Minimax approaches to Robust Control

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Abstract

This Thesis is a fundamental investigation of minimax approaches to robust control. The minimax games considered here are for bounded classes of uncertain plant where the performance is measured by a quadratic cost function. These games are between the controller and a group of uncertainty, disturbance and measurement noise signals with the possible inclusion of the initial condition of the plant.

An H_∞ with transients problem is presented where a non zero initial condition and structured uncertainty are permitted. Necessary and sufficient conditions for the existence of controllers that solve this problem for state feedback and measurement feedback are given. The optimal solution to the state feedback problem may be found by a convex optimisation. These results represent an extension of [Khargonekar et al., 1991].

A state feedback minimax problem is presented where the initial condition is known and multiple channels of uncertainty, each satisfying an integral quadratic constraint, are permitted. Necessary and sufficient conditions for the existence of a minimax controller are given and the design is shown to be the result of a convex optimisation. These results are an extension of [Savkin and Petersen, 1995]. Similar measurement feedback problems are also discussed. Comparisons and special cases of the minimax and H_∞ with transients and structure problems are presented. Also, expressions for the worst case uncertainty, disturbance and measurement noise signals are given.

Finally, a set valued estimation problem is considered for closed loop uncertain plants. The initial condition of the plant is constrained to lie in an ellipsoid and the uncertainty is permitted to be structured and satisfies a type of integral quadratic constraint. Given the history of measurements from the initial time to the current time, a method for determining the set of possible current states is presented. This result represents an extension of [Savkin and Petersen, 1995a] and [Bertsekas and Rhodes, 1971] to permit structured uncertainty. It is also shown how the set valued estimator may be used as a model invalidator for models with bounded uncertainty.

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Contents

Notation

The following notation will be used throughout the Thesis: Signal norms will be represented as follows

$\|a(t)\|^2$	$\int_0^\infty a(t)'a(t)\,dt$	A squared L_2 norm,
$\|a(t)\|_M^2$	$\int_0^\infty a(t)'Ma(t)\,dt$	A weighted squared L_2 norm,
$\|a(t)\|_{[b,c]M}^2$	$\int_b^c a(t)'Ma(t)\,dt$	A weighted time integral.

The L_2 induced norm, or infinity norm for a linear system, will be denoted using the same notation as for the L_2 norm of a signal;

$\|G\|$	$\sup_{w(t)\neq 0} \frac{\|G(w(t))\|}{\|w(t)\|}$	The L_2 induced norm.

The following acronyms will also be used

ARE		Algebraic Ricatti Equation,
RDE		Ricatti Differential Equation,
LQR		Linear Quadratic Regulator,
LQG		Linear Quadratic Gaussian,
IQC	$\|w_i(t)\|^2 \leq \|z_i(t)\|^2 + d_i$	Integral Quadratic Constraint,

along with some additional notation

$\dot{a}$	$\frac{da}{dt}$	Time derivative,
$(\dot{\cdot})$	$\frac{d}{d\epsilon}(\cdot)\mid_{\epsilon=0}$	A derivative of a scaling matrix τ, in some direction. See equations (3.24) to (3.27) for details.
$\mathbf{R}^n$		The set of real vectors of dimension $n \times 1$.

Chapter 1

Introduction

1.1 Robust control

Mathematical models are frequently used for Controller design. Inevitably, there will be a discrepancy between the plant and the mathematical model of the plant. This discrepancy means that a controller which performs well when applied to the plant model will not necessarily perform well when applied to the real plant. If the plant model is extended to a *class* of plant which contains a description of the real plant then a controller which guarantees a certain level of performance for all members of the class of plant is said to be *robust* since this level of performance is guaranteed for the control system corresponding to the real plant. This problem may be extended to find a controller which optimises the performance for the 'worst' or most badly performing member of the class of plant. The solution to such a problem is a minimax controller, the design of which is the subject of this Thesis.

1.2 The history of some existing methods for robust controller design

Robustness is considered in classical control by the specification of suitable gain and phase margins. The classical methods were satisfactory for the design of controllers for single input/single output systems but more flexible methods are required to deal with multivariable controller design. The Linear Quadratic Regulator (LQR) and Linear Quadratic Gaussian (LQG) multivariable controller design methods emerged in the 1960's, allowing the design of an optimal controller for a nominal linear plant [Kwakernaak and Sivan, 1972], [Anderson and Moore, 1989]. However, neither the LQR method, for state feedback controller design, nor the LQG method, for measurement feedback controller design, explicitly considered robustness. [Safonov and Athans, 1977] showed that a controller designed by the LQR method has some desirable robustness properties but [Doyle, 1978] showed that an LQG controller has no *a priori* guaranteed robustness properties. Doyle's result highlighted the need for a multivariable controller design method with *a priori* guaranteed robustness properties.

There are two main approaches to multivariable robust control which have been particularly successful; the operator theoretic approach and the guaranteed cost approach. The development of the operator theoretic methods and the guaranteed cost control methods will be treated separately.

1.2.1 The operator theoretic approach

The Small Gain Theorem of [Zames, 1963] provided the foundation for the operator theoretic approaches to robust control. The measure of the gain of a linear system used by Zames was the H_∞ norm which is the L_2 induced norm of a linear system; suppose a system $\mathcal{G}$ has input signal $w(t)$ so the output is $\mathcal{G}(w(t))$. The L_2 induced norm is defined as the ratio of the L_2 norm of the output signal to the L_2 norm of the input signal,

$$\|\mathcal{G}\| \triangleq \sup_{w(t)\neq 0} \frac{\|\mathcal{G}(w(t))\|}{\|w(t)\|}$$

If the system $\mathcal{G}$ is linear then it may be represented by its transfer function $G(s)$ and the H_∞ norm may be written as

$$\|G(s)\|_\infty \triangleq \sup_\omega \bar{\sigma}(G(j\omega))$$

where $\bar{\sigma}$ denotes the maximum singular value. Notice that, for a single input/single output system, the H_∞ norm is the maximum gain value that appears on the magnitude Bode plot.

The Small Gain Theorem states that the closed loop system of Figure 1.1 is stable if the product of the L_2 induced norms $\|G\|$ and $\|\Delta\|$ is less than 1. Thus, any signal injected into the loop will be attenuated.

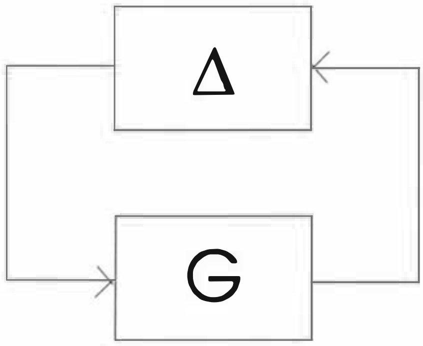

Figure 1.1

The H_∞ control problem was posed by [Zames, 1981]. The problem was to design a controller $\mathcal{K}$ such that the system shown in Figure 1.2 is sable for all Δ such that $\|\Delta\| \leq \gamma^{-1}$.

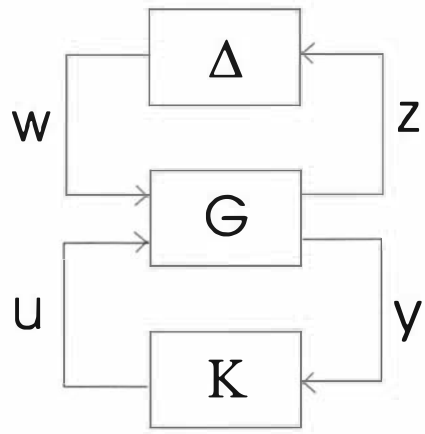

Figure 1.2

The system Δ may be used to represent uncertainty in the plant model so that the linear plant $\mathcal{G}$ and the uncertain system Δ, together, define a class of plant. Since Δ is constrained to satisfy an L_2 induced norm bound, the class of plant is a continuum of nonlinear systems. If the real plant is a member of this class then a controller which stabilises the system shown in Figure 1.2, for all members of the class, will also stabilise the real plant.

Elegant and tractable solutions to the state feedback and measurement feedback H_∞ problems were given by [Doyle et. al., 1989]. They presented necessary and sufficient conditions for the existence of a controller which stabilises the system shown in Figure 1.2. An algorithm to synthesise one such controller was also given for the case when a solution exists.

Following the solution of the H_∞ problem, extensions and generalisations of this problem emerged. One such extension allowed a structured uncertainty description where multiple channels of uncertainty are permitted. Such a structured uncertainty description is shown in Figure 1.3.

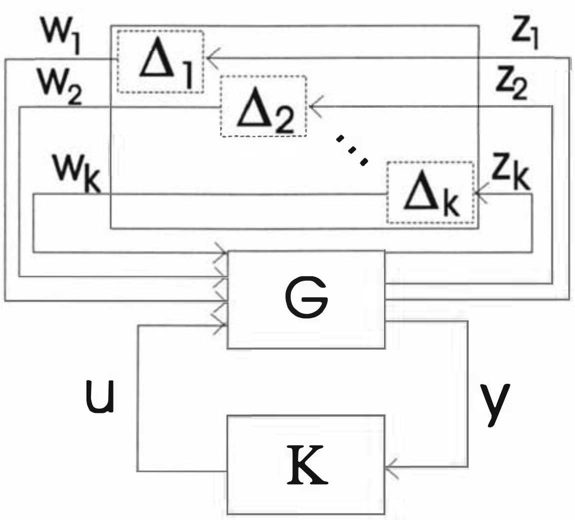

Figure 1.3

The structured uncertainty, Δ, is described by a finite number of L_2 induced norm bounded systems Δ_i. These systems satisfy the L_2 induced norm bounds $\|\Delta_i\| \leq \gamma$, for $i = 1 \ldots k$. The solution to this problem was given by [Shamma, 1994] in terms of the solution to a scaled H_∞ problem. Details of how this scaling works will be presented later in this Thesis during an investigation of similar problems, however, a brief explaination is offered here:

Firstly, notice that if the structure of the uncertainty blocks Δ_i, shown in Figure 1.3, is ignored such that the uncertainty operator is no longer restricted to be diagonal then a standard H_∞ problem is recovered. Therefore, a standard H_∞ problem may be solved to give a conservative solution to the H_∞ problem with structured uncertainty. To recover this conservatism consider a scaling applied to the system of Figure 1.3, as shown in Figure 1.4. For the class of scaled systems shown in Figure 1.4 to be equivalent to the class of systems shown in Figure 1.3, it is required that the operators τ_i are such that

$$\frac{\|\tau_i(w_i)\|}{\|\tau_i(z_i)\|} \leq 1 \quad \Leftrightarrow \quad \frac{\|w_i\|}{\|z_i\|} \leq 1 \tag{1.1}$$

This equivalence is valid if the operators τ_i are constant scalar multipliers.

The structured uncertainty in the scaled system of Figure 1.4 may also be ignored to give a conservative solution to the H_∞ problem with structured uncertainty. [Shamma, 1994] showed that there exists a set of scaling scalars $\tau_1, \tau_2 \ldots \tau_k > 0$ such that the solution to the H_∞ problem with structured uncertainty is equivalent to the (unstructured) H_∞ problem for the scaled system. That is, Shamma gave necessary and sufficient conditions for the existence of a solution to the structured H_∞ problem in terms of the solution to a scaled H_∞ problem.

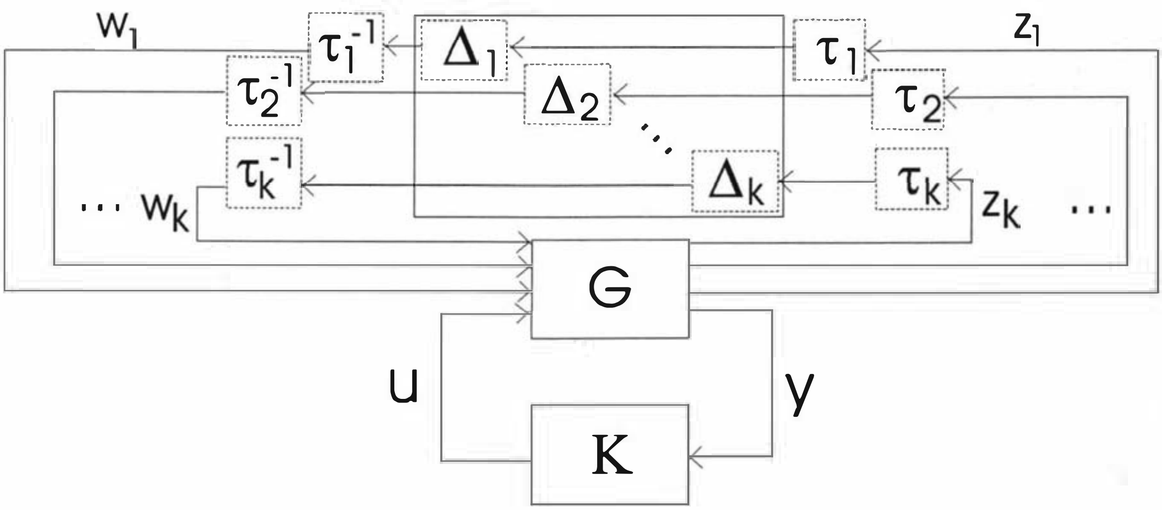

Figure 1.4

More restrictive classes of uncertainty have also been considered; the problem where the uncertain systems Δ_i (Figure 1.3) are constrained to be linear, time invariant and satisfy

$\|\Delta_i\|_\infty \leq 1$ is the complex structured singular value problem, also known as the complex μ problem. This problem has been solved conservatively using a bound on a system property known as μ, described in [Packard and Doyle, 1993]. The conservative solution to this problem requires an iterative procedure known as $D - K$ iteration, where D represents a scaling operator (similar to the operators τ_i of Figure 1.4) and K is a controller. Unfortunately, this iteration is not guaranteed to converge and, as the solution is approached, the order of the controller generally increases and may approach infinity.

Since the uncertain systems Δ_i are constrained to be linear, a class of scaling operators, larger than those used for the structured H_∞ problem, may be used. If there are k uncertainty channels then the scaling operator D is made up of k time invariant linear systems so $D\Delta = \Delta D$ which implies (1.1). [Poolla and Tikku, 1995] showed that the controller which optimises the bound on μ is the solution to a problem where the uncertain systems Δ_i are permitted to be arbitrarily slowly time varying linear systems rather than time invariant.

Another even more restrictive class of uncertainty is also permitted within the μ framework; repeated scalar uncertainty blocks are required to have the form $\Delta_i = \delta I$, where $\delta \in \mathbf{R}$. Thus, this uncertainty description may be used to represent memoryless uncertainty. Some progress has been made towards a solution to this problem, known as the real μ problem, however, it remains unsolved. Indeed, there are a number of ways to describe uncertainty and the appropriateness of each uncertainty class is problem specific.

Performance requirements may be encoded into robust controller design problems by the introduction of a fictitious uncertainty block which causes the plant to become unstable if the performance bound is violated. Such problems, with uncertain plant and performance requirements, are known as robust performance problems. For more information on posing and encoding robust performance problems see [Postlethwaite and Skogestad, 1996].

An optimal solution may be sought for all of the robust control problems described so far. The optimal controller permits the largest bound on the uncertainty or the strictest possible performance requirement. The solution is approached by successively enlarging the uncertainty class or strengthening the performance requirement until the (scaled) ARE criteria of [Doyle et. al., 1989] are no longer satisfied. This process is known as γ iteration and the optimal controller corresponds to the largest uncertainty class or the strongest performance requirement for which a solution exists.

Unfortunately, the H_∞ problem, with or without structured uncertainty, gives an impractical solution for the optimal case; for state feedback, as γ approaches its optimal value, the maximum singular value of the gain matrix of the controller approaches infinity. A similar problem occurs for measurement feedback; the filter gain matrix approaches infinity as γ approaches its optimal value.

The H_∞ with transients results of [Khargonekar et al., 1991] offered an interpretation of the benefit of using suboptimal values of γ; if a nonzero initial condition is included in the uncertainty description for the state feedback H_∞ problem then the optimal solution corresponds to a suboptimal solution of a standard H_∞ problem. However, the optimal solution for the measurement feedback H_∞ with transients problem specifies a filter which, initially, has finite gain but, at some time, the maximum singular value of the filter gain matrix approaches infinity. Although the H_∞ with transients problem of [Khargonekar et al., 1991] resulted in an impractical optimal measurement feedback controller, they presented a method for designing a time invariant suboptimal controller with finite filter gain.

Summarising the robust control methods discussed thus far, the H_∞ solution of [Doyle et. al., 1989] formed a foundation which has been generalised to allow a structured uncertainty description and various uncertainty classes. The H_∞ with transients results of [Khargonekar et al., 1991] offer some justification of the use of a suboptimal solution to the H_∞ problem in terms of robustness to initial conditions.

1.2.2 Guaranteed cost approaches

Concurrent to work stemming from the H_∞ problem of, for example, [Zames, 1981], extensions to the LQR and LQG methods to explicitly consider robustness were emerging. The LQR and LQG methods may be used to design a controller which minimises a linear quadratic cost function for a nominal plant model [Anderson and Moore, 1989]. A guaranteed cost controller guarantees that a cost function will be less than some upper bound for all members of a class of plant and, therefore, explicitly considers robustness.

[Chang and Peng, 1972] wrote a pioneering paper on guaranteed cost control where they described a state feedback guaranteed cost controller which guaranteed a bound on a cost function. The uncertainty was permitted to be time varying and satisfied a Euclidean norm bound, that is, an uncertain parameter in the model was instantaneously bounded in Euclidean norm at every point in time. [Petersen and McFarlane, 1994] also considered a state feedback guaranteed cost control problem where the uncertainty satisfied a Euclidean norm bound and, unlike the operator theoretic problems, the initial condition was not assumed to be unknown or zero but was, instead, given as problem data. They gave a sufficient condition for the existence of a guaranteed cost controller and a cost bound in terms of a scaled ARE with a single scalar scaling parameter. [Savkin and Petersen, 1995] showed that this guaranteed cost controller was minimax for a larger class of uncertainty described by the Integral Quadratic Constraint (IQC) of [Yakubovich, 1973]. The IQC is a generalisation of the L_2 induced norm bounded uncertainty description used in the H_∞ type designs and the optimality of the controller was proven using a result known as the 'S procedure' [Megretsky and Treil, 1990] and [Yakubovich, 1992]. The guaranteed cost approaches have also been extended to measurement feedback by [Petersen, 1995] and [Savkin and Petersen, 1996a]. These approaches culminated in the paper [Savkin and Petersen, 1997] where necessary and sufficient conditions for the existence of a guaranteed cost controller were given for a plant with IQC bounded uncertainty. The initial condition was treated as unknown but non zero intial conditions increased the allowable cost bound.

It is interesting that the H_∞ type methods, originating from the problem of [Zames, 1981] and the minimax and guaranteed cost problems which are, historically, extensions of the LQR and LQG problems to accomodate robustness have converged to very similar problems. The state feedback H_∞ problem with structured uncertainty specifies L_2 induced norm bounded uncertainty and a zero initial condition whereas the state feedback minimax problem of [Savkin and Petersen, 1995] allows an IQC uncertainty description which is a superset of the L_2 induced norm bound. This minimax problem also allows a non-zero initial condition. The solutions to both the state feedback H_∞ with structure problem and this state feedback minimax problem require the existence of a set of real positive scaling scalars such that a scaled ARE satisfies the state feedback H_∞ solution requirements of [Doyle et. al., 1989]. However, for the minimax problem, the scaling parameters should be optimised to minimise a cost bound which is a function of the initial condition of the state, $x(0)$. Also, the (impractical) optimal solution of the structured H_∞ problem is equivalent to a special case of the minimax problem of [Savkin and Petersen, 1995]. For this special case the uncertainty is described by an L_2 induced norm bound rather than the more general IQC except for one channel which satisfies an arbitrarily large L_2 norm bound such that the transient effect of the initial condition is negligible.

Similarly, for measurement feedback, the operator theoretic approach and the guaranteed cost approach have yielded similar solutions. The measurement feedback H_∞ with transients solution of [Khargonekar et al., 1991] is similar to the measurement feedback guaranteed cost results of [Savkin and Petersen, 1997]. Both problems allow an initial condition which incurs a cost depending on its distance from the origin in state space, however, the guaranteed cost controller allows an IQC uncertainty description whereas the uncertainty for the H_∞ with transients case satisfies an L_2 induced norm bound. The solutions to both problems require the existence of a set of real positive scaling scalars such that there exist suitable solutions to a scaled ARE and a scaled RDE.

1.2.3 A game approach

All of the problems described so far may be viewed as two player games, a concept which will be used throughout this Thesis. The H_∞ control problem represents a game between the controller u and the uncertainty/disturbance/measurement noise signal w. The w player seeks to maximise some cost function, denoted $J(u, w)$ while the controller u seeks to minimise this cost.

One player in a two player game plays first so, generally, the second player, knowing the first player's strategy, has an advantage. Therefore,

$$\inf_u \sup_w J \geq \sup_w \inf_u J$$

where the first infimum or supremum represents the player that plays first. A minimax problem is one where $\inf_u \sup_w J(u, w)$ is sought, so the controller plays first.

For some games the order of play is unimportant so $\inf_u \sup_w J = \sup_w \inf_u J$. Such games are said to have a saddle point. Saddle points are useful because the problems are generally more tractable and the solution is independent of the order of play. Saddle point solutions exist for the optimal case of the H_∞ problem with and without structure, the H_∞ with transients problem and the minimax problem of [Savkin and Petersen, 1995]. They also play an important role in this Thesis. For more information on the game approach to the H_∞ problem see [Basar and Bernhard, 1995].

1.3 Main contributions of this Thesis

There are two main contributions of this Thesis. Firstly, a controller synthesis method is presented for an H_∞ problem with structured uncertainty and transients. Secondly, a state feedback minimax controller synthesis method is presented which permits structured uncertainty. Convexity and existence results ensure the optimisation required to design the minimax controller is tractable. A Set Valued Estimator (SVE) is also presented for systems with structured uncertainty and may be applied as a model invalidator.

A method for controller synthesis for the problem of H_∞ with transients and structured uncertainty is presented in Chapter 3. It is a generalisation of the H_∞ with transients method of [Khargonekar et al., 1991] and the H_∞ with structure problem of [Savkin and Petersen, 1996]. The uncertainty is permitted to be structured and is described by an arbitrary number of L_2 induced norm bounds. The initial condition is permitted to be non-zero but incurs a cost which depends on its distance from the origin. Necessary and sufficient conditions for the existence of state feedback and measurement feedback controllers are given. A state feedback controller may be designed by a convex search but the design of a measurement feedback controller is the result of a search over a set of scaling parameters which are not necessarily convex or compact. The results presented in Chapter 3 are presented in [Milliken et al., 1999a]

The state feedback minimax results presented in Chapter 4 give a slightly strengthened version of [Savkin and Petersen, 1995] with a proof using only standard mathematical tools. Necessary and sufficient conditions for the existence of a minimax controller are given and an algorithm for designing a minimax controller is presented. Furthermore, it is shown that the design of the minimax controller may be achieved by the result of a convex search. This ensures the optimisation is tractable. The results of Chapter 4 were presented in [Milliken et al., 1999].

In Chapter 5, a Set Valued Estimator (SVE) is developed for systems with structured uncertainty where the uncertainty is represented by an arbitrary number of finite time Integral Quadratic Constraints (IQC's) which must be satisfied for all time. This result is an extension of [Bertsekas and Rhodes, 1971] and [Savkin and Petersen, 1996a] to permit structured uncertainty. The set valued estimator may be used (generally conservatively) as a model invalidator for models with bounded uncertainty. The results of Chapter 5 were presented in [Milliken and Marsh, 1998].

Chapter 2

Preliminaries

2.1 Preliminaries

For the sequel, consider uncertain systems of the form

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \quad (2.1)$$
$$z(t) = C_1 x(t) + D_{12} u(t) \quad (2.2)$$
$$y(t) = C_2 x(t) + D_{21} w(t) \quad (2.3)$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ the control input vector and $y(t) \in \mathbf{R}^r$ is the measurement vector. The disturbances, measurement noise and uncertainty are represented by $w(t) = (w_p(t)', w_1(t)', w_2(t)' \cdots w_k(t)')' \in \mathbf{R}^p$ with $w_p(t) \in \mathbf{R}^{p_p}$ the disturbance/measurement noise inputs and $w_i(t) \in \mathbf{R}^{p_i}$ the uncertainty inputs. The performance and uncertainty outputs are $z(t) = (z_p(t)', z_1(t)', z_2(t)' \cdots z_k(t)')' \in \mathbf{R}^p$ with $z_p(t) \in \mathbf{R}^{p_p}$ the performance output and $z_i(t) \in \mathbf{R}^{p_i}$ the uncertainty outputs.

The matrices B_1, C_1, D_{12} and D_{21} may also be written in terms of channels; $B_1 = [B_{1_p}\ B_{1_1}\ B_{1_2} \cdots B_{1_k}]$, $C_1 = [C_{1_p}'\ C_{1_1}'\ C_{1_2}' \cdots C_{1_k}']'$, $D_{12} = [D_{12_p}'\ D_{12_1}'\ D_{12_2}' \cdots D_{12_k}']'$ and $D_{21} = [D_{21_p}\ D_{21_1}\ D_{21_2} \cdots D_{21_k}]$.

It is also required that the nominal system $[A, B_2]$ be stabilisable and $[A\ C_2]$ be detectable.

2.2 An algebraic Riccati equation comparison theorem

In this Section an algebraic Riccati equation (ARE) comparison Theorem is presented. The Theorem presented here may be found in [Ran and Vreugdenhil, 1988] and is instrumental in the proofs of many of the results presented in this Thesis so is included for completeness. Firstly, some preliminaries are presented, followed by the ARE comparison Theorem without proof.

Suppose a function of X, $\mathcal{R}(X)$, is defined such that

$$\mathcal{R}(X) = A'X + XA - XBR^{-1}B'X + Q \quad (2.4)$$

where (A, B) is assumed to be stabilisable, $R > 0$ and Q is Hermitian.

Two sets M and N are considered such that

$$M = \{X = X' : \mathcal{R}(X) \geq 0\} \tag{2.5}$$

and a subset of M,

$$N = \{X = X' : \mathcal{R}(X) = 0\} \tag{2.6}$$

Theorem 2.2.1 *Assume M is not empty. Then there exists an $X_+ \in N$ such that $X_+ \geq X$ for all $X \in M$. Then, in particular, X_+ is the maximal Hermitian solution of (2.6). Moreover, all the eigenvalues of the matrix $A - BR^{-1}B'X_+$ are in the closed left half plane.*

Proof: For a proof of this Theorem consult [Ran and Vreugdenhil, 1988]. □

Theorem 2.2.1 will be used extensively in this Thesis to compare H_∞ type ARE's. An example of how this Theorem may be applied follows. Consider the two ARE's (2.7) and (2.8), where $\beta \geq 0$.

$$A'X + XA + X\left(\gamma^{-2}B_1B_1' - B_2B_2'\right)X + C_1'C_1 \tag{2.7}$$

$$A'X_\beta + X_\beta A + X_\beta\left((\gamma+\beta)^{-2}B_1B_1' - B_2B_2'\right)X_\beta + C_1'C_1 \tag{2.8}$$

Pre and post multiplying (2.7) by X^{-1}, pre and post multiplying (2.8) by X_β^{-1} and multiplying by -1 gives

$$0 = X^{-1}(-A') + (-A')X^{-1} - X^{-1}C_1'C_1X^{-1} - \gamma^{-2}B_1B_1' - B_2B_2' \tag{2.9}$$

and

$$0 = X_\beta^{-1}(-A') + (-A')'X_\beta^{-1} - X_\beta^{-1}C_1'C_1X_\beta^{-1} - (\gamma+\beta)^{-2}B_1B_1' - B_2B_2' \tag{2.10}$$

From Theorem 2.2.1, if there exists $X^{-1} > 0$ then there exists $X_\beta^{-1} \geq X^{-1} > 0$ with $-A' - C_1'C_1X_\beta^{-1}$ stable. Now, (2.10) may be rearranged to give

$$0 = X_\beta^{-1}(-A'-C_1'C_1X_\beta^{-1})+(-A'-C_1'C_1X_\beta^{-1})'X_\beta^{-1}+X_\beta^{-1}C_1'C_1X_\beta^{-1}-(\gamma+\beta)^{-2}B_1B_1'-B_2B_2' \tag{2.11}$$

Since $-A' - C_1'C_1X_\beta^{-1}$ is stable, by a Lyapunov argument, $X_\beta^{-1}C_1'C_1X_\beta^{-1} - (\gamma+\beta)^{-2}B_1B_1' - B_2B_2' > 0$. Immediately, $C_1'C_1 - X_\beta\left((\gamma+\beta)^{-2}B_1B_1' - B_2B_2' > 0\right)X_\beta > 0$. Now, (2.8) may be rearranged to give

$$\begin{aligned} 0 \;=\; & \left(A + \left((\gamma+\beta)^{-2}B_1B_1' - B_2B_2'\right)X_\beta\right)'X_\beta + X_\beta\left(A + \left((\gamma+\beta)^{-2}B_1B_1' - B_2B_2'\right)X_\beta\right) \\ & -X_\beta\left((\gamma+\beta)^{-2}B_1B_1' - B_2B_2'\right)X_\beta + C_1'C_1 \end{aligned} \tag{2.12}$$

So, by another Lyapunov argument, $A + \left((\gamma+\beta)^{-2}B_1B_1' - B_2B_2'\right)X_\beta$ is stable. Therefore, if there exists a solution $X > 0$ to (2.7) then there exists X_β satisfying (2.8) such that $X \geq X_\beta > 0$ and $A + \left((\gamma+\beta)^{-2}B_1B_1' - B_2B_2'\right)X_\beta$ is stable.

Chapter 3

H_∞ with Transients and Structure

3.1 Introduction

The objective of the standard H_∞ problem, introduced by [Zames, 1981], was to design a controller such that the infinity norm of the closed loop plant was less than some pre-specified value, γ. However, if the designer seeks to minimise γ, the H_∞ norm bound of the plant, the solution is not practical; for the state feedback case, as γ approaches the optimal value, the maximum singular value of the gain of the controller approaches infinity. A similar problem occurs for measurement feedback where the filter gain approaches infinity as γ approaches its optimal value. A controller with an infinite gain or infinite filter gain is not practical. This problem of an impractical solution for optimal γ is also experienced when the H_∞ problem is extended to allow structured uncertainty [Savkin and Petersen, 1996].

The problems associated with the H_∞ solution for optimal γ were typically avoided by choosing a sub-optimal value of γ. The 'H_∞ with transients' results of [Khargonekar et al., 1991] give an interpretation of the benefit of sub-optimal γ in terms of robustness to initial conditions. [Khargonekar et al., 1991] extended the game between the controller and the disturbance/measurement noise to a game between the controller and the disturbance/measurement noise and initial condition pair. For state feedback and optimal γ, this extension gave a solution for which the controller was static with finite gain. The solution for optimal γ for measurement feedback gave a filter with time varying gain. The filter gain is initially finite, however, at some time, the maximum singular value of the filter gain approaches infinity. Although the problem of [Khargonekar et al., 1991] resulted in an impractical controller for measurement feedback, they presented a method for designing time invariant sub-optimal controllers with finite filter gains.

The main contribution of this Chapter is to extend the 'H_∞ with Transients' method of [Khargonekar et al., 1991] to allow a structured uncertainty description. The structured uncertainty may be dynamic, time-varying and nonlinear and is represented by an arbitrary number of L_2 induced norm bounded operators in perturbation feedback loops around the plant. These results are also an extension of [Savkin and Petersen, 1996], allowing uncertainty to enter the measurement equation. This merger of the robust performance method with 'H_∞ with Transients' provides a powerful and flexible framework.

In Section 3.2, some additional preliminaries for state feedback are presented. Section 3.3 presents the main results for state feedback; the optimal controller involves the solution to a parameter dependent algebraic Ricatti equation which is a convex function of the scaling parameters. Section 3.4 is a discussion of the state feedback case. Section 3.5 presents additional preliminaries for the measurement feedback problem where uncertainty in the measurement equation and measurement noise are allowed. Section 3.6, presents the main results for measurement feedback which involve the solution to a parameter dependent ARE and a parameter dependent RDE. Finally, section 3.7 is a discussion of the measurement feedback results.

3.2 Preliminaries for state feedback

For state feedback, uncertain systems of the form (2.1), (2.2) are considered. The uncertainty inputs $w_i(t)$ are related to the uncertainty outputs $z_i(t)$ by L_2 induced norm bounded operators; thus $w(t) \in \mathcal{W}$ where

$$\mathcal{W} = \{w(t) : w_p(t) \in L_2, \|w_i(t)\| \leq \|z_i(t)\|, i = 1 \ldots k\} \tag{3.1}$$

For the sake of simplicity of exposition, it is required that $D'_{12_p} D_{12_p} > 0$ and $D'_{12} C_1 = 0$. These two standard assumptions may be relaxed with some additional complexity of the results presented in the sequel.

The performance is measured by the cost function

$$J_\gamma(u(t), w(t), x(0)) = \|z_p\|^2 - \gamma^2 \left(\|w_p\|^2 + x(0)' R x(0)\right) \tag{3.2}$$

where $R > 0$ is a specified weighting matrix.

The following algebraic Riccati equation plays an important role in the sequel:

$$0 = A'P + PA + P(B_1 \tau_\gamma B'_1 - B_2 \bar{R}^{-1} B'_2)P + C'_1 \tau^{-1} C_1. \tag{3.3}$$

The matrices A, B_2, B_1, C_1 and D_{12} are from the system (2.1), (2.2), $\bar{R}(\tau, \gamma) = D'_{12} \tau^{-1} D_{12}$, and $\tau_\gamma \in \mathbf{R}^{p \times p}$ is a scaling matrix of the form

$$\tau_\gamma = \begin{pmatrix} \gamma^{-2} I_{p_p} & 0 & & \\ 0 & \tau_1 I_{p_1} & 0 & \\ & 0 & \ddots & 0 \\ & & 0 & \tau_k I_{p_k} \end{pmatrix}, \tag{3.4}$$

where I_{p_i} is the $p_i \times p_i$ identity matrix and $\tau_i \in \mathbf{R} > 0$. Also, $\tau \triangleq \tau_\gamma$ with $\gamma \equiv 1$ and $\tau_0 \triangleq \tau_\gamma$ with $\gamma^{-2} \equiv 0$.

Definition 3.2.1 A solution P to (3.3) is stabilising if and only if $\tilde{A} + B_2 K$ is stable, where

$$\tilde{A} = A + B_1 \tau_\gamma B'_1 P \tag{3.5}$$
$$K = -\bar{R}^{-1} B'_2 P \tag{3.6}$$

Definition 3.2.2 For a given value of γ, the set of matrices $\tau > 0$ for which there exists a stabilising solution P to (3.3) such that $0 < P \leq \gamma^2 R$ will be denoted $\Gamma_{SF}(\gamma)$.

3.3 Main results for state feedback

Theorem 3.3.1 *Consider the system (2.1), (2.2) where the state is available for feedback. Given $\gamma > 0$, the following statements are equivalent*

1. *$\Gamma_{SF}(\gamma)$ is non-empty.*
2. *There exists a state feedback controller such that the cost function (3.2) $J_\gamma(u(t), w(t), x(0)) \leq 0$, for all $w(t) \in \mathcal{W}$ and for all $x(0)$. Furthermore, the static controller $u(t) = Kx(t)$ will achieve this performance where K is defined in (3.6).*

Proof: 1 $\Rightarrow$ 2: For brevity of notation, the dependence of the signals u, w and x on t will be omitted. A 'completing the squares' argument, similar to [Khargonekar et al., 1991], is used. Pre and post multiplying (3.3) by x' and x, respectively, and integrating from zero to infinity gives

$$0 = \int_0^\infty x' \left(A'P + PA + P(B_1\tau_\gamma B_1' - B_2\bar{R}^{-1}B_2')P + C_1'\tau^{-1}C_1 \right) x\, dt \tag{3.7}$$

which may be rewritten as

$$\begin{aligned} 0 \;=\; & \int_0^\infty x' \left(A'P + PA \right) x + x'PB_1w + w'B_1'Px + x'PB_2u + u'B_2'Px + \\ & x'P(B_1\tau_\gamma B_1' - B_2\bar{R}^{-1}B_2')Px + xC_1'\tau^{-1}C_1x - x'PB_1w - \\ & w'B_1'Px - x'PB_2u - u'B_2'Px\, dt \end{aligned} \tag{3.8}$$

This equation may be simplified using (2.1) ,

$$\begin{aligned} 0 \;=\; & \int_0^\infty \dot{x}'Px + x'P\dot{x} + x'P(B_1\tau_\gamma B_1' - B_2\bar{R}^{-1}B_2')Px + xC_1'\tau^{-1}C_1x - \\ & x'PB_1w - w'B_1'Px - x'PB_2u - u'B_2'Px\, dt. \end{aligned} \tag{3.9}$$

Since

$$\int_0^\infty \dot{x}'Px + x'P\dot{x}\, dt = \int_0^\infty \frac{d}{dt}(x'Px) = x'(\infty)Px(\infty) - x(0)'Px(0) \tag{3.10}$$

(3.9) becomes,

$$\begin{aligned} 0 \;=\; & x'(\infty)Px(\infty) - x(0)'Px(0) + \int_0^\infty x'P(B_1\tau_\gamma B_1' - B_2\bar{R}^{-1}B_2')Px + \\ & xC_1'\tau^{-1}C_1x - x'PB_1w - w'B_1'Px - x'PB_2u - u'B_2'Px\, dt. \end{aligned} \tag{3.11}$$

Now, define

$$\begin{aligned} \hat{w} &= \tau_\gamma B_1' P x & (3.12)\\ \hat{u} &= -\bar{R}^{-1} B_2' P x & (3.13)\\ X &= \gamma^2 R - P \geq 0 & (3.14) \end{aligned}$$

So, (3.11) may be written as

$$\begin{aligned} 0 &= x'(\infty)Px(\infty) - x(0)'\left(\gamma^2 R - X\right)x(0) + \|w-\hat{w}\|^2_{\tau_\gamma - 1} - \|w\|^2_{\tau_\gamma - 1} \\ &\quad - \|u-\hat{u}\|^2_{D_{12}'\tau^{-1}D_{12}} + \|z\|^2_{\tau^{-1}}\, dt \qquad (3.15) \end{aligned}$$

Therefore,

$$\begin{aligned} J_\gamma(u,w,x(0)) &= \|z_p\|^2 - \gamma^2\left(\|w_p\|^2 + x(0)'Rx(0)\right) \\ &= \|u-\hat{u}\|^2_{D_{12}'\tau^{-1}D_{12}} - \|w-\hat{w}\|^2_{\tau_\gamma^{-1}} - \left(\|z\|^2_{\tau_0^{-1}} - \|w\|^2_{\tau_0^{-1}}\right) \\ &\quad - x(0)'Xx(0) - x'(\infty)Px(\infty) \qquad (3.16) \end{aligned}$$

Now, consider the controller $u = \hat{u}$. Since $\|w_i\| \leq \|z_i\|$ then, from (3.16), $J_\gamma(u,w,x(0)) \leq 0$ for all $w \in \mathcal{W}$ and for all $x(0)$, as required.

$2 \Rightarrow 1$:

Since, for structured uncertainty there is no guarantee that $\lim_{\gamma\to\infty} \Gamma_{SF}(\gamma)$ is non empty, consider two cases separately:

Case 1. $\lim_{\gamma\to\infty} \Gamma_{SF}(\gamma)$ is non-empty.

Case 2. $\lim_{\gamma\to\infty} \Gamma_{SF}(\gamma)$ is empty.

Case 1:

Begin with an overview of the proof. Firstly an ordering on γ is established: for $\gamma_1 \geq \gamma_2$, if there exists $\tau \in \Gamma_{SF}(\gamma_2)$ then $\tau \in \Gamma_{SF}(\gamma_1)$. Secondly, it is shown that there does not exist $\tau \in \lim_{\gamma\to 0} \Gamma_{SF}(\gamma)$. This leads to the existence of γ_{min} such that $\Gamma_{SF}(\gamma_{min})$ is non empty but, for all $\gamma < \gamma_{min}$, $\Gamma_{SF}(\gamma)$ is empty. Next, the existence of $w \in \mathcal{W}$ and $x(0)$ such that $J_{\gamma_{min}}(u,w,x(0)) \geq 0$ for all controllers u is established from which it is shown that, for $\gamma < \gamma_{min}$, there exist $w \in \mathcal{W}$ and $x(0)$ such that $J_\gamma(u,w,x(0)) > 0$ for all u. Thus, it may be deduced that *not* $1 \Rightarrow$ *not* 2 so $2 \Rightarrow 1$.

Begin with the following Lemma.

Lemma 3.3.1 *Consider $\gamma_1 \geq \gamma_2$, if there exists $\tau \in \Gamma_{SF}(\gamma_2)$ then $\tau \in \Gamma_{SF}(\gamma_1)$.*

Proof: Define $\Pi = P^{-1}$. Pre and post multiplying (3.3) by Π and multiplying through by -1 gives

$$0 = \Pi(-A)' + (-A)\Pi - \Pi\left(C_1'\tau^{-1}C_1\right)\Pi - B_1\tau_\gamma B_1' + B_2\bar{R}^{-1}B_2' \qquad (3.17)$$

Defining $\Pi_2 = P(\gamma_2)^{-1} > 0$, and $\Pi_1 = P(\gamma_1)^{-1}$, if it exists, and using (3.17) gives two equations

$$0 = \Pi_1(-A)' + (-A)\Pi_1 - \Pi_1\left(C_1'\tau^{-1}C_1\right)\Pi_1 - B_1\tau_{\gamma_1}B_1' + B_2\bar{R}^{-1}B_2' \tag{3.18}$$

$$\begin{aligned} 0 &\leq B_{1p}\left(\gamma_2^{-2} - \gamma_1^{-2}\right)B_{1p}' \\ &= \Pi_2(-A)' + (-A)\Pi_2 - \Pi_2\left(C_1'\tau^{-1}C_1\right)\Pi_2 - B_1\tau_{\gamma_1}B_1' + B_2\bar{R}^{-1}B_2' \end{aligned} \tag{3.19}$$

By Theorem 2.1 of [Ran and Vreugdenhil, 1988], if there exists a solution Π_2 to (3.19) then there exists a solution Π_1 to (3.18) where $\Pi_1 \geq \Pi_2 > 0$ (so $P_2 \geq P_1 > 0$) and $-A - \Pi_1 C_1'\tau^{-1}C_1$ is stable. Since $0 < P_1 \leq P_2 \leq \gamma_2^2 R \leq \gamma_1^2 R$ then P_1 is allowable if it can be shown that $A + (B_1\tau_{\gamma_1}B_1' - B_2\bar{R}^{-1}B_2')P_1$ is stable.

Since $-A - \Pi_1 C_1'\tau^{-1}C_1$ is stable then, rearranging (3.18) and using a Lyapunov argument gives

$$\begin{aligned} \Pi_1\left(-A - \Pi_1\left(C_1'\tau^{-1}C_1\right)\right)' + \left(-A - \Pi_1\left(C_1'\tau^{-1}C_1\right)\right)\Pi_1 = -\Pi_1\left(C_1'\tau^{-1}C_1\right)\Pi_1 + \\ B_1\tau_{\gamma_1}B_1' - B_2\bar{R}^{-1}B_2' < 0 \end{aligned} \tag{3.20}$$

So, $-C_1'\tau^{-1}C_1 + P_1\left(B_1\tau_{\gamma_1}B_1' - B_2\bar{R}^{-1}B_2'\right)P_1 < 0$. Now with $\gamma = \gamma_1$, (3.3) may be written as

$$\begin{aligned} \left(A + (B_1\tau_{\gamma_1}B_1' - B_2\bar{R}^{-1}B_2')P_1\right)'P_1 + P_1\left(A + (B_1\tau_{\gamma_1}B_1' - B_2\bar{R}^{-1}B_2')P_1\right) = -C_1'\tau^{-1}C_1 + \\ P_1(B_1\tau_{\gamma_1}B_1' - B_2\bar{R}^{-1}B_2')P_1 < 0 \end{aligned} \tag{3.21}$$

So, by a Lyapunov argument, $A + (B_1\tau_{\gamma_1}B_1' - B_2\bar{R}^{-1}B_2')P_1$ is stable. This completes the proof of the Lemma. □

Now it will be shown that there does not exist $\tau \in \lim_{\gamma\to 0}\Gamma_{SF}(\gamma)$. Since it is required that $0 < P \leq \lim_{\gamma\to 0}\gamma^2 R$, then $0 < P \leq 0$ is required which contradicts the existence of a scaling matrix $\tau \in \lim_{\gamma\to 0}\Gamma_{SF}(\gamma)$.

Since $\lim_{\gamma\to\infty}\Gamma_{SF}(\gamma)$ is not empty by assumption in Case 1, $\lim_{\gamma\to 0}\Gamma_{SF}(\gamma)$ is empty and Lemma 3.3.1 gives an ordering, so there exists γ_{min} such that $\Gamma_{SF}(\gamma_{min})$ is not empty but $\Gamma_{SF}(\gamma)$ is empty for all $\gamma < \gamma_{min}$. Also, suppose the scaling matrix τ, corresponding to γ_{min}, is denoted $\hat{\tau}(\gamma_{min}) \in \Gamma_{SF}(\gamma_{min})$.

Lemma 3.3.2 *There exist $w \in \mathcal{W}$ and $x(0)$ such that $J_{\gamma_{min}}(u, w, x(0)) \geq 0$ for any controller u.*

Proof: For the proof of the Lemma, $P(\tau,\gamma)$ will be used to denote a solution to (3.3), explicitly showing the dependence on τ and γ. Now, some properties of γ_{min} will be established.

Consider a vector $e(\tau,\gamma)$ such that $e(\tau,\gamma)$ is an eigenvector corresponding to the minimum eigenvalue of $X(\tau,\gamma)$ (3.14). Also, suppose that, for γ fixed, $\tau = \hat{\tau}(\gamma)$ is such that

$$e(\hat{\tau},\gamma)'X(\hat{\tau},\gamma)e(\hat{\tau},\gamma) \geq e(\tau,\gamma)'X(\tau,\gamma)e(\tau,\gamma) \tag{3.22}$$

for all τ. So, $\hat{\tau}$ maximises the minimum eigenvalue of $X(\tau,\gamma)$. This is consistent with the definition of $\hat{\tau}(\gamma_{min})$. Since γ_{min} is the smallest value of γ such that $\Gamma_{SF}(\gamma)$ is non-empty then, from Lemma 3.3.1, $e(\hat{\tau},\gamma_{min})$ is an eigenvector corresponding to a zero eigenvalue of $X(\hat{\tau},\gamma_{min}) \geq 0$. Therefore,

$$e(\hat{\tau},\gamma_{min})'X(\hat{\tau},\gamma_{min})e(\hat{\tau},\gamma_{min}) = 0 \tag{3.23}$$

Now, consider the following representation of the scaling matrix $\tau = \hat{\tau} + \epsilon\eta$ where $\eta \in \Lambda$ defines an arbitrary direction in τ space with

$$\Lambda \triangleq \left\{ \eta : \eta = \begin{pmatrix} 0_{p_p} & 0 & & \\ 0 & \eta_1 I_{p_1} & 0 & \\ & 0 & \ddots & 0 \\ & & 0 & \eta_k I_{p_k \times p_k} \end{pmatrix}, \sum_{i=1}^{k} \eta_i = 1 \right\} \tag{3.24}$$

where 0_{p_p} is a $p_p \times p_p$ zero matrix.

Also, since the solution P to (3.3) is a continuous function of τ at γ_{min},

$$\frac{d}{d\epsilon}\left[e(\hat{\tau},\gamma_{min})'X(\hat{\tau},\gamma_{min})e(\hat{\tau},\gamma_{min})\right] |_{\epsilon=0} = 0 \tag{3.25}$$

for all directions η in τ space. Using (3.14), the product rule and the fact $e(\hat{\tau},\gamma_{min})$ is an eigenvector corresponding to the zero eigenvalue of $X(\hat{\tau},\gamma_{min})$, (3.25) may be written as

$$e(\hat{\tau},\gamma_{min})'\acute{P}(\hat{\tau},\gamma_{min})e(\hat{\tau},\gamma_{min}) = 0 \tag{3.26}$$

for all η, where $(\acute{\cdot})$ denotes $\frac{d}{d\epsilon}(\cdot) |_{\epsilon=0}$. Now, let the initial condition $x(0) = e(\gamma_{min},\hat{\tau})$. So,

$$x(0)'\acute{P}(\hat{\tau},\gamma_{min})x(0) = 0 \tag{3.27}$$

(3.27) may now be written as

$$x(\infty)'\acute{P}(\hat{\tau},\gamma_{min})x(\infty) - \int_0^\infty \dot{x}'\acute{P}(\hat{\tau},\gamma_{min})x + x'\acute{P}(\hat{\tau},\gamma_{min})\dot{x}\,dt = 0. \tag{3.28}$$

Using the fact that when $u = \hat{u}$ and $w = \hat{w}$

$$\dot{x} = A + \left(B_1\hat{\tau}_{\gamma_{min}}B_1' - B_2\bar{R}^{-1}B_2'\right)P(\hat{\tau},\gamma_{min}) \tag{3.29}$$

which is stable, then, (3.28) becomes

$$\begin{aligned} 0 &= \int_0^\infty x'\left(\left(A + \left(B_1\hat{\tau}_{\gamma_{min}}B_1' - B_2\bar{R}^{-1}B_2'\right)P(\hat{\tau},\gamma_{min})\right)'\acute{P}(\hat{\tau},\gamma_{min})\right. \\ &\quad \left. + \acute{P}(\hat{\tau},\gamma_{min})\left(A + \left(B_1\hat{\tau}_{\gamma_{min}}B_1' - B_2\bar{R}^{-1}B_2'\right)P(\hat{\tau},\gamma_{min})\right)\right)x\,dt \end{aligned} \tag{3.30}$$

Now, differentiating the ARE (3.3), with $\tau = \hat{\tau}$ and $\gamma = \gamma_{min}$, with respect to ϵ at $\epsilon = 0$ gives

$$\begin{aligned} 0 &= \left(A + \left(B_1\hat{\tau}_{\gamma_{min}}B_1' - B_2\bar{R}^{-1}B_2'\right)P\right)'\acute{P} + \acute{P}\left(A + \left(B_1\hat{\tau}_{\gamma_{min}}B_1' - B_2\bar{R}^{-1}B_2'\right)P\right) \\ &\quad + P\left(B_1\eta B_1' - B_2\bar{R}^{-1}D_{12}'\hat{\tau}^{-1}\eta\hat{\tau}^{-1}D_{12}\bar{R}^{-1}B_2'\right)P - C_1'\hat{\tau}^{-1}\eta\hat{\tau}^{-1}C_1 \end{aligned} \tag{3.31}$$

Pre and post multiplying by x' and x, respectively, integrating from zero to infinity and using (3.30) gives

$$0 = \int_0^\infty x'\left(P\left(B_1\eta B_1' - B_2\bar{R}^{-1}D_{12}'\hat{\tau}^{-1}\eta\hat{\tau}^{-1}D_{12}\bar{R}^{-1}B_2'\right)P - C_1'\hat{\tau}^{-1}\eta\hat{\tau}^{-1}C_1\right)x\,dt \tag{3.32}$$

so,

$$\int_0^\infty \hat{w}(\hat{\tau})'\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}\hat{w}(\hat{\tau})\,dt = \int_0^\infty z'\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}z\,dt \tag{3.33}$$

for all $\eta \in \Lambda$ for $u = \hat{u}(\hat{\tau}, \gamma_{min})$ (3.13), $w = \hat{w}(\hat{\tau}, \gamma_{min})$ (3.12) and $x(0) = e(\hat{\tau}, \gamma_{min})$. Therefore, $\hat{w} \in \mathcal{W}$. Using (3.33) and (3.23) in (3.16) gives $J_{\gamma_{min}}(\hat{u}, \hat{w}, e(\hat{\tau}, \gamma_{min})) = 0$.

Now, consider any controller u (not necessarily $\hat{u}$). Define $\tau = \check{\tau}(\gamma) : \|\hat{w}_i(\check{\tau}(\gamma), \gamma_{min})\| = \|z_i(\check{\tau}(\gamma), \gamma)\|$ so $\hat{w}(\check{\tau}(\gamma), \gamma_{min}) \in \mathcal{W}$ only if $\hat{w}_p(\check{\tau}(\gamma), \gamma_{min}) \in \mathcal{L}_2$. Letting $w = \hat{w}(\check{\tau}(\gamma_{min}), \gamma_{min})$ and $x(0) = e(\check{\tau}(\gamma_{min}), \gamma_{min})$, (3.16) becomes

$$J_{\gamma_{min}}(u, \hat{w}(\check{\tau}), e(\check{\tau}, \gamma_{min})) = \|u - \hat{u}\|_{\bar{R}}^2 - e(\check{\tau}, \gamma_{min})'X(\check{\tau}, \gamma_{min})e(\check{\tau}, \gamma_{min}) \tag{3.34}$$

where the dependence of $\check{\tau}(\gamma_{min})$ on γ_{min} has been omitted. Since $\hat{\tau}$ is such that

$$e(\hat{\tau}, \gamma_{min})'X(\hat{\tau}, \gamma_{min})e(\hat{\tau}, \gamma_{min})$$

is maximised,

$$e(\check{\tau}, \gamma_{min})'X(\check{\tau}, \gamma_{min})e(\check{\tau}, \gamma_{min}) \le e(\hat{\tau}, \gamma_{min})'X(\hat{\tau}, \gamma_{min})e(\hat{\tau}, \gamma_{min}) = 0$$

Also, $\|u - \hat{u}\|_{\bar{R}}^2 \ge 0$ so

$$J_{\gamma_{min}}(u, \hat{w}(\check{\tau}), e(\check{\tau}, \gamma_{min})) = \|u - \hat{u}\|_{\bar{R}}^2 - e(\check{\tau}, \gamma_{min})'X(\check{\tau}, \gamma_{min})e(\check{\tau}, \gamma_{min}) \ge 0$$

as required. □

Now, consider some $\gamma < \gamma_{min}$. Using Lemma 3.3.1 and Lemma 3.3.2,

$$\begin{aligned} J_\gamma(u, \hat{w}(\check{\tau}(\gamma_{min}), \gamma_{min}), e(\check{\tau}(\gamma_{min}), \gamma_{min})) &= \|z_p\|^2 - \gamma^2\|\hat{w}_p(\check{\tau}(\gamma_{min}), \gamma_{min})\|^2 - \\ &\quad \gamma^2 e(\check{\tau}(\gamma_{min}), \gamma_{min})'Re(\check{\tau}(\gamma_{min}), \gamma_{min}) \\ &> \|z_p\|^2 - \gamma_{min}^2\|\hat{w}_p(\check{\tau}(\gamma_{min}), \gamma_{min})\|^2 - \\ &\quad \gamma_{min}^2 e(\check{\tau}(\gamma_{min}), \gamma_{min})'Re(\check{\tau}(\gamma_{min}), \gamma_{min}) \\ &= J_{\gamma_{min}}(u, \hat{w}(\check{\tau}(\gamma_{min}), \gamma_{min}), e(\check{\tau}(\gamma_{min}), \gamma_{min})) \\ &\ge 0 \end{aligned} \tag{3.35}$$

Therefore, there does not exist a controller such that $J_\gamma(u, w, x(0)) \leq 0$ for all $w \in \mathcal{W}$ and $x(0)$ for $\gamma < \gamma_{min}$.

It has been shown, from the definition of γ_{min}, that *not* 1 $\Rightarrow \gamma < \gamma_{min}$ and for such γ there does not exist a controller u such that $J_\gamma(u, \hat{w}(\check{\tau}(\gamma_{min})), e(\check{\tau}(\gamma_{min}), \gamma_{min})) \leq 0 \Rightarrow$ *not* 2. So, *not* 1 $\Rightarrow$ *not* 2 which is equivalent to 2 $\Rightarrow$ 1, as required.

Case 2:

For Case 2, consider a system where the effect of the signal w is scaled-down by a factor of $\alpha \in \mathbf{R} \geq 1$ giving a system representation

$$\dot{x} = Ax + \alpha^{-1}B_1w + B_2u \tag{3.36}$$

and (2.2), with cost function

$$J_\gamma^\alpha(u, w, x(0)) = \|z_p\|^2 - \gamma^2\left(\|w_p\|^2 + \alpha^2 x(0)'Rx(0)\right) \tag{3.37}$$

where it is required that $w \in \mathcal{W}$ (3.1).

The ARE corresponding to this modified problem differs from (3.3) only through the scaling of the 'B_1' term,

$$0 = A'P_\alpha + P_\alpha A + C_1'\tau^{-1}C_1 + P_\alpha(\alpha^{-2}B_1\tau_\gamma B_1' - B_2\bar{R}^{-1}B_2')P_\alpha \tag{3.38}$$

Definition 3.3.1 Given α and γ, the set of scaling matrices $\tau > 0$ for which there exists a stabilising solution P_α to (3.38) such that $0 < P_\alpha \leq \alpha^2\gamma^2 R$ will be denoted $\Gamma_{SF}^\alpha(\gamma)$.

Since it is assumed that $\lim_{\gamma\to\infty}\Gamma_{SF}(\gamma)$ is empty then it may be inferred from the ordering on γ that $\Gamma_{SF}(\gamma)$ is empty for all finite γ. Thus, *not* 1. If it can be inferred that there does not exist a controller such that $J_\gamma^\alpha(u, w, x(0)) \mid_{\alpha=1} \leq 0$ for all $w \in \mathcal{W}$ and $x(0)$ then *not* 2 will be established and 2 $\Rightarrow$ 1 would follow for Case 2. Firstly an ordering on α, for any γ, is established. For $\alpha_2 \leq \alpha_1$, if there exists $\tau \in \Gamma_{SF}^{\alpha_2}(\gamma)$ then $\tau \in \Gamma_{SF}^{\alpha_1}(\gamma)$. Secondly, it is shown that an LQR ARE is recovered in the limit as $\alpha \to \infty$, so, $\lim_{\alpha\to\infty}\Gamma_{SF}^\alpha(\gamma)$ is non-empty. Since, for all γ, by assumption for Case 2, $\Gamma_{SF}^\alpha(\gamma) \mid_{\alpha=1}$ is empty then it follows that there exists $\alpha_{min}(\gamma) > 1$ such that, for all $\alpha < \alpha_{min}(\gamma)$, $\Gamma_{SF}^\alpha(\gamma)$ is empty. Then it is shown that there exist $w \in \mathcal{W}$ and $x(0)$ such that $J_\gamma^{\alpha_{min}}(u, w, x(0)) \geq 0$ for all u which, by a similar argument to the proof for Case 1, leads to the non existence of a controller such that $J_\gamma^\alpha(u, w, x(0)) \mid_{\alpha=1} \leq 0$ for all $w \in \mathcal{W}$ and $x(0)$ so *not* 2.

Begin with the following Lemma that performs a similar role to Lemma 3.3.1

Lemma 3.3.3 *Consider $\alpha_1 \geq \alpha_2$ and γ fixed, if there exists $\tau \in \Gamma_{SF}^{\alpha_2}(\gamma)$ then $\tau \in \Gamma_{SF}^{\alpha_1}(\gamma)$.*

Proof: The proof is similar to the proof of Lemma 3.3.1. Pre and post multiplying (3.38) by Π_α and multiplying through by -1, where $\Pi_\alpha = P_\alpha^{-1}$ gives

$$0 = \Pi_\alpha(-A)' + (-A)\Pi_\alpha - \Pi_\alpha\left(C_1'\tau^{-1}C_1\right)\Pi_\alpha - \alpha^{-2}B_1\tau_\gamma B_1' + B_2\bar{R}^{-1}B_2' \tag{3.39}$$

Denoting $P_{\alpha_2}^{-1} = \Pi_{\alpha_2}$, and $P_{\alpha_1}^{-1} = \Pi_{\alpha_1}$, if it exists, and using (3.39) gives two equations

$$0 = \Pi_{\alpha_1}(-A)' + (-A)\Pi_{\alpha_1} - \Pi_{\alpha_1}C_1'\tau^{-1}C_1\Pi_{\alpha_1} - \alpha_1^{-2}B_1\tau_\gamma B_1' + B_2\bar{R}^{-1}B_2' \quad (3.40)$$

$$\begin{aligned} 0 &\leq B_1\left(\alpha_2^{-2} - \alpha_1^{-2}\right)B_1' \\ &= \Pi_{\alpha_2}(-A)' + (-A)\Pi_{\alpha_2} - \Pi_{\alpha_2}C_1'\tau^{-1}C_1\Pi_{\alpha_2} - \alpha_1^{-2}B_1\tau_\gamma B_1' + B_2\bar{R}^{-1}B_2' \quad (3.41)\end{aligned}$$

From here the proof follows the same procedure as the proof of Lemma 3.3.1, resulting in the existence of a solution P_{α_1} such that $0 < P_{\alpha_1} \leq P_{\alpha_2} \leq \alpha_2^2\gamma^2 R \leq \alpha_1^2\gamma^2 R$ with $A + (\alpha_1^{-2}B_1\tau_\gamma B_1' - B_2\bar{R}^{-1}B_2')P_{\alpha_1}$ stable. □

Taking the limit as $\alpha \to \infty$ in (3.38), an LQR ARE is recovered for which there will exist a positive stabilising solution $\lim_{\alpha\to\infty} P_\alpha$ for all scaling matrices $\tau > 0$ [Anderson and Moore, 1989]. Since it has been established that, for all finite $\gamma > 0$, $\lim_{\alpha\to\infty}\Gamma_{SF}^\alpha(\gamma)$ is non-empty, $\Gamma_{SF}^\alpha(\gamma)\mid_{\alpha=1}$ is empty and Lemma 3.3.3 gives an ordering on α then there exists $\alpha_{min} > 1$ such that $\Gamma_{SF}^{\alpha_{min}}(\gamma)$ is non-empty and for all $\alpha < \alpha_{min}$, $\Gamma_{SF}^\alpha(\gamma)$ is empty. The dependence of $\alpha_{min}(\gamma)$ on γ has been omitted for brevity of notation.

Lemma 3.3.4 *There exists $w \in \mathcal{W}$ and $x(0)$ such that $J_\gamma^{\alpha_{min}} \geq 0$ for any controller u.*

Proof: Similar to the proof of Lemma 3.3.2, some properties of α_{min} will be established; there exists a vector $e(\tau,\alpha)$ such that, for fixed γ, α and τ,

$$e(\tau,\alpha)'X_\alpha(\tau,\alpha)e(\tau,\alpha) \quad (3.42)$$

is minimised, where $X_\alpha(\tau,\alpha)$ is given by

$$X_\alpha(\tau,\alpha) = \alpha^2\gamma^2 R - P_\alpha(\tau,\alpha) \quad (3.43)$$

Also, suppose that, for fixed α and fixed γ, $\tau = \hat{\tau}(\alpha)$ such that

$$e(\hat{\tau}(\alpha),\alpha)'X(\hat{\tau}(\alpha),\alpha)e(\hat{\tau}(\alpha),\alpha) \geq e(\tau,\alpha)'X(\tau,\alpha)e(\tau,\alpha) \quad (3.44)$$

for all scaling matrices τ.

Now, a perturbation of the scaling matrix τ may be represented as $\tau = \hat{\tau} + \epsilon\eta$, with $\eta \in \Lambda$ (3.24). A similar argument to the proof of Lemma 3.3.2 gives

$$x(0)'\acute{P}_{\alpha_{min}}(\hat{\tau}(\alpha_{min}))x(0) = 0 \quad (3.45)$$

for $x(0) = e(\hat{\tau}(\alpha_{min}),\alpha_{min})$ where $(\acute{\cdot})$ denotes $\frac{d}{d\epsilon}(\cdot)\mid_{\epsilon=0}$. Now, (3.45) may be written as

$$0 = x(\infty)'\acute{P}_{\alpha_{min}}(\hat{\tau}(\alpha_{min}))x(\infty) - \int_0^\infty \dot{x}'P_{\alpha_{min}}(\hat{\tau}(\alpha_{min}))x + x'P_{\alpha_{min}}(\hat{\tau}(\alpha_{min}))\dot{x}\,dt \quad (3.46)$$

Using the fact that when $u = \hat{u}(\hat{\tau}(\alpha_{min}))$ (3.13) and

$$w = \hat{w}_{\alpha_{min}}(\hat{\tau}(\alpha_{min})) \triangleq \alpha_{min}^{-1}\hat{\tau}_\gamma(\alpha_{min})B_1'P_{\alpha_{min}}(\hat{\tau}(\alpha_{min}))x$$

(3.36) becomes

$$\dot{x} = A + \left(\alpha_{min}^{-2}B_1\hat{\tau}_\gamma(\alpha_{min})B_1' - B_2\bar{R}^{-1}B_2'\right)P_{\alpha_{min}}(\hat{\tau}(\alpha_{min})) \tag{3.47}$$

Now, differentiating the ARE (3.38) with respect to ϵ at $\epsilon = 0$ gives

$$\begin{aligned} 0 &= \left(A + (\alpha^{-2}B_1\tau_\gamma B_1' - B_2\bar{R}^{-1}B_2')P_\alpha\right)'\acute{P}_\alpha + \acute{P}_\alpha\left(A + (\alpha^{-2}B_1\tau_\gamma B_1' - B_2\bar{R}^{-1}B_2')P_\alpha\right) \\ &\quad + P_\alpha\left(\alpha^{-2}B_1\eta B_1' - B_2\bar{R}^{-1}D_{12}'\tau^{-1}\eta\tau^{-1}D_{12}\bar{R}^{-1}B_2'\right)P_\alpha - C_1'\tau^{-1}\eta\tau^{-1}C_1 \end{aligned} \tag{3.48}$$

Now, pre and post multiply (3.48) by x' and x, respectively, let $\alpha = \alpha_{min}$ and let $\tau = \hat{\tau}(\alpha_{min})$. Integrating from zero to infinity and using (3.47) gives an equation similar to (3.33)

$$\begin{aligned} &\int_0^\infty \hat{w}_{\alpha_{min}}(\hat{\tau}(\alpha_{min}))'\hat{\tau}_0(\alpha_{min})^{-1}\eta\hat{\tau}_0(\alpha_{min})^{-1}\hat{w}_{\alpha_{min}}(\hat{\tau}(\alpha_{min}))\,dt = \\ &\qquad\qquad \int_0^\infty z'\hat{\tau}_0(\alpha_{min})^{-1}\eta\hat{\tau}_0(\alpha_{min})^{-1}z\,dt \end{aligned} \tag{3.49}$$

where $\hat{w}_\alpha(\tau) = \alpha^{-1}\tau B_1'P_\alpha x$. So, $\hat{w}_\alpha \in \mathcal{W}$.

Now, consider $\check{\tau}(\alpha) : \|\hat{w}_{\alpha_i}(\check{\tau}(\alpha))\| = \|z_i\|\,\forall i$, where i denotes the i^{th} channel of a signal. Letting $w = \hat{w}_{\alpha_{min}}(\check{\tau}(\alpha_{min}))$ and $x(0) = e(\check{\tau}(\alpha_{min}), \alpha_{min})$, (3.37) becomes

$$\begin{aligned} &J_\gamma^{\alpha_{min}}(u, \hat{w}_{\alpha_{min}}(\check{\tau}(\alpha_{min})), e(\hat{\tau}(\alpha_{min}), \alpha_{min})) = \\ &\quad \|u - \hat{u}\|_{\bar{R}}^2 - e(\check{\tau}(\alpha_{min}), \alpha_{min})'X_{\alpha_{min}}(\gamma, \check{\tau}(\alpha_{min}))e(\check{\tau}(\alpha_{min}), \alpha_{min}) \end{aligned} \tag{3.50}$$

Similar to the proof of Lemma 3.3.2, it may be established that

$$e(\hat{\tau}(\alpha_{min}), \alpha_{min})'X_{\alpha_{min}}(\hat{\tau}(\alpha_{min}))e(\hat{\tau}(\alpha_{min}), \alpha_{min}) = 0$$

and

$$e(\hat{\tau}(\alpha_{min}), \alpha_{min})'\acute{X}_{\alpha_{min}}(\hat{\tau}(\alpha_{min}))e(\hat{\tau}(\alpha_{min}), \alpha_{min}) = 0$$

Also, similar to the proof of Lemma 3.3.2, it may be estabished that

$$e(\check{\tau}(\alpha_{min}), \alpha_{min})'X_{\alpha_{min}}(\check{\tau}(\alpha_{min}))e(\check{\tau}(\alpha_{min}), \alpha_{min}) \leq 0$$

for all control signals u. Since $\|u - \hat{u}\|_{\bar{R}}^2 \geq 0$,

$$J_\gamma^{\alpha_{min}}(u, \hat{w}_{\alpha_{min}}(\check{\tau}(\alpha_{min})), e(\alpha_{min}, \check{\tau}(\alpha_{min}))) \geq 0,$$

as required. □

Now, consider some $\alpha < \alpha_{min}$. Using Lemma 3.3.4,

$$\begin{aligned} J_\gamma^\alpha(u, \hat{w}_{\alpha_{min}}(\check{\tau}(\alpha_{min})), e(\alpha_{min}, \check{\tau}(\alpha_{min}))) &= \|z_p\|^2 - \alpha^2\left(\gamma^2\|\hat{w}_{\alpha_{min_p}}(\check{\tau}(\alpha_{min}))\|^2 - \right. \\ &\quad \left. e(\alpha_{min}, \check{\tau}(\alpha_{min}))' Re(\alpha_{min}, \check{\tau}(\alpha_{min}))\right) \\ &> \|z_p\|^2 - \alpha_{min}^2\left(\gamma^2\|\hat{w}_{\alpha_{min_p}}(\check{\tau}(\alpha_{min}))\|^2 - \right. \\ &\quad \left. e(\alpha_{min}, \check{\tau}(\alpha_{min}))' Re(\alpha_{min}, \check{\tau}(\alpha_{min}))\right) \\ &= J_\gamma^{\alpha_{min}}(u, \hat{w}(\check{\tau}(\alpha_{min})), e(\alpha_{min}, \check{\tau}(\alpha_{min})) \\ &\geq 0 \end{aligned} \tag{3.51}$$

Therefore, for $\alpha < \alpha_{min}$, there does not exist a controller such that $J_\gamma^\alpha(u, w, x(0)) \leq 0$ for all $w \in \mathcal{W}$ and $x(0)$. Since $\alpha_{min} > 1$ then there does not exist a controller u such that $J_\gamma^\alpha(u, w, x(0)) \mid_{\alpha=1} = J_\gamma(u, w, x(0)) \leq 0$ for all $w \in \mathcal{W}$ and $x(0)$.

Therefore, it has been shown that *not* 1 $\Rightarrow$ *not* 2 which is equivalent to 2 $\Rightarrow$ 1, as required. This completes the proof of the Theorem. □

Theorem 3.1 gave necessary and sufficient conditions for the existence of a controller which solves the H_∞ with transients and structure problem. Furthermore, a formula for one such controller was given. However, the design of the controller involves a search over the scaling matrix τ. The following Theorem establishes the convexity of $\Gamma_{SF}(\gamma)$ which ensures that the optimisation is tractable.

Theorem 3.3.2 *The set $\Gamma_{SF}(\gamma)$ is convex.*

Proof: If τ is a member of the set $\Gamma_{SF}(\gamma)$ then $P(\tau)$ satisfies the ARE (3.3) and is such that $0 < P(\tau) \leq \gamma^2 R$. Therefore, to prove convexity of the set $\Gamma_{SF}(\gamma)$, it is sufficient to show $P(\tau)$ is a convex function of τ. To prove the convexity of $P(\tau)$ consider a perurbation of τ; $\tau + \epsilon\eta$. where $\eta \in \Lambda$ (3.24). So η represents an arbitrary direction in τ space and ϵ is the magnitude of the perturbation. The convexity of $P(\tau)$ will be established if it can be shown that $\frac{d^2P(\tau)}{d\epsilon^2} \mid_{\epsilon=0} \geq 0$ for all $\eta \in \Lambda$. Now, define

$$\Pi \triangleq P(\tau)^{-1} \tag{3.52}$$

Differentiating (3.52) twice with respect to ϵ and evaluating at $\epsilon = 0$ gives

$$\frac{d^2P}{d\epsilon^2} \mid_{\epsilon=0} = -P\frac{d^2\Pi}{d\epsilon^2} \mid_{\epsilon=0} P + 2\acute{P}\Pi\acute{P} \tag{3.53}$$

where $(\acute{\cdot})$ denotes $\frac{d}{d\epsilon}(\cdot) \mid_{\epsilon=0}$. From (3.53), if it can be shown that $\frac{d^2\Pi}{d\epsilon^2} \mid_{\epsilon=0} \leq 0$ then $\frac{d^2P}{d\epsilon^2} \mid_{\epsilon=0} \geq 0$ and the convexity of $P(\tau)$ will have been established.

Now, consider the ARE (3.3). Pre and post multiplying by Π gives

$$0 = \Pi A' + A\Pi + B_1\tau_\gamma B_1' - B_2\bar{R}^{-1}B_2' + \Pi C_1'\tau^{-1}C_1\Pi \tag{3.54}$$

Now, differentiating the ARE (3.54) twice with respect to ϵ and evaluating at $\epsilon = 0$ leads to

$$
\begin{aligned}
-\frac{d^2\Pi}{d\epsilon^2}\,|_{\epsilon=0} \left(A+\Pi C_1'\tau^{-1}C_1\right)' - \left(A+\Pi C_1'\tau^{-1}C_1\right)\frac{d^2\Pi}{d\epsilon^2}\,|_{\epsilon=0}= \\
\left(C_1\acute{\Pi}-\eta\tau^{-1}C_1\Pi\right)'\tau^{-1}\left(C_1\acute{\Pi}-\eta\tau^{-1}C_1\Pi\right) \\
+2B_2\bar{R}^{-1}D_{12}'\tau^{-1}\eta\left(\tau^{-1}-\tau^{-1}D_{12}\bar{R}^{-1}D_{12}'\tau^{-1}\right)\eta\tau^{-1}D_{12}\bar{R}^{-1}B_2', \qquad (3.55)
\end{aligned}
$$

after some manipulations. Since $\tau^{-1}-\tau^{-1}D_{12}\bar{R}^{-1}D_{12}'\tau^{-1}\geq 0$ then the right hand side of (3.55) is also greater than or equal to zero. Now, if it can be shown that $-A-\Pi C_1'\tau^{-1}C_1$ is stable then a Lyapunov argument may be used to show the required result $\frac{d^2\Pi}{d\epsilon^2}\,|_{\epsilon=0}\leq 0$.

The ARE (3.3) may be rearranged to give

$$
\begin{aligned}
0 &= \left(A+\left(B_1\tau_\gamma B_1'-B_2\bar{R}^{-1}B_2'\right)P\right)'P+P\left(A+\left(B_1\tau_\gamma B_1'-B_2\bar{R}^{-1}B_2'\right)P\right) \\
&\quad -P\left(B_1\tau_\gamma B_1'-B_2\bar{R}^{-1}B_2'\right)P+C_1'\tau^{-1}C_1 \qquad (3.56)
\end{aligned}
$$

Since P is the stabilising solution to (3.3) then $A+\left(B_1\tau_\gamma B_1'-B_2\bar{R}^{-1}B_2'\right)P$ is stable so, by a Lyapunov argument, $-P\left(B_1\tau_\gamma B_1'-B_2\bar{R}^{-1}B_2'\right)P+C_1'\tau^{-1}C_1\geq 0$. Immediately, $B_1\tau_\gamma B_1'-B_2\bar{R}^{-1}B_2'-\Pi C_1'\tau^{-1}C_1\Pi\leq 0$. Now, (3.54) may be rearranged to give

$$
0=\Pi\left(A+\Pi C_1'\tau C_1\right)'+\left(A+\Pi C_1'\tau C_1\right)\Pi+B_1\tau_\gamma B_1'-B_2\bar{R}^{-1}B_2'-\Pi C_1'\tau^{-1}C_1\Pi \qquad (3.57)
$$

Now, using another Lyapunov arugment $-\left(A+\Pi C_1'\tau C_1\right)$ is stable, as required. This completes the proof of the Theorem. □

3.4 Discussion for state feedback

The State feedback H_∞ with transients and structure problem may be solved by the solution of a parameter dependent ARE. The solution is a static state feedback controller which is the result of a convex optimisation problem.

To design a controller which approaches the optimal controller the following procedure may be followed: Begin with some (large) value of γ for which there exists $\tau\in\Gamma_{SF}(\gamma)$. It should be noted that $\Gamma_{SF}(\gamma)$ may be empty for all values of γ, in which case, $\lim_{\gamma\to\infty}\Gamma_{SF}(\gamma)$ would be empty so the system would not be robustly stabilisable. That is, there would not exist a state feedback controller which stabilises the system (2.1) for all $w(t)\in\mathcal{W}$ and $x(0)$. If the system is robustly stabilisable then once $\tau\in\Gamma_{SF}(\gamma)$ is found, successively reduce γ until $\Gamma_{SF}(\gamma)$ becomes empty. The smallest value of γ such that $\Gamma_{SF}(\gamma)$ is non-empty is the optimal value of γ. This optimisation is tractable since P, the solution to the ARE (3.3) is a convex function of the scaling matrix τ which is proven in Theorem 3.3.2.

The controller which is the solution to the state feedback H_∞ with transients and structure problem, for optimal γ, generally, has finite gain. This is because P, the solution to the ARE (3.3), must be such that $0<P\leq\gamma^2R$. Conversely, the maximum singular value of the gain

matrix of the controller which is the solution to the non-transients case (R approaching ∞I_n), for optimal γ, will tend to infinity. This is a consequence of the ordering on $P(\gamma)$ presented in the proof of Lemma 3.3.1.

For optimal γ, the class of worst case initial conditions $x(0)$ is any eigenvector corresponding to the eigenvalue of $\gamma^2 R - P$ which is approaching zero. The worst case uncertainty/disturbance signal is $\hat{w}(t) = \tau_\gamma B_1' P x$ and the optimal controller $u(t)$ is $\hat{u}(t) = -\bar{R}^{-1} B_2' P x$. The worst case uncertainty signal $\hat{w}$ is proportional to the state vector x. This suggests the method for H_∞ with transients and structured uncertainty may be applied to problems with smaller uncertainty classes, such as Euclidean norm bounded uncertainty, without incurring unacceptable amounts of conservatism.

The choice of R, the penalty on the initial condition, determines the robustness of the resulting design to non-zero initial conditions. The state feedback H_∞ with structure problem of [Savkin and Petersen, 1996] may be recovered by letting R approach ∞I_n in (3.2). Thus, in the game between $u(t)$ and the $[w(t), x(0)]$ pair, a non zero initial condition on the state would incur an infinite penalty for the $[w(t), x(0)]$ player. Therefore, the initial condition $x(0)$ would be forced to zero and the resulting design would have no *a priori* guaranteed robustness to non-zero initial conditions.

Also, the state feedback H_∞ with transients results of [Khargonekar et al., 1991] may be recovered by ignoring the structure in which case the scaling matrix τ_γ becomes γ^{-2} and τ becomes identity.

3.5 Preliminaries for measurement feedback

For the measurement feedback case, systems of the form (2.1), (2.2), (2.3) are considered where the measurement noise and uncertainty signals are bounded by (3.1). For simplicity of exposition it was assumed that $D_{21_p} D_{21_p}' > 0$ and $D_{21}' B_1 = 0$ where D_{21} can be written in terms of the channels of w as $D_{21} = [D_{21_p}\ D_{21_1}\ D_{21_2} \cdots D_{21_k}]$. These two standard assumptions may also be relaxed with some additional complexity of the resulting equations. The performance is measured by the cost function (3.2). It was also required that the nominal system $[A\ C_2]$ be detectable.

The following RDE is important in the sequel

$$0 = \dot{Z}(t) + \tilde{A}' Z(t) + Z(t) \tilde{A} + K' \bar{R} K - C_2' \bar{Q}^{-1} C_2 + Z(t) B_1 \tau_\gamma B_1' Z(t) \tag{3.58}$$

with initial condition $Z(0) = \gamma^2 R - P$ where $P > 0$ is the stabilising solution to the ARE (3.3), $\tilde{A}$ is defined in (3.5) and $\bar{Q} = D_{21} \tau_\gamma D_{21}'$.

Definition 3.5.1 *A solution $Z(t)$ to (3.58) is antistabilising if and only if the unforced time-varying system*

$$\dot{\xi}(t) = \left(\tilde{A} + B_1 \tau_\gamma B_1' Z(t) \right) \xi(t) \tag{3.59}$$

is antistable.

Definition 3.5.2 *For given γ the set of matrices $\tau \in \Gamma_{SF}(\gamma)$ for which there also exists an antistabilising solution $Z(t)$ to (3.58) such that $Z(t) > 0$ for all $t > 0$ will be denoted $\Gamma_{MF}(\gamma)$.*

3.6 Main results for measurement feedback

Theorem 3.6.1 *Consider the system (2.1), (2.2) , (2.3) where the measurement $y(t)$ is available for feedback. Given $\gamma > 0$, the following statements are equivalent*

1. *$\Gamma_{MF}(\gamma)$ is non-empty.*

2. *There exists a measurement feedback controller such that the cost function (3.2) $J_\gamma(u(t), w(t), x(0)) \leq 0$, for all $w(t) \in \mathcal{W}$ and for all $x(0)$. Furthermore, the dynamic controller*

$$\dot{\hat{x}}(t) = \left(\tilde{A} + B_2 K\right)\hat{x}(t) + Z(t)^{-1}C_2'\bar{Q}^{-1}\left(y(t) - C_2\hat{x}(t)\right) \quad (3.60)$$

$$u(t) = K\hat{x}(t) \quad (3.61)$$

with $\hat{x}(0) = 0$, achieves this robust performance.

Proof: $1 \Rightarrow 2$:

Again, a 'completing the squares' argument will be used, similar to [Khargonekar et al., 1991]. The existence of $\tau \in \Gamma_{MF}(\gamma)$ will be assumed and it will be shown that the controller (3.60), (3.61) satisfies the desired robust performance.

Pre and post multiplying the RDE (3.58) by $(\hat{x}-x)'$ and $\hat{x}-x$, respectively, and integrating from time zero to infinity gives

$$\begin{aligned} 0 = & \int_0^\infty (\hat{x}-x)'\left(\dot{Z}(t) + \tilde{A}'Z(t) + Z(t)\tilde{A} + K'\bar{R}K - \right. \\ & \left. C_2'\bar{Q}^{-1}C_2 + Z(t)B_1\tau_\gamma B_1'Z(t)\right)(\hat{x}-x)\,dt \end{aligned} \quad (3.62)$$

which may be rewritten as,

$$\begin{aligned} 0 = & \int_0^\infty (\hat{x}-x)'\left(\dot{Z}(t) + \tilde{A}'Z(t) + Z(t)\tilde{A}\right)(\hat{x}-x) - (\hat{x}-x)'ZB_1(w-\hat{w}) - \\ & (w-\hat{w})'B_1'Z(\hat{x}-x) + (\hat{x}-x)'C_2'\bar{Q}^{-1}(y-C_2\hat{x}) + (y-C_2\hat{x})'\bar{Q}^{-1}C_2(\hat{x}-x) + \\ & (\hat{x}-x)'ZB_1(w-\hat{w}) + (w-\hat{w})'B_1'Z(\hat{x}-x) - (\hat{x}-x)'C_2'\bar{Q}^{-1}(y-C_2\hat{x}) - \\ & (y-C_2\hat{x})'\bar{Q}^{-1}C_2(\hat{x}-x) + \\ & (\hat{x}-x)'\left(K'\bar{R}K - C_2'\bar{Q}^{-1}C_2 + Z(t)B_1\tau_\gamma B_1'Z(t)\right)(\hat{x}-x)\,dt \end{aligned} \quad (3.63)$$

Now, subtracting the state equation (2.1) from the filter equation (3.60) gives

$$(\dot{\hat{x}} - \dot{x}) = \tilde{A}(\hat{x} - x) - B_1(w - \hat{w}) + Z(t)^{-1}C_2'\bar{Q}^{-1}(y - C_2\hat{x}) \tag{3.64}$$

with initial condition $\hat{x}(0) - x(0) = -x(0)$. Using (3.64), (3.63) may be written as,

$$\begin{aligned} 0 &= \int_0^\infty (\dot{\hat{x}} - \dot{x})'Z(t)(\hat{x} - x) + (\hat{x} - x)'\dot{Z}(t)(\hat{x} - x) + (\hat{x} - x)'Z(t)(\dot{\hat{x}} - \dot{x})\,dt \\ &\quad + \|(w - \hat{w}) + \tau_\gamma B_1'Z(t)(\hat{x} - x)\|^2_{\tau_\gamma^{-1}} + \|u - \hat{u}\|^2_{\bar{R}} - \|w - \hat{w}\|^2_{\tau_\gamma^{-1}} \\ &\quad - \|C_2(\hat{x} - x) + (y - C_2\hat{x})\|^2_{\bar{Q}^{-1}} + \|y - C_2\hat{x}\|^2_{\bar{Q}^{-1}} \end{aligned} \tag{3.65}$$

Since,

$$\begin{aligned} &\int_0^\infty (\dot{\hat{x}} - \dot{x})'Z(t)(\hat{x} - x) + (\hat{x} - x)'\dot{Z}(t)(\hat{x} - x) + (\hat{x} - x)'Z(t)(\dot{\hat{x}} - \dot{x})\,dt \\ &= \int_0^\infty \frac{d}{dt}\left((\hat{x} - x)'Z(t)(\hat{x} - x)\right)\,dt \\ &= (\hat{x}(\infty) - x(\infty))'Z(\infty)(\hat{x}(\infty) - x(\infty)) - (\hat{x}(0) - x(0))'Z(0)(\hat{x}(0) - x(0)), \end{aligned} \tag{3.66}$$

$\hat{x}(0) = 0$ and $Z(0) = X = \gamma^2 R - P$, (3.65) becomes

$$\begin{aligned} 0 &= (\hat{x}(\infty) - x(\infty))'Z(\infty)(\hat{x}(\infty) - x(\infty)) - x(0)'Xx(0) \\ &\quad + \|(w - \hat{w}) + \tau_\gamma B_1'Z(t)(\hat{x} - x)\|^2_{\tau_\gamma^{-1}} + \|u - \hat{u}\|^2_{\bar{R}} - \|w - \hat{w}\|^2_{\tau_\gamma^{-1}} \\ &\quad - \|C_2(\hat{x} - x) + (y - C_2\hat{x})\|^2_{\bar{Q}^{-1}} + \|y - C_2\hat{x}\|^2_{\bar{Q}^{-1}} \end{aligned} \tag{3.67}$$

Now, using (2.3), $C_2(\hat{x} - x) + (y - C_2\hat{x}) = y - C_2x = D_{21}w$ so (3.67) may be written as

$$\begin{aligned} 0 &= (\hat{x}(\infty) - x(\infty))'Z(\infty)(\hat{x}(\infty) - x(\infty)) - x(0)'Xx(0) + \|u - \hat{u}\|^2_{\bar{R}} + \|(w - \hat{w}) + \\ &\quad \tau_\gamma B_1'Z(t)(\hat{x} - x)\|^2_{\tau_\gamma^{-1}} - \|w - \hat{w}\|^2_{\tau_\gamma^{-1}} - \|D_{21}w\|^2_{\bar{Q}^{-1}} + \|y - C_2\hat{x}\|^2_{\bar{Q}^{-1}} \end{aligned} \tag{3.68}$$

$D_{21}w$ may be written as $D_{21}\left((w - \hat{w}) + \tau_\gamma B_1'Z(t)(\hat{x} - x)\right)$ since, by assumption, $D_{21}B_1' = 0$. Therefore, (3.68) may be written as

$$\begin{aligned} 0 &= (\hat{x}(\infty) - x(\infty))'Z(\infty)(\hat{x}(\infty) - x(\infty)) - x(0)'Xx(0) + \|u - \hat{u}\|^2_{\bar{R}} + \|(w - \hat{w}) + \\ &\quad \tau_\gamma B_1'Z(t)(\hat{x} - x)\|^2_{\tau_\gamma^{-1} - D_{21}\bar{Q}^{-1}D_{21}'} - \|w - \hat{w}\|^2_{\tau_\gamma^{-1}} + \|y - C_2\hat{x}\|^2_{\bar{Q}^{-1}} \end{aligned} \tag{3.69}$$

Recognising (3.16), (3.69) becomes,

$$\begin{aligned} J &= \|z_p\|^2 - \gamma^2\left(\|w_p\|^2 + x(0)'Rx(0)\right) \\ &= -x(\infty)'Px(\infty) - (\hat{x}(\infty) - x(\infty))'Z(\infty)(\hat{x}(\infty) - x(\infty)) - \|y - C_2\hat{x}\|^2_{\bar{Q}^{-1}} - \\ &\quad \|(w - \hat{w}) + \tau_\gamma B_1'Z(t)(\hat{x} - x)\|^2_{\tau_\gamma^{-1} - D_{21}\bar{Q}^{-1}D_{21}'} - \left(\|z\|^2_{\tau_0^{-1}} - \|w\|^2_{\tau_0^{-1}}\right) \end{aligned} \tag{3.70}$$

Since $w \in \mathcal{W}$ (3.1) then $\|z\|^2_{\tau_0^{-1}} - \|w\|^2_{\tau_0^{-1}} \geq 0$. Also, $\tau_\gamma^{-1} - D_{21}\bar{Q}^{-1}D_{21}' \geq 0$ so $J \leq 0$ for all $w \in \mathcal{W}$ and all $x(0)$, as required.

$2 \Rightarrow 1$:

As for the state feedback case, for structured uncertainty there is no guarantee that $\lim_{\gamma\to\infty}\Gamma_{MF}(\gamma)$ is non empty, so, two cases will be considered seperately:

Case 1. $\lim_{\gamma\to\infty}\Gamma_{MF}(\gamma)$ is non empty.

Case 2. $\lim_{\gamma\to\infty}\Gamma_{MF}(\gamma)$ is empty.

Case 1:

Again, begin with an overview of the proof which follows a similar argument to the proof for state feedback. Firstly, an ordering on γ is established such that, for $\gamma_1 > \gamma_2$, if $\Gamma_{MF}(\gamma_2)$ is non-empty then $\Gamma_{MF}(\gamma_1)$ is also non-empty. Secondly, it is shown that $\lim_{\gamma\to 0}\Gamma_{MF}(\gamma)$ is empty. This leads to the existence of γ_{min} such that $\Gamma_{MF}(\gamma_{min})$ is non empty and $\Gamma_{MF}(\gamma)$ is empty for all $\gamma < \gamma_{min}$. Next, it is shown that there exists $w \in \mathcal{W}$ and initial condition $x(0)$ such that $J_\gamma(u, w, x(0)) > 0$ for all control signals u generated by a measurement feedback controller and for $\gamma < \gamma_{min}$. So, it is deduced that *not* $1 \Rightarrow$ *not* 2, thus $2 \Rightarrow 1$.

Begin with the following Lemma.

Lemma 3.6.1 *Consider $\gamma_1 \geq \gamma_2$, if $\Gamma_{MF}(\gamma_2)$ is non empty then $\Gamma_{MF}(\gamma_1)$ is also non empty.*

Proof: Begin by making a new Riccati differential Equation. Adding (3.58) and (3.3), multiplying through by γ^{-2} and defining

$$\Xi(t) \triangleq \gamma^{-2}\left(P + Z(t)\right) \tag{3.71}$$

gives

$$0 = \dot{\Xi}(t) + A'\Xi(t) + \Xi(t)A + \Xi(t)B_1\gamma^2\tau_\gamma B_1'\Xi(t) + C_1'\left(\gamma^2\tau\right)^{-1}C_1 - C_2'\left(D_{21}\gamma^2\tau_\gamma D_{21}'\right)^{-1}C_2 \tag{3.72}$$

Definition 3.6.1 A solution $\Xi(t)$ to (3.72) is antistabilising if and only if the unforced time-varying system

$$\dot{\xi}(t) = \left(A + B_1\tau_\gamma B_1'\Xi(t)\right)\xi(t) \tag{3.73}$$

is antistable.

Differentiating the RDE (3.72) with respect to ϵ at $\epsilon = 0$ gives

$$\begin{aligned} 0 \;=\;& \acute{\dot{\Xi}}(\hat{\tau},t) + \left(A + B_1\hat{\tau}_\gamma B_1'\Xi(\hat{\tau},t)\right)'\acute{\Xi}(\hat{\tau},t) + \acute{\Xi}(\hat{\tau},t)\left(A + B_1\hat{\tau}_\gamma B_1'\Xi(\hat{\tau},t)\right) + \\ & \gamma^2\Xi(\hat{\tau},t)B_1\eta B_1'\Xi(\hat{\tau},t) - \gamma^{-2}C_1'\hat{\tau}^{-1}\eta\hat{\tau}^{-1}C_1 + \gamma^{-2}C_2'\bar{Q}^{-1}D_{21}\eta D_{21}'\bar{Q}^{-1}C_2 \end{aligned} \tag{3.74}$$

where η, defined in (3.24), represents an arbitrary direction in τ space and $(\dot{\cdot})$ denotes $\frac{d}{d\epsilon}(\cdot)\mid_{\epsilon=0}$. Since a perturbation of γ is not allowed, then η is zero in the performance channel. Therefore $\tau_\gamma^{-1}\eta\tau_\gamma^{-1} = \tau_0^{-1}\eta\tau_0^{-1} = \tau^{-1}\eta\tau^{-1}$ so, (3.74) may be written as

$$\begin{aligned}\dot{\dot{\Xi}}(\hat\tau,t) + (A + B_1\hat\tau_\gamma B_1'\Xi(\hat\tau,t))'\dot{\Xi}(\hat\tau,t) + \dot{\Xi}(\hat\tau,t)\left(A + B_1\hat\tau_\gamma B_1'\Xi(\hat\tau,t)\right) = \\ \gamma^{-2}\Big(-(Z(\hat\tau,t)+P(\hat\tau))B_1\hat\tau_\gamma\hat\tau_0^{-1}\eta\hat\tau_0^{-1}\hat\tau_\gamma B_1'(Z(\hat\tau,t)+P(\hat\tau)) - \\ C_2'\bar{Q}^{-1}D_{21}\hat\tau_\gamma\hat\tau_0^{-1}\eta\hat\tau_0^{-1}\hat\tau_\gamma D_{21}'\bar{Q}^{-1}C_2 + C_1'\hat\tau^{-1}\eta\hat\tau^{-1}C_1\Big) \qquad (3.75)\end{aligned}$$

This RDE will be used later in the argument.

Now, an alternative definition for $\Gamma_{MF}(\gamma)$ is 'the set of matrices $\tau \in \Gamma_{SF}(\gamma)$ for which there exists an antistabilising solution $\Xi(t)$ to (3.72) with $\Xi(t) > \gamma^{-2}P$ for all $t > 0$'. Suppose there exists $\tau = \bar\tau \in \Gamma_{MF}(\gamma_2)$. Letting P_2 denote $P(\bar\tau_{\gamma_2})$ and multiplying (3.3) through by γ_2^{-2} gives

$$\begin{aligned}0 &= A'\left(\gamma_2^{-2}P_2\right) + \left(\gamma_2^{-2}P_2\right)A + C_1'\left(\gamma_2^2\bar\tau\right)^{-1}C_1 + \\ &\quad \left(\gamma_2^{-2}P_2\right)\left(B_1\gamma_2^2\bar\tau_{\gamma_2}B_1' - B_2\left(D_{12}'\left(\gamma_2^2\bar\tau\right)^{-1}D_{12}\right)^{-1}B_2'\right)\left(\gamma_2^{-2}P_2\right) \qquad (3.76)\end{aligned}$$

Define $\Pi_2 = \left(\gamma_2^{-2}P_2\right)^{-1}$. Pre and post multiply (3.76) by Π_2 and multiply by -1 to give

$$\begin{aligned}0 &= \Pi_2(-A)' + (-A)\Pi_2 - \Pi_2\left(C_1'\left(\gamma_2^2\bar\tau\right)^{-1}C_1\right)\Pi_2 - \\ &\quad B_1\gamma_2^2\bar\tau_{\gamma_2}B_1' + B_2\left(D_{12}'\left(\gamma_2^2\bar\tau\right)^{-1}D_{12}\right)^{-1}B_2' \qquad (3.77)\end{aligned}$$

Now, the ARE (3.3) is considered with $\gamma = \gamma_1$ and $\tau_{\gamma_1} = \frac{\gamma_2^2}{\gamma_1^2}\bar\tau_{\gamma_2}$. So, letting P_1 denote $P\left(\frac{\gamma_2^2}{\gamma_1^2}\bar\tau_{\gamma_2}\right)$ and multiplying through by γ_1^{-2} gives

$$\begin{aligned}0 = A'\left(\gamma_1^{-2}P_1\right) + \left(\gamma_1^{-2}P_1\right)A + C_1'\left(\gamma_2^2\bar\tau\right)^{-1}C_1 - C_{1_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)C_{1_p} + \left(\gamma_1^{-2}P_1\right)\times \\ \left(B_1\gamma_2^2\bar\tau_{\gamma_2}B_1' - B_2\left(D_{12}'\left(\gamma_2^2\bar\tau\right)^{-1}D_{12} - D_{12_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)D_{12_p}\right)^{-1}B_2'\right)\left(\gamma_2^{-2}P_2\right) (3.78)\end{aligned}$$

Define $\Pi_1 = \gamma_1^{-2}P_1$. Pre and post multiply (3.78) by Π_1 and multiply by -1 to get

$$\begin{aligned}0 &= \Pi_1(-A)' + (-A)\Pi_1 - \Pi_1\left(C_1'\left(\gamma_2^2\bar\tau\right)^{-1}C_1 - C_{1_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)C_{1_p}\right)\Pi_1 - \\ &\quad B_1\gamma_2^2\bar\tau_{\gamma_2}B_1' + B_2\left(D_{12}'\left(\gamma_2^2\bar\tau\right)^{-1}D_{12} - D_{12_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)D_{12_p}\right)^{-1}B_2' \qquad (3.79)\end{aligned}$$

(3.77) may be rewritten as

$$\begin{aligned} 0 \ &= \ \Pi_2(-A)' + (-A)\Pi_2 - \Pi_2\left(C_1'\left(\gamma_2^2\bar{\tau}\right)^{-1}C_1 - C_{1_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)C_{1_p}\right)\Pi_2 - B_1\gamma_2^2\bar{\tau}_{\gamma_2}B_1' + \\ &\quad B_2\left(D_{12}'\left(\gamma_2^2\bar{\tau}\right)^{-1}D_{12} - D_{12_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)D_{12_p}\right)^{-1}B_2' - Q \end{aligned} \tag{3.80}$$

where

$$\begin{aligned} Q \ &= \ B_2\left(\left(D_{12}'\left(\gamma_2^2\bar{\tau}\right)^{-1}D_{12} - D_{12_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)D_{12_p}\right)^{-1} - \right. \\ &\quad \left.\left(D_{12}'\left(\gamma_2^2\bar{\tau}\right)^{-1}D_{12}\right)^{-1}\right)B_2 + \Pi_2\left(C_{1p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)C_{1_p}\right)\Pi_2 \end{aligned}$$

Since $Q > 0$, then from Theorem 2.1 of [Ran and Vreugdenhil, 1988], there exists a solution Π_1 to (3.79) with $\Pi_1 \geq \Pi_2$ and

$$-A - \Pi_1\left(C_1'\left(\gamma_2^2\bar{\tau}\right)^{-1}C_1 + C_{1_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)C_{1_p}\right)$$

stable. Rearranging (3.79) gives

$$\begin{aligned} 0 = \Pi_1&\left(-A - \Pi_1\left(C_1'\left(\gamma_2^2\bar{\tau}\right)^{-1}C_1 - C_{1_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)C_{1_p}\right)\right)' + \\ &\left(-A - \Pi_1\left(C_1'\left(\gamma_2^2\bar{\tau}\right)^{-1}C_1 - C_{1_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)C_{1_p}\right)\right)\Pi_1 \\ &+\Pi_1\left(C_1'\left(\gamma_2^2\bar{\tau}\right)^{-1}C_1 - C_{1_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)C_{1_p}\right)\Pi_1 \\ &-B_1\gamma_2^2\bar{\tau}_{\gamma_2}B_1' + B_2\left(D_{12}'\left(\gamma_2^2\bar{\tau}\right)^{-1}D_{12} - D_{12_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)D_{12_p}\right)^{-1}B_2' \end{aligned} \tag{3.81}$$

So, by a Lyapunov argument,

$$\begin{aligned} -B_1\gamma_2^2\bar{\tau}_{\gamma_2}B_1' + B_2&\left(D_{12}'\left(\gamma_2^2\bar{\tau}\right)^{-1}D_{12} - D_{12_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)D_{12_p}\right)^{-1}B_2' \\ &+\Pi_1\left(C_1'\left(\gamma_2^2\bar{\tau}\right)^{-1}C_1 - C_{1_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)C_{1_p}\right)\Pi_1 \geq 0 \end{aligned}$$

So, immediately,

$$\begin{aligned} -\left(\gamma_1^2P_1\right)&\left(B_1\gamma_2^2\bar{\tau}_{\gamma_2}B_1' - B_2\left(D_{12}'\left(\gamma_2^2\bar{\tau}\right)^{-1}D_{12} - D_{12_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)D_{12_p}\right)^{-1}B_2'\right)\left(\gamma_1^2P_1\right) \\ &+\left(C_1'\left(\gamma_2^2\bar{\tau}\right)^{-1}C_1 - C_{1_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)C_{1_p}\right) \geq 0 \end{aligned} \tag{3.82}$$

Rearranging (3.78) gives

$$0 = A_c'\left(\gamma_1^{-2}P_1\right) + \left(\gamma_1^{-2}P_1\right)A_c + C_1'\left(\gamma_2^2\bar{\tau}\right)^{-1}C_1 - C_{1_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)C_{1_p} - \left(\gamma_1^{-2}P_1\right)\times \left(B_1\gamma_2^2\bar{\tau}_{\gamma_2}B_1' - B_2\left(D_{12}'\left(\gamma_2^2\bar{\tau}\right)^{-1}D_{12} - D_{12_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)^{-1}D_{12_p}\right)B_2'\right)\left(\gamma_2^{-2}P_2\right) \quad (3.83)$$

where

$$A_c = A + \left(B_1\gamma_2^2\bar{\tau}_{\gamma_2}B_1' - B_2\left(D_{12}'\left(\gamma_2^2\bar{\tau}\right)^{-1}D_{12} - D_{12_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)D_{12_p}\right)B_2'\right)\left(\gamma_1^{-2}P_1\right)$$

Using (3.82), (3.83) and using a Lyapunov argument, A_c is stable. Also, since $\Pi_1 \geq \Pi_2 > 0$ then $\gamma_2^{-2}P_2 \geq \gamma_1^{-2}P_1 > 0$.

Now, consider an ARE with $\gamma = \gamma_2$ and $\tau = \bar{\tau}$ and with solutions $\bar{\Xi}_2$. It will be shown later that one of the solutions $\bar{\Xi}_2$ approaches the solution to the RDE (3.72) with $\gamma = \gamma_2$ and $\tau = \bar{\tau}$ as $t \to \infty$.

$$0 = A'\bar{\Xi}_2 + \bar{\Xi}_2A + \bar{\Xi}_2B_1\gamma_2^2\bar{\tau}_{\gamma_2}B_1'\bar{\Xi}_2 + C_1'\left(\gamma_2^2\bar{\tau}\right)^{-1}C_1 - C_2'\left(D_{21}\gamma_2^2\bar{\tau}_{\gamma_2}D_{21}'\right)^{-1}C_2 \quad (3.84)$$

Consider the same ARE with $\gamma = \gamma_1$ and $\tau_{\gamma_1} = \frac{\gamma_2^2}{\gamma_1^2}\bar{\tau}_{\gamma_2}$ and with solution $\bar{\Xi}_1$. Multiplying through by -1 gives

$$\begin{aligned} 0 \;=\;& (-A)'\bar{\Xi}_1 + \bar{\Xi}_1(-A) - \bar{\Xi}_1B_1\gamma_2^2\bar{\tau}_{\gamma_2}B_1'\bar{\Xi}_1 - C_1'\left(\gamma_2^2\bar{\tau}\right)^{-1}C_1 + C_{1p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)C_{1_p} \\ & + C_2'\left(D_{21}\gamma_2^2\bar{\tau}_{\gamma_2}D_{21}'\right)^{-1}C_2 \end{aligned} \quad (3.85)$$

Multiplying (3.84) by -1 and rearranging gives

$$\begin{aligned} 0 \;=\;& (-A)'\bar{\Xi}_2 + \bar{\Xi}_2(-A) - \bar{\Xi}_2B_1\gamma_2^2\bar{\tau}_{\gamma_2}B_1'\bar{\Xi}_2 - C_1'\left(\gamma_2^2\bar{\tau}\right)^{-1}C_1 + C_{1_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)C_{1_p} + \\ & C_2'\left(D_{21}\gamma_2^2\bar{\tau}_{\gamma_2}D_{21}'\right)^{-1}C_2 - C_{1_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)C_{1_p} \end{aligned} \quad (3.86)$$

So, using Theorem 2.1 of [Ran and Vreugdenhil, 1988] to compare (3.85) and (3.86), if there exists $\bar{\Xi}_2$ then there exists a unique maximising solution to (3.85) $\bar{\Xi}_1 = \bar{\Xi}_1^+$ such that $\bar{\Xi}_1^+ \geq \bar{\Xi}_2$ and $-A - B_1\gamma_2^2\bar{\tau}_{\gamma_2}B_1'\bar{\Xi}_1^+$ is stable. Since $\bar{\tau} \in \Gamma_{MF}(\gamma_2)$ then $\bar{\Xi}_2 \geq \gamma_2^{-2}P_2$ so $\bar{\Xi}_1 \geq \bar{\Xi}_2 \geq \gamma_2^{-2}P_2 \geq \gamma_1^{-2}P_1 > 0$.

Now, it remains to be shown that $\lim_{t\to\infty}\Xi_1(t) = \bar{\Xi}_1^+$ and that $Z_1(t) \geq 0$ for all $t > 0$. Firstly, it will be shown that a sufficient condition for $Z_1(t) \geq 0$ for all $t > 0$ is $\Xi_1(t) \geq \Xi_2(t)$ for all $t > 0$.

Begin with $\Xi_1(t) \geq \Xi_2(t)\,\forall t > 0$. From the definition of $\Xi(t)$ (3.71),

$$\begin{aligned}\gamma_1^{-2}Z_1(t) &\geq \gamma_2^{-2}Z_2(t)+\gamma_2^{-2}P_2-\gamma_1^{-2}P_1\\ \Rightarrow \gamma_1^{-2}Z_1(t) &\geq \gamma_2^{-2}Z_2(t)>0\end{aligned}$$

since $\gamma_2^{-2}P_2-\gamma_1^{-2}P_1\geq 0$. Immediately, it would follow that $Z_1(t)>0$ for all $t>0$. For notational simplicity, $\Xi(\bar{\tau}_{\gamma_2},t)$ will be denoted $\Xi_2(t)$. Similarly, $\Xi\left(\frac{\gamma_2^2}{\gamma_1^2}\bar{\tau}_{\gamma_2},t\right)$ will be denoted $\Xi_1(t)$. Now, define $\Delta(t)=\Xi_1(t)-\Xi_2(t)$. Subtracting (3.72) with $\tau_\gamma=\bar{\tau}_{\gamma_2}$ from (3.72) with $\tau_\gamma=\frac{\gamma_2^2}{\gamma_2^2}\bar{\tau}_{\gamma_2}$ gives

$$\begin{aligned}0 &= \dot{\Delta}(t)+\left(A+B_1\gamma_2^2\bar{\tau}_{\gamma_2}B_1'\Xi_2(t)\right)'\Delta(t)+\Delta(t)\left(A+B_1\gamma_2^2\bar{\tau}_{\gamma_2}B_1'\Xi_2(t)\right)\\ &\quad +\Delta(t)B_1\gamma_2^2\bar{\tau}_{\gamma_2}B_1'\Delta(t)-C_{1_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)C_{1_p}\end{aligned}\tag{3.87}$$

with $\Delta(0)=0$. Evaluating (3.87) at $t=0$ gives

$$\dot{\Delta}(0)=C_{1_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)C_{1p}>0$$

Now, using a contradiction argument, suppose at some time $t=T_c$ there exists a vector g such that, $g'\Delta(T_c)g=0$ and $g'\dot{\Delta}(T_c)g<0$. Therefore, at some time $t>T_c$, $\Delta(t)g<0$. Substituting into (3.87) and pre and post multiplying by g' and g, respectively, gives

$$0=g'\dot{\Delta}(T_c)g-g'C_{1_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)C_{1_p}g\tag{3.88}$$

So,

$$g'\dot{\Delta}(T_c)g=g'C_{1_p}'\left(\gamma_2^{-2}-\gamma_1^{-2}\right)C_{1_p}g>0$$

which contradicts the assumption that $g'\dot{\Delta}(T_c)g<0$. Therefore, $\Delta(t)\geq 0$ for all $t>0$. Now, since (3.87) is an LQR ARE when $t=\infty$, then the positive solution $\lim_{t\to\infty}\Delta(t)$ is unique (see, for example, [Anderson and Moore, 1989]). Also,

$$\lim_{t\to\infty} A+B_1\gamma_2^2\bar{\tau}_{\gamma_2}B_1'\left(\Xi_2(t)+\Delta(t)\right)=\lim_{t\to\infty} A+B_1\gamma_2^2\bar{\tau}_{\gamma_2}B_1'\Xi_1(t)$$

is stabilising. Therefore, $\lim_{t\to\infty}\Xi_1(t)=\bar{\Xi}_1^+$, as required. Thus, $\frac{\gamma_2^2}{\gamma_1^2}\bar{\tau}_{\gamma_2}\in\Gamma_{MF}(\gamma_1)$ This completes the proof of the Lemma. □

Since, for the proof of Theorem 3.3.1, it was shown that $\lim_{\gamma\to 0}\Gamma_{SF}(\gamma)$ is empty then $\lim_{\gamma\to 0}\Gamma_{MF}(\gamma)$ is also empty. Also, by assumption for Case 1, $\lim_{\gamma\to\infty}\Gamma_{MF}(\gamma)$ is non empty. Lemma 3.6.1 gives an ordering so there exists γ_{min} such that $\Gamma_{MF}(\gamma_{min})$ is non empty but $\Gamma_{MF}(\gamma)$ is empty for all $\gamma<\gamma_{min}$.

Lemma 3.6.2 *There exist $w\in\mathcal{W}$ and $x(0)$ such that $J_\gamma(u,w,x(0))>0$ for any controller u and for $\gamma<\gamma_{min}$.*

Proof: Since $\gamma < \gamma_{min}$, there exists some time $t = T(\tau) > 0$ at which $Z(\tau, T(\tau))$ develops a zero eigenvalue. The eigenvector corresponding to the zero eigenvalue of $Z(\tau, T(\tau))$ will be refered to as $e(\tau)$. It is not necessary to consider the case where $Z(\tau, 0)$ has a zero or negative eigenvalue because $P(\tau) \not< \gamma^2 R$ so $\tau \notin \Gamma_{SF}(\gamma)$. Therefore, by Theorem 3.3.1, there does not exist a state feedback controller which gives the desired performance. Thus, there also does not exist a measurement feedback controller so, for the remainder of the argument, $T(\tau) > 0$ will be considered.

Notice from the definition of $T(\tau)$ that

$$e(\tau)'Z(\tau, T(\tau))e(\tau) = 0 \tag{3.89}$$

for all $\tau \in \Gamma_{SF}(\gamma)$

Now, $\hat{\tau}$ is defined such that $T(\hat{\tau}) \geq T(\tau)$ for all τ. Therefore, $\hat{\tau}$ is such that

$$\frac{dT(\hat{\tau})}{d\epsilon} \mid_{\epsilon=0} = 0 \tag{3.90}$$

for all η. Where η, defined in (3.24), represents an arbitrary direction in τ space, so the above expression defines a stationary point in τ space.

Since $\hat{\tau}$ maximises the time at which $Z(\tau, t)$ takes a zero eigenvalue, differentiating (3.89) with respect to ϵ at $\epsilon = 0$ at $\epsilon = 0$ when $\tau = \hat{\tau}$ gives

$$\frac{d}{d\epsilon} \left(e(\hat{\tau})'Z(\hat{\tau}, T(\hat{\tau}))e(\hat{\tau}) \right) \mid_{\epsilon=0} = 0 \tag{3.91}$$

for all $\tau \in \Gamma_{SF}(\gamma)$ and for all directions η in τ space. Expanding (3.91) gives

$$\acute{e}(\hat{\tau})'Z(\hat{\tau}, T(\hat{\tau}))e(\hat{\tau}) + e(\hat{\tau})'\acute{Z}(\hat{\tau}, T(\hat{\tau}))e(\hat{\tau}) + e(\hat{\tau})'Z(\hat{\tau}, T(\hat{\tau}))\acute{e}(\hat{\tau}) \tag{3.92}$$

where $(\acute{\cdot})$ denotes $\frac{d}{d\epsilon}(\cdot) \mid_{\epsilon=0}$. Using the fact that $Z(\tau, T(\tau))e(\tau) = 0$, (3.92) becomes

$$e(\hat{\tau})'\acute{Z}(\hat{\tau}, T(\hat{\tau}))e(\hat{\tau}) = 0 \tag{3.93}$$

Now, consider a disturbance/measurement noise/uncertainty signal

$$\begin{aligned} w(t) &= w_{MF}(\tau, t) \\ &\triangleq \begin{cases} \left(\tau_\gamma B_1'(Z(\tau,t) + P(\tau)) - \tau_\gamma D_{21}'(D_{21}\tau_\gamma D_{21}')^{-1}C_2\right)x(t), & t \in [0, T(\tau)) \\ \tau_\gamma B_1' P(\tau)x(t), & t \in [T(\tau), \infty) \end{cases} \end{aligned} \tag{3.94}$$

and an initial condition $x(0) = x_0(\tau)$ such that $x(T(\tau)) = e(\tau)$. Notice that $w_{MF}(\tau, t)$ is such that $y(t) \equiv 0$ for $t \in [0, T(\tau))$. Now, it will be shown that if $y(t) = 0$ for $t \in [0, T(\tau))$ then $u(t) \neq 0$ for $t \in [0, T]$ will not satisfy $J_\gamma \leq 0$ for all $w \in \mathcal{W}$ and $x(0)$ compatible with $y(t) = 0$ for $t \in [0, T(\tau))$. To show this consider $u \neq 0$ and $w = 0$, $x(0) = 0$. Substituting into (3.2) gives

$$J_\gamma(u \neq 0, w = 0, x(0) = 0) = \|u\|^2_{D_{12p}' D_{12p}} > 0 \tag{3.95}$$

Therefore, if $y(t) = 0$ for $t \in [0, T(\tau))$ then it is required that $u(t) = 0$ for $t \in [0, T(\tau))$ to have a possibility to satisfy robust performance. Now, consider $w(t) = w_{MF}(\tau, t)$, $x(0) = x_0(\tau)$ and any control signal $u(t)$ which is zero for $t \in [0, T(\tau))$ and which is such that the system (2.1), (2.3) is stable. An expression for the cost J_γ, defined in (3.2), that is separated into two parts will be considered; the cost incurred during the interval $t \in [0, T(\tau))$ and the cost incurred during $t \in [T(\tau), \infty)$. Still considering $\gamma < \gamma_{min}$ and using manipulations similar to those which gave (3.70) for the time interval $t \in [0, T(\tau))$ gives

$$\begin{aligned} J_\gamma &= -(\hat{x}(T(\tau)) - x(T(\tau)))' Z(\tau, T)(\hat{x}(T(\tau)) - x(T(\tau))) - x(T(\tau))' P(\tau) x(T(\tau)) - \\ &\quad \|w - \hat{w}(\tau_\gamma) + \tau_\gamma B_1' Z(\tau, t)(\hat{x}(t) - x(t))\|^2_{\tau_\gamma^{-1} - D_{21}\bar{Q}^{-1}D_{21}'} - \|y - C_2\hat{x}(t)\|^2_{[0,T(\tau)]\bar{Q}^{-1}} \\ &\quad - \left(\|z\|^2_{[0,T(\tau)]\tau_0^{-1}} - \|w\|^2_{[0,T(\tau)]\tau_0^{-1}} \right) + \|z_p\|^2_{(T(\tau),\infty)} - \gamma^2 \|w_p\|^2_{(T(\tau),\infty)} \end{aligned} \tag{3.96}$$

Using manipulations similar to those which gave (3.16) for the time interval $t \in [T(\tau), \infty)$ gives

$$\begin{aligned} J_\gamma &= -(\hat{x}(T(\tau)) - x(T(\tau)))' Z(\tau, T)(\hat{x}(T(\tau)) - x(T(\tau))) - \\ &\quad \|w - \hat{w}(\tau_\gamma) + \tau_\gamma B_1' Z(\tau, t)(\hat{x}(t) - x(t))\|^2_{\tau_\gamma^{-1} - D_{21}\bar{Q}^{-1}D_{21}'} - \|y - C_2\hat{x}(t)\|^2_{[0,T(\tau)]\bar{Q}^{-1}} \\ &\quad - \left(\|z\|^2_{[0,T(\tau)]\tau_0^{-1}} - \|w\|^2_{[0,T(\tau)]\tau_0^{-1}} \right) + \|u - \hat{u}(\tau_\gamma)\|^2_{(T(\tau),\infty)D_{12}'\tau^{-1}D_{12}} - \\ &\quad \|w - \hat{w}(\tau_\gamma)\|^2_{(T(\tau),\infty)\tau_\gamma^{-1}} - \left(\|z\|^2_{(T(\tau),\infty)\tau_0^{-1}} - \|w\|^2_{(T(\tau),\infty)\tau_0^{-1}} \right) \end{aligned} \tag{3.97}$$

Substituting for $w = w_{MF}(\tau)$ and $x(0) = x_0(\tau)$ gives

$$\begin{aligned} J_\gamma(u(t), w_{MF}(\tau), x_0(\tau)) &= -e(\tau)' Z(\tau, T) e(\tau) - \left(\|z\|^2_{[0,T(\tau)]\tau_0^{-1}} - \|w_{MF}(\tau)\|^2_{[0,T(\tau)]\tau_0^{-1}} \right) \\ &\quad + \|u - \hat{u}(\tau_\gamma)\|^2_{(T(\tau),\infty)D_{12}'\tau^{-1}D_{12}} - \left(\|z\|^2_{(T(\tau),\infty)\tau_0^{-1}} - \|w_{MF}(\tau)\|^2_{(T(\tau),\infty)\tau_0^{-1}} \right) \end{aligned} \tag{3.98}$$

Now, consider $\tau = \check{\tau}$ such that $\|z_i(u(t), w_{MF}(\check{\tau}, t))\| = \|w_{iMF}(\check{\tau}, t)\|$ for $x(0) = x_0(\check{\tau})$. Recognising that $e'(\check{\tau}) Z(\check{\tau}, T(\check{\tau})) e(\check{\tau}) = 0$, (3.98) becomes

$$J_\gamma(u(t), w_{MF}(\check{\tau}), x_0(\tau)) = \|u - \hat{u}(\check{\tau}_\gamma)\|^2_{(T(\check{\tau}),\infty)D_{12}'\check{\tau}^{-1}D_{12}} \tag{3.99}$$

Since it is not possible for a measurement feeback controller to produce the output of the optimal state feedback controller $\hat{u}(\check{\tau})$, at the instant $t = T(\check{\tau})$ when the measurement signal $y(t)$ becomes non zero, then

$$J_\gamma(u(t), w_{MF}(\check{\tau}, t), x_0(\check{\tau})) > 0$$

for all measurement feedback controllers.

Now, it must be shown that if $\Gamma_{SF}(\gamma)$ is non-empty then $\check{\tau}(u_{MF}(\check{\tau}), w_{MF}(\check{\tau}), x_0(\check{\tau})) \in \Gamma_{SF}(\gamma)$. To do this, it will be shown that $\check{\tau}(u_{MF}(\check{\tau}), w_{MF}(\check{\tau}), x_0(\check{\tau})) = \hat{\tau}$. Differentiating (3.71) with respect to ϵ at $\epsilon = 0$ and substituting for $\acute{Z}(\hat{\tau}, T(\hat{\tau}))$ in (3.93) gives

$$0 = \gamma^2 e(\hat{\tau})'\acute{\Xi}(\hat{\tau}, T(\hat{\tau}))e(\hat{\tau}) - e(\hat{\tau})'\acute{P}(\hat{\tau})e(\hat{\tau}) \tag{3.100}$$

Letting the initial condition $x(0) = x_0(\hat{\tau})$, $u(t) = u_{MF}(\hat{\tau}, t)$ and $w(t) = w_{MF}(\hat{\tau}, t)$, (3.100) may be written as

$$\begin{aligned} 0 &= \gamma^2 \int_0^T x'\dot{\acute{\Xi}}(\hat{\tau}, t)x + \dot{x}'\acute{\Xi}(\hat{\tau}, t)x + x'\acute{\Xi}(\hat{\tau}, t)\dot{x}\, dt + \int_T^\infty \dot{x}'\acute{P}(\hat{\tau})x + x'\acute{P}(\hat{\tau})x\, dt \\ &\quad + x(0)'\acute{\Xi}(\hat{\tau})(0)x(0) - x(\infty)'\acute{P}(\hat{\tau})x(\infty) \end{aligned} \tag{3.101}$$

and the state equation (2.1) becomes

$$\dot{x}(t) = \left(A + B_1\gamma^2\hat{\tau}_\gamma B_1'\Xi(\hat{\tau}, t)\right)x(t), \quad t \in [0, T(\hat{\tau})) \tag{3.102}$$

$$\dot{x}(t) = \left(A + \left(B_1\hat{\tau}_\gamma B_1' - B_2\bar{R}^{-1}B_2'\right)P(\hat{\tau})\right)x(t), \quad t \in [T(\hat{\tau}), \infty) \tag{3.103}$$

Recognise that since $\Xi(0) = R$ then $\acute{\Xi}(0) = 0$. Substituting (3.102) and (3.103) into (3.101) gives

$$\begin{aligned} 0 &= \gamma^{-2} \int_0^T x' \left(\dot{\acute{\Xi}}(\hat{\tau}, t) + \left(A + B_1\gamma^2\hat{\tau}_\gamma B_1'\Xi(\hat{\tau}, t)\right)' \acute{\Xi}(\hat{\tau}, t) + \right. \\ &\quad \left. \acute{\Xi}(\hat{\tau}, t)\left(A + B_1\gamma^2\hat{\tau}_\gamma B_1'\Xi(\hat{\tau}, t)\right)\right) x\, dt + \int_T^\infty x' \left(\left(A + \left(B_1\hat{\tau}_\gamma B_1' - B_2\bar{R}^{-1}B_2'\right)P(\hat{\tau})\right)' \times \right. \\ &\quad \left. \acute{P}(\hat{\tau}) + \acute{P}(\hat{\tau})\left(A + \left(B_1\hat{\tau}_\gamma B_1' - B_2\bar{R}^{-1}B_2'\right)P(\hat{\tau})\right)\right) x\, dt - x(\infty)'\acute{P}(\hat{\tau})x(\infty) \end{aligned} \tag{3.104}$$

Noticing that $x(\infty) = 0$ since (3.103) is stable and using the RDE (3.75) and the ARE (3.3), (3.104) may be written as

$$\begin{aligned} 0 &= \int_0^T x' \left(-(Z(\hat{\tau}, t) + P(\hat{\tau}))B_1\hat{\tau}_\gamma\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}\hat{\tau}_\gamma B_1'(Z(\hat{\tau}, t) + P(\hat{\tau})) - \right. \\ &\quad \left. C_2'\bar{Q}^{-1}D_{21}\hat{\tau}_\gamma\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}\hat{\tau}_\gamma D_{21}'\bar{Q}^{-1}C_2 + C_1'\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}C_1\right) x\, dt - \int_T^\infty x' \left(P(\hat{\tau}) \times \right. \\ &\quad \left. \left(B_1\hat{\tau}_\gamma\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}\hat{\tau}_\gamma B_1' - B_2\bar{R}^{-1}\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}\bar{R}^{-1}B_2'\right)P(\hat{\tau}) - C_1'\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}C_1\right) x\, dt \end{aligned} \tag{3.105}$$

Now, (3.105) may be written as

$$\begin{aligned} 0 &= \left\|z\left(w_{MF}(\hat{\tau}), x_0(\hat{\tau})\right)\right\|_{[0,T(\hat{\tau}))\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}} - \left\|w_{MF}(\hat{\tau})\right\|_{[0,T(\hat{\tau}))\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}} + \\ &\quad \left\|z\left(u_{MF}(\hat{\tau}), w_{MF}(\hat{\tau}), x_0(\hat{\tau})\right)\right\|_{(T,\infty)\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}} - \left\|w_{MF}(\hat{\tau}, t)\right\|_{(T,\infty)\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}} \end{aligned} \tag{3.106}$$

Therefore,

$$0 = \|z\left(u_{MF}(\hat{\tau}), w_{MF}(\hat{\tau}), x_0(\hat{\tau})\right)\|_{\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}} - \|w_{MF}(\hat{\tau})\|_{\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}} \quad (3.107)$$

which is equivalent to the definition of $\hat{\tau}$ for $u = u_{MF}(\hat{\tau})$, $w = w_{MF}(\hat{\tau})$ and $x(0) = x_0(\hat{\tau})$

Therefore, it has been shown that, for $\gamma < \gamma_{min}$, if there does not exist $\tau \in \Gamma_{SF}(\gamma)$ then, by Theorem 3.3.1, there does not exist a state feedback controller and therefore a measurement feedback controller which gives the required performance. Secondly, if there does exist $\tau \in \Gamma_{SF}(\gamma)$ then there also does not exist a controller which statisfies robust performance. This completes the proof of the Lemma. □

It has been shown, from the definition of γ_{min}, that *not* 1 $\Rightarrow \gamma < \gamma_{min}$ and for such γ there does not exist a measurement feedback controller such that $J_\gamma(u(t), \hat{w}_{MF}(\hat{\tau}), e(\hat{\tau}))) \leq 0 \Rightarrow$ *not* 2. So, *not* 1 $\Rightarrow$ *not* 2 which is equivalent to 2 $\Rightarrow$ 1, as required.

Case 2:

The modified problem that will be considered is a system where the effect of the signal w is scaled-down by a factor of $\alpha^{-1} \in \mathbf{R} > 0$ giving a system representation (3.36), (2.2) and

$$y = C_2 x + \alpha^{-1} D_{21} w \quad (3.108)$$

with cost function (3.37) where it is required that $w \in \mathcal{W}$ (3.1).

The ARE corresponding to this modified problem is (3.38) and the RDE corresponding to the modified problem differs from (3.58) through a scaling of the B_1 and D_{21} terms by α^{-1}.

$$0 = \dot{Z}_\alpha(t) + \tilde{A}_\alpha' Z_\alpha(t) + Z_\alpha(t)\tilde{A}_\alpha + K_\alpha' \bar{R} K_\alpha - \alpha^2 C_2' \bar{Q}^{-1} C_2 + \alpha^{-2} Z_\alpha(t) B_1 \tau_\gamma B_1' Z_\alpha(t) \quad (3.109)$$

where $\tilde{A}_\alpha = A + \alpha^{-2} B_1 \tau_\gamma B_1' P_\alpha$. The initial condition for the RDE is $Z_\alpha(0) = \alpha^2\gamma^2 R - P_\alpha$ where $P_\alpha > 0$ is the stabilising solution to the ARE (3.38) and where $K_\alpha = -\bar{R}^{-1} B_2' P_\alpha$.

Definition 3.6.2 Given α and γ, the set of scaling matrices $\tau \in \Gamma_{SF}^\alpha(\gamma)$ for which there exist antistabilising solutions $Z_\alpha(t)$ to (3.109) such that $Z_\alpha(t) > 0$ for all $t > 0$ will be denoted $\Gamma_{MF}^\alpha(\gamma)$.

Since it has been assumed that $\lim_{\gamma\to\infty} \Gamma_{MF}(\gamma)$ is empty then it is possible to infer from the ordering on γ that $\Gamma_{MF}(\gamma)$ is empty for all finite γ. So, automaticially *not* 1 is established. If it can be inferred that there does not exist a controller such that $J_\gamma^\alpha(u, w, x(0)) \mid_{\alpha=1} \leq 0$ for all $w \in \mathcal{W}$ and $x(0)$ then *not* 2 will be established and it will have been proven that 2 $\Rightarrow$ 1 for Case 2. Firstly an ordering on α will be established for any γ. For $\alpha_2 \leq \alpha_1$, if there exists $\tau \in \Gamma_{MF}^{\alpha_2}(\gamma)$ then $\tau \in \Gamma_{MF}^{\alpha_1}(\gamma)$. Secondly, it will be shown that, in the limit as $\alpha \to \infty$, an LQR ARE and an LQG RDE are recovered so $\lim_{\alpha\to\infty} \Gamma_{MF}^\alpha(\gamma)$ is non-empty. Since, for all γ, by assumption for Case 2, $\Gamma_{MF}^\alpha(\gamma) \mid_{\alpha=1}$ is empty then it follows that there exists $\alpha_{min}(\gamma) > 1$ such that $\Gamma_{MF}^{\alpha_{min}}(\gamma)$ is non-empty but, for all $\alpha < \alpha_{min}(\gamma)$, $\Gamma_{MF}^\alpha(\gamma)$ is empty. It is then shown that there exist $w \in \mathcal{W}$ and $x(0)$ such that $J_\gamma^\alpha(u, w, x(0)) > 0$ for $\alpha < \alpha_{min}$ and for all u which leads to the non-existence of a controller such that $J_\gamma^\alpha(u,w,x(0)) \mid_{\alpha=1} \leq 0$ for all $w \in \mathcal{W}$ and $x(0)$ so *not* 2.

Begin with the following Lemma which performs a similar role to Lemma 3.6.1.

Lemma 3.6.3 *Consider $\alpha_1 \geq \alpha_2$ and γ fixed, if there exists $\tau \in \Gamma_{MF}^{\alpha_2}(\gamma)$ then $\tau \in \Gamma_{MF}^{\alpha_1}(\gamma)$.*

Proof: The proof follows a similar argument to the proof of Lemma 3.5.1. Begin by considering a new RDE given by adding (3.38) and (3.109) and multiplying through by α^{-2}

$$0 = \dot{\Xi}_\alpha(t) + A'\Xi_\alpha(t) + \Xi_\alpha(t)A + \Xi_\alpha(t)B_1\tau_\gamma B_1'\Xi_\alpha(t) + \alpha^{-2}C_1'\tau^{-1}C_1 - C_2'\left(D_{21}\tau_\gamma D_{21}'\right)^{-1}C_2 \tag{3.110}$$

with $\Xi_\alpha(0) = 0$ and where

$$\Xi_\alpha(t) = \alpha^{-2}\left(P_\alpha + Z_\alpha(t)\right) \tag{3.111}$$

Now, an alternative definition for $\Gamma_{MF}^\alpha(\gamma)$ is 'the set of matrices $\tau_\gamma \in \Gamma_{SF}^\alpha(\gamma)$ for which there exist antistabilising solutions $\Xi_\alpha(t)$ to (3.110) with $\Xi_\alpha(t) > \alpha^{-2}P_\alpha$ for all $t > 0$.'

Differentiating the RDE (3.110) with respect to ϵ at $\epsilon = 0$ gives

$$\begin{aligned} 0 \;=\; & \dot{\acute{\Xi}}_\alpha(\tau,t) + \left(A + B_1\tau_\gamma B_1'\Xi_\alpha(\hat{\tau},t)\right)'\acute{\Xi}_\alpha(\tau,t) + \acute{\Xi}_\alpha(\tau,t)\left(A + B_1\tau_\gamma B_1'\Xi_\alpha(\tau,t)\right) + \\ & \Xi_\alpha(\tau,t)B_1\eta B_1'\Xi_\alpha(\tau,t) - \alpha^{-2}C_1'\tau^{-1}\eta\tau^{-1}C_1 + C_2'\bar{Q}^{-1}D_{21}\eta D_{21}'\bar{Q}^{-1}C_2 \end{aligned} \tag{3.112}$$

Since a perturbation of γ is not allowed, η is zero in the performance channel. Therefore $\tau_\gamma^{-1}\eta\tau_\gamma^{-1} = \tau_0^{-1}\eta\tau_0^{-1} = \tau^{-1}\eta\tau^{-1}$ so, (3.112) may be written as

$$\begin{aligned} & \dot{\acute{\Xi}}_\alpha(\tau,t) + \left(A + B_1\tau_\gamma B_1'\Xi_\alpha(\tau,t)\right)'\acute{\Xi}_\alpha(\tau,t) + \acute{\Xi}_\alpha(\tau,t)\left(A + B_1\tau_\gamma B_1'\Xi_\alpha(\tau,t)\right) = \\ & \quad \gamma^{-2}\Big(-(Z_\alpha(\tau,t) + P_\alpha(\tau))B_1\tau_\gamma\tau_0^{-1}\eta\tau_0^{-1}\tau_\gamma B_1'(Z_\alpha(\tau,t) + P_\alpha(\tau)) - \\ & \quad \alpha^2 C_2'\bar{Q}^{-1}D_{21}\tau_\gamma\tau_0^{-1}\eta\tau_0^{-1}\tau_\gamma D_{21}'\bar{Q}^{-1}C_2 + C_1'\tau^{-1}\eta\tau^{-1}C_1\Big) \end{aligned} \tag{3.113}$$

This RDE will be used later in the argument.

Now, suppose there exists $\tau \in \Gamma_{MF}^{\alpha_2}(\gamma)$. Immediately, $\tau \in \Gamma_{SF}^{\alpha_2}(\gamma)$, so from Lemma 3.3.3, $\tau \in \Gamma_{SF}^{\alpha_1}(\gamma)$.

Now, consider an ARE with $\tau \in \Gamma_{MF}^{\alpha_2}(\gamma)$ and $\alpha = \alpha_2$ and with solution $\bar{\Xi}_{\alpha_2}$. It will later be shown that one solution $\bar{\Xi}_{\alpha_2}$ approaches the solution to the RDE (3.110) with the same values of τ, γ and α for t approaching infinity.

$$0 = A'\bar{\Xi}_{\alpha_2}(t) + \bar{\Xi}_{\alpha_2}(t)A + \bar{\Xi}_{\alpha_2}(t)B_1\gamma^2\tau_\gamma B_1'\bar{\Xi}_{\alpha_2}(t) + \alpha_2^{-2}C_1'\left(\gamma^2\tau\right)^{-1}C_1 - C_2'\left(D_{21}\gamma^2\tau_\gamma D_{21}'\right)^{-1}C_2 \tag{3.114}$$

Also, consider the same ARE but with $\alpha = \alpha_1$ and with solution $\bar{\Xi}_{\alpha_1}$. Multiplying through by -1 gives

$$\begin{aligned}0 &= (-A)'\bar{\Xi}_{\alpha_1}(t) + \bar{\Xi}_{\alpha_1}(t)(-A) - \bar{\Xi}_{\alpha_1}(t)B_1\gamma^2\tau_\gamma B_1'\bar{\Xi}_{\alpha_1}(t) - \alpha_1^{-2}C_1'\left(\gamma^2\tau\right)^{-1}C_1 \\ &\quad +C_2'\left(D_{21}\gamma^2\tau_\gamma D_{21}'\right)^{-1}C_2 \end{aligned} \tag{3.115}$$

Multiplying (3.114) by -1 and rearranging gives

$$\begin{aligned}0 &= (-A)'\bar{\Xi}_{\alpha_2}(t) + \bar{\Xi}_{\alpha_2}(t)(-A) - \bar{\Xi}_{\alpha_2}(t)B_1\gamma^2\tau_\gamma B_1'\bar{\Xi}_{\alpha_2}(t) - \alpha_1^{-2}C_1'\left(\gamma^2\tau\right)^{-1}C_1 \\ &\quad +C_2'\left(D_{21}\gamma^2\tau_\gamma D_{21}'\right)^{-1}C_2 - \left(\alpha_2^{-2}-\alpha_1^{-2}\right)C_1'\left(\gamma^2\tau\right)^{-1}C_1 \end{aligned} \tag{3.116}$$

So, by Theorem 2.1 of Ran and Vreugdenhil, if there exists a solution $\bar{\Xi}_{\alpha_2}$ to (3.116) then the maximal solution $\bar{\Xi}_{\alpha_1} = \bar{\Xi}^+_{\alpha_1}$ to (3.115) is such that $\bar{\Xi}^+_{\alpha_1} \geq \bar{\Xi}_{\alpha_2}$ and $A + B_1\bar{\tau}_\gamma B_1'\bar{\Xi}^+_{\alpha_1}$ is antistable. Since $\tau \in \Gamma_{MF}(\gamma_2)$ then there exists $\bar{\Xi}_{\alpha_2} > \alpha_2^{-2}\gamma_2^{-2}P_{\alpha_2}$ so $\bar{\Xi}^+_{\alpha_1} \geq \bar{\Xi}_{\alpha_2} > \alpha_2^{-2}\gamma_2^{-2}P_{\alpha_2} \geq \alpha_1^{-2}\gamma_1^{-2}P_{\alpha_1} > 0$. Now, it remains to be shown that $\lim_{t\to\infty}\Xi_{\alpha_1}(t) = \bar{\Xi}^+_{\alpha_1}$ and that $Z_{\alpha_1}(t) > 0$ for all $t > 0$. It will be shown that $\Xi_{\alpha_1}(t) \geq \Xi_{\alpha_2}(t)$ for all $t > 0$ which implies $Z_{\alpha_1}(t) > 0$. From the definition of $\Xi_\alpha(t)$ (3.111), $\Xi_{\alpha_1} \geq \Xi_{\alpha_2} > 0$ so

$$\begin{aligned}\alpha_1^{-2}Z_{\alpha_1}(t) &\geq \alpha_2^{-2}Z_{\alpha_2}(t) + \alpha_2^{-2}P_{\alpha_2} - \alpha_1^{-2}P_{\alpha_1} \\ \Rightarrow \alpha_1^{-2}Z_{\alpha_1}(t) &\geq \alpha_2^{-2}Z_{\alpha_2}(t) > 0 \end{aligned} \tag{3.117}$$

since $\alpha_2^{-2}P_{\alpha_2} - \alpha_1^{-2}P_{\alpha_1} \geq 0$. Immediately, it would follow that $Z_{\alpha_1}(t) > 0$ for all $t > 0$. For notational simplicity, $\Xi_{\alpha 2}(\tau_\gamma, t)$ will be denoted $\Xi_{\alpha 2}(t)$. Similarly, $\Xi_{\alpha_1}(\tau_\gamma, t)$ will be denoted $\Xi_{\alpha_1}(t)$. Now, define $\Delta_\alpha(t) = \Xi_{\alpha_1}(t) - \Xi_{\alpha_2}(t)$. Subtracting (3.110) with $\alpha = \alpha_2$ from (3.110) with $\alpha = \alpha_1$ gives

$$\begin{aligned}0 &= \dot{\Delta}_\alpha(t) + \left(A + B_1\tau_\gamma B_1'\Xi_{\alpha 2}(t)\right)'\Delta_\alpha(t) + \Delta_\alpha(t)\left(A + B_1\tau_\gamma B_1'\Xi_{\alpha 2}(t)\right) \\ &\quad +\Delta_\alpha(t)B_1\tau_\gamma B_1'\Delta_\alpha(t) - \left(\alpha_2^{-2}-\alpha_1^{-2}\right)C_1'\tau^{-1}C_1 \end{aligned} \tag{3.118}$$

with $\Delta_\alpha(0) = 0$. Evaluating (3.118) at $t = 0$ gives

$$\dot{\Delta}_\alpha(0) = \left(\alpha_2^{-2}-\alpha_1^{-2}\right)C_1'\tau^{-1}C_1 > 0$$

Now, using a contradiction argument similar to the proof of Lemma 3.6.1, suppose at some time $t = T_c$ there exists a vector g such that, $g'\Delta_\alpha(T_c)g = 0$ and $g'\dot{\Delta}_\alpha(T_c)g < 0$. Therefore, at some time $t > T_c$, $g'\Delta_\alpha(t)g < 0$. Substituting into (3.118) and pre and post multiplying by g' and g, respectively, gives

$$0 = g'\dot{\Delta}_\alpha(T_c)g - g'\left(\alpha_2^{-2}-\alpha_1^{-2}\right)C_1'\tau^{-1}C_1 g \tag{3.119}$$

So,

$$g'\dot{\Delta}_\alpha(T_c)g = g'\left(\alpha_2^{-2} - \alpha_1^{-2}\right)C_1'\tau^{-1}C_1 g > 0$$

which contradicts the assumption that $g'\dot{\Delta}(T_c)g < 0$. Therefore, $\Delta_\alpha(t) \geq 0$ for all $t > 0$. Now, since (3.118) is an LQR ARE when $t = \infty$, then the positive solution $\lim_{t\to\infty}\Delta_\alpha(t)$ is unique (see for example [Anderson and Moore, 1989]). Also,

$$\lim_{t\to\infty} A + B_1\tau_\gamma B_1'(\Xi_{\alpha_2}(t) + \Delta_\alpha(t)) = \lim_{t\to\infty} A + B_1\tau_\gamma B_1'\Xi_{\alpha_1}(t)$$

is stabilising. Therefore, $\lim_{t\to\infty}\Xi_{\alpha_1}(t) = \bar{\Xi}^+_{\alpha_1}$, as required. Thus, $\tau \in \Gamma^{\alpha_1}_{MF}(\gamma)$ This completes the proof of the Lemma. □

Taking the limit as $\alpha \to \infty$ in (3.38) and (3.110), an LQR ARE and and LQG RDE are recovered, respectively. Therefore, since $[A\ B_2]$ is stabilisable and $[A\ C_2]$ is detectable, there exists a positive stabilising solution $\lim_{\alpha\to\infty} P_\alpha$ to (3.38) and a postive antistabilising solution $\lim_{t\to\infty}\Xi_\alpha(t)$ to (3.110) for all scaling matrices $\tau > 0$. See, for example [Anderson and Moore, 1989].

Since it has been established that, for all $\gamma > 0$, $\lim_{\alpha\to\infty}\Gamma^\alpha_{MF}(\gamma)$ is non-empty, $\Gamma^\alpha_{MF}(\gamma)\mid_{\alpha=1}$ is empty and Lemma 3.6.3 gives an ordering on α then there exists $\alpha_{min} > 1$ such that $\Gamma^{\alpha_{min}}_{MF}(\gamma)$ is non-empty but, for all $\alpha < \alpha_{min}$, $\Gamma^\alpha_{MF}(\gamma)$ is empty. For brevity of notation the dependence of α_{min} on γ has been omitted.

Lemma 3.6.4 *There exist $w \in \mathcal{W}$ and $x(0)$ such that $J^\alpha_\gamma(u, w, x(0)) > 0$ for any controller u, for any γ and for $\alpha < \alpha_{min}$.*

Proof: Since $\alpha < \alpha_{min}$, there exists some time $t = T_\alpha(\tau) > 0$ at which $Z_\alpha(\tau, T_\alpha(\tau))$ develops a zero eigenvalue. The eigenvector corresponding to the zero eigenvalue of $Z_\alpha(\tau, T_\alpha(\tau))$ will be refered to as $e_\alpha(\tau)$. It is not necessary to consider the case where $Z_\alpha(\tau, 0)$ has a zero or negative eigenvalue because $P_\alpha(\tau) \not< \alpha^2\gamma^2 R$ so $\tau \notin \Gamma^\alpha_{SF}(\gamma)$. Therefore, by the proof of Theorem 3.3.1, there does not exist a state feedback controller which gives the desired performance. Thus, there also does not exist a measurement feedback controller so, for the remainder of the argument, $T_\alpha(\tau) > 0$ may be considered.

Notice from the definition of $T_\alpha(\tau)$ that

$$e_\alpha(\tau)'Z_\alpha(\tau, T_\alpha(\tau))e_\alpha(\tau) = 0 \tag{3.120}$$

Now, define $\hat{\tau}(\alpha)$ such that $T_\alpha(\hat{\tau}(\alpha)) \geq T_\alpha(\tau)$ for all τ. Therefore, $\hat{\tau}(\alpha)$ is such that

$$\frac{dT_\alpha(\hat{\tau}(\alpha))}{d\epsilon}\mid_{\epsilon=0} = 0 \tag{3.121}$$

for all η, where η, defined in (3.24), represents an arbitrary direction in τ space, so the above expression defines a stationary point in τ space. Since $\hat{\tau}(\alpha)$ maximises the time at which $Z_\alpha(\tau, t)$ takes a zero eigenvalue, it follows that

$$\frac{d}{d\epsilon}\left(e_\alpha(\hat{\tau}(\alpha))'Z_\alpha(\hat{\tau}(\alpha), T_\alpha(\hat{\tau}(\alpha)))e_\alpha(\hat{\tau}(\alpha))\right)\mid_{\epsilon=0} = 0 \tag{3.122}$$

Expanding (3.122) gives

$$\begin{aligned} 0 &= \acute{e}_\alpha(\hat{\tau}(\alpha))' Z_\alpha(\hat{\tau}(\alpha), T_\alpha(\hat{\tau}(\alpha)))e_\alpha(\hat{\tau}(\alpha)) + e_\alpha(\hat{\tau}(\alpha))' \acute{Z}_\alpha(\hat{\tau}(\alpha), T_\alpha(\hat{\tau}(\alpha)))e_\alpha(\hat{\tau}(\alpha)) + \\ & \quad e_\alpha(\hat{\tau}(\alpha))' Z_\alpha(\hat{\tau}(\alpha), T_\alpha(\hat{\tau}(\alpha)))\acute{e}_\alpha(\hat{\tau}(\alpha)) \end{aligned} \tag{3.123}$$

Since $Z_\alpha(\hat{\tau}(\alpha), T(\hat{\tau}(\alpha)))e_\alpha(\hat{\tau}(\alpha)) = 0$, (3.123) becomes

$$e_\alpha(\hat{\tau}(\alpha))' \acute{Z}_\alpha(\hat{\tau}(\alpha), T_\alpha(\hat{\tau}(\alpha)))e_\alpha(\hat{\tau}(\alpha)) = 0 \tag{3.124}$$

For notational convenience, the dependence of $\hat{\tau}(\alpha)$ on α will not be shown explictly for the remainder of the argument.

Now, consider a disturbance/measurement noise/uncertainty signal

$$\begin{aligned} w(t) &= w_{MF_\alpha}(\tau, t) \\ &\triangleq \begin{cases} (\alpha^{-1}\tau_\gamma B_1'(Z_\alpha(\tau,t) + P(\tau)) - \alpha\tau_\gamma D_{21}'(D_{21}\tau_\gamma D_{21}')^{-1}C_2)\, x(t), & t \in [0, T_\alpha(\tau)) \\ \alpha^{-1}\tau_\gamma B_1' P_\alpha(\tau)x(t), & t \in [T_\alpha(\tau), \infty) \end{cases} \end{aligned} \tag{3.125}$$

and an initial condition $x(0) = x_0(\tau)$ which is such that $x(T_\alpha(\tau)) = e_\alpha(\tau)$. Notice that $w_{MF_\alpha}(\tau, t)$ is such that $y(t) \equiv 0$ for $t \in [0, T_\alpha(\tau))$. A similar argument to the proof of Lemma 3.6.2 may be used to show that if $y(t) = 0$ for $t \in [0, T_\alpha(\tau))$ then it is required that $u(t) = 0$ for $t \in [0, T_\alpha(\tau))$ to have a possibility to satisfy robust performance. So $u(t) = 0$ is the only sensible control strategy for $t \in [0, T_\alpha(\tau))$.

Now, consider $w(t) = w_{MF_\alpha}(\tau, t)$, $x(0) = x_0(\tau)$ and any control signal $u(t)$ which is zero for $t \in [0, T_\alpha(\tau))$ and which is such that the system (3.36) is stable. Also, consider an expression for the cost J_γ^α, defined in (3.37), which is separated into two parts; the cost incurred during the interval $t \in [0, T_\alpha(\tau))$ and the cost incurred during $t \in [T_\alpha(\tau), \infty)$. Still considering $\alpha < \alpha_{min}$ and using an argument similar to that which gave (3.99) gives

$$J_\gamma^\alpha(u, w_{MF_\alpha}(\check{\tau}), x_0(\check{\tau})) = \|u - \hat{u}(\check{\tau}(\alpha))\|^2_{(T_\alpha(\check{\tau}(\alpha)),\infty) D_{12}'\check{\tau}(\alpha)^{-1} D_{12}} \tag{3.126}$$

where $\tau = \check{\tau}(\alpha)$ is such that $\|z_i(u(t), w_{MF_\alpha}(\check{\tau}(\alpha), t))\| = \|w_{i\,MF_\alpha}(\check{\tau}(\alpha, t)\|$ for $x(0) = x_0(\check{\tau}(\alpha))$. For notational convenience the dependence of $\check{\tau}(\alpha)$ on α will not be shown explicitly. Since it is not possible to design a measurement feeback controller which produces the output of the optimal state feedback controller $\hat{u}(\check{\tau})$, at the instant $t = T_\alpha(\check{\tau})$ when the measurement signal $y(t)$ becomes non zero, then

$$J_\gamma^\alpha(u(t), w_{MF_\alpha}(\check{\tau}(\alpha), t), x_0(\check{\tau}(\alpha))) > 0$$

for all measurement feedback controllers.

Now, it must be shown that if $\Gamma_{SF}^\alpha(\gamma)$ is non-empty then $\check{\tau}(\alpha, u_{MF_\alpha}(\check{\tau}), w_{MF_\alpha}(\check{\tau}), x_0(\check{\tau})) \in \Gamma_{SF}^\alpha(\gamma)$. To do this it will be shown that $\check{\tau}(\alpha, u_{MF_\alpha}(\check{\tau}), w_{MF_\alpha}(\check{\tau}), x_0(\check{\tau})) = \hat{\tau}(\alpha)$. Begin with (3.124). Substituting for $Z_\alpha(\hat{\tau}, T_\alpha(\hat{\tau}))$ from the definition of Ξ_α (3.111) and differentiating with respect to ϵ at $\epsilon = 0$ gives

$$0 = \alpha^2 e_\alpha(\hat{\tau}(\alpha))'\acute{\Xi}_\alpha(\hat{\tau}, T_\alpha(\hat{\tau}))e_\alpha(\hat{\tau}) - e_\alpha(\hat{\tau})'\acute{P}_\alpha(\hat{\tau})e_\alpha(\hat{\tau}) \tag{3.127}$$

Letting the initial condition $x(0) = x_0(\hat{\tau})$, $u(t) = u_{MF_\alpha}(\hat{\tau}, t)$ and $w(t) = w_{MF_\alpha}(\hat{\tau}, t)$, (3.127) may be written as

$$\begin{aligned} 0 &= \alpha^2 \int_0^{T_\alpha} x'\dot{\acute{\Xi}}_\alpha(\hat{\tau}, t)x + \dot{x}'\acute{\Xi}_\alpha(\hat{\tau}, t)x + x'\acute{\Xi}_\alpha(\hat{\tau}, t)\dot{x}\, dt + \int_{T_\alpha}^\infty \dot{x}'\acute{P}_\alpha(\hat{\tau})x + x'\acute{P}_\alpha(\hat{\tau})x\, dt \\ &\quad + x(0)'\acute{\Xi}_\alpha(\hat{\tau})(0)x(0) - x(\infty)'\acute{P}_\alpha(\hat{\tau})x(\infty) \end{aligned} \tag{3.128}$$

and the state equation (3.36) becomes

$$\dot{x}(t) = (A + B_1\hat{\tau}_\gamma B_1'\Xi_\alpha(\hat{\tau}, t))\, x(t), \quad t \in [0, T_\alpha(\hat{\tau})) \tag{3.129}$$

$$\dot{x}(t) = \left(A + \left(\alpha^{-2}B_1\hat{\tau}_\gamma(\alpha)B_1' - B_2\bar{R}^{-1}B_2'\right)P_\alpha(\hat{\tau})\right)x(t), \quad t \in [T_\alpha(\hat{\tau}), \infty) \tag{3.130}$$

Substituting (3.129) and (3.130) into (3.128) gives

$$\begin{aligned} 0 = \alpha^2 \int_0^{T_\alpha} & x'\left(\dot{\acute{\Xi}}_\alpha(\hat{\tau}, t) + (A + B_1\hat{\tau}_\gamma B_1'\Xi_\alpha(\hat{\tau}, t))'\acute{\Xi}_\alpha(\hat{\tau}, t) + \right. \\ & \left.\acute{\Xi}_\alpha(\hat{\tau}, t)\left(A + B_1\hat{\tau}_\gamma(\alpha)B_1'\Xi_\alpha(\hat{\tau}, t)\right)\right) x\, dt \\ & + \int_{T_\alpha}^\infty x'\left(\left(A + \left(\alpha^{-2}B_1\hat{\tau}_\gamma(\alpha)B_1' - B_2\bar{R}^{-1}B_2'\right)P_\alpha(\hat{\tau})\right)'P_\alpha(\hat{\tau}) + \right. \\ & \left. P_\alpha(\hat{\tau})\left(A + \left(\alpha^{-2}B_1\hat{\tau}_\gamma(\alpha)B_1' - B_2\bar{R}^{-1}B_2'\right)P_\alpha(\hat{\tau})\right)\right) x\, dt - x(\infty)'\acute{P}_\alpha(\hat{\tau})x(\infty) \end{aligned} \tag{3.131}$$

Noticing that $x(\infty) = 0$ since (3.130) is stable and using the RDE (3.113) and the ARE (3.38), (3.131) may be written as

$$\begin{aligned} 0 = \int_0^{T_\alpha} & x'\left(-\alpha^{-2}(Z_\alpha(\hat{\tau}, t) + P_\alpha(\hat{\tau}))B_1\hat{\tau}_\gamma\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}\hat{\tau}_\gamma B_1'(Z_\alpha(\hat{\tau}, t) + P_\alpha(\hat{\tau})) - \right. \\ & \left.\alpha^2 C_2'\bar{Q}^{-1}D_{21}\hat{\tau}_\gamma\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}\hat{\tau}_\gamma D_{21}'\bar{Q}^{-1}C_2 + C_1'\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}C_1\right) x\, dt - \int_{T_\alpha}^\infty x'\left(P_\alpha(\hat{\tau}) \times \right. \\ & \left.\left(\alpha^{-2}B_1\hat{\tau}_\gamma\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}\hat{\tau}_\gamma B_1' - B_2\bar{R}^{-1}\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}\bar{R}^{-1}B_2'\right)P_\alpha(\hat{\tau}) - C_1'\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}C_1\right) x\, dt \end{aligned} \tag{3.132}$$

Now, (3.132) may be written as

$$\begin{aligned} 0 &= \|z\left(w_{MF_\alpha}(\hat{\tau}), x_0(\hat{\tau})\right)\|_{[0,T_\alpha(\hat{\tau}))\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}} - \|w_{MF_\alpha}(\hat{\tau})\|_{[0,T_\alpha(\hat{\tau}))\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}} + \\ & \|z\left(u_{MF_\alpha}(\hat{\tau}), w_{MF_\alpha}(\hat{\tau}), x_0(\hat{\tau})\right)\|_{(T_\alpha,\infty)\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}} - \|w_{MF_\alpha}(\hat{\tau}, t)\|_{(T_\alpha,\infty)\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}} \end{aligned} \tag{3.133}$$

Therefore,

$$0 = \|z\left(u_{MF_\alpha}(\hat{\tau}), w_{MF_\alpha}(\hat{\tau}), x_0(\hat{\tau})\right)\|_{\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}} - \|w_{MF_\alpha}(\hat{\tau})\|_{\hat{\tau}_0^{-1}\eta\hat{\tau}_0^{-1}} \tag{3.134}$$

which is equivalent to the definition of $\check{\tau}$ for $u = u_{MF_\alpha}(\check{\tau}(\alpha))$, $w = w_{MF_\alpha}(\check{\tau}(\alpha))$ and $x(0) = x_0(\check{\tau}(\alpha))$

Therefore, it has been shown that if there does not exist $\tau \in \Gamma_{SF}^{\alpha}(\gamma)$ then, by Theorem 3.3.1, there does not exist a state feedback controller and therefore a measurement feedback controller which gives the required performance for $\alpha < \alpha_{min}$. And, if there does exist $\tau \in \Gamma_{SF}^{\alpha}(\gamma)$ then there also does not exist a controller which statisfies robust performance. This completes proof of Lemma. □

It has been shown, from the definition of α_{min}, that *not* 1 $\Rightarrow \alpha < \alpha_{min}$ and for such α there does not exist a measurement feedback controller such that $J_\gamma^\alpha(u(t), \hat{w}_{MF}(\check{\tau}(\alpha)), e_\alpha(\check{\tau}(\alpha)) \leq 0 \Rightarrow$ *not* 2. So, *not* 1 $\Rightarrow$ *not* 2 which is equivalent to 2 $\Rightarrow$ 1, as required. This completes the proof of the Theorem. □

The ARE

$$0 = \tilde{A}'Z_\infty + Z_\infty\tilde{A} + K'\bar{R}^{-1}K - C_2'\bar{Q}^{-1}C_2 + Z_\infty B_1\tau_\gamma B_1'Z_\infty \tag{3.135}$$

which is the time invariant form of the RDE (3.58), is used in the following Corollary.

Corollary 3.6.1 *If γ is large enough such that there exists $\tau \in \Gamma_{SF}(\gamma)$ such that $Z(\tau, t = 0) \geq Z_\infty$ then there exist time invariant, dynamic, measurement feedback controllers which satisfy statement 2 of Theorem 3.6.1. One such controller is $u(t) = K\hat{x}(t)$ (3.61) where $\hat{x}(t)$ is given by the time invariant filter*

$$\dot{\hat{x}}(t) = \left(\tilde{A} - B_2\bar{R}^{-1}B_2'P\right)\hat{x}(t) + Z_\infty^{-1}C_2'\bar{Q}^{-1}\left(y(t) - C_2\hat{x}(t)\right)$$

with $\hat{x}(0) = 0$.

Proof: A 'completing the squares' argument is used, similar to the 1 $\Rightarrow$ 2 argument of Theorem 3.6.1. The proof is almost identical to this argument except the proof begins with the ARE (3.135) rather than the RDE (3.58). □

3.7 Discussion for measurement feedback

The measurement feedback H_∞ with transients and structure problem may be solved by the solution to a parameter dependent ARE and a parameter dependent RDE. The solution is generally a time varying dynamic controller and, unlike the state feedback problem, the measurement feedback H_∞ with transients and structure problem involves a non-convex optimisation.

To design a controller which approaches the optimal controller the following procedure may be followed. Begin with some (large) value of γ for which there exists $\tau \in \Gamma_{MF}(\gamma)$. Note that if $\lim_{\gamma\to\infty}\Gamma_{MF}(\gamma)$ is empty then the system is not robustly stabilisable, that is, there does not exist a measurement feedback controller which will stabilise the system (2.1), (2.2), (2.3) for all $w(t) \in \mathcal{W}$ and all $x(0)$. Once a $\tau \in \Gamma_{MF}(\gamma)$ is found, successively reduce γ

until $\Gamma_{MF}(\gamma)$ becomes empty. The smallest value of γ such that $\Gamma_{MF}(\gamma)$ is non-empty is the optimal value of γ. Since $\Gamma_{MF}(\gamma)$ is not generally a convex set in τ space, the optimisation may be computationally difficult and a local minimum of γ will not necessarily be a global minimum.

The maximum singular value of the filter gain for the H_∞ with transients and structure problem, for optimal γ, approaches infinity at some time. The solution for the non-transients problem, for optimal γ, gives a constant filter gain with an infinite maximum singular value, so, the optimal solutions, with and without transients, are impractical. A Corollary was presented which gave a suboptimal solution to the H_∞ with transients and structure problem with a time-invariant dynamic controller.

The worst case strategy for the $[w(t), x(0)]$ player, for optimal γ, has been considered in the limit as γ approaches γ_{min} from below. In this limit, the worst case uncertainty signal $w(t)$ is such that the measurement signal $y(t)$ is zero for all time up to time T when $Z(T)$, the solution to the RDE, takes a zero eigenvalue. From time T onwards the signal $\hat{w}(t)$ (3.12) is used which is the worst case signal for the state feedback case. The worst case class of initial conditions $x(0)$ is such that $x(T)$ is an eigenvector corresponding to the zero eigenvalue of $Z(T)$. Since $y(t) = 0$ from time zero to T, the controller has no information so the control signal $u(t)$ must be zero for this time interval. After time T the controller receives information and the control signal can become non zero.

Similar to the state feedback case, the choice of R, the penalty on the initial condition, determines the guaranteed level of robustness of the resulting design to non-zero initial conditions. The structured H_∞ solution may be recovered by letting R approach ∞I_n in (3.2). Thus, in the game between $u(t)$ and the $[w(t), x(0)]$ pair, a non zero initial condition on the state would incur an infinite penalty for the $[w(t), x(0)]$ player. Therefore, the initial condition $x(0)$ would be forced to zero and the resulting design would have no guaranteed robustness to non-zero initial conditions. This special case of Theorem 3.6.1 where $R \to \infty I_n$ is a generalisation of [Savkin and Petersen, 1996], allowing for an uncertain measurement equation.

Also, as was remarked for the state feedback case, the H_∞ with transients results of [Khargonekar et al., 1991] may be recovered by ignoring the structure, in which case, the scaling matrix τ_γ becomes γ^{-2} and τ becomes identity.

Chapter 4

Minimax controller design

4.1 Introduction

In this Chapter, a state feedback minimax design problem is considered which, like the H_∞ with transients problem of Chapter 3, explicitly considers the effect of a non zero initial condition of the state. However, in the minimax problem presented in this Chapter, the initial condititon is not a player in the game but is given as problem data.

The minimax problem considered in this Chapter has its roots in the work of [Chang and Peng, 1972]. Their work on guaranteed cost control provided an upper bound on the value of a standard quadratic cost functional for a class of linear systems. [Petersen and McFarlane, 1994] showed that it was possible to design controllers which optimised such bounds; optimal guaranteed cost controllers. The class of uncertainty was called Euclidean norm-bounded, so Euclidean, rather than L_2 norm bounds, were used to describe the uncertainty. The design involves the solution of an Algebraic Riccati Equation (ARE) dependent on a single scalar scaling parameter. The existence of a controller that would provide a guaranteed cost bound or robustly stabilise the system was dependent upon the existence of a suitable solution to a parametric ARE. Certain convexity and existence results for the ARE ensured the optimisation was tractable.

These guaranteed cost controllers were shown to be minimax for a larger class of uncertainty by [Savkin and Petersen, 1995]. The uncertainty class for which these controllers were minimax was descibed by an Integral Quadratic Constraint (IQC) ([Yakubovich, 1973] and [Rantzer and Megretsky, 1994]) and may be used to (conservatively) represent Euclidean norm bounded uncertainty. By allowing for multiple channels of feedback, they permitted structured uncertainty. Minimax optimality was proven using a result presented in [Megretsky and Treil, 1990] and [Yakubovich, 1992] known as the 'S procedure'.

The approach presented in this Chapter is based on that of [Savkin and Petersen, 1995]. The uncertainty inputs satisfy an arbitrary number of IQC's. Only standard techniques have been used to prove that the controllers are minimax for given initial conditions. The multivariable optimisation problem is shown to be convex with a compact set of feasible parameters, thus, the design is computationally tractable. The H_∞ with structure problem of [Savkin and Petersen, 1996] for state feedback and optimal γ is recovered as a special case of this minimax problem.

In Section 4.2 preliminaries for the state feedback minimax problem are presented. Section 4.3 details the main results; the design of a minimax controller which is shown to be a convex optimisation problem. Section 4.4 investigates ways to apply the minimax design including a receding horizon approach. Finally, a minimax problem is posed for measurement feedback, although it appears to be intractable.

4.2 Preliminaries for state feedback

Uncertain state feedback systems of the form (2.1), (2.2) are considered where $B_{1_p} = 0$. Note that B_{1_p} is defined in the preliminaries section of Chapter 2.

Each uncertainty input is bounded by an integral quadratic constraint, thus the uncertainty input vector must lie in the set $\mathcal{W}_{mx}$ where

$$\mathcal{W}_{mx} = \left\{ w(t) : \int_0^\infty w_i'(t) w_i(t)\, dt \leq \int_0^\infty z_i(t)' z_i(t)\, dt + d_i \right\} \tag{4.1}$$

with $d_i \in R > 0$. This set of uncertainty/disturbance inputs is the most general for which our results apply. 'Robust Performance' type problems may be presented in this framework by an appropriate delineation of the input channels into those representing uncertainty via L_2 induced-norm bounded feedback perturbations and those representing L_2 norm bounded exogenous disturbances. For channels representing uncertainty inputs one should set $d_i = 0$, and for channels representing L_2 norm bounded exogenous disturbances, one should set $z_i(t) = 0$ with d_i the L_2 norm bound on $w_i(t)$.

The system performance is measured with a quadratic cost function

$$J = \|z_p\|^2 \tag{4.2}$$

where z_p is one channel of the signal z, defined in (2.2) in the preliminaries section of Chapter 2. It is also required that $C_{1_p} > 0$ and $D_{12_p} > 0$. It may be possible to remove the requirement that all states be costed but this is of little practical significance since the costs may be arbitrarily small.

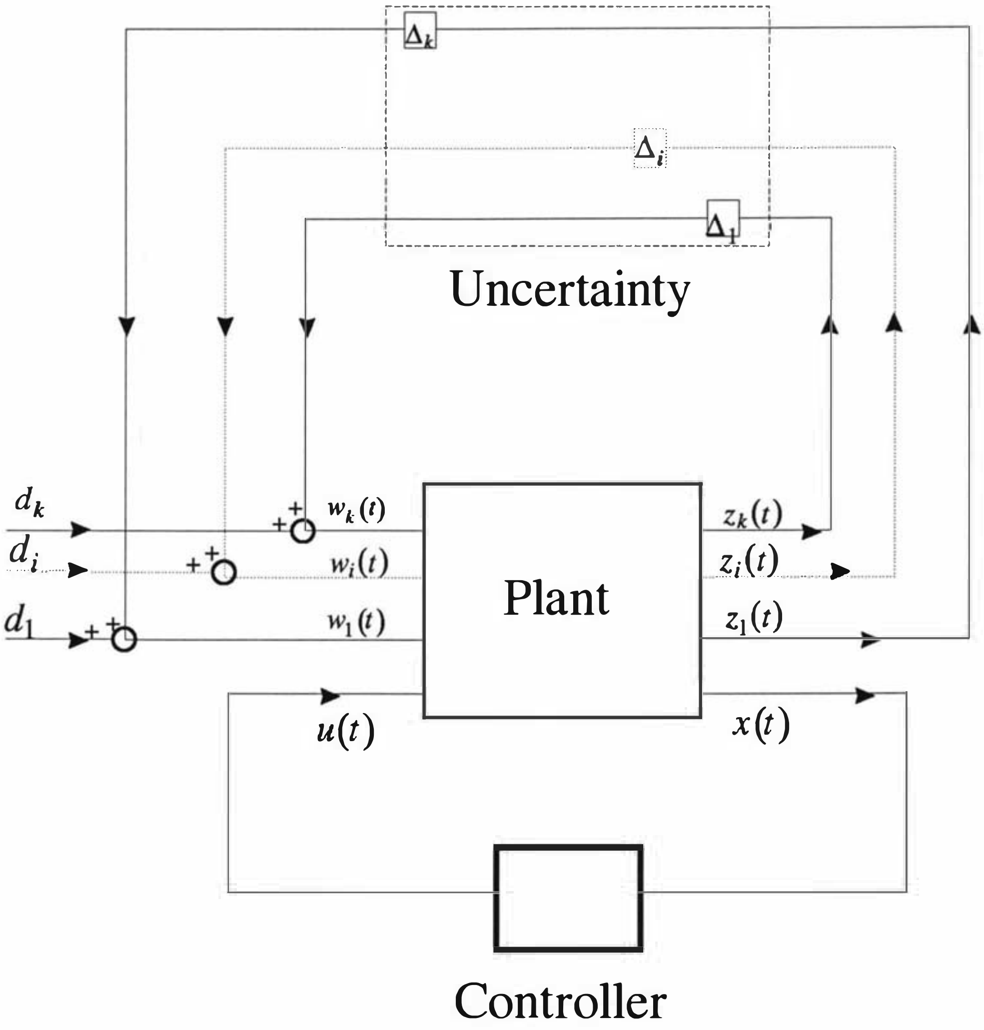

Figure 4.1

The following algebraic Riccati equation plays an important role in the sequel:

$$\begin{aligned} 0 &= (A - B_2\bar{R}(\tau)^{-1}D_{12}'\tau^{-1}C_1)'P + P(A - B_2\bar{R}(\tau)^{-1}D_{12}'\tau^{-1}C_1) \\ &\quad + P(B_1\tau_0 B_1' - B_2\bar{R}(\tau)^{-1}B_2')P + C_1'\tau^{-1}C_1 - C_1'\tau^{-1}D_{12}\bar{R}(\tau)^{-1}D_{12}'\tau^{-1}C_1, \end{aligned} \quad (4.3)$$

the matrices A, B_2, B_1, C_1, D_{12} are from the system (2.1), (2.2) and cost functional (4.2) and $\bar{R}(\tau) = D_{12}'\tau^{-1}D_{12}$. $\tau \in \mathbf{R}^{p\times p}$ is a scaling matrix, the definition of which is the same as that which was used for the H_∞ with transients and structure problem of Chapter 3;

$$\tau = \begin{pmatrix} I_{p_p} & 0 & & & \\ 0 & \tau_1 I_{p_1} & 0 & & \\ & 0 & \tau_2 I_{p_2} & 0 & \\ & & 0 & \ddots & 0 \\ & & & 0 & \tau_k I_{p_k} \end{pmatrix}, \tag{4.4}$$

where I_{p_i} is the $p_i \times p_i$ identity matrix and $\tau_i \in \mathbf{R} > 0$. To avoid confusion, it should be noted that these parameters are similar to those used by [Petersen and McFarlane, 1994] but are the reciprocal of those used by [Savkin and Petersen, 1995]. The set of matrices τ for which a solution $P > 0$ exists for (4.3) is denoted Γ. The matrix τ_0 is defined in the same way as was done for the H_∞ with transients and structure problem; the first element of τ_0 is a $p_p \times p_p$ zero matrix rather than identity.

For notational convenience, the ARE (4.3) may be rearranged to give the following equivalent ARE

$$0 = \bar{A}'P(\tau) + P(\tau)\bar{A} + P(\tau)\bar{X}P(\tau) + \bar{Q}, \tag{4.5}$$

where

$$\bar{A} = A - B_2\bar{R}(\tau)^{-1}D_{12}'\tau^{-1}C_1, \tag{4.6}$$
$$\bar{X} = B_1\tau_0B_1' - B_2\bar{R}(\tau)^{-1}B_2', \tag{4.7}$$
$$\bar{Q} = C_1'(\tau^{-1} - \tau^{-1}D_{12}\bar{R}(\tau)^{-1}D_{12}'\tau^{-1})C_1. \tag{4.8}$$

4.3 Main results for state feedback

In this section a solution is given for a minimax optimal controller for the class of uncertain systems (2.1), (2.2), (4.1) with given initial condition x_0 and cost functional (4.2). This problem has been studied by [Savkin and Petersen, 1995]. In this Chapter an alternative proof which uses only standard methods for linear systems is presented. [Basar and Bernhard, 1995] provide a good background to the techniques used. Furthermore, it is shown that an optimal controller may be designed by the solution of a convex optimisation problem.

Theorem 4.3.1 *(A small extension of [Savkin and Petersen, 1995]). The minimax cost for the class of systems (2.1), (2.2), (4.1) with cost functional (4.2) and initial condition x_0 is given by*

$$\inf_{u(t)} \sup_{w(t)\in\mathcal{W}_{mx}} J(u(t), w(t)) = \inf_{\tau\in\Gamma} \left(x_0'P(\tau)x_0 + \Sigma_{i=1}^{k}\tau_i^{-1}d_i\right) \tag{4.9}$$

where $P(\tau) > 0$ is the stabilising solution to (4.3). Furthermore, a minimax control strategy is given by the static controller

$$u(t) = \hat{u}_{mx}(\hat{\tau}) \triangleq -\bar{R}(\hat{\tau})^{-1}(B_2'P(\hat{\tau}) + D_{12}'\tau^{-1}C_1)x(t) \tag{4.10}$$

and the worst case uncertainty/disturbance input for this controller is given by

$$w(t) = \hat{w}_{mx}(\tau) \triangleq \hat{\tau}_0 B_1' P(\hat{\tau}) x(t). \tag{4.11}$$

where, $\hat{\tau}$ is the infimising value of τ for the right hand side of (4.9).

Moreover, if Γ is empty then there does not exist a controller which will guarantee a finite cost for all $w(t) \in \mathcal{W}_{mx}$ (4.1).

Proof: The proof begins with the presentation of two Lemma pertaining to the ARE (4.3). These two Lemma extend the convexity and existence results presented by [Petersen and McFarlane, 1994] to allow for a structured uncertainty description (4.1).

Lemma 4.3.1 *Suppose $\tau = \tilde{\tau} \in \Gamma$, then $\beta\tilde{\tau}_\beta \in \Gamma$, where*

$$\tau_\beta = \begin{pmatrix} \beta^{-1} I_{p_p} & 0 & & & \\ 0 & \tau_1 I_{p_1} & 0 & & \\ & 0 & \tau_2 I_{p_2} & 0 & \\ & & 0 & \ddots & 0 \\ & & & 0 & \tau_k I_{p_k} \end{pmatrix},$$

with $0 < \beta < 1$.

Proof: If, for some $\tau = \tilde{\tau} \in \Gamma$, there exists a solution $P(\tilde{\tau}) > 0$ to the algebraic Riccati equation (4.3) then it will be shown that there exists a stabilising solution $P(\beta\tilde{\tau}_\beta) > 0$ to the algebraic Riccati equation (4.3) for $0 < \beta < 1$ and thus $\beta\tilde{\tau}_\beta \in \Gamma$.

Define $\Pi_1 = P(\tilde{\tau})^{-1}$ and $\Pi_2 = \frac{1}{\beta} P(\beta\tilde{\tau}_\beta)^{-1}$. Pre and post multiplying (4.3) with $\tau = \tilde{\tau}$ by Π_1 gives

$$\begin{aligned} 0 \;=\; & \Pi_1 (A - B_2 \bar{R}(\tilde{\tau})^{-1} D_{12}' \tilde{\tau}^{-1} C_1)' + (A - B_2 \bar{R}(\tilde{\tau})^{-1} D_{12}' \tilde{\tau}^{-1} C_1) \Pi_1 + \\ & (B_1 \tilde{\tau}_0 B_1' - B_2 \bar{R}(\tilde{\tau})^{-1} B_2') \\ & + \Pi_1 \left(C_1' \tilde{\tau}^{-1} C_1 - C_1' \tilde{\tau}^{-1} D_{12} \bar{R}(\tilde{\tau})^{-1} D_{12}' \tilde{\tau}^{-1} C_1 \right) \Pi_1; \end{aligned} \tag{4.12}$$

similarly, pre and post multiplying (4.3) with $\tau = \beta\tilde{\tau}_\beta$ by Π_2 and multiplying through by β gives

$$\begin{aligned} 0 \;=\; & \Pi_2 (A - B_2 \bar{R}(\tilde{\tau}_\beta)^{-1} D_{12}' \tilde{\tau}_\beta^{-1} C_1)' + (A - B_2 \bar{R}(\tilde{\tau}_\beta)^{-1} D_{12}' \tilde{\tau}_\beta^{-1} C_1) \Pi_2 + \\ & (B_1 \tilde{\tau}_0 B_1' - B_2 \bar{R}(\tilde{\tau}_\beta)^{-1} B_2') \\ & + \Pi_2 \left(C_1' \tilde{\tau}_\beta^{-1} C_1 - C_1' \tilde{\tau}_\beta^{-1} D_{12} \bar{R}(\tilde{\tau}_\beta)^{-1} D_{12}' \tilde{\tau}_\beta^{-1} C_1 \right) \Pi_2, \end{aligned} \tag{4.13}$$

Equation (4.12) can be recast as

$$\begin{aligned}0 &= \Pi_1(A - B_2\bar{R}(\tilde{\tau}_\beta)^{-1}D_{12}'\tilde{\tau}_\beta^{-1}C_1)' + (A - B_2\bar{R}(\tilde{\tau}_\beta)^{-1}D_{12}'\tilde{\tau}_\beta^{-1}C_1)\Pi_1 + \\ &\quad (B_1\tilde{\tau}_0B_1' - B_2\bar{R}(\tilde{\tau}_\beta)^{-1}B_2') \\ &\quad +\Pi_1\left(C_1'\tilde{\tau}_\beta^{-1}C_1 - C_1'\tilde{\tau}_\beta^{-1}D_{12}\bar{R}(\tilde{\tau}_\beta)^{-1}D_{12}'\tilde{\tau}_\beta^{-1}C_1\right)\Pi_1 + H, \end{aligned} \tag{4.14}$$

where

$$\begin{aligned}H &= \left(B_2 + D_{12}'\tilde{\tau}_\beta^{-1}C_1\Pi_1\right)'\left(\bar{R}(\tilde{\tau}_\beta)^{-1} - \bar{R}(\tilde{\tau})^{-1}\right)\left(B_2 + D_{12}'\tilde{\tau}_\beta^{-1}C_1\Pi_1\right) + (1-\beta)\Pi_1C_{1_p}'C_{1_p}\Pi_1 \\ &\geq 0,\end{aligned}$$

and because (4.3) has a solution $P_1 > 0$, equation (4.12) and thus (4.14) have a solution $\Pi_1 > 0$. Theorem 2.2 of [Ran and Vreugdenhil, 1988] can be applied to compare (4.14) and (4.13); since (4.14) has a solution $\Pi_1 > 0$, (4.13) has a solution $\Pi_2 > 0$. Furthermore $\Pi_2 > \Pi_1$ and

$$\begin{aligned}\tilde{A} &\triangleq A - B_2\bar{R}(\tilde{\tau}_\beta)^{-1}D_{12}'\tilde{\tau}_\beta^{-1}C_1 + \\ &\quad \Pi_2\left(C_1'\tilde{\tau}_\beta^{-1}C_1 - C_1'\tilde{\tau}_\beta^{-1}D_{12}\bar{R}(\tilde{\tau}_\beta)^{-1}D_{12}'\tilde{\tau}_\beta^{-1}C_1\right)\end{aligned} \tag{4.15}$$

is antistable. Now, it remains to be shown that $\bar{A}(\beta\tilde{\tau}_\beta) + \bar{X}(\beta\tilde{\tau}_\beta)P(\beta\tilde{\tau}_\beta)$ is stable. Since $\tilde{A}$ is antistable then, by a Lyapunov argument for (4.13), it follows that

$$\beta^{-1}\bar{X}(\beta\tilde{\tau}_\beta) - \beta\Pi_2\bar{Q}(\beta\tilde{\tau}_\beta)\Pi_2 \leq 0 \tag{4.16}$$

where $\bar{X}$ and $\bar{Q}$ are defined in (4.7) and (4.8). Pre and post multiplying (4.16) by $P(\beta\tilde{\tau}_\beta)$ and multiplying through by β gives

$$\bar{Q}(\beta\tilde{\tau}_\beta) - P(\beta\tilde{\tau}_\beta)\bar{X}(\beta\tilde{\tau}_\beta)P(\beta\tilde{\tau}_\beta) \geq 0 \tag{4.17}$$

So, by using a Lyapunov argument for (4.3) with $\tau = \beta\tilde{\tau}_\beta$ it follows that

$$A - B_2\bar{R}(\beta\tilde{\tau}_\beta)^{-1}D_{12}'(\beta\tilde{\tau}_\beta)^{-1}C_1 + (B_1\beta\tilde{\tau}_0B_1' - B_2\bar{R}^{-1}(\beta\tilde{\tau}_\beta)^{-1}B_2')P(\beta\tilde{\tau}_\beta)$$

is stable. Therefore $\beta\tilde{\tau}_\beta$ is in Γ. This completes the proof of the Lemma. □

Lemma 4.3.2 *The function $x_0'P(\tau)x_0 + \Sigma\tau_i^{-1}d_i$ (3.10) is a convex function of $(\tau_1, \tau_2, \ldots, \tau_k)$ on Γ.*

Proof: As for the proof of Theorem 3.3.1, a small perturbation of τ is considered; $\tau + \epsilon\eta$, where $\eta \in \Lambda$. The function $x_0'P(\tau)x_0 + \Sigma\tau_i^{-1}d_i$ depends on the k independent variables $(\tau_1, \tau_2, \ldots, \tau_k)$ and the convexity of $x_0'P(\tau)x_0 + \Sigma\tau_i^{-1}d_i$ will be established if it can be shown that the quadratic form

$$\Xi(\eta) = \Sigma_{a=1}^{k}\Sigma_{b=1}^{k}\eta_a\eta_b \frac{\partial^2\left(x_0'P(\tau)x_0 + \Sigma\tau_i^{-1}d_i\right)}{\partial\tau_a\partial\tau_b}$$

is non-negative for all choices of $\tau \in \Gamma$ and all $\eta \in \Lambda$ (3.24). Since

$$\Xi(\eta) = \frac{d^2}{d\epsilon^2}\left(x_0'P(\tau)x_0 + \Sigma\tau_i^{-1}d_i\right)|_{\epsilon=0},$$

it suffices to show that the inequality

$$\frac{d^2\left(x_0'P(\tau)x_0 + \Sigma\tau_i^{-1}d_i\right)}{d\epsilon^2}|_{\epsilon=0}\geq 0$$

is satisfied for all $\tau \in \Gamma$ and all $\eta \in \Lambda$.

Now,

$$\frac{d^2\left(x_0'P(\tau)x_0 + \Sigma\tau_i^{-1}d_i\right)}{d\epsilon^2}|_{\epsilon=0}= x_0'\frac{d^2P(\tau)}{d\epsilon^2}|_{\epsilon=0} x_0 + 2\Sigma_{i=1}^{k}\eta_i^2\tau_i^{-3}d_i,$$

and since $\eta_i^2\tau_i^{-3}d_i \geq 0$, the convexity of $x_0'P(\tau)x_0 + \Sigma\tau_i^{-1}d_i$ on Γ can be established if it can be shown that $\frac{d^2P(\tau)}{d\epsilon^2}|_{\epsilon=0}\geq 0$ for all $\tau \in \Gamma$ and all $\eta \in \Lambda$.

Now, $P(\tau)$ satisfies (4.5) which may be pre and post multiplied by Π, where $\Pi = P(\tau)^{-1}$ giving

$$0 = \Pi\bar{A}' + \bar{A}\Pi + \bar{X} + \Pi\bar{Q}\Pi, \tag{4.18}$$

differentiating (4.18) twice with respect to ϵ gives

$$\begin{aligned}-\frac{d^2\Pi}{d\epsilon^2}|_{\epsilon=0}(\bar{A}+\Pi\bar{Q})' - (\bar{A}+\Pi\bar{Q})\frac{d^2\Pi}{d\epsilon^2}|_{\epsilon=0} &= 2\acute{\Pi}\acute{A}' + 2\acute{A}\acute{\Pi} + \Pi\frac{d^2}{d\epsilon^2}\bar{A}|_{\epsilon=0} + \frac{d^2}{d\epsilon^2}\bar{X}|_{\epsilon=0}\\ &\frac{d^2}{d\epsilon^2}\bar{A}|_{\epsilon=0}\Pi + \frac{d^2}{d\epsilon^2}\bar{X}|_{\epsilon=0} + 2\acute{\Pi}\acute{Q}\Pi + 2\Pi\acute{Q}\acute{\Pi} + 2\acute{\Pi}\bar{Q}\acute{\Pi} + \Pi\frac{d^2}{d\epsilon^2}\bar{Q}|_{\epsilon=0}\Pi,\end{aligned} \tag{4.19}$$

where, ‘ $(\acute{\cdot})$ ’ denotes $\frac{d}{d\epsilon}(\cdot)|_{\epsilon=0}$, and

$$\begin{aligned}\bar{A} &= A - B_2(D_{12_p}'D_{12_p})^{-1}\Sigma_{i=1}^{k}D_{12_i}'T^{-1}C_{1_i},\\ \bar{Q} &= C_{1_p}'C_{1_p} + \Sigma_{i=1}^{k}C_{1_i}'T^{-1}C_{1_i},\\ \acute{A} &= B_2(D_{12_p}'D_{12_p})^{-1}\Sigma_{i=1}^{k}D_{12_i}'T^{-1}\eta_i \times\\ &\quad T^{-1}C_{1_i},\\ \frac{d^2}{d\epsilon^2}\bar{A}|_{\epsilon=0} &= 2B_2(D_{12_p}'D_{12_p})^{-1}\Sigma_{i=1}^{k}D_{12_i}'T^{-1}\eta_iT^{-1}\eta_iT^{-1}C_{1_i},\\ \acute{Q} &= \Sigma_{i=1}^{k}C_{1_i}'T^{-1}\eta_iT^{-1}C_{1_i},\end{aligned}$$

$$\begin{aligned}\frac{d^2}{d\epsilon^2}\bar{Q}\mid_{\epsilon=0} &= 2\Sigma_{i=1}^{k}C_{1_i}'T^{-1}\eta_i T^{-1}\eta_i T^{-1}C_{1_i},\\ \bar{X} &= B_1\tau_0 B_1' - B_2\bar{R}(\tau)^{-1}B_2',\\ \frac{d^2}{d\epsilon^2}\bar{X}\mid_{\epsilon=0} &= B_2(D_{12_p}'D_{12_p})^{-1}\Sigma_{i=1}^{k}\left(D_{12_i}'T^{-1}\eta_i T^{-1}\eta_i T^{-1}D_{12_i}\right)(D_{12_p}'D_{12_p})^{-1}B_2'.\end{aligned}$$

where

$$T = \Sigma_{j=1}^{k}\tau_j I_{p_j} + D_{12_j}(D_{12_p}'D_{12_p})^{-1}D_{12_j}'$$

Note that alternative but equivalent expressions to those used in the proof of Theorem 4.3.1, for $\bar{A}$, $\bar{Q}$, $\acute{\bar{A}}$ and $\acute{\bar{Q}}$ are used here. $\bar{X}$ is identical but included for completeness. Factorising (4.19) gives

$$\begin{aligned}&-\frac{d^2\Pi}{d\epsilon^2}\mid_{\epsilon=0}(\bar{A}+\Pi\bar{Q})' - (\bar{A}+\Pi\bar{Q})\frac{d^2\Pi}{d\epsilon^2}\mid_{\epsilon=0} = 2\left(\sum_{i=1}^{k}C_{1_i}\acute{\Pi} + \eta_i T^{-1}(C_{1_i}\Pi + D_{12_i}\times\right.\\ &\left.(D_{12_p}'D_{12_p})^{-1}D_{12_i}')'T^{-1}\left(\sum_{i=1}^{k}C_{1_i}\acute{\Pi} + \eta_i T^{-1}\left(C_{1_i}\Pi + D_{12_i}(D_{12_p}'D_{12_p})^{-1}D_{12_i}'\right)\right)\right) +\\ &B_2(D_{12_p}'D_{12_p})^{-1}\Sigma_{i=1}^{k}\left(D_{12_i}'T^{-1}\eta_i T^{-1}\eta_i T^{-1}D_{12_i}\right)(D_{12_p}'D_{12_p})^{-1}B_2' \qquad (4.20)\end{aligned}$$

Recognise that the right hand side of (4.20) is positive semi definite and that $-\bar{A}-\Pi\bar{Q}$ is equal to $-\tilde{A}$, (4.15), with $\beta = 1$. Recall that $-\tilde{A}$ is stable, so, by a Lyapunov argument, $-\frac{d^2\Pi}{d\epsilon^2}\mid_{\epsilon=0}\geq 0$, so $\frac{d^2\Pi}{d\epsilon^2}\mid_{\epsilon=0}\leq 0$. Since $\Pi = P^{-1}$ then $\frac{d^2\Pi}{d\epsilon^2}\mid_{\epsilon=0} = 2\Pi\acute{P}\Pi\acute{P}\Pi - \Pi\frac{d^2P}{d\epsilon^2}\mid_{\epsilon=0}\Pi$ and, therefore, $\Pi\frac{d^2P}{d\epsilon^2}\mid_{\epsilon=0}\Pi = 2\Pi\acute{P}\Pi\acute{P}\Pi - \frac{d^2\Pi}{d\epsilon^2}\mid_{\epsilon=0}$. Since $\Pi > 0$ and $\frac{d^2\Pi}{d\epsilon^2}\mid_{\epsilon=0}\leq 0$ then $\frac{d^2P}{d\epsilon^2}\mid_{\epsilon=0}\geq 0$. This completes the proof of the Lemma. □

Now, for the proof of the Theorem, 2 Cases will be considered separately

Case 1. Γ is non-empty

Case 2. Γ is empty.

Case 1:

Begin with an overview of the proof for Case 1. Firstly, a 'completing the squares' argument is used to develop an expression for the cost (4.2) in terms of the solution to the ARE (4.3). Then a lower bound for $\inf_{u(t)}\sup_{w(t)\in\mathcal{W}_{mx}} J(u(t), w(t))$ is found and an upper bound for $\sup_{w(t)\in\mathcal{W}_{mx}}\inf_{u(t)} J(u(t), w(t))$ is developed. Finally, employing a standard inequality from game theory, the Theorem is proven for Case 1.

Begin with the ARE (4.3). Pre and post multiplying (4.3) by $x'(t)$ and $x(t)$ respectively, and integrating from $t = 0$ to $t = \infty$ gives

$$\begin{aligned}0 = \int_0^\infty x'(t)\Big(&(A - B_2\bar{R}(\tau)^{-1}D_{12}'\tau^{-1}C_1)'P + P(A - B_2\bar{R}(\tau)^{-1}D_{12}'\tau^{-1}C_1) +\\ &P(B_1\tau_0B_1' - B_2\bar{R}(\tau)^{-1}B_2')P + C_1'\tau^{-1}C_1 - C_1'\tau^{-1}D_{12}\bar{R}(\tau)^{-1}D_{12}'\tau^{-1}C_1\Big)x(t)\,dt \quad (4.21)\end{aligned}$$

Using 'completing the squares' manipulations similar to those used for the $1 \Rightarrow 2$ part of the proof of Theorem 3.3.1 gives a factorisation of $J(u(t), w(t))$. Completing the squares and noting that, for stabilising $u(t)$, $\int_0^\infty \frac{d}{dt}(x(t)'P(\tau)x(t))\, dt = -x_0'P(\tau)x_0$ gives

$$\begin{aligned} J(u(t), w(t)) &= x_0'P(\tau)x_0 + \|u(t) - \hat{u}_{mx}(\tau)\|^2_{\bar{R}(\tau)} - \|w(t) - \hat{w}_{mx}(\tau)\|^2_{\tau^{-1}} \\ &\quad - \left(\|z(t)\|^2_{\tau^{-1}} - \|w(t)\|^2_{\tau^{-1}}\right) \end{aligned} \quad (4.22)$$

where the right hand side may be evaluated for any $\tau \in \Gamma$, all of which yield the same value.

Now, given any stabilising control signal $u(t)$, consider

$$\tau = \check{\tau} : \|\hat{w}_{mx_i}(\check{\tau})\|^2 = \|z_i(t)\|^2 + d_i, \forall i \quad (4.23)$$

So (4.22) becomes

$$\begin{aligned} J(u(t), w(t)) &= x_0'P(\check{\tau})x_0 + \Sigma_{i=1}^k \check{\tau}_i^{-1} d_i + \\ &\quad \|u(t) - \hat{u}_{mx}(\check{\tau})\|^2_{\bar{R}(\check{\tau})} - \|w(t) - \hat{w}_{mx}(\check{\tau})\|^2_{\check{\tau}^{-1}} \end{aligned} \quad (4.24)$$

Now, a lower bound for the minimax cost will be developed;

$$\inf_{u(t)} \sup_{w(t) \in \mathcal{W}_{mx}} J \geq \inf_{u(t)} J(u(t), \hat{w}_{mx}(\check{\tau})) \quad (4.25)$$

Using (4.24)

$$\inf_{u(t)} J(u(t), \hat{w}_{mx}(\check{\tau})) = \inf_{u(t)} \left(x_0'P(\check{\tau})x_0 + \Sigma_{i=1}^k \check{\tau}_i^{-1} d_i + \|u(t) - \hat{u}_{mx}(\check{\tau})(t)\|^2_{\bar{R}(\check{\tau})}\right) \quad (4.26)$$

Since $\check{\tau}$ is a function of $u(t)$ then $x_0'P(\check{\tau})x_0 + \Sigma_{i=1}^k \check{\tau}_i^{-1} d_i$ is also a function of $u(t)$ so

$$\begin{aligned} \inf_{u(t)} \left(x_0'P(\check{\tau})x_0 + \|u(t) - \hat{u}_{mx}(\check{\tau})\|^2_{\bar{R}(\check{\tau})} + \Sigma_{i=1}^k \check{\tau}_i^{-1} d_i\right) &\geq \inf_{u(t)} \left(x_0'P(\check{\tau})x_0 + \Sigma_{i=1}^k \check{\tau}_i^{-1} d_i\right) \\ &\quad + \inf_{u(t)} \|u(t) - \hat{u}_{mx}(\check{\tau})(t)\|^2_{\bar{R}(\check{\tau})} \\ &= \inf_{u(t)} \left(x_0'P(\check{\tau})x_0 + \Sigma_{i=1}^k \check{\tau}_i^{-1} d_i\right) \\ &= x_0'P(\hat{\tau})x_0 + \Sigma_{i=1}^k \hat{\tau}_i d_i \end{aligned} \quad (4.27)$$

From (4.25), (4.26) and (4.27) it follows that

$$\inf_{u(t)} \sup_{w(t) \in \mathcal{W}_{mx}} J \geq x_0'P(\hat{\tau})x_0 + \Sigma_{i=1}^k \hat{\tau}_i d_i \quad (4.28)$$

Now, an upper bound for $\sup_{w(t) \in \mathcal{W}_{mx}} \inf_{u(t)} J(u(t), w(t))$ will be found. Firstly, it will be shown that $\check{\tau}(\hat{u}_{mx}, \hat{w}_{mx}) = \hat{\tau}$. Consider a perturbation of the scaling matrix $\tau + \epsilon\eta$ where η

in Λ defines an aribitrary direction in τ space with Λ defined by (3.24). Differentiating the ARE (4.5) with respect to ϵ at $\epsilon = 0$ gives

$$0 = \acute{\bar{A}}' P(\tau) + \bar{A}'\acute{P} + P(\tau)\acute{\bar{A}} + \acute{P}\bar{A} + \acute{P}\bar{X}P(\tau) + P(\tau)\acute{\bar{X}}P(\tau) + P(\tau)\bar{X}\acute{P} + \acute{\bar{Q}} \tag{4.29}$$

where '$(\acute{\cdot})$' denotes $\frac{d}{d\epsilon}(\cdot) \mid_{\epsilon=0}$ and

$$\begin{aligned}
\acute{\bar{A}} &= -B_2'\bar{R}(\tau)^{-1}D_{12}'\tau^{-1}\eta(\tau^{-1} + \tau^{-1}D_{12}\bar{R}(\tau)^{-1}D_{12}'\tau^{-1})C_1, \\
\acute{\bar{X}} &= B_1\eta B_1' - B_2\bar{R}(\tau)^{-1}D_{12}'\tau^{-1}\eta\tau^{-1}D_{12}\bar{R}(\tau)^{-1}B_2', \\
\acute{\bar{Q}} &= -C_1'(\tau^{-1} - \tau^{-1}D_{12}\bar{R}(\tau)^{-1}D_{12}'\tau^{-1})\eta(\tau^{-1} - \tau^{-1}D_{12}\bar{R}(\tau)^{-1}D_{12}'\tau^{-1})C_1.
\end{aligned}$$

Since $\hat{\tau}$ is the infimising value of τ then $\hat{\tau}$ is a stationary point for the right hand side of (4.9), so a zero gradient condition for $\tau = \hat{\tau}$ is satisfied,

$$0 = x_0'\acute{P}(\hat{\tau})x_0 - \Sigma_{i=1}^{k}\hat{\tau}_i^{-2}\eta d_i \tag{4.30}$$

Using the fact that $u(t)$ is assumed to be stabilising, (4.30) may be written as

$$0 = -\int_0^\infty \frac{d}{dt}\left(x'\acute{P}(\hat{\tau})x\right)\,dt - \Sigma_{i=1}^{k}\hat{\tau}_i^{-2}\eta d_i \tag{4.31}$$

Now, consider $u(t) = \hat{u}_{mx}(\hat{\tau})$ and $w(t) = \hat{w}_{mx}(\hat{\tau})$. Thus, from (2.1), $\dot{x} = (\bar{A}(\hat{\tau}) + \bar{X}(\hat{\tau})P(\hat{\tau}))x$ so (4.31) becomes

$$0 = -\int_0^\infty x'\left((\bar{A}(\hat{\tau}) + \bar{X}(\hat{\tau})P(\hat{\tau}))\,\acute{P}(\hat{\tau}) + \acute{P}(\hat{\tau})\,(\bar{A}(\hat{\tau}) + \bar{X}(\hat{\tau})P(\hat{\tau}))\right)x\,dt - \Sigma_{i=1}^{k}\hat{\tau}_i^{-2}\eta d_i \tag{4.32}$$

Substituting from the ARE (4.29) gives

$$0 = \int_0^\infty x'\left(\acute{\bar{A}}'P(\hat{\tau}) + P(\hat{\tau})\acute{\bar{A}} + P(\hat{\tau})\acute{\bar{X}}P + \acute{\bar{Q}}\right)x\,dt - \Sigma_{i=1}^{k}\hat{\tau}_i^{-2}\eta d_i \tag{4.33}$$

which may be written as

$$0 = \|z(\hat{u}_{mx}(\hat{\tau}))\|^2_{\hat{\tau}^{-1}\eta\hat{\tau}^{-1}} - \|\hat{w}_{mx}(\hat{\tau})\|^2_{\hat{\tau}^{-1}\eta\hat{\tau}^{-1}} + \Sigma_{i=1}^{k}\hat{\tau}_i^{-2}\eta_i d_i \tag{4.34}$$

which is the definition of $\check{\tau}(\hat{u}_{mx}, \hat{w}_{mx})$, (4.23), for all $\eta \in \Lambda$. Therefore $\check{\tau}(\hat{u}_{mx}, \hat{w}_{mx}) = \hat{\tau}$, the infimising value of τ.

Now, an upper bound for $\sup_{w(t)\in\mathcal{W}_{mx}} \inf_{u(t)} J(u(t), w(t))$ will be developed using (4.24).

$$\begin{aligned}
\sup_{w(t)\in\mathcal{W}_{mx}} \inf_{u(t)} J(u(t), w(t)) &\le \sup_{w(t)\in\mathcal{W}_{mx}} J(\hat{u}_{mx}(\hat{\tau}), w(t)) \\
&= \sup_{w(t)\in\mathcal{W}_{mx}} \Big(x_0'P(\hat{\tau})x_0 - \|w(t) - \hat{w}_{mx}(\hat{\tau})\|^2_{\hat{\tau}^{-1}} - \\
&\quad \left(\|z(\hat{u}_{mx}(\hat{\tau}))\|^2_{\hat{\tau}^{-1}} - \|w(t)\|^2_{\hat{\tau}^{-1}}\right)\Big)
\end{aligned} \tag{4.35}$$

Since it is required that $w(t) \in \mathcal{W}_{mx}$, the supremum is acheived when $w(t) = \hat{w}(\hat{\tau})$

$$\sup_{w(t)\in\mathcal{W}_{mx}} x_0'P(\hat{\tau})x_0 - \|w(t) - \hat{w}_{mx}(\hat{\tau})(t)\|^2_{\hat{\tau}^{-1}} - \left(\|z(\hat{u}_{mx}(\hat{\tau})\|^2_{\hat{\tau}^{-1}} - \|w(t)\|^2_{\hat{\tau}^{-1}}\right)$$
$$= x_0'P(\hat{\tau})x_0 + \Sigma_{i=1}^k \hat{\tau}_i^{-1} d_i \quad (4.36)$$

So, from (4.35) and (4.36)

$$\sup_{w(t)\in\mathcal{W}_{mx}} \inf_{u(t)} J(u(t), w(t)) \leq \sup_{w(t)\in\mathcal{W}_{mx}} J(\hat{u}_{mx}(\hat{\tau}), w(t)) = x_0'P(\hat{\tau})x_0 + \Sigma_{i=1}^k \hat{\tau}_i^{-1} d_i \quad (4.37)$$

A standard result from game theory (see, for example, [Basar and Bernhard, 1995]) states that

$$\sup_{w(t)\in\mathcal{W}_{mx}} \inf_{u(t)} J(u(t), w(t)) \leq \inf_{u(t)} \sup_{w(t)\in\mathcal{W}_{mx}} J(u(t), w(t)) \quad (4.38)$$

Substituting (4.28) and (4.37) into (4.38), gives

$$x_0'P(\hat{\tau})x_0 + \Sigma_{i=1}^k \hat{\tau}_i^{-1} d_i \leq \inf_{u(t)} \sup_{w(t)\in\mathcal{W}_{mx}} J(u(t), w(t)) \geq x_0'P(\hat{\tau})x_0 + \Sigma_{i=1}^k \hat{\tau}_i^{-1} d_i \quad (4.39)$$

Therefore,

$$\inf_{u(t)} \sup_{w(t)\in\mathcal{W}_{mx}} J(u(t), w(t)) = x_0'P(\hat{\tau})x_0 + \Sigma_{i=1}^k \hat{\tau}_i^{-1} d_i \quad (4.40)$$

as required. This completes the proof of the Theorem for case 1.

Case 2:

The modified problem that will be considered is a system where the effect of the signal w is scaled-down by a factor of $\beta \in \mathbf{R} > 0$ giving a system representation

$$\dot{x} = Ax + \beta^{-1}B_1 w + B_2 u \quad (4.41)$$

where it is still required that $w(t) \in \mathcal{W}_{mx}$ (4.1) and the cost (4.2) for this system will be denoted $J^\beta(u(t), w(t))$. The ARE corresponding to this modified problem differs from (4.3) only through the scaling of the 'B_1' term,

$$\begin{aligned} 0 &= (A - B_2\bar{R}(\tau)^{-1}D_{12}'\tau^{-1}C_1)'P_\beta + P_\beta(A - B_2\bar{R}(\tau)^{-1}D_{12}'\tau^{-1}C_1) + \\ &\quad P_\beta(\beta^{-2}B_1\tau B_1' - B_2\bar{R}(\tau)^{-1}B_2')P_\beta + C_1'\tau^{-1}C_1 - C_1'\tau^{-1}D_{12}\bar{R}(\tau)^{-1}D_{12}'\tau^{-1}C_1 \end{aligned} \quad (4.42)$$

Definition 4.3.1 *Given β, the set of scaling matrices $\tau > 0$ for which there exists a stabilising solution P_β to (4.42) will be denoted Γ^β.*

Now, it will be proven that, if Γ is empty then there does not exist a controller which will guarantee a finite cost for all $w(t) \in \mathcal{W}_{mx}$. Firstly an ordering on β is established; if there exists $\tau \in \Gamma^{\beta_2}$ then $\tau \in \Gamma^{\beta_1}$, for $\beta_1 \geq \beta_2$. Secondly, it is shown that, in the limit as $\beta \to \infty$, an LQR ARE is recovered so $\lim_{\beta\to\infty} \Gamma^\beta$ is non-empty. Since, by assumption for Case 2, $\Gamma^\beta \mid_{\beta=1}$ is empty then it follows that there exists $\beta_{min} > 1$ such that, $\Gamma^{\beta_{min}}$ is not empty but, for all $\beta < \beta_{min}$, Γ^β is empty. It is then shown that there exist $w \in \mathcal{W}_{mx}$ and $x(0)$ such that $\lim_{\beta\to\beta_{min}} J^\beta(u(t), w(t), x(0)) = \infty$ for all $u(t)$ which leads to the non-existence of a controller such that $J^\beta(u(t), w(t), x(0)) \mid_{\beta=1}$ is finite for all $w(t) \in \mathcal{W}_{mx}$ and $x(0)$.

Begin with the following Lemma which performs a similar role to Lemma 3.3.1

Lemma 4.3.3 *Consider $\beta_1 \geq \beta_2$. If there exists $\tau \in \Gamma^{\beta_2}$ then $\tau \in \Gamma^{\beta_1}$.*

Proof: The proof follows the same line of argument as the proof of Lemma 3.3.3 so will not be repeated. □

Recognise that, in the limit as β approaches infinity, (4.42) approaches an LQR ARE for which a positive stabilising solution is guaranteed since the system (2.1) was assumed to be stabilisable (see, for example, [Anderson and Moore, 1989]). Since $\lim_{\beta\to\infty} \Gamma^\beta$ is not empty, $\Gamma^\beta \mid_{\beta=1}$ is empty by assumption for Case 2 and Lemma 4.3.3 gives an ordering, there exists β_{min} such that $\Gamma^{\beta_{min}}$ is not empty but Γ^β is empty for all $\beta < \beta_{min}$ where $1 < \beta_{min} < \infty$. Furthermore, it follows from Lemma 4.3.1 that $\tau \in \lim_{\beta\to\beta_{min}} \Gamma^\beta$ is such that the parameters τ_i approach the origin for all i. Therefore, since $d_1 > 0$, $\lim_{\beta\to\beta_{min}} \tau_1^{-1} d_1 = \infty$, so, from (4.40),

$$\begin{aligned} \lim_{\beta\to\beta_{min}} \inf_{u(t)} \sup_{w(t)\in\mathcal{W}_{mx}} J^\beta(u(t), w(t)) &= \lim_{\beta\to\beta_{min}} \left(x_0' P(\hat{\tau}) x_0 + \Sigma_{i=1}^k \hat{\tau}_i^{-1} d_i \right) \\ &= \infty \end{aligned} \tag{4.43}$$

For β approaching $\beta_{min} > 1$, the disturbance/uncertainty signal $w(t)$ is bounded by the same class $\mathcal{W}_{mx}$ (4.1) but has less effect on the system(4.41) than for $\beta = 1$. Therefore,

$$\inf_{u(t)} \sup_{w(t)\in\mathcal{W}_{mx}} J^\beta(u(t), w(t)) \mid_{\beta=1} \geq \lim_{\beta\to\beta_{min}} \inf_{u(t)} \sup_{w(t)\in\mathcal{W}_{mx}} J^{\beta_{min}}(u(t), w(t)) = \infty \tag{4.44}$$

as required for the proof for Case 2. This completes the proof of the Theorem. □

4.4 Discussion

The minimax controller (4.10), described in [Savkin and Petersen, 1995], is a non-dynamic time invariant controller. The design of a minimax controller requires the minimisation of the right hand side of (4.9) which was shown to be a convex problem on Γ in Lemma 4.3.2 so any local minimum is a global minimum. To start the search a $\tau \in \Gamma$ must be located. Lemma 4.3.1 indicates that if such a τ exists then it may be found arbitrarily close to the origin, however, it is still necessary to consider the relative size of the parameters since not all positions close to the origin in parameter space are admissible. If the infimising value of τ is not found then the resulting controller, although not minimax, is a guaranteed cost controller.

The remainder of the discussion considers various ways of applying the minimax controller (4.10). The minimax controller may be applied once at time zero or redesigned continuously on the receding horizon.

The saddle point solution for the game between the controller and the uncertainty/ disturbance is initial condition dependent so the controller gain matrix also depends on the initial condition. Often the initial condition may not be known a priori, however, full state information is available to the controller so, at time zero, the initial state becomes known and the optimal controller may be designed. This minimax controller differs from H_∞ type controllers of, for example, Chapter 3 because the minimax controller gain matrix depends on the initial state. Discrete events, such as making a step change in reference may be regarded as incurring an initial condition on the plant at time zero and, in cases such as these, the minimax controller may be applicable.

Often the time that is chosen as zero is arbitrary and not necessarily the beginning of a significant transient effect. In this case, the H_∞ with transients results of Chapter 3 may be a more appropriate formulation. However, there is another possibility: a receding horizon implementation of the minimax state feedback controller also results in a controller that does not depend on the choice of the initial time. To apply this strategy the current state is regarded as the initial condition and the minimax controller (4.10) is recalculated at every instant in time (or at least frequently relative to the dynamics of the closed loop plant). The implementation of the minimax controller on the receding horizon is computationally intensive and there are few formal results regarding its performance. However, the author believes this control strategy will offer good 'near future' performance since the optimisation of the controller gain at each instant of time is a function of the (initial) state at that time. The compromise between focusing on regulating the transient effect caused by the current state (near future performance) and rejecting the worst possible future L_2 norm bounded disturbance is codified by the relative sizes of the d_i terms of (4.1) and the current state. Also note that, as the state approaches zero the controller will approach the optimal state feedback H_∞ controller which has infinite gain and is, therefore, impractical.

Rather than letting the initial condition approach zero to recover the optimal solution to the state feedback structured H_∞ problem of [Savkin and Petersen, 1996], their results may be recovered by allowing the d_1 term of the IQC (4.1) to approach infinity such that the L_2 norm bound for the plant disturbance approaches infinity. This makes the relative effect of the initial condition negligible so the minimax controller is independent of the initial condition. For this special case of the minimax problem, the optimal value of τ_1 is the minimal value of γ^{-2} for the equivalent structured H_∞ problem.

4.5 Illustrative example

This section presents a design example of a minimax controller for a given initial condition to regulate a system with integral quadratic constraint uncertainty description. The benefit of explicitly using knowledge of the structure of the uncertainty in the design is apparent.

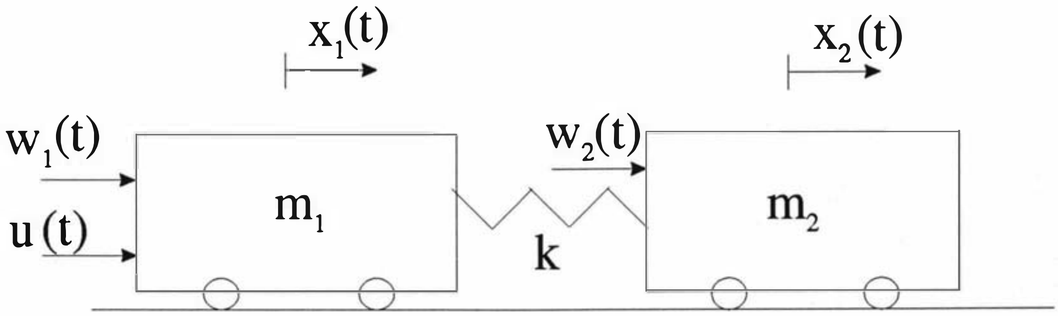

Figure 4.2

Figure 4.2 depicts a system similar to the benchmark problem proposed by [Wie and Berstein, 1992]. The control force u may be used to regulate the system from some perturbed initial condition, x_0. A spring of stiffness $k = 100\,Nm^{-1}$ connects two containers with uncertain masses m_1 and m_2. The system uncertainty may be represented by two channels of perturbation feedback, w_1, w_2 each satisfying an integral quadratic constraint with $d_1 = d_2 \equiv 0$. The nominal values of the masses are $m_{1nom} = 1\,kg$ and $m_{2nom} = .8\,kg$. The uncertainty class permits constant, static discrepancies in mass such that $0.59 < m_1 < 3.3$ and $0.57 < m_2 < 1.56$, however the integral quadratic constraint does not put any finite limits on the instantaneous values of the masses or their rates of change. Such dynamic uncertainty in the masses may represent, for example, the presence of liquid in the containers.

The system may be described in the form (2.1), (2.2) with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_{1nom}} & 0 & \frac{k}{m_{1nom}} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_{2nom}} & 0 & -\frac{k}{m_{2nom}} & 0 \end{pmatrix}, \quad B_2 = \left(0, \frac{1}{m_{1nom}}, 0, 0\right)',$$

$$B_1 = \begin{pmatrix} 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.56 \end{pmatrix}', \quad C_1 = \begin{pmatrix} -k & 0 & k & 0 \\ k & 0 & -k & 0 \end{pmatrix}, \quad D_{12} = (1, 0)'.$$

The uncertainty structure is such that the scaling matrix τ is of the form

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau_1 & 0 \\ 0 & 0 & \tau_2 \end{pmatrix},$$

and the system performance is measured with a quadratic cost functional (4.2) with $D_{12_p}' D_{12_p} = 1$ and $C_{1_p}' C_{1_p} = 10^3 I_4$, thus, all states and the control input are costed.

Consider the initial condition, $x_0 = 10^{-2}(1,0,1,0)'$; i.e. both carts are displaced 10 mm with zero initial speed. A minimax controller is given by (4.10) where the infimising τ is determined by a convex search to minimise $x_0'P(\tau)x_0 + \sum_{i=1}^{k} \tau_i^{-1} d_i$ for this initial condition. A plot of $x_0'P(\tau)x_0 + \sum_{i=1}^{k} \tau_i^{-1} d_i$ versus τ_1 and τ_2 is shown in Figure 4.3, the minimum occurs at

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.44 & 0 \\ 0 & 0 & 0.024 \end{pmatrix},$$

where $x_0'P(\tau)x_0 + \sum_{i=1}^{k} \tau_i^{-1} d_i = 0.253$. The resulting controller gain matrix is

$$(-616, \ -75.5, \ 558, \ 4.07)$$

If the structure for the uncertainty was ignored then τ would be constrained to be of the form $\tau = \tau_1 I_3$. Clearly the optimal value of τ is not of this form and a conservative solution would result with a higher cost.

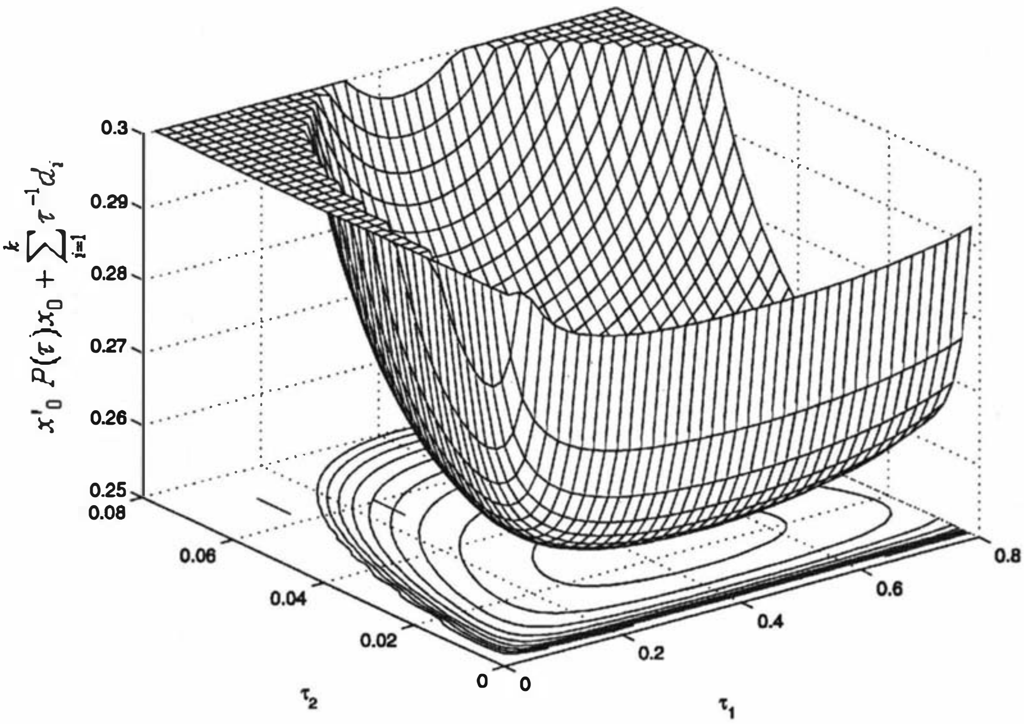

Figure 4.3

4.6 The Measurement feedback Minimax problem

In this section, a measurement feedback extension to the state feedback minimax problem of this Chapter is proposed. Unfortunately, the problem appears to be intractable. However, a similar problem is described which has been successfully tackled by [Savkin and Petersen, 1997]. The solution to their problem has some of the desirable 'near future' performance or robustness to transients properties of the state feedback minimax problem.

The minimax problem presented in Section 4.2 is a problem that generally cannot be accommodated within the H_∞ with transients and structure framework. An important difference between the minimax problem of Section 4.2 and the H_∞ with transients and structure problem is that a minimax controller gain is a function of the given non-zero initial condition. To extend this problem to measurement feedback, the following problem is posed:

Consider the system (2.1), (2.2), (2.3) with uncertainty description (4.1) and cost function (4.2). Suppose the initial condition is in an ellipsoid centred at x_0;

$$\chi_0 = \{x(0) : (x(0) - x_0)' X_0 (x(0) - x_0) \leq 1\} \tag{4.45}$$

The minimax, causal, measurement feedback controller $u(y(\tau) \mid_{\tau=[0,t)})$ will be sought. So, the minimax cost is

$$\inf_{u(y)} \sup_{w(t)\in\mathcal{W},\, x(0)\in\chi_0} J(u, w, x(0)) \tag{4.46}$$

Unfortunately, this problem appears to be intractable. This is perhaps not surprising since the location of the saddle point solution for the state feedback minimax problem is initial condition dependent. However, some work has been done on similar problems. [Savkin and Petersen, 1997] presented a necessary and sufficient condition for the existence of a guaranteed cost controller for the cost requirement

$$\int_0^\infty x'Qx + u'Ru\, dt \leq x(0)'Px(0) \tag{4.47}$$

where $R > 0$ and $Q > 0$, for all initial conditions $x(0) \neq 0$ and for $w(t)$ satisfying the single IQC.

$$\|w(t)\|^2 \leq \|z(t)\|^2 + x(0)'Dx(0) \tag{4.48}$$

where $D > 0$. Note that the left hand side of (4.47) is equivalent to $\|z_p\|^2$ where $C_{1_p}'C_{1_p} = Q$ and $D_{12_p}'D_{12_p} = R$. The standard simplifying assumptions similar to those made for the H_∞ with transients and structure are also made by [Savkin and Petersen, 1997]; $C_1'D_{12} = 0$, $D_{21}D_{21}' > 0$ $[A\ B_1]$ stabilisable, $[A\ B_2]$ stabilisable, $[A\ C_2]$ detectable and $B_1D_{12}' = 0$. Furthermore, [Savkin and Petersen, 1997] presented a guaranteed cost controller for the problem in terms of an ARE and an RDE scaled by a single scalar scaling parameter τ.

This minimax problem of [Savkin and Petersen, 1997] has some of the 'near future' performance properties of the state feedback solution discussed in Section 4.4. The matrices P (4.47) and D (4.48) work to restrict the gain of the controller and the intial filter gain, providing robustness to non zero initial conditions. For a more detailed discussion of this problem see [Savkin and Petersen, 1997].

The special case of this problem where the matrix D approaches zero in the IQC (4.48) causes the IQC to collapse to an L_2 induced norm bound. Therefore, this special case is equivalent to the H_∞ with transients and structure problem of Chapter 3 with one uncertainty channel. Comparing the two problems, the cost bound matrix P of (4.47) is $\gamma^2 R$ in (3.2) where $w_p = 0$. To avoid confusion, it should be noted the scaling parameter, τ, used by [Savkin and Petersen, 1997] is the inverse of τ_1 (3.4) of the H_∞ with transients and structure problem of Chapter 3.

Chapter 5

Set Valued Estimation and model invalidation

5.1 Introduction

In this Chapter, a set valued estimation problem is considered. The set valued estimator produces the current set of possible states consistent with a history of measurements, a set of possible initial conditions and an uncertain plant model. It is possible to apply the set valued estimator as a model invalidator; if the set of states consistent with the uncertain model, a set of possible initial conditions and the time history of measurements is empty then the model is invalidated.

Model invalidation for uncertain systems where the uncertainty is described by L_2 induced norm bounds or IQC's is not possible. The L_2 induced norm bound and the IQC that are used to describe the uncertainty in the H_∞ with transients and structure methods of Chapter 3 and the minimax methods of Chapter 4 are infinite horizon bounds. Therefore, uncertain models described by these bounds can not be invalidated within a finite time. However, if a finite time IQC that is required to be satisfied for all time is used to describe the uncertainty then set valued estimation and model invalidation are possible.

The class of uncertainty/disturbance/measurement noise considered in this Chapter is constrained by finite time IQC's which must be satisfied at every instant of time. Multiple channels of uncertainty/disturbance/measurement noise are allowed and the initial condition is constrained to lie in an ellipsoid, of arbitrary dimension, not necessarily centred at the origin. The set of current states consistent with the plant model, uncertainty bounds, set of possible initial conditions and measurements may be calculated on line. The results presented in this Chapter are an extension of [Bertsekas and Rhodes, 1971] and [Savkin and Petersen, 1996a] to allow structured uncertainty.

Uncertainty that is described by a finite time IQC that must be satisfied for all time is a subset of uncertainty described by infinite time IQC's as used in Chapter 4. Therefore, if a plant had uncertainty that could be well represented by the finite time IQC's that are used in this Chapter then the H_∞ with transients and structure methods of Chapter 3 and the minimax methods of Chapter 4 may be used to (conservatively) design a controller for such a plant.

5.2 Preliminaries

Consider linear systems of the form (2.1), (2.2), (2.3). Since only closed loop systems are being considered $D_{12} = 0$ and $B_2 = 0$, and the system representation simplifies to

$$\dot{x}(t) = Ax(t) + B_1 w(t) \tag{5.1}$$
$$z(t) = C_1 x(t) \tag{5.2}$$
$$y(t) = C_2 x(t) + D_{21} w(t) \tag{5.3}$$

It is assumed, for simplicity and with some loss of generality, that the process disturbance and measurement noise signals are distinct, i.e. $B_1 D_{21}' = 0$. This assumption may be relaxed with some additional complexity of the results. The uncertainty input signal $w(t)$ is bounded by finite time integral quadratic constraints which must be satisfied for all time,

$$\mathcal{W}_{SV} = \left\{ w(t) : \int_0^s w_i(t)' w_i(t)\, dt \leq \int_0^s z_i(t)' z_i(t)\, dt + d_i\, \forall i \in (1, 2 \ldots k)\, \forall s > 0 \right\} \tag{5.4}$$

where $d_i \geq 0$. To utilise a channel to describe a process disturbance or measurement noise, the appropriate C_{1_i} may be set to zero and to utilise a channel to describe an uncertainty input, the appropriate d_i may be set to zero.

The initial state is constrained to lie in the ellipsoid

$$\chi_0 = \left\{ x(0) : (x(0) - x_0)' X_0 (x(0) - x_0) \leq d_0 \right\} \tag{5.5}$$

where $X_0 \in R^{n \times n} > 0$ and $d_0 > 0$.

The set valued state estimation problem is to find the set, denoted χ_s, of possible states, $x(s)$, at any time $s > 0$ given measurements $y(t)$ for $0 \leq t \leq s$ and $x(0) \in \chi_0$.

5.3 Main results

The following Theorem extends the result of [Savkin and Petersen, 1995a] to allow for structured uncertainty.

Theorem 5.3.1 *$x(s) \in \chi_s\, \forall w \in \mathcal{W}_{SV}$ where*

$$\begin{aligned} \chi_s = & \left\{ x(s) : (x(s) - \hat{x}(s))' X(s) (x(s) - \hat{x}(s)) \leq d_0 + \sum_{i=1}^{k} \tau_i^{-1} d_i + \right. \\ & \left. \int_0^s \hat{x}(t)' C_1' \tau^{-1} C_1 \hat{x}(t) - (y(t) - C_2 \hat{x}(t))' D_{21} \tau^{-1} D_{21}' (y(t) - C_2 \hat{x}(t))\, dt\ \forall \tau > 0 \right\} \end{aligned} \tag{5.6}$$

where $X(s)$ is the solution to the Riccati differential equation

$$0 = \dot{X}(t) + A'X(t) + X(t)A + X(t)B_1\tau B_1'X(t) + C_1'\tau^{-1}C_1 - C_2'D_{21}\tau^{-1}D_{21}'C_2 \tag{5.7}$$

at time s with $X(0) = X_0$ and, $\hat{x}(t)$ is given by the filter

$$\dot{\hat{x}}(t) = \left(A + X(t)^{-1}\left(C_1'\tau^{-1}C_1 - C_2'D_{21}\tau^{-1}D_{21}'C_2\right)\right)\hat{x}(t) + X(t)^{-1}C_2'D_{21}\tau^{-1}D_{21}'y(t) \tag{5.8}$$

with initial condition $\hat{x}(0) = x_0$ and where

$$\tau = \begin{pmatrix} \tau_1 I_{p_1} & 0 & & \\ 0 & \tau_2 I_{p_2} & 0 & \\ & 0 & \ddots & 0 \\ & & 0 & \tau_k I_{p_k} \end{pmatrix}, \tag{5.9}$$

with $\tau_i \in R > 0\,\forall i \in (1, 2 \ldots k)$.

Notice that the scaling matrix (5.9) differs from those used for the H_∞ with transients and structure problem (3.4) and the minimax problem (4.4) since, for the set valued estimation problem, there is no performance channel.
Proof: The constructive proof for the Theorem begins with the following Lemma.

Lemma 5.3.1 *The following statements are equivalent*

1. $x(0) \in \chi_0$ *and* $w \in \mathcal{W}_{SV}$
2. $(x(0)-x_0)'X_0(x(0)-x_0) \leq d_0 + \sum_{i=1}^{k}\tau_i^{-1}d_i + \|z(t)\|^2_{[0,s)\tau^{-1}} - \|w(t)\|^2_{[0,s)\tau^{-1}}$ *for all* $\tau > 0$.

Proof: $1 \Rightarrow 2$:
Multiply (5.4) by any $\tau_i^{-1} > 0$ for $i = 1, 2 \ldots k$, add these inequalities to (5.5) which gives

$$(x(0) - x_0)'X_0(x(0) - x_0) \leq d_0 + \sum_{i=1}^{k}\tau_i^{-1}d_i + \|z(t)\|^2_{[0,s)\tau^{-1}} - \|w(t)\|^2_{[0,s)\tau^{-1}}, \forall\tau > 0 \tag{5.10}$$

which is statement 2 of the Lemma.

$2 \Rightarrow 1$: Using a contradiction two cases will be considered. Firstly, suppose (5.4) is violated for some $i = v \in \{1, 2, \ldots, k\}$ and consider the limit as τ_v tends to zero. Since (5.10) must be satisfied for all $\tau > 0$,

$$\begin{aligned}(x(0) - x_0)'X_0(x(0) - x_0) &\leq \lim_{\tau_v \to 0}\left(d_0 + \sum_{i=1}^{k}\tau_i^{-1}d_i + \|z(t)\|^2_{[0,s)\tau^{-1}} - \|w(t)\|^2_{[0,s)\tau^{-1}}\right) \\ &= \lim_{\tau_v \to 0}\tau_v^{-1}\left(\|z_v(t)\|^2_{[0,s)} - \|w_v(t)\|^2_{[0,s)} + d_v\right) \end{aligned} \tag{5.11}$$

Since (5.4) is violated for $i = v$, the right hand side of (5.11) is negative. The left hand side is clearly positive semi-definite which is a contradiction.

Secondly, suppose (5.5) is violated, a similar argument is used. Taking the limit as all of the scaling parameters τ_i tend to ∞ gives

$$\begin{aligned}(x(0)-x_0)'X_0(x(0)-x_0) &\leq \lim_{\tau_i \to \infty \, \forall i} d_0 + \sum_{i=1}^{k} \tau_i^{-1} d_i + \int_0^s z(t)'\tau^{-1}z(t) - w(t)'\tau^{-1}w(t)\,dt \\ &= d_0 \end{aligned} \tag{5.12}$$

which contradicts the violation of (5.5). Thus if Statement 2 is true then Statement 1 is true. This completes the proof of the Lemma. □

Now returning to the proof of the Theorem, it will be shown that $x(s) \in \chi_s$. Given $x(0) \in \chi_0$ and $w(t) \in \mathcal{W}_{SV}$, then, from Lemma 5.3.1, inequality (5.10) holds. Let the filter be of the form

$$\dot{\hat{x}} = A\hat{x} + f(\hat{x}, y) \tag{5.13}$$

with $\hat{x}(0) = x_0$. Starting from

$$\begin{aligned}&(x(s)-\hat{x}(s))'\,X(s)\,(x(s)-\hat{x}(s)) - (x(0)-x_0)\,X_0\,(x(0)-x_0) = \\ &\int_0^s \frac{d}{dt}(x(t)-\hat{x}(t))'\,X(t)\,(x(t)-\hat{x}(t))\,dt\end{aligned} \tag{5.14}$$

and making substitutions from the inequality (5.10), the filter equation (5.13), and the state equation (5.1) leads to

$$\begin{aligned}&(x(s)-\hat{x}(s))'X(s)(x(s)-\hat{x}(s)) \leq d_0 + \sum_{i=1}^{k}\tau_i^{-1}d_i + \int_0^s z(t)'\tau^{-1}z(t) - w(t)'\tau^{-1}w(t) + \\ &\quad (A(x(t)-\hat{x}(t)) + B_1 w(t) - f)'\,X(t)(x(t)-\hat{x}(t)) + (x(t)-\hat{x}(t))'X(t)\,(A(x(t)-\hat{x}(t)) \\ &\quad + B_1 w(t) - f) + (x(t)-\hat{x}(t))'\dot{X}(t)(x(t)-\hat{x}(t))\,dt\end{aligned} \tag{5.15}$$

Rearranging gives

$$\begin{aligned}&(x(s)-\hat{x}(s))'X(s)(x(s)-\hat{x}(s)) \leq d_0 + \sum_{i=1}^{k}\tau_i^{-1}d_i + \int_0^s (x-\hat{x})'\left(\dot{X}(t) + A'X(t)\right. \\ &\quad \left. + X(t)A + C_1'\tau^{-1}C_1 + X(t)B_1\tau B_1'X(t)\right)(x(t)-\hat{x}(t)) + \hat{x}(t)C_1'\tau^{-1}C_1\hat{x}(t) - \\ &\quad \hat{x}(t)'C_1'\tau^{-1}C_1(x(t)-\hat{x}(t)) - (x(t)-\hat{x}(t))'C_1'\tau^{-1}C_1\hat{x}(t) - \\ &\quad (w(t) - \tau B_1'X(t)(x(t)-\hat{x}(t)))'\,\tau^{-1}\,(w(t) - \tau B_1'X(t)(x(t)-\hat{x}(t))) \\ &\quad - f'X(t)(x(t)-\hat{x}(t)) - (x(t)-\hat{x}(t))'X(t)f\,dt\end{aligned} \tag{5.16}$$

Writing the term

$$\int_0^s (w(t) - \tau B_1' X(t)(x(t) - \hat{x}(t)))' \tau^{-1} (w(t) - \tau B_1' X(t)(x(t) - \hat{x}(t)))\, dt = T(s) +$$
$$\int_0^s (w(t) - \tau B_1' X(t)(x(t) - \hat{x}(t)))' D_{21}' D_{21} \tau^{-1} D_{21}' D_{21} (w(t) - \tau B_1' X(t)(x(t) - \hat{x}(t)))\, dt$$

where

$$T(s) \triangleq \|w(t) - \tau B_1' X(t)(x(t) - \hat{x}(t))\|^2_{[0,s)(\tau^{-1} - D_{21}' D_{21} \tau^{-1} D_{21}' D_{21})}$$

which is positive semi-definite, gives

$$\begin{aligned}
(x(s) - \hat{x}(s))' X(s)(x(s) - \hat{x}(s)) \le d_0 + \sum_{i=1}^{k} \tau_i^{-1} d_i + \int_0^s (x(t) - \hat{x}(t))' \Big(\dot{X}(t) + A' X(t) \\
+ X(t) A + C_1' \tau^{-1} C_1 + X(t) B_1 \tau B_1' X(t)\Big) (x(t) - \hat{x}(t)) + \hat{x}(t) C_1' \tau^{-1} C_1 \hat{x}(t) - \\
\hat{x}(t)' C_1' \tau^{-1} C_1 (x(t) - \hat{x}(t)) - (x(t) - \hat{x}(t))' C_1' \tau^{-1} C_1 \hat{x}(t) - \\
(w(t) - \tau B_1' X(t)(x(t) - \hat{x}(t)))' D_{21}' D_{21} \tau^{-1} D_{21}' D_{21} (w(t) - \tau B_1' X(t)(x(t) - \hat{x}(t))) \\
- f' X(t)(x(t) - \hat{x}(t)) - (x(t) - \hat{x}(t))' X(t) f\, dt - T(s). \qquad (5.17)
\end{aligned}$$

Using (5.3) gives $y(t) = C_2 x(t) + D_{21} w(t) = C_2(x(t) - \hat{x}(t)) + C_2 \hat{x}(t) + D_{21} w(t)$ which may be substituted into (5.17) to give

$$\begin{aligned}
(x(s) - \hat{x}(s))' X(s)(x(s) - \hat{x}(s)) \le d_0 + \sum_{i=1}^{k} \tau_i^{-1} d_i + \int_0^s (x(t) - \hat{x}(t))' \Big(\dot{X}(t) + A' X(t) \\
+ X(t) A + C_1' \tau^{-1} C_1 + X(t) B_1 \tau B_1' X(t)\Big) (x(t) - \hat{x}(t)) + \hat{x}(t) C_1' \tau^{-1} C_1 \hat{x}(t) \\
- \hat{x}(t)' C_1' \tau^{-1} C_1 (x(t) - \hat{x}(t)) - (x(t) - \hat{x}(t))' C_1' \tau^{-1} C_1 \hat{x}(t) \\
- (y(t) - C_2 \hat{x}(t) - C_2(x(t) - \hat{x}(t)))' D_{21} \tau^{-1} D_{21}' (y(t) - C_2 \hat{x}(t) - C_2(x(t) - \hat{x}(t))) \\
- f' X(t)(x(t) - \hat{x}(t)) - (x(t) - \hat{x}(t))' X(t) f\, dt - T(s). \qquad (5.18)
\end{aligned}$$

Now let $f(y(t), \hat{x}(t)) = X(t)^{-1} (C_2' \tau^{-1}(y(t) - C_2 \hat{x}(t)) - C_1' \tau^{-1} C_1 \hat{x}(t))$, so (5.18) becomes

$$\begin{aligned}
(x(s) - \hat{x}(s))' X(s)(x(s) - \hat{x}(s)) \le d_0 + \sum_{i=1}^{k} \tau_i^{-1} d_i + \int_0^s (x(t) - \hat{x}(t))' \Big(\dot{X}(t) + A' X(t) \\
+ X(t) A + C_1' \tau^{-1} C_1 + X(t) B_1 \tau B_1' X(t)\Big) (x(t) - \hat{x}(t)) + \hat{x}(t) C_1' \tau^{-1} C_1 \hat{x}(t) \\
- (y(t) - C_2 \hat{x}(t))' D_{21} \tau^{-1} D_{21}' (y(t) - C_2 \hat{x}(t))\, dt - T(s) \qquad (5.19)
\end{aligned}$$

Notice that the Riccati differential expression in (5.7) appears in (5.19). Substituting for (5.7) gives

$$(x(s)-\hat{x}(s))'X(s)(x(s)-\hat{x}(s)) \leq d_0 + \sum_{i=1}^{k} \tau_i^{-1} d_i + \int_0^s \hat{x}(t)'C_1'\tau^{-1}C_1\hat{x}(t) -$$
$$(y(t)-C_2\hat{x}(t))' D_{21}\tau^{-1}D_{21}' (y(t)-C_2\hat{x}(t))\, dt - T(s)\ \forall \tau > 0 \qquad (5.20)$$

and since, $T(s)$ is positive semi-definite then $x(s) \in \chi_s$. This completes the proof of the Theorem. □

5.4 Discussion and model invalidation using the set valued estimator

Model invalidation over a finite time interval is not possible for systems where the uncertainty description is measured over an infinite time interval. The H_∞ with transients and structure results of Chapter 3 and the minimax problem of Chapter 4 have uncertainty descriptions which are defined by infinite time integrals so, unfortunately, set valued estimation and model invalidation are not possible for these systems. However, if the infinite time IQC used in Chapter 4 is further restricted to a finite time IQC (5.4) which must be satisfied for all time then set valued estimation and model invalidation are possible.

The derivation of a set valued estimator has been presented. Given a linear nominal plant model, an uncertainty description, the set of possible initial states and the history of the measurement signal, the set of possible values for the current system state may be determined. This set, denoted χ_s, is the intersection of the continuum of sets created from the continuum of positive definite scaling matrices τ. In practice, a superset of χ_s can be determined by the intersection of a finite selection of sets corresponding to a selection of scaling matrices.

When $X(s)$ is positive definite, the set, χ_s, is an ellipsoid in state space. The set χ_s is characterised by three features: its centre $\hat{x}(s)$, its orientation $X(s)$ and its size determined by the right hand side of (5.6). The right hand side of (5.6) is dependent on the history of the measurement signal and provides a method of model invalidation. If the measurement $y(t)$, for $0 \leq t \leq s$, is such that the right hand side of (5.6) is negative then the set of states consistent with the uncertain plant model, the set of initial conditions and the measurement signal is the empty set so the model is invalidated. To fully test for model invalidation, an exhaustive search of the space of scaling matrices is required. In practice, it may be possible to test a model with acceptable reliability using a finite (small) selection of scaling matrices.

The set valued estimator may be used for control systems conservatively designed by the H_∞ with transients methods of Chapter 3. If the IQC uncertainty description (5.4) is further restricted such that the d_i terms are zero for $i = 1 \ldots k$ then the uncertainty description is a subset of the L_2 induced norm bound used in the H_∞ with transients and structure problem of Chapter 3. Therefore, the results of Chapter 3 may be used to (conservatively) design a controller for a system with such an uncertainty description.

Chapter 6

Conclusions

Three problems were considered in this Thesis, firstly, an H_∞ with transients and structured uncertainty controller synthesis problem was presented in Chapter 3 and also in [Milliken et al., 1999a]. Chapter 4 considered a state feedback minimax controller synthesis problem that has been presented in [Milliken et al., 1999]. Chapter 5 detailed a set valued estimator which may be used for model invalidation. The results of Chapter 5 also appeared in [Milliken and Marsh, 1998].

The solution to the H_∞ with transients and structured uncertianty problem provides a powerful and flexible framework for the design of state feedback and measurement feedback controllers. The uncertainty is permitted to be structured and is described by an arbitrary but finite number of L_2 induced norm bounds. The plant is perturbed by a disturbance signal in L_2 and a, possibly non zero, initial condition.

For state feedback, necessary and sufficient conditions for the existence of a controller that provides robust performance were given in terms of the solution to an ARE dependent on an arbitrary number of constant scalar scaling parameters. Such a controller is a static controller. For optimal γ the solution is a static state feedback controller with a finite gain matrix which differs from the optimal state feedback solution for the non-transients case where the controller gain has an infinite maximum singular value. For state feedback, the optimal controller may be designed by the solution to a convex optimisation problem.

For measurement feedback, necessary and sufficient conditions for the existence of a controller that provides robust performance were given in terms of the solutions to an ARE and an RDE, both of which depend on an arbitrary number of constant scalar scaling parameters. One such controller has been presented and is a dynamic controller with a time varying filter gain matrix. Unfortunately, for optimal γ, the maximum singular value of the filter gain matrix approaches infinity at some time giving an impractical solution. The optimal controller may be designed by the solution to an optimisation that is not generally convex. If γ is sufficiently suboptimal then a solution with a finite constant filter gain matrix may be found.

The H_∞ with transients and structure problem treats transients by imposing a cost on any non zero initial condition. As the penalty on the initial condition approaches infinity the H_∞ with transients and structure problem approaches the non transients problem.

If the initial condition on the state, for a robust controller design problem, is given as problem data rather than being a player in the game against the controller, as was the case for the H_∞ with transients and structure problem, then a minimax problem may be solved. For the state feedback minimax problem considered in Chapter 4, the plant uncertainty/disturbances were bounded by an arbitrary number of IQC's and the performance is measured by a quadratic cost function. The minimax solution to this problem is a static controller which generally has finite gain. The solution involves the solution of an ARE dependent on an aribitrary finite number of scaling matrices. The optimisation has been shown to be convex and the search may be begun arbitrarily close to the origin if a solution exists.

The state feedback H_∞ with structure problem for optimal γ may be recovered by allowing one of the bounds on the disturbances to approach infinity so that the effect of the initial condition becomes irrelevant.

The minimax controller gain matrix depends on the initial condition and, therefore, depends on the time that is chosen to be zero. To remove the dependence of the controller on the choice of zero time, the minimax controller may be applied on the receding horizon, which involves redesigning the controller for the current (initial) state at each instant of time. The author suspects that this implementation will focus on the rejection of transient behaviour more than the H_∞ with structure problem and will, therefore, provide better performance for the 'near future'.

It appears that, generally, extensions of this minimax problem to measurement feedback are intractable, however, [Savkin and Petersen, 1997] presented an extension to measurement feedback which provides robustness to intial conditions. A special case of this problem was shown to be a special case of the H_∞ with transients and structure problem.

A set valued estimator was considered which, given an ellipsoidal set of possible initial conditions, a nominal linear plant model and an uncertainty description produces the set of current states compatible with the uncertain model and the history of measurements. The uncertainty class is described by an arbitrary finite number of finite time IQC's which must be satisfied at each instant of time. The uncertainty described by this class is a subset of that described by infinite time IQC's. Also, the set valued estimator may be used as a model invalidator; if the set of states compatible with the uncertain model, the set of possible initial conditions and a history of measurements is the empty set then the model is invalidated.

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