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Minimising L^p Distortion for Mappings Between Annuli

A thesis presented in partial fulfillment of the requirements
for the degree of

Master of Science

in

Mathematics

at Massey University, Albany, New Zealand

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2006

ABSTRACT

When deforming or distorting one material object into another, for various physical reasons the final deformation is expected to minimise some sort of energy functional. Classically, the theory of quasiconformal mappings provides us with a theory of distortion, yielding some limited results concerning minimising the maximal distortion. The calculus of variations is aimed at extremising certain kinds of functionals (such as the integral of the gradient squared or of distortion over a region in the complex plane). This thesis investigates quasiconformal and related mappings between annuli, introduces some novel results, and outlines some conjectures for further research.

ACKNOWLEDGEMENTS

The writing of a Masters thesis is a practically insurmountable task if one does not have the necessary and wide support network of colleagues, friends, and family. The names I acknowledge here are but the tip of the proverbial iceberg; it is impossible for me to fully acknowledge the time and effort of everyone that has supported me, and it is inevitable that names will be omitted that should have been included. My apologies for this.

I would like to thank Gaven Martin, for making it possible for me to pursue this course of study, and for supervising this thesis and providing an endless stream of helpful advice. My parents, Theo and Els, thank you for being there for me when I needed it the most, and for giving me the confidence and upbringing to choose my own destiny. Alexandra, thank you for believing in my ability when I didn't, for making me feel proud of my achievements, for encouraging me when I did not feel I was achieving, and for your unconditional love. Graeme Wake, thank you for your tireless pursuit of solutions to some of the problems I have encountered in the course of this thesis, and your good advice. NDK, Joanne, David, Ron, Haydn, Qing and Mark, thank you for being friends when I needed it most, for your patience, support and advice. I also want to thank the staff at the Institute of Information and Mathematical Sciences at the Albany Campus for their support and for helping me throughout the entire process, and for the opportunities you have given me.

Thank you, all who have been involved in some way in helping me write this thesis. I could not have done it without you.

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INTRODUCTION

If we look at simply connected domains, then the Riemann Mapping Theorem (see Section 1.5) tells us that any simply connected region (which is not equal to the whole complex plane \mathbb{C} , or the extended complex plane (the Riemann sphere) $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$) is conformally equivalent to the unit disc \mathbb{D} . The exceptions to this theorem are obvious: the complex plane, because a bounded analytic function that is entire on the whole complex plane is constant (by Liouville's Theorem), and thus the image of the complex plane cannot be the unit disc; and the extended complex plane because it is both open and closed.

In light of Riemann's Mapping Theorem, a theorem by Schottky (see Section 2.3) and some other well-known results in quasiconformal theory, the problem of finding minimisers of distortion functionals (between doubly connected regions) can be simplified to looking at annuli.

Classical measures of distortion are investigated thoroughly, and a new way of calculating distortion is examined in some cases. In particular, the calculus of variations is applied to the problem, and gives sharp results for the classical distortion measures (with exception of some cases—see Section 3.2) as well as limited results for the new distortion measure. Some interesting results arise; in particular, there are cases where there are no minimisers, as well as cases where minimisers are difficult to find.

Throughout, the notation $\dot{\rho}$ will be used for the real derivative of ρ , to distinguish from the complex derivative, which uses the notation ρ' . Also, note that if a proposition or theorem number begins with “.0” instead of the number of the chapter that it is in, then it may be found in the appendix.

We have attempted to provide all information necessary to understand this thesis, and where not able to do so, provide reference to materials where further details may be found.

1. CONFORMAL MAPPINGS AND THE RIEMANN MAPPING THEOREM

The concept of a conformal mapping is built up through the basic ideas of analytic functions and the Cauchy-Riemann derivatives. Riemann's Mapping Theorem is examined in detail, and a proof given. The use of this important theorem is to specify precisely the first case where topology plays a role in finding extrema of distortion functionals. Other theorems, such as Schottky's Theorem on conformal mappings between annuli, will then be used to further simplify and isolate the problem.

1.1 Analytic functions

Definition 1.1.1. Let $\Omega \subset \mathbb{C}$ be an open set. Then $f : \Omega \rightarrow \mathbb{C}$ is called *(complex-)differentiable* at $z_0 \in \Omega$ if

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists or, equivalently, if

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. Furthermore, f is said to be **analytic** if f is complex-differentiable at each $z \in \Omega$; and f is said to be **analytic at** $z_0 \in \Omega$ if it is analytic on a neighborhood of z_0 . If f is analytic on \mathbb{C} , then f is said to be **entire**.

The next few propositions and theorems will be given without proof; it

is assumed the reader is familiar with these properties.

Proposition 1.1.2. *If $f'(z_0)$ exists, then f is continuous at z_0 .*

Proposition 1.1.3. *The following derivative rules hold:*

- $(cf)'(z_0) = cf'(z_0)$, where c is any constant;
- $(f + g)'(z_0) = f'(z_0) + g'(z_0)$, the **sum rule** for derivatives;
- $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$, the **product rule** for derivatives;
- Provided $g(z_0) \neq 0$, $\left(\frac{f}{g}\right)' = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{[g(z_0)]^2}$, the **quotient rule** for derivatives.

Proposition 1.1.4. *(The **Chain Rule** for complex differentiation)*

Let $f : \Omega \rightarrow \mathbb{C}$ and $g : \Omega' \rightarrow \mathbb{C}$ be such that $f(\Omega) \subset \Omega'$. For any $z_0 \in \Omega$, if f is differentiable at z_0 and g is differentiable at $f(z_0)$, then $g \circ f$ is differentiable at z_0 , and

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

It follows from the rules for differentiation that if f and g are analytic functions on some open domain $\Omega \subset \mathbb{C}$ that cf (where c is some constant), $f + g$, and fg are analytic on Ω , and that f/g is analytic in $\Omega \setminus \{z : g(z) = 0\}$ if this is not empty. Likewise the composition $g \circ f$, if $f(\Omega)$ is contained in the domain of g , is analytic on Ω .

Notable in the definition of the derivative of a function f at z_0 in some open region Ω is that the existence of

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

contains no constraint on how z_0 is approached; differentiability of f means that this limit is the same regardless of what sequence or curve we follow to get to z_0 . This leads us to the formulation of the Cauchy-Riemann Equations.

The Cauchy-Riemann equations

For ease of notation, we denote each complex number in the form $z = x + iy$, where x and y are real numbers, and we denote the partial derivative $\partial(f)/\partial(x)$, for example, by f_x . Suppose a function $f(z) = f(x, y) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$. Then by holding the imaginary component of z constant ($y = y_0$) and approaching z_0 along a line parallel to the real axis, we obtain (Palka, 1991, p. 69):

$$\begin{aligned} f'(z_0) &= \lim_{z \rightarrow z_0} \frac{f(x + iy_0) - f(x_0 + iy_0)}{x - x_0} \\ &= \lim_{z \rightarrow z_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} \\ &= \lim_{z \rightarrow z_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i \lim_{z \rightarrow z_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0} \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0) \\ &= f_x(z_0). \end{aligned}$$

Similarly, if we hold the real component of z constant ($x = x_0$) and approaching z_0 along a line parallel to the imaginary axis, we get

$$\begin{aligned} f'(z_0) &= \lim_{z \rightarrow z_0} \frac{f(x_0 + iy) - f(x_0 + iy_0)}{i(y - y_0)} \\ &= v_y(x_0, y_0) - iu_y(x_0, y_0) \\ &= -if_y(z_0). \end{aligned}$$

Combining the results from the previous paragraph, we conclude that

$$u_x = v_y \quad , \quad v_x = -u_y,$$

or, equivalently,

$$f_x = -if_y,$$

at (x_0, y_0) . These famous equations are known as the *Cauchy-Riemann equations*. A necessary condition for f to be complex-differentiable at z_0 is that f

satisfies the Cauchy-Riemann equations there. However, this is not in general a sufficient condition for differentiability; see, for example, Palka (1991, p. 75, Example 3.3). But this need not significantly reduce the utility of these equations: when there is more information available concerning f we can sometimes establish differentiability using the Cauchy-Riemann equations.

Theorem 1.1.5. (*The Cauchy-Riemann Theorem*)

Let $f = u + iv$ be defined in an open region $\Omega \subset \mathbb{C}$, and suppose that the partial derivatives of u and v with respect to x and y exist everywhere in Ω . If each of u_x, u_y, v_x, v_y is continuous at $z_0 \in \Omega$ and if the Cauchy-Riemann equations are satisfied at z_0 , then f is differentiable at z_0 and $f'(z_0) = f_x(z_0)$.

Proof. (Bak and Newman, 1997, pp. 36–37) Let $h = \xi + i\eta$. We are required to show that

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} = f_x(z_0).$$

By the Mean Value Theorem for functions of a real variable (Theorem .0.1),

$$\begin{aligned} \frac{u(z_0 + h) - u(z_0)}{h} &= \frac{u(x_0 + \xi, y_0 + \eta) - u(x_0, y_0)}{\xi + i\eta} \\ &= \frac{u(x_0 + \xi, y_0 + \eta) - u(x_0 + \xi, y_0)}{\xi + i\eta} \\ &\quad + \frac{u(x_0 + \xi, y_0) - u(x_0, y_0)}{\xi + i\eta} \\ &= \frac{\eta}{\xi + i\eta} u_y(x + \xi, y + t_1\eta) \\ &\quad + \frac{\xi}{\xi + i\eta} u_x(x + t_2\xi, y + \eta) \end{aligned}$$

and, similarly,

$$\begin{aligned} \frac{v(z_0 + h) - v(z_0)}{h} &= \frac{\eta}{\xi + i\eta} v_y(x + \xi, y + t_3\eta) \\ &\quad + \frac{\xi}{\xi + i\eta} v_x(x + t_4\xi, y + \eta) \end{aligned}$$

for some t_k with $0 < t_k < 1$ and $k \in \{1, 2, 3, 4\}$. Hence

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} &= \frac{\eta}{\xi + i\eta} [u_y(x + \xi, y + t_1\eta) + iv_y(x + \xi, y + t_3\eta)] \\ &\quad + \frac{\xi}{\xi + i\eta} [u_x(x + t_2\xi, y + \eta) + iv_x(x + t_4\xi, y + \eta)]. \end{aligned}$$

Since *ex hypothesi* $f_x = -if_y$, we can write $f_x(z_0)$ in the form

$$f_x(z_0) = \frac{\eta}{\xi + i\eta} f_y(z_0) + \frac{\xi}{\xi + i\eta} f_x(z_0).$$

Subtracting this from both sides we obtain

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} - f_x(z_0) &= \frac{\eta}{\xi + i\eta} [(u_y(x + \xi, y + t_1\eta) - u_y(x_0, y_0)) \\ &\quad + i(v_y(x + \xi, y + t_3\eta) - v_y(x_0, y_0))] \\ &\quad + \frac{\xi}{\xi + i\eta} [(u_x(x + t_2\xi, y + \eta) - u_x(x_0, y_0)) \\ &\quad + i(v_x(x + t_4\xi, y + \eta) - v_x(x_0, y_0))]. \end{aligned}$$

But, since $\xi, \eta \rightarrow 0$ as $h \rightarrow 0$, each of the bracketed expressions tends to 0. Furthermore,

$$\left| \frac{\eta}{\xi + i\eta} \right|, \left| \frac{\xi}{\xi + i\eta} \right| \leq 1,$$

so that

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} - f_x(z_0) = 0$$

as required. □

Before departing this section, it should be mentioned that there is a certain formalism associated with the preceding discussion. The *formal* z - and \bar{z} -*partial derivatives of f at z_0* (also known as the Cauchy-Riemann derivatives of f at z_0), denoted $f_z(z_0)$ and $f_{\bar{z}}(z_0)$ respectively, are defined by the formulae

$$f_z(z_0) = \frac{1}{2} (f_x(z_0) - if_y(z_0))$$

and

$$f_{\bar{z}}(z_0) = \frac{1}{2} (f_x(z_0) + if_y(z_0)).$$

If f satisfies the Cauchy-Riemann equations at z_0 , then it follows that $f_{\bar{z}}(z_0) = 0$; if f is differentiable at z_0 , it is immediate that $f'(z_0) = f_z(z_0)$. However, it must be emphasized the existence of $f_z(z_0)$ does not place the same strength of demands on f as does the existence of $f'(z_0)$.

1.2 Conformal mappings

Definition 1.2.1. A domain Ω in \mathbb{C} is called **simply connected** if $\widehat{\mathbb{C}} \setminus \Omega$ is connected.

Assuming that Ω is a region in the complex plane and that $f : \Omega \rightarrow \mathbb{C}$ is a member of the class $C^1(\Omega)$, we define the continuous real-valued function J_f on Ω by

$$J_f(z) = \det Df(z) = u_x(z)v_y(z) - u_y(z)v_x(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2.$$

Definition 1.2.2. A mapping $f : \Omega \rightarrow \mathbb{C}$ is called **conformal at** $z_0 \in \Omega$ if there exist $r > 0$ and $\theta \in [0, 2\pi)$ such that for any curve $\gamma(t)$ that is differentiable at $t = 0$, with $\gamma(t) \in \Omega$ and $\gamma(0) = z_0$, which satisfies $\gamma'(0) \neq 0$, the curve $\sigma = f \circ \gamma$ is differentiable at $t = 0$, we have

$$|\sigma'(0)| = r|\gamma'(0)|$$

and

$$\arg \sigma'(0) = \arg \gamma'(0) + \theta \pmod{2\pi}.$$

A mapping $f : \Omega \rightarrow \mathbb{C}$ is called **locally conformal** if it is conformal at every point of Ω . If, in addition to being locally conformal, f is one-to-one, then we say f is **conformal** on Ω .

See Marsden and Hoffman (1987, p. 71). Intuitively, being a simply connected domain means that there are no ‘holes’ in the domain; and to be

a conformal map is to preserve angles between tangent vectors of curves. In practice, at least in two dimensions the property of being conformal boils down to being analytic with a nonzero derivative. During our discussion, we shall use whichever definition of conformality is most convenient at the time.

Proposition 1.2.3. (*The Conformal Mapping Theorem*)

Let $f : \Omega \rightarrow \mathbb{C}$ be analytic, and $f'(z_0) \neq 0$. Then f is conformal at z_0 , with $\theta = \arg f'(z_0)$ and $r = |f'(z_0)|$.

Proof. Let $\gamma(t)$ be a curve in Ω , differentiable at $t = 0$, $\gamma(0) = z_0$, and $\gamma'(0) \neq 0$. Then by the chain rule, $\sigma'(0) = (f \circ \gamma)'(0) = f'(\gamma(0)) \cdot \gamma'(0) = f'(z_0) \cdot \gamma'(0)$. Therefore, taking $r = |f'(z_0)|$ and $\theta = \arg f'(z_0) \pmod{2\pi}$,

$$|\sigma'(0)| = |f'(z_0) \cdot \gamma'(0)| = |f'(z_0)| |\gamma'(0)| = r |\gamma'(0)|$$

and

$$\arg(\sigma'(0)) = \arg f'(z_0) + \arg \gamma'(0) = \arg \gamma'(0) + \theta \pmod{2\pi}$$

as required. □

Proposition 1.2.4. Suppose $f : \Omega \rightarrow \mathbb{C}$ is a conformal mapping. Then $f^{-1}(w)$ exists at each point $w \in f(\Omega)$, $(f^{-1})' = \frac{1}{f'}$, and f^{-1} is conformal on $f(\Omega)$.

Proof. Since f is conformal, $f' \neq 0$ on Ω . Since f is bijective (one-to-one and onto) on $f(\Omega)$, f^{-1} exists. By the Inverse Function Theorem (.0.3), f^{-1} is one-to-one and analytic, with derivative given by $\frac{1}{f'}$, on $f(\Omega)$. Furthermore, since f' is nonzero and finite on Ω , $(f^{-1})'$ is nonzero and finite on $f(\Omega)$. Thus f^{-1} is conformal on $f(\Omega)$. □

1.3 Normal families

Assuming familiarity with pointwise convergence, we define the following.

Definition 1.3.1. A sequence of functions $f_n : \Omega \rightarrow \mathbb{C}$ is said to **converge uniformly** to a function f if to each $\varepsilon > 0$ there corresponds a natural number N such that $|f_n(z) - f(z)| < \varepsilon$ for all $z \in \Omega$ whenever $n \geq N$. Also, a sequence of functions $(f_n)_{n \geq 1}$ in Ω is said to **converge normally** to a function f if (f_n) converges pointwise to f and if (f_n) converges uniformly to f on each compact set in Ω .

Uniform convergence is obviously a stronger condition than pointwise convergence. Normal convergence sometimes goes by other names in the literature, for example ‘locally uniform convergence’ or ‘uniform convergence on compacta in Ω ’. Also, to check for normal convergence it is not necessary that we check for uniform convergence on every compact set contained in Ω ; it is sufficient that (f_n) converges uniformly on each closed disk contained in Ω . For a proof, see Palka (1991, p. 247).

Definition 1.3.2. If Ω is an open subset of \mathbb{C} , a set \mathcal{F} of analytic functions on Ω is called a **normal family** if every sequence of functions in \mathcal{F} has a subsequence which converges uniformly on closed disks in Ω .

Definition 1.3.3. A family of functions \mathcal{F} defined on a region Ω is said to be **pointwise bounded in Ω** if for each fixed $z \in \Omega$ the set of values $\{f(z) : f \in \mathcal{F}\}$ is a bounded set of complex numbers. A family of functions \mathcal{F} defined on a region Ω is called **locally bounded** if its members are uniformly bounded on each compact set in Ω .

The latter means that for each compact $A \subset \Omega$ there exists a constant $m(A)$ with the property that $|f(z)| \leq m(A)$ for each $f \in \mathcal{F}$ and $z \in A$.

Definition 1.3.4. A family of continuous functions \mathcal{F} defined on some region Ω is called **normal in Ω** , or **pre-compact in Ω** , provided each sequence $(f_n)_{n \geq 1}$ from \mathcal{F} has at least one subsequence $(f_{n_k})_{k \geq 1}$ that converges normally in Ω .

Definition 1.3.5. A family of continuous functions \mathcal{F} defined on a region Ω is said to be **equicontinuous at $z_0 \in \Omega$** if to each $\varepsilon > 0$ there corresponds

a $\delta > 0$ such that $|z - z_0| < \delta$ implies that $|f(z) - f(z_0)| < \varepsilon$ for each $f \in \mathcal{F}$. The family \mathcal{F} is called **equicontinuous on Ω** (or just equicontinuous) if it is equicontinuous at every point of Ω .

Some texts, for example Marsden and Hoffman (1987, p. 225), also use the term *uniformly equicontinuous* to describe an equicontinuous family of functions. The next theorem is essential in proving the Arzelà-Ascoli Theorem, a necessary ingredient in proving the Riemann Mapping Theorem.

Theorem 1.3.6. (Palka, 1991, p. 279) *Let (f_n) be a sequence from an equicontinuous family of functions \mathcal{F} defined on a region Ω . Suppose that this sequence converges pointwise in Ω . Then it converges normally in Ω .*

Proof. Let f be the pointwise limit of (f_n) in Ω , and choose an arbitrary compact set K in Ω . It is required to show that (f_n) converges uniformly to f on K . In view of Theorem .0.5, we need to show that (f_n) is a uniform Cauchy sequence on K . Assume, on the contrary, that (f_n) is not uniformly Cauchy. Then there must exist some number $\varepsilon > 0$ such that there is no integer N for which

$$|f_m(z) - f_n(z)| < \varepsilon$$

for every $z \in K$ and all $m > n \geq N$. In particular, choose $N = k$ for some integer k . To this k , there must correspond some integers m_k, n_k and some point $z_k \in K$ with the property that

$$|f_{m_k}(z) - f_{n_k}(z)| \geq \varepsilon. \tag{1.1}$$

From this we obtain the sequence of points $(z_k)_{k \geq 1}$ in K , which has at least one accumulation point, z_0 , in K because K is compact. The family of functions \mathcal{F} is equicontinuous at z_0 (since it is equicontinuous everywhere in Ω), so we can select $\delta > 0$ such that for each n ,

$$|f_n(z) - f_n(z_0)| < \frac{\varepsilon}{3} \tag{1.2}$$

whenever $|z - z_0| < \delta$. Note that as $k \rightarrow \infty$, both $m_k, n_k \rightarrow \infty$ and hence

$$|f_{m_k}(z_0) - f_{n_k}(z_0)| \rightarrow |f(z_0) - f(z_0)| = 0.$$

Thus we can find k_0 with the property that

$$|f_{m_k}(z_0) - f_{n_k}(z_0)| < \frac{\varepsilon}{3} \quad (1.3)$$

whenever $k \geq k_0$. Furthermore, as z_0 is an accumulation point of (z_k) , we can choose an index $k \geq k_0$ such that $|z_k - z_0| < \delta$. Equations (1.1), (1.2), and (1.3), together with the triangle inequality yield, for this k :

$$\begin{aligned} \varepsilon &\leq |f_{m_k}(z_k) - f_{n_k}(z_k)| \\ &\leq |f_{m_k}(z_k) - f_{m_k}(z_0)| + |f_{m_k}(z_0) - f_{n_k}(z_0)| + |f_{n_k}(z_0) - f_{n_k}(z_k)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

which is absurd. Hence, in fact, (f_n) is a uniform Cauchy sequence in K , and therefore $f_n \rightarrow f$ uniformly on K . \square

The next section deals with some crucial results on which the proof of Riemann's Mapping Theorem rests.

1.4 The Arzelà-Ascoli Theorem and Montel's Theorem

Before establishing this piece of the Riemann Mapping Theorem jigsaw, we require two preliminary lemmata.

Lemma 1.4.1. (*Palka, 1991, p. 248*) *Suppose that each function in a sequence $(f_n)_{n \geq 1}$ is continuous in an open set Ω and that the sequence converges normally in Ω to the limit function f . Then f is continuous in Ω , and*

$$\int_{\gamma} f(z) dz = \lim_{n \rightarrow \infty} \int_{\gamma} f_n(z) dz$$

for every piecewise smooth path γ in Ω .

Proof. Fix a point $z_0 \in \Omega$. We verify that $f : \Omega \rightarrow \mathbb{C}$ is continuous at z_0 . Given $\varepsilon > 0$, we must find a $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $z \in \Omega$ and $|z - z_0| < \delta$. By the triangle inequality, for any point $z \in \Omega$ and any index n ,

$$|f(z) - f(z_0)| \leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)|. \quad (1.4)$$

Note that, since Ω is open, we can find $r > 0$ such that $\overline{D}(z_0, r) \subset \Omega$. Using the normal convergence of (f_n) on Ω , we find that (f_n) converges uniformly on $\overline{D}(z_0, r)$ by definition. Thus we can find an index n such that

$$|f_n(z) - f(z)| < \frac{\varepsilon}{3}$$

for each $z \in \overline{D}(z_0, r)$. In particular,

$$|f(z) - f_n(z)| + |f_n(z_0) - f(z_0)| < \frac{2\varepsilon}{3}$$

for every $z \in \overline{D}(z_0, r)$. Furthermore, since f_n is continuous on Ω for each n , it is by definition possible to choose $\delta > 0$ such that if $|z - z_0| < \delta$ then

$$|f_n(z) - f_n(z_0)| < \frac{\varepsilon}{3}.$$

Plugging these results into (1.4), we conclude that

$$|f(z) - f(z_0)| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

whenever $|z - z_0| < \delta$, which establishes the continuity of f on an arbitrary point z_0 in, and hence the entirety of, Ω .

To prove the second part of the lemma, let γ be a piecewise smooth path in Ω . Given $\varepsilon > 0$, using the normal convergence of (f_n) , we can find N such that

$$|f_n(z) - f(z)| < \frac{\varepsilon}{\ell(\gamma) + 1}$$

holds whenever $n \geq N$ for γ restricted to each compact set in Ω . Hence

$$\begin{aligned} \left| \int_{\gamma} f_n(z) dz - \int_{\gamma} f(z) dz \right| &= \left| \int_{\gamma} (f_n(z) - f(z)) dz \right| \\ &\leq \int_{\gamma} |(f_n)(z) - f(z)| |dz| \leq \int_{\gamma} \frac{\varepsilon}{\ell(\gamma) + 1} |dz| = \frac{\varepsilon \ell(\gamma)}{\ell(\gamma) + 1} < \varepsilon. \end{aligned}$$

whenever $n \geq N$, for γ restricted to compacta in Ω . As γ is piecewise smooth (smooth in a finite number of compacta in Ω), this proves the desired result. \square

Lemma 1.4.2. (*Palka, 1991, pp. 281–282*) *Let $(f_n)_{n \geq 1}$ be a sequence from an equicontinuous family of functions \mathcal{F} defined on Ω . Suppose that the sequence $(f_n(\xi))$ is convergent (to $f(\xi)$) for every ξ belonging to a dense subset Σ of Ω . Then (f_n) converges normally in Ω .*

Proof. Given Theorem 1.3.6, we are required to show that (f_n) converges pointwise in Ω . To this end, fix $z \in \Omega$, and choose $\varepsilon > 0$. By the equicontinuity of \mathcal{F} at z , we can choose $\delta > 0$ such that

$$|f_n(w) - f_n(z)| < \frac{\varepsilon}{3}$$

for all indices n , whenever $|w - z| < \delta$. Since Σ is dense in Ω , we can find $\zeta \in \Sigma$ such that $|\zeta - z| < \delta$. The sequence $(f_n(\zeta))$ converges to $f(\zeta)$, by hypothesis, and hence is a Cauchy sequence. Therefore, there is an integer N such that

$$|f_m(\zeta) - f_n(\zeta)| < \frac{\varepsilon}{3}$$

whenever $m > n \geq N$. Hence

$$\begin{aligned} |f_m(z) - f_n(z)| &\leq |f_m(z) - f_m(\zeta)| + |f_m(\zeta) - f_n(\zeta)| + |f_n(\zeta) - f_n(z)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

whenever $m > n \geq N$, and hence $(f_n(z))$ is a Cauchy sequence. Since the choice of $z \in \Omega$ was arbitrary, the pointwise convergence of (f_n) in Ω is

established, thus proving the lemma. \square

Theorem 1.4.3. (The Arzelà-Ascoli Theorem)

A family \mathcal{F} of functions that are defined and continuous on some region Ω is a normal family if and only if it is both equicontinuous and pointwise bounded in Ω .

Proof. (Palka, 1991, pp. 282–284) Assume that \mathcal{F} is equicontinuous and pointwise bounded in Ω . Note that the set $S = \{z \in \Omega : \Re(z) \in \mathbb{Q}, \Im(z) \in \mathbb{Q}\}$ is dense in Ω and that, being countable, it is possible to list the elements of this set in a sequence. Let (z_n) be such a listing. Consider the sequence $(f_n(z_1))$. Since \mathcal{F} is pointwise bounded, this sequence is bounded; by the Bolzano-Weierstrass Theorem (Theorem .0.4), there exists at least one accumulation point of $(f_n(z_1))$. Call this w_1 . The sequence $(f_n(z_1))$ has a subsequence converging to w_1 . That is, there exists a sequence of indices $m_{1,1} < m_{1,2} < m_{1,3} < \dots$ such that

$$\lim_{k \rightarrow \infty} f_{m_{1,k}}(z_1) = w_1.$$

Note that the sequence of integers $(m_{1,k})$ is associated with just z_1 (hence the subscript, $1, k$). Now, the sequence $(f_{m_{1,k}}(z_2))_{k \geq 1}$ is also a bounded sequence of complex numbers. Take one of its accumulation points, w_2 , and extract another subsequence of integers $m_{2,1} < m_{2,2} < m_{2,3} < \dots$ from the sequence we already had, $(m_{1,k})$, with the property that

$$\lim_{k \rightarrow \infty} f_{m_{2,k}}(z_2) = w_2.$$

Repeating this process, to each positive integer l we assign a strictly increasing sequence of positive integers $(m_{l,k})$ such that

$$\lim_{k \rightarrow \infty} f_{m_{l,k}}(z_l) = w_l$$

and such that $(m_{l+1,k})$ is a subsequence of $(m_{l,k})$. For $k \geq 1$, set $n_k = m_{k,k}$. By construction $n_1 < n_2 < \dots$. For fixed l , the subsequence (f_{n_k}) thus obtained is also a subsequence of $(f_{m_{l,k}})$, with the possible exception of the

first $l - 1$ terms. Therefore, for each l the sequence $(f_{n_k}(z_l))$ converges to w_l ; and hence the sequence $(f_{n_k}(\xi))$ has a limit point for each $\xi \in \Sigma$. By Lemma 1.4.2, (f_{n_k}) converges normally in Ω . Hence, by definition, \mathcal{F} is normal in Ω . \square

An important consequence of the Arzelà-Ascoli Theorem, and the result that is commonly used in proving Riemann's Mapping Theorem, is the following theorem due to Paul Montel (1876-1975):

Theorem 1.4.4. (Montel's Theorem)

(Palka, 1991, p. 285) Let \mathcal{F} be a family of functions that are analytic in an open set Ω . Suppose that \mathcal{F} is locally bounded in Ω . Then \mathcal{F} is a normal family in this set.

Proof. Since the family of functions \mathcal{F} is locally bounded in Ω , it is pointwise bounded in Ω . We prove that \mathcal{F} is equicontinuous in Ω ; for then, by the Arzelà-Ascoli Theorem, it is normal in Ω . Fix $z_0 \in \Omega$. Choose $r > 0$ such that the closed disk $K = \overline{D}(z_0, 2r) \subset \Omega$. Since \mathcal{F} is locally bounded, there exists $m = m(K) > 0$ such that $|f(\zeta)| \leq m$ whenever $f \in \mathcal{F}$ and $\zeta \in K$. Now, for $z \in D(z_0, r)$, we use Cauchy's integral formula, together with the fact that $|\zeta - z_0| = 2r$ implies that $|\zeta - z| \geq r$ for $r \in D$, to obtain the estimate

$$\begin{aligned} |f(z) - f(z_0)| &= \left| \frac{1}{2\pi i} \int_{|\zeta - z_0|=2r} \frac{f(\zeta)d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int_{|\zeta - z_0|=2r} \frac{f(\zeta)d\zeta}{\zeta - z_0} \right| \\ &= \frac{|z - z_0|}{2\pi} \left| \int_{|\zeta - z_0|=2r} \frac{f(\zeta)d\zeta}{(\zeta - z)(\zeta - z_0)} \right| \\ &\leq \frac{|z - z_0|}{2\pi} \int_{|\zeta - z_0|=2r} \frac{|f(\zeta)||d\zeta|}{|\zeta - z||\zeta - z_0|} \\ &\leq \frac{m|z - z_0|}{r}. \end{aligned}$$

Given $\varepsilon > 0$, set $\delta = \min\{r, \frac{r\varepsilon}{m}\}$. Then the above estimate gives us that

$|f(z) - f(z_0)| < \varepsilon$ for all $f \in \mathcal{F}$ whenever $|z - z_0| < \delta$. Hence \mathcal{F} is equicontinuous at z_0 . Since z_0 was arbitrary, the Arzelà-Ascoli Theorem now shows that \mathcal{F} is normal in Ω . \square

1.5 The Riemann Mapping Theorem

Theorem 1.5.1. (The Riemann Mapping Theorem)

Let Ω be a simply connected region such that $\Omega \neq \mathbb{C}, \widehat{\mathbb{C}}$, and choose $z_0 \in \Omega$. Then there exists a unique conformal mapping $f : \Omega \rightarrow \mathbb{D}$, from Ω onto the unit disk \mathbb{D} , such that $f(z_0) = 0$ and $f'(z_0) > 0$.

See Marsden and Hoffman (1987, p. 347). Before we establish the proof of this very useful theorem, there is one more preliminary lemma that needs to be proved; another lemma, which will be used, is cited without proof.

Lemma 1.5.2. Let Ω be a simply connected region properly contained in the complex plane and let z_0 be a point of Ω . Then there exists a conformal mapping $f : \Omega \rightarrow \mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.

Proof. It suffices to show that we can map Ω into the unit disc conformally. For, once that is accomplished, we need only compose with a linear fractional transformation that takes z_0 to 0 and then multiply by a constant $e^{i\theta}$ chosen such that the derivative of the resulting map at z_0 is positive. If Ω is bounded, say $|z - z_0| < \rho$ for all $z \in \Omega$, then the map $z \mapsto (z - z_0)/\rho$ will do.

If Ω is not bounded, then there is at least one point a that it omits. The translation f_1 under which $z \mapsto (z - a)$ takes Ω to a simply connected region Ω_1 not containing 0. We can now take any branch f_2 of $\log z$ in Ω_1 ; there is guaranteed to be at least one such branch by Proposition .0.6. Moreover, by definition of the logarithm, f_2 is a univalent analytic function which takes Ω_1 to a simply connected region Ω_2 . Fix $w_0 \in \Omega_2$, together with a radius ρ , such that $\overline{D}(w_0, \rho) \subset \Omega_2$. Setting $\tilde{w}_0 = w_0 + 2\pi i$, note that $\overline{D}(\tilde{w}_0, \rho)$ is disjoint from Ω_2 . For, supposing there exists a point \tilde{w} in $\overline{D}(\tilde{w}_0, \rho) \cap \Omega_2$, then $\tilde{w} = f_2(\tilde{z})$ for some point \tilde{z} in Ω_1 , and also $\tilde{w} = w + 2\pi i$ for some

$w \in \overline{D}(w_0, \rho)$. Furthermore there would exist a point $z \in \Omega_1$ such that $w = f_2(z)$, implying that

$$\tilde{z} = e^{f_2(\tilde{z})} = e^{\tilde{w}} = e^{w+2\pi i} = e^w = e^{f_2(z)} = z,$$

which leads to $w = f_2(z) = f_2(\tilde{z}) = \tilde{w} = w + 2\pi i$, an absurdity. Since $|z - \tilde{w}_0| > \rho$ for each $z \in \Omega_2$, the Möbius transformation given by $f_3(z) = \rho/(z - \tilde{w}_0)$ is a conformal mapping that takes Ω_2 to Ω_3 , a region contained in \mathbb{D} . The composition $f = f_3 \circ f_2 \circ f_1$ is thus a conformal map taking Ω into the unit disc. \square

A consequence of this lemma is the following:

Lemma 1.5.3. *Let Ω be a simply connected domain not equal to $\mathbb{C}, \hat{\mathbb{C}}$, let $z_0 \in \Omega$ and $f : \Omega \rightarrow \mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$. If $f(\Omega) \neq \mathbb{D}$, then there exists a conformal mapping $g : \Omega \rightarrow \mathbb{D}$ with $g(z_0) = 0$ and $g'(z_0) > f'(z_0)$.*

The existence of f in this lemma is guaranteed by Lemma 1.5.2. For a proof, see Palka (1991, pp. 418–419). We are now in a position to prove the Riemann Mapping Theorem.

Proof of the Riemann Mapping Theorem

Proof. Given a simply connected region Ω properly contained in \mathbb{C} and given $z_0 \in \Omega$, we must show that there is exactly one analytic function on Ω which maps Ω onto \mathbb{D} in a one-to-one fashion, with $f(z_0) = 0$ and $f'(z_0) > 0$. To this end, define

$$\mathcal{F} = \{f : \Omega \rightarrow \mathbb{D} \mid f \text{ is analytic and one-to-one on } \Omega, f(z_0) = 0, \text{ and } f'(z_0) > 0\}.$$

It is obvious that \mathcal{F} is locally bounded in Ω . Lemma 1.5.2 ensures that \mathcal{F} is nonempty. Suppose that $r > 0$ has the property that $D(z_0, r) \subset \Omega$. Then

Cauchy's estimate (Theorem .0.9) yields

$$f'(z_0) = |f'(z_0)| \leq r^{-1}$$

for every member of \mathcal{F} . Hence, the set

$$S = \{f'(z_0) : f \in \mathcal{F}\}$$

is bounded. Denote the supremum of S by s . For each positive integer n , select $f_n \in \mathcal{F}$ such that

$$s - \frac{1}{n} \leq f'_n(z_0) \leq s.$$

Now we use Montel's Theorem (1.4.4) to extract a subsequence (f_{n_k}) from (f_n) that converges normally in Ω to some limit function f , analytic in Ω . Note that, in particular,

$$f(z_0) = \lim_{k \rightarrow \infty} f_{n_k}(z_0) = 0 \quad \text{and} \quad f'(z_0) = \lim_{k \rightarrow \infty} f'_{n_k}(z_0) = s > 0.$$

Therefore f is non-constant in Ω , and hence (with a little work, using Hurwitz's Theorem (Theorem .0.11)) f is univalent in Ω . Since $f(\Omega) \subset \overline{\mathbb{D}}$, by the Open Mapping Theorem (Theorem .0.10) $f(\Omega)$ is a subset of \mathbb{D} , and hence $f \in \mathcal{F}$. Now, suppose that $f(\Omega) \neq \mathbb{D}$. Then, by Lemma 1.5.3, we could find $g \in \mathcal{F}$ with $g'(z_0) > f'(z_0) = s$, which is absurd. Hence f maps Ω onto the unit disk.

For uniqueness, suppose that g is a second such mapping. Consider $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ given by $\varphi = g \circ f^{-1}$. This is a conformal mapping of \mathbb{D} onto itself, with $\varphi(0) = 0$ and $\varphi'(0) = g'(0)/f'(0) > 0$. The only conformal mappings from \mathbb{D} onto itself are rotations about the origin (this is a well-established result in the theory of conformal mappings), and since $\varphi'(0) > 0$, we must have $\varphi(z) = z$. Therefore, $f(z) = g(z)$ for each $z \in \Omega$, and uniqueness is established. \square

Remarks

The foregoing proof of Riemann's Mapping Theorem suggests a general strategy for obtaining minima for certain classes of problems. Namely, we first establish equicontinuity in a certain class of maps (for instance a minimising sequence). The Arzelà-Ascoli Theorem then provides a limiting map. We must then establish the regularity of the minimising map, so establishing it belongs to the class of maps under consideration.

We will see that quite restrictive hypotheses are necessary to guarantee this equicontinuity (quasiconformality is enough, but having integrable distortion is not). Thus the ideas around quasiconformal maps and equicontinuity lead to the existence of mappings of smallest maximal distortion (see next section). When we look at integrable distortion we must find new tools since equicontinuity has not been established except for mappings with distortion which is exponentially integrable (Martin and Iwaniec, 2001).

2. QUASICONFORMAL MAPPINGS AND DISTORTION

In this chapter, the notion of quasiconformal mappings is introduced, and some crucial results are laid out, which (together with Riemann's Mapping Theorem) help to simplify the problem of finding minimisers of distortion of mappings between (some) regions in the complex plane.

2.1 Quasiconformal mappings

The idea of conformal mappings can be somewhat extended to include a much wider variety of functions that are "almost" conformal; these are the so-called *quasiconformal maps*.

Definition 2.1.1. Let Ω and Ω' be domains in $\overline{\mathbb{R}^n}$ and $f : \Omega \rightarrow \Omega'$ a homeomorphism. For $x \in \Omega$ and any $r > 0$ such that $B(x, r) \subset \Omega$ set

$$L(x, f, r) = \max_{|h|=r} |f(x+h) - f(x)|, \quad \text{and} \quad l(x, f, r) = \min_{|h|=r} |f(x+h) - f(x)|.$$

Note that $0 < l(x, f, r) \leq L(x, f, r) < \infty$, so that we can obtain the **linear dilatation of f at x** ,

$$H(x, f) = \limsup_{r \rightarrow 0} \frac{L(x, f, r)}{l(x, f, r)}.$$

Furthermore, if there is a number $K < \infty$ such that $H(x, f) \leq K$ for all $x \in \Omega \setminus \{\infty, f^{-1}(\infty)\}$ (i.e. if $H(x, f)$ is bounded throughout this domain), then we say that f is **(K -)quasiconformal**.

Obviously $1 \leq H(x, f) \leq \infty$. If $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear bijection, then

$H(x, A) = H(A)$ for all $x \in \mathbb{R}^n$, where

$$H(A) = \frac{\max_{|h|=1} |Ah|}{\min_{|h|=1} |Ah|}.$$

If f is differentiable at x and $J(x, f) \neq 0$ (i.e. if f is a diffeomorphism), then $H(x, f) = H(\dot{f}(x))$, and if $K = 1$ then f is conformal. An immediate consequence of this is that a mapping f is quasiconformal if and only if there exists $K \geq 1$ such that for each $x \in \Omega \setminus \{\infty, f^{-1}(\infty)\}$ there exists r_x such that $H(x, r) \leq K$ for all $0 < r \leq r_x$.

Take explicitly the example of the linear bijection $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$; this is a quasiconformal mapping. If $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ are the eigenvalues of $A^T A$, then $L(A) = \sqrt{\lambda_1}$, $l(A) = \sqrt{\lambda_n}$, and so $H(x, A) = H(A) = \sqrt{\frac{\lambda_1}{\lambda_n}}$. In particular, the bijection A is conformal if and only if all the eigenvalues of $A^T A$ are equal; that is, A is orthogonal or a multiple of an orthogonal matrix.

From here on, the notation for distortion will be K , to represent the smallest value of K that will do the job in the definition of quasiconformality. The notation H will figure only when it is important to the discussion; for the purposes of this chapter, we are interested in minimizing K .

Figure 2.1 displays the action of some quasiconformal function f (as example) on an *infinitesimal* level. That is, the circle that is being mapped can be arbitrarily small ($|r|$ small); we already know that if the circle and ellipse in Fig. 2.1 were of some fixed size, Riemann's Mapping Theorem would allow us to construct a conformal map (of no infinitesimal distortion) between them. Given this example, the number K will be (some nice function of) the ratio b/a .

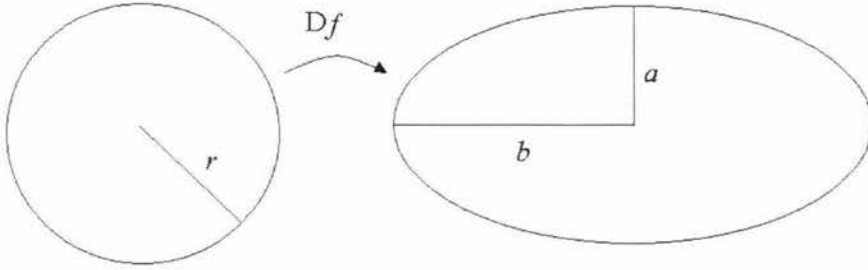


Fig. 2.1: The action of a quasiconformal mapping f on an “infinitesimal” circle.

2.2 The modulus of a curve family

This section outlines an important conformal invariant. The modulus of a family of curves is an important tool in the study of quasiconformal mappings, for (like dilatation) it gives a quantity that helps determine how nearly conformal a mapping is.

Suppose that Γ is a family of curves in \mathbb{R}^n . A (Borel measurable) function $\rho : \Gamma \rightarrow [0, \infty)$ is called *admissible* if for each $\gamma \in \Gamma$

$$\int_{\gamma} \rho \, ds \geq 1.$$

Denote by $\text{adm}(\Gamma)$ the set of all admissible functions. For each $p \geq 1$, set

$$M_p(\Gamma) = \inf_{\rho \in \text{adm}(\Gamma)} \int_{\mathbb{R}^n} \rho^p \, dm(x),$$

defining $M_p(\Gamma) = \infty$ if $\text{adm}(\Gamma) = \emptyset$. The number $M_p(\Gamma)$ is called the *p-modulus* of Γ . Then $0 \leq M_p(\Gamma) \leq \infty$. The most important case for our purposes is the case $p = n$. In this case, denote $M_n(\Gamma)$ simply by $M(\Gamma)$. In the literature, $\lambda(\Gamma) = 1/M(\Gamma)$ is referred to as the *extremal length* of Γ . Note that M_p is an outer measure in the space of all curves in \mathbb{R}^n (for a proof, see Väisälä (1971)).

To check that the modulus is a conformal invariant, consider a conformal

homeomorphism $f : \Omega \rightarrow \Omega'$, from a region Ω to another region Ω' in \mathbb{R}^n . Let Γ be the family of curves contained inside the region Ω , and likewise set $\Gamma' = f\Gamma$ (contained inside Ω'). The differential matrix Df gives rise to the Jacobian determinant of f :

$$|Df|^n = \det(Df) = J(x, f)$$

Since $|Df|$ is the largest eigenvalue of Df , observe that because $|\lambda_{\max}|^n = |\lambda_1 \lambda_2 \dots \lambda_n|$ the Jacobian matrix $J(x, f)$ must be (a multiple of) an orthogonal matrix. Choose $\rho \in \text{adm}(f\Gamma)$. Then $\rho(f(x)) : \Omega \rightarrow \mathbb{R}^+ \cup \{0\}$ (ρ is extended by 0 to \mathbb{R}^n , but this contributes nothing to the integral, so we may consider ρ to be on Ω only). Pick $\gamma \in \Gamma$. Then $\gamma \circ f = \gamma' \in \Gamma'$. Now

$$1 \leq \int_{\gamma'} \rho \, ds = \int_{\gamma \circ f} \rho \, ds \leq \int_{\gamma} \rho(f)|Df| \, ds$$

so that $\rho(f)|Df| \in \text{adm}(\Gamma)$. Hence, by definition,

$$M(\Gamma) \leq \int_{\Omega} \rho^n(f)|Df|^n \, dx = \int_{\Omega} \rho^n(f)J(x, f) \, dx = \int_{\Omega'} \rho^n \, dm(x)$$

by change of variables. Taking the infimum of this over all admissible ρ , we get that $M(\Gamma) \leq M(\Gamma')$. Now, we may repeat this process of obtaining an upper bound on the modulus $M(\Gamma')$, using f^{-1} instead. Since f is conformal, so is f^{-1} , and we obtain the other inequality, $M(\Gamma') \leq M(\Gamma)$. Hence, in fact,

$$M(\Gamma) = M(\Gamma').$$

This leads to another definition for a quasiconformal mapping: a mapping $f : \Omega \rightarrow \Omega'$ is K -quasiconformal if for every curve family Γ in Ω ,

$$\frac{1}{K}M(\Gamma) \leq M(f\Gamma) \leq KM(\Gamma). \quad (2.1)$$

For a proof that this is an equivalent definition to the one given in Section 2.1, see Väisälä (1971, pp. 46–48).

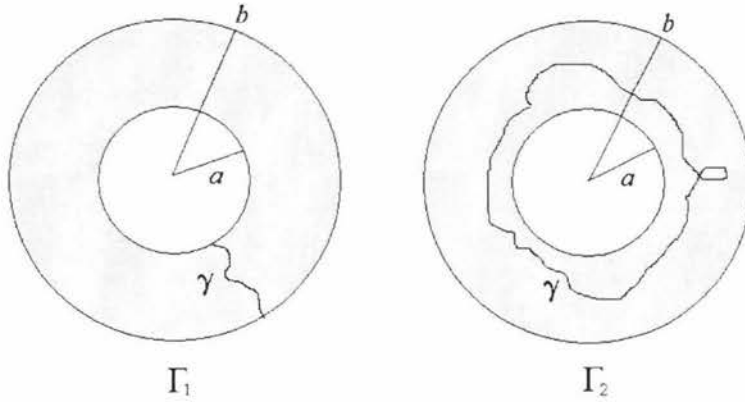


Fig. 2.2: The two curve families of an annulus.

Computing $M(\Gamma)$ can often be difficult, but considerations of the geometry of the situation often greatly simplify the task. Furthermore, an upper bound is often a lot easier to find. In particular, note that for $\rho \in \text{adm}(\Gamma)$,

$$M_p(\Gamma) \leq \int \rho^p dm.$$

Note that if $\ell(\gamma) \geq r > 0$ for each $\gamma \in \Gamma$ (where the γ all lie in some Borel set G), then

$$M_p(\Gamma) \leq \frac{m(G)}{r^p}.$$

For, defining $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\rho(x) = \frac{1}{r}$ for $x \in G$ and $\rho(x) = 0$ otherwise, then $\rho \in \text{adm}(\Gamma)$, and the required inequality follows.

Taking the example of the annulus, which is the important object in our study, we calculate the modulus of one of its possible curve families. With reference to Fig. 2.2, we calculate the modulus of the curve family Γ_1 . The reason for the choice of curve family will become apparent in Chapter 3; the geometry of the situation reduces the problem to dealing only with this curve family.

Let A be a spherical ring, with inner radius a and outer radius b . Take the family of curves Γ to be the set of all curves joining the sphere of radius

a to the sphere of radius b in A . Note that we could consider the curves that join these two spheres that are not contained entirely in A , but the size of $\int \rho$ on such curves is always at least as big as the size of $\int \rho$ on curves that are contained in A (since curves joining the two spheres must have at least some segment contained in A which also joins those spheres), so we restrict to the curves contained in A .

Let $\rho \in \text{adm}(\Gamma)$. Consider the lines $\gamma_u : [a, b] \rightarrow \mathbb{R}^n$, defined by $\gamma_u = tu$, where $t \in [a, b]$ and $u \in S^{n-1}$. We get

$$\int_{\gamma_u} \rho ds = \int_a^b \rho(tu) t^{\frac{n-1}{n}} t^{-\frac{n-1}{n}} dt$$

and therefore, by Hölder's inequality,

$$\begin{aligned} 1 &\leq \left(\int_{\gamma_u} \rho ds \right)^n \leq \int_a^b \rho^n(tu) t^{n-1} dt \left(\int_a^b t^{-1} dt \right)^{n-1} \\ &= \left(\log \frac{b}{a} \right)^{n-1} \int_a^b \rho^n(tu) t^{n-1} dt. \end{aligned}$$

Hence, as $b > a$,

$$\int_{S^{n-1}} \frac{1}{\left(\log \frac{b}{a} \right)^{n-1}} du \leq \int_{S^{n-1}} \int_a^b \rho^n(tu) t^{n-1} dt du = \int_A \rho^n dm$$

by change of variables (t^{n-1} is the Jacobian). Noting that

$$\int_{S^{n-1}} \frac{1}{\left(\log \frac{b}{a} \right)^{n-1}} du = \frac{\omega_{n-1}}{\left(\log \frac{b}{a} \right)^{n-1}},$$

where ω_{n-1} is the area of S^{n-1} , and taking the infimum over all ρ , we see that

$$M(\Gamma) \geq \frac{\omega_{n-1}}{\left(\log \frac{b}{a} \right)^{n-1}}.$$

On the other hand, define ρ by

$$\rho(x) = \begin{cases} \frac{1}{(|x| \log \frac{b}{a})^{n-1}} & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\rho \in \text{adm}(\Gamma)$, and so in fact

$$M(\Gamma) = \frac{\omega_{n-1}}{(\log \frac{b}{a})^{n-1}}.$$

In the case of the annulus in the complex plane, we have $n = 2$ so that

$$M(\Gamma) = \frac{2\pi}{\log \frac{b}{a}}.$$

The modulus of a curve family is important to our discussion for two reasons. The first and foremost is that it is a conformal invariant (i.e. a quantity that has the same value for any conformal $f \in \mathcal{S}(A_1, A_2)$), and so gives us geometric information about how close to conformal a mapping is. This is why it is natural that a quasiconformal map may be defined as in equation (2.1). The second reason is that the ratio of a to b is relevant in an important theorem from Schottky, which will help us to simplify the problem further; this is discussed in the next section.

2.3 Simplifying assumptions

Because of the Riemann Mapping Theorem, the first case where topology might play a role in determining functions of minimal distortion will be doubly connected regions (in the case of conformal mappings, there is no infinitesimal distortion). A well-known result in quasiconformal theory is that every doubly connected region is conformally equivalent to some annulus; see, for example, Ahlfors (1979, pp. 255–256). Using this result, the problem of finding the function of minimal distortion for doubly connected

regions reduces to that for functions between annuli. A further useful result is Schottky's Theorem:

Theorem 2.3.1. (Schottky, 1877; Astala et al., 2006) *An annulus $\mathbb{A} = \{z : r < |z| < R\}$ can be mapped conformally onto the annulus $\mathbb{A}' = \{z' : r' < |z'| < R'\}$ if and only if $\frac{R}{r} = \frac{R'}{r'}$. Moreover, every conformal mapping $f : \mathbb{A} \rightarrow \mathbb{A}'$ takes the form $f(z) = \lambda z^{\pm 1}$, where $|\lambda| = \frac{r'}{r}$ or $|\lambda| = r'R$ as the case may be.*

A simple proof of this theorem may be found in Astala et al. (2006); an outline which closely follows the proof presented in that paper is included here.

Consider the family $\mathcal{S}(A_1, A_2)$ of all orientation preserving homeomorphisms between annuli $A_1 = \{z \in \mathbb{C} : r_1 < |z| < R_1\}$ and $A_2 = \{z \in \mathbb{C} : r_2 < |z| < R_2\}$ in the complex plane, whose distributional first-order derivatives are locally integrable. Following Astala et al. (2006), we compute that for $f \in \mathcal{S}(A_1, A_2)$,

$$\Re \iint_{A_1} \frac{f_N}{rf} = \pm 2\pi \log \frac{R_2}{r_2} \stackrel{\text{def}}{=} \pm \text{Mod } A_2,$$

and

$$\Im \iint_{A_1} \frac{f_T}{rf} = \pm 2\pi \log \frac{R_1}{r_1} \stackrel{\text{def}}{=} \pm \text{Mod } A_1,$$

where $f_N \stackrel{\text{def}}{=} \frac{\partial f}{\partial r}$ and $f_T \stackrel{\text{def}}{=} \frac{1}{r} \frac{\partial f}{\partial \theta}$ are the normal and tangential derivatives of f , and $\text{Mod } A$ is the modulus of the annulus A . That is, there are invariable integrals in the curve family $\mathcal{S}(A_1, A_2)$.

Another method uses the Jacobian determinant $J(z, f)$. Letting dz and dz' denote Lebesgue area elements on A_1, A_2 respectively, then with f sufficiently smooth a change of variables shows that

$$\iint_{A_1} \frac{|J(z, f)| dz}{|f(z)|^2} \leq \iint_{A_2} \frac{dz'}{|z'|^2} = \text{Mod } A_2$$

for all $f \in \mathcal{S}(A_1, A_2)$. This is an inequality rather than an equality because Lusin's property (that sets of measure zero cannot be mapped onto sets of positive measure) may fail to hold for f .

Recall the Cauchy-Riemann equations (Section 1.1). In polar coordinates, they may be stated as

$$f_T(z) = i f_N(z) \quad \text{for almost every } z = re^{i\theta}.$$

For a conformal map, the tangential and normal derivatives of f are orthogonal vectors of the same length:

$$J(z, f) = |f_N|^2 = |f_T|^2.$$

Extending this to a quasiconformal mapping with distortion factor $1 \leq K < \infty$,

$$|f_N|^2 \leq K \cdot J(z, f) \quad \text{and} \quad |f_T|^2 \leq K \cdot J(z, f).$$

Putting all this information together and using Hölder's inequality, we obtain

$$\begin{aligned} (\text{Mod } A_1) \cdot (\text{Mod } A_2) &\geq \iint_{A_1} \frac{dz}{|z|^2} \cdot \iint_{A_1} \frac{|J(z, f)| dz}{|f(z)|^2} \geq \left(\iint_{A_1} \frac{\sqrt{J(z, f)} dz}{|z| |f(z)|} \right)^2 \\ &= \begin{cases} \left(\iint_{A_1} \left| \frac{f_N}{rf} \right| \right)^2 \\ \left(\iint_{A_1} \left| \frac{f_T}{rf} \right| \right)^2 \end{cases} \geq \begin{cases} \left(\Re \iint_{A_1} \frac{f_N}{rf} \right)^2 \\ \left(\Im \iint_{A_1} \frac{f_T}{rf} \right)^2 \end{cases} \geq \begin{cases} (\text{Mod } A_2)^2 \\ (\text{Mod } A_1)^2 \end{cases}, \end{aligned}$$

and therefore in order for a conformal map to exist from A_1 onto A_2 , it is necessary that $\text{Mod } A_1 = \text{Mod } A_2$. Equality is obtained when this condition is satisfied, and (with some further argument regarding the additional condition that $f(z) = \lambda z^{\pm 1}$) the proof is complete.

The reason the latter details are omitted is because the relevant part of the theorem is that we have now shown that there exist conformal mappings

(i.e. mappings with no infinitesimal distortion) between annuli with the same modulus. With this theorem, we further simplify the problem by needing only to look at annuli with a common parameter; inner radius equal to 1.

3. RADIAL MAPPINGS AND THE EULER-LAGRANGE EQUATION

3.1 Radial mappings

As a preliminary to studying mappings between annuli we first look at mappings from the unit disk (to the unit disk). Since rotations are conformal, the next simplest mapping to look at is a radial mapping. From considerations of the geometry of the problem (in particular, the radial symmetry), the minimisers of any distortion functional are likely to be radial mappings. The purpose of this section is largely motivational and the calculations here presented are heuristic (i.e. not rigorous).

Definition 3.1.1. A *radial mapping* of the unit disk is a one-to-one mapping f (from the unit disk onto the unit disk), given by

$$f(re^{i\theta}) = \rho(r)e^{i\theta},$$

where ρ is some real-valued function defined on $[0, 1]$, with $\rho(0) = 0$, $\rho(1) = 1$, and ρ is an increasing function on $[0, 1]$ (that is, $\dot{\rho} > 0$ whenever it exists).

Since $f : (r \cos \theta, r \sin \theta) \rightarrow (\rho(r) \cos \theta, \rho(r) \sin \theta)$, the Jacobian of f (in polar coordinates) is

$$J_f = \begin{pmatrix} \dot{\rho}(r) \cos(\theta) & -\frac{\rho(r)}{r} \sin(\theta) \\ \dot{\rho}(r) \sin(\theta) & \frac{\rho(r)}{r} \cos(\theta) \end{pmatrix}$$

and hence

$$\begin{aligned} J_f^T J_f &= \begin{pmatrix} \dot{\rho}(r) \cos(\theta) & \dot{\rho}(r) \sin(\theta) \\ -\frac{\rho(r)}{r} \sin(\theta) & \frac{\rho(r)}{r} \cos(\theta) \end{pmatrix} \begin{pmatrix} \dot{\rho}(r) \cos(\theta) & -\frac{\rho(r)}{r} \sin(\theta) \\ \dot{\rho}(r) \sin(\theta) & \frac{\rho(r)}{r} \cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} (\dot{\rho}(r))^2 & 0 \\ 0 & \frac{(\rho(r))^2}{r^2} \end{pmatrix}, \end{aligned}$$

whence the eigenvalues of $J_f^T J_f$ are $(\dot{\rho}(r))^2, \frac{(\rho(r))^2}{r^2}$. Since $\rho \geq 0, \dot{\rho} > 0$, and $r \geq 0$, the linear dilatation of f is given by

$$K(f) = \max \left\{ \frac{r\dot{\rho}(r)}{\rho(r)}, \frac{\rho(r)}{r\dot{\rho}(r)} \right\}.$$

For ease of notation, we set

$$K^+(f) = \frac{r\dot{\rho}(r)}{\rho(r)} \quad \text{and} \quad K^-(f) = \frac{\rho(r)}{r\dot{\rho}(r)}.$$

We are interested in the question: when does

$$\int_{\mathbb{D}} K(f) dm$$

exist? (Note that this is effectively measuring distortion in the L^1 -norm; the more general case will be discussed later.) Since ρ is independent of θ , integrating with respect to θ is identical to multiplication by the constant 2π . Thus $\int K(f)$ exists on the unit disk if and only if $\int_0^1 K r dr$ exists. There are three cases to consider: when $K = K^+$ exclusively, when $K = K^-$ exclusively, and when K is some mix of K^+, K^- . The case examined below is the case where $K = K^+$; the case where $K = K^-$ is covered after we have discussed the Euler-Lagrange equations (since integration is difficult in this case), but the mixed case is beyond the scope of this thesis and tagged for further research.

Suppose that $K = K^+$. Then, integrating by parts,

$$\int_0^1 \frac{r\dot{\rho}(r)}{\rho(r)} r dr = r^2 \log(\rho(r)) \Big|_0^1 - 2 \int_0^1 r \log(\rho(r)) dr.$$

Since $\rho(1) = 1$ and $\log(1) = 0$,

$$r^2 \log(\rho(r)) \Big|_0^1 = \lim_{r \rightarrow 0} r^2 \log(\rho(r)).$$

It is thus the behaviour of ρ near $r = 0$ that is the crucial factor in determining whether K is finite or not. If $\lim_{r \rightarrow 0} r^2 \log(\rho(r)) = L$ for some finite, nonzero number L , then (near $r = 0$) $\log(\rho(r))$ must behave like $\frac{L}{r^2}$. But in this case,

$$\int_0^a r \log(\rho(r)) dr \sim \int_0^a \frac{L}{r} dr$$

for small a , which diverges. Hence $\lim_{r \rightarrow 0} r^2 \log(\rho(r))$ cannot be any nonzero finite number. So in this case, the condition for $K(f)$ to be integrable on \mathbb{D} is that either

(a) $\lim_{r \rightarrow 0} r^2 \log(\rho(r)) = 0$ and $\int_0^1 r \log(\rho(r)) dr$ is finite, or

(b) $\lim_{r \rightarrow 0} r^2 \log(\rho(r)) = \pm\infty$ and $\int_0^1 r \log(\rho(r)) dr$ is infinite but commensurable with $\lim_{r \rightarrow 0} r^2 \log(\rho(r))$.

In case (a), the limit condition informs us that $\log(\rho(r))$ must behave like $-a(r)/r^2$, with $\lim_{r \rightarrow 0} a(r)/r^2 = \infty$ and $a(1) = 0$ in order to conform to the boundary conditions. Suppose therefore that $\rho(r) = \exp(-a(r)/r^2)$, where we assume $a(r)$ is continuously differentiable in order to simplify our reasoning. Plugging this into $\int_0^1 K^+ r dr$ and using the condition $a(1) = 0$,

we get

$$\begin{aligned} \int_0^1 K^+ r \, dr &= \int_0^1 \frac{r^2 \dot{\rho}(r)}{\rho(r)} \, dr = \int_0^1 \frac{r^2 \left(\frac{2ra(r) - r^2 \dot{a}(r)}{r^4} \right) \rho(r)}{\rho(r)} \, dr \\ &= \int_0^1 \left(\frac{2a(r)}{r} - \dot{a}(r) \right) \, dr \\ &= 2 \int_0^1 \frac{a(r)}{r} \, dr + a(0). \end{aligned}$$

Now, suppose $|a(0)| \neq 0$. Then $\frac{a(r)}{r}$ behaves (up to a change of sign) like $\frac{|a(0)|}{r}$ near $r = 0$, whence $\int_0^1 \frac{a(r)}{r} \, dr$ diverges; again we are in a situation where K is not finite. Therefore $|a(0)| = 0$, and thus $a(0) = 0$, and

$$\int_0^1 K^+ r \, dr = 2 \int_0^1 \frac{a(r)}{r} \, dr.$$

All the preceding facts then suggest that $a(r)$ is something like $r^\alpha(1-r)^\beta$, where $\alpha, \beta \neq 0$. Also, considering we require $\dot{\rho} > 0$, we must have $(2ra(r) - r^2\dot{a}(r))/r^4 > 0$, which implies

$$\dot{a}(r) - \frac{2a(r)}{r} < 0.$$

If this were an equality, we would obtain $a(r) = Cr^2$ (for some constant C ; solving for C given the boundary conditions would show $C = 0$ so this is in fact the trivial solution), so we see then that the borderline case is where $\alpha + \beta = 2$. Fixing $\beta = 1$ gives

$$\int_0^1 \frac{a(r)}{r} \, dr = \int_0^1 r^{\alpha-1}(1-r) \, dr = \int_0^1 r^{\alpha-1} \, dr - \int_0^1 r^\alpha \, dr$$

which converges for $\alpha > 0$ (and, since $\alpha + \beta = 2$ is the borderline case, we get $\alpha < 1$ with $\beta = 1$). Hence $\rho(r)$ looks something like

$$\rho(r) = \exp(-r^{\alpha-2}(1-r)^\beta)$$

with $1 < \alpha + \beta < 2$.

Case (b) may be a little easier to deal with. As before, we get

$$\int_0^1 K^+ r \, dr = 2 \int_0^1 \frac{a(r)}{r} \, dr + a(0),$$

this time under the presumption that $0 < |a(0)| < \infty$ (since this will satisfy the limit condition). We can thus presume that $a(r)$ may (up to multiplication by a constant) be a function like $(1-r)^\alpha$, $\alpha \neq 0$. We obtain

$$\int_0^1 \frac{a(r)}{r} \, dr \sim \int_0^1 \frac{1}{r^{1-\alpha}} \, dr,$$

which converges for $\alpha > 0$. Therefore we may guess that

$$\rho(r) = \exp\left(\frac{(1-r)^\alpha}{r^2}\right)$$

These are our tentative first guesses at what $\rho(r)$ might look like for the unit disk; of course, it is obvious that for *minimal* distortion only the identity function will do. However, the preceding discussion gives us some insight into what the radial stretch functions that minimise distortion between annuli might look like.

Now, these are only preliminary results and heuristic arguments. We shall see later that by using a different approach (from the calculus of variations) we are able to compute ρ exactly in some cases (especially in the L^p -norms, with $p > 1$, given a further restriction on the relative sizes of R and S).

3.2 The Euler-Lagrange Equation

The general process that is used to find extremals (for example, functions that extremize the integral of K over the annulus being mapped from) is part of the calculus of variations. Again, radial stretch functions are our candidates here, by considerations of symmetry of the problem. We look to

minimise

$$\int_1^R K^p(f)r \, dr.$$

Recall that for a linear bijection, K was just the ratio of the largest to the smallest eigenvalue. In trying to minimise the integral of K , it is expected that K is largest near the inner boundary of the annulus, and smallest near the outer boundary (since small rings near the inner boundary contribute the smallest area). It may be, therefore, that if we look at $K = K^+$, that we encounter some point where $K^+ < 1$ on the annulus. But this then is not the distortion; K must be greater than or equal to 1. So there is a point where K may change from being K^+ to K^- . This may cause a lack of differentiability of K at that point, so we introduce a new measure of distortion. Define

$$\mathbb{K} = \frac{1}{2} (K^+ + K^-).$$

This measure of distortion, provided K^+ and K^- are themselves smooth, will also be smooth, and therefore easier to minimise. So we also look at the integral

$$\int_1^R \mathbb{K}^p(f)r \, dr.$$

(In practice we take $\int_1^R (2\mathbb{K})^p(f)r \, dr$, to eliminate the factor of $\frac{1}{2}$ in the definition of \mathbb{K} , making the algebra a little easier.)

First, two lemmas (van Brunt, 2004).

Lemma 3.2.1. *Let $\alpha < \beta$ be two real numbers. Then there exists a function $\nu \in C^2(\mathbb{R})$ such that $\nu(x) > 0$ for all $x \in (\alpha, \beta)$ and $\nu(x) = 0$ for all $x \in \mathbb{R} \setminus (\alpha, \beta)$.*

Proof. Define

$$\nu(x) = \begin{cases} (x - \alpha)^3(\beta - x)^3 & \text{if } x \in (\alpha, \beta) \\ 0 & \text{otherwise.} \end{cases}$$

Then ν is a function which satisfies the required properties. Most of these

properties are obvious, except perhaps continuous derivatives at α and β . We will briefly address this; for a more detailed proof, see van Brunt (2004, pp. 31–32).

Observe that

$$\lim_{x \rightarrow \alpha^+} \frac{\nu(x) - \nu(\alpha)}{x - \alpha} = \lim_{x \rightarrow \alpha^+} (x - \alpha)^2 (\beta - x)^3 = 0,$$

and that

$$\lim_{x \rightarrow \alpha^-} \frac{\nu(x) - \nu(\alpha)}{x - \alpha} = 0,$$

so that $\nu'(\alpha) = 0$. Similar argument shows that $\nu''(\alpha) = \nu'(\beta) = \nu''(\beta) = 0$. Therefore,

$$\nu''(x) = \begin{cases} 6(x - \alpha)(\beta - x) [(x - \alpha)^2 + (\beta - x)^2 - 3(x - \alpha)(\beta - x)] & \text{if } x \in (\alpha, \beta) \\ 0 & \text{otherwise.} \end{cases}$$

Obviously,

$$\lim_{x \rightarrow \alpha} \nu''(x) = \nu''(\alpha) = 0 = \nu''(\beta) = \lim_{x \rightarrow \beta} \nu''(x),$$

whence $\nu \in C^2(\mathbb{R})$. □

Lemma 3.2.2. *Define*

$$\langle \nu, g \rangle = \int_{x_1}^{x_2} \nu(x)g(x) dx.$$

Let

$$H = \{\varphi \in C^2[x_1, x_2] : \varphi(x_1) = \varphi(x_2) = 0\},$$

and let $g : [x_1, x_2] \rightarrow \mathbb{R}$ be continuous. If $\langle \varphi, g \rangle = 0$ for all $\varphi \in H$, then

$$g(x) = 0$$

for all $x_1 < x < x_2$.

Proof. Suppose that $g(c) \neq 0$ for some $c \in [x_1, x_2]$. Without loss of generality, $c \in (x_1, x_2)$ and $g(c) > 0$. By continuity, there exist $c_1, c_2 \in [x_1, x_2]$ such that $c_1 < c < c_2$ and $g(x) > 0$ for all $x \in (c_1, c_2)$. By Lemma 3.2.1, there exists $\nu \in C^2[x_1, x_2]$ such that $\nu(x) > 0$ for all $x \in (c_1, c_2)$ and $\nu(x) = 0$ otherwise. Note that $\nu \in H$. But then

$$\langle \nu, g \rangle = \int_{x_1}^{x_2} \nu(x)g(x) dx = \int_{c_1}^{c_2} \nu(x)g(x) dx > 0$$

which contradicts the assumption that $\langle \varphi, g \rangle = 0$ for all $\varphi \in H$. Therefore, $g = 0$ on (x_1, x_2) (and hence, by continuity, on $[x_1, x_2]$). \square

We are now in a position to tackle the problem of finding a function that minimises distortion. Suppose that

$$I = \int_{r_1}^{r_2} f(r, \rho, \dot{\rho}) dr,$$

where $\rho = \rho(r)$, is a functional that we are looking to minimise (the integral of the distortion term is of this form). It is assumed that f is a twice-differentiable function with respect to any combination of its arguments, and that ρ is a twice-differentiable function. We are interested in the function $\rho(r)$ which minimizes this integral (and, of course, $f = rK^p$ or $r(2\mathbb{K})^p$).

We can represent any twice-differentiable functions R on $[r_1, r_2]$ with $R(r_1) = \rho(r_1)$ and $R(r_2) = \rho(r_2)$ in an ϵ -neighborhood of ρ by

$$R(r) = \rho(r) + \epsilon\varphi(r),$$

where φ is some twice-differentiable function such that $\varphi(r_1) = \varphi(r_2) = 0$. Thus

$$\dot{R}(r) = \dot{\rho}(r) + \epsilon\dot{\varphi}(r),$$

and by replacing ρ and $\dot{\rho}$ by R and \dot{R} , respectively, in I we obtain the integral

$$I(\epsilon) = \int_{r_1}^{r_2} f(r, R, \dot{R}) dr.$$

If ρ is an extremal of I , then $I(\varepsilon)$ must take on its extreme value when $\varepsilon = 0$. Note that $I'(0) = 0$ necessarily in this case, no matter what function φ is (provided it conforms to the constraints). Taking derivatives

$$I'(\varepsilon) = \frac{dI}{d\varepsilon} = \int_{r_1}^{r_2} \left[\frac{\partial f}{\partial R} \frac{\partial R}{\partial \varepsilon} + \frac{\partial f}{\partial \dot{R}} \frac{\partial \dot{R}}{\partial \varepsilon} \right] dr = \int_{r_1}^{r_2} \left[\frac{\partial f}{\partial R} \varphi + \frac{\partial f}{\partial \dot{R}} \dot{\varphi} \right] dr.$$

Integrating the last term of the integrand by parts and setting $\varepsilon = 0$,

$$I'(0) = \frac{\partial f}{\partial \dot{R}} \varphi \Big|_{r_1}^{r_2} + \int_{r_1}^{r_2} \left[\frac{\partial f}{\partial R} - \frac{d}{dr} \left(\frac{\partial f}{\partial \dot{R}} \right) \right] \varphi dr = 0$$

or, since $\varphi(r_1) = \varphi(r_2) = 0$,

$$\int_{r_1}^{r_2} \left[\frac{\partial f}{\partial R} - \frac{d}{dr} \left(\frac{\partial f}{\partial \dot{R}} \right) \right] \varphi dr = 0.$$

Remember that φ is any arbitrary twice-differentiable function. It now follows from Lemma 3.2.2 that

$$\frac{\partial f}{\partial R} - \frac{d}{dr} \left(\frac{\partial f}{\partial \dot{R}} \right) = 0$$

for all $r \in [r_1, r_2]$. This is called the **Euler-Lagrange equation**.

In the case of interest—radial stretch functions— $f(x) = \rho(r)$ (and $x = r$), and so we have

$$L(r) = \frac{\partial F}{\partial \rho} - \frac{d}{dr} \left(\frac{\partial F}{\partial \dot{\rho}} \right) = 0, \quad (3.1)$$

with $F = rK^p$ or $F = r(2\mathbb{K})^p$.

Turning attention to the problem at hand, for radial stretch mappings recall that

$$K = \max \left\{ \frac{r\dot{\rho}(r)}{\rho(r)}, \frac{\rho(r)}{r\dot{\rho}(r)} \right\} \quad \text{and} \quad 2\mathbb{K} = \frac{r\dot{\rho}(r)}{\rho(r)} + \frac{\rho(r)}{r\dot{\rho}(r)}.$$

For ease of notation, once again we use $K^+ = \frac{r\dot{\rho}(r)}{\rho(r)}$, and $K^- = \frac{\rho(r)}{r\dot{\rho}(r)}$.

Theorem 3.2.3. *For each $p > 1$, the radial mapping of minimal L^p -distortion*

between annuli $A_1 = \{z : 1 < |z| < R\}$ and $A_2 = \{z : 1 < |z| < S\}$, provided that $K = K^+$ throughout A_1 , is given by $f(re^{i\theta}) = \rho(r)e^{i\theta}$, where

$$\rho(r) = S^{-((r^{-q}-1)R^q)/(R^q-1)}, \quad (3.2)$$

with $q = \frac{2}{p-1}$.

Proof. Setting $F = r(K^+)^p$ in $L(r)$ yields

$$L(r) = -\frac{1}{\rho^2}pr(\dot{\rho})^{p-2} [(p-1)r\rho\ddot{\rho} - (p-1)r(\dot{\rho})^2 + (p+1)\rho\dot{\rho}]$$

which is a non-linear second-order differential equation for $\rho(r)$. Setting $L(r) = 0$ gives

$$\ddot{\rho} - \frac{1}{\rho}(\dot{\rho})^2 + \frac{p+1}{r(p-1)}\dot{\rho} = 0.$$

Substituting $u(r) = \dot{\rho}(r)/\rho(r)$ gives the linear differential equation

$$r\dot{u} + \frac{p+1}{p-1}u = 0.$$

Separating the variables gives

$$\frac{du}{u} = -\left(\frac{p+1}{p-1}\right)\frac{dr}{r},$$

which, upon integrating and solving for u , yields

$$u = Cr^{-\left(\frac{p+1}{p-1}\right)},$$

where C is a constant. Substituting back to eliminate u gives

$$\frac{d}{dr}(\ln(\rho(r))) = Cr^{-\left(\frac{p+1}{p-1}\right)}$$

and hence, upon integrating (and observing that $\frac{p+1}{p-1} + 1 = -\frac{2}{p-1}$) we get

$$\ln(\rho(r)) = -\frac{1}{2}C(p-1)r^{\left(-\frac{2}{p-1}\right)} + D,$$

with D as the constant of integration. Rearranging and substituting the constant $A = e^D$, we get

$$\rho(r) = Ae^{\left(-\frac{1}{2}C(p-1)r^{\left(-\frac{2}{p-1}\right)}\right)}.$$

Using the boundary condition $\rho(1) = 1$ determines A in terms of C , and mapping from the annulus $A(1, R)$ to the annulus $A(1, S)$ fixes $\rho(R) = S$, and hence C is determined. The resulting formula for ρ is

$$\rho(r) = S^{(-((r^{-q}-1)R^q)/(R^q-1))}$$

where $q = \frac{2}{p-1}$. □

This is all on the assumption that $\frac{r\dot{\rho}(r)}{\rho(r)} \geq 1$ (since then $K = \frac{r\dot{\rho}(r)}{\rho(r)}$). If, however, $\frac{\rho(r)}{r\dot{\rho}(r)} > 1$, then the calculation must be repeated for $K = K^-$. This yields

Theorem 3.2.4. *For each $p \in \mathbb{N}$, the radial mapping of minimal L^p -distortion between annuli $A_1 = \{z : 1 < |z| < R\}$ and $A_2 = \{z : 1 < |z| < S\}$, provided that $K = K^-$ throughout A_1 , is given by $f(re^{i\theta}) = \rho(r)e^{i\theta}$, where*

$$\rho(r) = S^{((r^q-1)/(R^q-1))}, \tag{3.3}$$

with $q = \frac{2}{p+1}$.

Proof. Analogous to Theorem 3.2.3, replacing K^+ by K^- throughout. □

Thus we have explicit solutions in the cases where $K = K^+$ ($p > 1$) and $K = K^-$ (any value of p) solely; as mentioned before, the mixed case will be looked at in research yet to come. Also, the Nitsche conjecture plays a role here; this will be discussed in the next chapter.

4. THE NITSCHKE CONJECTURE, SOME CALCULATIONS AND BOUNDS

4.1 Relationship to the Nitsche conjecture

Notice that while equation (3.3) is seemingly valid for all p , equation (3.2) is problematic when $p = 1$. This problem can be circumvented by taking \mathbb{K} instead. Following through the same calculation as at the end of the previous section, using \mathbb{K} in $L(r)$ yields the differential equation

$$\begin{aligned}
 L = \frac{p(r^2\dot{\rho}^2 + \rho^2)^{p-2}}{r^p\rho^{p+1}\dot{\rho}^{p+2}} & \left(-r^4\rho\dot{\rho}^5 - 4r^2\rho^3\dot{\rho}^3 + pr^5\dot{\rho}^6 + 4r^3\rho^2\dot{\rho}^4 - p\rho^5\dot{\rho} \right. \\
 & - r\rho^5\ddot{\rho} + \rho^5\dot{\rho} - pr^4\rho\dot{\rho}^5 - 2pr^3\rho^2\dot{\rho}^4 + 2pr^2\rho^3\dot{\rho}^3 \\
 & + r^5\rho\dot{\rho}^4\ddot{\rho} - 4r^3\rho^3\dot{\rho}^2\ddot{\rho} + pr\rho^4\dot{\rho}^2 - pr\rho^5\ddot{\rho} - pr^5\rho\dot{\rho}^4\ddot{\rho} \\
 & \left. + 2pr^3\rho^3\dot{\rho}^2\ddot{\rho} - r^5\dot{\rho}^6 + r\rho^4\dot{\rho}^2 \right)
 \end{aligned}$$

for p in general. Substituting $p = 1$ and setting $L0$ yields (after some simplifying)

$$r\dot{\rho}^3 - \rho\dot{\rho}^2 + \ddot{\rho}\rho^2 = 0,$$

which gives

$$\frac{r}{\rho(r)} - \frac{C_1}{2\rho(r)^2} - C_2 = 0$$

for some constants C_1, C_2 . Solving for $\rho(r)$ yields

$$\rho(r) = \lambda_1 \left(r + \sqrt{r^2 - \lambda_2} \right) \quad \text{or} \quad \rho(r) = \lambda_1 \left(r - \sqrt{r^2 - \lambda_2} \right),$$

where $\lambda_1 = \frac{1}{2C_2}$ and $\lambda_2 = 2C_1C_2$. The first solution preserves the order of the boundaries (that is, maps the inner boundary of the original annulus to the

inner boundary of the resulting annulus), the second solution reverses them. These mappings are inverses to the harmonic Nitsche mappings; see Nitsche (1962).

Observe also that (given the boundary condition $\rho(R) = S$) while equation (3.2) is valid when $S \gg R$, equation (3.3) is the appropriate solution for $S \ll R$ (and, presumably, some mix of the two for $S \sim R$). The case when $S < R$ is further constrained by the Nitsche conjecture, which says that minimisers of the distortion functional don't exist if the annulus that is being mapped onto is too thin; a specific relationship between R and S (called the *Nitsche bound*) for this conjecture may be found below (equation (4.2)).

In the case $p = 1$, the Nitsche conjecture has been proved (Iwaniec and Martin, 2006) for finite energy minimisers. For a mapping g , energy is given definitionally by

$$\int \|\nabla g\|^2.$$

In Iwaniec and Martin (2006), it is shown that if we consider g as an inverse of the quasiconformal mapping f between annuli, then $\int \|\nabla g\|^2 = \int \mathbb{K}(f)$. So finding the mapping of minimal distortion here corresponds to minimising an energy functional. It is also shown that $\nabla^2 g$, the Laplacian of g , is the Euler-Lagrange equation in this situation; setting it equal to 0 therefore shows that g is harmonic; the inverse Nitsche map (which is the inverse of g) is then the extremal mapping.

In particular, I refer to (a shortened version of) Theorem 3.1 of Iwaniec and Martin (2006);

Theorem 4.1.1. *Let $A = A(r, R)$ and A' be annuli of modulus σ and γ respectively. Suppose that*

$$\cosh(\gamma) \leq e^\sigma \tag{4.1}$$

Then the inverse Nitsche map is a representative for the extremal mapping of mean distortion.

We know what the moduli of the annuli in the theorem are; in the case we

have looked at in this thesis, we have $r = r' = 1$, $R = R$, and $R' = S$. Thus $\sigma = 2\pi/\log(R)$, $\gamma = 2\pi/\log(S)$, and so the Nitsche bound (4.1) becomes

$$\exp\left(\frac{2\pi}{\log(S)}\right) + \exp\left(\frac{-2\pi}{\log(S)}\right) \leq 2 \exp\left(\frac{2\pi}{\log(R)}\right) \quad (4.2)$$

This is quite surprising! The implication is that in the L^1 -norm, there are no minimisers of distortion if the target annulus is much thinner than the domain annulus. For example, for a domain annulus with $R = 10$, in order to have minimisers of the distortion problem S cannot be smaller than 6.276 (approximately). It is very surprising that this is the case; the order of magnitude of S is about the same as that of R , so visually, one might expect minimisers to exist, but (when taken in light of Nitsche's conjecture, proved for $p = 1$) this theorem proves that this is not the case. That is, if equation (4.1) fails to hold, there are no minimisers.

4.2 Main results

The case $p = 1$ has already been discussed; the mappings of minimal distortion are inverse Nitsche maps. Note that we had to calculate these using \mathbb{K} only—when attempting to calculate using K , there is the problem of division by zero. Given that we already have a solution for the case $p = 1$,

Proposition 4.2.1. *In the L^1 -norm, provided the order of the boundary components is respected the mappings which minimise mean distortion are given by*

$$\rho(r) = \lambda_1 \left(r + \sqrt{r^2 - \lambda_2} \right),$$

an inverse to a harmonic Nitsche map, where

$$\lambda_1 = \left(1 + \left| \frac{1 - 2RS + S^2}{S^2 - 1} \right| \right)^{-1}, \quad \lambda_2 = \frac{4S(S - R)(RS - 1)}{(S^2 - 1)^2},$$

whenever such minimisers exist. If $R = S$, then ρ reduces to the identity map.

Proof. The first part is a re-statement of Theorem 4.1.1. We can solve for the constants λ_1 and λ_2 in terms of R and S , using the boundary conditions $\rho(1) = 1, \rho(R) = S$. This gives the desired result, provided the Nitsche bound is respected. \square

Theorems 3.2.3 and 3.2.4 give results for the case where $K = K^+$ or $K = K^-$ entirely across the domain annulus; thus they are valid when R and S are quite different quantities (especially for low values of p). If R is nearly the same as S , then the deformation that minimises distortion (if it exists) is expected to be something close to the identity function—however, this is not what the equations in section 3.1 suggest. This is because if R is approximately the same size as S , then the distortion may be a mix of K^+ and K^- . It is also possible that there are no minimisers in this case—it may be some kind of phenomenon much like Nitsche’s conjecture. This is another reason to prefer \mathbb{K} to K as a measure of distortion suitable for minimising. Fortunately, however, we can determine exactly when the mixed case presents itself; this is summarized in Theorem 4.2.2.

Before we state the formal theorem, let us take a look at some specific results. Take the case $p = 2, R = 2$. Assume that $K = K^+$ throughout the domain annulus. Define $\rho_1(r)$ using (3.2) with $S = 6$ (and ρ_2 with $S = 4$). Use this function to compute $K_1 = K^+(r, \rho_1(r))$ (and $K_2 = K^+(r, \rho_2(r))$). The results are displayed in Fig. 4.1.

Note that $K_2(r)$ drops below 1 before r reaches 2; this means that it is not reasonable to assume that $K = K^+$ throughout, in the case presented by the preceding paragraph with $S = 4$. Therefore, the mixed case presents itself here.

Theorem 4.2.2. *Theorems 3.2.3 and 3.2.4 are valid for mappings between annuli $A_1 = \{z : 1 < |z| < R\}, A_2 = \{z : 1 < |z| < S\}$ if and only if*

$$S \geq \exp\left(\frac{1}{2}(p-1)\left(R^{\frac{2}{p-1}} - 1\right)\right). \quad (4.3)$$

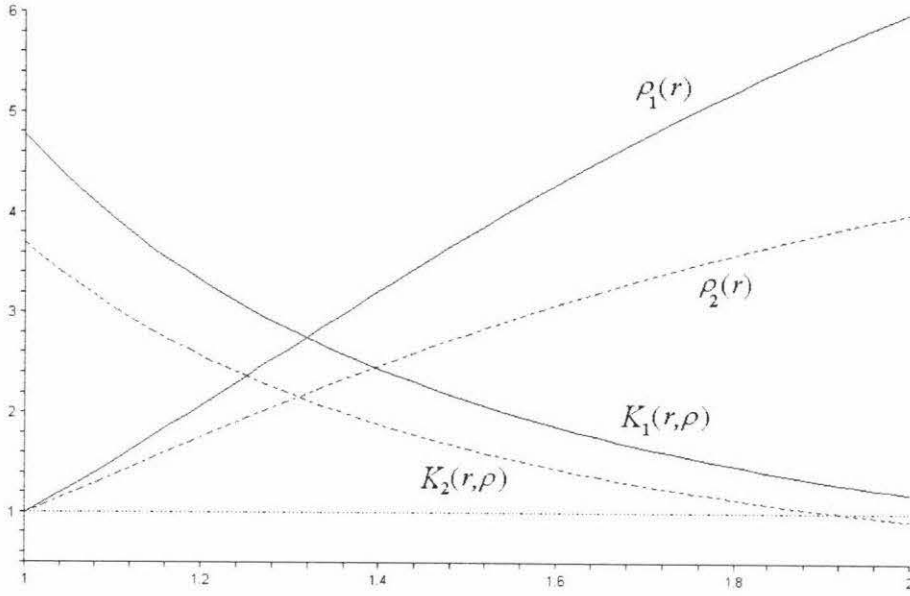


Fig. 4.1: Some specific results for $\rho(r)$ and $K(r, \rho)$.

or

$$S \leq \exp\left(\frac{1}{2}(p+1)\left(1 - R^{\frac{2p}{p+1}-2}\right)\right). \quad (4.4)$$

respectively.

Proof. In the case where $K = \frac{r\hat{\rho}(r)}{\rho(r)}$, equation 3.2 obtains as the minimiser. Substituting into K ,

$$K(r) = \frac{2 \ln S}{(p-1)\left(R^{\frac{2}{p-1}} - 1\right)} \left(\frac{R}{r}\right)^{\frac{2}{p-1}}$$

Setting $K = 1$ and solving for r yields

$$r_K = R \left(\frac{2 \ln S}{(p-1)\left(R^{\frac{2}{p-1}} - 1\right)} \right)^{\frac{p-1}{2}}$$

In order for $K \geq 1$ on $[1, R]$, it is required that $r_K \geq R$. Rearranging shows

that

$$S \geq \exp\left(\frac{1}{2}(p-1)\left(R^{\frac{2}{p-1}} - 1\right)\right).$$

In a similar way the restriction on S and R for the case $K = \frac{\rho(r)}{r\dot{\rho}(r)}$ may be obtained as

$$S \leq \exp\left(\frac{1}{2}(p+1)\left(1 - R^{\frac{2p}{p+1}-2}\right)\right).$$

□

For fixed $R > 1$, the case of equality for each p gives a sequence of values of $S = S(p)$, leading to the question whether S is increasing or decreasing with respect to p , and whether it has a limit as $p \rightarrow \infty$.

Theorem 4.2.3. *The envelope of values for S within which the mixed case represents itself is uniformly convergent, with respect to increasing p , to $S = R$ from both above and below.*

Proof. First, the case of equality in (4.3) is a decreasing sequence in p whenever $R > 1$. To see this, note that (for $S \geq 1$) S is decreasing with respect to p if and only if the function

$$f(p) = (p-1)\left(R^{\frac{2}{p-1}} - 1\right)$$

decreases with respect to p . Note the derivative

$$\dot{f}(p) = R^{\frac{2}{p-1}} - 1 - \frac{2R^{\frac{2}{p-1}} \ln R}{p-1}.$$

Then, since $R^{\frac{2}{p-1}} > 1$ for $R, p > 1$, we can substitute $x = R^{\frac{2}{p-1}}$ (and using the rules of logarithms) to get

$$\dot{f}_x = x - 1 - x \ln x.$$

Note that $\dot{f}_x = 0$ if and only if $x \ln x = x - 1$; that is, if $x = 1$ (to see that this is the only possible root of \dot{f}_x , taking a derivative will show that $\frac{d}{dx} \dot{f}_x \neq 0$ for $x > 1$). But $x > 1$ for all $R, p > 1$, and so $\dot{f} \neq 0$. Since \dot{f} is a continuous

function of p given $R > 1$, it follows that \dot{f}_x does not change sign; that is, the sequence given by (4.3) is monotonic. Furthermore, substituting some values in for p, R shows that $\dot{f}_x(p) < 0$ for all $p > 1$. Hence the sequence of bounds given by (4.3) decreases monotonically with p . Since it is bounded below by 0, it converges. Furthermore, by the continuity of the exponential function and L'Hôpital's Rule,

$$\begin{aligned} \lim_{p \rightarrow \infty} \exp\left(\frac{1}{2}(p-1)\left(R^{\frac{2}{p-1}} - 1\right)\right) &= \exp\left(\lim_{p \rightarrow \infty} \frac{\left(R^{\frac{2}{p-1}} - 1\right)}{\left(\frac{2}{p-1}\right)}\right) \\ &= \exp\left(\lim_{p \rightarrow \infty} \frac{\frac{-2}{(p-1)^2} R^{\frac{2}{p-1}} \ln R}{\left(\frac{-2}{(p-1)^2}\right)}\right) \\ &= \exp\left(\lim_{p \rightarrow \infty} R^{\frac{2}{p-1}} \ln R\right) \\ &= \exp(\ln R) \\ &= R. \end{aligned}$$

Similarly, an analogous argument shows that the limit of the lower bound on the envelope of S values where K is not purely K^+ or K^- is also R . \square

When $p = \infty$, if $R = S$ then both K^+ and K^- will give the same minimising function: the identity.

Figure 4.2 displays quite clearly what happens to the distortion term K (in fact, this is K^+ , with $R = 2$ and $S = 6$ as before) as p tends to infinity; it flattens out across the annulus. This is to be expected for, in the L^∞ -norm, we are interested just in the supremum of values for K , so to minimize this value it is expected to be distributed evenly across the entire annulus—the minimiser has constant distortion.

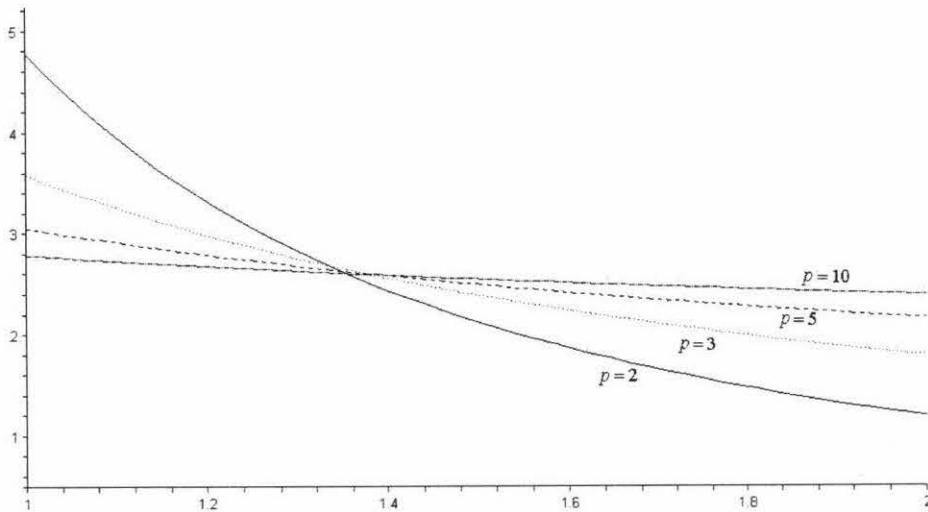


Fig. 4.2: The distortion $K(r, \rho)$ for various values of p .

4.3 Further research—open questions

It is as yet unknown what the correct formulation of the Nitsche conjecture is for norms other than the L^1 -norm. The preceding calculations have not respected the Nitsche bound in all cases (in particular, for $S \ll R$).

The distortion term \mathbb{K} also may be investigated for values of p other than 1, although the nature of the Euler-Lagrange equation in this case is highly nonlinear and not likely to easily yield outcomes. Computational investigations may lead to some insights.

The window of S values for which the distortion is given by a mixture of K^+ and K^- is to be investigated further; it might be that as S gets close to R for low p values, solutions will flip back and forth between having K^+ and K^- as their distortion value, in concentric ‘bands’ in the annulus (or ‘sub-annuli’). It is yet unclear what the situation is in this case, however it is interesting to note that when K^+ reaches a value of 1, and the solution changes from a K^+ solution to a K^- solution, that continuity and differentiability properties may yield some conclusions about the nature of the solution.

The question of existence and uniqueness of solutions need to be looked

at in more detail; especially whether the Nitsche bound can offer any insight into higher p -norms, and whether there may in fact be multiple solutions of minimal distortion. Some of the uniqueness question is touched upon by Väisälä (1971). The geometrical assumptions of symmetry that have simplified the problem so far may influence this problem.

APPENDIX

COMMON DEFINITIONS AND NOTATION

Throughout this thesis, familiarity with some topological notions and some aspects of analysis will be assumed, and certain notational conventions will be used:

- $\widehat{\mathbb{C}}$ denotes the extended complex plane, also known as the Riemann sphere. That is, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
- If $z = x + iy$ is a complex number, x denotes the real part of z , written as $\Re(z)$, and y denotes the imaginary part of z , written as $\Im(z)$.
- $D(z_0, r)$ denotes an *open disc* in \mathbb{C} (or $\widehat{\mathbb{C}}$ as appropriate), centered at z_0 , with radius $r > 0$. That is, $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$. Similarly, a closed disc centered at z_0 with radius $r > 0$ is denoted by $\overline{D}(z_0, r)$.
- In \mathbb{R}^n (or $\overline{\mathbb{R}^n}$ as appropriate), $B(c, r)$ denotes an *open ball*, centered at c , with radius $r > 0$. That is, $B(c, r) = \{x \in \mathbb{R}^n : \rho(c, x) < r\}$, where ρ is the metric associated with \mathbb{R}^n . Similarly, a closed ball centered at c with radius $r > 0$ is denoted by $\overline{B}(c, r)$.
- \mathbb{D} denotes the *open unit disc*; that is, $\mathbb{D} = D(0, 1)$.
- A set S containing z_0 is called a *neighborhood of z_0* if there exists a real number $r > 0$ such that $D(z_0, r) \subset S$.
- A set S is said to be *open* if for each $z \in S$ there exists $r > 0$ such that $D(z, r) \subset S$.
- For any set S , \widetilde{S} denotes the complement of S . That is, $\widetilde{S} = \mathbb{C} \setminus S$.

- A set S is said to be *closed* if \tilde{S} is open.
- A sequence $(z_n)_{n \geq 0}$ of complex numbers is said to be a *Cauchy sequence* if for each $\varepsilon > 0$ there exists a natural number N such that $m, n > N$ implies $|z_m - z_n| < \varepsilon$.
- A function $f : X \rightarrow Y$ is called a *homeomorphism* from the space X to the space Y if it is a continuous bijection with a continuous inverse.
- A sequence of functions $(f_n)_{n \geq 1}$ is called a *uniform Cauchy sequence on the set U* if for each $\varepsilon > 0$ there exists an index $N(\varepsilon)$ such that $m > n \geq N$ implies that $|f_m(z) - f_n(z)| < \varepsilon$ for all $z \in U$.
- A point z is in the *boundary of S* , ∂S , if every neighborhood of z intersects both S and \tilde{S} .
- The *closure of S* , \bar{S} , is given by $\bar{S} = S \cup \partial S$.
- S is called *bounded* if there exists some $r > 0$ such that $S \subset D(0, r)$.
- *Compact* sets are sets that are both closed and bounded.
- S is said to be *disconnected* if there exist open sets A and B such that:
 - (i) $A \cap S \neq \emptyset$;
 - (ii) $B \cap S \neq \emptyset$;
 - (iii) $S \subset A \cup B$; and
 - (iv) $(A \cap S) \cap (B \cap S) = \emptyset$.

A set that is not disconnected is called *connected*.

- A *region* (of the complex plane) is an open connected set.
- If $f(x, y) = u(x, y) + iv(x, y)$ is a function of a complex variable $z =$

$x + iy$, then the matrix of partials

$$Df(x, y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

is known as the *Jacobian matrix* (or simply the *Jacobian*) of f .

- A sequence of functions $f_n : \Omega \rightarrow \mathbb{C}$ is said to *converge pointwise* if and only if the sequence $f_n(z)$ converges for each $z \in \Omega$.
- A point z is an *accumulation point* of the sequence $(z_n)_{n \geq 1}$ if for each $\varepsilon > 0$ the disc $D(z, \varepsilon)$ contains z_n for infinitely many values of n .
- A *path* in \mathbb{R}^n is a continuous mapping $\gamma : I \rightarrow \mathbb{R}^n$ where I is an interval in \mathbb{R} . The path is said to be *open* or *closed* according to whether I is open or closed.
- The *locus* of a path is the point set $\gamma I \subset \mathbb{R}^n$. A *subpath* of a path $\gamma : I \rightarrow \mathbb{R}^n$ is the restriction of γ to a subinterval of I .
- Let $\gamma : [a, b] \rightarrow \mathbb{R}^n$ be a closed path, and let $a = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_k = b$ be a subdivision of $[a, b]$. The supremum of the sums

$$\sum_{i=1}^k |\gamma(t_i) - \gamma(t_{i-1})|$$

over all subdivisions of $[a, b]$ is called the *length* of γ and denoted by $\ell(\gamma)$. Note that $0 \leq \ell(\gamma) \leq \infty$, where $\ell(\gamma) = 0$ if and only if γ is constant. Furthermore, if $\ell(\gamma) < \infty$, γ is *rectifiable*; otherwise γ is *non-rectifiable*.

- A path given by $\gamma(t) = x(t) + iy(t)$ for $a \leq t \leq b$ is called a *smooth path* if its derivative $\dot{\gamma}(t) = \dot{x}(t) + i\dot{y}(t)$ with respect to the real parameter t exists for each t in $[a, b]$ and if the function $\dot{\gamma}$ is continuous on the

interval $[a, b]$. (The dot notation, $\dot{\gamma}$, is used to prevent confusion with complex differentiation, which uses the notation γ' .)

- A path $\gamma : [a, b] \rightarrow \mathbb{C}$ is said to be *piecewise smooth* if there exists a partition $P : a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a, b]$ with the property that the restriction of γ to each $[t_{k-1}, t_k]$ for $1 \leq k \leq n$ is a smooth path.
- If $A \subset \mathbb{R}^n$, then $m_n^*(A)$ denotes the Lebesgue outer measure of A . If A is measurable the star may be omitted; the n is omitted when there is no danger of misunderstanding.
- By $C^n(\Omega)$ we denote the class of all complex-valued functions that are n times continuously differentiable on the region Ω , and $C(\Omega)$ denotes the class of all complex-valued functions that are continuous on the region Ω .
- A *diffeomorphism*, $f : \Omega \rightarrow \Omega'$ is a C^1 -homeomorphism whose Jacobian $J(x, f)$ does not vanish.
- S^n is a unit n -sphere. That is to say, a unit sphere in \mathbb{R}^n .
- A complex valued function f , defined on some neighborhood of z_0 , is said to be *continuous at z_0* if $f(z) \rightarrow f(z_0)$ whenever $z \rightarrow z_0$. Alternatively, f is continuous at z_0 if to each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$. We call f *continuous in a region Ω* if f is continuous at every point of Ω .
- Let γ be a closed curve in \mathbb{C} and $z_0 \in \mathbb{C}$ be a point not on γ . Then the *index* of γ with respect to z_0 , also called the *winding number* of γ with respect to z_0 , is given by

$$n(\gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0}$$

We say that γ *winds around* z_0 $n(\gamma, z_0)$ times.

THEOREMS AND PROPOSITIONS NOT FOUND IN THE TEXT

Theorem .0.1. (*The Mean Value Theorem*)

Let f be differentiable on (a, b) and continuous on $[a, b]$. Then there is at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

For a proof, see Anton (1995, p. 235).

Theorem .0.2. (*The Real-Variable Inverse Function Theorem*)

If $f : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is continuously differentiable and $Df(x_0, y_0)$ has a nonzero determinant, then there are neighborhoods U of (x_0, y_0) and V of $f(x_0, y_0)$ such that $f : U \rightarrow V$ is a bijection, $f^{-1} : V \rightarrow U$ is differentiable, and

$$Df^{-1}(f(x, y)) = [Df(x, y)]^{-1}.$$

For a proof, see Marsden (1974, Ch. 7).

Theorem .0.3. (*The Inverse Function Theorem*)

Let $f : \Omega \rightarrow \mathbb{C}$ be analytic, with f' continuous and $f'(z_0) \neq 0$. Then there exists a neighborhood U of z_0 and a neighborhood V of $f(z_0)$ such that $f : U \rightarrow V$ is a bijection and the inverse function f^{-1} is analytic, with derivative given by

$$\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(z_0)} \quad \text{where} \quad w = f(z_0).$$

For a proof, see Marsden and Hoffman (1987, pp. 77–78).

Theorem .0.4. (The Bolzano-Weierstrass Theorem)

Suppose that $(z_n)_{n \geq 1}$ is a bounded sequence in \mathbb{C} . Then (z_n) has at least one accumulation point. Moreover, this sequence has exactly one accumulation point if and only if it is a convergent sequence with the unique accumulation point as its limit.

For a proof, see Palka (1991, p. 53), or Bridges (1998, p. 48).

Theorem .0.5. (The Cauchy Criterion for Uniform Convergence)

Suppose that each function in a sequence (f_n) is defined on a set U . The sequence converges uniformly on U if and only if it is a uniform Cauchy sequence on U .

For a proof, see Palka (1991, p. 246).

Proposition .0.6. *If Ω is a simply connected region of the complex plane, then for every function f that is both analytic and free of zeros in Ω there exists a branch of $\log f(z)$ in this region.*

For a proof, see Palka (1991, pp.196–197).

Theorem .0.7. (Liouville's Theorem)

The only bounded entire functions on \mathbb{C} are constant.

For a proof, see Marsden and Hoffman (1987, pp. 171–172).

Theorem .0.8. (Cauchy's Integral Formula)

Let f be analytic in an open disk D and γ a closed, piecewise smooth path in D . Then

$$n(\gamma, z)f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - z}$$

for every z in $D \setminus |\gamma|$.

For a proof, see Palka (1991, pp. 161–162)

Theorem .0.9. (*One form of Cauchy's Estimate*)

Suppose that a function f is analytic in an open disk $D = D(z_0, r)$ and that there is some constant m for which $|f(z)| \leq m$ holds throughout D . Then for each positive integer, the estimate

$$|f^{(k)}(z_0)| \leq \frac{k! mr}{(r - |z - z_0|)^{k+1}}$$

is valid for every $z \in D$. In particular, $|f^{(k)}(z_0)| \leq k! mr^{-k}$.

This is an extended consequence of Cauchy's Integral Formula. For a proof, see Palka (1991, p. 167).

Theorem .0.10. Let $\Omega \subset \mathbb{C}$ be a domain and $f : \Omega \rightarrow \mathbb{C}$ a nonconstant analytic mapping. Then $f(\Omega)$ is also a domain (in particular, it is open).

For a proof, see Marsden and Hoffman (1987, pp 435–436).

Theorem .0.11. (*Hurwitz's Theorem*)

Suppose that each function in a sequence (f_n) is analytic and zero-free in a domain Ω and that $f_n \rightarrow f$ normally in Ω . Then either f is free of zeros in Ω or it is identically zero there.

For a proof, see Palka (1991, pp.348–349).

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