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ORIGINAL PAPER

The use of cultural contexts for patterning tasks: supporting young diverse students to identify structures and generalise

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Abstract
A key aspect of young children's development of algebraic reasoning is the process of visualising and identifying structures to both abstract and generalise. There has been a growing body of research focused on how students form generalisations, this article adds to the existing body of research by examining how young culturally diverse students identify mathematical structures in contextual growing patterns and the teaching and learning actions that assist them to generalise. Data were collected from one classroom of 29 Year Two (6 years old) students in a low socio-economic school in New Zealand. Results from the analysis of lessons related to two tasks showed that the contextual tasks led students to notice different mathematical structures. Specific pedagogical actions were used to facilitate students' engagement with the growing patterns. These included positioning students to engage with different representations (pictorial and numerical, tabular, and natural language) to represent thinking, the use of classroom discussions, noticing and responding to student thinking, and pressing students to find far terms. The findings highlight that both the contextual patterning tasks and teacher actions supported the young students to develop a range of sophisticated generalisations related to the underlying mathematical structure and functional relationships of the growing patterns.

Keywords Early algebra · Contextual tasks · Patterning · Structure · Centers of focus

1 Introduction

Over the past decades, there has been calls for a greater emphasis on the teaching and learning of algebra in primary classrooms (Blanton et al., 2018). Patterning activities and functions offer an opportunity to integrate early algebraic reasoning into the existing mathematics curriculum. The importance of visualising and identifying structures to assist students to abstract and generalise has been established as key for developing students' algebraic reasoning. We know from earlier studies (e.g., Rivera, 2018; Wijns et al., 2019) that young children can abstract and generalise mathematical patterns although they often initially notice recursive relationships rather than functional relationships. Generally, these studies have used figural or geometric growing patterns (e.g., Moss & Beatty, 2010; Radford, 2010). The use of familiar contextual patterns from students' home and community contexts may be an alternative means to provide young students with access to early algebraic reasoning while also addressing their cultural knowledge and using this as a strength within mathematics teaching and learning (Makonye, 2020; Wijns et al., 2019). We argue that there is a need to determine how young culturally diverse students notice and identify structures within contextual patterns as part of social interactions in a classroom setting. Specifically, we ask how do young diverse students see mathematical structures in contextual growing patterns and articulate their generalisations about these? And what teaching actions assist students to identify and generalise these growing pattern structures?

2 Literature

2.1 Mathematical structures and patterning

Important to mathematics is the ability to see structure. Mathematical structures can be defined as the ways

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that concepts are organised and the integral relationships between these features which are often expressed as generalisations (Mulligan et al., 2020). Mason et al. (2009) explain that mathematical structures are the 'general properties which are instantiated in particular situations as relationships between elements or subsets of elements of a set' (p.10). Students need to identify and recognise the general properties to build mathematical connections as they form generalities of structure. Early algebra, and in particular mathematical patterning, is one way of engaging young students to notice, reason, and abstract mathematical structures (Blanton & Kaput, 2011).

Exploration of growing patterns supports the development of functional thinking (Warren & Cooper, 2008). Both linear and nonlinear functions can be represented in growing patterns, however typically in the primary curriculum linear functions are explored. Growing patterns can be represented in the following ways: (i) patterns with two visual variables where the relationship between variables is visually explicit; (ii) patterns with one visual variable with the second variable being the position of each term in a sequence; and, (iii) neither variable is visually explicit (a number pattern) (Miller, 2015). In addition to this, where growing patterns are depicted either figuratively or geometrically, they can 'grow' in unidirectional ways. Research has indicated that the way growing patterns are represented can impact on students accessing the underlying structure. Often primary school students use recursive thinking rather than seeing the structure and forming generalisations as a functional relationship (Radford, 2010).

2.2 Contextual patterning tasks

There has been increasing interest in the use of contextual tasks to both improve mathematics teaching and learning and to connect with students' out of school experiences and cultural knowledge (Makonye, 2020; Reinke & Casto, 2020). Previous studies focused on early algebra and patterning have typically involved tasks with de-contextualised geometric shape patterns or generic examples such as the number of eyes and tails on animals, or tile combinations (e.g., Pinto & Cañadas, 2021; Rivera, 2018; Twohill, 2018). However, there is growing evidence that mathematics educators should consider the factors that influence the difficulty of identifying how the pattern grows and of developing a generalisation. Among other factors this includes the nature of the pattern and students' experiences with different types of patterns in the home (Wijns et al., 2019). Familiar patterns potentially provide students with opportunities to recognise the structure of the pattern more readily and engage in more sophisticated generalisation (Hunter & Miller, 2022; Wijns et al., 2019).

Use of contextual patterns within the classroom offers an opportunity to introduce abstract mathematical ideas to students in relatable ways. Context in mathematical tasks can be a conceptual anchor to allow learners to make sense of new ideas (Reinke & Casto, 2020). Within early algebra, we have evidence from studies (Blanton & Kaput, 2011; Miller & Hunter, 2017) that contextual patterns can increase accessibility by providing a familiar visual aspect from students' known worlds before moving to an abstract representation of the functional pattern. For example, Miller and Hunter (2017) highlighted that prior to instruction, young Pacific students were able to generalise a contextualised growing pattern from a cultural artefact, tapa, patterned bark cloth, at a greater level than a de-contextualised growing pattern of squares. Another study by Hunter and Hunter (2019) with Pacific students noted the benefits that both teachers and students identified when cultural contexts were used. For example, the teacher referred to tapa patterns and highlighted that while the students knew the pattern, they did not readily see the maths within, however, by using that as the context of the problem "you have a foundation for the discussion, and we can then expand the maths" (p. 250). Similarly, the students noted "the problems relate to our cultures and celebration which makes it more understandable" (p. 250). Also of note, is that contextual patterns provide opportunities for students to make connections between their cultural heritage and mathematics (Hunter & Miller, 2022).

2.3 Teaching actions to support generalisation

Research studies show the key role of the teacher in supporting students to move beyond pattern spotting to developing generalisations (Blanton et al., 2018; Twohill, 2018). Of importance is that teachers facilitate students to move beyond recursive generalisation to developing explicit generalisations. Recursive generalisation describes the relationship between successive pattern terms and often links to additive thinking, while explicit generalisations require students to see the relationship between two variables, for example, the pattern quantity and the pattern term. Growing patterns are often presented that limit students' awareness and accessibility to generalise multiplicative pattern structures (Pinto & Cañadas, 2021). This leads to students articulating recursive thinking to express generalisations rather than generalising the functional relationship. Moss and Beatty (2010) explain 'while recursive strategies allow students to predict what comes in the next couple of positions of a series, it does not foster the ability to perceive the (structural) relationship across the two data sets to find the underlying rule' (p. 16), nor to see these sets as domain and co-domain of a function. Thus, often students need to be scaffolded to recognise the pattern term number (independent variable) in geometric patterns. To overcome such an issue, independent and dependent variables need to be explicitly represented (Moss, et al., 2008).

A review of research literature highlights specific actions and pedagogical strategies that can be utilised by teachers. Teachers can support students to explore relationships in patterns by using questioning to position them to engage in multifaceted observations of structure. This includes both figural aspects of the pattern and numerical quantities of the components (Becker & Rivera, 2008; Twohill, 2018). In Twohill's (2018) study, the interviewer actively supported students to consider figural aspects of the pattern to generalise. Representations including the use of natural language, numerical, pictorial, tabular, and graphical representations are an important pedagogical tool in supporting students to generalise (Pinto & Cañadas, 2021; Stephens et al., 2017). The purposeful use of questioning to prompt students to find different terms and the use of quasi-variables have been noted as helpful in supporting students to generalise and to shift from arithmetic to algebraic thinking (Fujii & Stephens, 2001; Pinto & Cañadas, 2021; Twohill, 2018). Work by Cooper and Warren (2008), expanded on this notion of quasi-variable (Fujii & Stephens, 2001) to define a quasi-generalisation, that is the bridging step towards a generalisation. Research indicates using a far term supports students to move from a physical context to a mathematical context. Similarly, two different studies (Pinto & Cañadas, 2021; Twohill, 2018) found that pressing students to find values for far terms contributed to the development of explicit generalisations.

3 Theoretical framework

Two key theoretical frameworks underpin this study and support the analysis of the findings, the focusing framework (Lobato et al., 2013) and the levels of thinking of primary school students as they generalise functional relationships developed by Blanton et al. (2015) and then later refined by Stephens et al. (2017).

3.1 Focusing framework

To understand the teaching and learning actions that support young students to generalise, we draw on the *focusing framework* (Lobato et al., 2013) to examine the processes of noticing during classroom interactions. This framework was aligned with the collaborative context developed in the classroom (discussed later in the article) and our focus on collective learning. We draw on the focusing framework to analyse classroom interactions as students demonstrate how they 'see' or notice mathematical structures of contextual growing patterns and how the teacher adapts their practice to support student learning.

Past research with older students emphasised that this framework offers an opportunity to capture the complexities of student noticing, recognising that social interactions influence individual cognition (Lobato et al., 2013). There are four key components to the framework: first, *centers of focus* require focusing on student noticing that elicits properties, features, and regularities of mathematics. Second, *focusing interactions* includes the discourse (conversations, diagrams, gesture, self-talk) and teaching actions that assist students to attend to particular centers of focus. In the context of this study, this refers to the ways students and teachers facilitate each other to see the mathematical structures that assist students to reach generalisations. Third, *mathematical tasks* are considered to determine how the features of the tasks influence what students notice. Finally, the *nature of the mathematical activity* is examined to ascertain how the participatory organization of the students and teacher contributes to students accessing and reasoning their mathematics.

3.2 Young students' generalisations

Research studies identify different categories to explain how students generalise mathematical structures in algebraic contexts. Recently, Blanton's et al. (2015) presented eight levels of thinking displayed as young primary students generalise functional relationships and Stephens et al. (2017) built on from this work to develop a model consisting of 10 levels. These levels of thinking demonstrate the interrelatedness between seeing structure and generalising functional relationships consisting of variational thinking, co-variational thinking, and correspondence thinking. We draw on Blanton et al (2015) and Stephens et al. (2017) levels of sophistication of thinking with regards to young children moving towards generalisations as they engage with functional thinking. These levels of generalisations are utilised in this study to understand the ways in which young children reason and generalise growing pattern structures from contextual tasks. Figure 1 displays a type of functional task and Table 1 provides an illustrative example of the types and levels of functional thinking young students display.

These levels of generalisation will support the analysis process in this study.

4 Research questions

Drawing from the literature and theoretical frameworks, the following research questions were posed:

- How do young diverse students see mathematical structures in contextual growing patterns and articulate their generalisations about these?

Fig. 1 Example of a functional task

Example of the task: In our classroom, two children can sit at a desk. When there are three desks, six children can sit in class. How many children can sit in the class if there are 9 desks? How many children sit in the class if there are 35 desks? What if there were 82 children sitting in the class, how many desks would there be? Can you write or explain a rule for this pattern?

Table 1 Levels of thinking and sophistication of generalising functional relationships (Blanton et al., 2015; Stephens et al., 2017)

Thinking	Levels	Illustrative example
Variational thinking	Level 1 Recursive-particular	Recursive pattern in either or both variables by referring to particular numbers only. *It goes 2, 4, 6…*
	Level 2 Recursive-general	Correct recursive pattern in either or both variables. *The number of people goes up by 2 each time*
Covariational thinking	Level 3 Covariational relationship	Covariational relationship where the two variables are coordinated rather than separate. *Every time you add a desk, you add two more people*
Correspondence thinking	Level 4 Single instantiation:	Expression or equation with numbers and/or unknowns that provides one instantiation of the function rule but does not generally relate the two variables. $2\times2=4$
	Level 5 Functional-particular	Functional relationship using particular numbers but does not make a general statement relating the variables
	Level 6 Functional basic	General relationship between variables but not the transformation between them. *Times 2*
	Level 7/8 Functional-emergent	Incomplete function rule in variables (L7, e.g., *dx2*) or words (L8, e.g., *You multiply the desks by 2*), often describing transformation on one variable but not explicitly relating to other
	Level 9/10 Functional condensed	Function rule in variables (L9, e.g., *p = dx2*) or words (L10, e.g., *If you multiply the number of desks by 2, you get the number of people who can sit.*) that describes a generalised relationship between the two variables

- What teaching actions assist students to identify and generalise these growing pattern structures?

5 Research design

This article reports on one aspect of a larger study, underpinned by classroom teaching experiment methodology, focusing on the use of contextual patterns from Pacific and Māori culture to develop young culturally diverse students' understanding and generalisation of functional patterns.

5.1 Participants

Given the nature of this study and the use of indigenous Pacific and Māori patterns and contexts, we note that one of the researchers (first author) comes from a Kuki Airani (Cook Island) background. Similarly, the teacher who agreed to participate in the study was an experienced teacher of Māori heritage. The study involved a classroom of Year Two (aged 6 years) students in a low socio-economic school in New Zealand. Consent was sought from both parents and children and 29 students agreed to participate in the study, including 17 male and 12 female students. The students were predominantly of Pacific descent (n = 24), with three students from an indigenous Māori background, and two students from South East Asia.

5.2 Classroom teaching experiment

The teaching episodes took place over a four-week period in Term 3 of the schooling year. Drawing on the design of a classroom teaching experiment (Steffe & Thompson, 2000), students participated in ten 30-min lessons exploring and developing their understandings of functional growing pattern generalisation. Each lesson involved a similar structure with the launch of the task, paired work, and whole class discussion. Students in this classroom had previously engaged with tasks involving repeating patterns but growing patterns were unfamiliar as this is not a curriculum expectation until Year Four. In this article, we focus on four lessons involving two different pattern tasks.

The tasks (see Table 2) were collaboratively developed by the research team and teacher in regular meetings. The process for task development involved looking at existing familiar patterns related to the students' cultural heritage or typical family/community activities and interrogating these for existing patterns. The task was then developed by taking the existing pattern and framing it within a problematic context. A key aspect of planning the tasks was also anticipating possible student responses to the tasks so that these could be used to collectively move student reasoning towards generalisations during the whole class discussions.

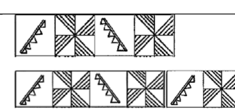
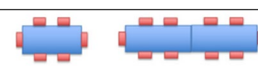
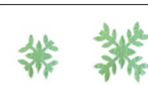
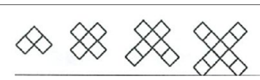
Each lesson typically followed the format of having students: (i) discuss the pattern in pairs; (ii) continue the pattern for the next three terms; (iii) solve for a far term/quasi-generalisation; (iv) identify a general rule.

We focus on two lessons related to task three and two lessons related to task four. These were preceded by four lessons involving repeating multiplicative patterns (see Hunter & Miller, 2022). Given that this was the first linear growing pattern in the lesson series, the research team and teacher selected a context accessible across the class. Students often attended family/community events at church halls or community centres. *Kaikai* (feast) is part of events and extended family and community sit together to share food around tables arranged in a communal way. Children help set up the tables and arrange seating. The second task drew on a Cook Island pattern used in tivaevae quilting. Tivaevae is a traditional form of Cook Island quilting involving groups of women designing, cutting, and embroidering quilts. These are *taonga* (treasures) given as gifts on special occasions such as weddings or significant birthdays. Designs are developed from nature and frequently incorporate forms of growing patterns.

Teacher pedagogical support was provided throughout the classroom teaching experiment in two ways. First, the teacher had been involved in ongoing professional development as part of the Developing Mathematical Inquiry Communities (DMIC) research project (Hunter and Hunter 2019). As part of this, the teacher was familiar with connecting mathematics teaching to the students' cultural identity and using

Table 2 Lesson, pattern name, cultural context and pattern image

Lesson, Pattern Name, Cultural Context and Pattern image

Lesson	Pattern name	Cultural Context	Pattern (image)
1 & 2	Task 1: Tapa Cloth	A Tongan artist has been working on their tapa cloth design.	
3 & 4	Task 2: Samoan Sasa	At the Pāsifika festival they perform a Samoan sasa (slap legs two times and clap once).	
5 & 6	Task 3: Tables and chairs	You are having a family reunion at your church hall and need to set up the tables.	
7 & 8	Task 4: Tivaevae	A group of Mamas are working on a Tivaevae design.	
9	Task 5: Māori tuku tuku pattern	Tuku tuku panel.	
10	Task 6: Māori tuku tuku pattern	Tuku tuku panel.	

tasks that allowed multiple levels of differentiated outcomes and the use of different representations Second, one of the researchers (author 1) was a participant observer for each lesson and worked alongside the teacher providing support for questioning to position students to notice both figural and numerical aspects in the pattern. This allowed for centers of focus and focusing interactions to be drawn out more consistently across lessons.

5.3 Data collection

To capture student–teacher and student–student interactions, all lessons were video-recorded. This included the use of video cameras that focused both on the teacher and pairs of students working together. The aim of using the video-recordings was to support triangulation within the study and provide a broader scope for data collection. More specifically, it documented teacher actions, how students discussed and shared their knowledge, and the whole class discussions that promoted student engagement with the generalisation process. All video-recordings were downloaded at the conclusion of the lesson and transcribed for analysis.

5.4 Data analysis

We drew on a thematic approach using a coding scheme which we developed from Lobato et al. (2013) focusing framework and the levels of sophistication of generalisation (Blanton et al., 2015; Stephens et al., 2017) to analyse the lesson data. The initial coding scheme drew on four components of the focusing framework: centers of focus; focussing interactions; mathematical tasks; nature of the mathematics activity. Each transcript was analyzed using iterative cycles to evidence the focusing framework. Table 3 presents the four cycles of data analysis and alignment to the focusing framework.

All transcripts were coded drawing on the focusing framework, and then coded for levels of generalisations displayed during lessons (see Table 1). To ensure reliability of the coding, both researchers coded the data independently using both coding systems and then cross checked the analysis. During this process, when coding did not align or there were contradictions, the researchers discussed the differences until a consensus was reached. The researchers then reviewed the example of the transcript and the examples in the theoretical frames to determine alignment. Insights gained from the lessons are presented in the following sections.

6 Results

First, we present the findings in relation to the mathematical tasks for both lessons. Following this, each pattern task is presented with an in-depth analysis of the centers of focus

Table 3 Data analysis cycles for the focusing framework with examples

Focusing framework	Description	Examples of analysis
Cycle 1: Centers of focus (CF)	Student noticing of properties, features, and regularities of mathematics	Student move: articulates structure of pattern including pre-structural (parts of the general structure); structural (full structure of the pattern); additive; multiplicative
Cycle 2: Focusing interactions (FI)	Discourse and teaching actions that assist students to attend to particular centers of focus	Teacher supported move to facilitate noticing of the structure of the pattern or generalisation: or promote mathematical discussion, question posing, representations, gesture
Cycle 3: Mathematical tasks (MT)	Features of the tasks that influence students mathematical noticing and discourse	Connections between the pattern and lived experiences; cultural context as a conceptual anchor before moving to abstract reasoning
Cycle 4: Nature of the mathematical activity (NMA)	Participatory organization of the students and teacher contributes to students accessing and reasoning their mathematics	Moving between small group and whole class discussions. Shifting between mathematical discussions and mathematical argumentation (making claims, using evidence, justifying, reasoning)

and focusing interactions. Finally, we provide a summary of the analysis of the nature of the mathematical activity. Weaved within each of these sections are students' levels of thinking with regards to forming their generalisations.

6.1 Features of the mathematical tasks

The tasks were purposefully designed to connect with the children's experiences outside of school and their cultural contexts. To launch the tables and chairs task, the teacher began with a discussion about family celebrations and eating together at halls. She encouraged the students to share their own personal experiences of sitting together at church or family feasts and arranging tables. This task context provided an opportunity to physically model the situation with the tables and chairs in the classroom during the launch and focusing interactions, and as it built from a familiar visual structure, students were able to make connections with their own experiences. Similarly, the context of the second task, tivaevae, drew on a pattern structure that was familiar to many of the students. The teacher began the lesson by showing students pictures of tivaevae quilts. She then made an explicit connection to a Cook Island student and asked her to share her experiences of her Mama and Aunties making quilts. She asked other students to share where they had seen tivaevae. Although tivaevae is specific to the Cook Islands, the motifs and patterns are used in craftwork across the Pacific. Student responses evidenced their familiarity with these patterns: "I've seen those at church" (Tiare). Another student referenced cultural festivals in Auckland that are part of community life: "You can see those at a festival" (Hone).

Both tasks were designed to position students to recognise the structure of the pattern and engage in functional thinking to develop generalisations. The tables and chairs pattern had a structure of two visual variables (tables and chairs) with the relationship between the variables visually explicit. The structure of this task grew in a horizontal manner to the left which could be viewed as an additive or recursive pattern growth, however, the need to "move" the end chair resulted in a requirement to explicitly identify the pattern structure and growth. In contrast, the tivaevae pattern grew in multiple directions. The structure of this task required students to engage in seeing the pattern grow in multiple directions while holding a constant structure in the centre (four leaves). Both tasks were also structured with a prompt to press students to consider a far term or quasi-generalisation to further support the students to engage in correspondence thinking and articulate a more sophisticated generalisation.

A key feature of the table and chair task was the opportunity for students to generate multiple representations to explore the relationships and the structure of the pattern. This included being able to visualise and work physically with the tables and chairs in the classroom and then re-create this by drawing. In Sect. 6.2 (focusing interaction 1), we highlight how the students were able to engage in argumentation by referencing the physical model and their existing knowledge of seating arrangements and contrasting this with a student developed partial representation. We also highlight how students used the context as a conceptual anchor moving from representations to exploration of structure. In contrast, the tivaevae pattern was more difficult for the students to draw given the complexity of the pattern. This meant that the task along with the familiarity of the leaf motif pattern positioned students to use different representations to represent structure. Although, as shown in Sect. 6.3 (center of focus 1 and 2), students initially approached the task by counting, they readily moved towards numerical representations that exemplified varying structures. We conjecture again that the context provided an anchor for the students to move to more abstract algebraic reasoning.

6.2 Tables and chairs task

Centers of focus 1: Continuing the pattern without recognising the structure of the pattern

When students began to work in pairs to find the number of chairs for three tables, many began by attempting to copy the pattern from the first table to continue the pattern, without recognising the structure. In this center of focus, the students attended to the number of chairs for each table without considering how they were to be joined together. Some students built on the representation of the first table representing three tables with six chairs each to a total of 18 chairs. Other pairs started counting the number of chairs around the two tables and then developed a pre-structural representation of the three tables (see Fig. 2).

In both examples, students were beginning to demonstrate pre-structural thinking and were starting to notice the recursive nature of the pattern. While the students were starting to explore the relationship between the chairs and tables, they had not seen the underlying relationship between the quantities of each variable.

Focusing interaction 1: Orienting to the structure of pattern through discussion and examination of the representation

Noticing that students were having difficulty in accessing the structure of the pattern, the first focusing interaction was used to orient students to the structure. To achieve this, the teacher led an examination of a purposefully selected student developed partial representation (see Fig. 2) and positioned

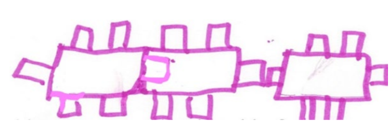


Fig. 2 Pre-structural representation

students to engage in mathematical argumentation (make claims, provide evidence, justify thinking). The teacher redrew the representation on the board and asked students to: "have a look at it and have a think, do you agree with their plan? Talk to your buddy about why you agree or don't agree". After collaborative talk, the teacher stepped in to focus attention on the relationship between the two visual variables (tables and chairs): "look at where they have put the chairs? Do you agree or disagree with where they've put the chairs?". At this point, all the students in the group expressed disagreement with the placement of the chair.

Teacher: Why do you think it's false?

Diora: Hmmm, cos there is a chair inside.

Teacher: And why can't we have one there?

Diora: Cos you will get squashed.

Teacher: Who wants to change their plan?

Arielle: I want to change mine, I disagree.

The teacher then provided students with an opportunity to re-work their representation and solution for three tables. Interestingly, many of the students continued to find it difficult to move beyond a partial representation and as shown in Fig. 3, they removed some of the chairs but continued to view the relationship between the chairs and tables as a multiplicative growing pattern that increased by five chairs with each table. One suggested reason for this is that the previous two lessons focused on multiplicative repeating patterns and students may be transferring this thinking to this task (see Table 2).

Noticing that students continued to develop partial representations demonstrating pre-structural thinking, the teacher again engaged students in a whole class discussion. At this point, the teacher referred to the physical actions used during the launch and took the following actions to assist students to shift their thinking:

Teacher: But what did you think about these ones? [points to the chairs in between the tables].

Arielle: Hmmm it's going to get squashed.

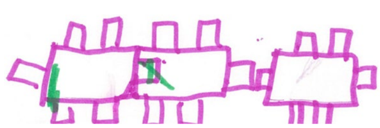


Fig. 3 Pre-structural representation

Teacher: Look when I pushed the two tables together, what did I do with the extra chairs?

Arielle: You put them over.

Teacher: So, what do you think you could do with this [teacher gestures to the chair on the drawing].

Tiana: Put them over and cross them out.

Arielle: Five over here [points to Table 1] then four over here [points to Table 2] and five over here [points to Table 3].

This extended focusing interaction allowed the teacher to refocus the students' center of focus so that they could now see that the pattern was made of structures that stayed the same and structures that changed.

Centers of focus 2: Recognising and representing the structure of the pattern

At this point, students were able to recognise the structure of the pattern and manipulate the representation so that each table had a correct number of chairs. Student developed representations (see Fig. 4) of the structure of the pattern showed varying levels of sophistication. In the first example below, the pattern for three tables was simply extended without explicitly identifying the structure. Other students developed pictorial representations in which they explicitly identified and notated the structure as 5/4/5 and others who noticed the structure as 2/2/2 with additional chairs at each end.

These students were developing variational thinking, specifically, they were demonstrating recursive particular thinking (L1), representing for a particular instance how the pattern was increasing in a recursive way. It was important for the teacher to recognise that students had different noticings in relation to the pattern structure as this had the potential for students to express their generalisations in different ways.

Focusing interaction 2: Connecting representations and generalisation

In this focusing interaction, the teacher moved students from variational thinking to developing correspondence thinking through connecting representations and generalisation. In the next section of the lesson, students were working in pairs to solve the number of chairs for eight tables. The teacher transitioned students from paired work to whole class discussion. To give all students access to a representative form that would support generalisation, she purposely drew on a representation developed by a pair of students that explicitly identified the structure of

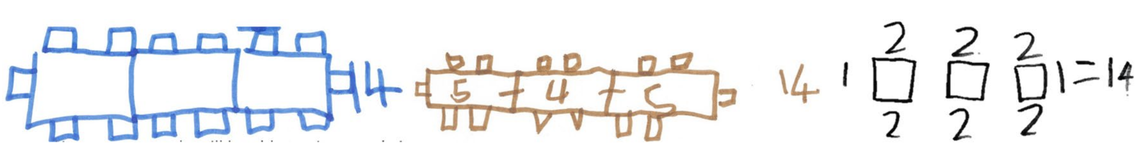


Fig. 4 Structural representations

the pattern and drew on covariational thinking (L3), where the students were beginning to relate the two variables:

Teacher: Sose and Ana thought of a quicker way to record. I'm going to write it the way they did, instead of doing each chair. They wrote fives here [indicates end tables], see that, and then what number went on all these tables? [indicates tables in the middle]

Sebastian: Four, but then there are only six tables? [indicates tables in the middle]

Teacher: Why are there only six fours? That's a great question, Ana, can you explain?

Ana: Because six plus two equals eight.

Teacher: Can you explain why six plus two equals eight is helpful? [long pause] So there is the six tables… [indicates tables in the middle]

Ana: And the ends

Sose: One there [indicates table at one end] and one there [indicates table at the other end]

Following extended discussion of the generalisations, the teacher pressed students to consider a far term of 85 tables (quasi-generalisation). Notably, given the young age of the students, they were not asked to calculate the quantity for the far term but instead were asked to describe how it could be solved or what the pattern would look like by referring to the structures that had been developed.

Teacher: Can you explain that to us for the 85 tables?

Sebastian: Five at that end and then five there [indicating ends of the tables] and fours.

Teacher: How many fours would there be?

Hamuera: Eighty-five.

Sebastian: Eighty-three.

Teacher: Sebastian and Mere, can you explain why there would be 83?

Sebastian: Because they take away five.

Mere: They take away the two fives.

At this point, analysis of the broader classroom discussion indicated that more students were displaying a functional-particular (L5) generalisation. This discussion continued with students shifting their thinking to articulate the relationship in more general terms: "you could multiply the tables in the middle by four to find the number of chairs". This is an example of a generalisation which sits between functional emergent (L8) generalisation in words and a functional condensed (L10) generalisation in words. The students are using correspondence thinking to show the relationship between two variables (number of chairs and number of tables), however, students did not articulate what happens with the end tables so thus have not attended to the full pattern structure. The teacher continued to work on connecting the representation to language used by the students in their expressed generalisations.

Centers of focus 3: Generalizing and developing a rule

The third center of focus in this lesson was developing a function rule in words and variables and more sophisticated correspondence thinking. The earlier focusing interactions, where the teacher asked students to connect the representation and language used in student generalisations, appeared to support students to develop a function rule. For example, some students articulated their generalisation (L10) as "number of chairs equals any number of tables, take off the five from each end and then multiply the remaining tables by four and add on the two sets of five chairs you took off". Others explained this as "the number of tables take away two, times the four chairs, and then add ten".

6.3 Tivaevae pattern task

Centers of focus 1: Continuing the pattern moving from noticing the particular to the recursive structure

Students initially began solving the tivaevae task by attempting to draw or count to find the number of leaves for pattern positions. For example, Sebastian and Cruz began by counting the twelve leaves for position one beginning with the four leaves in the centre (4) and two leaves up each of the four stems (8) and for position two counted another eight leaves up to 20 leaves. After the third position, although they did not explicitly discuss this, it was apparent that they had noticed the regularity of the increase by eight. They then used the increase by eight to continue to count by visualising the next pattern position. This is an example of students moving from pre-structural understanding of the pattern to recursive particular thinking (L1) to recursive general thinking (L2), each time the pattern grew students could see another eight had been added.

Focusing interaction 1: Orienting to the structure of pattern through discussion, table of values, and drawn representations

The teacher noticed that many students were counting all leaves rather than using multiplicative reasoning or using the 'add 8' to continue the pattern. To assist students to shift their thinking, the teacher introduced a table of values to record the pattern and re-orientate the way students saw the structure beyond the count all approach. However, despite the introduction of the table of values, the teacher noticed that most students continued either drawing or counting all to find the answer for subsequent positions. Potentially, the introduction of the tabular representation was unsuccessful because it focused on the total number of leaves rather than the underlying structure. To address this, the teacher next used specific pedagogical actions. Firstly, building on student thinking during small group work, she asked students to discuss how they were seeing the pattern grow:

> I heard Jaya and Teresia having a really interesting conversation, instead of counting they were talking

about how the pattern is getting bigger [gestures circular shape with hands] so have a think about how does it get bigger, what happens each time, how do the Mamas make it bigger each time?

This teaching moment was key to shift student attention to explicitly noticing the structure of the pattern rather than using a count all technique. This included a verbal prompt along with the use of gesture combined with examination of the pattern.

Following paired discussion, the teacher then initiated another whole class discussion and specifically modelled a pictorial and numerical representation that built on student explanations of how they saw the pattern structure (see Fig. 5). At the same time students were explaining how they saw the pattern structure; the teacher began to orient students to attend to a constant variable (four leaves in the centre of the pattern).

Hamuera: The pattern on the outside is, they are putting two each.

Teacher: So just before we say about the outside, can someone remind us how many leaves there are on the inside? What about in that middle bit?

Students: Four.

Teacher: Okay so it's four in the middle, does everyone understand where that four came from?

Ned: Yes from the middle of the tivaevae pattern.

Teacher: Ok and Hamuera, what did you say happened each time?

Hamuera: They're putting two each on the outside.

Teacher: Two leaves, just two or two on every?

Hamuera: Side.

Similarly, the teacher modelled a representation of the structure for the third pattern for students who saw the pattern increasing in eight (see Fig. 5). Again, the teacher focused students on the constant despite 'adding eight' to each pattern position:

Ngaire: Every time you add on leaves you add eight.

Teacher: So can I just check how many were in the middle on your one?

Sima: Four.

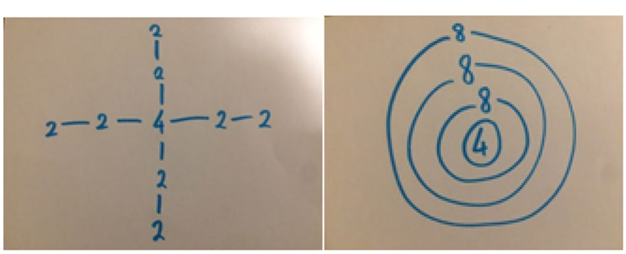


Fig. 5 Teacher generated representations

Teacher: It's still four in the middle, you said, every time we add eight. Turn and tell your buddy why they're saying they'll always add eight [draws diagram]

This focusing interaction provided students with a representation which they could use to show how they visualised the structure of the pattern. Furthermore, this gave access for students to understand their peers' perception of the structure of the growing pattern.

Centers of focus 2: Recognising and representing the structure of the pattern

Students worked in pairs and built on the representation that the teacher had introduced to construct their own representation for how they saw the pattern growing for the 7th position. Analysis of the data indicates that students visually represented the pattern as growing in four distinct ways: (i) total number of leaves for each stem (14 leaves for each stem); (ii) growing two (pair of leaves) up the stem (seven sets of 2); (iii) growing in sevens up each side of the stem (eight groups of seven); and (vi) growing in multiples of eight represented as seven circles around the centre of the pattern (see Fig. 6). While there were four different representations, it appeared the latter three provided an opportunity for students to think about the multiplicative structures and the variables within the pattern which could assist students to move from variational thinking to correspondence thinking.

In this moment, not only is the teacher having students attend to structure, but it is also evident that students are displaying the visual contractions of the pattern, aligned more closely to the structure they are seeing. In this instance, there has been a transfer from a contextual pattern into a mathematical context. Following collaborative work to draw the structure, the teacher provided an opportunity for all students to access the ways in which their peers visualised the pattern growth.

Focusing interaction 2: Using far terms and quasi-generalisation to press for more sophisticated generalisation across pattern structures and reaffirming mathematical language

The second focusing interaction included the use of a far term to extend students to articulate the general structure of the pattern across other instances. The teacher asked students to explain what the pattern structure would look like at position 76 moving them beyond particular instances and encouraging students to see the underlying multiplicative structure as it would be inefficient for the students to use a count all method for position 76. Also, the teacher drew on the different ways students were seeing the growing pattern structure. For example, the teacher asked all students to describe how the growth in eights (circle structure) would be represented:

Sose: Seventy-six.

Teacher: Seventy-six what's?

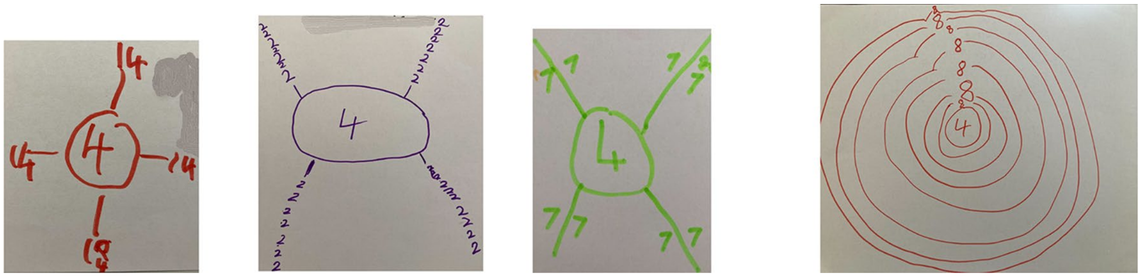


Fig. 6 Student developed structural pictorial and numerical representations

Alannah: Eight 76 times.

Teacher: So we would have to write eight 76 times and add those together.

Ana: What do you mean eight 76 times?

Teacher: Turn and ask her.

Ana: What do you mean eight 76 times?

Sose: We had to do eight 76 times with the leaves.

Following this interaction, the teacher again reminded students about the constant variable and how this could be included in a more general statement about their pattern. She encouraged students to attend to the mathematical language that was needed to make the statement by exploring far terms (e.g., position 100 and 1000):

Teacher: How many eights would it be [at position 1000]?

Esteban:One thousand times eight.

Teacher: We've got all the eights, what haven't we put in? What's in the middle?

Esteban: Four.

Teacher: Four so we have to say…

Esteban: Four.

Teacher: What's the mathematical way of saying that?

Esteban: Add four.

This focused teaching interaction supported students to move from an incomplete statement about the pattern rule in relation to a particular relationship (number of leaves at position 1000) to a functional-particular generalisation (L5) "1000 times eight add four".

Centers of focus 3: Developing a generalisation

The final center of focus was providing a general rule for the pattern. One student pair articulated, "the four stays the same, like the eight". Here they were attempting to generalise the multiplicative nature of the pattern (×8) and the constant variable (+4) as two aspects of the general rule that remain the same for each pattern position. When examining this statement, against the levels of thinking (Blanton et al., 2015; Stephens et al., 2017), it is difficult to determine where this statement fits. The students have clearly moved beyond variational thinking (L1/L2) as they are no longer considering the pattern to be a recursive relationship, and it could be argued they are now demonstrating co-variational thinking (L3) or even functional basic thinking (L6) as the students are making generalised statements about the constant variable (4 leaves in the center) and the multiplicative structure of eight saying the same (what you multiply the pattern position by to determine the number of leaves) but the students are not attending to the relationship between each of the variables. A different student group articulated emergent functional thinking in words (L8) to describe the general rule as "Eight times plus four." It was clear there were key attributes of the generalised relationship between the two variables, however, the students had missed articulating how multiplying by eight related to the pattern position number (e.g., 8×___+4). At the end of the lesson the teacher facilitated students to collectively form a rule in words (L10): "for any position you take the pattern position number times by eight plus four equals the number of leaves)".

6.4 Nature of mathematical activity

All lessons shared similar features in relation to the nature of mathematical activity. The participatory organisation guiding the roles and expectations for the students and teacher drew on notions of collectivism and collaboration. Students were expected to work collaboratively in pairs and to develop solutions collectively both by sharing their own ideas and listening to others. Across all lessons, the teacher began by setting up norms for small group work with the students, asking them to discuss why they were working together and how it helped them in their learning. Paired work was alternated with whole class discussion and the teacher consistently revisited the expectations of collaborative work. Collectivism was evident in the ways that students would support each other, question, and build explanations as a collective and this supported students to further develop understanding of structure and make generalisations. For example, students attended to the ideas of others and questioned their explanations, and this led to the further generalisation and collective explanations (see Sect. 6.3, center of focus 3).

During whole class discussions, the teacher utilised pedagogical actions that developed discourse and required the students to reflect on each other's ideas and agree or disagree with these (see Sect. 6.2, focusing interaction 1). This builds on the notion of productive lingering (Russell et al., 2017) when the topic of discussion is complex and abstract providing opportunities for students (and teachers) to make connections. The teacher continued to engage the class (see Sect. 6.2, focusing interaction 2) to deepen the discussion and probe further considerations. These pedagogical actions supported students to engage in mathematical argumentation, where students would agree or disagree with mathematical ideas and provide reasons for this. Again, this can be directly linked to the development of generalisation (see Sect. 6.3, center of focus 3).

7 Discussion and conclusion

In this article, we sought to examine both how young culturally diverse students see mathematical structures and develop generalisations related to contextual growing patterns and the teaching actions that assisted the students to identify and generalise the patterns. We used the theoretical framing of centers of focus (Lobato et al., 2013) to analyse two lessons focused on growing patterns and the generalising process. Importantly, the mathematical task and related pattern structure was shown to have the potential to impact on student's center of focus and, the students' center of focus directly linked to how they generalised. The focusing framework was a useful tool to examine collective noticing and development that occurred. While we used the focusing framework as an analytical tool, we believe that this also has potential as a teaching framework for developing early algebraic thinking.

7.1 Features of contextual growing patterns tasks and their influence on students' recognition of mathematical structure

This study contributes to the growing set of research literature that shows young students can demonstrate a range of sophisticated generalisations to articulate the underlying mathematical structure and functional relationships of growing patterns. Past research has established that the way early algebra tasks are presented and taught impacts on students' ability to generalise (Cañadas et al., 2016; Moss et al., 2008). The tasks selected for this research were contextual to the students' everyday lives and this meant that they could draw on situational knowledge to consider the underlying structures of the patterns. The teacher purposefully launched the tasks to help students connect to their experiences at home and in the community, providing a contextual anchor. As Wijns et al. (2019) highlight students' prior experiences with patterns can provide access to growing patterns. In this study, we conjecture that cultural knowledge and familiarity with the patterns supported the students to identify the structures of the pattern, a key factor in the beginning stages of developing generalisations. Additionally, the use of contextual patterns offered an opportunity to highlight how patterns are part of the everyday life and experiences of culturally diverse communities.

We argue that contextual growing pattern tasks offer access to algebra but of importance in the design of these tasks is considering the underlying structure of the pattern. This includes: (a) whether both variables are explicit (e.g., number of tables; number of chairs); (b) if the variables were embedded in the pattern (e.g., tivaevae); and, (c) if the multiplicative structure could be easily identified. There is potential that the sequencing of tasks impacted on the ways in which students were identifying structures and generalising. Both patterns provided an opportunity for these young students to engage with correspondence thinking and it was evident that the students saw the growing pattern structures in different ways for both tasks. This impacted on their representations of the general pattern structure and generalisations. There appeared to be more commonalities in the ways in which students saw the structure of the table and chair task compared to the tivaevae pattern. While the context of the patterns were accessible for both tasks, we conjecture the difference in the way students saw the underlying structures of the patterns could have occurred for the following reasons: (i) there was opportunity to physically experience the table and chairs task and this could have impacted on how students saw the structure; (ii) the complexity of the Tivaevae growing in multiple directions thus having multiple ways of seeing the underlying structure; and, (iii) the different multiplicative structures for each pattern (table pattern $y=4x+2$; tivaevae pattern $y=8x+4$). Thus, further investigation is warranted.

7.2 Teachers' pedagogical strategies to foster students' recognition of mathematical structure

Analysis of the data indicates that a range of pedagogical strategies were required to orient students to the structure of the pattern. Key to these strategies was facilitating students to connect to different forms of representations. Earlier research studies (e.g., Pinto & Cañadas, 2021; Stephens et al., 2017; Twohill, 2018) have shown the benefits of a range of representational forms. In this study, students were initially provided with opportunities to collaboratively develop pictorial representations of the pattern structure. This then became a tool that the teacher could use to engage the wider group of students in argumentation and reflection on the structure. In both lessons, the teacher facilitated students to represent the growing pattern using drawing and numbers coupled with paired and whole class discussion.

This supported the students to "see" differing structures and at times refocused the students' center of focus.

Careful monitoring of student thinking and analysis of their representations enabled the teacher to be responsive to student reasoning. Previous studies have shown how function tables can support students to record patterns and make generalisations (Cañadas et al., 2016; Moss & Beatty, 2010). In the tivaevae lesson, the teacher introduced a tabular form to support students to access the structure of the pattern. Interestingly, this was not effective in moving student reasoning forward. This highlights the need for educators to both be responsive to student reasoning and to consider how different representational forms link to the structure of the pattern. Many curricula provide teachers with static suggestions of tables or graphs as appropriate representations related to the age of students. In contrast, the findings of this study suggest that a more nuanced approach should be taken. An important consideration is how specific representations provide opportunities for orienting students to structure particularly for patterns that grow in multiple directions.

7.3 Dynamic stages of functional thinking revealed in classroom discourse

By building from seeing the underlying structures, students were able to move through the different stages of functional thinking (variational, covariational, and correspondence) to articulate their generalisations. For the table and chairs task, analysis of the classroom discussion and student artefacts indicated that students appeared to move through different levels of functional thinking throughout the lesson in the following way: L1, L3, L5, L8 and L10. Similarly, in the tivaevae task students moved through the levels as follows: L2, L1, L5, L6, L8, L10. In the second task there was a fold back from L2 to L1, this potentially highlights how the progressions of thinking are not linear in classroom contexts rather students can move between levels as they form their generalised rules. There were few instances of students using letters as variables as in L7 and L9. The teacher took an important role to facilitate students to engage in the generalising process. Using student generated representations, she provided all students with access to the thinking of others and consistently refocused student attention on the constant within the patterns. When she was confident that students were able to visualise the pattern growth, she introduced a quasi-generalisation and asked students to find far terms for the pattern. This was undertaken in an accessible way for the age of the students where they could use the pictorial representation to frame their generalisation.

Of note, is that the type of generalisations developed by these young students align with other researcher's work with middle years students (Radford, 2010). This demonstrates the potential for engagement with patterning tasks and generalisations from the early years of schooling which potentially can accelerate students' progress and provide opportunities for them to engage with abstracting and generalising. By attending to these structures, students began to see a common feature of the pattern, which has been termed as 'grasping' (Radford, 2010). In turn, this 'grasping' assisted students to display correspondence thinking. While it can be argued that some generalisations (i.e. symbolic generalisation or alphanumeric notation) are more sophisticated than others, it is suggested that in the early years greater importance lies in the ability to initially determine contextual generalisation or generalisations articulated in words. This involves moving beyond particular pattern figures and identifying a relationship between pattern figures and pattern terms. Additionally, what provides students with the ability to move between these different types of generalisations is of importance and impacts the way in which teachers engage students in the learning experiences.

7.4 Concluding remarks

In conclusion, this article highlights the possibilities afforded by providing young culturally diverse students access to contextual patterning tasks and opportunities for generalisation. Generalisation is an important aspect of mathematical competency across schooling and a key part of algebra. This article contributes to the growing field of research related to growing patterns and highlights the importance of representations to position students to consider structure. We note that this is a growing area of research and further research is needed to build on these initial findings including: research with a larger sample of students and a range of data sources; and considering the role of local and global visualisation through a comparison study of how students/teachers see the structure of the pattern, to name a few. Finally, we acknowledge the ongoing challenges of teachers supporting young learners to engage in algebraic thinking as well as aligning mathematics to a cultural context. We anticipate that the tasks included in this research and teacher actions highlighted may be one way forward to providing avenues that address these issues in culturally diverse primary school settings.

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