Some Aspects of Graph Theory

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This thesis on graph theory is submitted in conformity with the requirements for the degree of Doctor of Science at Massey University. The work it contains is original and has not previously been accepted for another qualification at this or any other university.

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Contents

Chapter 1

Introduction

This thesis embodies most of the research I have undertaken since the completion of my Ph.D. dissertation in 1972. It seems appropriate, and is indeed a pleasure, at this point to acknowledge the influences of others on my work over the years. My interest in graph theory was originally stimulated in 1968 by Alfred Lehman, who supervised my masterate at the University of Toronto. The resulting thesis consisted of the development and implementation of an algorithm for testing the planarity of a graph. Although this masterate was in computer science, I realised that it was the mathematical aspects of the work that most interested me. Consequently I enrolled for a doctorate in graph theory at the University of Waterloo, where I studied under the supervision of Dan Younger. There I focussed on 1-factors of graphs, investigating in particular Kasteleyn's method for enumerating them. I attempted to characterise those graphs for which Kasteleyn's method succeeds. Since the method was known to succeed for planar graphs, this work was not far removed from my earlier interest in planarity. Consequently the two themes of topological graph theory and matchings in graphs were prominent from the early stages of my research career.

Following the completion of my degree at the University of Waterloo, in 1973 I secured an appointment at the Royal Melbourne Institute of Technology Ltd.. Here I had the great good fortune to be in contact with Derek Holton, now a professor at the University of Otago. His office at the University of Melbourne was only about a 15-minute walk from my own, and we often worked together. I interested him in my ideas on planarity and together we succeeded in discovering a characterisation of planar graphs. Alas Chemyak, working in Russia, beat us to that result, but by the time his proof appeared Derek and I were already collaborating on a generalisation. Derek was really a mentor for me at that early stage in my career, and I owe him a great deal.

My interests in topological graph theory and 1-factors of graphs have been maintained since my arrival at Massey University in 1982. My collaborators since that time have included my masterate student Janet McCall, doctoral students Paul Bonnington, Allister Campbell and Serguei Norine and postdoctoral fellows Hong Wang and Feng Ming Dong. With Paul I wrote my first book, *The Foundations of Topological Graph Theory*. Dong inspired many of us in the mathematics discipline at Massey with his enthusiasm for chromatic polynomials, and this topic became a third theme in my research. More recently I have collaborated with Bruce van Brunt and Kee Teo in writing a second book, *The Number Systems of Analysis*. Of my other collaborators, special mention should be made of Use Fischer, with whom I worked at the University of Klagenfurt during my period of study leave in 1999. Together with her I characterised Pfaffian near bipartite graphs, and this result represents what is probably my best work.

Chapter 2

Synopsis

As explained in the introduction, there have been three basic strands to my research since the completion of my Ph.D. —matchings in graphs, topological graph theory and chromatic polynomials. However there are also some papers that do not fit into any of these categories. Accordingly the 60 papers presented in this thesis have been placed into four classes according to topic. The papers are therefore presented in four appendices. The contents of these appendices are listed in the next chapter. In addition to the papers, this thesis also contains the two books mentioned in the introduction. They are presented in two further appendices. I wish to thank the publishers of these books, Springer and World Scientific respectively, for their kind permission for me to include the books in this thesis.

I now proceed to give an overview of the content of each appendix.

2.1 Appendix A: Papers on Matchings in Graphs

The papers in this appendix pertain to matchings in graphs, especially those matchings that are 1-factors. My interest in this subject dates back to my Ph.D. thesis, the topic of which arose from the work of the Dutch physicist Piet Kasteleyn on crystal physics. The problem on which he was working required him to enumerate the 1-factors of certain planar graphs.

Before going any further, I need to explain the terms I am using. A graph consists of a set of vertices and a set of edges joining certain pairs of vertices. Such a graph may be drawn by representing each vertex by a point and each edge by a line connecting the points that represent the vertices joined by the edge. A graph is said to be planar if it can be drawn in the plane without lines crossing.

An edge is said to be incident on the vertices it joins. A matching is a set of edges of which no two are incident on a common vertex. Such a matching is a 1-factor if every vertex has an edge of the matching incident on it.

Kasteleyn achieved his goal of enumerating the 1-factors of all planar graphs. The method he devised also succeeds in enumerating the 1-factors of certain non-planar graphs. The obvious question then arises: which non-planar graphs? Because of the nature of Kasteleyn's method and the tools needed to implement it, the graphs for which his method works are referred to as Pfaffian. I attempted to characterise Pfaffian graphs in my Ph.D. thesis, but succeeded only for bipartite graphs. (Bipartite graphs are those whose vertices can be painted in two colours so that vertices joined by an edge receive different colours.) The resulting paper appeared in 1975 but is not included here. Instead, paper [17] in this appendix gives a shorter proof than in the earlier paper. It is interesting to note that it was 29 years before this shorter proof was discovered despite the fact that the 1975 paper attracted considerable attention.

The topic of my Ph.D. thesis continued to interest me, and I studied the problem of characterising Pfaffian graphs from several different points of view. The results of this study are found in the papers presented in this appendix. The main achievement is found in paper [14] where Pfaffian near bipartite graphs are characterised.

2.2 Appendix B: Papers on Topological Graph Theory

The relationship between Pfaffian graphs and planarity intrigued me from an early stage in my career, and I continued to wonder whether embedding graphs in surfaces other than the plane would provide any clue as to how to enumerate their 1-factors. These reflections led naturally to a consideration of how to characterise planar graphs. Many such characterisations are in fact known, including a famous theorem of Kuratowski, but during the 1970s I began to think that yet another might exist. I had in my mind a picture of some weird kind of machine consisting of a set of cogs arranged in a circle, each cog being interlocked with the two adjacent to it. The cogs are set spinning simultaneously. Because of the way the cogs are interlocked, each cog must spin in the opposite sense (clockwise or anticlockwise) from both its neighbours. It follows that if the machine is to work then the number of cogs will have to be even. It seemed to me that this fact is a manifestation of a fundamental property of the plane. Because of Kuratowski's theorem, I could see that any non-planar graph contains within it a structure that is analogous to a machine with an odd number (actually 3 or 5) of cogs. The converse of this observation (the statement that no planar graph has such a structure), however, was not so clear to me. The main difficulty lay in trying to decide just what sort of structure in a graph is analogous to the kind of machine I had in mind. I worked on this problem for a long period of time with Derek Holton. We made two erroneous attempts to find the right structure before succeeding. We then published a conjecture which was proved by Chernyak, thus giving a new characterisation of planar graphs.

By the time Chernyak's proof appeared, however, Derek and I were aware of the possibility of generalising our work so as to prove a theorem that is valid for all graphs, not just planar ones. The mechanism is supplied by the concept of duality. We may imagine a drawing of a planar graph as dividing a region of the plane into countries. If we now endow each such country with a capital city and link each capital city to those of the neighbouring countries by highways, we obtain a new graph whose vertices are the capital cities and whose edges are the highways. The new graph is said to be dual to the original one. Any theorem that is proved for a planar graph will then have a dual theorem that holds in the dual graph. Derek and I knew that if we could prove our conjecture for planar graphs then of course its dual would also hold for planar graphs, but there appeared to be no obvious reason why this dual should not hold for all graphs, planar or not. In fact it does: the proof is found in papers [81 and [12] of this appendix.

Later on I became interested in another possible way to characterise planar graphs. My interest was aroused by a paper of Rosenstiehl and Read in which they showed that the edges of any graph can be partitioned into three classes in a very natural way. Their proof involved only very elementary linear algebra. It could have been discovered by undergraduates, and it is almost embarrassing that this tripartition was not discovered much earlier in the history of graph theory. Rosenstiehl and Read went on to use this concept to derive what they claimed to be another characterisation of planar graphs. However an examination of their paper reveals that their theorem characterises only those planar graphs whose tripartitions satisfy a certain condition (a specific class of the three being empty). I began to wonder whether their theorem could be modified to be a true characterisation of all planar graphs. In 1990 I was working with Dan Archdeacon at the University of Vermont, and we studied this problem using ideas that I had already worked on with my doctoral student, Paul Bonnington. Several attempts to find the right generalisation failed before we found the idea that succeeded. The resulting paper is number [16] in this appendix.

Much of my work in topological graph theory has therefore concerned planarity. However I have also been interested in the logical foundations of the subject. Most of my papers in that area are not included in this appendix as they were subsumed in a book written with Paul. This book is presented in Appendix E. A fuller account of this work therefore appears in the synopsis of that appendix.

2.3 Appendix C: Papers on Chromatic Polynomials

In 1998 a new impetus to my research arrived in the person of Feng Ming Dong, a postdoctoral fellow working with Kee Teo and me. Dong's interests were quite different from my own. He studied what are called proper colourings of the vertices of a graph. These are colourings that satisfy the property that adjacent vertices (vertices joined by an edge) receive distinct colours.

Proper vertex colourings have aroused a great deal of interest in graph theory. Their study dates back to a famous problem originating in the nineteenth century called the four colour conjecture. Given a map consisting of an area of land divided into countries, it is natural to try to colour the countries so that those sharing a common border are coloured differently. With how many colours should a mapmaker equip himself to be able to guarantee success no matter what map is presented to him? This innocuous sounding question stimulated immense cerebration. The answer was easily seen to be 4 or 5, but which number is correct remained a mystery for about a century despite the fact that some of the world's best mathematicians spent their entire careers trying to settle the matter. Eventually it was discovered by Appel and Haken that four colours suffice, but the proof required enormous computing resources and even modern refinements of it are beyond human capacity to check without the assistance of technology.

The relationship between this map colouring problem and proper vertex colourings is easy to explain. The map can be regarded as a drawing of a graph in the plane, and this graph has a dual as described earlier. The vertices of the dual graph represent the capital cities of the countries. Given a colouring of the map, we may colour each capital city with the colour of its country. This procedure gives a colouring of the vertices of the dual graph. It is a proper vertex colouring: since two capital cities are joined by a highway if and only if their countries have a common border and are therefore coloured differently, it follows that any two vertices of the dual graph that are joined by an edge must receive distinct colours.

Thus a great deal of interest attaches to proper vertex colourings of graphs.

One problem that has attracted a great deal of attention is the enumeration of the proper vertex colourings of a graph when a given number of colours are available. It is not difficult to see that the formula for the number of proper vertex colourings, for any given graph, will be a polynomial in the number of colours. This polynomial is called the chromatic polynomial of the graph. It is then natural to study the roots of this polynomial. For example, if it could be shown that 4 is never a root of a chromatic polynomial of a planar graph, then it would follow that any planar graph has at least one proper vertex colouring in no more than four colours. Accordingly one could then conclude that our hypothetical mapmaker would require only four hues. Note however that a root of a polynomial need not be an integer. In this sense the study of chromatic polynomials is undertaken in a setting that is more general than that of vertex colourings of graphs.

Problems involving chromatic polynomials often relate to their calculation, to the determination of their roots and to the question of whether a given chromatic polynomial defines a graph uniquely. These are all questions that have interested Dong and his coworkers, and several of us addressed them during Dong's time at Massey. Most of the papers in this appendix are the result of work on these three topics. Historically, progress in the area of chromatic polynomials has been rather slow and is often the outcome of very detailed calculations. Much of our effort was devoted to the development of new techniques, and this aspect of our work is particularly evident in papers [3] and [4],

2.4 Appendix D: Other Papers

From time to time in my career I have met other people with interests a little different from mine but nevertheless close enough to mine to enable effective collaboration. Such collaborations have resulted in several single papers or small sets of papers on topics that do not bear much relation to each other or to those covered in the first three appendices. I have included in this appendix those that have appealed to me the most.

Of these papers, [13] is the one to which I am most likely to return in the future course of my research. It is related to the edge tripartition, due to Rosenstiehl and Read, described in Appendix B. Together with Mike Henning I modified their idea and discovered that this modification led to a partition of the edges of a graph into two classes instead of three. Encouraged by the success of my joint work with Archdeacon and Bonnington in using the tripartition to characterise planar graphs, we wondered whether this bipartition would enable us to characterise those graphs that can be embedded in the projective plane. Though we did not succeed in finding such a characterisation, this idea remains an intriguing possibility which I would like to investigate further.

2.5 Appendix E: The Foundations of Topological Graph Theory

This appendix consists of a book written jointly with Paul Bennington. The concept of the book arose from my experiences teaching graph theory to advanced students at both the Royal Melbourne Institute of Technology Ltd. and Massey University. Specifically, I taught Kuratowski's theorem to such students but was unable to discuss a proof because of the amount of topology required. I was aware that there existed a method, pioneered by Jack Edmonds, for studying embeddings of graphs in surfaces (not necessarily the plane or sphere) by using cyclic orderings of the edges incident on a given vertex. This idea made it possible to study embeddings of graphs in surfaces without using any topology. Although the concepts used in this approach are elementary, the attractiveness of the approach is circumscribed by the complexity of some of the arguments involved.

However, during the 1980s an idea due to Sostenes Lins caught my attention. He studied a concept called a graph-encoded map, or gem. Gems were introduced by Neil Robertson in his doctoral dissertation, but Lins pointed out their applicability to some aspects of topological graph theory. A gem is actually a special case of a more general concept called a 3-graph. In order to describe what a 3-graph is, we need some more definitions. A graph is said to be cubic if every vertex has just three edges incident on it. A colouring of the edges of a graph is proper if any two edges incident on a common vertex receive distinct colours. A 3-graph is essentially a cubic graph endowed with a proper edge colouring using just three colours. Most theorems about gems generalise to 3-graphs, and so the more general concept attracts more attention.

It was Lins's insight that a gem could be used to model an embedding of a graph in a surface. In some ways it seems a more attractive concept to work with. For instance, switching from a graph drawn in the plane to its dual amounts simply to interchanging two colours in the corresponding gem.

I began to wonder how far this idea could be taken. For example, could it be used to provide another proof of Kuratowski's theorem, one that does not depend on topology? Could it be used to prove the Jordan curve theorem? This theorem, notoriously difficult to prove, encapsulates the intuition that a closed curve drawn in the plane without crossing itself divides the plane into two regions that have the curve as their common boundary. I interested Paul Bonnington in these questions and we obtained an affirmative answer to both. Stimulated by these results, we decided to write a book to demonstrate the extent to which topological graph theory can be studied by means of gems. This appendix is that book.

2.6 Appendix F: The Number Systems of Analysis

The history of this book bears some resemblance to the story behind the writing of the book in the previous appendix. Once again it arose from my teaching. The original idea, suggested by Bruce van Brunt, was to write a text to cover the material taught to undergraduate students of analysis at Massey. Following discussions that also included Kee Teo, we decided to expand this concept to include the construction of the real number system. It was felt that some of the difficulties that students often have with analysis arise from an inadequate understanding of what real numbers are. We then realised that the development of the other number systems (natural numbers, integers, rational numbers and complex numbers) should also be included. A thorough treatment of this development requires an understanding of sets. Sets, in turn, cannot be treated adequately without a grounding in logic. By this time we decided to postpone our plan to write a book on analysis as we realised that we already had plenty of material to cover in one book.

The book therefore begins with an account of the elementary logic required. A chapter on sets and functions follows, and then the development of the various number systems begins. A question still remained: where do we stop? We decided that the obvious goal was a proof of the fundamental theorem of algebra. This theorem states that any complex polynomial must have a complex root. It does not remain true if we restrict numbers to natural numbers, integers, rational numbers or real numbers. Indeed it was this realisation that motivated the development first of the integers and then, in turn, the rational, real and complex numbers. We were then left with the problem of finding a way to prove the fundamental theorem of algebra that did not require a great deal of material that had not already been introduced for the construction of the number systems. Fortunately we were able to find such a method and consequently this book includes a proof that we believe to be new.

Chapter 3

List of Publications

3.1 Appendix A

1. "On defect-d matchings in graphs", *Discrete Math.* **13** (1975), 41-54 (with D.D. Grant and D.A. Holton) (MR54 5043)
2. "A short proof of a recent theorem of G. Szekeres", *J. Austral. Math. Sac.* **21** (1976), 277 (MR57 16144)
3. "On the number of parity sets in a graph", *Can. J. Math.* **28** (1976), 1167-71 (MR55 2644)
4. "Complete bipartite graphs and the Hadamard conjecture", *Proceedings of the Tenth Southeastern Conference on Combinatorics, Graph Theory and Computing*, 1979, pp. 721-6 (with D. Thuente) (MR81f.05034)
5. "The Hadamard conjecture and circuits of length four in a complete bipartite graph", *J. Austral. Math. Soc.* **A31** (1981), 252-6 (with D. Thuente) (MR84d:05122)
6. "The Hadamard conjecture and bases for the cycle space of $K_{is >4s}$", *Ars Combinatoria* **A17** (1984), 231-40 (with M.D. Hendy) (MR85k:05028)
7. "Another characterisation of Pfaffian bipartite graphs", *J. Austral. Math. Soc.* **A38** (1985), 132 42 (MR86j:05113)
8. "An analogue of the Hadamard conjecture for n x n matrices with $n = 2$ (mod 4)", *J. Austral. Math. Soc.* **A42** (1987), 287-95 (MR88i:05041)

9. “The Hadamard conjecture and integer lattices”, *J. Austral. Math. Soc.* **A43** (1987), 257-67 (with J. McCall) (MR89a:05043)
10. “An algorithm for the ear decomposition of a 1-factor covered graph”, *J. Austral. Math. Soc.* **A46** (1989), 296-301 (with F. Rendl) (MR89m:05093)
11. “Maximal matchings in graphs with large neighborhoods of independent vertices”, *J. Graph Theory* **14** (1990), 167-71 (with I. Rinsma and D.R. Woodall) (MR91c:05150)
12. “Operations preserving the Pfaffian property of a graph”, *J. Austral. Math. Soc.* **A50** (1991), 248-57 (with F. Rendl) (MR92d:05127)
13. “Evolutionary families of sets”, *Electronic J. Combinatorics* 7(1) (2000), R10 (with A.E. Campbell) (MR2001b:68098)
14. “A characterisation of Pfaffian near bipartite graphs”, *J. Combinatorial Theory* **B82** (2001), 175-222 (with I. Fischer) (MR2002d:05103)
15. “Towards a characterisation of Pfaffian near bipartite graphs”, *Discrete Math.* **244** (2002), 279-97 (with F. Rendl and I. Fischer) (MR2002m:05169)
16. “Even circuits of prescribed clockwise parity”, *Electronic J. Combinatorics* **10(1)** (2003), R45 (with I. Fischer)
17. “A new proof of a characterisation of Pfaffian bipartite graphs”, *J. Combinatorial Theory* **B91** (2004), 123-6 (with S. Norine and K.L. Teo)

3.2 Appendix B

1. “A conjecture about circuits in planar graphs”, *Combinatorial Mathematics III*, Lecture Notes in Mathematics, vol. 452, Springer-Verlag, New York, 1975, pp. 171-5 (MR52 2954)
2. “A theorem on planar graphs”, *Combinatorial Mathematics IV*, Lecture Notes in Mathematics, vol. 560, Springer-Verlag, New York, 1976, pp. 136-41 (MR55 157)
3. “On rings of circuits in planar graphs”, *Combinatorial Mathematics V*, Lecture Notes in Mathematics, vol. 622, Springer-Verlag, New York, 1977, pp. 133 40 (MR57 16136)

4. "An additivity theorem for maximum genus of a graph", *Discrete Math.* **21** (1978), 69-74 (with R.D. Ringeisen) (MR80d:05022)

5. "On the strong graph embedding conjecture", *Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory and Computing,* 1978, pp. 479-87 (with R.D. Ringeisen) (MR81f:05067)

6. "A characterization of planar cubic graphs", *J. Combinatorial Theory* **B29** (1980), 185-94 (MR81k:05045)

7. "Elegant odd rings and non-planar graphs", *Combinatorial Mathematics VIII,* Lecture Notes in Mathematics, vol. 884, Springer-Verlag, New York, 1981, pp. 234-68 (with D.A. Holton) (MR83a:05057)

8. "Rings of bonds in graphs", *J. Combinatorial Theory* **B33** (1982), 1-6 (with D.A. Holton) (MR83m:05084)

9. "Regular odd rings and non-planar graphs", *Combinatorica* **2** (1982), 149-52 (with D.A. Holton) (MR84f:05039)

10. "An interesting decomposition of $K_{4n\,4n}$ into planar subgraphs", *Combinatorial Mathematics IX,* Lecture Notes in Mathematics, vol. 952, Springer-Verlag, New York, 1982, pp. 353-7 (MR83m:05109)

11. "A generalisation of a recent characterisation of planar graphs", *Bull. Austral. Math. Soc.* **27** (1983), 225-30 (MR85b:05076)

12. "No graph has a maximal 3-ring of bonds", *J. Combinatorial Theory* **B38** (1985), 139 42 (with D.A. Holton) (MR86h:05047)

13. "Cubic combinatorial maps", *J. Combinatorial Theory* **B44** (1988), 44—63 (MR89a:05058)

14. "Discrete Jordan curve theorems", *J. Combinatorial Theory* **B47** (1989), 251-61 (with A. Vince) (MR90m:05118)

15. "Embedding schemes and the Jordan curve theorem", in *Topics in Combinatorics and Graph Theory,* R. Bodendiek and R. Henn (ed.), Physica-Verlag, Heidelberg, 1990, pp. 479-89 (with A. Vince) (MR92a:05043)

16. "An algebraic characterization of planar graphs", *J. Graph Theory* **19** (1995), 237-50 (with D. Archdeacon and C.P. Bonnington) (MR96a:05046)

17. "Nonhamiltonian triangulations with large connectivity and representativity", *J. Combinatorial Theory* **B68** (1996), 45-55 (with D. Archdeacon and N. Hartsfield) (MR98g:57035)

18. "A condition for a normal semicycle to separate an orientable 3-graph", in *Combinatorics, Complexity and Logic*, D.S. Bridges, C.S. Calude, J. Gibbons, S. Reeves, I.H. Witten (ed.), Springer, Singapore, 1997, pp. 330-7 (with B. van Brunt) (MR99g:05116)

19. "Obstruction sets for outer-projective-planar graphs", *Ars Combinatoria* **49** (1998), 113-27 (with D. Archdeacon, N. Hartsfield and B. Mohar) (MR2000a:05065)

3.3 Appendix C

1. "Chromatic equivalence classes of certain generalized polygon trees", *Discrete Math.* **172** (1997), 103-14 (with **Y.-H.** Peng, **K.L.** Teo and **H.** Wang) (MR99f:05040)

2. "An attempt to classify bipartite graphs by chromatic polynomials", *Discrete Math.* **222** (2000), 73-88 (with F.M. Dong, K.M. Koh, K.L. Teo and M.D. Hendy) (MR2001g:05085)

3. "Chromatically unique bipartite graphs with low 3-independent partition numbers", *Discrete Math.* **224** (2000), 107-24 (with F.M. Dong, K.M. Koh, K.L. Teo and M.D. Hendy) (MR2001f:05063)

4. "Sharp bounds for the number of 3-independent partitions and the chromaticity of bipartite graphs", *J. Graph Theory* **37** (2001), 48-77 (with F.M. Dong, K.M. Koh, K.L. Teo and M.D. Hendy) (MR2001m:05140)

5. "Some inequalities on chromatic polynomials", *New Zealand J. Math.* **30** (2001), 111-8 (with F.M. Dong, K.L. Teo and M.D. Hendy) (MR2002i:05042)

6. "Two invariants for adjointly equivalent graphs", *Australasian J. Combinatorics* **25** (2002), 133 43 (with F.M. Dong, K.L. Teo and M.D. Hendy) (MR2003f:05045)

7. "Zeros of adjoint polynomials of paths and cycles", *Australasian J. Combinatorics* **25** (2002), 167-74 (with F.M. Dong, K.L. Teo and M.D. Bendy) (MR2002j:05061)
8. "The vertex-cover polynomial of a graph", *Discrete Math.* **250** (2002), 71-8 (with F.M. Dong, M.D. Bendy and K.L. Teo) (MR2003c:05173)
9. "Chromaticity of some families of dense graphs", *Discrete Math.* 258 (2002), 303-21 (with F.M. Dong, K.L. Teo and M.D. Bendy)
10. "Chromatically unique multibridge graphs", *Electronic J. Combinatorics* **11(1)** (2004), R12 (with F.M. Dong, K.L. Teo, M.D. Bendy and K.L. Koh)
11. "Graph-functions associated with an edge-property", *Australasian J. Combinatorics* **30** (2004), 3-20 (with F.M. Dong, M.D. Bendy and K.L. Teo)

3.4 Appendix D

1. "A characterization of graphs with a generalized Tait coloring", *Proceedings of the Eighth Southeastern Conference on Combinatorics, Graph Theory and Computing*, 1977, pp. 467-72 (with M.J. Lipman) (MR58 16366)
2. "On graphs as unions of Eulerian graphs", *Combinatorial Mathematics*, Lecture Notes in Mathematics, vol. 686, Springer-Verlag, New York, 1978, pp. 206-9 (MR80m:05076)
3. "Cycles in bipartite tournaments", *J. Combinatorial Theory* **B32** (1982), 140-5 (with L.W. Beineke) (MR83e:05057)
4. "Cycles through six vertices excluding one edge in 3-connected cubic graphs", *University of Melbourne Mathematics Research Report* **11** (1982), 1-38 (with M.N. Ellingham and D.A. Bolton)
5. "Comparing trees with pendant vertices labelled", *SIAM J. Applied Math.* **44** (1984), 1054-65 (with M.D. Bendy and D. Penny) (MR85m:92004)
6. "Cycles through ten vertices in 3-connected cubic graphs", *Combinatorica* **4** (1984), 265-73 (with M.N. Ellingham and D.A. Bolton) (MR86j:05087)
7. "A theorem on integer flows", *Ars Combinatoria* **A26** (1988), 109—12 (with D. Younger and W.T. Tutte) (MR90g:90062)

8. “Cycles and paths in multigraphs”, *Australasian J. Combinatorics* **5** (1992), 3-1 1 (with R.B. Eggleton and D.A. Holton) (MR93a:05077)
9. “Vertex disjoint cycles in a directed graph”, *Australasian J. Combinatorics* **12** (1995), 113-9 (with H. Wang) (MR96c:05109)
10. “Partition of a directed bipartite graph into two directed cycles”, *Discrete Math.* **160** (1996), 283-9 (with H. Wang and K.L. Teo) (MR97h:05103)
11. “On a conjecture on directed cycles in a directed bipartite graph”, *Graphs and Combinatorics* **13** (1997), *2C1-2T3* (with H. Wang and K.L. Teo) (MR98e:05052)
12. “Fusion in bipartite graphs”, *New Zealand J. Math. 28 (1999),* 225-36 (with H. Wang and K.L. Teo) (MR2001i:05125)
13. “On the principal edge bipartition of a graph”, *Australasian J. Combinatorics* **20** (1999), 91-6 (with M.A. Henning) (MR2000g:05089)

3.5 Appendix E

The Foundations of Topological Graph Theory (with C.P. Bennington), Springer-Verlag, New York, 1995 (MR97e:05071)

3.6 Appendix F

The Number Systems of Analysis (with K.L. Teo and B. van Brunt), World Scientific, Singapore, 2003