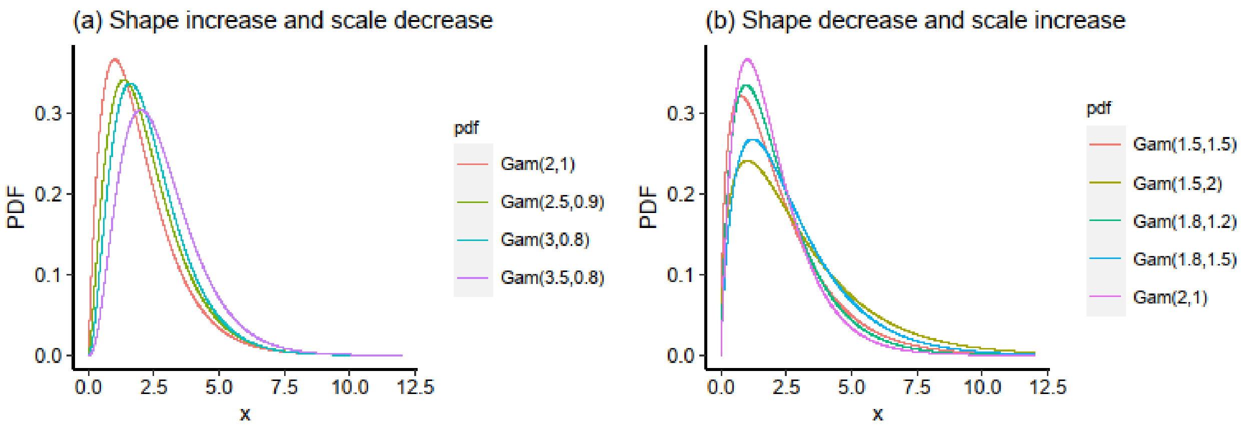

Figure 3.26: In-control and out-of-control pdfs for upward mean shift - change in both parameters - opposite direction - Gam(2,1)

As depicted in Table 3.23, the $\tilde{X}_{cdf}$ and $\tilde{X}_{haz}$ control charts showed lower ARL_1 compared to $\bar{X}$ control chart when the mean decreases by increasing the shape parameter and decreasing the scale parameter for sample size 10. $\tilde{X}_{max}$ and $\tilde{X}_{(1-cdf)}$ control charts gave small ARL_1 than $\bar{X}$ control chart when the scale parameter increased and the shape parameter decreased from their in-control values to detect a decrease in the process mean. Figure 3.27 shows the corresponding pdf. It was observed that when the parameters shift to the opposite direction resulting a mean decrease, the pdf was shrunk to the left.

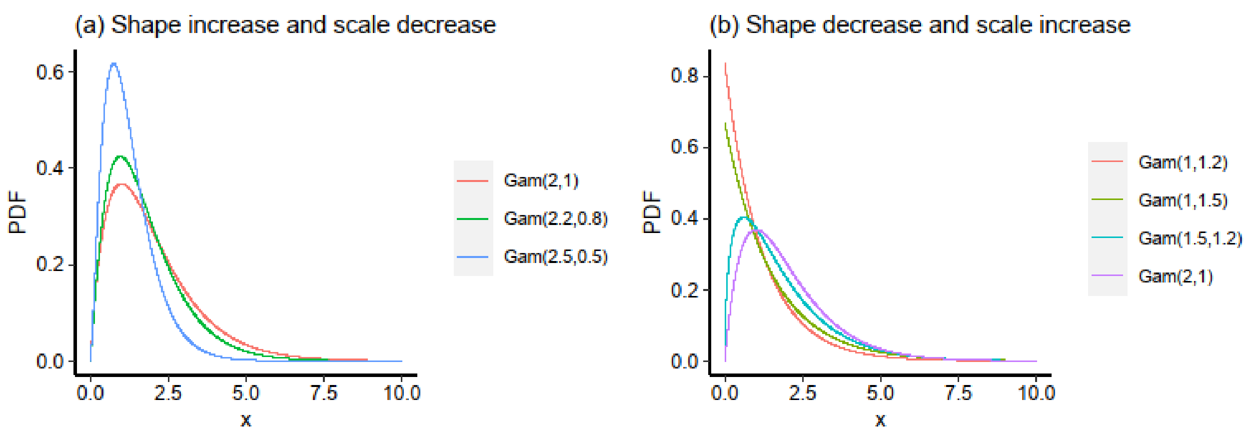

Figure 3.27: In-control and out-of-control pdfs for downward mean shift - change in both parameters - opposite direction - Gam(2,1)

Table 3.23: ARL and $SDRL$ for downward mean shift - changing both parameters - opposite direction - Gam(2,1)

Sample size	Shape	Scale	Mean	Variance	Skewness	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart		$\tilde{X}_{haz}$ Chart	
						ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**2.00**	**1.00**	**2.00**	**2.00**	**1.41**	**99.75**	**100.11**	**99.13**	**99.28**	**99.82**	**100.11**	**101.22**	**100.17**	**99.02**	**99.10**	**100.33**	**100.29**
	2.20	**0.80**	1.76	1.41	1.35	112.81	111.28	130.25	128.78	118.43	117.65	100.06	100.20	150.46	148.89	104.07	103.99
	2.50	**0.50**	1.25	0.63	1.26	18.06	17.41	34.70	34.19	23.59	23.28	15.97	15.46	46.76	45.68	15.98	15.51
	1.50	**1.20**	1.80	2.16	1.63	36.25	35.87	30.94	30.07	42.91	42.16	38.68	38.05	30.86	30.01	39.10	38.82
	1.00	**1.50**	1.50	2.25	2.00	9.23	8.64	6.31	5.85	12.02	11.47	13.73	13.34	6.12	5.65	13.00	12.48
	1.00	**1.20**	1.20	1.44	2.00	4.93	4.41	3.79	3.26	7.19	6.63	8.89	8.35	4.09	3.58	7.66	7.14
n=10	**2.00**	**1.00**	**2.00**	**2.00**	**1.41**	**99.73**	**98.32**	**99.61**	**98.73**	**100.00**	**99.64**	**99.96**	**99.93**	**100.77**	**100.72**	**101.21**	**100.47**
	2.20	**0.80**	1.76	1.41	1.35	72.16	70.55	91.91	90.55	93.10	92.44	63.63	63.03	132.44	132.44	66.26	65.39
	2.50	**0.50**	1.25	0.63	1.26	5.00	4.44	8.95	8.49	8.45	7.89	4.73	4.14	19.15	18.69	4.51	3.96
	1.50	**1.20**	1.80	2.16	1.63	29.40	29.34	23.63	23.09	30.86	30.16	37.94	37.44	19.68	19.01	37.64	36.88
	1.00	**1.50**	1.50	2.25	2.00	6.27	5.74	4.07	3.52	6.64	6.08	13.73	13.29	3.12	2.57	11.97	11.60
	1.00	**1.20**	1.20	1.44	2.00	2.75	2.20	2.14	1.57	3.47	2.96	7.33	6.81	2.04	1.45	5.16	4.60

Table 3.24 gives the ARL and $SDRL$ for shifts in the parameters which do not change the mean. All the control charts were insensitive for shifts when the shape parameter increases and the scale parameter decreases. $\tilde{X}_{cdf}$ and $\tilde{X}_{haz}$ control charts gave less ARL_1 than the $\bar{X}$ control chart when the shape parameter decreases and the scale parameter increases, but the mean does not change. It was observed that the pdfs were shrunk to the left (Figure 3.28).

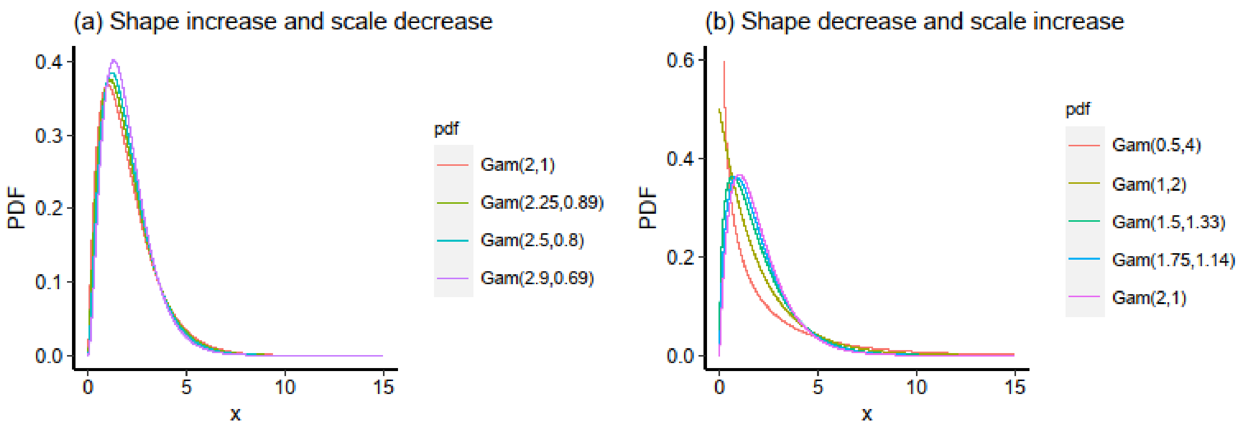

Figure 3.28: In-control and out-of-control pdfs for both parameters shift, mean in-control - Gam(2,1)

3.4 Summary

The ARL and the $SDRL$ obtained for N(0,1), exp(1), Gam(0.5,1), Gam(1.5,2) and Gam(2,1) were summarised in this chapter. The mean shifted by changing parameters in several possible scenarios were considered when calculating the ARL and $SDRL$. The next chapter will discuss the findings of the study.

Table 3.24: ARL and $SDRL$ for parameters shift, mean in-control - Gam(2,1)

	Shape	Scale	Mean	Variance	Skewness	$\bar{X}$ Chart		$\tilde{X}_{max}$ Chart		$\tilde{X}_{pdf}$ Chart		$\tilde{X}_{cdf}$ Chart		$\tilde{X}_{(1-cdf)}$ Chart		$\tilde{X}_{haz}$ Chart	
						ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
n=5	**2.00**	**1.00**	**2.00**	**2.00**	**1.41**	**99.75**	**100.11**	**99.13**	**99.28**	**99.82**	**100.11**	**101.22**	**100.17**	**99.02**	**99.10**	**100.33**	**100.29**
	2.25	**0.89**	2.00	1.78	1.33	157.53	157.11	143.54	141.11	129.25	127.70	164.99	164.57	136.85	135.62	161.84	160.20
	2.50	**0.80**	2.00	1.60	1.26	251.28	248.44	193.26	195.73	155.64	157.95	259.22	259.79	169.50	170.43	265.08	267.28
	2.90	**0.69**	2.00	1.38	1.17	513.12	514.64	278.97	276.65	191.72	193.26	489.46	484.97	200.42	201.99	537.47	536.02
	1.75	**1.14**	2.00	2.27	1.51	62.18	62.08	64.30	64.17	74.55	74.48	60.55	60.06	64.59	64.38	61.20	60.40
	1.50	**1.33**	2.00	2.65	1.63	38.36	37.68	38.80	38.34	51.80	51.47	35.05	34.38	36.68	35.75	36.13	35.56
	1.00	**2.00**	2.00	4.00	2.00	14.30	13.84	12.26	11.69	20.80	20.04	11.13	10.69	9.56	8.95	12.08f	11.59
	0.50	**4.00**	2.00	8.00	2.83	4.90	4.38	3.46	2.94	7.06	6.58	3.59	3.03	2.36	1.79	**-**	-
n=10	**2.00**	**1.00**	**2.00**	**2.00**	**1.41**	**99.73**	**98.32**	**99.61**	**98.73**	**100.00**	**99.64**	**99.96**	**99.93**	**100.77**	**100.72**	**101.21**	**100.47**
	2.25	**0.89**	2.00	1.78	1.33	157.27	157.41	141.10	140.90	121.79	122.04	156.69	154.36	123.44	124.24	161.62	161.11
	2.50	**0.80**	2.00	1.60	1.26	254.60	256.40	190.13	189.40	132.52	131.18	220.13	218.08	123.44	124.24	249.65	250.29
	2.90	**0.69**	2.00	1.38	1.17	514.10	505.57	264.83	266.04	132.14	130.07	322.58	323.57	124.56	123.65	455.88	454.58
	1.75	**1.14**	2.00	2.27	1.51	62.08	61.70	63.05	62.21	72.51	71.89	58.27	57.64	59.78	58.94	60.30	59.62
	1.50	**1.33**	2.00	2.65	1.63	38.38	37.97	38.00	37.91	46.28	46.27	30.77	30.03	28.35	27.89	34.36	33.90
	1.00	**2.00**	2.00	4.00	2.00	14.46	14.13	11.87	11.44	15.82	15.40	8.33	7.87	5.64	5.09	10.22	9.70
	0.50	**4.00**	2.00	8.00	2.83	4.99	4.46	3.38	2.80	4.90	4.38	2.52	1.96	1.47	0.83	**-**	-

Chapter 4 - Discussion

4.1 Introduction

Chapter 4 consists of a discussion about the weighted averages and the results presented in Chapter 3. Further, the control charts for Phase I for the probability distributions considered in this study are included. Finally, joint monitoring of the mean and variance of the normally distributed process is discussed.

4.2 Weighted Averages

The skewness of the empirical distribution of the control statistic discussed in this study for sample sizes five and ten are given in Tables 4.1 and 4.2, respectively. All the averages are positively skewed in exponential and gamma distributions. The simple average ($\bar{X}$) and the weighted average based on PDF-weight ($\tilde{X}_{pdf}$) are symmetric in a normal distribution, and the weighted averages based on CDF-weight ($\tilde{X}_{cdf}$) and Max-weight ($\tilde{X}_{\max}$) are negatively skewed. In contrast, Haz-weight ($\tilde{X}_{haz}$) and CoCDF-weight ($\tilde{X}_{(1-cdf)}$) based weighted averages are positively skewed for normally distributed data. The unweighted average is highly skewed in the Gam (0.5,1) distribution.

Table 4.1: Observed skewness of control statistics for sample size five

Weight	N (0,1)	Exp (1)	Gam (0.5,1)	Gam (1.5,2)	Gam (2,1)
Unweighted	0	0.9	1.3	0.7	0.6
Max	-0.2	1.2	1.8	1.0	0.8
PDF	0	1.3	2.3	0.9	0.8
Haz (1-PDF) *	0.1	0.9	1.7	0.7	0.6
CDF	-0.1	0.9	1.3	0.7	0.6
1- CDF	0.1	1.3	1.9	1.1	1.0

* for exponential distribution only

Table 4.2: Observed skewness of control statistics for sample size ten

Weight	N (0,1)	Exp (1)	Gam (0.5,1)	Gam (1.5,2)	Gam (2,1)
Unweighted	0	0.6	0.9	0.5	0.5
Max	-0.1	0.7	1.1	0.6	0.5
PDF	0	0.8	1.7	0.6	0.5
Haz (1-PDF) *	0.1	0.6	1.3	0.5	0.5
CDF	-0.1	0.6	0.9	0.5	0.4
1- CDF	0.1	0.8	1.1	0.7	0.6

* for exponential distribution only

4.2.1 Unweighted Average - $\bar{X}$

The histograms plotted for different pdfs are shown in Figure 4.1 for subgroup size ten. The distribution of the unweighted or the simple average of the symmetric distribution is also symmetric. The distribution of the unweighted average is positively skewed when the underlying data distribution is positively skewed.

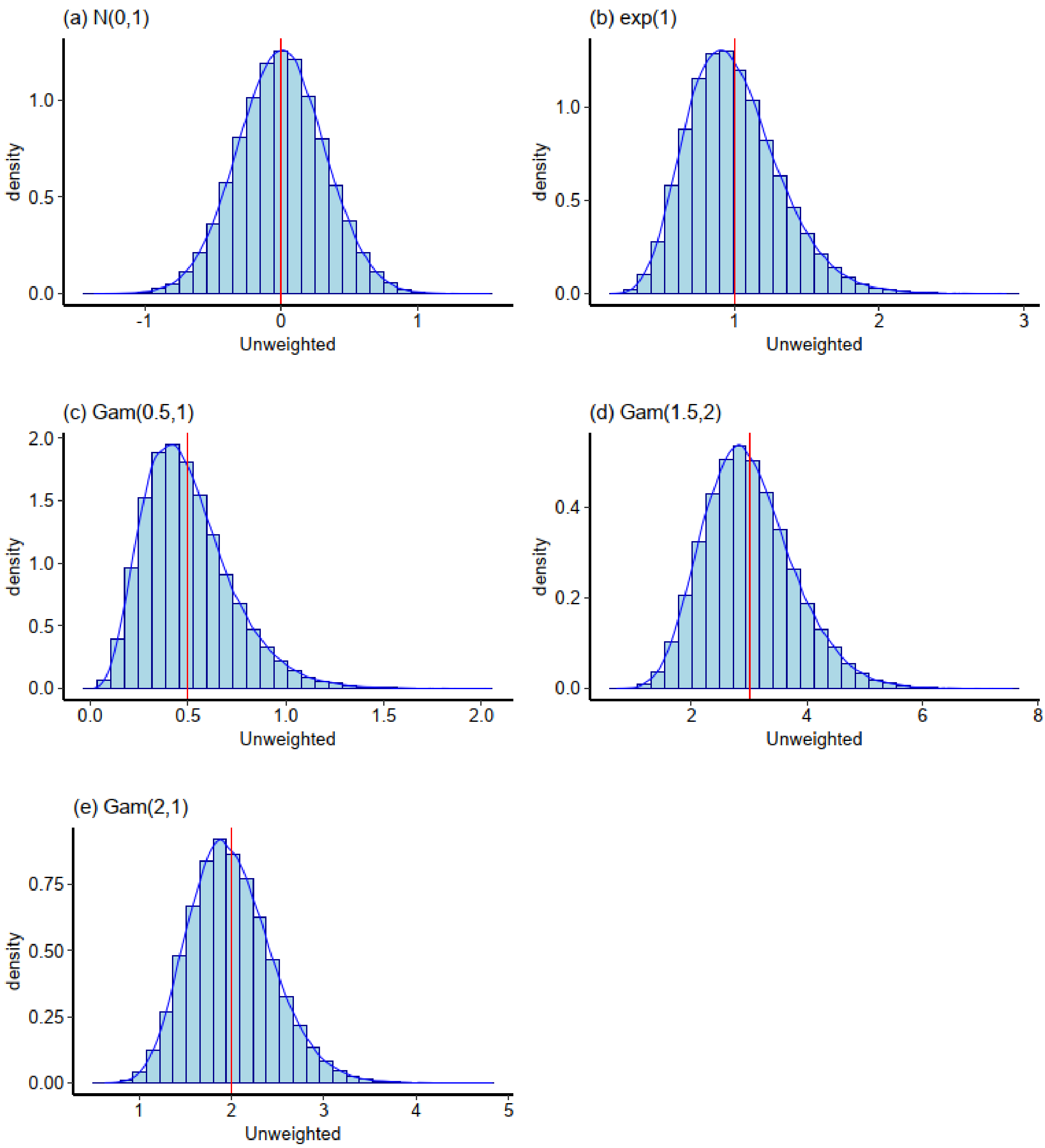

Figure 4.1: Histograms of empirical distributions of $\bar{X}$

4.2.2 Maximum Distance Based Weighted Average - $\tilde{X}_{max}$

The Max-weight considered the distance from the maximum value to the particular observation. The highest weight is assigned to the minimum value in the subgroup, and the maximum value is avoided since it got a zero weight. Therefore, the distribution of $\tilde{X}_{max}$ is positively skewed. Figure 4.2 shows the histograms obtained from the empirical distribution of $\tilde{X}_{max}$ for subgroup size 10.

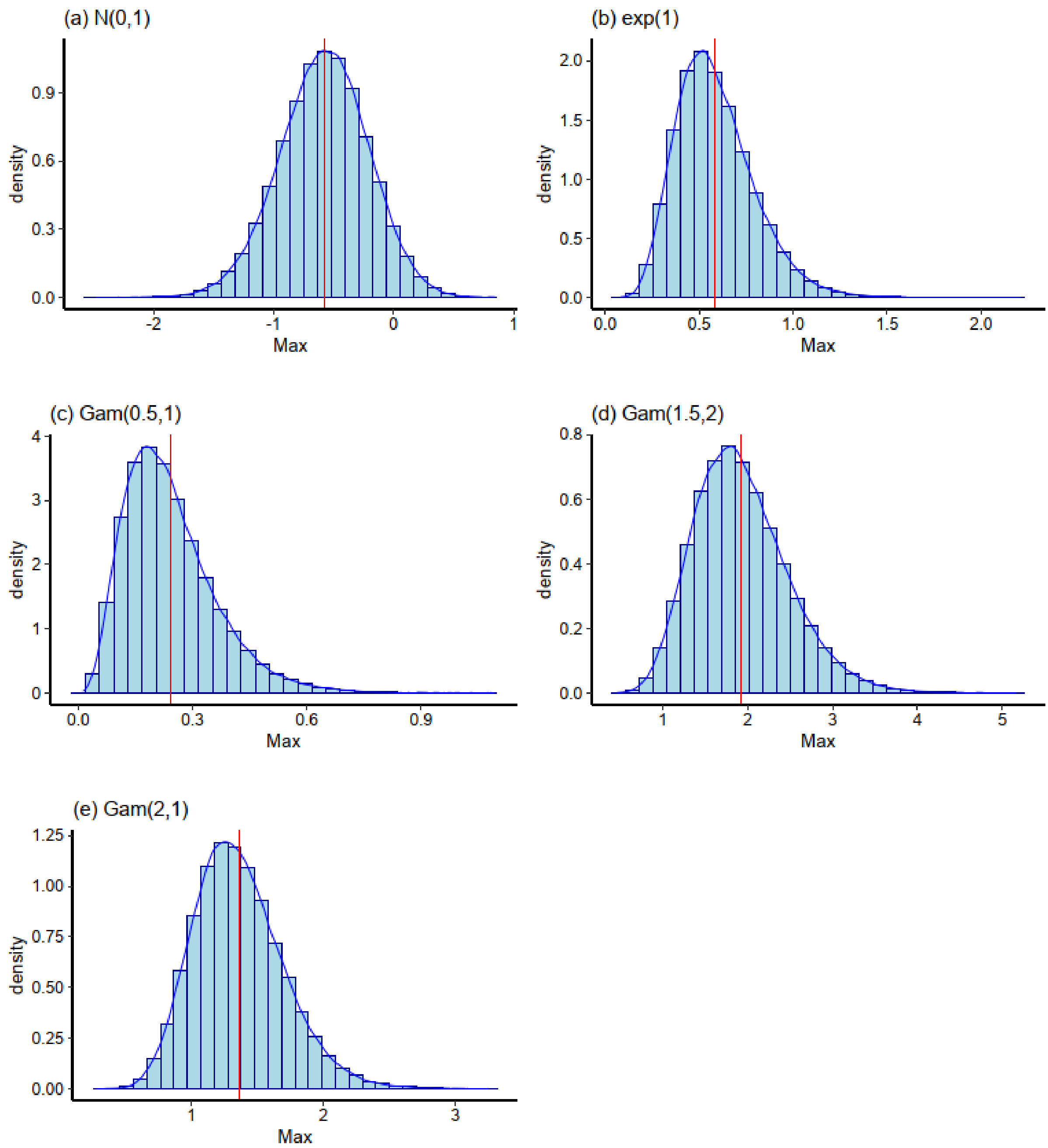

Figure 4.2: Histograms of empirical distributions of $\tilde{X}_{max}$

4.2.3 Probability Density Based Weighted Average - $\widetilde{X}_{pdf}$

The PDF-weight assigned the corresponding probability for observations according to the underlying distribution. The highest weight is given to the data which have the highest probability of occurring. When the underlying distribution is symmetric, the distribution of the $\tilde{X}_{pdf}$ is also symmetric and skewed otherwise. The histograms for the different probability distributions drawn for subgroup size ten are shown in Figure 4.3.

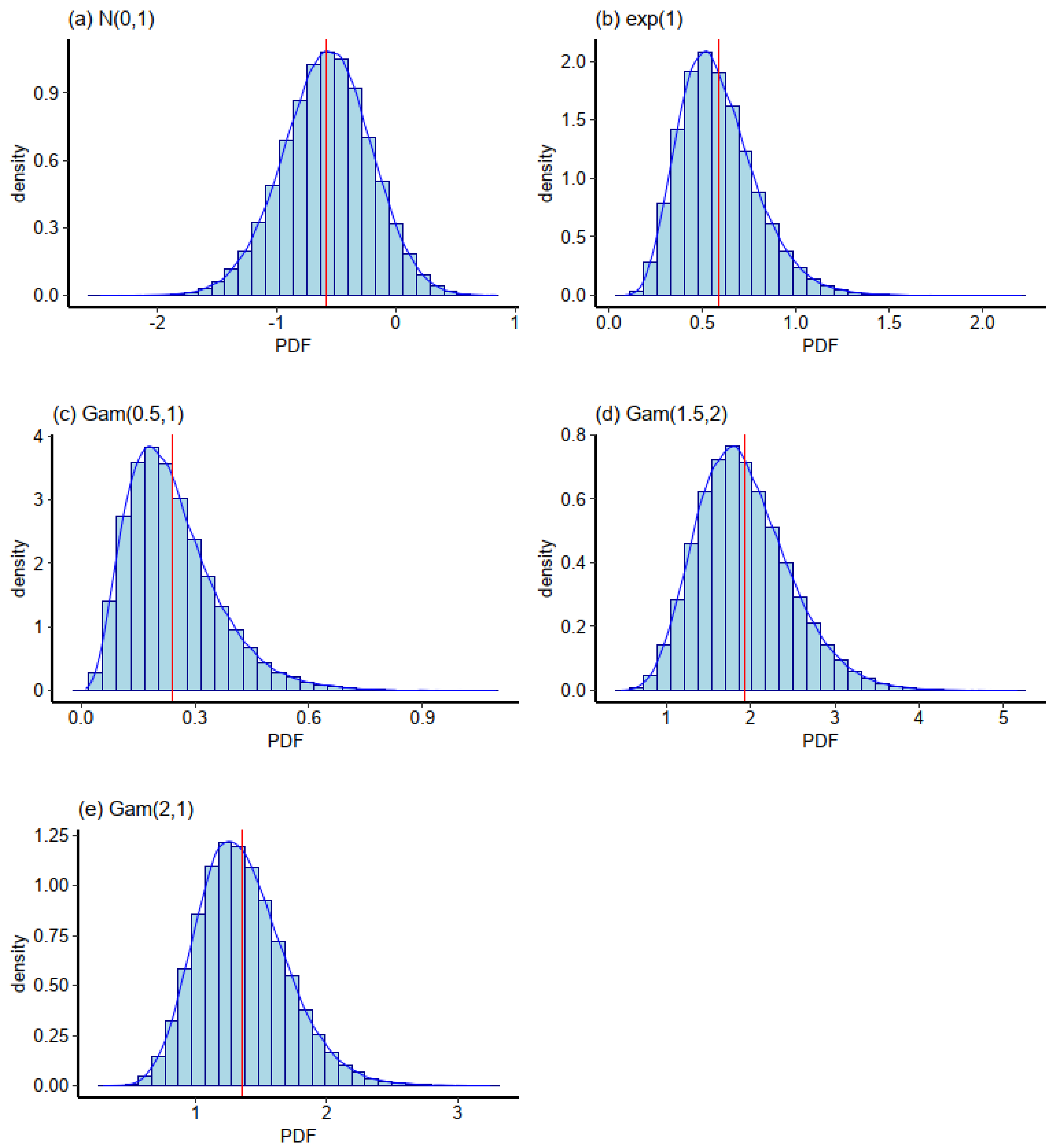

Figure 4.3: Histograms of empirical distributions of $\tilde{X}_{pdf}$

4.2.4 Hazard Function Based Weighted Average - $\tilde{X}_{haz}$

The Haz-weight is based on the hazard function. The distribution of the $\tilde{X}_{haz}$ is skewed. The highest weight is assigned to the highest data in the subgroup for symmetric distributions. In contrast, the highest weight is given to the lowest data in the subgroup when the distribution is positively skewed. The histograms for the different probability distributions drawn for subgroup size ten are shown in Figure 4.4.

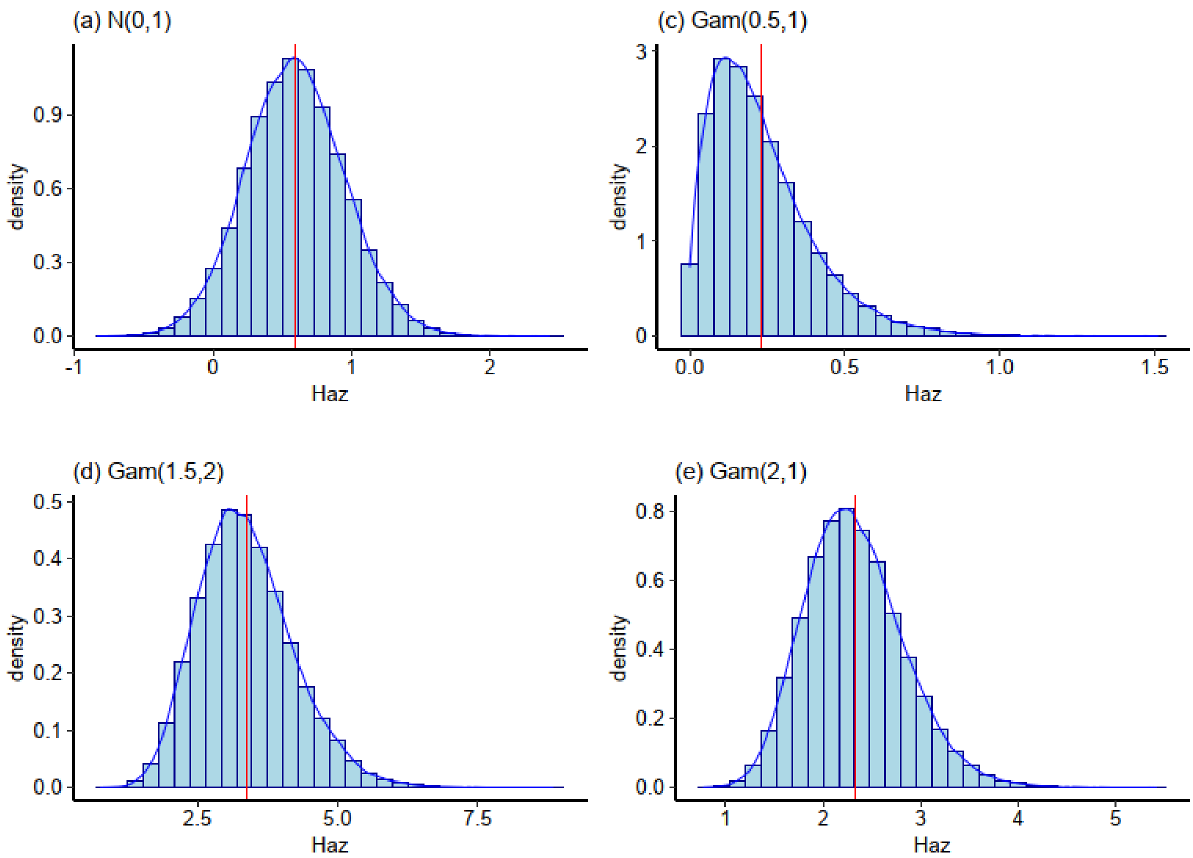

Figure 4.4: Histograms of empirical distributions of $\tilde{X}_{haz}$

4.2.5 Cumulative Density Function Based Weighted Average - $\tilde{X}_{cdf}$

The CDF-weight assigned the cumulative probability under the data in the subgroup. The highest weight is given to the most extensive data in the subgroup. The distribution of $\tilde{X}_{cdf}$ is skewed. The histograms drawn for subgroup size ten for different probability distributions are shown in Figure 4.5.

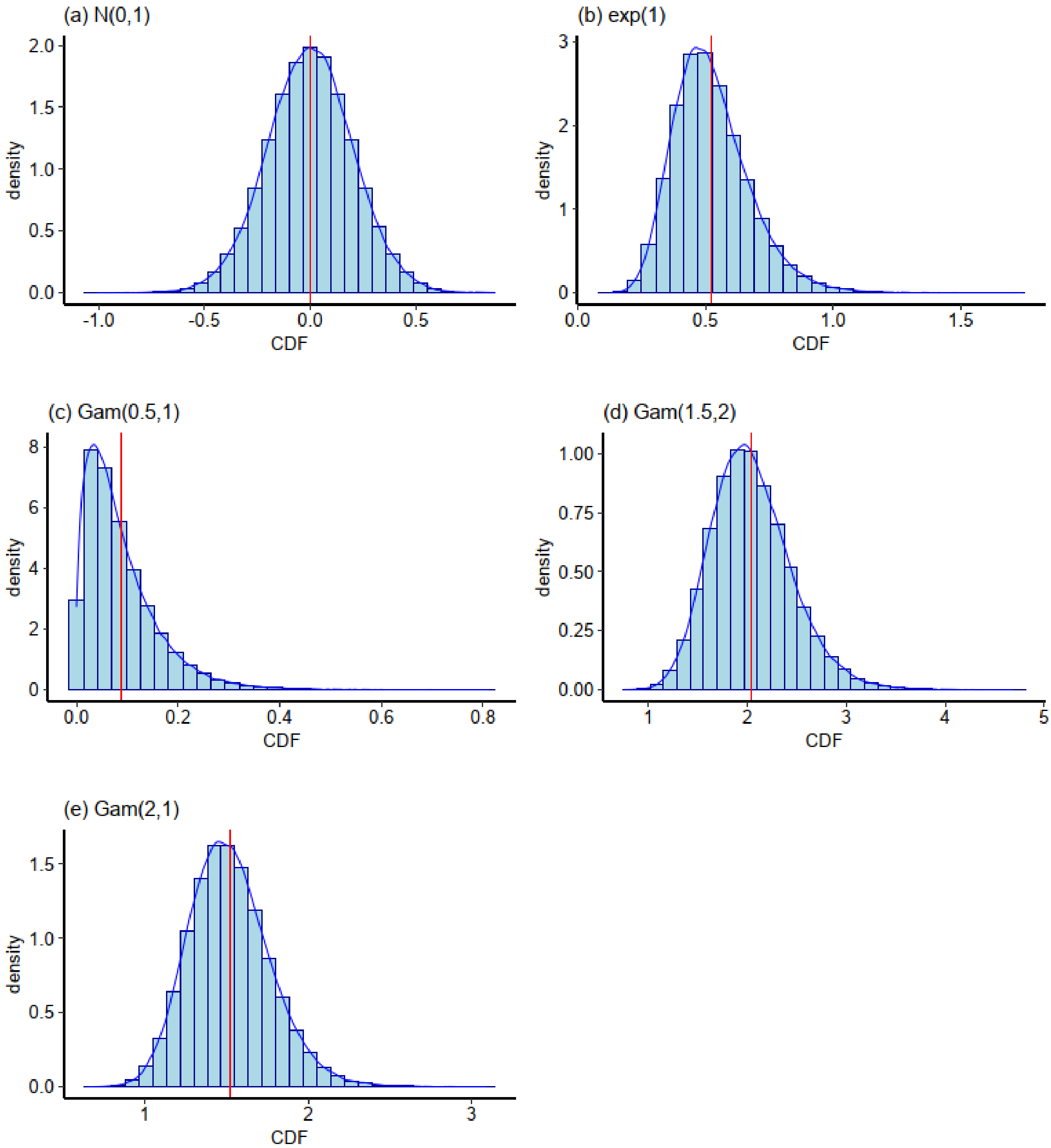

Figure 4.5: Histogram of empirical distributions of $\tilde{X}_{cdf}$

4.2.6 Complement of Cumulative Density Function Based Weighted Average - $\tilde{X}_{(1-cdf)}$

The CoCDF-weight gives the probability greater than observation in the subgroup. The highest weight is given to the smallest observation in the subgroup. The distribution of $\tilde{X}_{(1-cdf)}$ is skewed. The histograms drawn for subgroup size ten for different probability distributions are shown in Figure 4.6.

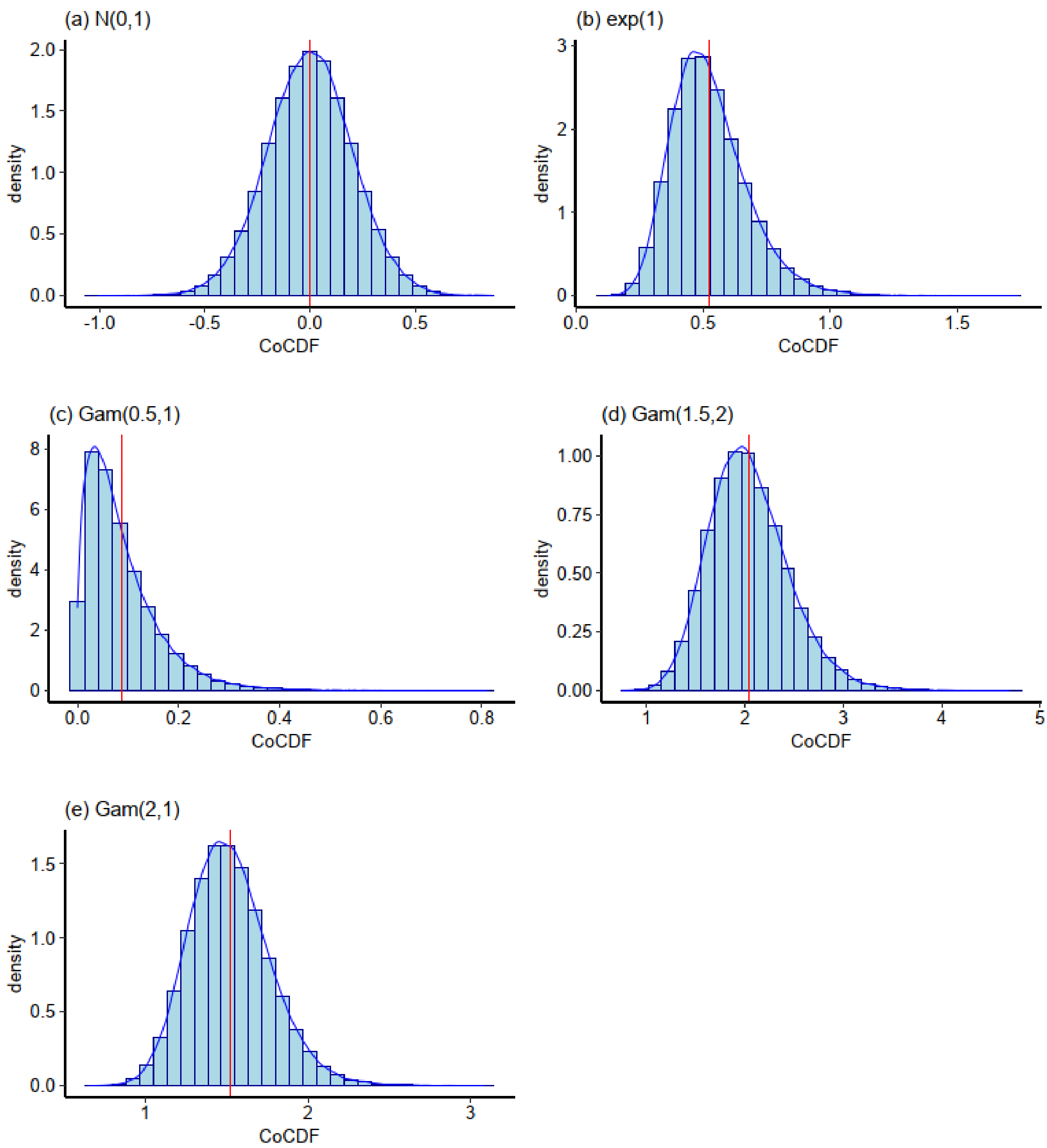

Figure 4.6: Histograms of empirical distributions of $\tilde{X}_{1-cdf}$

4.3 Performance of Weighted Average Control Charts

A general discussion based on the results obtained in Chapter 3 is included in this section. Importantly, the performance of the weighted average control charts will be discussed.

4.3.1 Symmetric Distributions

When the underlying distribution is normal, the Shewhart $\bar{X}$ control chart performs better than the weighted average control charts in detecting mean shifts. Interestingly, the variance increases from the in-control state when the mean is in its in-control value, the weighted average control charts, $\tilde{X}_{max}$, $\tilde{X}_{haz}$, $\tilde{X}_{cdf}$ and $\tilde{X}_{(1-cdf)}$ are superior to the $\bar{X}$ control chart. $\tilde{X}_{cdf}$ and $\tilde{X}_{(1-cdf)}$ control charts outperform the other control charts to detect the upward and downward mean shifts when the variance increases, respectively. A decrease in the variance cannot be identified from these control charts because all charts are insensitive to variance decreases.

4.3.2 Positively Skewed Distribution

The $\bar{X}$ chart is superior to the weighted average control charts discussed in this study to detect mean shifts in the exponential distribution. However, the variance also shifts in the same direction as the mean shifts in an exponential distribution. Therefore, this study does not focus on variance shifts in the exponential distribution.

The gamma distributions were selected to measure the performance of the weighted average control charts, where the shape parameter is less than one, the shape parameter is greater than one, and the shape parameter less than or greater than the scale parameter. Figure 2.1 shows that the shape of the distribution depends on the value of the shape parameter. A change in the parameters affects either mean or variance, or skewness. Therefore, mean, variance and skewness are considered in describing the performance of the control charts with the shifts that occur in shape and scale parameters.

The $\bar{X}$ control chart performs better than the weighted average control charts to identify a mean shift that occurred by a change in the scale parameter when the underlying distribution of the process is gamma. A change in the scale parameter does not affect the skewness. Therefore, when the skewness of the gamma distribution is in its in-control value the $\bar{X}$ control chart detects an out-of-control signal quicker than the proposed weighted average control charts.

When the shape parameter is less than one, weighted average control charts can detect the mean shifts in various situations. The weighted average control charts $\tilde{X}_{max}$, $\tilde{X}_{pdf}$, $\tilde{X}_{haz}$ and $\tilde{X}_{(1-cdf)}$ performs better in detecting the mean shifts that occurred by changing the shape parameter. The skewness of the gamma distribution depends on the shape parameter. Therefore, the weighted average control charts perform better than the $\bar{X}$ control chart when the skewness of the process changed.

$\tilde{X}_{haz}$ control chart performs well for identifying the mean shifts when the process mean shifted by changing the scale and shape parameters to the same direction. Also, the mean shift that occurs by moving the shape and scale parameters in opposite directions can be identified using weighted average control charts. However, the performance of the control charts depends on the variance and the skewness of the process. All the control charts are insensitive when the variance decreases while the mean is in its in-control value regardless of the skewness. On the other hand, the $\tilde{X}_{pdf}$, $\tilde{X}_{haz}$ and $\tilde{X}_{cdf}$ control charts show better performance in detecting an increase in the variance, but the mean does not change. The $\tilde{X}_{cdf}$ control chart can detect an increase in the mean, the variance, and the skewness when shape and scale parameters shift in the opposite directions. Although the $\tilde{X}_{max}$, $\tilde{X}_{pdf}$, $\tilde{X}_{haz}$ and $\tilde{X}_{(1-cdf)}$ control charts perform better when the skewness decreases as the mean and the variance increases. Further, the $\bar{X}$ chart is insensitive and $\tilde{X}_{pdf}$ and $\tilde{X}_{(1-cdf)}$ control charts perform well for detecting the mean increase when the variance and skewness decreases. The $\tilde{X}_{max}$, $\tilde{X}_{pdf}$, $\tilde{X}_{haz}$ and $\tilde{X}_{cdf}$ control charts outperform the $\bar{X}$ control chart when the variance and skewness increase while mean decrease occurs as the parameters shift oppositely. Moreover, the performance of the $\tilde{X}_{max}$, $\tilde{X}_{pdf}$, $\tilde{X}_{haz}$ and $\tilde{X}_{(1-cdf)}$ control charts are better in monitoring the downward mean shift when variance decreases while the skewness increases.

Weighted average control charts show better performance than the $\bar{X}$ chart, when the underlying distribution is gamma, and the shape parameter is greater than one. The $\tilde{X}_{max}$ control chart is superior in identifying mean shifts that occurred by a change in the shape parameter solely, where the skewness of the process is changed from the in-control value. When the mean is in its in-control state and the variance decreases, all the control charts are insensitive. Despite that, the $\tilde{X}_{cdf}$ and $\tilde{X}_{haz}$ control charts can monitor the process when the variance increases while the mean is in the in-control value for any degree of skewness. The performance of weighted average control charts depends on whether the shape parameter is less than or greater than the scale parameter.

When the shape parameter is less than the scale parameter, the weighted average control charts perform according to the changes in the variance and skewness from the in-control value. The $\tilde{X}_{max}$ and $\bar{X}$ control charts performed well for mean decrease and increases when the mean shift occurred by shifting the shape and scale parameters in the same direction, respectively. The $\tilde{X}_{cdf}$ and $\tilde{X}_{haz}$ control charts outperform when the mean increases with the variance and skewness by shifting parameters oppositely. The other three weighted average control charts $\tilde{X}_{max}$, $\tilde{X}_{pdf}$ and $\tilde{X}_{(1-cdf)}$ performs better than the $\bar{X}$ control chart when mean and variance increase while skewness decreases. The control charts discussed in this study are insensitive when the variance and skewness decrease, but the mean increases by shifting the parameters oppositely. The $\tilde{X}_{max}$ and $\tilde{X}_{(1-cdf)}$ control charts perform better when the mean decrease with skewness increases, either variance increases or decreases. When the mean is reduced while the variance and skewness decreases, the $\tilde{X}_{cdf}$ and $\tilde{X}_{haz}$ control charts showed the best performance.

When the shape parameter is greater than the scale parameter, $\bar{X}$ control chart performed better for the mean shifts when both parameters shift in the same direction and mean increases with variance and skewness when the parameters change in opposite directions. The $\tilde{X}_{max}$, $\tilde{X}_{pdf}$ and $\tilde{X}_{(1-cdf)}$ control charts perform better when the mean and the variance increase while skewness decreases. Also, the $\tilde{X}_{max}$ and $\tilde{X}_{(1-cdf)}$ control charts perform well in identifying a mean decrease while the skewness and the variance increases occurred by shifting parameters oppositely.

4.4 Implementation of Control Charts for Phase I

4.4.1 Symmetric Distributions

Assuming the determined distribution is standard normal, the control limits were found using bootstrapping. The bootstrapped control limits for sample size ten are given in Table 4.3. The control limits of $\tilde{X}_{cdf}$ and $\tilde{X}_{haz}$ are almost the same, and the control limits of $\bar{X}$ and $\tilde{X}_{pdf}$ are approximately equal.

Table 4.3: Control limits for Phase I - N(0,1) distributed data

Weight	**Lower Control Limit (LCL)**	**Center Line (CL)**	**Upper Control Limit (UCL)**
Unweighted	-0.28	0.01	0.30
Maximum distance (max)	-0.52	-0.17	0.17
Probability density function (pdf)	-0.26	0.01	0.28
Cumulative density function (cdf)	-0.22	0.08	0.34
Complement of cumulative function (1-cdf)	-0.35	-0.05	0.24
Hazard function (haz)	-0.22	0.08	0.35

The control charts for Phase I is constructed using the control limits in Table 4.3. The $\bar{X}$ control chart and the weighted average control charts; $\tilde{X}_{max}$, $\tilde{X}_{pdf}$, $\tilde{X}_{cdf}$, $\tilde{X}_{(1-cdf)}$, $\tilde{X}_{haz}$ for the data from the N(0,1) distribution are shown in Figure 4.7. All the averages demonstrated the same pattern. All the control charts are in control, and the derived control limits can be used for Phase II for monitoring the process.

4.3.1.1 Example

An example (6.1 - page 239) is considered from Montgomery (2020) and found the bootstrapping control limit for $\bar{X}$ control chart. The bootstrapped control limits are (1.3349, 1.6836) and the control limits found in the textbook are (1.3179, 1.6932). The actual control limits are approximately equal to the bootstrap control limit. The $\bar{X}$ chart drawn for the data using the bootstrap control limits is shown in Figure 4.8. The process is in control since all the points lie inside the control limits.

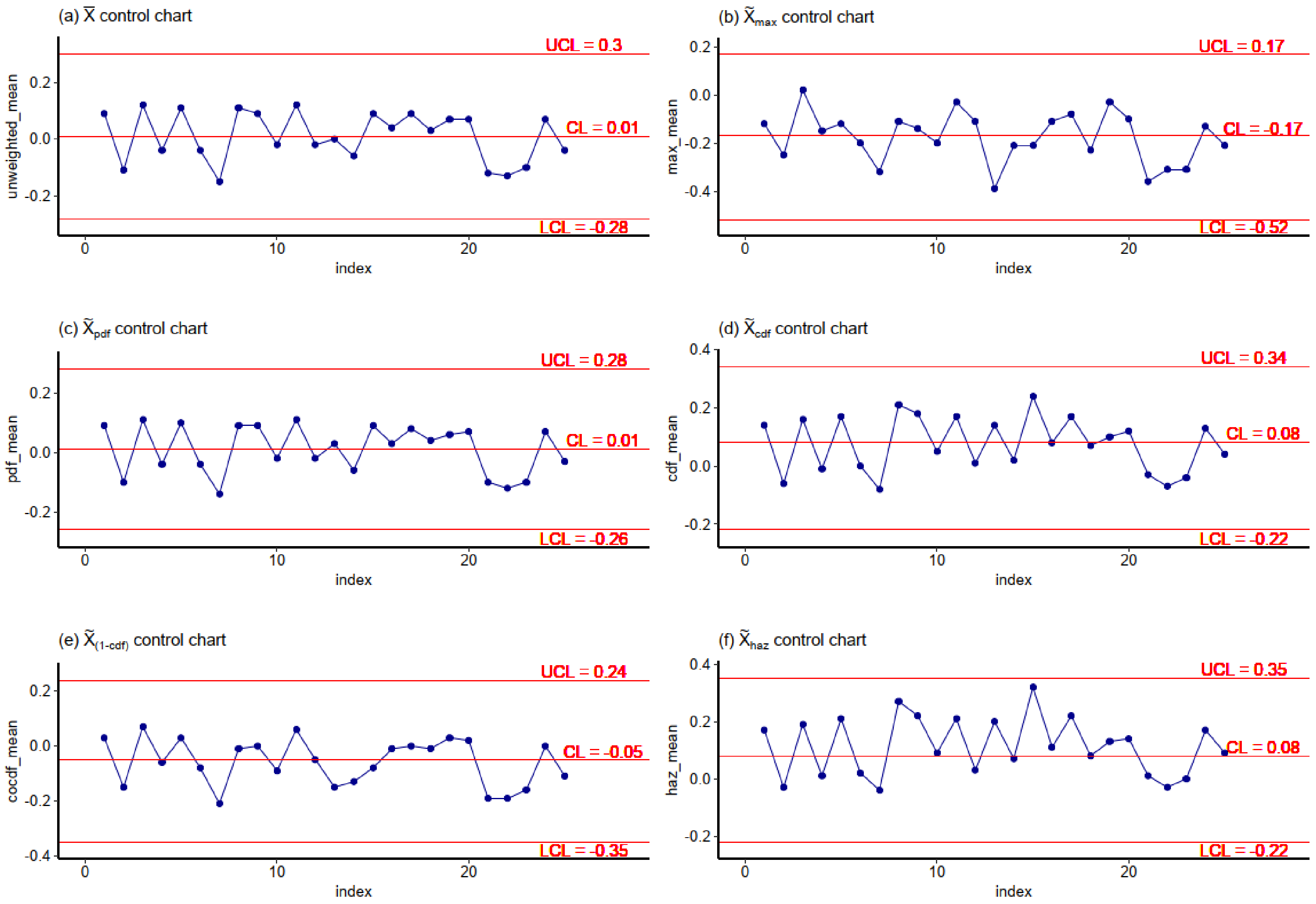

Figure 4.7: Control charts for Phase I - N(0,1) distributed data

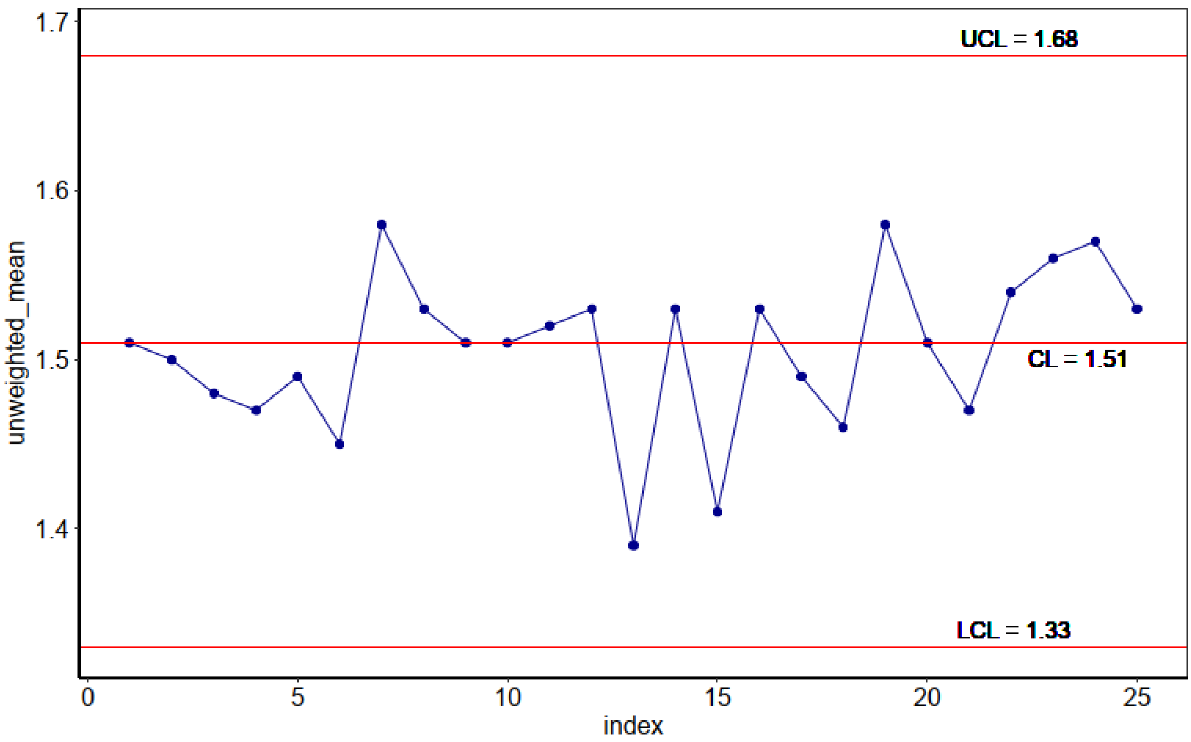

Figure 4.8: $\bar{X}$ control chart – example

4.4.2 Positively Skewed Distributions

The Phase I applications of exponential and gamma distributions are discussed in this section.

4.4.2.1 Exponential Distribution

The control limits are found using bootstrapping when the data comes from an exponential distribution, and the estimated rate is 1. The bootstrapped control limits for sample size ten are given in Table 4.4.

Table 4.4: Control limits for Phase I - exp(1) distributed data

Weight	**Lower Control Limit (LCL)**	**Center Line (CL)**	**Upper Control Limit (UCL)**
Unweighted	0.75	1.01	1.31
Maximum distance (max)	0.63	0.85	1.14
Probability density function (pdf)	0.71	0.93	1.22
Complement of density function (1-pdf)	0.78	1.05	1.35
Cumulative density function (cdf)	0.78	1.05	1.35
Complement of cumulative function (1-cdf)	0.71	0.93	1.22

The $\bar{X}$ control chart and the weighted average control charts; $\tilde{X}_{max}$, $\tilde{X}_{pdf}$, $\tilde{X}_{1-pdf}$, $\tilde{X}_{cdf}$, $\tilde{X}_{(1-cdf)}$ for the exp(1) data are shown in Figure 4.9. All the points lie inside the control limits for all control charts. Hence, all the control charts are in control, and the estimated control limits can be used for Phase II application to monitor the process. Further, the control charts $\tilde{X}_{pdf}$ and $\tilde{X}_{(1-cdf)}$ and $\tilde{X}_{1-pdf}$ and $\tilde{X}_{cdf}$ are identical because $\lambda = 1$.

4.3.2.2. Gamma Distribution

Gam(0.5,1) Distribution

The data distribution was identified as gamma, and the shape and scale parameters were estimated as 0.5 and 1, respectively. The control limits for the control charts discussed in this study were identified using the bootstrapping method given in Table 4.5.

Table 4.5: Control limits for Phase I - Gam(0.5,1) distributed data

Weight	**Lower Control Limit (LCL)**	**Center Line (CL)**	**Upper Control Limit (UCL)**
Unweighted	0.30	0.47	0.69
Maximum distance (max)	0.23	0.37	0.57
Probability density function (pdf)	0.26	0.40	0.61
Cumulative density function (cdf)	0.32	0.50	0.73
Complement of cumulative function (1-cdf)	0.28	0.42	0.62
Hazard function (haz)	0.29	0.45	0.67

Figure 4.10 shows the control charts constructed for control statistic for Phase I. The averages from the gamma distribution are positively skewed. Therefore, most averages are between the center line and the lower control limit. However, all the points in each control chart are between the control limits. Hence the control charts are in control, and the estimated control limits can be used for monitoring the process for Phase II applications.

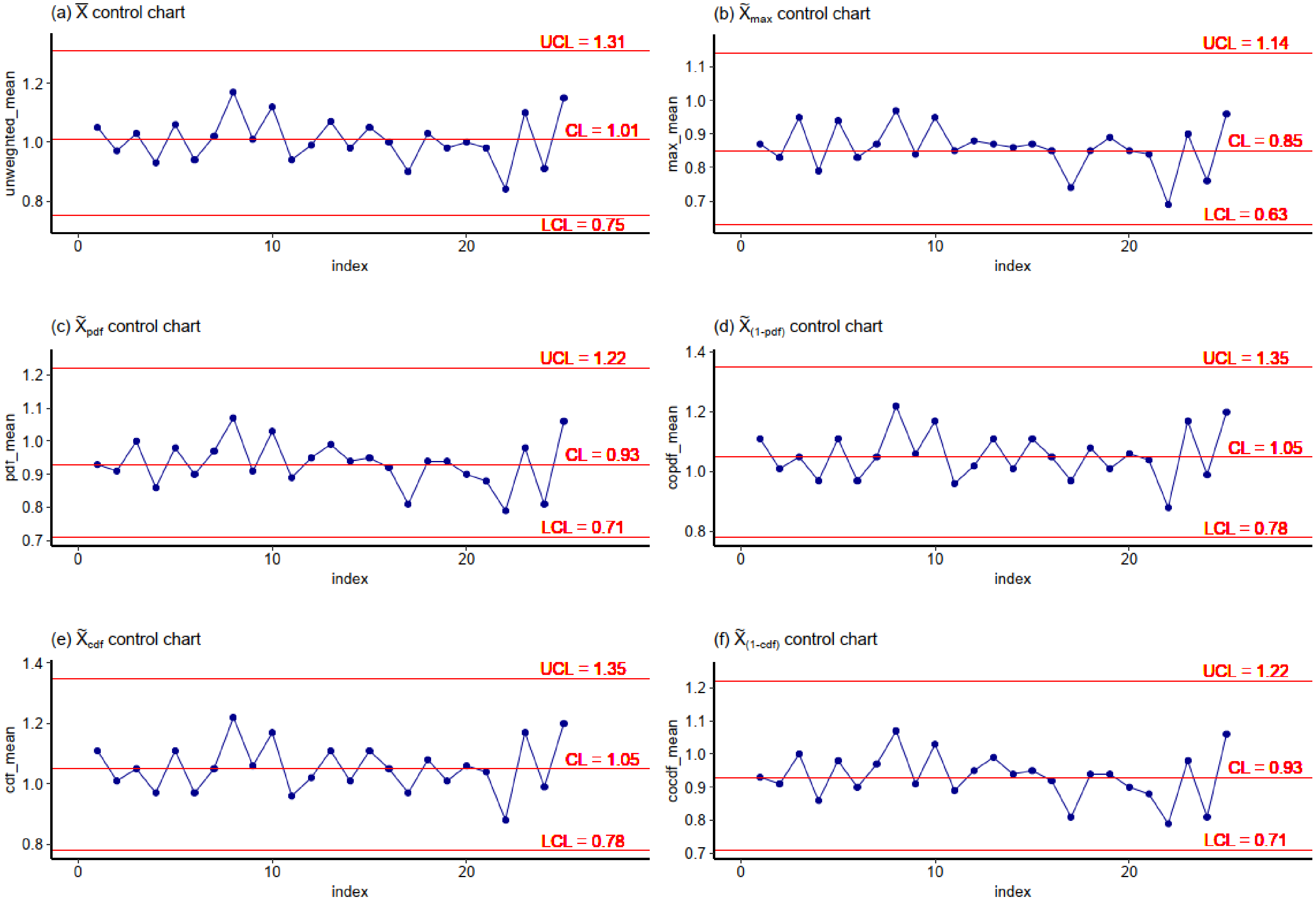

Figure 4.9: Control charts for Phase I - exp(1) distributed data

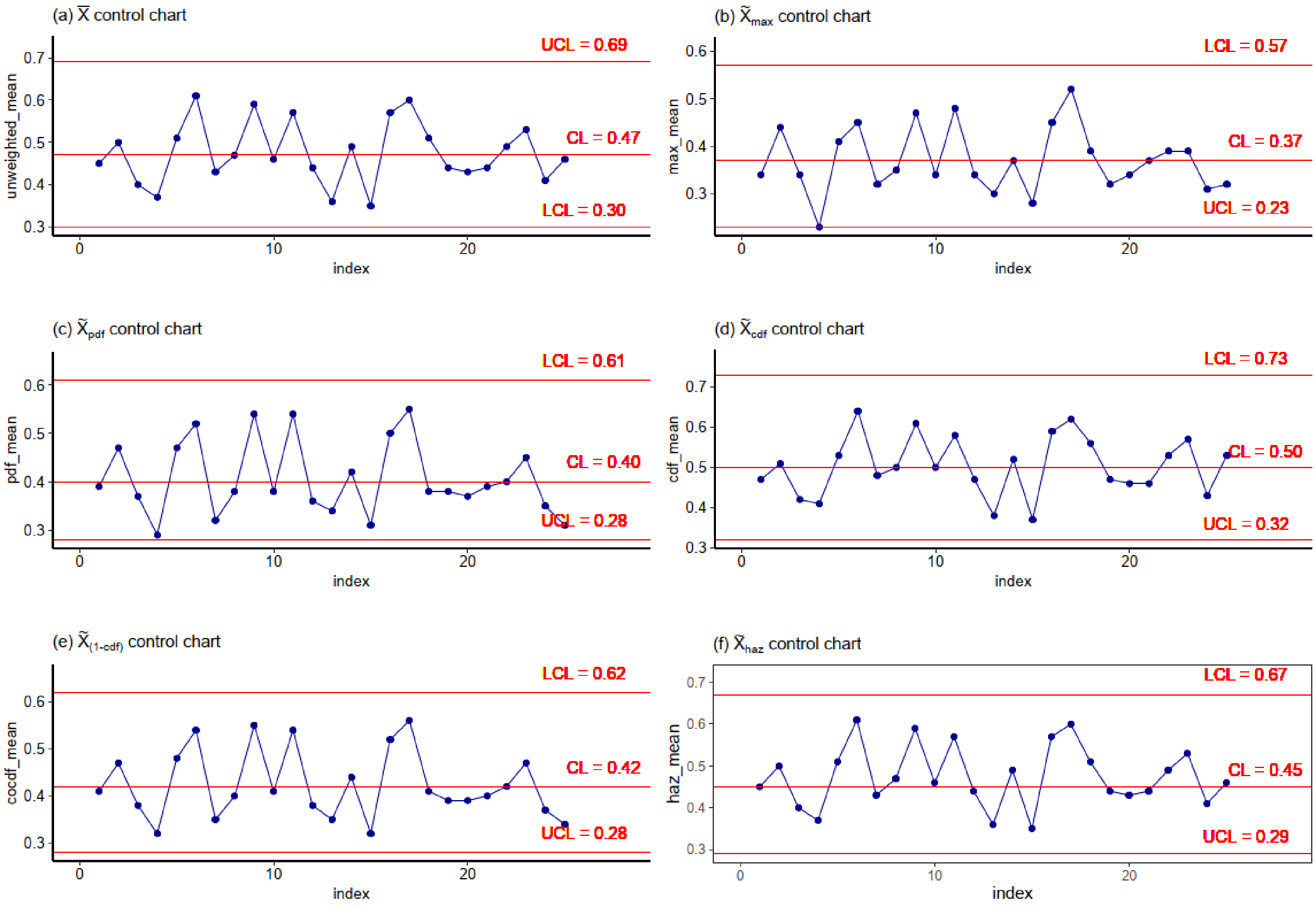

Figure 4.10: Control chart for Phase I - Gam(0.5,1) distributed data

Gam(1.5,2) Distribution

When the data is identified to follow a Gam(1.5,2) distribution, the control limits estimated using the bootstrap method given in Table 4.6.

Table 4.6: Control limits for Phase I - Gam(1.5,2) distributed data

Weight	Lower Control Limit (LCL)	Center Line (CL)	Upper Control Limit (UCL)
Unweighted	2.25	2.89	3.60
Maximum distance (max)	1.85	2.51	3.22
Probability density function (pdf)	2.16	2.75	3.43
Cumulative density function (cdf)	2.35	3.02	3.74
Complement of cumulative function (1-cdf)	2.13	2.72	3.41
Hazard function (haz)	2.27	2.92	3.64

The simple and weighted averages are positively skewed when the data is from Gam (1.5,2) distribution, and the control charts constructed for Phase I are shown in Figure 4.11. In addition, there was a point outside the upper control limit in each control chart. This data is removed from the data set, and the control limits were calculated and shown in Table 4.7.

Table 4.7: Revised control limits for Phase I - Gam(1.5,2) distributed data

Weight	Lower Control Limit (LCL)	Center Line (CL)	Upper Control Limit (UCL)
Unweighted	2.24	2.86	3.54
Maximum distance (max)	1.84	2.48	3.18
Probability density function (pdf)	2.14	2.72	3.39
Cumulative density function (cdf)	2.34	2.98	3.66
Complement of cumulative function (1-cdf)	2.10	2.70	3.38
Hazard function (haz)	2.26	2.89	3.56

The control chart constructed using the revised control limits are shown in Figure 4.12. All the points were inside the control limits, so the process is in-control, and the estimated control limits can be used for Phase II applications.

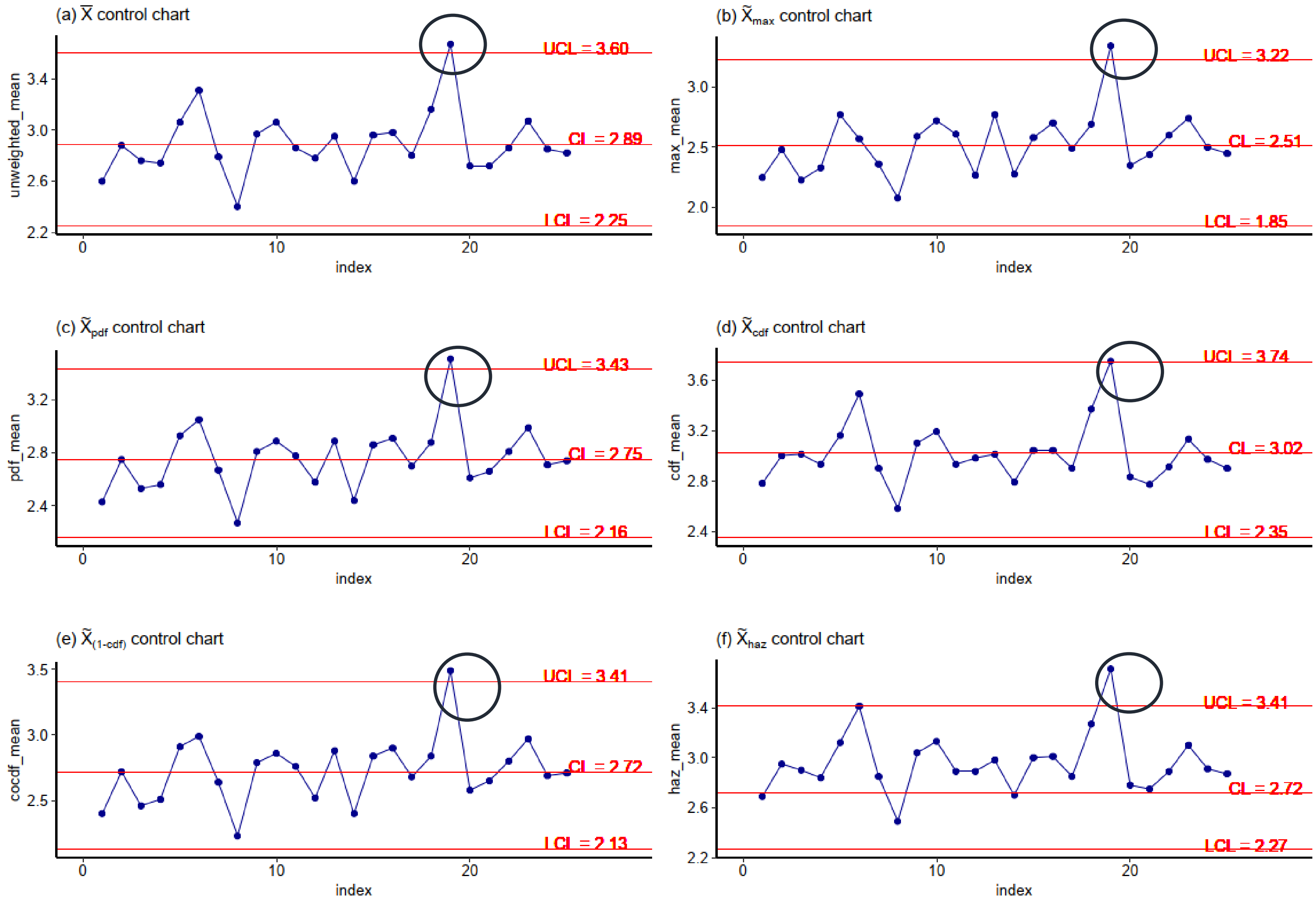

Figure 4.11: Control chart for Phase I - Gam(1.5,2) distributed data

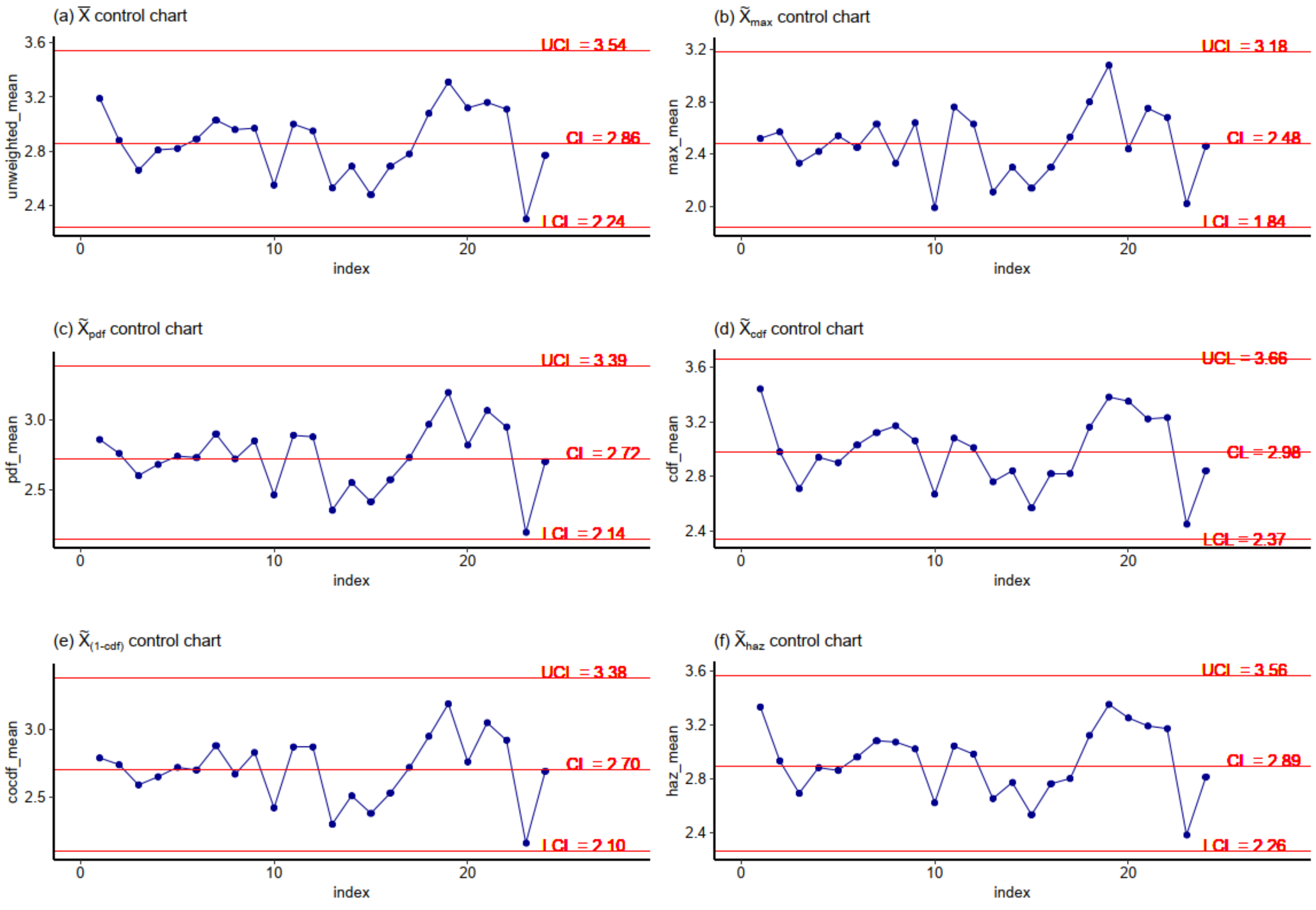

Figure 4.12: Revised control charts for Phase I - Gam(1.5,2) distributed data

Gam(2,1) Distribution

Table 4.8 gives the Phase I control limits estimated using bootstrapping for a process that follows a gamma distribution with the shape and scale parameters 2 and 1, respectively.

Table 4.8: Control limits for Phase I - Gam(2,1) distributed data

Weight	Lower Control Limit (LCL)	Center Line (CL)	Upper Control Limit (UCL)
Unweighted	1.59	1.93	2.31
Maximum distance (max)	1.39	1.72	2.11
Probability density function (pdf)	1.56	1.87	2.24
Cumulative density function (cdf)	1.64	1.99	2.38
Complement of cumulative function (1-cdf)	1.53	1.85	2.22
Hazard function (haz)	1.61	1.95	2.34

Figure 4.13 shows the Phase I control charts constructed for the control statistic discussed in this study for the underlying distribution Gam(2,1). According to the Figure 4.13, the $\tilde{X}_{1-cdf}$ control chart has a point lying on the upper control limit and the $\tilde{X}_{max}$ control chart shows a point lying outside the upper control limit. Therefore $\tilde{X}_{1-cdf}$ and $\tilde{X}_{max}$ control charts are out-of-control, and hence the control limits are revised and shown in Table 4.9. All the points lie within the control limits for $\overline{X}$, $\tilde{X}_{pdf}$, $\tilde{X}_{cdf}$ and $\tilde{X}_{haz}$ control charts, and hence they are in-control.

Table 4.9: Revised control limits for Phase I - Gam(2,1) distributed data

Weight	Lower Control Limit (LCL)	Center Line (CL)	Upper Control Limit (UCL)
Maximum distance (max)	1.38	1.71	2.09
Complement of cumulative function (1-cdf)	1.53	1.84	2.20

Figure 4.14 shows the revised control charts for $\tilde{X}_{max}$ and $\tilde{X}_{(1-cdf)}$ control charts. All the points were inside the control limits. Therefore, these control charts are in control, and the revised control limits can be used for Phase II in monitoring the process.

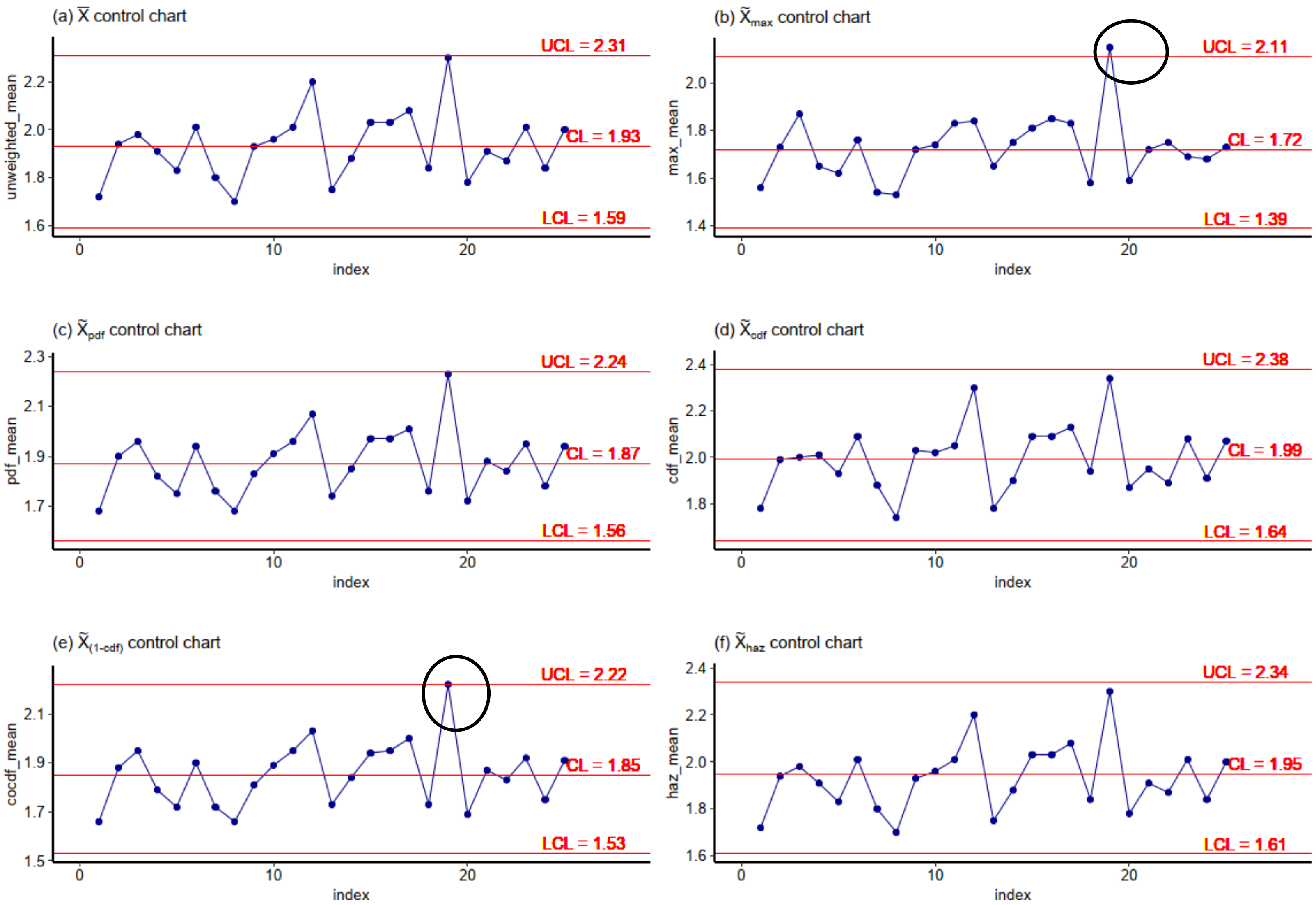

Figure 4.13: Control charts for Phase I - Gam(2,1) distributed data

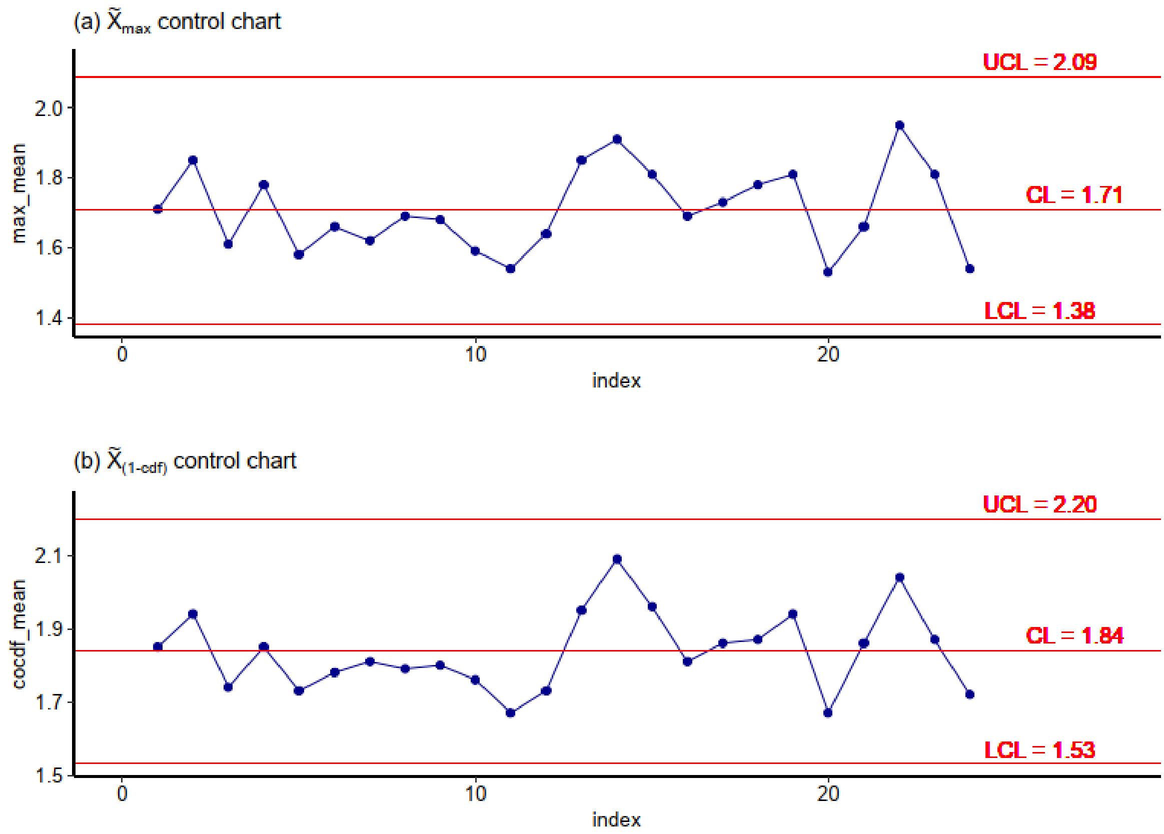

Figure 4.14: Revised control charts for Phase I - Gam(2,1) distributed data

4.5. Detecting Shifts in Variance of the Normal Distribution

This study also identified another use of weighted averages. An increase in the variance of normally distributed data can be detected by using weighted average control charts. The advantage of this finding is that this weighted average can be used with the simple average $\bar{X}$ to detect the mean and variance in a process. Moreover, these two control statistic can be drawn in the same graph since the measures are on the same scale, make it easy to detect when a process has gone out-of-control.

4.5.1 Joint Monitoring of Mean and Variance

Usually, in SPC, two control charts are used simultaneously for monitoring the process, such as using $\bar{X}$ and R charts for monitoring the mean and the variance of a process. However, this is timely and expensive. Therefore, research was conducted to design a control chart that monitors the mean and variance in the same chart. This concept is handy when both the mean and variance are changed. For example, a change in scale or shape parameter of the gamma distribution changes both the mean and the variance.

One type is the simultaneous control chart which plots two statistics in the same chart. A control chart that discusses the mean and the standard deviation of a process in one graph is proposed by White and Schroeder (1987). Since the mean and variance are monitored simultaneously, the chart is referred to as a simultaneous control chart. In this study, the sample median and the interquartile range (Q) are used to measure location and variability. Here, the control limits were found using the distribution of median and Q. A box and whiskers diagram is drawn with the control limits. Another method using one statistic to monitor both quality characteristics, referred to as single variable control chart was proposed by Cheng and Thaga (2006). Chao and Cheng (2008) discussed a two-dimensional control chart, which combines two statistics into one chart and is plotted using two dimensions. Chen and Cheng (1998) combined the $\bar{X}$ chart with R or S chart, which is referred to as MAX chart. The major problem in these concepts is to keep the chart simple. For more information see, Mccracken and Chakraborti (2013).

A shift in the mean impacts only the mean chart, and a change in variance impacts both the variance chart and the mean chart. A reduction in the variance is beneficial if the mean remains in the in-control state. However, a decrease in the variance is difficult to detect. When the variance is increased, while the mean is in-control or shifted from the in-control, the weighted average control charts show a lower ARL than the $\bar{X}$ control chart.

4.5.2 Monitoring an Increase in Variance When Mean In-control

Figure 4.15 shows the ARL for different control statistic for subgroup size 10, and it revealed that the control statistic based on the CDF-weight and CoCDF-weight have a lower ARL than the unweighted average. Therefore, the $\tilde{X}_{cdf}$ or $\tilde{X}_{1-cdf}$ control chart can be used in detecting the increasing variance together with the $\bar{X}$ control chart where the two statistics can be drawn in the same graph.

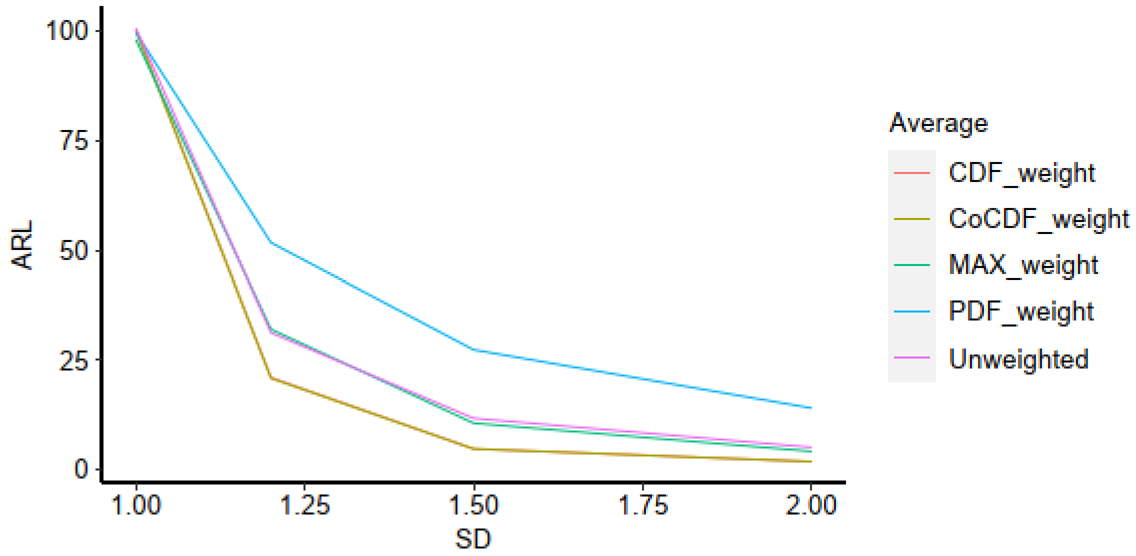

Figure 4.15: ARL curve for variance increase, mean in-control - N(0,1)

The control charts constructed to determine the variance increase while the mean is in its in-control states as shown in Figures 4.16 and 4.17.

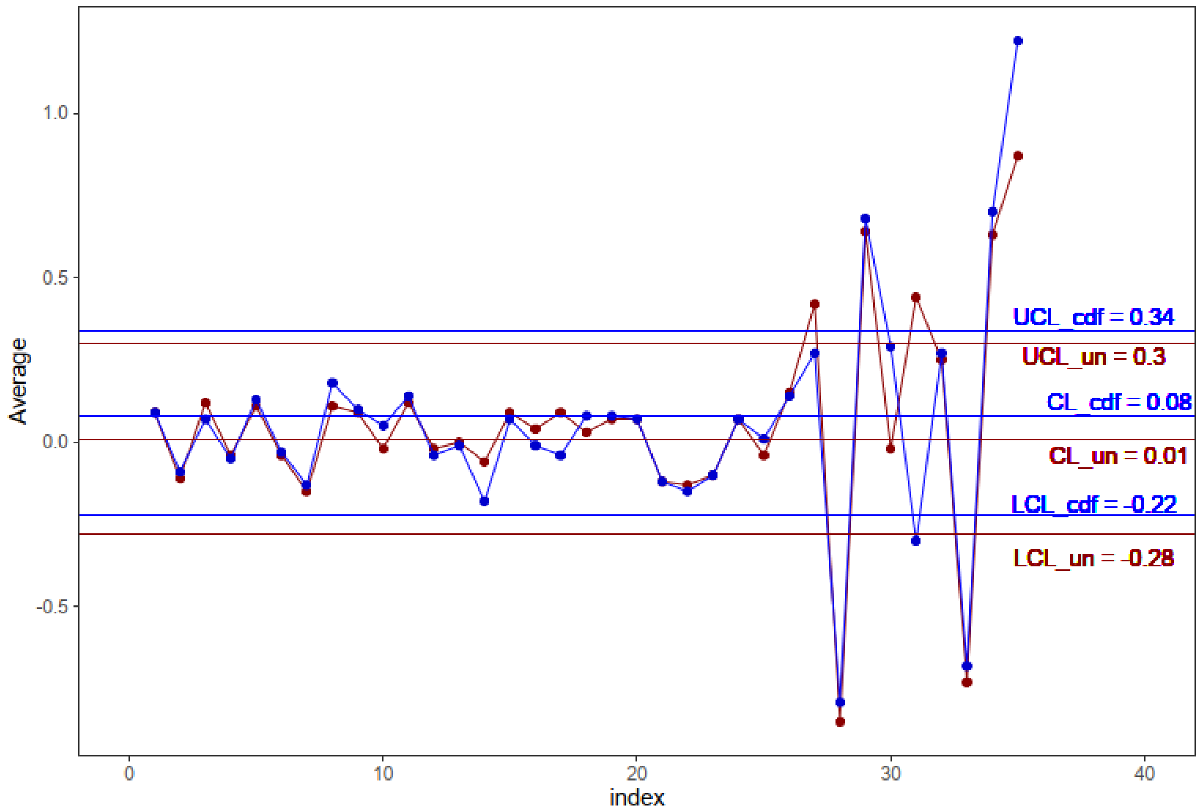

Figure 4.16: $\bar{X}$ and $\tilde{X}_{cdf}$ joint control chart - N(0,1)

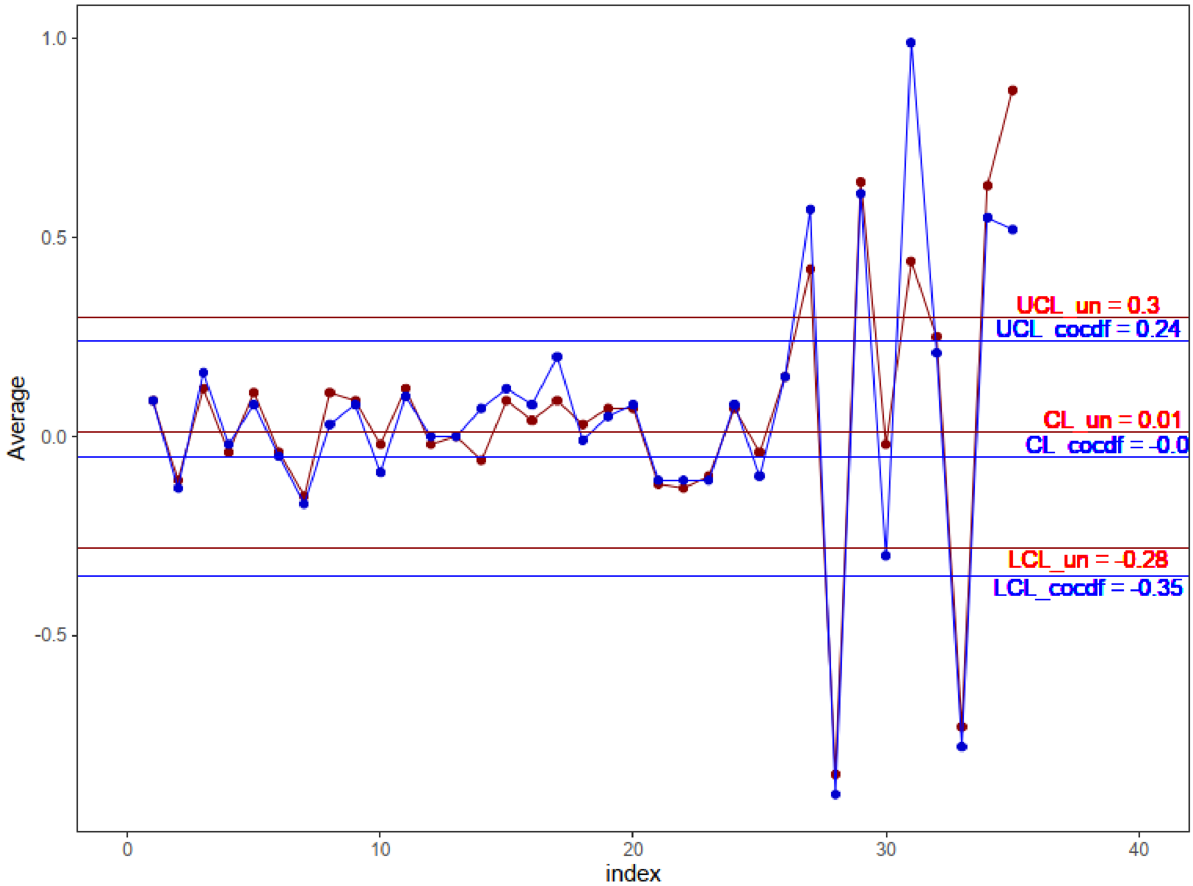

Figure 4.17: $\bar{X}$ and $\tilde{X}_{(1-cdf)}$ joint control chart - N(0,1)

4.6 Summary

This chapter discussed the findings of the study in detail and highlighted the usefulness of the findings for joint monitoring where simultaneous detection of the mean and the variance is possible. The next chapter will outline the conclusions made based on this study and future research opportunities.

Chapter 5 - Conclusions and Future Study

5.1 Introduction

The outcomes of this study and how these outcomes addressed the research objectives are presented in this Chapter. Further, the opportunities for future research are also discussed.

5.2 General Conclusion

5.2.1 Weighted Average Control Charts

5.2.1.1 Symmetric Distributions

If the underlying distribution is identified as normal in Phase I, the Shewhart $\bar{X}$- chart can be used to detect shifts in the mean. An increase in the variance when the mean is in its in-control value can be detected using the weighted average control charts, $\tilde{X}_{max}$, $\tilde{X}_{haz}$, $\tilde{X}_{cdf}$ and $\tilde{X}_{(1-cdf)}$. Major findings were that $\tilde{X}_{cdf}$ control chart can be used to identify when the mean and variance increases from their in-control value while the $\tilde{X}_{(1-cdf)}$ control chart is better in detecting downward mean shifts when the variance increases. A decrease in the variance can not be detected using any control chart discussed in this study, however a decrease in variance would not normally be seen as a problem.

5.2.1.2 Positively skewed distribution

When an underlying distribution is not symmetric and if it is identified as an exponential distribution, then the $\bar{X}$- chart can be used to detect mean shifts that occur in the process. However, any of the weighted average control charts discussed in this study cannot be used to identify mean changes when the underlying process variable is exponential.

If the underlying distribution of the process is identified as gamma in Phase I estimation,, then the $\bar{X}$ chart is the best control chart to identify the mean shift when the skewness of the process does not change. When the shape parameter is found to be less than one, and the skewness shifted as a result of a change in the shape parameter, the weighted average control charts $\tilde{X}_{max}$, $\tilde{X}_{pdf}$, $\tilde{X}_{haz}$ and $\tilde{X}_{(1-cdf)}$ can be used to detect the mean shift. Also, when the skewness is shifted due to changes in both parameters to the same direction, the $\tilde{X}_{max}$ control chart is helpful to detect the mean shifts. However, no control charts can detect a variance decrease while the mean is in its in-control value regardless of the skewness shifts. In contrast, the weighted average control charts $\tilde{X}_{pdf}$, $\tilde{X}_{haz}$ and $\tilde{X}_{cdf}$ are good in detecting an increase in the variance when the mean is in control. On the other hand, $\tilde{X}_{cdf}$ control chart can detect an increase in mean, variance, and skewness by shifting the shape and scale parameter to the opposite directions. Further, $\tilde{X}_{max}$, $\tilde{X}_{pdf}$, $\tilde{X}_{haz}$ and $\tilde{X}_{(1-cdf)}$ control charts detect a mean and variance increase while skewness decreases. $\tilde{X}_{pdf}$ and $\tilde{X}_{(1-cdf)}$ control charts can be used in detecting the mean increase when the variance and skewness decreases. To identify the mean decrease that occurred as the parameters shifted oppositely, $\tilde{X}_{max}$, $\tilde{X}_{pdf}$, $\tilde{X}_{haz}$ and $\tilde{X}_{cdf}$ control charts can be used when the variance and skewness increases. Further, $\tilde{X}_{max}$, $\tilde{X}_{pdf}$, $\tilde{X}_{haz}$ and $\tilde{X}_{(1-cdf)}$ control charts can be used to monitor the variance decreases while the skewness increases.

When the underlying distribution is identified as gamma and the shape parameter is greater than 1, the weighted average control chart $\tilde{X}_{max}$ can be used to identify mean shifts that occurred by changing the shape parameter, where the skewness changes from the in-control value. Despite that, $\tilde{X}_{cdf}$ and $\tilde{X}_{haz}$ control charts can monitor the process when the variance increases while the mean is in the in-control value for any skewness.

When the shape parameter is less than the scale parameter, the $\tilde{X}_{max}$ control chart can be used to detect a mean decrease while $\bar{X}$ control chart can be used to detect mean increases when both parameters shift to the same direction. $\tilde{X}_{cdf}$ and $\tilde{X}_{haz}$ control charts can effectively detect a mean increase while the variance and skewness increase by shifting parameters oppositely. The other three weighted average control charts $\tilde{X}_{max}$, $\tilde{X}_{pdf}$ and $\tilde{X}_{(1-cdf)}$ can be used in detecting the mean increase with the variance while skewness decreases. $\tilde{X}_{max}$ and $\tilde{X}_{(1-cdf)}$ control charts are useful in identifying a mean decrease with skewness increases, either variance increases or decreases. The mean, variance and skewness decrease can be detected using $\tilde{X}_{cdf}$ and $\tilde{X}_{haz}$ control charts.

When the shape parameter is greater than the scale parameter, $\bar{X}$ control chart can detect mean shifts when the parameters shift in the same direction. Also, $\bar{X}$ control chart is good at detecting a mean increase with the variance and the skewness when the parameters change in opposite directions. The $\tilde{X}_{max}$, $\tilde{X}_{pdf}$ and $\tilde{X}_{(1-cdf)}$ control charts can detect the mean increase effectively with the variance increases while skewness decreases. When the mean decrease while the skewness and the variance increases when shifting parameters oppositely, the $\tilde{X}_{max}$ and $\tilde{X}_{(1-cdf)}$ control charts can be used to detect out-of-control signals efficiently.

5.2.2 Summary of Research Objectives and Achievements

The preceding chapters have comprehensively discussed the weighted average control charts as a control statistic in constructing control charts for positively skewed data. In chapter 2, six locally weighted averages were proposed. The response for symmetric distributions from the weighted average control charts was discussed in chapter 3. Finally, it was identified that the variance of normally distributed data could be monitored using the cumulative and complement of cumulative function based weightings, which is discussed in detail in chapter 4. The performance of the weighted average control charts and the Shewhart $\bar{X}$ control chart was compared based on the average run length (ARL) and the standard deviation of the run length ($SDRL$) of the control charts in chapter 3. The performance was discussed for Phase II control charts assuming that the distribution of the data is known. The establishment of control limits was discussed in chapter 4. The findings were encouraging and there were many situations identified where the weighted average control charts outperformed Shewhart $\bar{X}$ control chart. In several situations, the locally weighted averages have been effective as a control statistic.

5.3 Future Research Opportunities

This study was focused on identifying a locally weighted average as the control statistic that outperforms in quality control charts for positively skewed distributions. The possible extensions to the current study are described in this section.

Six weighted averages were discussed in this study. However, there are numerous ways of defining locally weighted averages. Therefore, there is still room to explore new weighted averages can be proposed to perform better for right skewed processes. Also, the performance of these control statistic can be identified for negatively skewed distributions since this study considered the positively skewed distributions only.

As illustrated in chapters 3 and 4, the weighted average was good at detecting the variance of symmetric distributions. In this case, both the unweighted and weighted averages can be plotted in the same graph, which is an extra advantage. This concept can be explored further.

On the other hand, the discussion of the performance of these proposed control statistics can be extended to exponentially weighted moving average control charts (EWMA) and cumulative sum (CUSUM) control charts for positively skewed processes. Further, weighted averages can be discussed for acceptance sampling procedures.

Another aspect is that the weighted average can be used to measure the variance of a process. If so, the mean and the variance of a process can be detected using only one control chart will be a handy advantage.

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