NUMBER SENSE IN SECONDARY SCHOOL PUPILS

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ABSTRACT

Number sense refers to a quantitative appreciation of numbers and an ability to perform calculations in creative and original ways. A pupil with number sense will not be bound by learnt procedures when calculating, instead he/she will be able to create self-made solutions that reflect a good understanding of numbers and flexible ways of using operations on them. Number sense development is now a recognised as an integral component in the syllabi of many western countries and is promoted as an appropriate and productive way of teaching mathematics. Much of number sense revolves around computational mathematics, at which New Zealand pupils, on a comparative international basis, do not rate highly.

This study examines the number sense of a range of New Zealand secondary school-aged pupils by means of a questionnaire, and the processes that pupils use when performing calculations by a series of interviews. Affective factors that have influenced a pupil's mathematical development have also been studied. The qualitative and quantitative data generated enables judgements to be made about the development of number sense and what factors affect this development.

Results suggest that many pupils have had little experience with some of the computational aspects of number sense (estimation and mental arithmetic), only partly appreciate the potential of the distributive principle in numerical calculations and are too locked into algorithmic type solutions. By contrast, pupils with well developed number sense demonstrate good number operation knowledge which they employ in a variety of creative ways when solving problems. Indications are that the foundations for this expertise are put in place at an early age by examining the properties of numbers, performing number drill exercises and validating the processes so that a sense making aspect is built into mathematics education.

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CHAPTER 1.

INTRODUCTION

Number sense refers to an intuitive feeling for numbers and their various uses and interpretations; an appreciation for various levels of accuracy when figuring; the ability to detect errors, and a common-sense approach to using numbers....Above all, number sense is characterised by a desire to make sense of numerical situations. (Reys, 1992a, p.3)

1.1 BACKGROUND

Results from the Second and Third International Mathematics Study conducted in 1981 and 1994 show that, compared with pupils from other developed western countries, New Zealand third formers do not perform well at arithematic computations and estimation exercises. Mediocre performances by our pupils suggest that the curriculum goals of developing problem solving skills and attaining confidence at mathematical calculations are barely being attained. Given that our intended curricula are similar to other countries, it must be assumed that the way the content is delivered results in this disappointing performance. Pupils who are algorithmic-bound, who are unable to call upon a flexible repertoire of arithematic operations or have no broad knowledge of number facts are unlikely to perform well at numeric calculations. A preliminary report of participating schools in the Third I.E.A International Mathematics and Science Study (1996) has number sense and estimation listed as examined aspects of mathematics knowledge for junior pupils. Preliminary findings show that, compared to other countries, our performance at number sense questions was average only.

The development of number sense now occupies a prominent position in the curricula of New Zealand and other western countries. However to date there has been very little investigation into the extent of this sense among New Zealand secondary school pupils or any identification of factors that promote or inhibit its development.

The number sense approach to mathematics education provides a path to mathematical competency by stressing that calculations should make sense. For too many pupils the algorithmic approach to performing mathematics calculations has left them ill-equipped for manipulating and applying mathematics to problems. This research exercise is in two parts; in the first the number sense of a range of secondary-aged pupils is examined using a questionnaire, and the second, based on interviews, is an in-depth examination of how pupils perform calculations and what attitudes and expectations they have developed. The study provides an opportunity to analyse mathematics education and to identify attributes that are sympathetic to the development of number sense.

> *When we know why we do something in the classroom and what effect it will have on our students, we shall be able to claim that we are contributing to the clarification of our understanding of our activity as if it were a science.* (Caleb Gattegno, cited in Jaworski, 1994. p. 2)

The advent of cheap, versatile and sophisticated calculators has meant that reliance on the paper and pencil approach to teaching mathematics is no longer appropriate. In the past many school curricula emphasised efficiency and speed in numerical computation with the result that pupils considered that the answer itself was more important that the process by which it was achieved, and guessing rather than proper calculations became common place. The focus on procedural knowledge is unproductive and does not in itself lead to good conceptual understanding (Narode, Board & Davenport, 1993). Pupils who do not have full understanding of numbers and what they mean must develop an extensive repertoire of rules to enable them to solve problems (Ekenstam, 1977). For example, a pupil who does not know that 0.45, 0.450 and $\frac{9}{20}$ are the same, or that $\frac{5}{8}$ is less than $\frac{4}{5}$, or that the relative difference between 1000 and 10,000 is much less than between 10,000 and 100,000, will find performing numerical calculations awkward and unproductive.

After twenty years of involvement with teaching secondary school I question whether sufficient numbers of pupils are leaving school with mathematics skills that allow them to function productively in the world. We need to develop instructional approaches which lead to greater competency at mathematical calculations and numeracy skills. Factors that influence pupils' achievement at mathematics are many and varied. This study examines number sense, what it is, how it can be developed, nurtured and refined. To achieve this a holistic consideration of the subject is presented, incorporating historical and philosophical development of mathematics, a framework for teaching mathematics that acknowledges constructivist learning and details of the attributes of number sense.

1.2 INTRODUCTION TO NUMBER SENSE

Contrary to the normal expectations of things mathematical, number sense does not lend itself to a precise all-embracing definition. Number sense emphasises number knowledge, operations on numbers and creative and flexible ways that the two can be incorporated for solving problems. Pupils with number sense possess in-depth number and number operation knowledge, can confidently perform calculations and are able to reflect on and evaluate answers. Number sense emphasises numbers as meaningful entities, and encourages a quantitative appreciation of numbers and an expectation that mathematical calculations give results which make sense and are reasonable. Thus pupils with number sense are able to call up checking procedures to judge the reasonableness of numerical calculations. The following two commentaries exemplify the spirit and essence of number sense. The first identifies the components of number sense.

> *Most characteristics of number sense focus on its intuitive nature, its gradual development, and the ways it is manifested. Manifestations include using numbers flexibly when mentally computing, estimating, judging number magnitude, and judging reasonableness of results; moving between number representations; and relating numbers, symbols, and operations, all stemming from a disposition to make sense of numerical situations.* (Markovits and Sowder, 1994, p.4)

The second focuses on the importance of understanding, how mathematical processes function and the need to be able to communicate mathematically.

> *Number sense refers to a person's understanding of number and the operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and to develop useful strategies for handling numbers and operations. It reflects an inclination and an ability to use numbers and quantitative methods as a means of communicating, processing and interpreting information. It results in an expectation that numbers are useful and that mathematics has a certain regularity.* (McIntosh, A., Reys, B.J. and Reys, R.E, 1992, p. 3)

A pupil with developed number sense will not be restricted to a single approach with a computation but will employ a variety of processes that use the properties of numbers and operations in creative ways. The following non-algorithmic approaches to the problem of 38 x 24 demonstrate thinking that is indicative of someone with developed number sense.

- 38 x 24 is approximately 40 x 25 which is the same as 4 x 250 which is 1000
- 38 x 24 is 38(20 + 4) which is the same as (38 x 20) + (38 x 4) which is the same as (380 x 2) + (38 x 2) +(38 x 2) which gives 760 + 76 + 76 = 912.
- 38 x 24 will give an even answer as both the numbers being multiplied are even.
- 38 x 24 will have an answer that ends with a 2 because 4 x 8 gives 32.
- 38 x 24 is 2 x 19 x 2 x 2 x 2 x 3 which is the same as 2^4 x 3 x 19 using its prime factors.
- 38 x 24 is 2 x 24 x 19 which is 48 x 19 which is(48 x 20) - (48 x 1) which is (480 x 2) - 48 which gives 912.
- Would know that the answer should lie between 800 and 1000.

There may well be other ways of performing this computation. Taken together the above responses demonstrate an *at homeness with numbers* (Cockroft Report, 1982), as well as an approach to calculations which relies on the properties of the numbers, an understanding of the effects of operations and a realization of acceptable limits for the answer. A pupil with number sense could be expected to give any of the above as a response to the problem and would certainly comprehend and appreciate all the replies listed.

This contrasts with many students' reliance on using algorithms to solve problems. This procedural and memory-dependent approach discourages understanding and inhibits self-regulated approaches to solving problems. The result is that many pupils end up with compartmentalised knowledge and do not know how to put this knowledge together in successful ways for solving problems. The number sense approach to mathematics instruction encourages flexibility with calculations as opposed to traditional instruction that is algorithmic in design.

1.3 HISTORICAL DEVELOPMENT OF THE NUMBER SENSE APPROACH

The historical derivation of the term number sense is difficult to precisely determine. As early as the 1930s William Brownell advocated the need for meaningful learning to be an integral part of mathematics education. Brownell stressed that pupils should possess an:

> *intelligent grasp of number relations and the ability to deal with arithmetic situations with proper comprehension of their mathematical as well as their practical significance.* (p. 3)

The term *number sense* made its first appearance in print in the Cockroft Report (1982). Up to that time and subsequently, though to a lesser extent, the term *numeracy* was used in relation to working with numbers. *Numeracy* has been defined as those mathematical skills that enable an individual to cope with the practical demands of everyday life - comprehending and interpreting numbers, calculating and estimating.

The term *numeracy*, coined by Crowther in 1959, came to represent for mathematics education what the term *literacy* did for English teaching. Together, numeracy and literacy provided convenient references points for the popular *back to basics* calls of that time. *Numeracy* was to be developed, mainly by repeated practise of algorithms that would enable pupils to function in the everyday world. Little importance was attached to understanding or meaning nor to the benefits of considering alternate ways of using properties of numbers and number operations to solve numerical calculations. While the term *numeracy* is less frequently used in modern texts, a number of countries - America, Australia, England and New Zealand have incorporated the objectives of *number sense* into their resent curricula, justifying its inclusion on the basis that it promotes sense making in numerical calculations.

Number sense and problem solving are two of the modern thrusts in mathematics education. Both defy exact definition and are better considered as approaches to teaching rather than as actual topics that can be precisely structured and taught. Comprehensive number sense is beneficial for problem solving (Dougherty & Crites 1989), useful for establishing the magnitude and the expected number type for an answer and with helping to select the appropriate computation procedures for solving problems. While problem solving necessitates a high degree of number sense, the converse is not necessarily true. Problem solving helps to cement the conceptual and procedural aspects of number sense.

1.4 DECONSTRUCTION OF NUMBER SENSE

Despite previous suggestions that a precise definition of number sense is not possible, a number of authors have proposed that it is characterised by three principal attributes: number knowledge, knowledge of operations on numbers and an understanding of how to apply them in computational settings (Howden, 1989; Rowan & Thompson, 1989; Sowder, 1992a). McIntosh et al. (1992) has taken these three broad headings, teased out the fundamental components and elements that constitute each heading and presented them in a detailed, well organised framework that provides a very comprehensive overview of what constitutes number sense. For this study McIntosh's framework (Appendix A) has been modified to put more emphasis on the sense making attributes of number sense, focusing on pupils' grasp of the inherent logical implications of number and operation knowledge.

Figure 1 details the components of number sense that have been used in this study. The following text expands this framework in order to give a full description of the composition of number sense:

Number Knowledge
- **Sense of orderliness** - place value, ordering numbers
- **Number representation** decomposition/recomposition, equivalent numerical forms
- **Sense of magnitude** - comparative sizes, sense of size
- **Awareness of benchmarks** - both mathematical and physical

Operations Knowledge
- **Effect of operation** - appropriate use of operands, effect of operands on fractions and decimals
- **Properties of operations** - commutative, associative, distributive, use of inverses
- **Interrelationship of operations** - order of operations, use of repeated addition for multiplication and division for repeated subtraction

Numerical Computation
- **Computational estimation** - calculating with exact and approximate data
- **Mental computation** - use of multiple strategies, interpreting problems in context
- **Metacognition** - reflection on results and procedure
- **Use of logic** - awareness of the logical implication of number properties and operations on them

Fig 1. A number sense framework

The following is an in-depth consideration of the characteristics of the three components of number sense.

Number knowledge

Number knowledge involves the workings of the base 10 system, the structure of the number line, the numerical value of numbers, ways a numbers can be represented and an awareness of numerical benchmarks that will be useful for making everyday quantitative judgements. Specifically, important features include:

Sense of orderliness

- An understanding of the place value of numerals in a number including decimals and fractions. For example knowing what the 4 in 1.04 signifies.
- An awareness of the structure of the number system, so that 11 more than 689 is no more difficult to comprehend than 1 less than 20,000.
- An awareness of the patterns that numbers generate, particularly on the number line, for example that there are values between ⅓ and ⅔. Pupils should be able to generate patterns and understand the notion of density on the number line.
- An ability to order numbers, for example, ⅔, 0.61, ½, 0.49 and 0.492.

Number representation

- An ability to represent numerical values in a variety of ways, for example forty minutes can be thought of as ⅔ or 0.667 or 66.7% of an hour or that forty minutes implies that there is ⅓ of an hour left.
- An ability to express a number in other equivalent forms, for example 0.5 as ½ so as to simplify the calculation of a problems like 346 x 0.5.
- The ability to decompose and recompose a number so that it can be more easily manipulated. For example 452 x 71 could be re-expressed as (400 + 50 + 2) x 71 where the decomposition of 452 provides values that are easier to compute. Often exchanges of money involve manipulations that limit the number of coins

used, for example a fee of \$7.25 may be paid out with a \$10 note and 25c, so that the minimum number of coins is given.

Sense of magnitude

- Best exemplified with reference to a number line where relative numerical values can be easily demonstrated. For example that the relative sizes of 420 and −418 are almost the same or that 10,000 is ten lots of 1000.
- An awareness that the relative differences between 1000 and 100 are considerably less than that for 10,000 to 1000 or that by age fifteen years one has lived for just a little less than five and a half thousand days.
- Knowing that the difference between 53 and 78 is the same as that between 453 and 478.

Awareness of benchmarks

- Benchmarks provide a mental reference for making meaningful comparisons. For example, a door height is just under two metres, a normal walking pace is four kilometres per hour, a comfortable room temperature is seventeen degrees, and a popular sports stadium will accommodate fifty thousand people are all useful benchmarks. In a mathematical setting a realization that $\frac{5}{8}$ is close to $\frac{1}{2}$ or 2.98 is nearly 3 are examples of benchmarks that would be useful in estimations calculations.
- The use of knowledge of powers of ten for multiplication and division.

Operation on Numbers

By secondary school pupils are familiar with the algorithms for the four basic operations with integers, decimals and to some extent fractions. An understanding of the properties and interrelationships of the four basic operations enables a pupil to be flexible in the performance of mathematical computations. An appreciation of the effect of operations on numbers, for example, the fact that multiplication does not always give a greater result, as well as how they actually function, is vital for performing computations effectively. An understanding of how the properties of operations require them to be applied in a particular order and an awareness of the interrelationship between operations (multiplication can be considered as repeated addition or that division is equivalent to repeated subtraction) are examples of knowledge of operations on numbers.

Effect of operations

- Knowing that multiplying by 2 will increases the result faster than adding by 2, or that 5 + 5 + 5 + 5 + 5 is more conveniently calculated as 5 x 5 or that dividing by 10 is the same as moving the decimal point to the left.
- Multiplication by a number in the domain 0 to 1 will decrease the total while division with the same values increases the result.
- Effect of operating on numbers by the identities 1 and 0.
- Understanding the working of operations, for example to increase an amount by $12\frac{1}{2}$ % is the same as multiplying that value by 1.125.

Properties of operations

- Know the order of operations of the operands.
- The commutative, and associative principles and how they can be used in calculations.
- A sound knowledge of the distributive principle will provide a pupil with a powerful tool for carrying out estimation and mental calculations. The calculation 12 x 54 becomes 12 x (50 + 4) which moves to (12 x 50) + (12 x 4) which ends as (12 x 5 x 10) + (12 x 4) to give 648.

Interrelationship of operations

- That multiplication is repeated addition or that division can be considered as an application of subtraction.
- Knowledge of inverse relationship with operations - addition with subtraction and multiplication with division, for example computations like 84 - 50 is the same as asking *"50 plus what value"* gives 84, or 420 ÷ 6 can be evaluated as

"6 times what gives 420".

Numerical Computations

To engage in a mathematical activity with confidence it is important to have a well developed appreciation of numbers and the outcome of operating on them. The level of performance at estimation and mental computation problems is indicative of a pupil's depth of understanding of how numbers and operations on them can be applied and provides insights into their intuition for mathematical calculations.

Computational Estimation

- Approximating a number within the implied limits of a problem. For example, 0.94 is roughly 1, or $^{17}/_{32}$ is close to ½.
- Knowledge of strategies that can be used to perform a calculation such as reformation, translation or compensation. These strategies are listed in Appendix B and are derived from those proposed by Reys et al., (1982).
- An ability to make quantitative judgements about length, time, weight and volume.
- Be able to approximate shapes with standard ones. For example, that the South Island of New Zealand is approximately rectangular.

Mental Computation

- Be able to perform calculations without recourse to paper, pencils or calculator.
- Be able to translate a verbal problem into a mathematical one.
- Analyse the content and adopt strategies to perform the calculation mentally. These strategies entail being able to put together number and operation on numbers knowledge in creative ways. Mental computation strategies (Hope, 1987) are listed in Appendix B.

Metacognition

- Self-reflection on mathematical processes and correcting or adjusting strategies being used so as to improve the outcome of a calculation. Pupils can be taught to consider the appropriateness of what they are calculating and to reflect on their answer to see it *makes sense,* that it lies within bounds that are suggested by the problem.

Use of logic

- The properties of numbers and the way that operations on them function imposes logical consequences. A pupil with number sense will appreciate this logic and be able to use it to validate a calculation. He or she will be able to explain why subtraction cannot be commutative, why division by zero is not determinable or why an odd number times an even number is always even.

Number sense arises out of combining number knowledge, knowledge of operations on numbers and numerical computational knowledge and looking for the interrelationships between these components. While each of these considered in isolation will not foster the development of number sense, the three components taken together will.

Greeno (1991) has put forward a psychological perspective for the development of number sense. Greeno proposes a metaphor for number sense, in which a person knows the layout of a workshop (mathematics), where the materials (number facts and operation knowledge) are kept, and how to use the tools (combine conceptual and procedural knowledge) to make objects. The metaphor also extended to instruction so that the person would have enough confidence (accepted into a mathematics community) to consult a manual (invent new strategies) to produce new creations. Greeno's theoretical consideration of how cognitive activity can occur is a useful model for considering what number sense is. It contrasts well with the practical and tangible definitions proposed by other authors who identify attributes or dispositions of people with number sense.

1.5 NUMBER SENSE IN THE CURRICULUM

The genesis of the development of the ideas behind number sense can be traced to the 1982 Cockroft report *Mathematics Counts* which promoted a more expansive approach to how mathematics should be taught. The reference to an *at homeness* with numbers indicates a more humanist view of learning mathematics that encourages familiarity and flexibility with numbers and operations.

In America the 1989 National Council of Teachers of Mathematics report *Curriculum and Evaluation Standards for School Mathematics* includes number sense as a major theme throughout its recommendations. The tone and intent of the NCTM statement is sympathetic with the aims of the Cockroft report.

> *The greatest revisions to be made in the teaching of computation include the following: fostering a solid understanding of, and proficiency with, simple calculations; abandoning the teaching of tedious calculations using paper and pencil algorithms in favour of exploring more mathematics; fostering the use of a wide variety of computation and estimation techniques - ranging from quick mental calculations to those using computers - suited to different mathematical settings; developing the skills necessary to use appropriate technology and then translating computed results to the problem setting; and providing students with ways to check the reasonableness of computations promotes number and algorithmic sense, estimation skills* (p.95).

In 1991 the Australian Education Council published *A National Statement on Mathematics for Australian Schools* which adopts a constructivist philosophy to learning mathematics and encouraging teaching styles that recognize pupils' personal constructions of their own mathematics knowledge. Emphasis is given to encouraging pupils to reflect on their mathematical calculations and to the need to link new knowledge to existing knowledge, so that mathematical ideas can be refined and consolidated. In all strands and levels the importance of number sense is explicit.

> *All people need to develop a good sense of numbers, that is, ease and familiarity with and intuition about numbers. This requires a sound grasp of number concepts and notation, familiarity with number patterns and relationships, a working repertoire of number skills and, most importantly, confidence in one's capacity to deal with numerical situations* (p. 107).

Teachers are encouraged to promote the historical and cultural aspects of the subject and provide opportunities for pupils to appreciate the structure and logical interconnection that occurs in mathematical calculations.

The *Mathematics in the New Zealand Curriculum* (Ministry of Education, 1992) suggests a considerable change of emphasis in how mathematics is to be taught. Problem solving is extolled as the means by which the content should be learnt. Although there is no explicit reference to number sense in the introduction, aims or objectives, the need to foster numeracy skills, to reflect critically on methods used and to develop an understanding of number are mentioned. One of the five strands is devoted to number. In the first five of these levels number sense and its development is directly referred to under the heading *Suggested Learning Experiences.* The following quotes concerning number sense are from strand statements:

Exploring number

- *developing number sense by exploring number in the context of their own experiences and the world around them;* (Level 1, p.33)
- *developing a number sense by exploring number in the context of their everyday experience and the world around them;* (Level 2, p.37)
- *developing number sense by exploring number in the context of their everyday experiences and the world around them, and using number to explore events in their own lives;* (Level 3; p.41)

Exploring computation and estimation

- *developing number sense by exploring estimation and computation, in the context of their own experiences and the world around them, using concrete* *materials, mental strategies, and calculators;* (Level 1, p.33)

- *developing a number and computation sense by exploring estimation and computation in the context of their everyday lives;* (Level 3, p.41)

These objectives go some way to capturing the essence of number sense. Further reference to components of number sense that occur within the level descriptors include:

- *place value* (Level 2, p.37)
- *investigating odd and even numbers* (Level 2 p. 37)
- *extending their understanding of the number system* (Level 4 p.450)
- *developing instant recall of basic addition, subtraction, multiplication and division facts*. (Level 2 & 3, p.37 & 41)

1.6 THE RESEARCH OBJECTIVES

The principle aim of this study was to examine the level of number sense in secondary school pupils. In addition to measuring pupils' performance at the various components of number sense, the study examines affective factors which influence mathematical development.

The following objectives determined the enquiry of this research

1) To examine the number sense of a range of secondary school pupils and to identify any development.

A questionnaire was used to arrive at a quantitative basis upon which comparisons of number sense could be made. Identification of the development of number sense was examined by comparing year nine, year ten, year eleven and year twelve students knowledge of numbers, proficiency at operations and competency at computational arithmetic.

2) To investigate pupil's inclination to make sense of their mathematical thinking.

Attitudinal responses from a questionnaire and interviews were used to examine the extent of this inclination, whether pupils expected to make sense of calculations and how inclined they were to question the appropriateness of their answers.

3) To identify activities, attitudes and perceptions that contribute to or impair the development of a pupil's number sense.

Pupils with a range of mathematical abilities were interviewed to determine factors that contributed to or detracted from the development of their number sense. An examination of teaching approaches, classroom cultures and levels of base numerical knowledge was conducted to determine what factors contribute to number sense development.

1.7 JUSTIFICATION OF THE STUDY

Studies of the relationship between the acquisition of number concepts and pupils' proficiency in mathematics have been carried out in New Zealand by Graham (1991) and Young-Loveridge (1987, 1991), but these have been confined to primary and intermediate aged children. A number of studies into the mathematical needs of the *average* New Zealander point to the importance of computational skills and proficiencies at manipulating numbers. The Massey University report *The Mathematical Needs of New Zealand School Leavers* (Knight et al., 1992) noted that :

> *Most of the mathematics used in everyday lives can fairly be described as elementary. Arithmetic and basic geometry will suffice for most purposes. What is important is to be able to apply these elementary mathematical tools in a variety of situations in order to solve the many different kinds of problems that arise in everyday life* (p.28).

and that:

> *All school leavers should be familiar with the properties and manipulations of the number system*

Which leads on to the following recommendations;

> *Our education system should ensure that most of our students are competent in:*

(a) numeracy

(b) data measurement, storage, retrieval, manipulation and interpretation

(c) use of formula

(d)the basic arithmetical operations of $+$, $-$, x, $\div$ *and the use of %* (p. 78)

The Massey findings are endorsed by a 1994 Lifelong Learning survey by Chartwell Consultants, involving eighty nine randomly selected adults from the Manawatu region. When questioned about the mathematics that they actually used in their every day life it was noted that:

> *the ability to make sensible quantitative judgements based on mental arithmetic and estimation was very important* (p. 30).

It seems that one of the most important functions of a mathematics education is to provide people with a wide knowledge of numbers, a sound appreciation of the ways that they can be manipulated so that everyday mathematics type problems can be calculated with confidence and ease. Numerical judgements of the appropriateness of a result (especially in computations involving quantities and money) are important and a useful skill for anyone wanting to function fluently in the everyday world.

The general lack of ability at making judgements about relative magnitudes has been well documented by Paulos (1988) in his text *Innumeracy*. Familiarity with commonly used benchmarks, like the average walking pace is 4 km/hr or a rugby ground is about half a hectare in area, enable pupils to apply mathematics to everyday problems in useful ways. Mental calculation has been identified by Biggs (1967) as one of the most commonly used mathematics techniques by the general populace and deserves to have a prominent place in pupils mathematics curricula. Estimation and mental calculations are integral components of number sense.

Number sense, flexibly taught and properly grasped will not only provide a sturdy platform for advanced mathematical studies but will also serve the general populace well with their day-to-day requirements for dealing with numbers.

SUMMARY

A consideration of the depth of development of number sense in secondary school pupils and the ways that it can be nurtured and developed is both timely and necessary. There can be little doubt that number sense plays an important role in post-secondary-school life and its development is crucial for people wanting to take an active part in everyday life.

Number sense is made up of number knowledge, number operations knowledge and numerical computation techniques. Each of these three can be divided up into a number of identifiable aspects which can be used as a basis for examining the extent of number sense in school pupils. There is sufficient literature to show that pupils with well developed number sense will be able to apply this knowledge to out-of-school problems. Chapter 2 will propose a historical, philosophical and theoretical basis upon which number sense can be considered, as well as ways to integrate its development into mathematics education which are sympathetic to its intent and goals.

CHAPTER 2

A THEORETICAL FRAMEWORK

The major reform recommendations from the *Cockroft Report (1982)*, the *American National Council of Teachers of Mathematics (1989 & 1991), the National Statement on Mathematics for Australian Schools (1991)* and our own *Mathematics in the New Zealand Curriculum* (1992) were for a thorough re-evaluation of *how* mathematics is taught rather than *what* is taught. The focus on developing number sense is one of the shifts in emphasis recommended in the various curricula. However, implementing curriculum changes is dependent on many factors and often requires a change of perception of how learning occurs, and a reconsideration of the historical and philosophical perceptions that underpin the subject.

The following discussion situates the development of number sense within a constructivist paradigm, a progressive classroom environment and gives a psychological perspective on the cognitive workings of number sense. A short historical/philosophical overview creates a basis for considering number sense in the continuum of the development of mathematical education.

2.1 AN HISTORICAL FRAMEWORK

An acquaintance with the various philosophical foundations of mathematics, especially ones that are flexible and humanist in outlook, is necessary for any teacher who wants to be confident about the place of number sense in the whole scheme of mathematics. Ernest (1989, 1994) has written extensively on the need for mathematics teachers to have a philosophical understanding of the basis of the subject. Fennema and Franke (1992), Cobb, Wood & Yackel (1992) have shown that pupils' achievement is dependent on teachers' mathematical knowledge, understanding and beliefs. The following brief historical resumé traces some of the thinking that has shaped the philosophical foundation of mathematics.

The western perspective of mathematics is still very much influenced by what the ancient Greeks thought about the subject. Many teachers would view mathematics as a body of knowledge founded upon a few self-evident truths, held together by systems of deductive and inductive proofs. This austere but beautifully logical and infallible body of knowledge was the Greeks' legacy, a rather constraining and potentially elitist one that served mankind well for about one and a half thousand years. The growth of interest in analysis, stimulated by the desire to attach precision and understanding to the finer points of Differential Calculus, showed that there were shortcomings and gaps in our knowledge of mathematics. The development of Non-Euclidean Geometry in the nineteenth century by Riemann and others suggested that Euclidean Geometry was not the only approach that could be taken to describe the physical environment and predicate other systems upon which mathematical knowledge could be based.

An axiomatic construction of *Set Theory* offered a promising basis for compiling a foundation for mathematics until it was cut to shreds by Russell's *set of all sets* paradox in the early part of this century. Various 'patch-ups' of Frege's theory have been proposed and are in use today. However, with the publication of Gödel's *Incompleteness Theorem* in 1931 the search for any system of axioms or propositions, upon which a logical foundation for mathematics could be built, would appeared to be fruitless.

A modern philosophy of mathematics is a cocktail of Intuitionism, Platonism and Formalism. Guided by the ideas of Lakatos (cited in Worrall & Zahar, 1976), mathematics is viewed as a body of fallible knowledge in which concepts can grow in a dialectic of proof and refutation, argument and counter-argument, as opposed to a mechanical system constrained by the logical consequences of a few axioms. Many modern philosophers share the following views on mathematics:

- *Mathematics is human. It is a part of and fits into human culture. It is not, as Frege proposed, a timeless, tenseless, objective and abstract reality.*
- *Mathematical knowledge isn't by nature infallible.*
- *There are different versions of proof and rigour, depending on time, place, for*

example, computers can be used for proving mathematical statements.

- *Mathematical objectives are a certain variety of Social-Cultural-Historical objectives, they're distinctive and are shared much as "Moby Dick" is a part of our culture and not simply just a story* (Hersh, cited in Ernest, 1994. p. 14).

From a cultural perspective it is interesting to contrast the Western and Indian disposition towards the importance and use for formal proofs. The Indians' *aim was not to build up an edifice of geometry on a few self evident axioms, but to convince the intelligent student of the validity of the theorems so that the visual demonstration was quite an acceptable from of proof* (S. Amma, cited in Ernest, 1994. p. 200).
The Indian mathematicians' attempt to make sense of the subject and relate it to some purpose, is the antithesis of the Greek tradition with its expectation of having to formally derive every step in a mathematical argument.

The four-thousand-year-old history of mathematical knowledge should be appreciated by all who intend to teach it. Pupils can benefit from knowing about the important figures in the history of the development of mathematics. The loose, imprecise yet knowable aspects of number sense are compatible with modern philosophical views of mathematics. Mathematics is perceived as a fallible body of information whose meaning has to be negotiated, constructed, developed by experience and reflection rather than as a precise, closed and definitive system. This view is in keeping with the intent and spirit of number sense.

2.2 CONSTRUCTIVISM : AN OVERVIEW

Constructivist principles, central to recent reforms in mathematics education, (Fenmnema et al., 1996; Yackel & Cobb, 1996) embrace the development of number sense. The constructivist paradigm is a departure from the behaviourist model popularized by Thorndike in the early part of this century. Behaviourists view learning as a passive process, stressing drill and practice routines, maintaining that pupils respond and conform to outside stimuli. This model was altered slightly to account for varying abilities but remains the accepted learning perspective in many New Zealand classrooms. It is justified by the assertion that learning involves a change in performance, and so changed scores on standard tests constitute evidence of learning.

Constructivism on the other hand :

> *derives from a philosophical position that we human beings have no access to an objective reality, that is, a reality independent of our way of knowing it. Rather, we construct our knowledge of our world from our perceptions and experiences, which are themselves mediated through our previous knowledge. Learning is the process by which human beings adapt to their experiential world.* (Simon 1996, p.2)

According to the constructivist model each child will construct his/her own mathematical knowledge. He/she may well compare it with others and alter or amend it as a result of an encounter. A teacher can never know if a pupil has precisely the same construction of a piece of mathematical knowledge that he/she has.

Von Glaserfeld (cited in Jaworski, 1994. p.15 - 16) has proposed the following two defining principles for constructivism:

> *1. knowledge is not passively received but actively built up by the cognising subject;*
>
> *2. the function of cognition is adaptive and serves the organisation of the experiential world, not the discovery of ontological reality.*

The first principle, recognizing that the learner is responsible for what is learnt, has been accepted since the time of Socrates. The learner actively constructs his or her own meaning in ways that are unique. Clements and Battista (1990) have shown that pupils construct their own mathematical knowledge, building their own interruption and do not receiving *intact* any package of knowledge delivered by a third party.

The second principal, more commonly called the *Radical Constructivist Principle*, implies that there is no separate ontological reality against which an objective comparison can be made. On this basis there is no *real world* to be discovered, rather a person's intelligence organizes its own experiential world, that is, the only way a person knows the world is by experiencing it. This model is in accord with Piaget's biological model of the chronological development of a child's mind.

Obviously radical constructivism is merely a model, there is no objective 'real world', against which comparisons can be made. In this sense constructivism is subject to its fidelity, in a recursive way as Confrey (1995) claims:

> *Thus, (constructivism) is only true to the extent that it is shown useful in allowing us to make sense of our experience.*

Number sense is not knowledge that fits neatly into *packets*, but rather is knowledge that will continuously be added to, amended and expanded as a pupil encounters new mathematical ideas and experiences. In so far as number sense is something that is built, added to and expanded upon with experience, its development would be enhanced by a constructivist-based learning environment, rather than one that functions on the behaviourist principles. As such, number sense cannot be presented as a particular entity, but will evolve and mature as a learner engages in mathematical activity.

2.3 A COGNITIVE PERSPECTIVE ON NUMBER SENSE

Constructivism is a theory of learning and as such does not provide insights into the actual thinking process, that is, how cognition occurs. In this section I review Greeno's (1991) model of cognition that views thinking as knowing one's way around an environment of knowledge. This model of knowing in a conceptual domain developed by Greeno provides insights into how number sense develops and functions at the intellectual level. According to this model a conceptual domain or subject matter domain *is a structure of facts, concepts, principles, procedures and phenomena that provide resources for the cognitive activities of knowing, understanding, and reasoning* (Greeno, 1991, p.174).

In this model the domain, using a physical metaphor, can be thought of:

> *as an environment, with resources at various places in the domain. In this metaphor, knowing the domain is knowing your way around in the environment and knowing how to use its resources as well as being able to find and use those resources for understanding and reasoning. Knowing the domain also includes knowing what resources are in the environment that can be used to support your individual and social activities and the ability to recognize, find, and use those resources productively. Learning the domain, in this view, is analogous to learning to live in an environment: learning your way around, learning what resources are available, and learning how to use those resources* (Greeno, 1991, p.175).

Constructivism differs from the information-processing model, in that knowing a concept is analogous to being able to find and use it *in* the domain rather than recalling isolated and unconnected information to perform calculations.

> *A person's knowledge, however is in his or her ability to find and use the resources, not in having mental versions of maps and instructions as the basis for all reasoning and action* (Greeno, 1991, p.175).

Experience is viewed as a necessary function of learning. Knowing can be compared to reading and understanding the instructions in a recipe book and using them to produce a successful dish. Reasoning involves the construction of *mental models,* corresponding to objects in the situation. Inferences made with mental models are based upon the results of simulations of objects rather than on the outcome of applying rules to expressions or propositions. The model simulates the behaviour of objects that it represents. So reasoning with models that have appropriate constraints is a crucial part of learning the domain. Taking the example of the kitchen again, knowing the location of the implements and the ingredients, the quantities to mix together and the appropriate conditions for combining them should guarantee that whatever is cooked will be edible. The reasoning process is knowing where the kitchen resources are and how they can be put together. This model is flexible enough to allow for experimentation and the construction of new knowledge.

Applying this model to number sense Greeno maintains that:

> *People with number sense know where they are in the environment, which things are nearby, which things are easy to reach from where they are, and how routes can be combined flexibly to reach other places efficiently. They also know how to transform the things in the environment to form other things by combinations, separations, and other operations* (p.185).

A pupil with number sense will know the environment of numbers and quantities, will know where to find conceptual objects that are present and will know the relations that exist between them. For example, consider the computation 35 x 56. A pupil with number sense would construct a mental model containing representations of 35 and 56. Included would be knowledge of appropriate multiplication properties, so that 35 x 56 moves to (5 x 7) x (8 x 9), which becomes (5 x 8) x (7 x 7) = 40 x 49 = (40 x 50) - 40. Knowledge of the domain of multiplication properties offers an *affordance* [1] that makes moving from 35 x 56 to (40 x 50) - 40 a matter of course. The problem drives the performer to conjure up this approach which is mentally easy to calculate.

Students should be taught mathematics so they construct mental models of numbers that highlight their properties so calculations then centre around finding the number operations that allow for easy and flexible calculations. Most of what matters in expert or fluid calculations is learnt without being deliberately taught. The depth of acquisition of number sense is analogous to the ways that driving skills improve with experience, once an initial body of skills has been mastered, or as Greeno would have it, the novice is familiar with the components of learning to drive; experience becomes the guide to gaining competency.

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[1]Affordance, a term coined by J.J. Gibson, 1986, to designate how aspects of a situation contribute to the ease or difficulty of activities.

2.4 A PEDAGOGY FOR LEARNING NUMBER SENSE

The following model of teaching is sympathetic to the constructivist view of active learning. To accommodate this model of learning the teacher has to direct instruction to what is perceived to be occurring in the learner's mind.

Traditional modes of instruction focus on the acquisition of one skill or idea at a time using routine practices to reinforce them. This leads to an accumulation of ideas that may or may not be interconnected. If they are, then it will be through complex networks that are not readily accessible nor automatically available to the learner; here teaching is like training. In a constructivist paradigm learning is more likely to occur by challenging a learner's conceptions using a variety of contexts. Instruction that compels students to examine their views of mathematical concepts and to amend them where necessary, will be more effective than merely supplying ready-made *packets* of knowledge (Simon, 1995).

The challenge to the mathematics teacher is encapsulated by Richards (1991):

> *It is necessary (for the mathematics teacher) to provide a structure and a set of plans that support the development of informed exploration and reflective inquiry without taking that initiative or control away from the student. The teacher must design tasks and projects that stimulate students to ask questions, pose problems, and set goals. Students will not become active learners by accident, but by design, through the use of the plans that we structure to guide exploration and enquiry* (p.38).

Instruction is built on the principle that students will construct their own understanding and will not simply absorb the understanding of the teacher (Simon, 1995). Yackel and Cobb (1993), Nodding (1990) have argued that recognition of the *norms* that exist in a classroom have a direct bearing on how well pupils learn mathematics. In a traditional classroom the expectation is that the teacher is the source of all knowledge and mathematics is viewed as a strict rule-bound discipline where the answer is all important. In a room that functions on constructivist principles there is an atmosphere where reflection on concepts, discussion of alternate approaches and negotiation of meaning is cultivated, nurtured and encouraged. Thus, when Simon teaches the topic of *area* he examines it from a number of viewpoints by considering aspects of area that will cause pupils to question their own understanding. Simon's approach to teaching proceeds by examining

> *the reflective relationship between the teacher's design of activities and consideration of the thinking that students might engage in as they participate in those activities. The consideration of the learning goal, the learning activity, and the thinking and learning in which students might engage make up the hypothetical learning trajectory* (Simon, p.33).

Behind any teaching decision there has to be an ultimate goal and decisions on ways (trajectories) to arrive at the goal, rationalised in terms of contributing to pupils' knowledge. To accommodate the parameters of imparting knowledge Simon (1995) proposes that a *Mathematics Teaching Cycle* be employed - a schematic model of the cyclical interrelationship of aspects of teacher knowledge, thinking, decision making, and activities that assess pupils' knowledge. (Jones et al., (1995), Cobb (1988), Cobb, Yackel & Wood (1992) have also considered the benefits of considering learning as an interactive as well as a constructive activity). This model has a cycle which is completed only after a number of reflections and assessments. Figure 2 below is a representation of this proposed cycle.

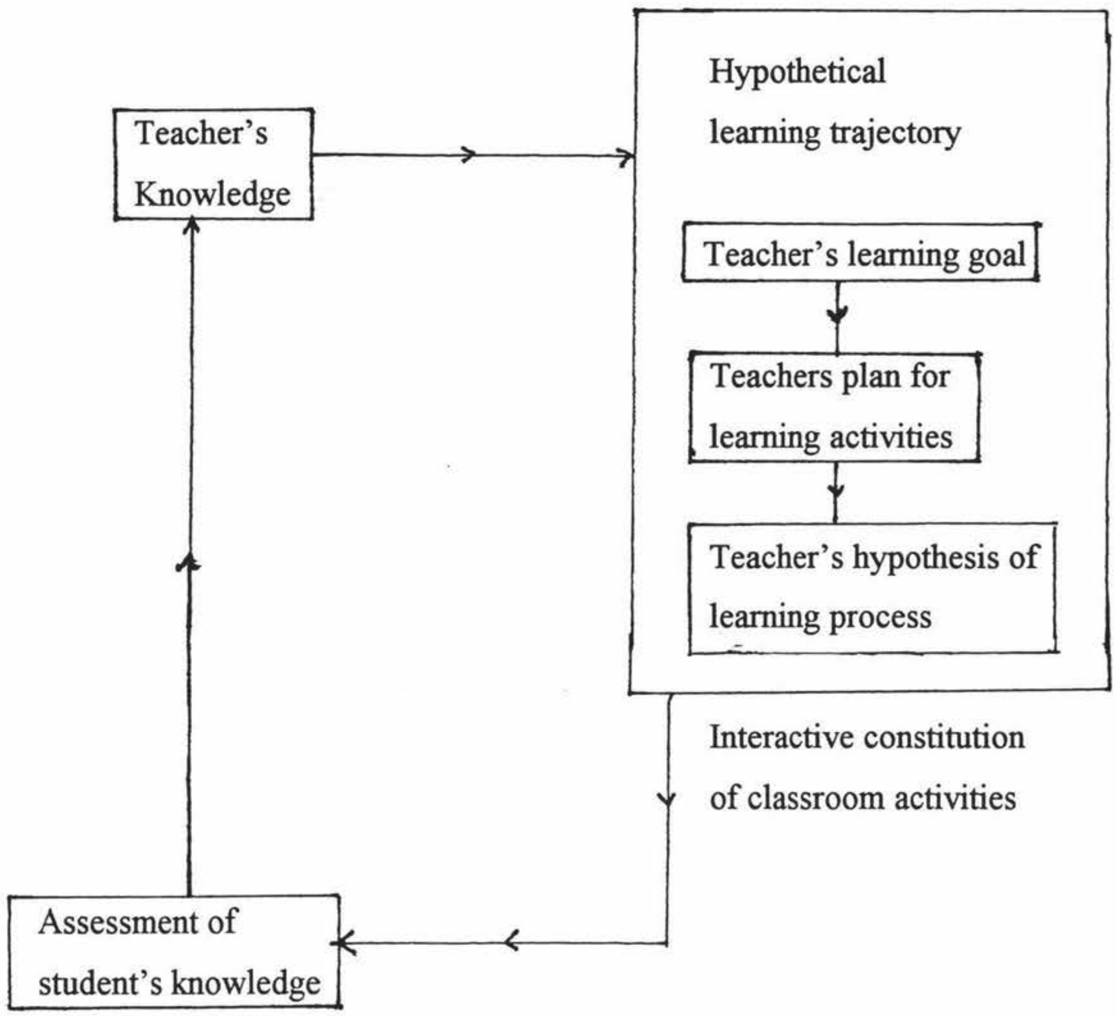


Figure 2. Mathematics teaching cycle

The cycle illustrates the processes by which the teacher makes decisions about the content, design, and sequences of mathematical tasks, by posing problems and tasks and encouraging reflection on the outcomes and if necessary using another cycle (trajectory) to clarify or add to a concept. The objectives of number sense acquisition are more likely to be realized with the use of this model, rather than with the traditional information processing method.

2.5 THE ROLE OF INTUITIVE KNOWLEDGE

No matter which philosophy a teacher brings to mathematics or the models of pedagogy he or she embraces, pupils' intuitive knowledge of mathematics has to be recognised and accommodated. Resnick (1989a) and Gelman and Gallistel (1978) have demonstrated that children as young as three years of age have well developed ideas of quantity, magnitude, number as well as an awareness of numeric properties like equivalence, identity and transitivity. It has been argued that the concept of unit is innate, arising from the association of a singularity with one's body. Plurality comes from a recognition of a particular object being repeated. The concepts of unit and plurality form the basis for the further development of number.

From this perspective a pupil is not an empty vessel to be filled with information, but rather an intelligent being wanting to construct a meaningful mental representation of the world. On this innate knowledge a raft of mathematical knowledge is built, and the more self-evident it becomes the more we say it is intuitive. Intuition is not the primary source of true, certain, cognition but it appears to be so, because this is exactly its role, to create the appearance of certitude, to attach to various interpretations or representations the attributes of intrinsic unquestionable certitude. Without it no behaviour, practise or intellectual endeavour would be possible.

Experience is a fundamental factor in shaping intuition (Fischbein, 1987). There is much theoretical analysis supporting the view that the basic source of intuitive cognition is the experience accumulated in relatively constant conditions. An intuition is a complex cognition structure, organising the available information, however incomplete, into coherent, internally consistent, meaningful representations. Fischbein proposes that intuition expresses the fundamental need of human beings to avoid uncertainty, that is, intuition is vital for any functioning being. We need to feel that we know what to do in normal day-to-day situations. I contend that the thrust of Greeno's model, the cultivation of familiar territory is a recognition of the role that intuition plays when mathematical calculations have to be performed.

The *sense* of number sense implies an attachment of meaning and understanding to the domain of numbers. Inherent implications of the juxtaposition of number and sense implies that the numbers themselves, taken together have themes of meaning, patterns of consistency and a logical structure about them. A sense in the normal use of the term is an intuitive, cognitive component of the rational aspects of thinking. A sense can be altered, developed and amended but it will always remain intuitive rather than precisely definable and knowable.

SUMMARY

Number sense associates a humanist dimension with mathematics. Implying that numbers have an intuitive logical construction. I feel that pupils' mathematics education can benefit from Greeno's model of how thinking occurs, how learning is perceived in a constructivist framework and how teaching can be enhanced using Simon's pedagogy.

Viewing mathematics as a fallible body of knowledge that is learnt by a cyclic process of reflection, negotiation and testing is in keeping with the thrust of number sense. The constructivist approach to learning accommodates this view and allows for theoretical considerations of how knowledge is acquired and put to use. These theories reflect the human condition and allow for everyone to be mathematically creative. Through these models the notion of intuition gives the *substance* that the mind relies upon for engaging in mathematical activity.

Chapter 3 will examine literature on number sense under five headings, ranging from the results of studies of pupils' number sense to an examination of what constitutes number sense, how it develops and how it can be cultivated.

CHAPTER 3

RESEARCH STUDIES INVOLVING NUMBER SENSE

3.1 INTRODUCTION

The research literature concerning number sense primarily originates from America, with some contributions from Australia and England and one New Zealand publication. Research on number sense has tended to focus on examining its existence and development in primary-aged pupils, with little consideration given to secondary school pupils. Early references to number sense are contained in the 1982 Cockroft report *Number Counts*, where the term is used to imply an *at homeness* and familiarity with numbers. Emphasis was placed on examining the properties of numbers and encouraging flexible ways of using the basic operations when performing calculations. The drive behind this approach to mathematics, particularly for the junior school, was the realization that mathematics instruction, based around mastering algorithms had been unsuccessful. Literature promoting and analysing number sense and research findings into how to teach, assess and encourage number sense is discussed in the following four categories:

- Number sense: characteristics and pupil performance
- Instruction promoting number sense
- Relationship between number sense and other mathematics topics
- Complementary studies

3.2 NUMBER SENSE: CHARACTERISTICS AND PUPIL PERFORMANCE

What precisely constitutes number sense is difficult to capture in a concise definition. However, several research-based studies have identified common characteristics and performance indicators of number sense. Sowder (1990a) has used the studies of

Greeno (1989), Markovits (1989), Resnick (1989b) and Tranfton (1989) to compile a list of mathematical abilities that characterise number sense:

- *Ability to compose and decompose numbers; to move flexibly among different representations; to recognize when one representation is more useful than another.*
- *Ability to recognize the relative magnitude of numbers - to be able to compare and order numbers, to appreciate the density property of rational numbers.*
- *Ability to deal with the absolute magnitude of numbers.*
- *Ability to use benchmarks.*
- *Ability to link numeration, operation, and relation symbols in meaningful ways.*
- *Understanding the effect of operations on numbers.*
- *Ability to perform mental computations through invented strategies that take advantage of numerical and operational properties.*
- *Ability to use numbers flexibly to estimate numerical answers to computations, and to recognize when an estimate is appropriate.*
- *A disposition towards making sense of numbers* (Sowder 1990a, page 4, 5, 6).

These nine attributes are grouped together under the three broad headings, **Numeration, Mental Computation** and **Computational Estimation**. McIntosh et al. (1992) has taken these divisions and devised a framework extensively detailing the composition of number sense. See Appendix A. The dimensions of numeration, mental calculation, computational estimation and pupil performance are detailed below.

Numeration

Numeration encompasses place value, cardinality and ordering of numbers. Fuson and Hall (1983), Gelman and Gallistel (1978) detail the extent of informal number knowledge pre-school pupils possess and their ability to use it to solve simple addition and subtraction word problems. This informal knowledge can provide the foundation for the development of further number knowledge (Carpenter & Moser, 1984; Riley, Greeno & Heller, 1983). Place value understanding is built on the part-whole schema which allows children to think of number as being composed of other numbers (Resnick, 1983). Despite its importance children's understanding of place value remains inadequate. A British survey of eleven to sixteen-year-olds' understanding of place value (Brown, 1981) found that 68% of twelve-year-olds and 88% of fifteen-year-olds knew what one more than 6399 was, only 42% of twelve-year-olds and 57% of fifteen-year-olds could numerate four hundred thousand and seventy two, and a mere 22% of the same twelve-year-olds and 43% of the same fifteen-year-olds knew what the 2 in 521,400 stood for. Incomplete place value knowledge continues to create difficulties, particularly in recognizing the value of numerals in decimals (Leonard & Sackur-Grisward, 1985). For example, pupils will say 3.53 is larger than 3.7 because 53 is larger than 7.

Mental Computation

These calculations are performed mentally rather than with pencil and paper or with calculators. An intervention study (Markovits & Sowder, 1988) of primary-aged pupils encouraging the use of non-algorithmic solutions produced a 43% increase in success at mental arithmetic type problems. A follow-up survey six months later showed that the pupils were still using self-made, non-algorithmic methods. A similar study by Hope (1987) also showed an increase by secondary-aged pupils use of properties of numbers and operations in flexible and creative ways for solving mental problems, rather than reliance on taught algorithms. The ability to decompose and recompose numbers is the base skill necessary for flexible mental computation (Greeno 1989, Resnick 1989a, Trafton 1989).

Mental computations, an integral component of number sense, has been found to be the most often used method of calculation by the general populace. However, a survey of 5000 primary-aged pupils from 69 Australian classrooms (Biggs, 1967) found that on average students spent between 0 and 11 minutes per day on mental calculations, that the main object of exercises was speed performance and that as a result, pupils developed number anxiety and little improvement in attainment.

Computational Estimation

From a study of mathematical processes that adults use Reys (1984) found that approximately 80% of them used estimation skills in their every day lives. Estimation involves numerical judgements about the size of objects or the value of the result of operating on numbers. Reys, Rybolt, Bestgen and Wyatt (1982) identified the three key procedural aspects of estimation as **Reformation:** altering the values but maintaining the structure of the problem; **Translation:** altering the structure of the problem; and **Compensation:** adjusting the calculation to produce an estimate that is acceptably close to the exact one. Good estimators need a good knowledge and understanding of place value, arithmetic properties, ability at mental computation, appreciation of relative size of numbers and the ability to use a variety of strategies when calculating (Reys et al., 1982; Rubenstein, 1985; Levin, 1982; Sowder & Markovits, 1994). In contrast, poor estimators are tied to algorithms and lack agility with calculations. Reys et al., (1982) has compiled a comprehensive framework of the dimensions of estimation, included in Appendix B.

The need to be able to make numerical judgements about the magnitude of physical situations has been well argued by Paulos (1988) in his text *Innumeracy*. Familiarity with commonly used benchmarks make numerical judgements of length, weight, area and capacity possible, and enable pupils to apply mathematics to everyday problems in a useful way. In an examination of estimation abilities of secondary school-aged pupils Johnston (1995) demonstrated that pupils were not confident at approximating physical shapes with standard geometric forms, and lacked usable gauges of height, area, walking pace and ambient temperatures. A Threadgill-Sowder (1984) study of primary-aged pupils showed that there was considerable variation in pupils' ability to make estimations concerning attendances at a sports stadium.

McIntosh argues that unless there are well established interconnections between these components, flexible and creative methods for solving problems may not be found and an expectation that numbers are useful, and that mathematics makes sense, may not develop.

3.3 INSTRUCTION AND NUMBER SENSE

Appropriate instruction can have a direct impact on the acquisition of number sense. Instruction in number sense requires an alteration in approach and design of teaching (Resnick, 1989b; Reys, 1994; Rowan & Thompson, 1989; Sowder et al., 1989). Examination of the outcomes from traditional teaching with its emphasis on practice and routines, was questioned as early as 1935 by Brownell. Brownell (1935) claimed that although drill succeeded in producing people who could calculate faster, learners did not necessarily develop conceptual understanding of what they were doing. Willis (1992) conjectured that promotion of an algorithmic approach in teaching provided an automatic and speedy response to a calculation, at the expense of understanding.

McIntosh (1996) questions the reliance on using algorithms to teach mathematics, and cites the following research studies as supporting evidence.

- Pupils rarely use algorithms. Plunkett (1979) examined the methods that eighty eleven-year-olds used to perform four standard operations in the classroom and found that over half of the calculations were by non-standard methods.
- School taught algorithms were used in *school-type* problems but not in real life situations. Carraher et al., (1987) found that with the exception of addition, school taught algorithms did not necessarily give appropriate answers when used in out of school context. Wearne and Hiebert (1988) noted that once the algorithms have been learnt, rightly or wrongly they are difficult to change and tended to limit the calculation options.
- In a study of mathematical processes used by adults Wandt and Brown (1957) found that there was a 3 to 1 differential in the use of mental process over paper and pencil type algorithms, suggesting that school learnt methods stay at school and do not transfer into adult usage.
- Jones (1988) argues that algorithms do not teach a person to think, are context-bound and do not encourage flexible enquiry. Reliance on memory and rules has been shown to be counter-productive. Teaching mathematics in such a manner and not emphasising the sense-making aspect does not produce competent calculators, Silver (1989). Moreover, Resnick (1989b) has found that

teaching by algorithms results in children compartmentalising knowledge and not constructing appropriate interrelationships. In a study of elementary mathematics teachers by Markovits (1989) many not only performed poorly at numerical calculations but were at a loss to explain how the calculations worked. The following extracts are taken from a report on the study indicate the pitfalls of a dependance on algorithmic type learning in mathematics.

> *When asked if the answer to 264 ÷ 0.79 would be more than, equal to, or less than 264, 49% said it would be less, because you divide. But if asked to do the computation, all of them could find the correct answer.*
> *When asked if the answer to 52 x 28 would be more than, exactly, or less than 1500, 38% said it would be exactly 1500.*
> *Only 31% were able to order 0.53, 14/13, 5/12 and 0.993 correctly.*

Although commonly used, the procedural (algorithmic) approach is not effective and is out of step with the intent of good number sense development.

Brown and Palincsar (1989) have considered various useful approach to teaching mathematics and classroom atmosphere/culture that will contribute towards a pupil's mathematical performance. Their findings are summarised in the following extract:

> *Environments that encourage questioning, evaluating, criticizing, and generally worrying knowledge, taking it as an object of thought, are believed to be fruitful breeding grounds for restructuring. Change is more likely when one is required to explain, elaborate, or defend one's position to others, as well as to oneself; striving for an explanation often makes a learner integrate and elaborate knowledge in new ways* (p.395).

Reys (1992) encouraged teachers to direct their teaching towards *sense-making* activities by using open-ended questions. Reys promotes and encourages the use of mental computation and estimation questions as approaches that could be used to achieve this understanding. Greeno (1989) has compiled evidence of people using number sense to perform calculations in a school situation. For example, a flexible approach to a mental calculation like 25 x 48 would be 100/4 x 48 to 100 x 48/4 to

100 x 12. Greeno and McIntosh et al. (1997) argue that number sense can be developed and nurtured by using estimation and mental computation techniques in the classroom. At the same time Greeno cautioned against teaching them as separate topics, and argues for a conceptual approach rather than a procedural approach with instruction.

Doyle (1983) characterised traditional instruction as *direct* – pupils are lead through explicit, systematic lessons leading to the mastery of content. Direct instruction produces to two types of expertise which Hatano (1988) terms *routine* and *adaptive*. Routine expertise is demonstrated when familiar problems are quickly and accurately solved while adaptive expertise is where rich conceptual knowledge enables novel or unusual problems to be solved by inventing new procedures. The adaptive procedures arise from instruction that focuses on discovering number relations and encouraging the production of one's own algorithms for solving mathematical problems, an aim of number sense instruction.

An investigation by Helme, Clarke and Kessel (1995) found that meaningful learning occurs in a classroom less often than teachers expected, and when it does occur it is more likely to be as a result of instruction that uses outside class experiences as a focussing mechanism. These findings complement and support longitudinal investigation into instruction by Fennema et al. (1996), which argues that teachers must constantly monitor pupils' knowledge construction.

Whitin (1989) encouraged teachers to create mathematics classrooms where pupils ask "*why*" rather than simply take properties and concepts as given. Teachers were encouraged to follow a pupil's reasoning, encouraging pupils to construct meaning with appropriate questioning, to use concrete examples in their explanations and to expect pupils to be able to justify processes used. For example, an examination of why the sum of any even number of odd numbers has to be even is consistent with the way that number sense can be developed.

Sowder (1992b) proposes that number sense is best learnt using a constructivist learning environment where learners are viewed as becoming acculturated members of a mathematics community. Similarly, Lampert (1990) proposes that to teach number sense a class needs to become - *a community of discourse engaged in development of a common culture of ideas about mathematical topics* (p. 55). Lampert argues that individuals learn by actively constructing their own knowledge and that this is a social and public process when it takes place in a school. Constructivism is the model that puts the most emphasis on understanding in the conceptual domain and so is most successful for instruction in number sense (Gadanidis 1994).

The importance of creating a culture of mathematics discourse or a mathematical environment rather than by concentrating on direct instruction is emphasised by Reys (1992), Resnick (1989), Sowder (1992b) and Willis (1992). They maintain that direct instruction in number sense is counter-productive and that it is better to view its development as a by-product of instruction that promotes meaning and understanding of mathematics symbols, concepts and operations. Specifically, Sowder (1992a) lists four instruction focuses that may promote number sense:

- Sense-making is emphasised in all aspects of instruction.
- A climate of sense-making pervades the room so that all participants are relaxed and forthcoming about exposing their thinking.
- Mathematics is viewed as socially constructed, and learning involves more than just acquiring skills and information. It means an intellectual practice is taking place.
- Instruction operates so that pupils are at the edge of their understanding - at what Vygotsky call *the zone of proximal development*. To do this pupils must be interacting with their peers under the guidance of an expert adult.

Markovits and Sowder (1994) instructed a group of eleven and twelve-year-olds on numeration (place value, magnitude of numbers, ordering and density of rationals), mental computation and estimation. Pupils were encouraged to consider alternate approaches to performing mental computation (decomposing and recomposing numerals, possible uses of the distributive principle) and estimation (rounding, compensating and using benchmarks). Results of the pre- and post-instruction assessments clearly demonstrated that pupils were able to produce creative and original methods for solving problems and a six month follow-up review showed that they were still able to invent useful algorithms for solving problems.

Recent publication of ideas and approaches to teaching number sense have been numerous (eg. Howden, 1989; Reys, 1992 & 1994). The recommendation is that teachers be more creative and investigative in their approaches to teaching mathematics and include the following instructional suggestions:

- Look for patterns in numbers. Examine interesting sequences, for example possible distribution of prime numbers throughout the counting numbers, or examine the comparative pictorial representations of even and odd numbers
- For each day of a week construct ways of writing the date - for example, the number 13 = 10 + 3 or 13 = (5 x 2) + 3 or 13 = 21 - 8 etc.
- Look for alternate ways of rewriting addition problems. For example: 253 + 421 = 200 + 50 + 400 + 20 + 4.
- Use open ended sentences. For example - "Find a numbers that when squared gives the same result as when added to itself?"
- Display a number line and ask pupils to place fractions on it.
- Use calculators to examine the effect of multiplying by numbers less than one.
- Use concrete problems that require reflection. For example, a bath has two outlets. One empties it in 10 minutes the other in 4 minutes, discuss possible timings for when both outlets are used.

The issue of context is important in the development of number sense. Hope (1989) argues that all instruction on number sense should involve calculations that have a relevance to the real world and that results should be interrupted with real possibilities in mind, for example, the result of 1855 ÷12 is 154.5833, but would be interpreted as 154 if this divider had been a baker or as $154.58 if this was to be one's share of a lottery prize. Reys (1992) urges teachers to use instruction that compels students to generate their own solutions, rather than rely on those supplied by the teacher.

Although there is general agreement on instruction focuses that promote development of number sense, evaluation procedures are problematic. Reys (1992) claims that paper and pencil tests are inappropriate as number sense is a qualitative process. Resnick (1989b) proposes that traditional forms of assessment are founded on two possibilities, one that mathematics is mastered by learning its parts and the other that, once a fact is known, like $100 \div 4 = 25$, then the fact is abstracted to all such possibilities, for example, four quarters make a dollar or $4 \times 0.25 = 1$. While all of these abstractions may not be used consciously, it means that potentially all mathematical knowledge can be known. Neither of these assessment methods allow for different ways in which the components of mathematics can be used, or combined to produce personal and interesting new constructions in specific situations.

The constructivist model of learning provides the most meaningful overview for planning instruction on number sense. Meaningful evaluation procedures are yet to be agreed upon and await further development.

3.4 RESEARCH ON NUMBER SENSE COMPONENTS

This section will review literature which examins the development of number sense through aspects of mathematics education such as problem solving, mental computation, number knowledge and estimation.

Number Knowledge

A study of the use of numbers in magazines by Joram, Resnick and Gabriele (1995) showed that adult magazines used rational numbers in many settings, particularly percentages in interconnected relationships involving increase/decrease and part/whole situations. Teenage literature use rational numbers as well, though not to the same degree of complexity. Many authors have argued that meaningful instruction will occur when mathematical concepts are taught in context.

Mental Computation

Mental computation is nurtured by encouraging pupils to reflect on the problems and to invent their own strategies for solving them (Sowder 1990b). Plunkett (1979) has described mental algorithms as procedures having the following properties:

- They are variable. For example, 7 x 28 = 7 x 7 x 4 or 49 x 4 or 7^2 x 2^2. As such these algorithms are fleeting, mostly invented on the spot.
- They are flexible and can be adapted to suit the numbers involved. For example, the procedures to calculate 83 - 76 is different to the one for 83 - 47, which is different again for the one used with 83 - 9.
- They are active in the sense that the user makes a definite, if not always conscious, choice of methods and is in control of his/her own calculations.
- They are holistic, in that they work with complete numbers rather than separate tens and units digits. For example, 4 x 35 = 2 x 70 = 140.
- They are frequently conservative and predictable, working from one part of the question towards the answer. For example, 37 + 28 goes 37 then 47 then 57 then 67 then 65 by adding in 10s then removing 2.
- They require understanding throughout the calculation. No matter what procedure is used pupils will only produce the correct answer if his/her number and operational knowledge is sound.
- Often students give an early approximation to the correct answer. This is usually because a leftmost digit is calculated first, but in the context of complete numbers. For example, 145 + 37 = 175, then 182. Or 34 x 4 = 120 then 136.

Flavell, Friederichs and Hoyt (1970) have evidence that pupils' invented strategies enhance their understanding, while Lindquest (1989) has demonstrated that it is possible to teach pupils techniques for inventing mental strategies. Sowder (1992), Sowder and Kelin (1993) promote number sense by using instruction on mental computation which discourages paper and pencil algorithms and promoting self made solutions.

Estimation

Estimation is best performed by those students with well-developed quantitative intuition and a feel for quantities represented by numbers (Sowder, 1992). Good estimators have a good grasp of basic number facts, place value, and arithmetic operations, are skilled at mental computation, are self-confident, are tolerant of error, and flexible in their use of strategies (Sowder & Kelin, 1993).

Reys, Rybolt, Bestgen and Wyatt (1982) identified the strategies of reformation, translation and compensation employed to perform estimation calculations. Sowder and Wheeler (1989) have added further components necessary for successful estimation, mainly, conceptual components, affective components and related concepts and skills such as mental computation skills, place value knowledge, ability to work with powers of ten and knowledge of properties of operations.

Measurement estimation is a process of arriving at a measurement or a measure without the aid of a measuring tools. Although it is a mental process, there are often visual or manipulative components (Bright 1979). Measurement estimation often uses benchmarks, knowledge that a pupil directly learns by memorising standards, like the height of a door is two metres, or the capacity of Carisbrook stadium is forty thousand or Mount Taranaki is roughly conical and is two thousand five hundred metres high. Not surprisingly, adults outperform pupils on measurement estimation problems (Swan & Jones, 1980). Hildreth (1983) has developed strategies for teaching length and area estimates using comparisons of known quantities with the unknown ones, and Markovits (1987) has shown that instruction using these strategies is effective.

Problem Solving

Dougherty and Crites (1989) claim that number sense helps pupils with problem solving by establishing the magnitude of the anticipated answer and identifying integral aspects of the numbers involved. Sowder (1988) identified knowledge of the magnitude of numbers as an important component in making the right decisions about how to proceed in problem solving situations and being able to judge the reasonableness of an answer.

Problem solving, estimation and mental calculations all require well developed number sense if they are to be performed effortlessly and efficiently. McIntosh (1997) claims that instruction on estimation and mental calculations is the most effective way of promoting a pupil's number sense.

3.5 COMPLEMENTARY STUDIES

This review concentrates on research related to mathematics education which impacts on the development of number sense. The findings and recommendations influence the procedures used to teach number sense and reinforce its usefulness and importance.

Teacher/Pupil Beliefs about Learning Mathematics

There is a consensus that a pupil's mathematical knowledge is related to a teacher's understanding of how a pupil learns mathematics. A number of researchers have examined the effect of teachers' increased understanding of their students' thinking (Cobb, Yackel & Wood, 1993; Fennema & Franke, 1992 Simon & Schifter, 1991). Such studies provide evidence that consideration of students' thinking motivated instructional changes by the teachers, which in turn lead to an improvement in pupil performance. A long-term study of an active programme to identify and use pupils' mathematical thinking by primary teachers, so as to improve instruction has shown success (Fennema et al., 1996) and resulted in more effective development of pupils' conceptual knowledge.

Schoenfeld (1989) examined the mathematical beliefs and expectations of 230 fifteen and sixteen year old pupils. From the survey the following pupil beliefs emerged:

- *Good teaching is where many examples are used.*
- *Mathematics is a rule-bound subject.*
- *Pupils believe in natural ability as opposed to cultivated ability.*
- *Mathematics is a creative subject.*
- *Mathematics is a valuable subject (especially for career prospects).*

Schoenfield noted that those pupils who were successful at mathematics did not view the subject as rule-bound or memory-dependent, whereas those with mediocre ability tended to associate success with luck. Pupils who viewed mathematics as a conceptual challenge were more successful than those who looked for procedural paths to solving problems. Number sense flourishes when conceptual approaches to teaching mathematics are allowed to dominate the procedural ones.

Cross-cultural Comparative Studies

A comparison of mental computation performance by Australian, Japanese and American pupils aged from seven to fourteen (McIntosh, Nohda, B. Reys & R. Reys, 1995), indicated early differences in performance which evened out in later years. Visually presented problems were solved using standard algorithms, or variations on them, whereas the orally presented problems tended to be solved by a variety of solution strategies. The indications were that oral questioning leads to the formation of much deeper conceptual understanding than occurs when the same material is presented visually. The suggestion is that instruction centred on pupil-generated solution methods are more beneficial than those that rely on algorithmic-based methods.

Stevenson, Lee and Stigler (1986) examined the mathematical achievement of Japanese, Chinese and American pupils from kindergarten to the end of primary school. A considerable difference in performance, mainly in favour of the Japanese and Chinese, was attributed to the differences in instruction design, content and parent/pupil expectations. Contrasts in instruction delivery is significant; the Japanese and Chinese teachers not only spent a greater period of time on content instruction, but also used far less exemplar problems. Moreover, the examples are approached from a number of perspectives. This contrasts with the American approach where only one or two approaches are used with many examples. Stevenson et al. concluded that the Japanese and Chinese favour conceptual development while the Americans favour procedural. The conceptual approach requires an in-depth understanding of number operations and number knowledge that is the essence of number sense.

Informal Developing Mathematical Knowledge

Resnick (1989a) compiled findings on pre-school children's mathematical knowledge, detailing numerosity judgements that six-month-old children can make (Starkey, Spelke & Gelman, 1989) and the proto-quantitative schema that children use for making numerical judgements on size. The implicit knowledge that three and four-year-olds have which allows for the development of counting has been examined by Gelman and Gallistel (1978), who argue that teachers should use this knowledge when instructing on numbers. Pre-schoolers have informal strategies for performing addition and subtraction, which Resnick suggests are not sufficiently utilised as the basis for instructing pupils on elementary mathematical operations.This could help explain their later poor performances at mathematics tests and exams.

SUMMARY

This chapter has focused on the characteristics of number sense, pupils' performance at and knowledge of these characteristics and the instructional design - both content and affective factors - that educators need to be conscious of, if instruction is to be successful. Teaching that focuses on developing conceptual knowledge has been shown to foster richer understanding of mathematics. There is considerable evidence that teaching that emphasises sense making, examines the properties of numbers and the operations that can be performed on them, promotes number sense and engenders a positive attitude towards mathematics.

CHAPTER 4

RESEARCH DESIGN

4.1 INTRODUCTION

This study was designed to examine number sense and evidence of its development in a cross-section of secondary-aged pupils, and also to gain insights into how pupils perform numerical calculations and how they know *what to do* and *how to* proceed in a mathematical situation. An examination of attitudinal dispositions towards mathematics studies and what influence these have on the development of pupils' number sense and mathematics education.

To examine the development of number sense in a range of pupils from two secondary schools a questionnaire was produced, this being the most efficient method of producing enough data (quantitative) from which an objective judgement could be made. To gain insights into pupils' thought processes when solving problems, interviews of selected pupils were conducted. The qualitative data produced allowed for an evaluation of the effectiveness of mathematics education programmes. The use of a mixture of data types in an enquiry rests on the premise that the weakness in each single method will be compensated by the counterbalancing strengths of the other. Cronbash (1982) argues that the two research paradigms are complementary, reinforced by Campbell's (1979) observation that *quantitative knowledge has to trust and build upon the qualitative to produce an applied epistemology.* The mix of empirical data and theoretical knowledge produces a more rounded analysis by allowing subjective observations to be compared with objective ones to provide a more complete view of an enquiry (Todd, 1979). A combination of qualitative and quantitative data was successfully used by Koloto (1995) in a study of estimation in Tongan schools. The two approaches should produce a more holistic and conceptual portrayal of number sense development.

The constructivist approach to learning has been championed throughout this study as the approach most likely to produce a comprehensive nurturing of number sense. The interview protocol used with selected students provided an opportunity to tap into what knowledge a pupil has, how it has been constructed and how it is used (Nodding, 1990). Carr and Kemmis (1986) note that:

> *the aim of inquiry is understanding and reconstruction of the constructions that people (including the inquirer) initially hold, aiming towards consensus but still open to new interpretation as information and sophistication improve. The criterion for progress is that over time, everyone formulates more informed and sophisticated constructions and becomes more aware of the content and meaning of competing constructions. The inquirer is cast in the role of participant and facilitator.*

The intent is that by considering the various constructions that pupils place on information, some consensus can be reached about optimum conditions and approaches to knowledge construction that could create sound and useful number sense.

In the remainder of this chapter a brief overview of qualitative and quantitative research methods is presented and the rationale for using a questionnaire and interviews is discussed. The subject setting, the ethical considerations, the design of the questionnaire and the interview tasks are detailed. The mechanics of the research are given and finally how data was analysed along with considerations of limitations of the study are discussed.

4.2 QUANTITATIVE AND QUALITATIVE RESEARCH

Quantitative research is a traditional approach (Crowl, 1993), characterised by findings that are expressed in a numerical form and which can be analysed statistically to test the validity or applicability of a theory. This form of research is designed so that any other researcher following the same design should produce the same or very similar results. The approach has been used in many studies of mathematical abilities

(Carpenter et al., 1981; McIntosh et al., 1997; etc). In order to make judgements about the number sense exhibited by secondary school pupils and whether there is any noticeable increase in the dimension of this sense across the age levels, a questionnaire was designed, completed by selected pupils and analysed, producing considerable empirical quantitative data.

Reliance on empirical data, although objective and useful for making generalizations does not examine the working of the mind, only the end product of an enquiry. Qualitative research has evolved from anthropologists wanting to gain an ethnographic understanding of their subjects (Guba & Lincoln, 1994). The goal of qualitative research, as Schofield (1993) notes, is not to produce a set of results that any other researcher in the same situation would replicate, rather it is to produce a coherent and illuminating description of and perspective on a situation. Burns (1990) notes that qualitative investigation is:

> *based upon a recognition of the importance of the subjective, experiential life world of the human being....*
>
> *that it provides avenues that can lead to the discovery of these deeper levels of meaning....*
>
> *directed at understanding events from the viewpoint of the participants.*

Since part of this study is about gaining insights into what number and operation knowledge pupils use when performing numerical calculations, a qualitative approach using interviews was chosen.

Questionnaire

Questionnaires have long been used in educational research (Anderson, 1990), mathematics education research lends itself to their use. According to Wolf (1997) a well made questionnaire is highly deceptive. It appears to be well organized, the questions are well-drawn and exhaustive, and there is a natural ordering or flow to the questions that keeps the respondent moving forward towards completion. Number sense questionnaires have been used successfully by a number of researchers, Hope, 1989; Markovits et al., 1989; McIntosh et al., 1997; Sowder, 1992a.

Interviews

According to Keats (1997) clinical interviews, exploring a pupil's reasoning, are dynamic and allow for the interviewer and the respondent to react to the enquiry. The benefits of using the interview protocol are well summed up by Huinker (1993, p.80):

> *Advantages of using interviews include the opportunity to delve deeply into students' thinking and reasoning, to better determine their levels of understanding, to diagnose misconceptions and missing connections, and to assess their verbal ability to communicate mathematical knowledge.*

Kitwood (cited in Cohen & Manion, 1977) claims that if the interviewer does his/her job well and if the respondent is sincere and well motivated then accurate data will be obtained. Ginsberg et al. (1983), Fontana and Frey (1994) have detailed the requirements for conducting structured interviews and have cautioned that the interviewer must be aware of respondent differences and must be flexible enough to make adjustments for unanticipated developments. The interview protocol has been used successfully in examining pupils' estimation strategies (Reys et al., 1984; Threadgrill-Sowder, 1984) and to delve into the processes that a pupil uses when performing mental calculations (Hope, 1987). This study used a structured approach with the same set of questions for each pupil, but deviations into a pupil's reasoning paths were allowed for, that is, there was a realization that if one was to gain insights into how a pupil has constructed knowledge, and the influences behind his/her understanding of a concept, then the interviewer will have to deviate from the script at times.

4.3 SUBJECTS AND SETTING

It was the intention of the research to examine number sense across a range of secondary-aged pupils. Specifically, the focus was pupils whose mathematical knowledge was still at a developmental stage, pupils who were still acquiring knowledge and understanding, rather than with those who had already begun to specialize in mathematics. For the questionnaire 179 pupils comprising third, fourth, fifth and sixth formers from two schools were selected and apart from the sixth formers the pupils were of mixed ability. The sixth formers were graded in that they had all passed their school certificate exam so were considered to have displayed a recognized level mathematical attainment.

The two schools, which service a middle to lower socio-economic cross-section, are single-sex and achieve average pass rates in School Certificate and Bursary examinations. Both schools enjoy considerable public support and have had climbing rolls for a number of years. The mathematics staff at the two institutions are well qualified and highly motivated.

4.4 ETHICAL CONSIDERATIONS

All 179 respondents to the questionnaire had been informed by letter of the nature of the study, that their participation was voluntary and that they were able to terminate their involvement at any time. Volunteers for the interviews communicated their consent in writing. Interviewees' parents were provided with research information and in turn produced written consent to allow their child to be part interviewed. Both school principals were consulted about the study and their permission for it to proceed was received.

The 11 pupils who agreed in writing to be interviewed comprised 3 third formers, 3 fourth formers, 2 fifth formers and 3 sixth formers. The interviewees were guaranteed complete confidentiality and assured that no direct reference to them would be made in the report. A similar undertaking was given to the principals as regards the identify of their schools.

4.5 THE QUESTIONNAIRE

The questionnaire, included in Appendix C, was made up of 61 items, 45 examined numerical computations skills and 16 related to attitudinal investigations. The numerical computation questions examined number knowledge, knowledge of operations on numbers and numerical computation abilities. These three divisions form the basis of number sense. Details of what questions relate to which aspect of number sense are detailed in Figures 3, 4 and 5 below.

Number Knowledge

		Question
	place value	30, 31, 32, 33
Orderliness of Numbers	ordering numbers	22, 23, 25, 28, 37, 41
	equivalent numerical forms	21
Representation of numbers	decomposition/recomposition	43
Number Magnitude	sense of size	27, 38, 39, 40, 45
Applying Benchmarks		26, 42

Fig. 3: Components of number knowledge with associated questions

Operations on Numbers

		Question
	use of appropriate operation.	29
Effect of Operations.	multiplication by less than one	24
Properties of Operations	distributive principle	34

Fig. 4: Components of number operation knowledge with associated questions

Numerical Computations

		Question
	ability to approximate	12, 16, 17, 18
Computational Estimation	use of operations	13, 14, 15
	use of benchmarks	19, 20
Mental Computation	use of number properties	1, 2, 3, 4, 5, 6, 8, 9, 10, 11
	use of multiple strategies	7
Computational Deductions	understanding	35, 36, 44

Fig. 5: Numerical computation attributes with associated questions

In designing the questionnaire previous studies were consulted for questions that would mirror the aspects of number sense that were to be tested. In Figure 6 below acknowledgement of sources used and number sense component examined are detailed.

Number Sense component tested	Source
Estimation	Threadgill-Sowder 1984, Sowder & Wheeler 1989
Place value	Sowder 1992a
Relative size of numbers	Sowder 1992b, Markovits et al., 1989
Number types	Markovits et al., 1989
Operation on numbers	Hope 1989, Markovits et al., 1989
Attitudinal responses	Schoenfeld 1985

Fig. 6: Source for number sense questions

The attitudinal questions in the last section of the questionnaire addressed pupils' disposition towards making sense of their mathematics education as well as their perception of the importance and relevance of mathematics. In Figure 7 below questions and the attitudinal aspect addressed are listed.

	Question
Disposition towards the subject	46, 47, 48, 49
Perception of the subject	50, 51, 52, 54, 57
Sense making attributes	53, 55, 56
Evaluation of personal ability	58, 59, 60, 61

Fig. 7: Composition of the attitudinal questions

THE INTERVIEWS

The interview questions served two main purposes. One was to enquire into the foundations of pupils' mathematics knowledge, how they have constructed this knowledge, and how they use it to solve problems. The second was to examine pupils' consciousness of the sense-making aspect of mathematical calculations and how an appreciation of this affected their disposition towards studying the subject. By combining these two goals an attempt was made to gauge what threshold of mathematics knowledge a pupil requires so that responses to calculations seem intuitive. The interview questions that were given to each of the 11 interviewees are listed in Appendix D.

The interview protocol was made up of 15 questions. Questions 1 to 9 plus 13 and 14 examined the pupil's procedural and conceptual knowledge, Questions 10, 11 and 12 were attitudinal enquiries. The responses to these three questions are to be compared with those from the attitudinal section of the questionnaire (Questions 48, 49 and 60). Questions 15 examines pupils' basic number manipulation knowledge, how they developed this knowledge and how it influences their confidence to tackle numerical calculations. The interviews are an opportunity to get inside a pupil's head and make judgements about how familiarity with numbers and number operations equip a pupil to performing in a mathematical situation.

The interviews provided an insight into pupils' intuitive mathematical knowledge and the role that it plays in solving problems. In the attitudinal section the enquiries were directed at investigating how this intuitive knowledge had been built up.

PILOT STUDY

A pilot questionnaire containing 69 questions examining the various aspects of number sense was trialed by two pupils. The ambiguous nature of some questions and the length of time that the calculations took suggested that some modification was necessary. The questionnaire was revised by removing 24 numerical calculations and adding in 16 attitudinal ones. The ratio of components of number sense examined was not altered.

The interview questions were given a student who had recently left school, and her responses to the questions and comments concerning the degree of difficulty of the questions, lead to alterations to two of the computation questions. The attitudinal questions appeared appropriate and answerable.

4.6 QUESTIONNAIRE - FORMAT AND PROCEDURE

Reys (1984) demonstrated that restricting pupils to a fixed, short time for answering questions is more likely to produce a response that is in keeping with the intent of examining procedural and factual knowledge. For this reason, and from a practical point of view, pupils were given 8 to 10 seconds in which to answer each question. They were directed to the next question once time was up. The questions were read to the pupils twice. The mental computation questions were read out aloud and displayed on a board. The rationale for reading out the computation questions was to eliminate confusion. The researcher was also able to expand each question a little, where necessary, so that pupils would be clear about its intent. The attitudinal questions were not read aloud, rather pupils were given a fixed time of ten minutes in which to complete them. The attitudinal questions are of a personal nature and it was thought to be more effective to allow the pupils to reflect on their responses in quiet, without the continual and distracting pressure of being moved to the next question. Answers were written in the space provided on the questionnaire, and pupils were encouraged to put down all calculations on the answer sheets provided. Calculators were not permitted.

The questionnaires were given to pupils in their normal class at the start of a period, and on most occasions the teacher left the room. On average, it took between 18 and 20 minutes to complete the computational task and between 8 and 10 minutes to work through the attitudinal section. Pupils were reminded at the start of the exercise that the questionnaire was not a test and that the results would not be made available for school assessment purposes nor disclosed to their teachers. Once each class had completed the questionnaire, they were thanked for their co-operation and efforts. Pupils expressed no negative comments about the content of the questionnaire, or about having to take part in the exercise, though some third formers remarked that they found it difficult to complete the mental and estimation questions in the time allowed.

4.7 INTERVIEW - FORMAT AND PROCEDURE

As far as was possible pupils with a range of mathematical abilities were approached to take part in the interviews. Mutually convenient times were agreed to and the interviews were recorded on a tape in a quiet room. The objectives of the interview were explained to the pupil, it was stressed that this was not a test rather an enquiry about their mathematical knowledge, attitudes to mathematics and the methods that they use when performing numerical calculations. A certain amount of informal discussion took place to relax the pupils before the interviews were started. During the tasks each pupil was asked to talk out aloud their method of answering the questions. Pupils were cautioned that this was foreign to their normal mode of problem solving and that the interviewer would encourage them to *speak their thoughts* if bouts of silence occurred while they were working problems through. It was stressed that it was more important to know how they were performing the calculation than what the actual answer was. Most of the interviews were completed in around about 30 minutes.

Each pupil was given a copy of the questions and an answer sheet. The questions were read to the pupils, and they were instructed not only to write down their answer, but also to talk out aloud how the result was calculated. Reactions to alternate strategies were noted, particularly any acknowledgement by the pupil that the strategies were mathematically more appropriate or helped make the problem easier to calculate.

Similarly, the attitudinal section of the interview involved considerable probing as to how pupils' attitudes had developed and what had influenced them. The questions were designed to get pupils to articulate their thoughts, impressions and attitudes related to their mathematics education. At the completion of the interview each pupils was thanked for their efforts and praised for their honesty. Most of the responses to questions were frank, and pupils were forthright about their attitudes and expectations of mathematics.

4.8 DATA ANALYSIS

Questionnaires

The questionnaire booklets were analysed in class lots and the results summarized on a single sheet. Questions were marked either right (one mark) or wrong (zero mark). Those containing more than one part are rewarded either the one mark or none depending upon whether all the parts to the question were correct or not (Questions 38, 41, 42 and 43). The estimation answers were graded on a scale used by Leven (1982): full marks were awarded, provided the result was within ten percent of the actual answer. On each scoring sheet consistent errors were recorded as for Question 9 and popular ranges of answers as occurred with Question 43. Replies to the attitudinal questions 46 to 61 were recorded as a percentage of pupils from each class who had opted for each of the four possible option responses. The results from the nine score sheets were summarised on a single sheet so that comparisons about number sense development between the class levels could be made.

The results have been used for comparisons, not for making judgements about actual levels of number sense. Graphs have been used to give a pictorial representation of pupils' achievement by class lots. The four graphs, one each for number knowledge, operation knowledge, numerical computation ability and overall achievement at number sense have been used to highlight differences in performance by each of the class levels. The results from the questionnaire survey are presented in Chapter 5.

Interviews

The primary aim of Questions 1 to 9, 13 and 14 was an in-depth analysis of pupils' mathematical knowledge and the procedures they use for solving numerical calculations. The interest was in how the answer was calculated. The ease with which pupils solved problems and their reaction to alternate solution paths were noted. For the remainder, Questions 10 to 15, excluding 13 and 14, the enquiry was mainly centred around discovering how attitudes towards mathematics had developed and what was the extent of familiarity with basic numerical and operational facts. Below is an overview of how the questions were analysed.

Computation Questions

- For Questions 1 to 5 a pupil's use of algorithmic or non-algorithmic processes to solve problems and comments on ease of calculation and receptiveness to the use of alternate procedures were recorded.
- For Questions 6 to 9 pupils' number knowledge, understanding of effect of operations and their ability to reason mathematically have been recorded along with observations on the ease with which pupils performed calculation and their reactions to the appropriateness or usefulness of other suggested solution methods.
- Questions 13 and 14 explore the foundations of pupils intuitive knowledge.

Attitudinal tasks

- Questions 10 to 12 explored pupils' attitudes and dispositions to mathematics.
- Question 15 delves into pupils' perceptions of their personal mathematics ability and that of their parents. If pupils viewed their ability as being average

or below average then further questions were directed at determining what fundamental components of number sense they were lacking and how this had occurred. An opportunity for pupils to relate anecdotal experiences about their mathematics education were offered at the end of task 15.

Analysis of the eleven interviews are presented in Chapter Five.

4.9 LIMITATIONS

As the two schools involved in the research sit side by side and serve the same clientele, extrapolation of the results to the general population of secondary-aged school pupils may not be completely valid. However, considering the performance of the pupils from both schools in national exams I think that those who took part in this study are representative of mathematically average ability New Zealand secondary school pupils.

Getting pupils to explain how they ended up with incomplete or wrong mathematics information can be a little touchy and awkward. Pupil's attitudes to and understandings of mathematics are often related to their primary or intermediate schooling and judgements made about lessons during these early schooling years are often very superficial and simplistic. All enquiries concerning what had been taught to a pupil at intermediate or primary school were kept to content only, and pupils' recollections on the staff who taught them were not solicited.

SUMMARY

This chapter has focused on the processes and procedures that were used for carrying out this study. A mixture of quantitative and qualitative data was considered the most appropriate way to arrive at an overall consideration of number sense in a cross-section of secondary school pupils. The questionnaire gives an overview of the extent of number sense and also allows judgements about its development over the secondary class years to be made. The interviews allowed for a more detailed enquiry into pupils' number sense and the factors that had influenced its development.

The questionnaire was largely put together from tasks that had been used in others studies of number sense. A pilot study resulted in some minor changes to the questionnaire and the interview tasks. The interview offered the opportunity to construct questions that would help illuminate the processes that pupils use to solve problems and what influences have shaped their disposition towards the subject. Use of the interview protocol was validated on the basis that this study approaches pupils' learning from a constructivist paradigm, so to find out what is happening at the cognitive level a selected number of pupils were interviewed.

The mechanics of how the study was conducted and how the compiled data was analysed have been covered. It was noted that all involved with the study were most supportive. Pupils and staff were willing to help with the investigation and gave freely of their time. The pupils were co-operative with the questionnaire, and the interviewees were very open and honest with their replies. Two of the interviewees were conscious of their lack of mathematics ability and it is to their credit that they did not allow this to colour their willingness to partake. The next chapter contains the results from the questionnaires and the interviews along with discussion and comment.

CHAPTER 5

RESULTS AND DISCUSSION

5.1 INTRODUCTION

In this study data on pupils' number sense and attitudinal disposition was gathered from two sources, a questionnaire completed by 179 secondary school pupils and interviews of 11 of the questionnaire respondents. The questionnaire results are divided into two sections - Pupils' Number Sense (5.2) and Attitudinal Responses (5.3). The Interview Results are also considered under two headings, Computational Knowledge (5.4), and Pupils' Attitudinal Development (5.5).

QUESTIONNAIRE DATA

5.2 PUPILS' NUMBER SENSE

The results are grouped under the three main components of number sense: **Number Knowledge, Operation Knowledge** and **Numeric Computation Performance.**

Number Knowledge

The aspects of number knowledge reported on in this section include sense of size; ordering numbers; knowledge of equivalent forms; use of benchmarks; and understanding of place value.

Sense of Size

These questions examined pupils' appreciation of the relative sizes of numbers. Specifically, Questions 27 and 38 focus on comparing the sizes of numbers, Questions 39 and 40 consider the size of a number presented in word form and Question 45 involves comparing the relative sizes of three numbers. The percentage correct by each class level is given in Table 1 below. Across the form levels there is an increase in pupil ability at making correct judgements about the relative sizes of numbers. This skill is taught to pupils from an early age and appears to be well developed.

Question	Form 3 (n=36)	Form 4 (n=42)	Form 5 (n=61)	Form 6 (n=40)
27	72	58	80	95
38	70	59	78	92
39	61	40	54	80
40	78	69	82	98
45	50	60	77	93
Average	**66**	**57**	**74**	**92**

Table 1: Sense of size.

Ordering

These questions examine the ability of pupils to put numbers in numerical order, whether presented as decimals, Question 22 and 41, as fractions, Question 23, as values on a number line, Questions 25 and 28 or as integers, Question 37. The results given in table 2 below are the percentage of pupils correct at each form level.

Question	Form 3 (n=36)	Form 4 (n=42)	Form 5 (n=61)	Form 6 (n=40)
22	67	79	85	98
23	39	69	59	80
25	39	52	72	85
28	75	57	79	85
37	81	76	89	95
41	58	42	69	86
Average	**60**	**63**	**76**	**88**

Table 2: Ordering numbers

Pupils have an above average success rate with ordering numbers, and correspondingly there is an overall improvement at ordering the various types of numbers across the age levels. Third formers have struggled with ordering fractions (Questions 23 and 25). Many junior secondary school pupils experience difficulties with fractions.

Equivalent Forms

Question 21 examined the equivalence between a number given as a word and its numerical value. The percentage correct by each level are listed in Table 3.

Question	Form 3 (n=36)	Form 4 (n=42)	Form 5 (n=61)	Form 6 (n=40)
21	100	88	98	100

Table 3: Equivalent numerical forms

All the form levels performed well at this question.

Benchmarks

Questions 26 and 42 examined pupils' knowledge of commonly used physical quantities. In Question 42 where all four parts had to be answered correctly to attain a credit, most attempted all four parts. While many were successful with the approximate value of a door height, few had little quantitative appreciation of the ambient classroom temperature, the length of time to walk one kilometre or an approximate length for the South Island. Numerical results in Table 4 represent percentage correct by each form level.

Question	Form 3 (n=36)	Form 4 (n=42)	Form 5 (n=61)	Form 6 (n=40)
26	64	69	77	98
42	37	28	40	48
Average	**51**	**49**	**59**	**73**

Table 4: Benchmarks

Question 26 examines the capacity of a well known sports ground. Although there is an overall improvement, general knowledge of every day physical dimensions is fairly low.

Place value

The actual value of numerals in a number was examined in questions 30, 31, 32 and 33. Numerical responses listed in Table 5 below show the percentage correct by each form level.

Question	Form 3 (n=36)	Form 4 (n=42)	Form 5 (n=61)	Form 6 (n=40)
30	83	76	92	100
31	25	40	49	83
32	61	52	51	83
33	39	38	66	75
Average	**52**	**52**	**65**	**85**

Table 5: Place value

The overall trend shows an improvement in ability at place value questions. Questions 31 and 33 examine understanding of the difference between numerical values in a number.

Question 31. To the number 3,200 is added as many hundreds as there are thousands. The result is...

Question 33. For the number 5,678,143 the numerical difference between the number represented by 6 and the one represented by 7 is...

Only the sixth formers answered these two questions well, indicating that junior secondary pupils have an incomplete knowledge of place value.

Decomposition/Recomposition

This aspect was more thoroughly examined in the interviews. Question 43 asked pupils to re-express a number in four different ways. Credit was only given if all four equivalent representations were correct. Table 6 shows the correct responses expressed as a percentage for each form level.

Question	Form 3 (n=36)	Form 4 (n=42)	Form 5 (n=61)	Form 6 (n=40)
43	81	69	93	98

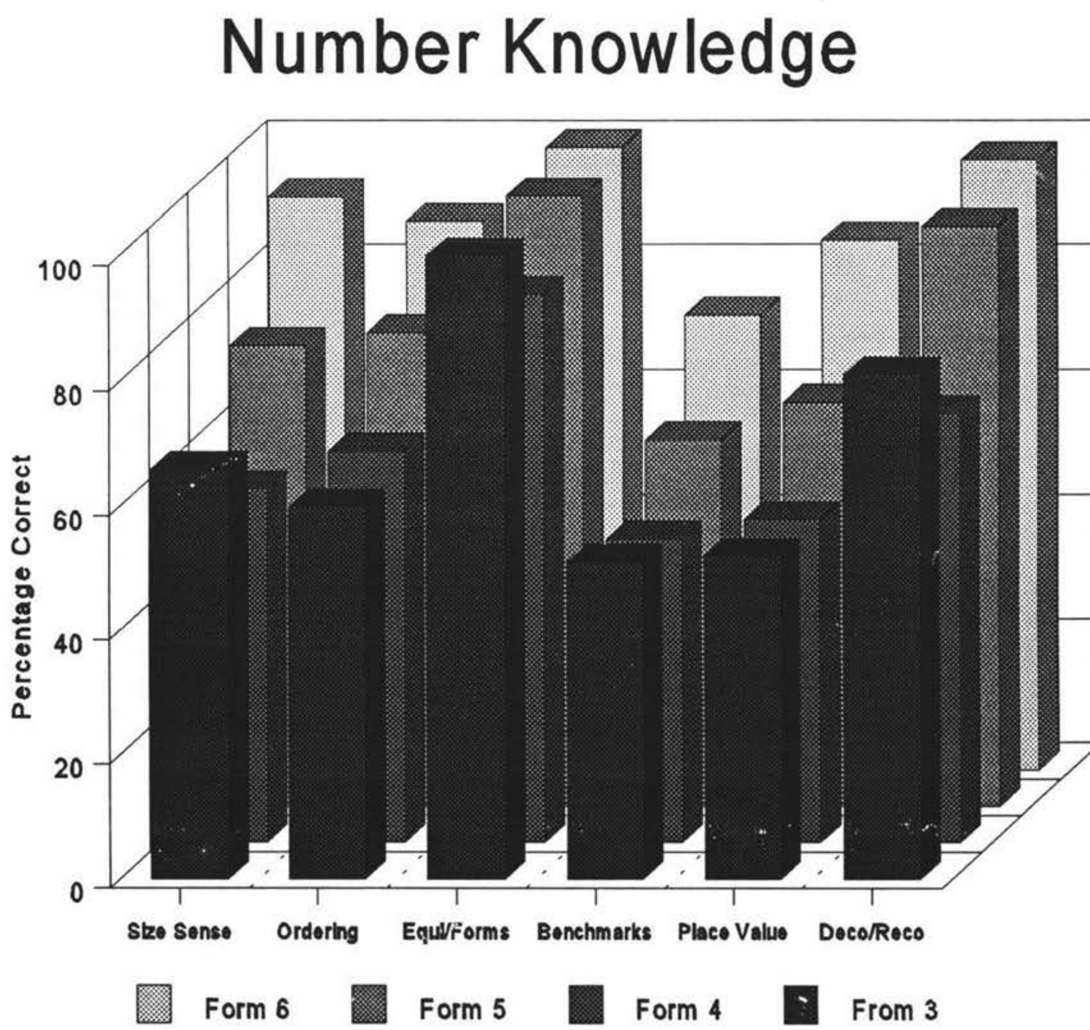
Table 6: Decomposition/Recomposition

Although pupils were not specifically asked to be creative or expansive with their attempts, only 20 out of the 179 who completed the questionnaire produced replies using the distributive principle. The majority gave replies based around re-expressing 56 using numerals added, subtracted or multiplied together, for example

30 + 26, 1 + 55, 59 - 3 or 4 x 14. The only other variation occurred with one pupil expressing 56 in exponent form, $2^3 \times 7$. The results could be summarized as unimaginative.

Summary

Pupils have a reasonable appreciation of the components of number knowledge, although the use of benchmarks and understanding of aspects of place value are below expectations for all forms except the sixth. Graph 1 depicts class performance at the components of number knowledge.



Graph 1: Number Knowledge

OPERATION KNOWLEDGE

Knowledge of how operations on numbers function and creative ways of performing numerical computations is examined more intently in the interviews. In this section procedural knowledge of the effect of operations and the properties of operations are examined.

Effect of Operations

In Question 24 the effect of multiplication on numerals is examined, while Question 29 considers the effect of subtraction and division on numbers.

Question	Form 3 (n=36)	Form 4 (n=42)	Form 5 (n=61)	Form 6 (n=40)
24	31	31	38	75
29	53	38	41	55
Average	**42**	**35**	**40**	**65**

Table 7: Effect of operations

The result of multiplication on numbers less than 1 (Question 24) is not well understood by most pupils. The responses to Question 29 suggests that many pupils are unable to gauge the results of dividing or subtracting two numbers that are numerically very close.

Properties of Operations

In Question 34 the operation of multiplication on numbers expressed in different forms is considered.

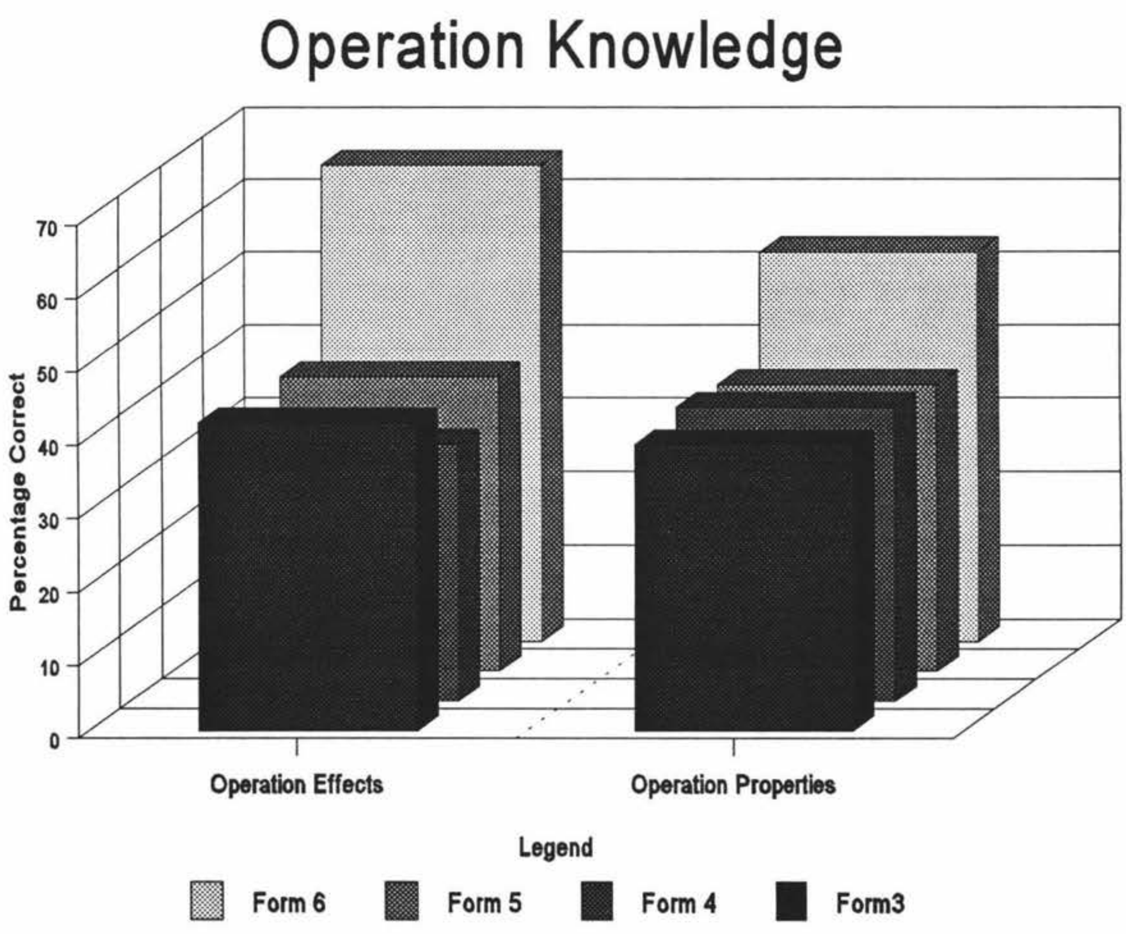
Question	Form 3 (n=36)	Form 4 (n=42)	Form 5 (n=61)	Form 6 (n=40)
34	39	40	39	53

Table 8: Properties of operations

Results indicate a lack of knowledge of the effect of multiplication on numbers expressed in different forms, that is, pupils have difficulty applying the distributive principle.

Summary

Results from the three questions indicate that pupils lack an in-depth understanding of the effect of operations on numbers. Pupils at this stage in their mathematics education could be expected to perform at a higher level than they have with these three questions, especially those in Form 6. Graph 2 below represents pupil performance on operation knowledge for the various form levels.



Graph 2: Operation Knowledge

NUMERICAL COMPUTATION PERFORMANCE

In this section the questions examined pupils ability with mental computations, estimation calculations and logical deductions questions.

Mental Computation

For these questions pupils perform the calculations mentally. The percentage correct for each of the mental calculations is given by form level in table 9 below.

Question	Form 3 (n=36)	Form 4 (n=42)	Form 5 (n=61)	Form 6 (n=40)
1	50	38	56	73
2	25	29	61	80
3	19	19	41	65
4	58	55	62	85
5	28	29	52	65
6	6	14	20	53
7	39	26	38	83
8	39	52	56	95
9	8	5	13	15
10	78	74	89	95
11	33	33	43	60
Average	**35**	**34**	**48**	**70**

Table 9: Mental calculations

Questions 3, 6 and 9 were poorly answered:

Question 3. Find two numbers that differ by 16 but add to 24

Question 6. What is the square root of 256?

Question 9. 24 cm^2 is the same as how many mm^2 ?

The results of Questions 3 and 9, which involve performing more that one calculation, suggest that pupils have difficulty in finding solution to multi step problems. Although the square root of 256 (Question 6) is outside the commonly used ones, a moment's reflection by pupils should have given the correct value. Only the sixth formers can be said to exhibit overall proficiency at mental computation questions, indicating that this type of calculation is not performed very regularly by pupils. Performance levels across the 11 questions are generally low.

Computational Estimation

The estimation questions examined pupils' number rounding ability and their knowledge of the effects of operations on numbers. The results in Table 10 are the percentage correct by each form.

Question	Form 3 (n=36)	Form 4 (n=42)	Form 5 (n=61)	Form 6 (n=40)
12	47	26	66	65
13	11	10	30	50
14	17	21	25	50
15	19	7	33	53
16	3	5	13	28
17	8	7	23	53
18	8	10	10	58
19	64	76	85	68
20	33	29	39	50
Average	**23**	**21**	**36**	**53**

Table 10: Estimation calculations

Throughout this section many of the questions were left unanswered. Some questions, such as numbers 16 and 18:

Question 16. Estimate $39.8\ (52.6 + 187.4)$

Question 18. Estimate $(28.73)^2$

produced answers containing decimal points, suggesting that pupils do not understand the intent of estimation type questions. The results show that pupils have difficulty with this type of calculation, sixth formers also have only average success. Indications are that pupils do not use estimation calculations very often and are not familiar with techniques used for solving these types of problems.

Computational Deductions

The three questions required a degree of reasoning rather than just the mechanical application of an arithematic operation. Table 11 below is the percentage correct by each form.

Question	Form 3 (n=36)	Form 4 (n=42)	Form 5 (n=61)	Form 6 (n=40)
35	39	45	48	68
36	39	43	57	65
44	81	69	85	98
Average	**53**	**52**	**63**	**77**

Table 11: Computational deductions

Third, fourth and fifth form pupils experienced difficulty with Questions 35 and 36.

Question 35. If X and Y are two numerals such that $X + Y < 4$, $X > 0$ *and* $Y > 0$ *then how many pairs of whole numbers could X and Y be?*

Question 36. If A is a digit and 0.A is a number shown by

$0.A = 0.AAAAAAAA....$

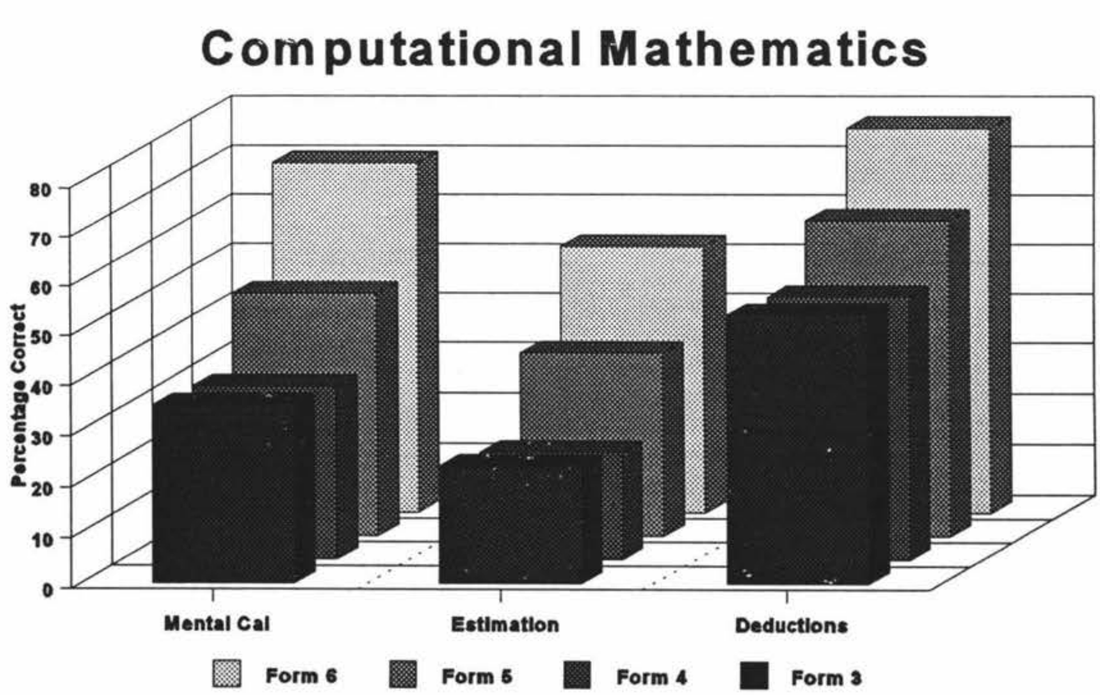
What is the value of $0.8 + 0.6$

The three questions were attempted by most of the pupils. Question 35 was algebraic in nature, which may explain the low success for the third, fourth and fifth formers. Many were unable to fully comprehend the implications of the definition given in Question 36. The degree of difficulty of Question 44 is considerably less than that of Question 35. Overall there is an increase in the performance level by the pupils.

Summary

Pupils have only rudimentary ability with mental and estimation computations, indicating a lack of factual and procedural knowledge for manipulating numbers. The results suggest that pupils are not familiar with this type of calculation and lack the skills and procedures required for performing them. A number of pupils attempted to give exact answers to the estimation questions. Technically pupils were deficient in the use of conceptual and procedural knowledge of multiplication and division by ten, rounding of numbers and using the proximity principle. The execution of Questions 12 to 18 was based around the use of this knowledge as well as an understanding of how multiple strategies can be effectively used.

Contrasting the results for Question 2 and 17 indicates that pupils are unfamiliar with the use of equivalent forms for numbers. An appreciation of standard physical quantities (benchmarks) was considered in Questions 19 and 20 and, apart from the incorrect use of the units of millilitres instead of grams, these were reasonably well answered. The deduction questions results also suggest that pupils are not exposed to these types of questions very regularly. The class trends for each of the components of computational mathematics are presented in Graph 3 below for comparative analysis. The overall indications are that third, fourth and fifth form pupils lack experienced at these types of calculations. Only the sixth formers have any competency, but this is less than could be expected of pupils who have had up to eleven years of mathematics education.



Graph 3: Computational Mathematics

SUMMARY AND DISCUSSION OF THE QUESTIONNAIRE RESULTS

A consideration of pupil performance in the three principle components of number sense indicates that pupils' number knowledge is well developed but the effect of operations on numbers is only partly understood. The computational components of mental and estimation calculations are considered by a number of authors, McIntosh et al., (1997) and Sowder (1992a), to be the most effective instruments for testing a pupil's number sense and on this basis only the sixth formers can be said to have developed sufficient number sense to any degree. Results from the junior forms reveal a lesser development of number sense, in many cases below expectations. In particular, these pupils are not familiar with mental and estimation calculations techniques.

5.3 THE ATTITUDINAL RESPONSES

The attitudinal responses fall into five categories; making sense of mathematics, disposition towards studying mathematics, perceptions of the usefulness of mathematics, a gauge of changes in performance levels at mathematics and an assessment of personal ability. In each case the numerical values are the percentage support for each option.

Making Sense of Mathematics

Questions 53, 55 and 56 are based around examining whether pupils view learning mathematics as a sense making or a memorising exercise. Table 12 below gives the percentage support for each options.

Question	Yes	Maybe/Sometimes	Don't Know	No
53	47	34	8	12
55	34	36	9	22
56	21	39	13	31

Table 12: Making sense or memorising.

These replies are somewhat contradictory; Question 53 responses indicate that pupils expect to make sense of their mathematics lessons where as Question 55 suggests that the majority see the subject as dominated by rules. There is a split in the replies to Question 56 which shows that about half of the pupils expect to be able to assimilate a mathematical idea, without having to have it explained more than once, suggesting that they expect mathematical ideas to make sense. A further analysis of results for Question 56 shows that over 60% of the fifth and sixth formers opted for the yes and maybe/sometimes response, but only 40% of third and fourths, suggesting that the sense making attribute develops with exposure to mathematics.

Disposition towards Mathematics

These questions examine how pupils perceive studying mathematics and what importance they attach to the subject. Table 13 below details the overall percentage response for each option.

Question	Yes	Maybe/Sometimes	Don't Know	No
46	18	66	4	11
47	11	47	17	26
48	73	21	4	2
49	69	22	4	2

Table 13: Disposition to studying mathematics.

The responses indicate that pupils regard mathematics as both a useful subject and one that can help them to gain employment. However the luke-warm responses to attending mathematics classes suggests that the reality of acquiring this knowledge is not universally enjoyed.

Perception of Mathematics

These questions examine pupils perceptions of the mental attributes required for successfully studying mathematics and value judgements pupils have about the nature of the subject. Table 14 below is the percentage responses for each question.

Question	Yes	Maybe/Sometimes	Don't Know	No
50	43	21	19	18
51	29	45	9	17
52	6	29	16	49
54	28	39	9	23
57	57	31	3	9

Table 14: Perception of mathematics

Many pupils perceive mathematics as memory-dependent, mathematical ability as innate, with success coming from practice. However, interestingly, the subject is not considered as a jumble of unrelated ideas. Most respondents feel that calculators can be used to perform most school mathematics problems, suggesting that the study is procedural rather than conceptual.

Personal ability expectations

These questions examine pupils perception of parental expectation, as well as reflections on mathematics performance while in primary or intermediate schools. Table 15 details the overall percentage responses to the options in each question.

Question	Yes	Maybe/Sometimes	Don't Know	No
58	49	21	12	18
59	82	11	3	4
60	56	14	22	7

Table 15: Personal ability expectations

Most pupils indicate that they want to be more competent at mathematics, and many feel that their parents also desire this. The majority consider that they were successful at maths when they were in primary school. Generally pupils responded positively with many indicating a desire to master the subject.

Assessment of personal ability

This question asked pupils to rank their ability. Table 16 shows the overall percentage responses to each of the options

Question	Very Good	Good	Average	Poor
61	7	39	43	12

Table 16: Assessment of ability

These results are consistent with a normal distribution of pupils' mathematical abilities. However, the fact that 55% consider themselves to be average to poor suggests that only half of the pupils regard themselves as marginally successful at the subject.

SUMMARY

Mathematics is perceived as a useful, though not necessarily interesting, subject to study. Most pupils want to be more successful and recognise that employment prospects are considerably enhanced by having a good grounding in the subject.

Many of the pupils consider that competency at mathematics is not attainable by everyone. It is perceived that those who are successful will need a good memory and must be prepared to practise with many problems. Most of the pupils want to be able to make sense of their mathematics learning, but feel that the subject is rule-bound; many expect to have to have concepts and ideas explained to them a number of times.

There appears to be a student perception of a loss of mathematical ability with the transition from primary to secondary school. Many of the pupils consider that their present ability is lower than it was primary school.

In general, respondents expressed both positive and negative attitudes about studying mathematics. Many want to make sense of the subject and perform better at it but feel that success is a reflection of natural ability rather than a result of personal endeavour. They do not consider mathematics to be a jumble of unrelated ideas, but do feel that only practice will make them successful, indicating that success is more memory-dependent than based on good conceptual understanding. The perception that mathematics knowledge is useful and will enhance their employment prospects contrasts with an acknowledgement by more than half of the pupils that they have *average to poor* mathematics abilities and so will have discarded a number of employment options. There is an overall desire by pupils, supported by parents, to perform better at mathematics, many indicating that they find the subject interesting, expect it to make sense, but that the learning process is not enjoyable.

THE INTERVIEWS

Results from the interviews have been divided into two sections, **Computational knowledge** and **Pupils' attitudinal development,** with the layout of the results varying to suit the responses. Pupil's verbal responses have often been used, as the intent was to identify how they perceive mathematics and how they tackle numerical problems. A summary will accompany each section. Eleven pupils were interviewed, their responses recorded on tape and in writing.

5.4 Computation Questions

The computation questions are divided into three categories, the first examining pupil reliance on the use of algorithms to solve problems, the second looking at pupils' number knowledge, operation knowledge, their ability to reason deductively and the final questions investigating pupils' intuitive mathematics knowledge.

Questions 1 to 5.

The first five questions examined pupils' procedural knowledge for performing mathematical computations and noted their reactions to using alternate procedures. Pupils were encouraged to explain the processes they had used to perform the calculations. In Table 17 below the original question is given along with a summary of pupil responses. In all cases pupils were given the questions in writing, asked to calculate the problems out aloud as well as record the result on the supplied answer sheet.

Questions 1 to 5 and summarised pupils' responses

Question	**Performed using Algorithms**	**Non Algorithms solutions**	**Evaluation and or Comments**
1) 5 x 472	6 used 472 $\underline{\times 5}$ not always correctly, 3 incorrect answers.	5 used the distributive principle, of which 2 calculated wrong answers.	1 only was fluent and comfortable with the distributive method 4 accepted that $\frac{472 \times 10}{2}$ is an easier possibility.
2) 420 ÷ 140	5 used $140\overline{)420}$ 2 with numbers transposed to 420)120. 3 incorrect answers.	4 attempted other methods. 1 calculated wrong answer. 1 used 42÷7 then halved.	2 had no procedure at all. 2 accepted that 42÷14 would work. 2 others insisted that this result multiplied by 10 would work.
3) \$72 - \$28.43	7 used 72.00 $\underline{-28.43}$ 3 with numbers transposed to 28.43 $\underline{-72.00}$ 5 incorrect answers.	4 used \$72 - \$28 then take off 43c. 1 added up to \$78 from \$29 and subtracted 57c.	3 did quick mental estimations of \$45 so to compare it with their calculated answer. 1 was unable to use subtraction. 1 would accept using associative principle
4) 38 + 52 + 127	4 used 38 52 $\underline{+ 127}$ 6 used 38 + 52 then added 127. 1 calculated the above incorrectly.	1 decomposed as 50 + 30 then add 120 then add 2 + 8 + 7.	1 approximated as 220 then did exact calculation. 4 acknowledged that it would be easier to decompose the problem.
5) 20% of \$48	6 tried variations on 48 ÷ 20 x 100. 4 incorrect answers.	2 used 10% of \$48 and double. 2 used $^1/_5$ of \$48	1 couldn't perform the calculation.

Table 17: Use of algorithms

The responses show that many of the pupils performed the calculations by attempting to slot the numbers into learnt algorithms. Three of the pupils were visibly uncomfortable performing the calculations, wrongly applying algorithms or failing to carry out the calculations. When it was pointed out that the procedure they were following was incorrect, the pupils claimed that this was how they had been taught. Most of the students were ponderous and slow at performing the calculations, often saying that they would normally use a calculator. This was in marked contrast to two pupils who carried out the calculations using non-algorithmic processes, looking for ways that suited the problem, rather than relying on standard algorithmic procedure, and were receptive to other suggested approaches. For example, they acknowledged that Question 3 could be tackled as \$72 - \$22 - \$6 - \$0.43, provided a left-to-right strategy was used. These two pupils used estimations to check if their answer was within acceptable limits - they were able to perform a quantitative evaluation of the appropriateness of their calculation.

For most pupils alternate approaches to the calculations were not considered useful nor an improvement on what had already been used. For example, only four pupils considered that Question 1 could be calculated as 472 x 10 and then divided by 2 and that this was an improvement on how they had calculated it. Two pupils believed that this method would give an answer that was out by a factor of 10.

Two pupils who were unable to calculate Question 2 or 4, maintained that they had never been taught how to do division, nor to calculate percentages. They were not confident at handling calculations, both admitting that they lacked procedural knowledge. For them calculating mathematics problems was arduous whereas those with a sound knowledge of number properties and how operations on numbers can be used calculated the results with ease

Results show that those pupils with incomplete number and operation knowledge applied algorithms, not always correctly to solving problems. Pupils who used non-algorithmic processes possessed good number knowledge and were able to use operations on numbers in creative and successful ways.

Questions 6 to 9

Questions 6 to 9 required pupils to use procedural knowledge along with an understanding of number properties and the logical consequences of these properties to perform the calculations. Table 18 below (pp 79, 80) summarizes pupils' responses under headings that suit each question.

Summary of results for Questions 6 to 9

Question	**Addition Subtraction**	**Factors**	**Totalling 28 2 + 8 = 10**	**Comment**
6) Find properties of 28 other than being even.	6 gave replies like 28 is: 17 + 11, 29 - 1, 20 + 8, 13 + 15' 50 - 32. 1 pupil did not know what even meant.	10 listed some factors. 3 listed 5 factors.	2 noted that the numerals totalled 10.	Little creativity shown. No-one observed that 28 = 4 weeks or is the shortest length in days for a month or that its factors total 28.
	Result	**Reasoning**	**Thoroughness**	**Comment**
7) What number squared equals twice itself ?	4 found both answers. 6 found one answer. 1 couldn't do the problem.	Most recognised that answer had to be small, and worked from there.	4 argued that 2 was largest so looked below it for other result.	Most were quick to find 2 as an answer. Intuition appeared to operate in getting started.
	Result	**Ease of Procedure**	**Number Manipulation**	**Use of Benchmark**
8) Find the length of 1 million tooth picks. Relate this to a known distance.	10 arrived at a result. 1 was unable to complete the calculation.	5 found the calculation awkward. All appreciated what had to be calculated. 2 were quick and fluent.	3 found division by 10s complicated. 1 did not know how many zeros in 1 million.	5 gave places that were close to 65 km from Dunedin.

	Results	Number Knowledge	Use of Number Properties	Comment
9) Examine the possible sum of the odd numbers 1 to 11, each placed on the faces of a dice thrown six times.	All found at least 4 out of the five.	All recognized that 6 is smallest total. All found combination s for 8 and 28, none for 31. Most figured 66 as highest score.	All found one only possible combination for 8. 2 found one only combination for 28. 2 demonstrated flexibility at calculating combinations. None discovered the rational for not obtaining 31.	3 only were fluent at the combinations. 2 demonstrated originality and or dexterity. All demonstrated some deductive reasoning. 1 struggled with the number manipulations. 4 appreciated the explanation for no combinations that give 31.

Table 18: Number knowledge and use of reasoning

The solutions to these problems are non-algorithmic in nature, relying instead on number knowledge, an understanding of number properties and the logical consequences of these properties. There is an element of creativity and conceptual visualisation necessary with the solving mechanism used. Questions 7 and 9, in particular, have a metacognitive aspect to their solutions, evidenced by the way pupils deliberated on their responses. Some of the pupils appeared to enjoy the problems, employed a range of strategies and were receptive to alternate suggestions for solving the problem.

The number properties of 28 (Question 6) did not elicit any notable original or surprising results. Most of the replies were rather workman-like and predictable. This is probably the first time pupils had been asked to examine numbers from other than a magnitude aspect. When the researcher pointed out that the factors will sum to 28 they were surprised and interested. Two of the participants were nonplused about the question, and one did not appreciate what an even number was, or how to identify it.

Pupils used a *guess and try* strategy for Question 7. It was interesting that some argued that once values above 2 were considered, the two operations would give results that diverge very quickly. With some prompting four were able to find that zero will also satisfy the requirements of the question. Seven were unable to find zero or did not bother attempting to look further for another solution. Two also noted that with numbers less than zero the two operations produced values that also diverge. Again some were comfortable and flexible with their answers while others laboured through the problem.

Question 8 was the most straightforward and mechanical of the problems. Of interest was the number who were not fluent at dividing by tens, hundreds etc. They all were able to begin the problem but some struggled with the need to convert the measurement into a value that they could relate to. I informed the pupils that the distance from Dunedin to Port Chalmers (a well known part of the Otago Harbour) was about 15 kilometres. Using this information could they relate the 65 km they had calculated for the length of the picks to a distance from the city. One was unprepared to attempt this even though he had performed the calculation correctly and quickly, as he was unable to relate the measurement with a known distance.

Some pupils had to have Question 9 explained to them a number of times, particularly that the numbers on the cubes were the odds 1, 3, 5, 7, 9, and 11. Some were ponderous in their calculations of possible combination totals. Of interest here was the creative strategies used by two of the pupils to find alternate combinations for 28. Starting with a combination of 11, 9, 5 , 1, 1, 1 other possible combinations are found by, for example taking 11 and 5 which total 16, then finding other totals that give 16 like 9 and 7. Most pupils looked for other combinations by trial and error.

Only a few of the pupils appreciated why a total 31 cannot be obtained by adding any even combination of odd numbers. Pupils appear to have differing levels of understanding of the properties of numbers and the logical effect of operations on them. Most established that the totals 5 and 68 were not attainable by trial and error, though two were very slow.

The results from the successful pupils suggest that sound number knowledge and ease at performing calculations are interdependent. For these pupils errors in calculating were self-corrected and did not dent their computational confidence, or cause them to doubt their line of thought. These pupils were able to easily tap into the logical implications of the problem, marrying number knowledge with appropriate operations, so that the whole solution process made sense. Those who plodded through the problems lacked number knowledge and knowledge of the effect of operations on numbers. They were slow at the numerical calculations and lacked appreciation of the inherent logical framework of the number system. For example, they found it difficult to understand that a total of 31 is not possible for Question 9. It seems that their base mathematical knowledge is not sufficiently well developed to allow for a more expansive review of possible problem solution paths. These pupils would, for example, find a combination that sums to 28, and then state that this is the only possible one and not bother to look for others. The pupils who struggled with these problems did have some factual and procedural knowledge but the relationships between them were limited.

Results for Questions 13 and 14

These are the last of the computational problems used in the interviews. Results from these two questions give an insight into the level of pupils' intuitive knowledge, that is, mathematical knowledge that they have quick and effortless access to, and gives them confidence to tackle numerical problems. The two questions are listed along with responses and discussions.

Question 13

What is the value of 9 x 8?

How do you get that?

How do you know the result is true?

What is the value of 8 x 9?

Again, can you tell me why this is true?

Seven gave quick correct replies and said that they "just know it", adding that they were well versed in tables and remember learning and being tested on them while at primary school. Three made a number of attempts before arriving at the correct response for example, offering "54, no 56, no 75, no 72", indicating a guess is made which is compared with a benchmark. One pupil was not able to reply immediately but added that he could find the answer by adding 8 nine times. Two used 10 x 8 then subtracted 8 to get the correct result.

Six were able to substantiate that 9 x 8 = 72, by proposing that counting a total of nine sets of eight objects will give seventy two. They recall using sets of objects (*sticks*) in primary school and counting the total present. Three were not able to justify their answer by any means.

Most, but not all, knew that multiplication was commutative, and also revealed that addition was, while subtraction and division were not. The pupil who could not calculate 9 x 8 recognised that 8 x 9 was 72 (I had informed him that 9 x 8= 72), and maintained that all arithematic operations were commutative. There is an appreciation of the interrelationship between multiplication and addition by those with a knowledge of their multiplication tables, whereas those who have incomplete knowledge of multiplication tables appear to lack this awareness.

Question 14

Consider the problem 61 + 58 −17 + 22. Can you think of ways of rewriting this question so that it could be easier to calculate, for example 60 + 60 + 1- 2 - 17 + 22.

Five pupils would perform the additions first then the subtraction. One gave, *"Add 60* *and 50, producing 110, then add 1 and 8 to give 119, then subtract 17 giving 102. Now add 22 giving an answer of 124".* He maintained this was the way he often did such problems. When asked where or from whom he had acquired this method, he was not able to identify specifically how he had come across this procedure. Two pupils reworked the problem as 22 + 58 + 61 - 17, to take advantage of the *tens* aspect in 22 + 58. One re-wrote the problem as 17 - 22 + 58 + 61 while an another proposed 61 + 58 then 17 + 22 but was unsure what to do from here.

Flexibility was demonstrated only by pupils who had been fluent and successful with the other questions. Likewise, the pupil who wrongly altered the order of the operations had experienced procedural problems with previous questions.

The above results indicate that those with a good procedural knowledge also have sound knowledge of number facts and properties. Those with difficulties lack sufficient basic number and operation knowledge upon which judgements concerning computation could be made. These same pupils were hesitant and uneasy about alternate approaches to the problems suggested for Questions 1 to 5.

Fluency at number manipulation appears to be related to the automatic responses associated with reciting tables, and there appears to be a relationship between computational ability and drills using operations on numbers. Pupils with fluency were also able to justify their answers indicating conceptual understanding

5.5 THE ATTITUDINAL QUESTIONS

Questions 10 to 12 enquire into factors that are influential in shaping a pupil's attitude towards learning mathematics. The questions were concerned with examining a combination of attitudinal dispositions to mathematics and personal evaluation of mathematical competency. To achieve this reflection a number of problems are worked through. Question 15 investigated how pupils approach mathematics problems, what their expectation is and what strategies they employ if they encounter difficulties. The role of intuitive knowledge and how it is acquired is also considered. Some anecdotal observations on pupils' experience of learning mathematics are included at the end. The results of these questions will be contrasted to similar ones from the questionnaire survey - Questions 48, 49 and 60.

Results for Attitude Questions 10 to 12

Question 10

Do you think that learning mathematics is important? Could you explain your answer?
All pupils unreservedly agreed that learning mathematics was important. Mathematics was perceived as being useful for career or work opportunities and for assistance with everyday problems. The inference was that mathematics is part of our everyday life. Comments such as "need it in everyday life" or "use maths every day" were common. The indications in these replies is that a knowledge of mathematics is necessary for anyone wanting to take an active and full part in the workings of the world. One of the sixth formers added a non utilitarian aspect to the justification: "Mathematics teaches you to think", recognizing the cognitive potential of the subject. Another sixth former maintained that mathematics could be broken up into two branches, "everyday maths for life and other maths, like calculus, for special things".

The intent in these responses are similar to those given to Questions 48 and 49 from the questionnaire.

Question 11

What do the other members of your family think about mathematics?
Pupil's responses suggest that mothers are the primary source of encouragement, have the most influence and take the most interest in their progress. However, it was noted that both parents often helped them with homework, considered mathematics important and were keen for their child to perform well.

Parental ability was often mentioned; pupils made judgements of how well their parents handled mathematics, " my mother was no good at mathematics " or " Dad can only add ". A few stated that their parents were not able at the subject, but did not use this as a explanation for their own difficulties. Parents do support and encourage their children with their mathematics studies. Results from Question 60 of the questionnaire supports these findings.

Question 12

Are there any areas of Mathematics that you find difficult or have trouble understanding?

The third and fourth formers reported difficulties with division, multiplication, percentages, integers and fractions. The two pupils who rated multiplication and division as difficult stated - "have learnt my tables but have forgotten them" and "it takes me a while to do the multiplications". This lack of familiarity at using number operations in turn gives incomplete computation knowledge.

For the fifth and sixth formers areas of concern were algebra and graphs. Those who had found certain areas difficult admitted to having limited knowledge of multiplication tables, suggesting that there is a base level of competency and knowledge of the operations on numbers which, once reached, empowers a learner to be able consider himself/herself mathematically capable and confident with numerical calculations.

Question 15

Would you say that you were good at mathematics?

If yes I) Have you always felt that you were good at the subject?

ii) Would you use mathematics to solve problems you come across, say around the house or in jobs you have had?

iii) Do you expect to be able to do maths problems that you are given at school?

iv) Do you enjoy mathematics? What is it that you enjoy about mathematics?

v) If you come across a problem that you can't do what do you do?

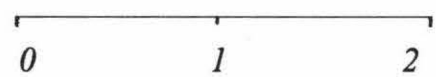
vi) Could you recite your 9 times table? Up to what number can you *do the multiplication table?*

Question 15 examined pupils' perception of their mathematical ability and identified aspects that they think contribute to or detract from their mathematics abilities. Those who considered themselves *good* at mathematics had always felt they were, would use it to solve out of school problems, enjoyed the subject and expected to be able to cope with mathematics problems that they meet at school. They have a good knowledge of number facts and appreciate the effect of operations on numbers. All said that they knew their multiplication tables up to the twelve times. One pupil stated that there were no areas that were difficult, provided that the ideas were well explained. The same pupil had worked through the interview questions successfully, was interested in alternate approaches to solving the problems and said that if he was having difficulty with a mathematics problem he would consult a text, a friend or the teacher for assistance and expected to be able to eventually complete the problem. This pupil had a high level of confidence about his mathematics ability, a reflection of his in-depth number and operation on number knowledge.

If no I) Can you recite your 9 times table?

ii) If you were asked to total 27 + 63 - 18 + 12 how would you do it?

iii) On the number line where would you place $1\frac{1}{5}$.



iv) Which of the following is the largest, 1.1473 or 1.1475 ? Why?

v) Which of the following will give the same values.

a) 20% of 10

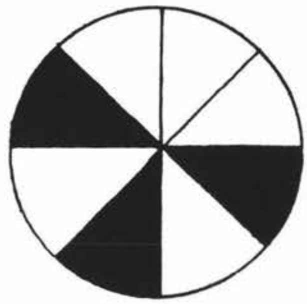
b) $\frac{6}{2}$

c) 52 ÷ 27

d) 2

e) $\frac{40}{20}$

vi) What fraction of the following diagram is shaded?



vii) In the number 2678.04 what value does the number 6 represent?

viii) If you are unable to do a problem what do you do?

Leave it

Look for a similar problem in your notes or text book to help.

Ask someone else.

Leave it and go back to it later?

Those that considered they were *not capable* at the subject exhibited some or all of the following

* either couldn't or were slow at reciting the nine times table.
* five could not place $1^4/_5$ correctly between 1 and 2 on a number line
* three identified 1.1473 as being larger than 1.1475 because 3 is closer to 1 than 5 is.
* three could not recognize that 20% of 10, 2 and $^{40}/_{20}$ are equivalent.
* All could calculate the fraction of shaded area in a circle correctly.
* one stated that 6 in 2678.04 represented 6000.
* two only would look for a similar problem in their text or notes first, the others would ask someone for assistance.

Pupils who stated they found mathematics difficult experienced difficulty with the following areas: number knowledge; place value; sense of size of numbers; and equivalent forms of numbers. These pupils had a limited understanding of the effect of operations on numbers and were hesitant or unable to devise alternate approaches to numerical calculations. They all have had less experience with number drill exercises, particularly multiplication tables, than the more able ones. Other comments of interest from the pupils concerning their experiences of learning mathematics include:

* *I have never learnt how to do division.*

* *I am not good at doing the problems quickly but can work them out if I am given time.*

* *I find mental addition, multiplication and division difficult.*

When asked about experiences of their primary school mathematics education, a couple replied

**I felt I had a good background in maths, I did lots of problems and would go home and do my tables.*

A comment about an experience of assistance from a parent with homework.

**I was told by my teacher not to use the way my mother had showed me.*

Summary

Pupils who have difficulty with mathematics lack in-depth number knowledge, do not appreciate how operations on numbers can be used to make calculations simpler and are unable to recall number facts quickly or correctly. For able pupils this knowledge is easily and quickly retrieved and gives the appearance of being intuitive. The able respondents not only know the results of a calculation, but are also confident that their calculation will be correct. These pupils are able to reflect on their calculations and make appropriate judgements concerning the results, altering it if necessary. Respondents who staggered through the calculations do not have access to immediate number and operation knowledge but can if given time and little assistance find the correct answer to a problem.

A constant theme in recollections about early mathematics education by the less successful pupils is that little emphasis was placed on mastering multiplication tables or on exercises that involved applying the standard operations to numbers. None of these pupils have had much experience of drill exercises involving manipulating numbers with the result that they do not have a good reservoir of number or operation knowledge to call upon.

All eleven of the interviewees considered mathematics was a useful subject to study, and said they needed it for everyday activities and that it enhanced their employment prospects. Pupils who consider themselves successful have always felt confident about the subject, enjoy it and remember doing drill exercises with number manipulations in their early schooling.

SUMMARY OF RESULTS

Analysis of the questionnaire responses show pupils have well developed number knowledge but are unable to apply it productively for solving mental and estimation problems. There is a an upward trend for pupils' number sense from the third forms to the sixth forms, but the value of the actual numerical levels achieved is below expectations of pupils who have spent up to nine years learning mathematics.

Results from the interviews and the attitudinal section of the questionnaire show that pupils do want to make sense of their mathematics education but consider that the subject is a memory-dependent and rule-bound study. Interviewees with good number sense had sound number knowledge, were inventive with the procedures they used with numerical calculations, were able to gauge the appropriateness of their answers and if necessary adjust the results of the calculation. For these pupils solving problems was an exercise in thinking, not one of simply applying a learnt algorithm, and the whole process of calculation appears to be driven by the logical consequences of each step in the calculation. Interviewees who found the calculations difficult were unable to quickly call up appropriate strategies, but instead they relied on trying to apply a learnt procedure, often incorrectly. Indications were that these pupils have limited number and operation knowledge, have insufficient experience at applying this knowledge and have not been able to develop number and operation fluency.

Results from the final interview questions showed that pupils and their parents have a positive attitude towards studying mathematics, recognize its potential usefulness and want to be successful at it. Those who experience difficulty do not have quick access to number manipulation skills and have had little experience with drill exercises. There is some evidence that pupils can end up accepting that there is a *correct way* or that *this is how you solve these problems* when instruction is focused around using algorithms. This is a very constricting method of teaching mathematics, one that denies the creative potential of pupils and does not allow them to develop useful number and operation relationships.

In Chapter 6 the results and implications of the findings will be discussed, recommendations for appropriate strategies to improve pupils number sense will be examined and finally the limitations of this study and further areas of relevant enquiry will be presented.

CHAPTER 6

CONCLUSIONS

6.1 INTRODUCTION

Pupils from two secondary schools completed number sense questionnaires ($n = 179$) examining number knowledge, operation knowledge and computational knowledge. Pupils were interviewed ($n = 11$) about how they perform number calculations, what strategies they use and how number and operation properties are used to expedite calculations.

The questionnaire and interviews were also used for examining pupils' attitudes, particularly their inclination to make sense of their mathematics education, the importance they attach to mastering the subject, and what factors have promoted or hindered their mathematical development.

In this chapter the main findings from the questionnaire and interviews will be presented so that the objectives of this research study can be addressed. The results are summarized under three headings. The order of presentation is; summary of findings on pupils' number sense (6.2); summary of findings on pupils' inclinations to make sense of their mathematics studies (6.3); summary of findings on pupils' performance on numerical computations (6.4). Following the summaries and reflections the overall implications of this research are discussed (6.5). Suggestions for further research are noted in (6.6).

6.2 SUMMARY OF FINDINGS ON PUPILS' NUMBER SENSE

The results are summarised either as average numerical performances for third through to sixth formers or as a list of descriptors. All findings are related to those detailed in Chapter 5.

RESULTS FROM THE QUESTIONNAIRE

Objective 1: **To examine the number sense of a range of secondary school pupils and to identify any development.**

Number Knowledge

The average achievement by the pupils at the six aspects of number knowledge were in the range 68% for third formers to 90% for sixth formers. Use of benchmarks and place value understanding were below expectations for all class levels. (See Graph 1.)

Operation Knowledge

The average achievement range was 40% for third formers through to 59% for sixth formers. Indications are that the range of possible use of operations on numbers is not fully understood, particularly operations on numbers in the 0 to 1 range, and the workings of the distributive principle. (See Graph 2.)

Mental Computation

The range of achievement was from 35% for third to 70% for sixth formers. (See Graph 3.) Many questions were left unattempted by third and fourth formers, who experienced difficulty with:

- two or more step problems
- general computation knowledge
- calculations using powers of 10
- using equivalent forms of numbers.

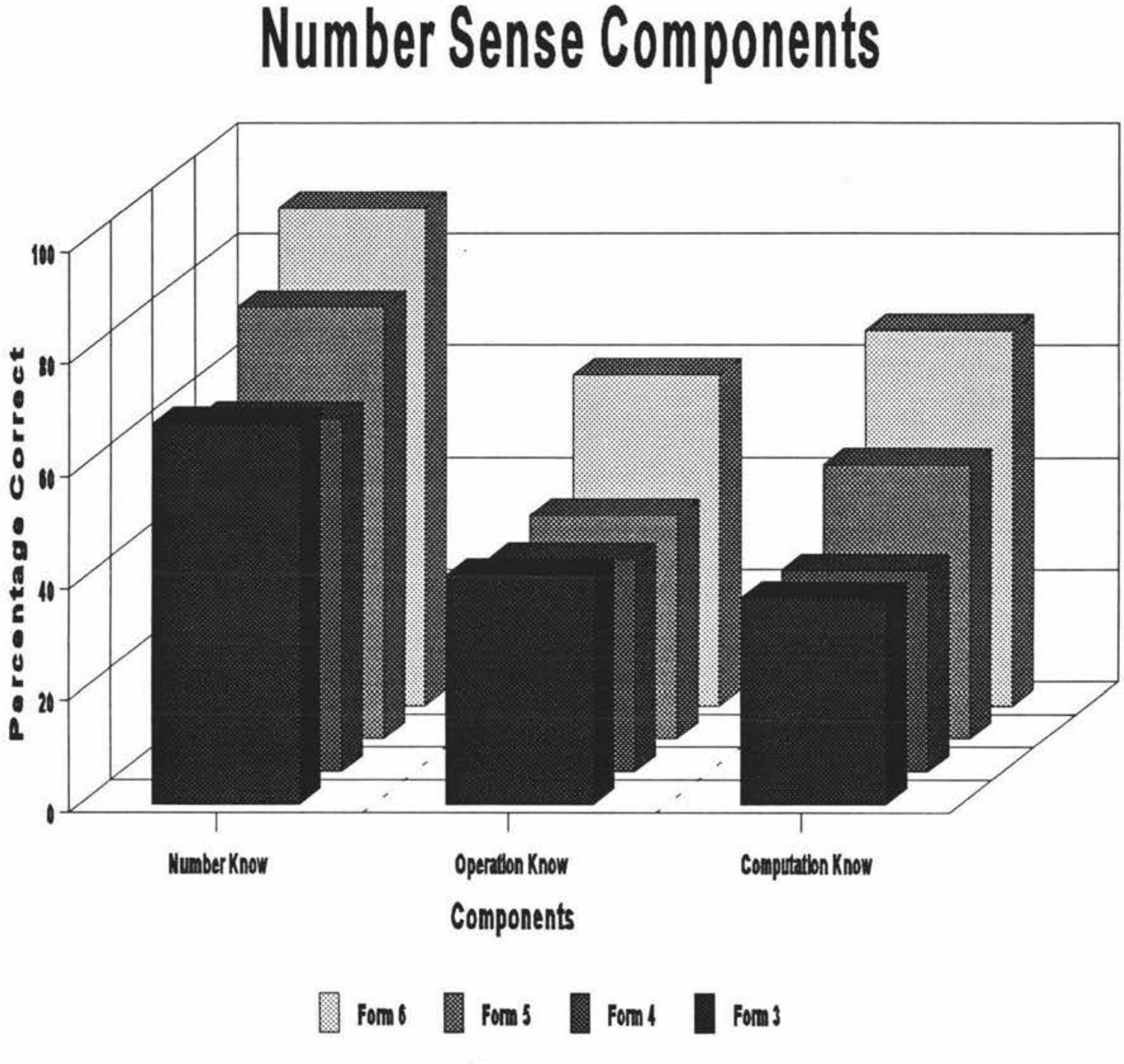
Estimation Calculations

Achievement levels are low, third formers averaged 23% to the sixth formers 53%. (See graph 3). A significant number of pupils supplied answers containing decimal points. Areas presenting difficulty were:

- rounding numbers and using the proximity principle
- multiplication and division by powers of 10
- use of multiple strategies with calculations.

A comparison of achievement by pupils with the three components of number sense – number knowledge, operation knowledge and computation ability is shown in Graph 4 below. The graph was compiled from averaging pupil performance at the three main components of number sense. The overall trend is for improved performance, however only the sixth formers record a performance level of around 50%, well below what could be expected for pupils at this level.

Number sense performances can be gauged from performance at mental computations (McIntosh et al., 1997) and estimation calculations (Sowder, 1992). To competently perform these two calculating processes a pupil requires sound knowledge of numbers, good understanding of the mechanics of number operations and a developed intuition of how these two can be combined to assist numerical calculations. The above results, for mental and estimation calculations indicate that the number sense of third, fourth and fifth former is low, that there is an improved attainment level over the classes, but the overall average achievement is below expectations for secondary school pupils. Pupils demonstrated incomplete place value, benchmark knowledge, as well as only a superficial understanding of how number operations can be re-expressed so as to ease a calculation.



Graph 4: Number Sense Components

Reflections on the development of pupils' number sense

A number of third, fourth and fifth formers either left some of the mental calculations blank or gave answers to estimation calculations containing decimals indicating little experience with these types of calculations. It is not surprising, therefore, that their number sense is underdeveloped. Although the sixth formers, on average, demonstrated familiarity with the two processes their achievement levels are low.

Pupils need more exposure to problems involving multiple operations strategies using the distributive principle. Pupils should be encouraged to construct their own solutions, to experiment with numbers and operations and so build a repertoire of options that can be applied to problems.

Pupils were able to use the standard operations, though division was a problem for some, but they could not transpose the operations flexible, that is, when asked to re-express a multiplication very few were able to use the distributive principle successfully. Pupils lack experience at re-expressing operations when exploring alternative solution paths. The interviews reinforce this narrowness of options for solving problems.

Pupils' place value knowledge is blurred for large numbers and for decimals with four or more places, indicating that the concepts have been only partially developed. There is a lack of understanding about the number line, where values sit on it, particularly fractions. The notion of number density on the number line is not well understood. Equivalent forms for expressing numbers requires further extension and development, knowledge of equivalent forms of numbers encourages flexibility, helping to reinforce the notion of multiple approaches and creative application of operation knowledge, vital for good number sense development.

Pupil performance at the components of number sense indicate a level of understanding below what secondary school-aged pupils should possess, suggesting limited exposure to self-generated answers and an over reliance on algorithmic solution strategies.

6.3 PUPILS' INCLINATION TO MAKE SENSE OF THEIR MATHEMATICAL STUDIES.

The most commonly chosen options for pupil expectations, perceptions and reactions to their mathematics education is summarized below The majority of pupils indicated that they:

- wanted to make sense of their mathematics education
- considered the subject to be dominated by rules
- perceived mathematical ability to be innate
- considered success to come from practise
- wanted to perform better at the subject
- thought mathematics is useful and necessary for employment
- did not enjoy mathematics lessons.

There is a strong inclination to make sense of mathematics study, but at the same time many indicated that they find the subject rule-bound and memory-dependent, indicating a procedural rather than a cognitive expectation for mathematical study. Mathematical ability is considered to be innate and any success is perceived to be due to practice with problems, rather than an acknowledgement of personal ability. The subject has a high status, is considered useful both in everyday application and for gaining employment. Pupils are positive about the subject, wanting to be more successful with it, however many do not enjoy mathematics lessons suggesting that approaches to mathematics teaching need to be examined. These findings were complemented by the interview results.

Reflections on the sense-making aspect of mathematics

The above results are similar to Schoenfeld's (1989) findings. What is of interest is the support given to the notion that ability is innate. In a number of the interviews pupils who regarded themselves as successful at numerical calculations had always felt this way and were confident about their ability, the implications being that unless pupils receive a good grounding in number manipulations they can come to view ability and success as natural rather than learnt. The emphasis on practice is more than likely a reflection of what teachers have convinced pupils is the key to mathematical success. Again, from the interviews, successful pupils were able to justify the calculations that they had used in solving a problem, were able to apply non- algorithmic solutions to questions and gave the impression of being in charge of the whole solution strategy. Their early mathematics lessons had not only contained number drills but the processes used had been validated, introducing a sense component to their education from an early age.

Objective 2: **To investigate pupils' inclination to make sense of their mathematical thinking.**

Though pupils do want to make sense of their mathematics study, the reality is that for many the study has deteriorated into a procedural process with little emphasis on the intellectual level. What support the subject has is as a result of societal pressure, where mathematical competency equates to employment prospects, and this support is not matched by pupils' achievement. Results showed that mathematics lessons are uninteresting for a majority of pupils and that the subject is perceived as being rule-bound. (This was supported from the interview results where most solved problems by slotting them into learnt algorithms.) This suggests there needs to be an examination of how mathematics is taught and what emphasises are placed on the sense aspects of the subject.

6.4 PUPIL PERFORMANCE ON NUMERICAL COMPUTATIONS

The results presented are from interviewing pupils *($n = 11$)*, examining the procedures they use for solving numerical problems, as well as attitudes and activities that foster number sense in a pupil. The results fall into two categories, computational results and attitudinal responses.

Computational Results

These results are presented in three sections, each examining procedural and cognitive knowledge. They give an insight into how pupils, correctly or incorrectly solve problems.

Questions 1 to 5 focused on pupils' use of algorithms for solving numerical calculations (See Figure 17).

- Most tried to solve problems by applying a learnt algorithm, not always correctly.
- Pupils who used non-algorithmic methods demonstrated sound knowledge of equivalent forms of numbers, could easily decompose/recompose numbers and were confident at multiplying and dividing by powers of ten.
- Successful pupils were able to judge the appropriateness of their answer by thoughtful introspection about the values used in the calculation.
- Unsuccessful pupils were not interested in alternate solution to problems. Successful pupils recognized the relevance of alternate approaches.

Questions 6 to 9 and 13 and 14 examined pupils' reasoning about the processes they used for solving computational problems presented in word form rather than numerically.

- Successful pupils have an intuition about performing calculations.
- Successful pupils are more likely to look for other more than one solutions to a problem. They have a greater sense of allowable results and will check for them. Unsuccessful pupils do not look beyond a single answer and do not investigate the possibility of other relevant results.
- Successful pupils were able to explain the logical consequences of a problem.
- Unsuccessful pupils have limited number knowledge, are unable to apply number operations creatively and tend to labour through calculations. They often are unable to conceive an overall solution strategy and have to be lead through a problem.

Question 13 and 14 examined pupils' operation knowledge. Results are summarized below.

- Successful pupils gave quick correct replies to multiplication problems which they can confidently validate. They have had extensive experience with number drill exercises.

- Successful pupils can rearrange a problem, calling on number and operation knowledge, to ease its calculation.
- Unsuccessful pupils can not quickly and accurately tap into number and operation facts

Attitudinal Responses

Responses to three attitudinal questions are listed below.

- Pupils consider mathematics as a useful subject and important for a career.
- Parents are supportive and, where possible, help with homework
- Pupils who considered themselves mathematically competent had always felt confident about the subject and would use mathematics around the home.
- Pupils who admitted to experiencing difficulty with mathematics had difficulty with placing numbers correctly on a number line, ordering numbers, recognizing equivalent forms of numbers, understanding place value of numerals in a number and confidently and accurately reciting their nine times table.

Objective 3: **To identify activities, attitudes and perceptions that contribute to or impair the development of a pupil's number sense.**

Pupils who were successful with calculations would use procedures that suited the question rather than trying to fit it into a learnt algorithms. They had a good quantitative understanding of problems and used it to make judgements about the validity of their answers. These pupils had received a thorough grounding at manipulating numbers, were fluent at multiplication tables, could justify calculations, knew the properties of operations and had experienced number drill exercises. They were confident about their abilities and considered mathematics to be a useful and interesting subject to study. Unsuccessful pupils were algorithmic-dependent, some displaying incorrect or faulty procedural knowledge. They were often very mechanical with their calculations, rarely applying any checks or balances as they proceeded.

These pupils did not exhibit any overall understanding of the procedures that they were using, the answer coming from applying the algorithm rather than from any creative cognitive activity. Unsuccessful pupils had underdeveloped number and operation knowledge, little experience with number drills and were unable to apply number operations flexibly to numerical calculations.

Reflections on activities, attitudes and perceptions that affect a pupil's number sense. Pupils who have well developed number sense exhibit mathematical knowledge that is intuitive in nature - they are can effortlessly call up operation and number facts. The impression is that all this knowledge is well ingrained, can be easily tapped into and has been so constructed that it is interconnected, that is the number and operation knowledge have been formed so that useful connections between the two can be easily made. Pupils who laboured through the calculations were too reliant on learnt procedures and have not been encouraged to look at a range of possible solution paths. It is as though there is one way of performing a calculation and the challenge is to find that way, rather than allowing the numbers and operations in the calculation to dictate how the problem is to be solved. Pupils have been encouraged to have a single method for solving problems, the possibility of alternative is not considered. In some cases using alternate methods have been actively discouraged.

Pupils with well developed number sense are able to confidently and creatively apply number and operation knowledge to solve problems. Their approach to numerical problems, the processes they use, the explanations of how and why their calculations proceed as they do, and their ability to continually monitor the calculation demonstrates mathematical aptitude and confidence.

6.5 IMPLICATIONS OF THIS STUDY

Number drill exercises and learning multiplication tables appear to equip pupils with a critical mass of knowledge upon which they are able to build a useful body of mathematical information. This knowledge takes on the appearance of being intuitive and provides a good base for acquiring further mathematical understanding. Pupils who can recall tables easily also perform calculations with ease and have a good grasp of number knowledge. It is important therefore that the role of memory in the development of number sense is acknowledged, promoted and fostered, particularly for primary school pupils.

Pupils who have developed good number sense have a distinct advantage over those who have not. They are mathematically more successful and have confidence about their mathematics ability. They will devise their own solution to problems and have developed sufficient numbers knowledge to be able to gauge the validity of an answer. It is important that pupils are exposed to a number of solution paths for a problem and that they are shown how operations can be re-expressed so as to make a calculation more manageable.

Pupils' understanding of the six components of number knowledge and the mechanics and properties of number operations need to be refined and extended in structured teaching programmes. Numerical calculations should be viewed as creative rather than procedural exercises. Secondary pupils need more exposure to place value, use of benchmarks and the structure of the number line. Estimation and mental calculations can be used for reinforcing the components of number sense, but it would appear that pupils have had insufficient exposure to these calculation methods.

Number sense in the *Number* levels of *Mathematics in the New Zealand Curriculum* (1992) is specific about the components that are to be taught. What is lacking is an overall statement about the nature of number sense, how number knowledge gives pupils a quantitative appreciation of numbers, how operation knowledge should be used creatively to solve problems, how developing good number sense will benefit a pupil's mathematical development by encouraging him/her to produce his/her own algorithms. Estimation and mental calculations are encouraged, however, apart from techniques for rounding numbers, little indication is given as to how to teach these two important aspects of number sense. In Appendix B a framework specifies cognitive and procedural processes for instructing estimation and mental computations. Efforts championing *problem solving* as a vehicle for teaching mathematics need to be balanced by a similar statement promoting the benefits of developing pupils' *number sense.*

6.6 IMPLICATIONS FOR FURTHER RESEARCH AND CONCLUDING THOUGHTS

A pupil will have number sense if it has been systematically developed. Like problem solving number sense is an approach to mathematics education, one that promotes learning mathematics as a sense-making-exercise. Mathematics is not a body of knowledge to be memorised as rules and applied in learnt algorithms. There is evidence that pupils who score well in traditional pencil and paper type examinations do not necessarily have good number sense (Sowder, 1992; Yang, 1995), indicating a different expertise, based on procedural mastery. The number sense approach differs from this traditional mastery expectation, offering a more expansive view of learning mathematics, one that aims to leave a pupil able to devise his/her own ways of solving numerical problems.

Teachers' Views of Number Sense.

For the number sense approach to teaching mathematics to be successful, research is needed on how teachers perceive number sense, and what their attitude towards the number sense approach is.

Assessment Problems.

Number sense does not lend itself to traditional forms of assessment. Tests that focus on recall fail to acknowledge the creative dimension of number sense. Research into assessment procedures that evaluate how a pupil solves a problem, that reward the creative ways the distributive principle is used in mental calculations, for example, or consider pupils' inclinations to reflect on their calculation, require careful investigation and thought. Similarly, resource packages on number sense exercises need to be developed. Often the most constructive way of having a new approach or idea accepted is to circulate exercises on the types of problems pupils are expected to master.

Calculators and Number Sense.

There is a view that calculator use hinders the development of numeracy and computation skills, that pupils become calculator-dependent and are unable to perform basic numerical calculations on their own. Recent studies of the use of calculators in Western Australian schools (Sparrow & Swan, 1997) and the use of technology in mathematics education in a number of New Zealand schools (Auckland University, 1997) indicate that their inclusion in the classroom, following appropriate instruction, will help with the development of pupil estimation skills, foster their numeracy skills and free pupils to create their own solution algorithms.

A closer consideration of how calculators can be used in developing a quantitative understanding of numbers and an in-depth appreciation of number operations needs to be undertaken, so that teachers will be able to integrate their use into classrooms constructively. The spirit behind the number sense approach to teaching mathematics is based upon pupils having an in-depth knowledge of numbers and number operations so that they can develop their own solutions to numerical problems. These solutions are not necessarily based on some learnt procedure but rather are as a result of bringing together the relevant knowledge in appropriate ways. There is a recursive implication to the number sense approach, in that the more a pupil manufactures his/her own answers the more they come to know and understand numbers and operations on them. Can calculators be used to assist this development ?

Conclusion.

This research study was, in part, a response to the mediocre performance results of secondary and primary pupils at international and national examinations. Too many school pupils are unable to numerically calculate confidently and accurately. The traditional teaching programmes have left many with incomplete and sometimes incorrect number and operation knowledge. Number sense is a refreshing and promising approach to mathematics with its focus on the specific components of number knowledge and on creative ways of applying operations, so that pupils feel that they are *in charge* of the solution that the whole process makes sense. As Schoenfeld (1992, p.343) observes on the traditional rule-dominated approach;

> *Students may simply give up trying to make sense of mathematics...they may come to believe that mathematics is not something they can make sense of, but rather something almost completely arbitrary which must be memorized without looking for meaning.*

The present study has shown that pupils do want to be able to make sense of their learning, that those with good number sense are able to confidently solve numerical problems, while those with underdeveloped number sense struggle with mathematical calculations. A well developed number sense approach to mathematics education will not only assist pupils to perform numerical calculations but will also set them up to take an active part in life beyond school.

APPENDIX A

NUMBER SENSE FRAMEWORK

The original framework for analysing number sense is illustrated below. An overall definition of number sense is followed by three columns, in which the components of number sense are analysed in greater detail moving from left to right. For a full description and analysis see McIntosh, Reys and Reys (1992).

Number Sense: A propensity for and an ability to use numbers and quantitative methods as a means of communicating, processing and interpreting information. It results in an expectation that numbers are useful and that mathematics has a certain regularity (makes sense).

1. Knowledge of and facility with NUMBERS
   - 1.1 Sense of orderliness of numbers
     - 1.1.1 Place value
     - 1.1.2 Relationship between number types
     - 1.1.3 Ordering numbers within and among number types
   - 1.2 Multiple representations for numbers
     - 1.2.1 Graphical/symbolic
     - 1.2.2 Equivalent numerical forms (including decomposition/ recomposition)
     - 1.2.3 Comparison to benchmarks
   - 1.3 Sense of relative and absolute magnitude of numbers
     - 1.3.1 Comparing to physical referent
     - 1.3.2 Comparing to mathematical referent
   - 1.4 System of benchmarks
     - 1.4.1 Mathematical
     - 1.4.2 Personal
2. Knowledge of and facility with OPERATIONS
   - 2.1 Understanding the effect of operations
     - 2.1.1 Operating on whole numbers
     - 2.1.2 Operating on fractions/decimals
   - 2.2 Understanding mathematical properties
     - 2.2.1 Commutativity
     - 2.2.2 Associativity
     - 2.2.3 Distributivity
     - 2.2.4 Identities
     - 2.2.5 Inverses
   - 2.3 Understanding the relationship between operations
     - 2.3.1 Addition/Multiplication
     - 2.3.2 Subtraction/Division
     - 2.3.3 Addition/Subtraction
     - 2.3.4 Multiplication/Division
3. Applying knowledge of and facility with numbers and operations to COMPUTATIONAL SETTINGS
   - 3.1 Understanding the relationship between problem context and the necessary computation
     - 3.1.1 Recognise data as exact or approximate
     - 3.1.2 Awareness that solutions may be exact or approximate
   - 3.2 Awareness that multiple strategies exist
     - 3.2.1 Ability to create and/or invent strategies.
     - 3.2.2 Ability to apply different strategies
     - 3.2.3 Ability to select an efficient strategy
   - 3.3 Inclination to utilise an efficient representation and /or method
     - 3.3.1 Facility with various methods (mental, calculator, paper/pencil)
     - 3.3.2 Facility choosing efficient number(s)
   - 3.4 Inclination to review data and result for sensibility
     - 3.4.1 Recognise reasonableness of data
     - 3.4.2 Recognise reasonableness of calculation

APPENDIX B

ESTIMATION FRAMEWORK

The following estimation framework has been compiled from studies by Reys et al (1982), Sowder and Wheeler (1989). The first column is the types of knowledge required, the second processes and strategies that successful estimators employ. Mental computations employ many of the same procedures.

- Conceptual Knowledge
  - The simplification process (ways a number can be modified to produce a solvable problem)
  - Use of multiple strategies.
  - Proximity principle (appropriateness of answer)
- Procedural Knowledge
  - Reformulation (structure of problem is retained, the numbers are changed)
    - — by Rounding
    - — by Truncating
    - — by Substitution
  - Translation (change the structure of the problem by altering the operations)
  - Compensation (amend the result to suit the numbers used)
- Factual Knowledge
  - Powers of ten
  - Place value
  - Magnitude of numbers
  - Ability to compute mentally
  - Knowledge of number and operation facts
- Affective Factors
  - Confidence in own mathematical ability
  - Tolerance of errors
  - Recognition of the usefulness of estimation
- Mental Computation
  - Factorising for the Distributive Principle

3333_{3} 444444 41.752 rounds to 41.8

$4! = 4 \times 3 \times 2 \times 1.$

* % + = -- --

NUMBER SENSE QUESTIONNAIRE

YOUR COOPERATION IN COMPLETING THIS STUDY IS MUCH APPRECIATED.

INSTRUCTIONS: This Questionnaire is in five parts.

PART ONE Consists of 11 Mental Arithmetic questions which will be read to you.

PART TWO Contains 9 Estimation Questions. These will be read out to you and a space is provided on the sheet for your answer.

PART THREE Contains 16 Multi Choice questions.
Put a cross $\boxed{X}$ in the square below the letter that you consider to be correct.

$8^0 = 1$

PART FOUR Contains 9 Written Response answers. There is space provided on the sheet for answers and working.

PART FIVE Contains 16 personal responses.
Please put a cross $\boxed{X}$ in the square below the response that you think best describes how you feel.

All calculations are to be performed **without** using a calculator

$1125 \div 25 = (1000 \div 25) + (100 \div 25) + (25 \div 25)$

$1 \times 2 \times 3 \times 10 = 60 = \frac{100}{2} + 10$

has factors of 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60

$16/2/{}^{1}/_{2}/{}^{1}/_{4} = 64$

1 2 3 4 5 6 7 8 9 0

$2/3 \times 18 = 12$

$299 - 125 = (200 - 100) + (90 - 20) + (9 - 5)$

$32 = 2^5$

$180 + 68 = 180 + 60 + 8$

$1270 = (130 \times 10) - (130 \times 1)$

PART ONE : Mental Computation.
(The following questions are to be read to the pupils and answered on the sheet provided in the booklet, at the same time each will be displayed on an overhead transparency)

Calculate the following.

1) 1005 − 784.

2) 520 x 0.5.

3) Find two numbers that differ by 16 but add to 24.

4) What number is added to 700,500 to give 901,500?

5) If you travel at 60 km/h for three hours and twenty minutes what distance will you cover?

6) $\sqrt[2]{256}$.

7) 1 ÷ 1/8.

8) Express 2472 grams in Kilograms.

9) 24 cm^2 is the same as how many mm^2?

10) A palindromic number is one that reads the same both ways, eg. 1991. Write down a five digit palindromic number.

11) Find the next number in the sequence 3, 4, 7, 11, 18,.......

PART ONE : Mental Computation Answer Sheet.

Place your answers on the line provided for each question. You can use the blank space for rough working.

1) ____________________

2) ____________________

3) ____________________

4) ____________________

5) ____________________

6) ____________________

7) ____________________

8) ____________________

9) ____________________

10) ____________________

11) ____________________

PART TWO: Estimation.
These questions will be read to you. Write down your answer in the space provided. In all cases your answer should be an estimate. If you are unable to answer a question leave it.

12) There are 623 books to be placed on shelves. If a shelf can hold 32 books estimate how many shelves are going to be used

Number of shelves = ______________

13) 38 people each won $ 32.45c. Estimate the total pay out

Total pay out = ____________

14) 47 boxes of fruit at a cost of $12.45c per box were bought by 19 families. Estimate the cost for each family
Cost per family = ___________

15) Estimate $\dfrac{4.98 \times 32.41}{6.1}$ Estimate = ____________

16) Estimate 39.8 (52.6 + 187.4) Estimate = ____________

17) Estimate 162.7 x $\frac{5}{6}$ Estimate = ____________

18) Estimate $(28.73)^2$ Estimate = ____________

19) Estimate the length of time it would take you to walk from school to the South Dunedin New World Supermarket.

Estimate Time = ______________

20) Estimate the weight of a can of Coke.

Estimate Weight = ______________

PART THREE: Multi-Choice. Put a $\boxed{X}$ in the square you consider is correct.

21) The number sixty four thousand and fifty two is shown by

A. 64520
B. 64052
C. 640520

A	B	C
☐	☐	☐

22) Which of the following is the largest number

A. 0.011
B. 0.0043
C. 0.0101

A	B	C
☐	☐	☐

23) Which is larger $\frac{5}{6}$ or $\frac{7}{9}$

A. $\frac{5}{6}$
B. $\frac{7}{9}$
C. Both the same

A	B	C
☐	☐	☐

24) For the following number line

the result of multiplying the value at D by that at E will be

A. Above 1
B. Below 1
C. Above 2

A	B	C
☐	☐	☐

25) For the following number line 0 $\frac{1}{4}$ $\frac{1}{2}$ 1 2

A fraction between $\frac{1}{4}$ and $\frac{1}{2}$ with a denominator of 10 is

A. $\frac{6}{10}$

B. $\frac{8}{10}$

C. $\frac{4}{10}$

A ☐ B ☐ C ☐

26) Which of the following is a possible value for the maximum number of people that can attend a match at Carisbrook?

A. 40,000

B. 80,000

C. 8,000

A ☐ B ☐ C ☐

27) The number 43,724,189 is closest to which of the following

A. 42,000, 189

B. 43,724,389

C. 43,624,189

A ☐ B ☐ C ☐

28) Which of the following statements is correct for the two points labelled A and B as shown on the number line

A -3 B -1 0 1

A. B > A

B. B = A

C. B < A

A ☐ B ☐ C ☐

29) For the numbers 1.4237 and 1.42371 which of the following is NOT true

A. 1.42371 is larger than 1.4237

B. $1.42371 \div 1.4237$ is greater than 1

C. $1.4237 - 1.42371$ is greater than 0

A ☐ B ☐ C ☐

30) The numeral 2 in 428157 represents

A. Two hundred

B. Twenty Thousand

C. Two Thousand

A ☐ B ☐ C ☐

31) To the number 3,200 is added as many hundreds as there are thousands. The result is

A. 5,200

B. 3,400

C. 3,500

A ☐ B ☐ C ☐

32) The numeral 4 in the decimal 1.2347 has the same value as

A. 4/100   A   B   C
B. 4/1000   ☐   ☐   ☐
C. 4/10

33) For the number 5,678,143 the numerical difference between the number represented by 6 and the one represented by 7 is

A. 600,000   A   B   C
B. 530,000   ☐   ☐   ☐
C. 70,000

34) Which of the following will give the same result

i. 8 x 45 and 40 x 9   A   B   C
ii. 1.8 x 3.5 and 18 x 0.35   i & ii   ii & iii   i & ii iii
iii. 88 x 87 and 91 x 85   ☐   ☐   ☐

35) If X and Y are two numerals such that $X + Y < 4$, $X > 0$ and $Y > 0$ then how many pairs of whole numbers could X and Y be

A. 3   A   B   C
B. 4   ☐   ☐   ☐
C. 5

36) If A is a digit and $0.\overline{A}$ is a number shown by

$0.\overline{A} = 0.AAAAAAA......$

What is the value of $0.\overline{8} + 0.\overline{6}$

A. 1.5 A   B   C
B. $1.\overline{5}$ ☐   ☐   ☐
C. $\overline{1.5}$

PART FOUR: Written Responses.

In this section wherever possible please show how you have calculated your answer.

37) Put the following numbers, -570, -40, -9200, -3 in order, smallest through to largest

____

smallest   largest

38) In the empty boxes below write a two digit number with the second digit a zero so the statements are correct

A. $\square < 90$

B. $28 + \square > 68$

C. $\square - 4 > 23 + 6$

39) In the space below write down the number that is 24 less than the largest possible four digit number

Answer ____________

40) Which is larger A) a six digit number of all nines or

B) a seven digit number of all sevens

The largest is: A ☐ or B ☐

41) The following are the heat times for two 100 metre races

Heat 1	Athlete	Time	Heat 2	Athlete	Time
	A	9.948		G	9.875
	B	9.845		H	9.904
	C	9.914		I	9.813
	D	9.837		J	9.943
	E	9.967		K	9.895
	F	9.800		L	9.805

a) To qualify a runner must record a time under 9.945 seconds. How many of the athletes qualified? ____________

b) Which athlete has a time that is closest to the qualifying time?
Closest qualifier is ____________

c) Which athlete has a time that is furthest from the qualifying time?

____

42) For each of the following give a measurement that you think approximates the situation described.

a) The height of the classroom door is________________

b) The length of time to walk one kilometre is____________

c) The length of the South Island of NZ is______________

d) The temperature in your classroom is_______________

43) The number 56 can be rewritten in a variety of ways for example

$56 = 8 \times 7$ or $40 + 16$

In the space below write 4 other representations for 56

56 = ________

56= ________

56 = ________

56 = ________

44) This year the 7th of August is a Wednesday . Write down the dates of the other Wednesdays in August

________ ________ ________

45) From the numbers 1168, 1487, 1279 which would you add together to give an answer that is between 1700 and 1900

______ + ______

PART FIVE : Personal Responses.

For each of the following I want you to put a cross in the box [X] of the response that you think best describes how you feel.

	Yes	Maybe / Sometimes	Don't Know	No
46) I find Mathematics interesting.	☐	☐	☐	☐
47) I look forward to Maths lessons.	☐	☐	☐	☐
48) I think Mathematics is a useful subject.	☐	☐	☐	☐
49) I think that learning Mathematics will help me to get a job.	☐	☐	☐	☐
50) I think that some people can do Mathematics and some can't.	☐	☐	☐	☐
51) To be good at Mathematics you have to practise on lots of problems.	☐	☐	☐	☐

	Yes	Maybe / Sometimes	Don't Know	No
52) To me Mathematics is a jumble of unrelated ideas.	☐	☐	☐	☐
53) I expect to be able to make sense of Mathematics lessons.	☐	☐	☐	☐
54) I think that to be good at Mathematics you need to have a good memory.	☐	☐	☐	☐
55) I think that there are too many rules to be learnt in Mathematics.	☐	☐	☐	☐
56) I always need to have the Mathematics ideas explained to me a number of times.	☐	☐	☐	☐
57) I think that calculators can be used to do most school Mathematics problems.	☐	☐	☐	☐
58) I was good at Mathematics in primary school.	☐	☐	☐	☐
59) I would like to be better at Mathematics.	☐	☐	☐	☐
60) My parents want me to be good at Mathematics.	☐	☐	☐	☐

	Very good	Good	average	poor
61) Put a cross $\boxed{X}$ in the box that best describes your mathematical ability.	☐	☐	☐	☐

INTERVIEW QUESTIONS.

" For each of the following I would like you to answer the question and to also explain the ways that you have used to get the answer. Are there other ways of doing these problems?, if so please state them".

1) 5 x 472.

2) 420 ÷ 140

3) \$72 – \$28.43c.

4) 38 + 52 + 127.

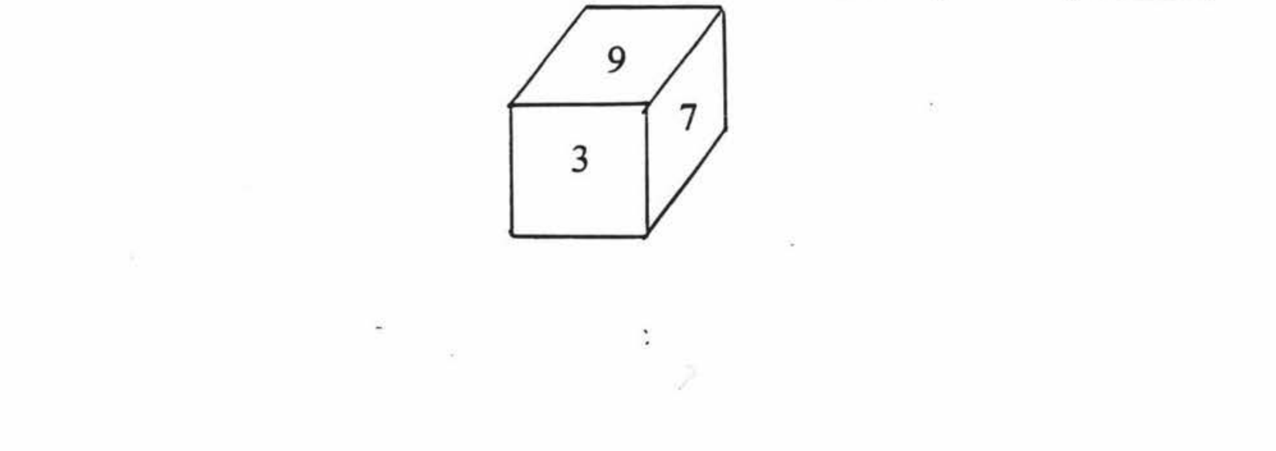
5) Find 20% of \$48.

6) The number 28 is an even number. It is also the same as $2^4 + 2^3 + 2^2$ or 15 + 13. It also has the property that it is divisible by 4. What other properties are you able to find for 28?

7) Can you find a number which when squared will give the same result as when it is added to itself?

8) Take this tooth pick and this ruler, now if you were to lay one million (1,000,000) of these tooth picks end to end how far would they reach? Can you relate this distance to any other?

9) A six face dice has the numbers 1,3,5,7,9,11 on each face. Six of these dice are rolled together. What combinations of the six rolled dice can be put together to give the following totals 5, 8, 28, 31, 56, 68 ?



ATTITUDE QUESTIONS:

10) Do you think that learning about mathematics is important? Could you explain your answer?

11) What do the other members of your family think about mathematics?

12) Are there any areas of mathematics that you find difficult or have trouble understanding?

13) What is the value of 9 x 8?
How do you get that?
How do know that the result is true?
What is the value of 8 x 9?
Again, can you tell me why this is true?

14) Consider the problem 61 + 58 - 17 + 22
Can you think of ways of rewriting this question so that it could be easier to calculate, for example : 60 + 60 - 1 - 2 - 17 + 22
or 22 + 58 - 17 + 61

15) Would you say that you were good at mathematics?
If yes i) Have you always felt that you were good at the subject?
ii) Would you use mathematics to solve problems you come across, say around the house or in jobs you have had?
Please give examples.
iii) Do you expect to be able to do maths problems that you are given at school?
iv)Do you enjoy doing mathematics? What is it that you enjoy about Mathematics?
v) If you come across a problem that you can't do what, do you do?
vi) Could you recite your 9 times table? Up to what number can you do the multiplication table?

If no i) Can you recite your 9 times table?

ii) If you were asked to total 27 + 63 - 18 + 12 how would you do it?
iii) On the number line given below could you place the number $1\frac{4}{5}$

0 1 2

iv) Which of the following numbers is the largest, 1.1473 or 1.1475? Why?

v) Which of the following will give the same value?

a) 20% of 10

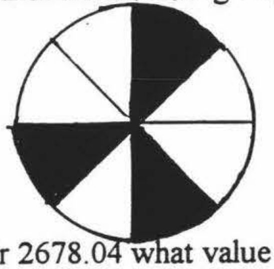
b) $\frac{6}{2}$

c) $52 \div 27$

d) 2

e) $\frac{40}{20}$

vi) What fraction of the following diagram is shaded?



vii) In the number 2678.04 what value does the numeral 6 represent?

viii) In this problem you are considering the multiplication 14 x 12. The following is an attempt to find an easier way to do the problem. Examine the working, correct it if you think it is wrong then complete it.

$$\begin{aligned} 14 \times 12 &= 14(10 + 2) \\ &= 14 \times 10 - 12 \\ &= ? \end{aligned}$$

Could you think up a way of doing 42 x 37 ?

ix) If you are unable to do a problem what do you do?

Leave it

Look for a similar problem in your notes or text book to help

Ask someone else

Leave it and go back to it later.

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FACULTY OF
INFORMATION
AND
MATHEMATICAL
SCIENCES

DEPARTMENT OF
MATHEMATICS

QUESTIONNNAIRE CONSENT FORM
MATHEMATICS EDUCATION RESEARCH PROJECT

"Number Sense"

I have read the accompanying letter regarding the research project on Number Sense and understand the purpose of the investigation.

I understand that I have the right to withdraw from the study any time and to decline to answer any particular questions in the study.

I agree to provide information to the researcher on the understanding that it is held in complete confidentiality and that I am not identified in the report.

I AGREE TO PARTICIPATE IN THE RESEARCH

Student's Name: ________________________________

Signature: ________________________________

Parent/Guardian Signature: ______________ ______________

Date: ________________________________

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FACULTY OF INFORMATION AND MATHEMATICAL SCIENCES

DEPARTMENT OF MATHEMATICS

INTERVIEW CONSENT FORM
FOR A MATHEMATICS EDUCATION RESEARCH PROJECT

"NUMBER SENSE"

Your son/daughter has recently completed a written questionnaire on Number Sense. So that I can gain a deeper appreciation of the processes involved I would like to interview your son/daughter about their understanding and appreciation of working with numbers.

It is understood that he/she has the right to withdraw and that he/she may decline to answer any question put to him/her.

This interview is being taped and it is understood that he/she has the right to ask for the tape to be turned off at any time.

Responses provided are held in complete confidentiality and students' identity will not be disclosed in any research report.

I AGREE TO PARTICIPATE IN THE INTERVIEW.

Students Name: ______________________________

Signature: ______________________________

Parents/Guardian Signature: ____________ ____________

Date: ______________________________.

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