Legendre Foliations on Contact Metric Manifolds

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ABSTRACT

This thesis develops the theory of Legendre foliations on contact manifolds by associating a contact metric structure with a contact manifold and investigating Legendre foliations on the resultant contact metric manifold. The contact metric structure introduces a metric for the Legendre foliation which enables us to study the curvature properties of a Legendre foliation, furthermore when this metric is bundle-like we have a semi-Riemannian foliation hence we can define a semi-Riemannian Legendre foliation and study its properties.

We use the invariant Π as defined by Pang to define a family of contact metric structures for a non-degenerate Legendre foliation and from this family we pick out a unique contact metric structure the canonical contact metric structure. Furthermore a canonical contact metric structure is identified for a flat Legendre foliation and shown to be a Sasakian structure.

Under some circumstances a Legendre foliation on a contact metric manifold has a second Legendre foliation, the conjugate Legendre foliation, associated with it. We investigate the conditions for the existence and the properties of the conjugate Legendre foliation.

By using a definition similar to that of a Legendre foliation on a contact metric manifold we conclude this thesis by defining a complex Legendre foliation on a complex contact metric manifold and beginning an investigation of its properties.

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CHAPTER 1. INTRODUCTION

1 (a). INTRODUCTION

A contact manifold as introduced by J. W. Gray [9] is an odd dimensional differential manifold which carries a contact structure. This contact structure is closely related to pseudogroups of contact transformations. The second volume of Sophus Lie's Theorie der transformation gruppen, 1890, is devoted to the local study of contact transformations in general and to the study of global contact transformations in the special case of euclidean space. The study of contact manifolds is a generalization of Lie's results to odd dimensional spaces.

Since the early 1960's the theory of contact manifolds has been developed by many authors, notably S. Sasaki for whom Sasakian manifolds are named, S. Tanno and D. E. Blair.

Foliation theory has its origins in the study of the solution curves of systems of ordinary differential equations. C. Ehresmann and G. Reeb in the 1940's defined the general notion of a foliation. Since then both the geometric theory and the differential geometry of foliations has been widely developed. An important class of foliations are the Riemannian foliations introduced by B.Reinhart [24] in 1959. Riemannian foliations have a metric for which the leaves are locally a constant distance apart.

A Legendre foliation as introduced by Pang [2] is a particular foliation of a contact manifold. So the study of Legendre foliations combines the theory of contact manifolds with foliation theory.

The purpose of this thesis is to investigate the properties of Legendre foliations on contact metric manifolds. Both, M. Y. Pang in his paper "The structure of Legendre foliations" and P. Libermann in her paper "Legendre foliations on contact manifolds", investigated the properties of Legendre foliations on contact manifolds, without a contact metric structure. We extend this work by associating a contact

metric structure with a contact manifold, and investigating the properties of Legendre foliations on the resultant contact metric manifold.

One of the main results of M. Y. Pang's paper is that any Legendre foliation is locally equivalent to a particular Legendre foliation on a unit cotangent bundle. To classify Legendre foliations he introduced two invariants, the fundamental quadratic form $\prod$ which is a symmetric 2-form on the tangent bundle of the foliation and the invariant G, a symmetric 3-form on the tangent bundle of the foliation. When $\prod$ is non-degenerate we have a non-degenerate Legendre foliation in which case Pang defined a canonical set of coframes and a unique connection. Furthermore when $\prod$ is positive definite and $G \equiv 0$ a Riemannian Legendre foliation is defined.

In her paper Libermann used Jacobi structure methods to recover many of Pang's results. She also introduces a semi-Riemannian metric but does not actually define a contact metric structure.

We show that the leaves of a Legendre foliation are anti-invariant submanifolds of a contact metric manifold. Formulae for $\prod$ and G are given in terms of the contact metric structure, although $\prod$ and G do not depend on the contact metric structure on the contact manifold. This form of $\prod$ enables classification of a Legendre foliation in terms of the Lie bracket $[X, \xi]$ where X is a tangential vector field of the foliation.

For a contact manifold there exist many contact metric structures. Among the possible contact metric structures on a contact manifold which carries a flat Legendre foliation, we identify a canonical contact metric structure and show it to be a Sasakian structure.

Furthermore when $\prod$ is non-degenerate, we have a non-degenerate Legendre foliation and the semi-Riemannian metric introduced by Libermann can be used to define a canonical contact metric structure that is related to the canonical coframe given by Pang. We show that this canonical contact metric structure is unique. It belongs to a family of contact metric structures defined by $\prod$, which we call the canonical family of contact metric structures. The unique connection given by Pang is related to the Levi-Civita connection of the canonical contact metric structure.

To enable the canonical contact metric structure to be defined in a natural way we have widened the definition of a contact metric structure to include cases where the metric g is not positive definite. A canonical contact metric structure for a non-degenerate Legendre foliation can be defined where g is positive definite and used to obtain similar results for Legendre foliations on the class of contact metric manifolds where g is positive definite. But the notation needed is complicated so we study Legendre foliations on the wider class of contact metric manifolds.

For a non-degenerate Legendre foliation on a unit cotangent bundle we obtain expressions for contact metric structures from the canonical family in terms of the local coordinates on the cotangent bundle.

Under some circumstances a Legendre foliation has a second Legendre foliation associated with it, which we call the conjugate Legendre foliation. We investigate the conditions for its existence which are related to the nature of the contact metric structure as well as to the classification of the Legendre foliation. We show that the conjugate of a flat Legendre foliation on the canonical Sasakian manifold exists and is also flat. Conditions for the existence of a conjugate Legendre foliation of a non-degenerate Legendre foliation on a manifold from the canonical family of contact metric manifolds are found. Furthermore, if a Legendre foliation of a K-contact manifold has a conjugate Legendre foliation then both Legendre foliations have the same classification.

We show that a Riemannian Legendre foliation, as defined by Pang, is a Riemannian foliation on any manifold from the canonical family of contact metric manifolds. An alternative definition of a Riemannian Legendre foliation in terms of a bundle-like contact metric structure is given. This definition of Riemannian Legendre foliations is then extended to semi-Riemannian Legendre foliations.

A complex contact manifold is a complex manifold of odd complex dimension. Using a definition similar to that of a Legendre foliation of a contact manifold, we define a complex Legendre foliation on a complex contact manifold. We show that the invariant Π can be defined naturally, but can not be used to define a complex contact metric structure. We begin an investigation of the properties of complex Legendre foliations, especially where they differ from Legendre foliations.

Throughout this work examples of Legendre foliations on contact metric manifolds are given and referred to. These are used to illustrate the properties of Legendre foliations that are being explored. These examples are produced by techniques developed in this work.

The first chapter is a general introduction to this work, giving the basic definitions and properties of contact manifolds and of foliations. Legendre foliations are introduced in the second chapter along with the invariants Π and G. Flat Legendre foliations in are studied in chapter three.

Chapter four investigates non-degenerate Legendre foliations. The canonical family of contact metric structures and the canonical contact metric structure are defined and their properties investigated. The conjugate Legendre foliation is defined in chapter five, and conditions for its existence found.

The introduction of a contact metric structure enables a semi-Riemannian Legendre foliation to be defined in terms of a bundle-like contact metric structure and its properties are studied in chapter six. Also a family of semi-Riemannian Legendre foliations whose conjugates exist and are flat is given.

In chapter seven Legendre foliations on K-contact and Sasakian manifolds are investigated. By investigating the second fundamental form of a Legendre foliation we also study totally umbilic, totally geodesic, isoparametric and harmonic Legendre foliations.

An extension of this work to complex Legendre foliations on complex contact manifolds is expounded in chapter eight.

1 (b). NOTATION

Throughout this work the following notation is used:

i) The summation convention is used, where a repeated index indicates a summation,
The Latin indices i, j, ... range from 1 to n+1 and the Greek indices α, β,... range from 1 to n.

ii) If $B \to M$ is a vector bundle, then ΓB denotes the set of all smooth sections of B.

iii) If B is a vector subbundle of TM then p_B is the natural projection onto B

$$p_B : TM \to B.$$

iv) The local coordinates (x_i) on a (n+1)-dimensional manifold $\tilde{M}$ induce local coordinates (x_i, y_i) on the cotangent bundle $T^*\tilde{M}$. If F is a function on $T^*\tilde{M}$, then $F_i = \frac{\partial F}{\partial y_i}$, $F_{ij} = \frac{\partial^2 F}{\partial y_i \partial y_j}$, $F_{ij} = \frac{\partial^2 F}{\partial y_i \partial y_j}$ and so on.

v) [X, Y] denotes the Lie bracket of the vector fields X and Y.

vi) $\mathcal{L}_X$ denotes the Lie derivative with respect to the vector field X while i(X) denotes the interior product with respect to the vector field X.

The exterior derivative and the interior product are defined with the p+1 factor included for example if ω is a 2-form on a manifold M then for any $X, Y, Z \in \Gamma TM$ we have

$$d\omega(X, Y, Z) = \frac{1}{3}\{X(\omega(Y, Z)) + Y(\omega(Z, X)) + Z(\omega(X, Y)) - \omega([X,Y], Z) - \omega([Y,Z], X) - \omega([Z,X], Y)\}$$

and

$$\{i(X)\omega\}(Y) = 2\omega(X, Y).$$

Hence

$$\mathcal{L}_X \omega = i(X)d\omega + d(i(X)\omega).$$

This is in line with the work of Blair [3], [4], [5], [6] and Yano and Kon [16] but not with that of Pang [2] and Libermann [1].

1 (c). CONTACT MANIFOLDS

The following is a brief review of contact manifolds, to introduce the properties and notation used in this work.

A (2n+1)-dimensional manifold M, is a *contact manifold* if it carries a global 1-form η such that

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M, where the exponent denotes the nth exterior power.

η is a *contact form* of M and (M, η) is the corresponding contact manifold.

A contact manifold (M, η) has a *contact metric structure* (ϕ, ξ, η, g) where ϕ is a tensor of type (1,1), ξ is a global vector field and g is a semi-Riemannian metric such that

(1.1) $\eta(\xi) = 1$

(1.2) $\phi^2 = - I + \eta \otimes \xi$

(1.3) $d\eta(W, W') = g(W, \phi W')$ $\forall\ W, W' \in \Gamma TM.$

A contact metric structure (ϕ, ξ, η, g) usually has g positive definite but we do not make this restriction and study the larger class of contact manifolds. Duggal [7] also lifts this restriction.

The *characteristic vector field*, ξ, is a unique global vector field determined by η for the contact manifold (M, η).

Yano and Kon [16] have shown that any contact manifold (M, η) has a contact metric structure (ϕ, ξ, η, g). This contact metric structure is not unique. (M, η) is used to denote a contact manifold when the contact metric structure is not specified while (M, ϕ, ξ, η, g) is used to denote the contact manifold (M, η) with the contact metric structure (ϕ, ξ, η, g), (M, ϕ, ξ, η, g) is called a *contact metric manifold*.

From properties 1.1, 1.2 and 1.3, (see Yano and Kon [16] and Duggal [7]), it can be shown that,

(1.4) $\phi\xi = 0$

(1.5) $\eta\phi = 0$

(1.6) $\text{rank}\,\phi = 2n$

(1.7) $g(\xi, \xi) = \varepsilon$
where $\varepsilon = +1$ or -1 depending on whether ξ is spacelike or timelike

(1.8) $\eta(W) = \varepsilon g(W, \xi)$ $\quad \forall\, W \in \Gamma TM$

(1.9) $g(\phi W, \phi W') = g(W, W') - \varepsilon\eta(W)\eta(W')$ $\quad \forall\, W, W' \in \Gamma TM$

(1.10) $g(W, \phi W') + g(\phi W, W') = 0$ $\quad \forall\, W, W' \in \Gamma TM.$

The following lemma shows that given a contact metric structure with ξ timelike we can always find a contact metric structure such that ξ is spacelike, or vice versa.

Lemma 1.11
If (M, ϕ, ξ, η, g) is a contact metric manifold such that ξ is timelike then there exists a contact metric structure $(\phi, \xi, \eta, \tilde{g})$ such that ξ is spacelike.

Proof
Let

$$\tilde{g} = g + 2\eta \otimes \eta,$$

then

$$\tilde{g}(\xi, \xi) = g(\xi, \xi) + 2\eta(\xi)\eta(\xi) = 1.$$

So ξ is spacelike.

It can be easily shown that $(\phi, \xi, \eta, \tilde{g})$ is a contact metric structure. ¤

Lemma 1.11 shows that there is no loss of generality in assuming that $\varepsilon = +1$, which we do in the following work.

The *contact distribution*, H, of a contact manifold (M, η) is defined by

$$\begin{aligned} H &= \text{ann}(\eta) \\ &= \{W \in TM \mid \eta(W) = 0\} \end{aligned}$$

and a line bundle E is defined by

$$E = \text{span}(\xi).$$

Thus the tangent bundle of (M, η) has the natural orthogonal decomposition

$$TM = H \oplus E.$$

The following relationships will be used later in this work.

For any $U \in \Gamma H$ we have

$$\begin{aligned} g([U, \xi], \xi) &= \eta([U, \xi]) && \text{by 1.8} \\ &= -2d\eta(U, \xi) && \text{since } U \in \Gamma H \\ &= -2g(U, \phi\xi) && \text{by 1.3} \\ &= 0 && \text{by 1.4.} \end{aligned}$$

Thus we have

$$(1.12) \qquad [U, \xi] \in \Gamma H \qquad \forall\, U \in \Gamma H.$$

Similarly for any $U, U' \in \Gamma H$ we have

$$(1.13) \qquad g([U, \phi U'], \xi) = 2g(U, U').$$

Also for any $Z \in \Gamma E$ and $W \in \Gamma TM$ we have

$$\begin{aligned} (1.14) \qquad (i(Z)d\eta)(W) &= -2d\eta(W, Z) \\ &= -2g(W, \phi Z) && \text{by 1.3} \\ &= 0 && \text{by 1.4.} \end{aligned}$$

For a contact metric manifold (M, ϕ, ξ, η, g) where g is positive definite the structure group of the tangent bundle, TM, reduces to $U(n) \times 1$, Blair [3]. Similarly in the case where g is a semi-Riemannian metric the structure group of the tangent bundle of the contact metric manifold (M, ϕ, ξ, η, g), reduces to $U(p, q) \times 1$, where (p, q) is the signature of $g|_H$. Using this we get the following lemma, which shows that it is possible to build up a contact metric structure on a manifold from an open cover of contact submanifolds.

Lemma 1.15

Let $\{U_s\}$ $s \in I$ be an open cover of a contact manifold (M, η) such that for each $s \in I$, (ϕ_s, ξ, η, g_s) is a contact metric structure on U_s. Then the union of all such ϕ_s and the union of all such g_s, for $s \in I$, provide a contact metric structure $(\cup\phi_s, \xi, \eta, \cup g_s)$ on (M, η).

An important class of submanifolds of a contact metric manifold are the anti-invariant submanifolds which we now define.

Definition 1.16

Let (M, η) be a contact manifold with contact metric structure (ϕ, ξ, η, g). A submanifold $\bar{M}$ of M is *anti-invariant* if $\phi T_x(\bar{M}) \subset T_x(\bar{M})^{\perp}$ for all $x \in \bar{M}$.

Torsion Tensors on a Contact Manifold

For a contact metric manifold (M, ϕ, ξ, η, g) the *Nienhaus torsion* N_ϕ of ϕ is defined by

(1.17) $$N_\phi(W, W') = \phi^2[W, W'] + [\phi W, \phi W'] - \phi[\phi W, W'] - \phi[W, \phi W'] \qquad \forall\ W, W' \in \Gamma TM.$$

Now we define the following tensors

(1.18) (i) $$N^1(W, W') = N_\phi(W, W') + 2d\eta(W, W')\xi \qquad \forall\ W, W' \in \Gamma TM$$

(ii) $$N^2(W, W') = (\mathcal{L}_{\phi W}\eta)W' - (\mathcal{L}_{\phi W'}\eta)W \qquad \forall\ W, W' \in \Gamma TM$$

(iii) $$N^3(W) = (\mathcal{L}_\xi \phi)W \qquad \forall\ W \in \Gamma TM$$

(iv) $$N^4(W) = (\mathcal{L}_\xi \eta)W \qquad \forall\ W \in \Gamma TM.$$

The following lemmas, 1.19 and 1.20, are proved in Yano and Kon [16], chapter V, section 3.

Lemma 1.19
If $N^1 = 0$, then $N^2 = N^3 = N^4 = 0$.

A vector field W on M is a *Killing vector field* if and only if the Lie derivative

$$\mathcal{L}_W\, g = 0.$$

Lemma 1.20
Let (M, ϕ, ξ, η, g) be a contact metric manifold, then N^2 and N^4 vanish. Moreover, N^3 vanishes if and only if ξ is a killing vector field with respect to g.

These torsion tensors are used to define two important classes of contact metric manifolds, the K-contact manifolds and the Sasakian manifolds.

Definition 1.21

(i) A *K-contact manifold* is a contact metric manifold (M, ϕ, ξ, η, g) with $N^3 = 0$, that is ξ is a Killing vector field.

(ii) A *Sasakian manifold* is a contact metric manifold (M, ϕ, ξ, η, g) with $N^1 = 0$.

Thus a Sasakian manifold is also a K-contact manifold, by lemma 1.19.

As $\phi\xi = 0$ we have ϕ: $TM \to H$, furthermore using 1.2, 1.4 and 1.6 it can be shown that $\phi|_H$: $H \to H$ is an isomorphism with $(\phi|_H)^{-1} = -\phi|_H$. Therefore since $\phi^2|_H = -I$, $\phi|_H$ is an almost complex structure on H. Hence it follows that $\phi|_H$ will be a complex structure on H if and only if the projection,

$$p_H(N_\phi(U, U')) = 0 \qquad \forall\ U, U' \in \Gamma H.$$

In the following lemma and theorem, we will find an alternative condition for $\phi|_H$ to be a complex structure on H.

Lemma 1.22

For a contact metric manifold (M, ϕ, ξ, η, g) the projection

$$p_E(N^1(U, U')) = 0 \qquad \forall\ U, U' \in \Gamma H.$$

Proof

As E is a line bundle spanned by ξ, the projection onto E of any vector field W on M can be written as $p_E(W) = g(W, \xi)\xi$.

Furthermore for any $U, U' \in \Gamma H$ we have

$$\begin{aligned} g([U, U'], \xi) &= -2g(U, \phi U') && \text{by 1.13} \\ &= g([\phi U, \phi U'], \xi) && \text{by 1.10 and 1.13.} \end{aligned}$$

Which as $p_E(W) = g(W, \xi)\xi$ implies that

$$p_E([U, U']) = p_E([\phi U, \phi U']) \qquad \forall\ U, U' \in \Gamma H. \tag{1.23}$$

Hence by 1.17 and 1.18 (i) we have for any $U, U' \in \Gamma H$ that

$$\begin{aligned} p_E(N^1(U, U')) &= p_E(\phi^2[U, U'] + [\phi U, \phi U'] - \phi[\phi U, U'] \\ &\qquad - \phi[U, \phi U'] + 2d\eta(U, U')\xi) \\ &= p_E(-[U, U'] + [\phi U, \phi U'] - \phi[\phi U, U'] \\ &\qquad - \phi[U, \phi U']) && \text{by 1.2 and 1.8} \\ &= 0 && \text{by 1.5 and 1.23.} \end{aligned}$$

□

Theorem 1.24

For a contact manifold (M, η) with contact metric structure (ϕ, ξ, η, g), $\phi|_H$ is a complex structure on H if and only if

$$N^1(U, U') = 0 \qquad \forall\ U, U' \in \Gamma H.$$

Proof

From lemma 1.22 for any $U, U' \in \Gamma H$ we have

$$\begin{aligned} N^1(U, U') &= p_H(N^1(U, U')) \\ &= p_H(N_\phi(U, U')) \qquad \text{by 1.18 (i).} \end{aligned}$$

Hence $\phi|_H$ is a complex structure on H if and only if $N^1(U, U') = 0$. ¤

This implies that for a Sasakian manifold (M, ϕ, ξ, η, g), $\phi|_H$ is a complex structure on H.

Corollary 1.25

A K-contact manifold (M, ϕ, ξ, η, g) is a Sasakian manifold if and only if $\phi|_H$ is a complex structure on H.

Proof

As E is a line bundle, for any $Z \in \Gamma E$ there exists a function f such that $Z = f\xi$.

For any $Z = f\xi \in \Gamma E$ and $W \in \Gamma TM$ we have

$$\begin{aligned} N^1(Z, W) &= N_\phi(Z, W)) + 2d\eta(Z, W)\xi && \text{by 1.18 (i)} \\ &= \phi^2[Z, W] - \phi[Z, \phi W] && \text{by 1.4, 1.14 and 1.17} \\ &= -[Z, W] + \eta([Z, W])\xi - \phi[Z, \phi W] && \text{by 1.2} \\ &= f(-[\xi, W] + \eta([\xi, W])\xi - \phi[\xi, \phi W]) && \text{by 1.1 and 1.4} \\ &= -f\phi N^3 W && \text{by 1.2 and 1.18 (iii)} \\ &= 0 && \text{for a K-contact manifold by definition 1.21(i).} \end{aligned}$$

Hence we have

$N^1 \equiv 0$ if and only if $N^1(U, U') = 0$ $\forall\ U, U' \in \Gamma H$
if and only if $\phi|_H$ is a complex structure on H by theorem 1.24.

¤

Since $N^2 = 0$ on a contact metric manifold Lemma 3.4 of Yano and Kon [16], chapter V, section 3 gives the following:

Lemma 1.26
For a contact metric manifold (M, ϕ, ξ, η, g) and any $W, W', W'' \in \Gamma TM$, we have

$$2g((\nabla_W \phi)W', W'') = g(N^1(W', W''), \phi W) + 2d\eta(\phi W', W)\eta(W'') - 2d\eta(\phi W'', W)\eta(W')$$

where ∇ is the Levi-Civita connection of g.

By substituting $W = \xi$ in lemma 1.26 and using 1.4 and 1.14 we get

(1.27) $\nabla_\xi \phi = 0,$

and from this we get

(1.28) $\nabla_\xi \xi = 0$ and $\nabla_\xi U \in \Gamma H$ $\forall\ U \in \Gamma H.$

Also for any $U \in \Gamma H$ we have

(1.29) $\nabla_U \xi \in \Gamma H$ by 1.12 and 1.28.

The Tensor h on a Contact Manifold

For a contact metric manifold (M, ϕ, ξ, η, g) Blair [4] defines a tensor of type (1,1) by

(1.30) $$h = \frac{1}{2}(\mathcal{L}_{\xi}\, \phi).$$

So for any $W \in \Gamma TM$

$$hW = \frac{1}{2}([\xi, \phi W] - \phi[\xi, W]).$$

From definition 1.30 it follows that h has the following properties, see Blair [4],

(1.31) h and ϕh are symmetric operators

(1.32) $$\phi h + h\phi = 0$$

(1.33) $$\eta \circ h = 0$$

(1.34) $$h\xi = 0.$$

By definition

$$h = \frac{1}{2} N^3.$$

In the following work we will use h instead of N^3, in line with the work done by Blair. The following property of h, (which is proved by Blair [4],) is used later in this work.

(1.35) $$\nabla_W \xi = -\phi hW - \phi W \qquad \forall\, W \in \Gamma TM.$$

From 1.34, h has a maximum rank of 2n and by definition $h \equiv 0$ if and only if (M, ϕ, ξ, η, g) is a K-contact manifold. The following theorem obtained from Blair [4] gives a case where h has rank 2n.

Theorem 1.36

For a contact metric manifold (M, ϕ, ξ, η, g) if R the curvature tensor field satisfies

$$R(\xi, W)\xi = 0 \qquad \forall\, W \in \Gamma TM$$

then h has rank 2n and M admits three mutually orthogonal distributions defined by the eigenspaces of h: namely D(0), D(1) and D(-1).

That is

$$D(1) = \{W \in TM \mid hW = W\}, \qquad D(-1) = \{W \in TM \mid hW = -W\}$$

and

$$D(0) = \{W \in TM \mid hW = 0\} = E \qquad \text{by 1.34.}$$

It can be shown that D(1) and D(-1) both are of dimension n.

Olszak [12] proved the following relationship, which will be used in later work.

(1.37) $$(\nabla_{\phi W}\phi)\phi W' + (\nabla_{W}\phi)W' = 2g(W, W')\xi - \eta(W')(W + hW + \eta(W)\xi) \qquad \forall\, W, W' \in \Gamma TM.$$

From which we can obtain the projection

(1.38) $$p_H((\nabla_{\phi U}\phi)\phi U') = -\, p_H((\nabla_{U}\phi)U') \qquad \forall\, U, U' \in \Gamma H.$$

1 (d). FOLIATIONS

The following is a brief review of foliations to introduce the properties and notation used in this work.

A *foliation* $\mathcal{F}$ of a n-dimensional smooth manifold M is a partition of M into connected submanifolds of dimension p. These submanifolds are called the *leaves* of the foliation. The foliation is said to have dimension p and codimension q = n - p.

A foliation, $\mathcal{F}$, of a manifold (M, g) provides an orthogonal decomposition of the tangent bundle, TM, of M as

$$TM = L \oplus L^{\perp}$$

where L is the subbundle of TM consisting of all vectors tangent to the leaves of $\mathcal{F}$, L is called the *tangent bundle* of the foliation. $L^{\perp}$ the *transverse bundle* of the foliation is the subbundle of TM defined by

$$L^{\perp} = \{ Y \in TM \mid g(Y, X) = 0, \forall X \in L \}.$$

For a manifold (M, g) let L be a p-dimensional subbundle of TM. L is completely integrable if for any X, X' $\in$ L the Lie bracket [X, X'] $\in$ L. By the Frobenius theorem, Molino [22], L is completely integrable if and only if it is the tangent bundle of a p-dimensional foliation, $\mathcal{F}$, of (M, g). In which case the leaves of $\mathcal{F}$ are the maximal connected integral submanifolds of L.

Bott Partial Connection

For any foliation $\mathcal{F}$, the Bott partial connection $\nabla^{L^{\perp}}$ in the transverse bundle $L^{\perp}$ is defined by

$$\nabla^{L^{\perp}}_{X} V = p_{L^{\perp}}([X, V]) \qquad \forall X \in \Gamma L, \forall V \in \Gamma L^{\perp}.$$

The Bott partial connection $\nabla^{L^\perp}$ in $L^\perp$ defines a Bott partial connection $\nabla^{L^{\perp *}}$ in the dual space $L^{\perp *}$ by

$$(\nabla^{L^{\perp *}}_X \nu)V = X\nu(V) - \nu([X,V]) \qquad \forall\, X \in \Gamma L,\ \forall\, V \in \Gamma L^\perp,\ \forall\, \nu \in \Gamma L^{\perp *}$$
$$= 2d\nu(X,V)$$
$$= (\mathcal{L}_X \nu)V$$

where $\mathcal{L}_X$ is the Lie derivative.

Hence we have

$$\nabla^{L^{\perp *}}_X \nu = \mathcal{L}_X \nu = i(X)d\nu. \qquad \forall\, X \in \Gamma L,\ \forall\, \nu \in \Gamma L^{\perp *} \tag{1.39}$$

The Bott partial connection $\nabla^{L^{\perp *}}$ is flat since for any $X, X' \in \Gamma L$, and any $\nu \in \Gamma L^{\perp *}$ its curvature tensor satisfies

$$R^{L^{\perp *}}(X, X')\nu = [\mathcal{L}_X, \mathcal{L}_{X'}]\,\nu - \mathcal{L}_{[X,\,X']}\nu$$
$$= 0.$$

Second Fundamental Form and Mean Curvature

For a foliation on a semi-Riemannian manifold (M, g) with ∇ the Levi-Civita connection of g we have

$$\nabla_X X' = p_L(\nabla_X X') + B(X, X') \qquad \forall\, X, X' \in \Gamma L$$

and

$$\nabla_X V = - A_V X + p_{L^\perp}(\nabla_X V) \qquad \forall X \in \Gamma L, \forall V \in \Gamma L^\perp,$$

where

$$B(X, X') = p_{L^\perp}(\nabla_X X')$$

is the *second fundamental form* and

$$A_V X = - p_L(\nabla_X V)$$

is the *Weingarten map*.

A foliation on a semi-Riemannian manifold (M, g) is *totally geodesic* if its second fundamental form vanishes identically, that is

$$B \equiv 0 \text{ which is equivalent to } A \equiv 0.$$

The leaves of a totally geodesic foliation are totally geodesic submanifolds of (M, g).

We will use the following equivalences in later work, they are easily proved using the geodesic condition and the properties of ∇ the Levi-Civita connection of g, (see Tondeur [26]).

Theorem 1.40
Let $\mathcal{F}$ be a foliation on (M, g). Then the following conditions are equivalent:

i) $\mathcal{F}$ is totally geodesic, that is
$g(\nabla_X X', V) = 0 \qquad \forall X, X' \in \Gamma L, \forall V \in \Gamma L^\perp;$

ii) $\mathcal{L}_V g|_L = 0 \qquad \forall V \in \Gamma L^\perp;$

iii) $g(\nabla_X V, X') + g(X, \nabla_{X'} V) = 0 \qquad \forall X,\ X' \in \Gamma L, \forall V \in \Gamma L^{\perp}$;

iv) $g([V, X], X) = 0 \qquad \forall X \in \Gamma L$ such that $|X| = 1$ and $\forall V \in \Gamma L^{\perp}$.

The *mean curvature vector field*, τ, of a foliation, $\mathcal{F}$, on the manifold (M, g) is defined by O'Neill [32]

$$\tau = g(X_i, X_j)B(X_i, X_j)$$

where $\{X_i\}$, i = 1....p, is a local orthonormal frame of L with respect to g.

Similarly the *mean curvature form*, κ, is the one form defined by

$$\kappa(V) = g(\tau, V) \qquad \text{for } V \in \Gamma L^{\perp},$$

and

$$i(X)\kappa = 0 \qquad \text{for } X \in \Gamma L.$$

Therefore for the local orthonormal frame $\{X_i\}$ we have

$$\kappa(V) = g(X_i, X_j)g(B(X_i, X_j), V) \qquad \text{for } V \in \Gamma L^{\perp}.$$

A foliation on a semi-Riemannian manifold (M, g) is *harmonic* if the mean curvature form vanishes that is

$$\kappa \equiv 0 \text{ which is equivalent to } \tau \equiv 0.$$

The leaves of a harmonic foliation are minimal submanifolds of the manifold (M, g). A totally geodesic foliation is also a harmonic foliation.

Riemannian Foliations

For a foliation, $\mathcal{F}$, on a Riemannian manifold (M, g) the Riemannian metric g is *bundle-like* with respect to $\mathcal{F}$ if

$$\mathcal{L}_X g(V, V') = 0 \qquad \forall X \in \Gamma L, \forall V, V' \in \Gamma L^{\perp}.$$

A foliation, $\mathcal{F}$, on (M, g) is a *Riemannian foliation* if g is a bundle-like metric with respect to $\mathcal{F}$.

Locally the leaves of a Riemannian foliation are a constant distance from each other.

The following equivalences, which we will use later, are easily proved using the bundle-like condition and the properties of ∇ the Levi-Civita connection of g, (see Tondeur [26]).

Theorem 1.41
Let $\mathcal{F}$ be a foliation on the manifold (M, g). Then the following conditions are equivalent:

i) $\mathcal{F}$ is a Riemannian foliation that is g is bundle-like;

ii) $g(\nabla_V X, V') + g(V, \nabla_{V'} X) = 0 \qquad \forall X \in \Gamma L, \forall V, V' \in \Gamma L^{\perp}$;

iii) $g(\nabla_V V' + \nabla_{V'} V, X) = 0 \qquad \forall X \in \Gamma L, \forall V, V' \in \Gamma L^{\perp}$;

iv) $g([X, V], V) = 0 \qquad \forall V \in \Gamma L^{\perp}$ such that $|V| = 1$ and $\forall X \in \Gamma L$.

CHAPTER 2. LEGENDRE FOLIATIONS

In this chapter we define a Legendre foliation on a contact metric manifold and give its basic properties. We also investigate how the foliation structure interacts with the contact metric structure of the manifold.

For a Legendre foliation on a contact manifold (M, η) there exists a bilinear form Π, whose value does not depend on the contact metric structure on (M, η). We use Π to classify Legendre foliations and then the related invariants $\overline{\Pi}$ and G are introduced.

An important class of contact manifolds are the unit cotangent bundles. A Legendre foliation is defined on a unit cotangent bundle and an example is given.

2 (a). LEGENDRE FOLIATIONS

A *Legendre foliation*, $\mathcal{F}$, is a foliation of a (2n+1)-dimensional contact manifold (M, η) by n-dimensional integral submanifolds of η. Since the leaves of $\mathcal{F}$ are integral submanifolds of η, the tangent bundle, L, of $\mathcal{F}$ is a subbundle of the contact distribution, H, of (M, η). Let (ϕ, ξ, η, g) be a contact metric structure on (M, η), then the transverse bundle of $\mathcal{F}$ is defined by

$$L^{\perp} = \{Y \in TM \mid g(Y, X) = 0, \forall X \in L\}.$$

Define Q a subbundle of H by

$$Q = H \cap L^{\perp}.$$

Then the foliation gives the following natural orthogonal decomposition of the tangent bundle, TM, of (M, η)

$$TM = L \oplus Q \oplus E$$

where $H = L \oplus Q$ and $L^{\perp} = Q \oplus E$.

The leaves of a Legendre foliation $\mathcal{F}$ on a contact metric manifold (M, ϕ, ξ, η, g) are anti-invariant submanifolds of (M, η) with respect to ϕ, as the following theorem shows.

Theorem 2.1
Let $\mathcal{F}$ be a Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) then

$$\phi L = Q \quad \text{and} \quad \phi Q = L$$

where $\phi L = \{\phi X \mid X \in L\}$ and $\phi Q = \{\phi Y \mid Y \in Q\}$.

Proof
For any $X, X' \in L$ we have

$$\begin{aligned} g(X, \phi X') &= d\eta(X, X') && \text{by 1.3} \\ &= -\frac{1}{2}\eta([X, X']) && \text{as L is a subbundle of H} \\ &= 0 && \text{as L is completely integrable.} \end{aligned}$$

Thus

$$\phi X' \in L^{\perp} = Q \oplus E,$$

but by 1.4 and 1.6 we know that

$$\phi X' \in H$$

which implies that

$$\phi X' \in Q \qquad \forall\ X' \in L.$$

As $\phi|_H$ is an isomorphism the theorem follows. ¤

As a result of theorm 2.1 the tangential and transverse metrics of a Legendre foliation on a contact metric manifold are not independent.

Also as a consequence of theorem 2.1 for any $Y, Y' \in \Gamma Q$ we have

$$\begin{aligned} 0 &= g(Y, \phi Y') \\ &= d\eta(Y, Y') && \text{by 1.3} \\ &= -\frac{1}{2}\eta([Y, Y']) && \text{as } Y, Y' \in \Gamma H. \end{aligned}$$

Thus for any $Y, Y' \in \Gamma Q$ we have

(2.2) $\qquad [Y, Y'] \in \Gamma H.$

For a Legendre foliation the transverse vector fields are related to the tangential vector fields by the contact metric structure as shown in the following lemma, which is proved directly using 1.9 and theorem 2.1.

Lemma 2.3

Let $\mathcal{F}$ be a Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) and

$\{X_\alpha\}$ $\alpha = 1,.....,n$ be a local orthonormal frame for L with respect to g

then

$\{\phi X_\alpha, \xi\}$ $\alpha = 1,....,n$ is a local orthonormal frame for $L^\perp$ with respect to g.

Due to the contact metric structure, the tangential vector fields of a Legendre foliation satisfy the following properties.

Theorem 2.4

Let $\mathcal{F}$ be a Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) with $\{X_\alpha\}$ $\alpha = 1,.....,n$ a local orthonormal frame for L with respect to g and ∇ the Levi-Civita connection of g. Then

i) For any α, β the projection

$$p_E([X_\alpha, \phi X_\beta]) = 2k_{\alpha\beta}\xi$$

where $k = [k_{\alpha\beta}] = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$ and (p, q) is the signature of $g|_L$.

ii) For any α, β we have

$$g((\nabla_X \phi)X_\alpha, X_\beta) = -\, g((\nabla_X \phi)X_\beta, X_\alpha) \qquad \forall\, X \in \Gamma L.$$

iii) For any α, β the projection

$$p_E(\nabla_{X_\alpha} X_\beta) = -\, p_E(\nabla_{\phi X_\alpha} \phi X_\beta).$$

Proof

i) For any α, β we have

$$g([X_\alpha, \phi X_\beta], \xi) = 2g(X_\alpha, X_\beta) \qquad \text{by 1.13}$$
$$= 2k_{\alpha\beta}.$$

Hence for any α, β the projection

$$p_E([X_\alpha, \phi X_\beta]) = 2k_{\alpha\beta}\xi.$$

ii) Using the properties of ϕ and of the Levi-Civita connection ∇ together with the complete integrability of L, we have for any α, β and any $X \in \Gamma L$ that

$$\begin{aligned} 0 &= g(\phi X_\alpha, [X, X_\beta]) - g(\phi X_\beta, [X, X_\alpha]) \qquad \text{by theorem 2.1} \\ &= g(\phi X_\alpha, \nabla_X X_\beta - \nabla_{X_\beta} X) - g(\phi X_\beta, \nabla_X X_\alpha - \nabla_{X_\alpha} X) \\ &= g(X_\beta, \phi\nabla_X X_\alpha - \nabla_X \phi X_\alpha) + g(X, \nabla_{X_\beta}\phi X_\alpha - \nabla_{X_\alpha}\phi X_\beta) \\ &= - g(X_\beta, (\nabla_X \phi)X_\alpha) + g(X, \nabla_{X_\beta} \phi X_\alpha - \nabla_{X_\alpha}\phi X_\beta). \end{aligned}$$

Hence for any α, β we have

$$\begin{aligned} g((\nabla_X \phi)X_\alpha, X_\beta) &= g(X, \nabla_{X_\beta} \phi X_\alpha - \nabla_{X_\alpha}\phi X_\beta) \\ &= - g((\nabla_X \phi)X_\beta, X_\alpha). \end{aligned}$$

iii) For any α, β we have

$$\begin{aligned} g(\nabla_{X_\alpha} X_\beta, \xi) &= - g(\nabla_{X_\alpha} \xi, X_\beta) \\ &= g(\phi X_\alpha + \phi h X_\alpha, X_\beta) \qquad \text{by 1.35} \\ &= \frac{1}{2}\, g(\phi(\phi[X_\alpha, \xi] - [\phi X_\alpha, \xi]), X_\beta) \\ &\qquad \text{by 1.30 and theorem 2.1} \\ &= - \frac{1}{2}\, g([X_\alpha, \xi] + \phi[\phi X_\alpha, \xi]), X_\beta) \qquad \text{by 1.2 and 1.12} \\ &= - \frac{1}{2}\, g(\nabla_{X_\alpha} \xi - \nabla_\xi X_\alpha + \phi\nabla_{\phi X_\alpha} \xi - \phi\nabla_\xi \phi X_\alpha, X_\beta) \\ &= - \frac{1}{2}\, g(\nabla_{X_\alpha} \xi + \phi\nabla_{\phi X_\alpha} \xi, X_\beta) \qquad \text{by 1.27} \\ &= \frac{1}{2}\, g(\xi, \nabla_{X_\alpha} X_\beta) - \frac{1}{2}\, g(\xi, \nabla_{\phi X_\alpha} \phi X_\beta) \qquad \text{by 1.10.} \end{aligned}$$

Hence for any α, β we have

$$g(\nabla_{X_\alpha} X_\beta, \xi) = - g(\nabla_{\phi X_\alpha} \phi X_\beta, \xi). \qquad \square$$

The contact metric structure associated to a Legendre foliation induces a coframe, which satisfies the following properties.

Lemma 2.5

Let $\mathcal{F}$ be a Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g). If $\{X_\alpha, \phi X_\alpha, \xi\}$ is a local orthonormal frame for TM with respect to g then $\{\mathcal{E}^\alpha, \theta^\alpha, \eta\}$ is the corresponding local coframe of T^*M which satisfies

i) $\theta^\alpha \in \Gamma Q^*$ and $\mathcal{E}^\alpha \in \Gamma L^*$.

ii) $d\eta = 2\delta_{\alpha\beta}\, \theta^\alpha \wedge \mathcal{E}^\beta$.

iii) $g|_Q = k_{\alpha\beta}\, \theta^\alpha \otimes \theta^\beta$, $g|_L = k_{\alpha\beta}\mathcal{E}^\alpha \otimes \mathcal{E}^\beta$, $g|_E = \eta \otimes \eta$.

where

(2.6) (i) $\theta^\alpha = -\frac{1}{2} i(X_\alpha)\, d\eta$,

(ii) $\mathcal{E}^\alpha = \frac{1}{2} k_{\alpha\beta} i(\phi X_\beta)\, d\eta$

and

$$k = [k_{\alpha\beta}] = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix},$$ (p, q) is the signature of $g|_L$.

Note that the subbundles L^*, Q^* and E^* of T^*M are defined by

$L^* = \mathrm{ann}(Q \oplus E)$, $Q^* = \mathrm{ann}(L \oplus E)$,

$E^* = \mathrm{ann}(L \oplus Q) = \mathrm{ann}(H) = \mathrm{span}\{\eta\}$.

Proof

Using 1.3, 1.9, 1.10 and 2.6 we have obtain any α and any $W \in \Gamma TM$

(2.7) $\theta^{\alpha}(W) = - d\eta(X_{\alpha}, W) = g(W, \phi X_{\alpha})$ and $\mathcal{E}^{\alpha}(W) = k_{\alpha\beta} d\eta(\phi X_{\beta}, W) = k_{\alpha\beta} g(W, X_{\beta})$.

Using theorem 2.1 for any $X \in \Gamma L$ we have,

$$\theta^{\alpha}(X) = 0,$$

furthermore by 1.14

$$\theta^{\alpha}(\xi) = 0$$

which implies that

$$\theta^{\alpha} \in \Gamma Q^{*}.$$

Similarly we have for any α that

$$\mathcal{E}^{\alpha} \in \Gamma L^{*}.$$

For any α, β using 1.9 and 2.7 we have

(a) $\theta^{\alpha}(\phi X_{\beta}) = g(\phi X_{\alpha}, \phi X_{\beta}) = g(X_{\alpha}, X_{\beta}) = k_{\alpha\beta}$ $\quad$ $\mathcal{E}^{\alpha}(X_{\beta}) = k_{\alpha\gamma} g(X_{\alpha}, X_{\gamma}) = \delta_{\alpha\beta}$

which with 1.1 implies that

$$g|_{Q} = k_{\alpha\beta} \theta^{\alpha} \otimes \theta^{\beta}, \qquad g|_{L} = k_{\alpha\beta} \mathcal{E}^{\alpha} \otimes \mathcal{E}^{\beta}, \qquad g|_{E} = \eta \otimes \eta.$$

By 1.3 we have for all α, β that

$$\begin{aligned} d\eta(\phi X_\alpha, X_\beta) &= g(\phi X_\alpha, \phi X_\beta) \\ &= g(X_\alpha, X_\beta) \qquad \text{by 1.9} \\ &= k_{\alpha\beta}, \end{aligned}$$

and for all $X, X' \in \Gamma L$, $Y, Y' \in \Gamma Q$, $Z \in \Gamma E$, and $W \in \Gamma TM$ we have that

$$d\eta(X, X') = d\eta(Y, Y') = d\eta(Z, W) = 0 \qquad \text{by 1.4 and theorem 2.1.}$$

From which using (a) and part (i) it can be shown that

$$d\eta = 2\delta_{\alpha\beta}\, \theta^\alpha \wedge \mathcal{E}^\beta. \qquad \square$$

For a Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) we call $\{\mathcal{E}^\alpha, \theta^\alpha, \eta\}$ *the local coframe of* T^*M *induced by the contact metric structure* (ϕ, ξ, η, g).

A vector field W on M is *foliate* if for all vector fields $X \in \Gamma L$ the Lie bracket $[X, W]$ belongs to ΓL, Molino [22].

Theorem 2.8

If $\mathcal{F}$ is a Legendre foliation on a contact manifold (M, η), then the non-zero sections of Q are not foliate vector fields.

Proof

Let (ϕ, ξ, η, g) be a contact metric structure on (M, η). Then by theorem 2.1 for $Y \neq 0$, $Y \in \Gamma Q$ we have $\phi Y \neq 0$ and $\phi Y \in \Gamma L$.

So for a $Y \in \Gamma Q$, $Y \neq 0$ we have

$$\begin{aligned} g([Y, \phi Y], \xi) &= 2g(Y, Y) \qquad \text{by 1.13} \\ &\neq 0. \end{aligned}$$

This implies that

$$[Y, \phi Y] \notin \Gamma L. \qquad \square$$

Thus if Z is a foliate vector field with respect to a Legendre foliation, then Z is a section of $E \oplus L$.

2 (b). MEAN CURVATURE OF A LEGENDRE FOLIATION

For any Legendre foliation the Weingarten map in the direction of the characteristic vector field, ξ, can be expressed in terms of the tensors ϕ and h. Furthermore, when $h: L \to L$ the mean curvature vector field is a section of Q.

Lemma 2.9
For a Legendre foliation, $\mathcal{F}$, on a contact metric manifold (M, ϕ, ξ, η, g)

$$A_\xi = p_L(\phi h).$$

Proof
For any $X, X' \in \Gamma L$ we have

$$\begin{aligned} g(A_\xi X, X') &= - g(\nabla_X \xi, X') \\ &= g(\phi hX + \phi X, X') && \text{by 1.35} \\ &= g(\phi hX, X') && \text{by theorem 2.1.} \end{aligned}$$

Hence

$$A_\xi = p_L(\phi h). \qquad \square$$

Corollary 2.10

For a Legendre foliation, $\mathcal{F}$, on a contact metric manifold (M, ϕ, ξ, η, g) such that h: L $\rightarrow$ L the mean curvature vector field is given by

$$\tau = g(X_\alpha, X_\beta) p_Q(\nabla_{X_\alpha} X_\beta),$$

where $\{X_\alpha\}$ $\alpha = 1,.....,n$ is a local orthonormal frame for L with respect to g.

Proof

For any α, β we have

$$\begin{aligned} g(\nabla_{X_\alpha} X_\beta, \xi) &= - g(X_\beta, \nabla_{X_\alpha} \xi) \\ &= g(X_\beta, \phi h X_\alpha + \phi X_\alpha) && \text{by 1.35} \\ &= 0 && \text{by theorem 2.1 as h: L} \rightarrow \text{L.} \end{aligned}$$

This implies that

$$\begin{aligned} \tau &= g(X_\alpha, X_\beta) p_{L^\perp}(\nabla_{X_\alpha} X_\beta) \\ &= g(X_\alpha, X_\beta) p_Q(\nabla_{X_\alpha} X_\beta). \end{aligned}$$ ¤

2 (c). THE INVARIANTS Π, $\overline{\Pi}$ AND G ON A LEGENDRE FOLIATION

For a Legendre foliation, $\mathcal{F}$, on a contact manifold (M, η), Pang [2] introduced on L a symmetric, bilinear form Π which can be defined by

$$\Pi(X, X') = - \{(\mathcal{L}_X \mathcal{L}_X \eta)(\xi)\} \qquad \forall\ X, X' \in \Gamma L.$$

Thus for any $X, X' \in \Gamma L$ we have

$$\begin{aligned}\Pi(X, X') &= -\{(\mathcal{L}_X\mathcal{L}_{X'}\eta)(\xi)\} \\ &= -\{X(\mathcal{L}_{X'}\eta)(\xi) - (\mathcal{L}_{X'}\eta)([X, \xi])\} \\ &= -\{X\{X'\eta(\xi) - \eta([X', \xi])\} - X'\eta([X, \xi]) \\ &\qquad + \eta([X', [X, \xi]])\} \\ &= -\eta([X', [X, \xi]]) \qquad \text{by 1.1 and 1.12.}\end{aligned}$$

Since $X', [X, \xi] \subset H$ this shows that Π can also be written as

$$\begin{aligned}(2.11)\qquad \Pi(X, X') &= 2d\eta(X', [X, \xi]) \qquad \forall\ X, X' \in \Gamma L \\ &= 2g([\xi, X], \phi X') \qquad \text{by 1.3 and 1.10,}\end{aligned}$$

where (ϕ, ξ, η, g) is any contact metric structure on (M, η).

It is proved later that Π is symmetric. By definition Π is an invariant of a Legendre foliation, so that its value does not depend on the contact metric structure on (M, η).

Π may be used to classify Legendre foliations as follows.

Definition 2.12
Let $\mathcal{F}$ be a Legendre foliation on a contact manifold (M, η) then

(i) $\mathcal{F}$ is a flat Legendre foliation if $\Pi \equiv 0$,

(ii) $\mathcal{F}$ is a non-degenerate (degenerate) Legendre foliation if Π is non-degenerate (degenerate),

(iii) $\mathcal{F}$ is a positive definite Legendre foliation if Π is positive definite.

As the value of Π doesn't depend on the contact metric structure definition 2.12 implies that though the Legendre foliation $\mathcal{F}$ defined by the subbundle $L \subset H$ on the contact metric manifold (M, ϕ, ξ, η, g) is different from the Legendre foliation $\mathcal{F}'$ defined by the subbundle $L \subset H$ on the contact metric manifold $(M, \phi', \xi, \eta, g')$, $\mathcal{F}$ and $\mathcal{F}'$ are of the same classification.

It follows from 2.11 that a Legendre foliation can be classified in terms of the Lie bracket $[\xi, X]$, where X is any tangential vector field.

Lemma 2.13
Let $\mathcal{F}$ be a Legendre foliation on a contact manifold (M, η) then

(i) $\mathcal{F}$ is a flat Legendre foliation if and only if ξ is a foliate vector field, that is the projection

$$p_Q([\xi, X]) = 0 \qquad \forall X \in \Gamma L.$$

(ii) $\mathcal{F}$ is a non-degenerate Legendre foliation if and only if the projection

$$p_Q([\xi, X]) \neq 0 \qquad \forall X \in \Gamma L.$$

(iii) $\mathcal{F}$ is a positive definite Legendre foliation if and only if

$$g([\xi, X], \phi X) > 0 \qquad \forall X \in \Gamma L,$$

where (ϕ, ξ, η, g) is any contact metric structure on (M, η).

For a non-degenerate Legendre foliation it will be useful to extend Π to a bilinear form defined on TM. To do this we define a tensor, λ, of type (1,1) by

(2.14) $$\Pi(\lambda W, X) = 4d\eta(W, X) \qquad \forall X \in \Gamma L, \forall W \in \Gamma TM,$$

then

$$\lambda: TM \to L \quad \text{and} \quad \text{the kernel of } \lambda \text{ is } L \oplus E.$$

The existence of λ is shown by Libermann [1], who also shows that λ is a surjective map satisfying $\lambda^2 = 0$.

For any $X \in \Gamma L$, we will show that λ satisfies

(2.15) $\lambda[\xi, X] = 2X.$

For any $X, X' \in \Gamma L$ we have

$$d\eta([\xi, X], X') = \frac{1}{2} \Pi(X, X') \qquad \text{by 2.11,}$$

and

$$d\eta([\xi, X], X') = \frac{1}{4} \Pi(\lambda[\xi, X], X') \qquad \text{by 2.14.}$$

Since λ: $TM \rightarrow L$ this implies that

$$\lambda[\xi, X] = 2X.$$

As λ is a surjective map and $L \subset$ kernel λ, for a non-degenerate Legendre foliation on a contact manifold (M, η), we can use 2.15 to give a local frame of TM.

Lemma 2.16

Let $\mathcal{F}$ be a non-degenerate Legendre foliation; if $X_1, \ldots\ldots, X_n$ generate L in an open set U of M, then $X_1, \ldots\ldots, X_n, [\xi, X_1], \ldots\ldots, [\xi, X_n]$ generate H in U.

This is proposition 2.16 of Libermann [1].

Thus if $\{X_\alpha\}$ $\alpha = 1, \ldots\ldots, n$ is a local frame of L then $\{X_\alpha, [\xi, X_\alpha], \xi\}$ is a local frame of TM.

For a non-degenerate Legendre foliation λ enables us to define a symmetric, bilinear form $\bar{\Pi}$ on TM related to the bilinear form Π on L as follows

$$\bar{\Pi}(W, W') = \frac{1}{4}\Pi(\lambda W, \lambda W') \qquad \forall\ W, W' \in \Gamma TM. \tag{2.17}$$

$\bar{\Pi}$ will be useful in our work on non-degenerate Legendre foliations.

For a Legendre foliation, $\mathcal{F}$, on a contact manifold (M, η), Pang [2] also introduced on L a symmetric 3-form G which for any $X, X', X'' \in \Gamma L$ can be defined by

$$G(X, X', X'') = \frac{1}{2}\{X\Pi(X', X'') + X'\Pi(X, X'') + X''\Pi(X, X') + (\mathcal{L}_{X'}\mathcal{L}_{X}\mathcal{L}_{X''}\eta + \mathcal{L}_{X''}\mathcal{L}_{X}\mathcal{L}_{X'}\eta)(\xi)\}$$

By definition G is an invariant of a Legendre foliation, so that its value does not depend on the contact metric structure on (M, η). Furthermore we will see later that G can be used to define semi-Riemannian Legendre foliations.

A simpler formula for G is given by

$$G(X, X', X'') = \frac{1}{2}\{- X\Pi(X', X'') - X'\Pi(X, X'') - X''\Pi(X, X') - \eta([X'', [X, [X', \xi]]]) - \eta([X', [X, [X'', \xi]]])\}$$
$$\forall\ X, X', X'' \in \Gamma L.$$

Which is obtained from the definition of G by using the following formula

$$(\mathcal{L}_{X'}\mathcal{L}_{X}\mathcal{L}_{X''}\eta)(\xi) = - X'\Pi(X, X'') - X\Pi(X', X'') - X''\Pi(X, X') - \eta([X'', [X, [X', \xi]]])$$
$$\forall\ X, X', X'' \in \Gamma L.$$

This is calculated as follows, for any $X, X', X'' \in \Gamma L$ we have

$$
\begin{aligned}
(\mathcal{L}_{X'}\mathcal{L}_{X}\mathcal{L}_{X''}\eta)(\xi) &= X'\{(\mathcal{L}_{X}\mathcal{L}_{X''}\eta)(\xi)\} - (\mathcal{L}_{X}\mathcal{L}_{X''}\eta)([X',\xi]) \\
&= X'\{X\{X''\eta(\xi) - \eta([X'', \xi])\} - X''\eta([X, \xi]) \\
&\quad + \eta([X'', [X, \xi]])\} - X\{X''\{\eta([X', \xi])\} - \eta([X'', [X', \xi]])\} \\
&\qquad + X''\{\eta([X, [X', \xi]]) - \eta([X'', [X, [X', \xi]]]) \\
&= X'\{\eta([X'', [X, \xi]]\} + X\{\eta([X'', [X', \xi]])\} \\
&\qquad + X''\{\eta([X, [X', \xi]]) - \eta([X'', [X, [X', \xi]]]) \\
&\qquad\qquad \text{by 1.1 and 1.12} \\
&= -2X'\{d\eta(X'', [X, \xi])\} - 2X\{d\eta(X'', [X', \xi])\} \\
&\qquad - 2X''\{d\eta(X, [X', \xi]) - \eta([X'', [X, [X', \xi]]]) \\
&\qquad\qquad \text{by 1.12} \\
&= -X'\Pi(X, X'') - X\Pi(X', X'') - X''\Pi(X, X') \\
&\qquad - \eta([X'', [X, [X', \xi]]]) \\
&\qquad\qquad \text{by 2.11.}
\end{aligned}
$$

From this simpler formula for G we obtain the following alternative formula for G, in terms of any contact metric structure on (M, η) which we will use in later work.

For all $X, X', X'' \in \Gamma L$ we have

$$
\begin{aligned}
(2.18)\quad G(X, X', X'') &= -Xg([\xi, X'], \phi X'') + g([X, [\xi, X']], \phi X'') \\
&\qquad + g([X, [\xi, X'']], \phi X')
\end{aligned}
$$

where (ϕ, ξ, η, g) is any contact metric structure on (M, η).

To prove 2.18 for any $X, X', X'' \in \Gamma L$ and any contact metric structure (ϕ, ξ, η, g) on (M, η), we first need the following relationships,

$$
\begin{aligned}
2g([\xi, X], \phi X') &= \Pi(X, X') && \text{by 2.11} \\
&= \Pi(X', X) && \text{as } \Pi \text{ is symmetric} \\
&= 2g([\xi, X'], \phi X) && \text{by 2.11,}
\end{aligned}
$$

and

$$\eta([X'', [X, [X', \xi]]]) = - 2d\eta(X'', [X, [X', \xi]]) + X''\eta([X, [X', \xi]])$$
$$\text{as } X'' \in \Gamma L$$
$$= - 2d\eta(X'', [X, [X', \xi]]) - 2X''d\eta(X, [X', \xi])$$
$$\text{as } X, [X', \xi] \in \Gamma L$$
$$= - 2\{g(X'', \phi[X, [X', \xi]]) + X''g(X, \phi[X', \xi])\}$$
$$\text{by 1.3,}$$

Then by substituting these into the simpler formula for G and using 2.11 we obtain 2.18.

2 (d). PARTIAL CONNECTIONS ON LEGENDRE FOLIATIONS

For a Legendre foliation, $\mathcal{F}$, on a contact manifold (M, η), the Bott partial connection, $\nabla^{L^{\perp *}}$, given by 1.39 induces a partial connection ∇^{Q^*} in Q defined by

$$\nabla^{Q^*}_X \nu = p_Q(\nabla^{L^{\perp *}}_X \nu) \qquad \forall\, X \in \Gamma L, \forall\, \nu \in \Gamma Q^*.$$

Let $\mathcal{F}$ be a Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) with $\{X_\alpha\}$ $\alpha = 1,.....,n$ a local orthonormal frame for L with respect to g and $\{\theta^\alpha, \eta\}$ the local coframe for $L^{\perp *}$ induced by (ϕ, ξ, η, g).

By evaluating the Bott partial connection on this frame and the corresponding coframe we will obtain the following:

(2.19) (i) $$\nabla^{L^{\perp *}}_{X_\alpha} \eta = - 2\theta^\alpha.$$

(ii) $$\nabla^{L^{\perp *}}_{X_\alpha} \theta^\beta = - \theta^\beta([X_\alpha, \phi X_\delta])k_{\delta\gamma}\theta^\gamma - \theta^\beta([X_\alpha, \xi])\eta.$$

(iii) $$\nabla^{Q^*}_{X_\alpha}\theta^\beta = -\,\theta^\beta([X_\alpha, \phi X_\delta])k_{\delta\gamma}\theta^\gamma.$$

For any α we have

$$\begin{aligned}(\nabla^{L^{\perp *}}_{X_\alpha}\eta) &= i(X_\alpha)d\eta && \text{by 1.39}\\ &= -\,2\theta^\alpha && \text{by 2.6 (i),}\end{aligned}$$

hence 2.19 (i).

For any α, β and γ we have

$$\begin{aligned}(\nabla^{L^{\perp *}}_{X_\alpha}\theta^\beta)(\phi X_\gamma) &= (i(X_\alpha)d\theta^\beta)(\phi X_\gamma) && \text{by 1.39}\\ &= 2d\theta^\beta(X_\alpha, \phi X_\gamma)\\ &= -\,\theta^\beta([X_\alpha, \phi X_\gamma]),\end{aligned}$$

and

$$\begin{aligned}(\nabla^{L^{\perp *}}_{X_\alpha}\theta^\beta)(\xi) &= (i(X_\alpha)d\theta^\beta)(\xi) && \text{by 1.39}\\ &= 2d\theta^\beta(X_\alpha, \xi)\\ &= -\,\theta^\beta([X_\alpha, \xi]).\end{aligned}$$

From which 2.19 (ii) and (iii) follow.

The relative partial connection form $\omega = (\omega^i_j)$ of $\nabla^{L^{\perp *}}$ is determined by the equation

$$\nabla^{L^{\perp *}}\begin{bmatrix}\theta^\beta\\ \eta\end{bmatrix} = -\begin{bmatrix}\omega^\beta_\gamma & \omega^\beta_{n+1}\\ \omega^{n+1}_\gamma & 0\end{bmatrix} \otimes \begin{bmatrix}\theta^\gamma\\ \eta\end{bmatrix}$$

From 2.19 we have

(2.20) (i) $\omega^{\beta}_{\gamma}(X_\alpha) = \theta^{\beta}([X_\alpha, \phi X_\delta])k_{\delta\gamma}$.

(ii) $\omega^{\beta}_{n+1}(X_\alpha) = \theta^{\beta}([X_\alpha, \xi])$.

(iii) $\omega^{n+1}_{\gamma}(X_\alpha) = 2\delta_{\alpha\gamma}$.

$\nabla^{L^{\perp *}}$ is flat so $\Omega = d\omega + \omega \wedge \omega$ vanishes and the relative connection form (ω^i_j) satisfies the equation

$$d\begin{bmatrix} \omega^{\beta}_{\gamma} & \omega^{\beta}_{n+1} \\ \omega^{n+1}_{\gamma} & 0 \end{bmatrix} + \begin{bmatrix} \omega^{\beta}_{\delta} & \omega^{\beta}_{n+1} \\ \omega^{n+1}_{\delta} & 0 \end{bmatrix} \wedge \begin{bmatrix} \omega^{\delta}_{\gamma} & \omega^{\delta}_{n+1} \\ \omega^{n+1}_{\gamma} & 0 \end{bmatrix} = 0$$

From which we get the following equations

(2.21) (i) $\omega^{n+1}_{\delta} \wedge \omega^{\delta}_{n+1} = 0$.

(ii) $d\omega^{\beta}_{n+1} + (\omega^{\beta}_{\delta} \wedge \omega^{\delta}_{n+1}) = 0$.

(iii) $d\omega^{n+1}_{\gamma} + (\omega^{n+1}_{\delta} \wedge \omega^{\delta}_{\gamma}) = 0$.

Evaluating 2.21 (i) on X_α, X_β with $\alpha \neq \beta$ gives

$$\begin{aligned} 0 &= \omega^{n+1}_{\delta} \wedge \omega^{\delta}_{n+1}(X_\alpha, X_\beta) \\ &= \frac{1}{2}\{\omega^{n+1}_{\delta}(X_\alpha)\,\omega^{\delta}_{n+1}(X_\beta) - \omega^{n+1}_{\delta}(X_\beta)\omega^{\delta}_{n+1}(X_\alpha)\} \\ &= (\theta^{\alpha}([X_\beta, \xi]) - \theta^{\beta}([X_\alpha, \xi])) && \text{by 2.20 (ii) and (iii)} \\ &= (-\, g([\xi, X_\beta], \phi X_\alpha) + g([\xi, X_\alpha], \phi X_\beta)) && \text{by 2.7} \\ &= -\frac{1}{2}\Pi(X_\beta, X_\alpha) + \frac{1}{2}\Pi(X_\alpha, X_\beta) && \text{by 2.11.} \end{aligned}$$

This shows that Π given by 2.11 is symmetric. Furthermore from this we obtain

$$\begin{aligned}\omega^{\beta}_{n+1}(X_\alpha) &= \theta^{\beta}([X_\alpha, \xi]) && \text{by 2.20 (ii)}\\ &= \theta^{\alpha}([X_\beta, \xi]) && \text{by above}\\ &= \omega^{\alpha}_{n+1}(X_\beta) && \text{by 2.20 (ii).}\end{aligned}$$

Evaluating 2.21 (iii) on X_α, X_β gives

$$\begin{aligned}0 &= (d\omega^{n+1}_{\gamma} + \omega^{n+1}_{\delta} \wedge \omega^{\delta}_{\gamma})(X_\alpha, X_\beta)\\ &= \frac{1}{2}\{- \omega^{n+1}_{\gamma}([X_\alpha, X_\beta]) + \omega^{n+1}_{\delta}(X_\alpha)\, \omega^{\delta}_{\gamma}(X_\beta) - \omega^{n+1}_{\delta}(X_\beta)\omega^{\delta}_{\gamma}(X_\alpha)\}\\ &= \frac{1}{2}\{- \omega^{n+1}_{\gamma}([X_\alpha, X_\beta]) + 2\omega^{\alpha}_{\gamma}(X_\beta) - 2\omega^{\beta}_{\gamma}(X_\alpha)\} && \text{by 2.20 (iii).}\end{aligned}$$

From the above with 2.20 (i) we obtain

$$\begin{aligned}(\theta^{\alpha}([X_\beta, \phi X_\delta]) - \theta^{\beta}([X_\alpha, \phi X_\delta]))k_{\delta\gamma}X_\gamma &= \frac{1}{2}\,\omega^{n+1}_{\gamma}([X_\alpha, X_\beta])X_\gamma\\ &= [X_\alpha, X_\beta]\end{aligned}$$

by 2.20 (iii), as L is completely integrable.

Using 2.7 this can be shown to be equivalent to

(2.22) $$g(\phi X_\alpha, [X_\beta, \phi X_\gamma]) - g(\phi X_\beta, [X_\alpha, \phi X_\gamma]) = g([X_\alpha, X_\beta], X_\gamma).$$

Let Φ: $L \to Q^*$ be the isomorphism defined by

$$\begin{aligned}\Phi(X_\alpha) &= \frac{1}{2}\, i(X_\alpha)d\eta\\ &= - \theta^{\alpha}.\end{aligned}$$

It follows that Φ^{-1}: $Q^* \to L$ and is defined by

$$\Phi^{-1}(\theta^\alpha) = - X_\alpha.$$

Now let ∇^L be the partial connection on L defined by

$$\nabla^L_X X' = \Phi^{-1} \circ \nabla^{Q^*}_X \circ \Phi(X') \qquad \forall\ X, X' \in \Gamma L.$$

Hence for any α, β

(2.23)
$$\begin{aligned} \nabla^L_{X_\alpha} X_\beta &= \Phi^{-1} \circ \nabla^{Q^*}_{X_\alpha} \circ \Phi(X_\beta) \\ &= - \Phi^{-1} \circ \nabla^{Q^*}_{X_\alpha} \theta^\beta \\ &= \Phi^{-1} (\theta^\beta([X_\alpha, \phi X_\delta])k_{\delta\gamma}\theta^\gamma) \qquad \text{by 2.19 (iii)} \\ &= - \theta^\beta([X_\alpha, \phi X_\delta])k_{\delta\gamma}X_\gamma \end{aligned}$$

It can be shown (see Pang [2]) that ∇^L is torsion free and has curvature

(2.24)
$$R^L(X,X')X'' = \prod(X', X'')X - \prod(X, X'')X' \qquad \forall\ X, X', X'' \in \Gamma L.$$

Thus if $\mathcal{F}$ is a flat Legendre foliation then $R^L \equiv 0$, so the leaves of a flat Legendre foliation have a natural flat affine structure.

Theorem 2.25

Let $\mathcal{F}$ be a Legendre foliation on a contact manifold (M, η) of dimension ≥ 5 then $R^L \equiv 0$ if and only if $\mathcal{F}$ is a flat Legendre foliation.

Proof

Suppose $R^L \equiv 0$ and $\prod \neq 0$, then there exists $X \in \Gamma L$ such that $\prod(X, X) = 1$ and as the dimension of (M, η) is ≥ 5, L is of dimension ≥ 2 so there exists $X' \neq 0 \in \Gamma L$ such that $\prod(X, X') = 0$.

For this X, X' we have

$$R^L(X', X)X = \Pi(X, X)X' - \Pi(X', X)X$$
$$= X' \neq 0.$$

Thus $R^L \equiv 0 \Rightarrow \Pi \equiv 0 \Rightarrow \mathcal{F}$ is a flat Legendre foliation. ¤

2 (e). UNIT COTANGENT BUNDLES

An important class of examples of Legendre foliations comes from the unit cotangent bundles which we will now investigate.

Let M be a (n+1)-dimensional manifold. Then the cotangent bundle T^*M is a (2n+2)-dimensional manifold, with a symplectic structure defined by the canonical 1-form Ψ, which has local expression

$$\Psi = y_i dx_i$$

where (x_i, y_i) denote the local coordinates on T^*M induced from the local coordinates (x_i) on M.

Let F be a function on T^*M which is homogeneous of degree one, that is

$$F: T^*M \to R \qquad \text{and} \qquad F(tv) = tF(v) \qquad \forall\, v \in T^*M, \forall\, t \in R.$$

We now define the unit cotangent bundle, $S^*_F M$, by

$$S^*_F M = F^{-1}(1).$$

$S^*_F M$, is a subbundle of T^*M. It is also a (2n +1)-dimensional contact manifold, with contact 1-form η given by the pull-back of the canonical 1-form Ψ on T^*M.

In local coordinates we have

(2.26) (i) $\eta = y_i dx_i$.

(ii) $$\xi = \frac{\partial F}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_j}\frac{\partial}{\partial y_j}.$$

The foliation given by the fibres of the projection map

$$p: S_F^*M \to M$$

is a Legendre foliation. We call this foliation the Legendre foliation $\mathcal{F}_F$ on S_F^*M.

Two Legendre foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ on contact manifolds (M_1, η_1) and (M_2, η_2) respectively are *equivalent* if there exists a diffeomorphism $\rho: M_1 \to M_2$ satisfying the conditions

$$\rho^*\eta_2 = \eta_1 \qquad \text{and} \qquad \rho^*\mathcal{F}_2 = \mathcal{F}_1$$

where $\rho^*\mathcal{F}_2$ is the foliation of M_1 whose leaves are inverse images under ρ of the leaves of $\mathcal{F}_2$.

Theorem 2.27
Any Legendre foliation is locally equivalent to one of the form $\mathcal{F}_F$. The foliation $\mathcal{F}_F$ is called the *normal form* of $\mathcal{F}$.

This is the normal form theorem of Pang [2].

Since F is homogeneous of degree one, we have on T^*M that

(2.28) $F = y_i F_i$ and $y_i F_{ij} = 0$.

where $F_i = \dfrac{\partial F}{\partial y_i}$, $F_{ij} = \dfrac{\partial^2 F}{\partial y_i \partial y_j}$.

Since $F = 1$ on S_F^*M for the Legendre foliation $\mathcal{F}_F$ we have

(2.29) $$dF(X) = F_i dy_i(X) = 0 \qquad \forall\, X \in \Gamma L.$$

This implies that the tangent bundle of $\mathcal{F}_F$ is given by

$$L = \{a_i \frac{\partial}{\partial y_i} \mid F_i a_i = 0\}$$

and the *normal bundle* N is given by

$$N = \frac{TS_F^*M}{L}$$
$$= \text{span}\{\frac{\partial}{\partial x_i}\}.$$

So we get a natural decomposition of TS_F^*M as

(2.30) $$TS_F^*M = L \oplus N$$
$$= \{a_i \frac{\partial}{\partial y_i} \mid F_i a_i = 0\} \oplus \text{span}\{\frac{\partial}{\partial x_i}\}.$$

Pang [2] defines Π on a Legendre foliation of the form $\mathcal{F}_F$ as

$$\Pi = F_{ij} dy_i \otimes dy_j$$

and proves in proposition 3.3 that it is equivalent to the definition of Π we have given in 2 (c).

Now since, see Pang [2] page 442,

$$\frac{1}{2}(F^2)_{ij}\, dy_i \otimes dy_j = (F_iF_j + FF_{ij})\, dy_i \otimes dy_j$$
$$= F_{ij}\, dy_i \otimes dy_j \qquad \text{by 2.28 and 2.29,}$$

we get the following alternative definitions of Π for a Legendre foliation $\mathcal{T}_F$ on S_F^*M.

(2.31)
$$\Pi = F_{ij}\, dy_i \otimes dy_j$$
$$= \frac{1}{2}(F^2)_{ij}\, dy_i \otimes dy_j$$
$$= (F_iF_j + FF_{ij})\, dy_i \otimes dy_j.$$

In the following example we give a Legendre foliation on a unit cotangent bundle of R^3 and introduce two different contact metric structures. Then we examine the properties of the resultant non-degenerate Legendre foliations on the different contact metric manifolds.

Example 2.1: Non-degenerate Legendre foliation on a unit cotangent bundle

Let

$$M = R^3$$
$$= \{(x_1, x_2, x_3)) \mid x_i \in R\}$$

and consider the function F: $T^*M \to M$ defined by

$$F(x, y) = \begin{cases} \dfrac{(y_1)^2 - (y_2)^2}{y_3} & y_3 \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$S_F^*M = F^{-1}(1)$$
$$= \{(x_1, x_2, x_3, y_1, y_2, y_3) \mid (y_1)^2 - (y_2)^2 = y_3\}.$$

So by 2.26 (i) and (ii) we have

$$\eta = y_1 dx_1 + y_2 dx_2 + y_3 dx_3$$

$$\xi = \frac{1}{y_3}\left(2y_1\frac{\partial}{\partial x_1} - 2y_2\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}\right).$$

Let $\mathcal{F}_F$ be the Legendre foliation given by the fibres of the projection

$$p: S_F^*M \to M.$$

By 2.31 we have

$$\begin{aligned}\Pi &= F_{ij}\, dy_i \otimes dy_j \\ &= \frac{2}{(y_3)^2}\{y_3 dy_1 \otimes dy_1 - y_1 dy_1 \otimes dy_3 - y_3 dy_2 \otimes dy_2 \\ &\quad + y_2 dy_2 \otimes dy_3 + dy_3 \otimes dy_3 - y_1 dy_3 \otimes dy_1 + y_2 dy_3 \otimes dy_2\}.\end{aligned}$$

Hence Π is non-degenerate so $\mathcal{F}_F$ is a non-degenerate Legendre foliation.

The tangent bundle, L, of $\mathcal{F}_F$ is given by

$$L = \text{span}\{X_1, X_2\}$$

where

$$X_1 = \sqrt{2}\left(y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} + 2y_3\frac{\partial}{\partial y_3}\right)$$

$$X_2 = \sqrt{2}\left(y_2 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial y_2} \right)$$

As $(y_1)^2 - (y_2)^2 = y_3$ and $y_3 \neq 0$ everywhere on S_F^*M, X_1 and X_2 are well defined, global vector fields on S_F^*M.

In (a) and (b) below we give two different contact metric structures for (S_F^*M, η), and investigate the non-degenerate Legendre foliations defined by L.

(a) We introduce a contact metric structure (ϕ^c, ξ, η, g^c) for (S_F^*M, η) so that L defines a non-degenerate Legendre foliation $\mathcal{F}_F^c$ of $(S_F^*M, \phi^c, \xi, \eta, g^c)$.

Let Q be the subbundle of H given by $Q = \phi^c L$ where ϕ^c is the tensor of type (1,1) given by

$$\phi^c = \frac{1}{(y_3)^2} \begin{bmatrix} 0 & 0 & 0 & -y_3 & 0 & y_1 \\ 0 & 0 & 0 & 0 & y_3 & -y_2 \\ 0 & 0 & 0 & y_1 & -y_2 & -1 \\ (y_3)^3 & 0 & 2y_1(y_3)^3 & 0 & 0 & 0 \\ 0 & -(y_3)^3 & 2y_2(y_3)^3 & 0 & 0 & 0 \\ 2y_1(y_3)^3 & 2y_2(y_3)^3 & 4(y_3)^4 & 0 & 0 & 0 \end{bmatrix}$$

Thus

$$\begin{aligned} Y_1 &= \phi^c X_1 \\ &= \frac{\sqrt{2}}{y_3}\left(y_1 \frac{\partial}{\partial x_1} - y_2 \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} \right) \end{aligned}$$

$$\begin{aligned} Y_2 &= \phi^c X_2 \\ &= \frac{\sqrt{2}}{y_3}\left(-y_2 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial x_2} \right). \end{aligned}$$

Furthermore let g^c be the semi-Riemannian metric defined by

$$g^c|_L = \frac{1}{4}\Pi$$

$$= \frac{1}{2(y_3)^2}\begin{bmatrix} y_3 & 0 & -y_1 \\ 0 & -y_3 & y_2 \\ -y_1 & y_2 & 1 \end{bmatrix} dy_i \otimes dy_j,$$

$$g^c|_Q = \frac{y_3}{2}\begin{bmatrix} 1 & 0 & 2y_1 \\ 0 & -1 & 2y_2 \\ 2y_1 & 2y_2 & 4y_3 \end{bmatrix} dx_i \otimes dx_j,$$

$$g^c|_E = \eta \otimes \eta,$$

$$g^c = 0 \qquad \text{elsewhere.}$$

It can be shown that (ϕ^c, ξ, η, g^c) is a contact metric structure on $(S^*_F M, \eta)$, with $\{X_1, X_2, Y_1, Y_2, \xi\}$ an orthonormal frame for $TS^*_F M$ with respect to g^c.

Calculating the Lie brackets we get

$$[X_1, X_2] = 0 \qquad [Y_1, Y_2] = 0$$
$$[X_1, \xi] = -2Y_1 \qquad [X_2, \xi] = -2Y_2$$
$$[Y_1, \xi] = 0 \qquad [Y_2, \xi] = 0$$
$$[X_1, Y_1] = -3\sqrt{2}Y_1 + 2\xi \qquad [X_1, Y_2] = -\sqrt{2}Y_2$$
$$[X_2, Y_1] = -\sqrt{2}Y_2 \qquad [X_2, Y_2] = \sqrt{2}Y_1 - 2\xi.$$

So L defines a non-degenrate Legendre foliation $\mathcal{F}^c_F$ on the contact metric manifold $(S^*_F M, \phi^c, \xi, \eta, g^c)$.

By calculation we have for $\alpha = 1, 2$

$$h^c X_\alpha = X_\alpha \qquad \text{and} \qquad h^c Y_\alpha = - Y_\alpha.$$

Also by calculation we have

$$\nabla_{X_\alpha} X_\beta = 0 \qquad \text{for any } \alpha, \beta.$$

Which implies that $\mathcal{F}_F^c$ is a totally geodesic Legendre foliation.

It can also be shown by calculation that

$$\nabla_{Y_1} Y_1 = - 3\sqrt{2} X_1 \qquad \text{and} \qquad \nabla_{Y_2} Y_2 = \sqrt{2} X_1.$$

(b) We now select a different contact metric structure (ϕ', ξ, η, g') on (S_F^*M, η) and use L to define a non-degenerate Legendre foliation $\mathcal{F}_F'$ on the contact metric manifold $(S_F^*M, \phi', \xi, \eta, g')$.

Let X_1' and X_2' be the global vector fields defined by

$$X_1' = \left(\frac{y_1}{y_3} \frac{\partial}{\partial y_1} + \frac{y_2}{y_3} \frac{\partial}{\partial y_2} + 2 \frac{\partial}{\partial y_3} \right)$$

$$= \frac{1}{\sqrt{2} y_3} X_1$$

$$X_2' = \left(\frac{y_2}{y_3} \frac{\partial}{\partial y_1} + \frac{y_1}{y_3} \frac{\partial}{\partial y_2} \right)$$

$$= \frac{1}{\sqrt{2} y_3} X_2.$$

Note that $L = \text{span}\{X_1, X_2\} = \text{span}\{X_1', X_2'\}$ and that $\{X_1', X_2'\}$ is an orthonormal frame of L with respect to g' , where g' is the semi-Riemannian metric defined by

$$g'|_L = \begin{bmatrix} y_3 & 0 & -y_1 \\ 0 & -y_3 & y_2 \\ -y_1 & y_2 & 1 \end{bmatrix} dy_i \otimes dy_j,$$

$$g'|_{Q'} = \frac{1}{4y_3} \begin{bmatrix} 1 & 0 & 2y_1 \\ 0 & -1 & 2y_2 \\ 2y_1 & 2y_2 & 4y_3 \end{bmatrix} dx_i \otimes dx_j,$$

$$g'|_E = \eta \otimes \eta,$$

$$g' = 0 \qquad \text{elsewhere,}$$

and where Q' is the subbundle of H given by $Q' = \phi' L$ and ϕ' is the tensor of type (1,1) given by

$$\phi' = \begin{bmatrix} 0 & 0 & 0 & -2y_3 & 0 & 2y_1 \\ 0 & 0 & 0 & 0 & 2y_3 & -2y_2 \\ 0 & 0 & 0 & 2y_1 & -2y_2 & 2 \\ \frac{1}{2y_3} & 0 & \frac{y_1}{y_3} & 0 & 0 & 0 \\ 0 & -\frac{1}{2y_3} & \frac{y_2}{y_3} & 0 & 0 & 0 \\ \frac{y_1}{y_3} & \frac{y_2}{y_3} & 2 & 0 & 0 & 0 \end{bmatrix}$$

Thus

$$\begin{aligned} Y_1' &= \phi' X_1' \\ &= 2\left(y_1 \frac{\partial}{\partial x_1} - y_2 \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} \right) \\ &= \sqrt{2} y_3 Y_1 \end{aligned}$$

$$\begin{aligned} Y_2' &= \phi' X_2' \\ &= 2\left(-y_2 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial x_2} \right) \\ &= \sqrt{2} y_3 Y_2. \end{aligned}$$

It can be shown that (ϕ', ξ, η, g') is a contact metric structure on (S_F^*M, η), with $\{X_1', X_2', Y_1', Y_2', \xi\}$ an orthonormal frame for TS_F^*M with respect to g'.

Calculating the Lie brackets we get

$$[X_1', X_2'] = -\frac{2}{y_3} X_2' \qquad [Y_1', Y_2'] = 0$$

$$[X_1', \xi] = -\frac{1}{(y_3)^2} Y_1' \qquad [X_2', \xi] = -\frac{1}{(y_3)^2} Y_2'$$

$$[Y_1', \xi] = 0 \qquad [Y_2', \xi] = 0$$

$$[X_1', Y_1'] = -\frac{1}{y_3} Y_1' + 2\xi \qquad [X_1', Y_2'] = \frac{1}{y_3} Y_2'$$

$$[X_2', Y_1'] = -\frac{1}{y_3} Y_2' \qquad [X_2', Y_2'] = \frac{1}{y_3} Y_1' - 2\xi.$$

So L defines a non-degenrate Legendre foliation $\mathcal{F}_F'$ on the contact metric manifold $(S_F^*M, \phi', \xi, \eta, g')$.

By calculation we have for $\alpha = 1, 2$

$$h'X_\alpha' = -\frac{1}{2(y_3)^2} X_\alpha' \qquad \text{and} \qquad h'Y_\alpha' = -\frac{1}{2(y_3)^2} Y_\alpha'.$$

Also by calculation we have

$$\nabla_{X_2'} X_2' = \frac{2}{y_3} X_1', \qquad \nabla_{X_2'} X_1' = \frac{2}{y_3} X_2',$$

$$\nabla_{X_1'} X_2' = \nabla_{X_1'} X_1' = 0$$

Which implies that $\mathcal{F}_F'$ is a totally geodesic Legendre foliation.

It can also be shown by calculation that

$$\nabla_{Y_1'} Y_1' = -\frac{1}{y_3} X_1' \qquad \text{and} \qquad \nabla_{Y_2'} Y_2' = -\frac{1}{y_3} X_1'.$$

CHAPTER 3. FLAT LEGENDRE FOLIATIONS

In this chapter we investigate the properties of flat Legendre foliations and give some examples of these foliations.

Among the possible contact metric structures on a contact manifold which carries a flat Legendre foliation we identify a canonical contact metric structure and show that with this structure the contact manifold is Sasakian.

We know by lemma 2.13 (i) that a Legendre foliation $\mathcal{F}$ on a contact manifold (M, η) is a flat Legendre foliation if and only if ξ is a foliate vector field, that is $[X, \xi] \in \Gamma L$ for all $X \in \Gamma L$. We also know that if $\mathcal{F}$ is a flat Legendre foliation then $R^L \equiv 0$, where R^L is the curvature of the partial connection ∇^L defined in section 2 (d) and is independent of any contact metric structure on (M, η). Thus the leaves of a flat Legendre foliation have a natural flat affine structure.

For a contact manifold (M, η) there exist local coordinates (x_α, y_α, z) such that, (see Blair [3]),

$$\eta = \frac{1}{2}(dz - y_\alpha dx_\alpha).$$

Also, for a p-dimensional foliation $\mathcal{F}$ on a n-dimensional manifold M there exist local coordinates (x_i, y_j), i = 1,....., q, j = 1,....., p, where q = n - p, such that the foliation is locally defined by x_i = constant, i = 1,....., q, (see Molino [22]).

A Legendre foliation $\mathcal{F}$ on a contact manifold (M, η) has both the contact metric structure and the foliation structure. Thus there exist local coordinates (x_α, y_α, z) such that

$$\eta = \frac{1}{2}(dz - y_\alpha dx_\alpha)$$

and there exist local coordinates $(x'_\alpha, y'_\alpha, z')$ such that the foliation is defined by x'_α = constant, z' = constant. In general these natural local coordinates do not coincide, but they do for a flat Legendre foliation as the next theorem shows.

Theorem 3.1

Let $\mathcal{F}$ be a Legendre foliation on a contact manifold (M, η). Then $\mathcal{F}$ is a flat Legendre foliation if and only if there exist local coordinates (x_α, y_α, z) such that

(i) $$\eta = \frac{1}{2}(dz - y_\alpha dx_\alpha).$$

(ii) The foliation is locally defined by z = constant, x_α = constant.

Proof

Let $\mathcal{F}$ be a flat Legendre foliation then by theorem 4.1 of Pang [2] there exist local coordinates (x_α, y_α, z) such that (i) and (ii) are satisfied.

Suppose there exist local coordinates (x_α, y_α, z) such that (i) and (ii) are satisfied then

$$\xi = 2\frac{\partial}{\partial z} \quad \text{and} \quad L = \text{span}\{X_\alpha\}$$

where $X_\alpha = 2\dfrac{\partial}{\partial y_\alpha}$.

Thus for any α the Lie bracket

$$[X_\alpha, \xi] = \left[2\frac{\partial}{\partial z}, 2\frac{\partial}{\partial y_\alpha}\right] = 0$$

hence $\mathcal{F}$ is a flat Legendre foliation by lemma 2.13 (i). ¤

Theorem 3.1 shows that all flat Legendre foliations are locally equivalent.

From the natural local coordinates given by theorem 3.1 we can derive a contact metric structure which is Sasakian.

Theorem 3.2

Let $\mathcal{F}$ be a flat Legendre foliation on a contact manifold (M, η) then there exists a contact metric structure (ϕ^c, ξ, η, g^c) such that $(M, \phi^c, \xi, \eta, g^c)$ is a Sasakian manifold.

Furthermore for this Sasakian structure

$$\nabla^L_X X' = \nabla_X X' \qquad \forall\ X, X' \in \Gamma L,$$

where ∇ is the Levi-Civita connection of g^c.

Proof

By theorem 3.1 for any point $m \in M$ there exists an open neighbourhood U of m with local coordinates (x_α, y_α, z) such that

(a) $$\eta = \frac{1}{2}(dz - y_\alpha dx_\alpha).$$

(b) $$X_\alpha = 2\frac{\partial}{\partial y_\alpha} \qquad \text{where } L = \text{span}\{X_\alpha\}.$$

Let (ϕ_u, ξ, η, g_u) be a contact metric structure on U defined by

$$g_u = \frac{1}{4}\begin{bmatrix} \delta_{\alpha\beta} + y_\alpha y_\beta & 0 & -y_\alpha \\ 0 & \delta_{\alpha\beta} & 0 \\ -y_\alpha & 0 & 1 \end{bmatrix}$$

$$\phi_u = \begin{bmatrix} 0 & \delta_{\alpha\beta} & 0 \\ -\delta_{\alpha\beta} & 0 & 0 \\ 0 & y_\beta & 0 \end{bmatrix}$$

$$\xi = 2\frac{\partial}{\partial z}.$$

(ϕ_u, ξ, η, g_u) is a Sasakian structure on U, (See Yano and Kon [16] page 276.).

So we can find an open cover $\{U_s\}$ $s \in I$ of (M, η) such that (ϕ_s, ξ, η, g_s) is a Sasakian structure on U_s, thus $(\phi^c = \cup_{s\in I} \phi_s, \xi, \eta, g^c = \cup_{s\in I} g_s)$ is a Sasakian structure on M, by lemma 1.15.

It can be shown that

$$\left\{X_\alpha = 2\frac{\partial}{\partial y_\alpha}\right\}$$

is a local orthonormal frame for L with respect to g^c therefore

$$\left\{\phi^c X_\alpha = 2\left(\frac{\partial}{\partial x_\alpha} + y_\alpha \frac{\partial}{\partial z}\right)\right\}$$

is a local orthonormal frame for $Q = \phi^c L$ with respect to g^c.

Calculating the Lie brackets for any α, β we obtain

$$[X_\alpha, X_\beta] = 0 \qquad [\phi^c X_\alpha, \phi^c X_\beta] = 0$$

$$[X_\alpha, \xi] = 0 \qquad [\phi^c X_\alpha, \xi] = 0$$

$$[X_\alpha, \phi^c X_\beta] = 2\delta_{\alpha\beta}\xi.$$

From these Lie brackets it is easily seen for any α, β that

$$\nabla_{X_\alpha} X_\beta = 0.$$

Furthermore from 2.23 for any α, β we have that

$$\nabla^L_{X_\alpha} X_\beta = 0.$$

Thus

$$\nabla^L_X X' = \nabla_X X' \qquad \forall\, X, X' \in \Gamma L.$$

¤

Therefore we have that

$$R(X, X')X'' = R^L(X, X')X'' = 0 \qquad \forall\, X, X', X'' \in \Gamma L,$$

where R is the curvature tensor field of g^c. Hence the leaves of a flat Legendre foliation on the Sasakian manifold $(M, \phi^c, \xi, \eta, g^c)$ are flat.

We call the contact metric structure, (ϕ^c, ξ, η, g^c), given by theorem 3.2 the *canonical contact metric structure of a flat Legendre foliation* and $(M, \phi^c, \xi, \eta, g^c)$ is the *canonical Sasakian manifold of a flat Legendre foliation*. Note that g^c is a Riemannian metric since it is positive definite.

For a flat Legendre foliation there are many other contact metric structures, but as ξ is a foliate vector field they must all satisfy the following lemma.

Lemma 3.3
Let $\mathcal{F}$ be a Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) then $\mathcal{F}$ is a flat Legendre foliation if and only if there exists an associated $(n + 1)$ - dimensional foliation $\tilde{\mathcal{F}}$ on (M, ϕ, ξ, η, g) defined by $L \oplus E$.

Proof
By Frobenius Theorem $L \oplus E$ is the tangent bundle of a foliation if and only if $L \oplus E$ is completely integrable.

As L is the tangent bundle of the Legendre foliation $\mathcal{F}$ it is completely integrable. Thus as E is a line bundle we have that $L \oplus E$ is completely integrable if and only if

$$[X, \xi] \in L \oplus E \qquad \forall\, X \in L.$$

As for any $X \in L$, $[X, \xi] \in H$ by 1.12 we have that

$$[X, \xi] \in L \oplus E \qquad \forall X \in L,$$

if and only if ξ is a foliate vector field of $\mathcal{F}$. That is by lemma 2.13 (i) if and only if $\mathcal{F}$ is a flat Legendre foliation. ¤

The local coordinates of a flat Legendre foliation with an arbitrary contact metric structure can also be defined.

Lemma 3.4
Let $\mathcal{F}$ be a flat Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) then there exist local coordinates (x_α, y_α, z) such that

$$E = \text{span}\{\frac{\partial}{\partial z}\}, \qquad L = \text{span}\{\frac{\partial}{\partial y_\alpha}\}, \qquad Q = \text{span}\{Y_\alpha\}$$

where

$$Y_\alpha = \frac{\partial}{\partial x_\alpha} + f_{\alpha\beta}\frac{\partial}{\partial y_\beta} + y_\alpha \frac{\partial}{\partial z}$$

and $f_{\alpha\beta} = f_{\beta\alpha}$ are functions on M.

Note: $\{Y_\alpha\}$ need not be a local orthonormal frame for Q with respect to g, that is Y_α need not satisfy $Y_\alpha = \phi X_\alpha$ where $\{X_\alpha\}$ is an orthonormal frame for L with respect to g.

Proof
Let (x_α, y_α, z) be the local coordinates given by theorem 3.1.

From theorem 3.1 (i) we have

$$\eta = \frac{1}{2}(dz - y_\alpha dx_\alpha).$$

This implies that

$$d\eta = -\frac{1}{2} dy_\alpha \wedge dx_\alpha, \qquad \text{and} \qquad \xi = 2\frac{\partial}{\partial z}.$$

Thus we have

$$E = \text{span}\{\frac{\partial}{\partial z}\}.$$

Furthermore theorem 3.1 (ii) implies that

$$L = \text{span}\{\frac{\partial}{\partial y_\alpha}\}.$$

We now define a local frame $\{X_\alpha, Y_\alpha, \xi\}$ of TM, which need not be an orthonormal frame of TM with respect to g.

Let $X_\alpha = \frac{\partial}{\partial y_\alpha}$, note that while $\{X_\alpha\}$ need not be an orthonormal frame for L with respect to g, it is a local frame of L.

Let

$$\begin{aligned}\theta^\alpha &= -\frac{1}{2}(i(X_\alpha)d\eta) \\ &= \frac{1}{4}(i(\frac{\partial}{\partial y_\alpha})dy_\beta \wedge dx_\beta) \\ &= \frac{1}{4} dx_\alpha.\end{aligned}$$

Clearly $\{\theta^\alpha\}$ is a local frame of $Q^* = \text{ann}\{L \oplus E\}$.

Define local vector fields Y_α, $\alpha = 1, \ldots, n$ by

$$Y_\alpha = \frac{\partial}{\partial x_\alpha} + f_{\alpha\beta}\frac{\partial}{\partial y_\beta} + y_\alpha \frac{\partial}{\partial z}$$

for some functions $f_{\alpha\beta}$.

Then for any α, we have

$$\begin{aligned}\eta(Y_\alpha) &= \frac{1}{2}\left(dz - y_\gamma dx_\gamma\right)\left(\frac{\partial}{\partial x_\alpha} + f_{\alpha\beta}\frac{\partial}{\partial y_\beta} + y_\alpha\frac{\partial}{\partial z}\right)\\ &= \frac{1}{2}(y_\alpha - y_\alpha)\\ &= 0,\end{aligned}$$

which implies that $Y_\alpha \in H$, for any α.

Furthermore for any α, β we have

$$\begin{aligned}\theta^\alpha\left(Y_\beta\right) &= \frac{1}{4}dx_\alpha\left(\frac{\partial}{\partial x_\beta} + f_{\beta\gamma}\frac{\partial}{\partial y_\gamma} + y_\alpha\frac{\partial}{\partial z}\right)\\ &= \frac{1}{4}\delta_{\alpha\beta},\end{aligned}$$

which implies that $Y_\alpha \notin L$, for any α.

So there exist functions $f_{\alpha\beta}$ such that $Y_\alpha \in \Gamma Q$. Thus $Y_1,\ldots,Y_n$ are n linearly independent vector fields that span Q.

$f_{\alpha\beta} = f_{\beta\alpha}$ as for any α, β, we have

$$\begin{aligned}0 &= d\eta(Y_\alpha, Y_\beta)\\ &= -\frac{1}{2}(dy_\gamma \wedge dx_\gamma)\\ &\qquad\left(\frac{\partial}{\partial x_\alpha} + f_{\alpha\delta}\frac{\partial}{\partial y_\delta} + y_\alpha\frac{\partial}{\partial z}, \frac{\partial}{\partial x_\beta} + f_{\beta\varepsilon}\frac{\partial}{\partial y_\varepsilon} + y_\beta\frac{\partial}{\partial z}\right)\\ &= -\frac{1}{4}(f_{\alpha\beta} - f_{\beta\alpha}).\end{aligned}$$

□

Clearly

$$[Y_\alpha, \frac{\partial}{\partial y_\beta}],\ [Y_\alpha, \frac{\partial}{\partial z}], [Y_\alpha, Y_\beta] \in L \oplus E.$$

Furthermore when $f_{\alpha\beta} = 0$ for any α, β and g is a Riemannian metric with $\{X'_\alpha = 2X_\alpha\}$ a local orthonormal frame of L with respect to g then (M, ϕ, ξ, η, g) is the Sasakian structure given by theorem 3.2 and $\phi X'_\alpha = 2Y_\alpha$.

The following theorem gives a family of flat Legendre foliations on contact metric manifolds which are not Sasakian.

Theorem 3.5

If (M, ϕ, ξ, η, g) is a contact metric manifold such that the curvature tensor field R satisfies

(i) $R(\xi, W)\xi = 0$ $\quad \forall\, W \in \Gamma TM.$

(ii) $R(X, X')\xi = 0$ $\quad \forall\, X, X' \in D(-1).$

Then D(-1) defines a flat Legendre foliation, where D(-1) is the distribution defined by the -1 eigenspace of h as in theorem 1.36.

Proof

For any $X \in D(-1)$ we have

$$hX = -X.$$

Thus for any $X \in D(-1)$ we have

(a) $$\nabla_X \xi = -\phi X - \phi hX \qquad \text{by 1.35}$$
$$= -\phi X + \phi X$$
$$= 0.$$

To show that D(-1) is completely integrable, we note that for any $X, X' \in D(-1)$

$$0 = R(X, X')\xi$$
$$= \nabla_X \nabla_{X'} \xi - \nabla_{X'} \nabla_X \xi - \nabla_{[X, X']} \xi$$
$$= \phi[X, X'] + \phi h[X, X'] \qquad \text{by (a) and 1.35,}$$

and

$$\begin{aligned} p_E([X, X']) &= g([X, X'], \xi) \\ &= g(X, \nabla_{X'} \xi) - g(X', \nabla_X \xi) \\ &= 0 \qquad \text{by (a).} \end{aligned}$$

From the above we obtain

$$\begin{aligned} & \phi h[X, X'] = - \phi[X, X'] \\ \Rightarrow \quad & \phi^2 h[X, X'] = - \phi^2[X, X'] \\ \Rightarrow \quad & h[X, X'] = - [X, X'] \qquad \text{by 1.2 and 1.33 as } p_E([X, X']) = 0. \end{aligned}$$

Thus we have

$$[X, X'] \in D(-1) \qquad \forall\, X, X' \in D(-1).$$

Hence D(-1) is completely integrable. Furthermore by theorem 1.36 D(-1) is a n-dimensional subbundle of H, thus D(-1) defines a Legendre foliation $\mathcal{F}$ on (M, ϕ, ξ, η, g).

$\mathcal{F}$ is flat as for any $X \in D(-1)$ we have

$$\begin{aligned} 0 &= R(X, \xi)\xi \\ &= \nabla_X \nabla_\xi \xi - \nabla_\xi \nabla_X \xi - \nabla_{[X, \xi]} \xi \\ &= \phi[X, \xi] + \phi h[X, \xi] \qquad \text{by 1.28, (a) and 1.35.} \end{aligned}$$

Therefore by 1.2, 1.12 and 1.33 we have

$$h[X, \xi] = - [X, \xi]$$

which implies that

$$[X, \xi] \in D(-1) \qquad \forall\, X \in D(-1).$$

Thus ξ is a foliate vector field so $\mathcal{F}$ is flat by lemma 2.13 (i). □

In the following we give an example of a contact manifold and introduce five different contact metric structures on it. We then define a Legendre foliation on each of the resultant contact metric manifolds, show that they are all flat Legendre foliations and examine their properties.

Example 3.1 : Flat Legendre foliation

Let

$$M = R^5 - \{y_1 \text{ axis}\}$$
$$= \{(x_1, x_2, y_1, y_2, z) \mid x_i, y_i, z \in R, y_1 \neq 0\}$$

and

$$\eta = \frac{1}{2}(dz - y_1 dx_1 - y_2 dx_2).$$

Then we have

$$d\eta = -\frac{1}{2}(dy_1 \wedge dx_1 + dy_2 \wedge dx_2)$$

which by calculation gives

$$\eta \wedge (d\eta)^2 = -\frac{1}{8}(dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2 \wedge dz)$$
$$\neq 0 \text{ everywhere on } M$$

thus (M, η) is a contact manifold.

Consider the vector fields

$$X_1 = 2\frac{\partial}{\partial y_1} \qquad \text{and} \qquad X_2 = 2\frac{\partial}{\partial y_2}.$$

These are global vector fields and belong to H the contact distribution of (M, η).

Let L be the subbundle of H given by

$$L = \text{span}\{X_1, X_2\}.$$

L is a completely integrable 2-dimensional subbundle of H therefore L defines a Legendre foliation on (M, η). This is a flat Legendre foliation as the coordinates (x_1, x_2, y_1, y_2, z) satisfy (i) and (ii) of theorem 3.1. We now define five different contact metric structures on (M, η) and use L to define a Legendre foliation on each of the different contact metric manifolds.

(a) Let (M, ϕ^c, ξ, η, g^c) be the canonical Sasakian manifold given by theorem 3.2. Thus (ϕ^c, ξ, η, g^c) is given by :

$$g^c = \frac{1}{4}\begin{bmatrix} 1+(y_1)^2 & y_1y_2 & 0 & 0 & -y_1 \\ y_1y_2 & 1+(y_2)^2 & 0 & 0 & -y_2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -y_1 & -y_2 & 0 & 0 & 1 \end{bmatrix}$$

$$\phi^c = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & y_1 & y_2 & 0 \end{bmatrix}$$

$$\xi = 2\frac{\partial}{\partial z}.$$

Hence

$$Y_1^c = \phi^c X_1 = 2\left(\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial z}\right) \qquad Y_2^c = \phi^c X_2 = 2\left(\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial z}\right),$$

so $\{X_1, X_2, Y_1^c, Y_2^c, \xi\}$ is an orthonormal frame for TM with respect to g^c.

Calculating the Lie brackets we get

$$[X_1, X_2] = 0 \qquad [Y_1^c, Y_2^c] = 0$$
$$[X_1, \xi] = 0 \qquad [X_2, \xi] = 0$$
$$[Y_1^c, \xi] = 0 \qquad [Y_2^c, \xi] = 0$$
$$[X_1, Y_1^c] = 2\xi \qquad [X_1, Y_2^c] = 0$$
$$[X_2, Y_1^c] = 0 \qquad [X_2, Y_2^c] = 2\xi.$$

Thus $L = \text{span}\{X_1, X_2\}$ defines a flat Legendre foliation $\mathcal{F}^c$ on the Sasakian manifold $(M, \phi^c, \xi, \eta, g^c)$. It can be shown that $\mathcal{F}^c$ is totally geodesic.

(b) We now select a different contact metric structure (ϕ', ξ, η, g') on (M, η), so that L defines a flat Legendre foliation $\mathcal{F}'$ on $(M, \phi', \xi, \eta, g')$.

Let (ϕ', ξ, η, g') be the contact metric structure given by

$$g' = \frac{1}{4}\begin{bmatrix} 1+(y_1)^2+z^2 & y_1y_2 & z & 0 & -y_1 \\ y_1y_2 & 1+(y_2)^2+z^2 & 0 & z & -y_2 \\ z & 0 & 1 & 0 & 0 \\ 0 & z & 0 & 1 & 0 \\ -y_1 & -y_2 & 0 & 0 & 1 \end{bmatrix}$$

$$\phi' = \begin{bmatrix} z & 0 & 1 & 0 & 0 \\ 0 & z & 0 & 1 & 0 \\ -1-z^2 & 0 & -z & 0 & 0 \\ 0 & -1-z^2 & 0 & -z & 0 \\ y_1z & y_2z & y_1 & y_2 & 0 \end{bmatrix}$$

$$\xi = 2\frac{\partial}{\partial z}.$$

Thus we have

$$Y_1' = \phi' X_1 \qquad\qquad Y_2' = \phi' X_2$$
$$= 2(\frac{\partial}{\partial x_1} - z\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial z}) \qquad = 2(\frac{\partial}{\partial x_2} - z\frac{\partial}{\partial y_2} + y_2\frac{\partial}{\partial z})$$
$$= - zX_1 + Y_1^c \qquad\qquad = - zX_2 + Y_2^c,$$

so $\{X_1, X_2, Y_1', Y_2', \xi\}$ is an orthonormal frame for TM with respect to g'.

Calculating the Lie brackets we get

$$[X_1, X_2] = 0$$
$$[Y_1', Y_2'] = 2y_2X_1 - 2y_1X_2$$
$$[X_1, \xi] = 0 \qquad\qquad [X_2, \xi] = 0$$
$$[Y_1', \xi] = 2X_1 \qquad\qquad [Y_2', \xi] = 2X_2$$
$$[X_1, Y_1'] = 2\xi \qquad\qquad [X_1, Y_2'] = 0$$
$$[X_2, Y_1'] = 0 \qquad\qquad [X_2, Y_2'] = 2\xi.$$

Thus $L = \text{span}\{X_1, X_2\}$ defines a flat Legendre foliation $\mathcal{F}'$ on the contact metric manifold $(M, \phi', \xi, \eta, g')$. It can be shown that $\mathcal{F}'$ is totally geodesic.

By calculation for $\alpha = 1, 2$

$$h'X_\alpha = X_\alpha, \qquad\qquad h'Y_\alpha = - Y_\alpha$$

hence $(M, \phi', \xi, \eta, g')$ is not a K-contact manifold.

c) We now select a different contact metric structure (ϕ'', ξ, η, g'') on (M, η), so that L defines a flat Legendre foliation $\mathcal{F}''$ on $(M, \phi'', \xi, \eta, g'')$.

Let (ϕ'', ξ, η, g'') be the contact metric structure given by

$$g'' = \frac{1}{4}\begin{bmatrix} 1+(y_1)^2+(y_2)^2 & y_1y_2 & y_2 & 0 & -y_1 \\ y_1y_2 & 1+(y_1)^2+(y_2)^2 & 0 & y_1 & -y_2 \\ y_2 & 0 & 1 & 0 & 0 \\ 0 & y_1 & 0 & 1 & 0 \\ -y_1 & -y_2 & 0 & 0 & 1 \end{bmatrix}$$

$$\phi'' = \begin{bmatrix} y_2 & 0 & 1 & 0 & 0 \\ 0 & y_1 & 0 & 1 & 0 \\ -1-(y_2)^2 & 0 & -y_2 & 0 & 0 \\ 0 & -1-(y_1)^2 & 0 & -y_1 & 0 \\ y_1y_2 & y_1y_2 & y_1 & y_2 & 0 \end{bmatrix}$$

$$\xi = 2\frac{\partial}{\partial z}.$$

Thus

$$Y_1'' = \phi X_1 = 2\left(\frac{\partial}{\partial x_1} - y_2\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial z}\right) = -y_2X_1 + Y_1^c$$

$$Y_2'' = \phi X_2 = 2\left(\frac{\partial}{\partial x_2} - y_1\frac{\partial}{\partial y_2} + y_2\frac{\partial}{\partial z}\right) = -y_1X_2 + Y_2^c,$$

so $\{X_1, X_2, Y_1'', Y_2'', \xi\}$ is an orthonormal frame for TM with respect to g".

Calculating the Lie brackets we get

$$[X_1, X_2] = 0$$
$$[Y_1'', Y_2''] = 2y_2X_2 - 2y_1X_1$$
$$[X_1, \xi] = 0 \qquad [X_2, \xi] = 0$$
$$[Y_1'', \xi] = 0 \qquad [Y_2'', \xi] = 0$$
$$[X_1, Y_1''] = 2\xi \qquad [X_1, Y_2''] = -2X_2$$
$$[X_2, Y_1''] = -2X_1 \qquad [X_2, Y_2''] = 2\xi$$

Thus L = span$\{X_1, X_2\}$ defines a flat Legendre foliation $\mathcal{F}''$ on the contact metric manifold $(M, \phi'', \xi, \eta, g'')$. $\mathcal{F}''$ is a harmonic but not a totally geodesic Legendre foliation as

$$\nabla_{X_1} X_1 = 0, \qquad \nabla_{X_1} X_2 = Y_1'' + Y_2''$$
$$\nabla_{X_2} X_1 = Y_1'' + Y_2'' \qquad \nabla_{X_2} X_2 = 0.$$

Furthermore we have

$$\nabla_{X_1} Y_2'' = - X_2 - y_1 Y_1'' \qquad \nabla_{Y_2''} X_1 = X_2 - y_1 Y_1''$$
$$\nabla_{X_2} Y_1'' = - X_1 - y_2 Y_2'' \qquad \nabla_{Y_1''} X_2 = X_1 - y_2 Y_2''.$$

Thus

$$A_{Y_1''} X_2 = - p_L(\nabla_{X_2} Y_1'') = X_1 \qquad A_{Y_2''} X_1 = - p_L(\nabla_{X_1} Y_2'') = X_2$$

which shows that

$$A_{Y_1''} X_2 \neq A_{Y_2''} X_1.$$

By calculation we have

$$h''W = 0 \qquad \forall\, W \in \Gamma TM.$$

hence $(M, \phi'', \xi, \eta, g'')$ is a K-contact manifold.

But $(M, \phi'', \xi, \eta, g'')$ is not a Sasakian manifold as

$$N^1(X_1, X_2) = - [X_1, X_2] + [\phi''X_1, \phi''X_2] - \phi''[\phi''X_1, X_2] - \phi''[X_1, \phi''X_2]$$
$$= 2y_2X_1 - 2y_1X_2 - 2Y_1'' + 2Y_2''.$$

d) We now select a different contact metric structure (ϕ, ξ, η, g) on (M, η), so that L defines a flat Legendre foliation $\mathcal{F}$ on (M, ϕ, ξ, η, g).

Let (ϕ, ξ, η, g) be the contact metric structure given by

$$g = \frac{1}{4}\begin{bmatrix} 1+(y_1)^2 & y_1y_2 & 0 & 0 & -y_1 \\ y_1y_2 & -1+(y_2)^2 & 0 & 0 & -y_2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -y_1 & -y_2 & 0 & 0 & 1 \end{bmatrix}$$

$$\phi = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & y_1 & -y_2 & 0 \end{bmatrix}$$

$$\xi = 2\frac{\partial}{\partial z}.$$

It can be shown that (ϕ, ξ, η, g) is a Sasakian contact metric structure.

Thus

$$\begin{aligned} Y_1 &= \phi X_1 \\ &= 2\left(\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial z}\right) \\ &= Y_1^c \end{aligned} \qquad \begin{aligned} Y_2 &= \phi X_2 \\ &= -2\left(\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial z}\right). \\ &= -Y_2^c, \end{aligned}$$

so $\{X_1, X_2, Y_1, Y_2, \xi\}$ is an orthonormal frame for TM with respect to g. Note as $g(Y_2, Y_2) = -1$, g is not a Riemannian metric.

Calculating the Lie brackets we get

$$[X_1, X_2] = 0 \qquad [Y_1, Y_2] = 0$$
$$[X_1, \xi] = 0 \qquad [X_2, \xi] = 0$$
$$[Y_1, \xi] = 0 \qquad [Y_2, \xi] = 0$$
$$[X_1, Y_1] = 2\xi \qquad [X_1, Y_2] = 0$$
$$[X_2, Y_1] = 0 \qquad [X_2, Y_2] = -2\xi$$

Thus $L = \text{span}\{X_1, X_2\}$ defines a flat Legendre foliation $\mathcal{F}$ on the contact metric manifold (M, ϕ, ξ, η, g). It can be shown that $\mathcal{F}$ is totally geodesic.

e) We now select a different contact metric structure $(\phi^{\#}, \xi, \eta, g^{\#})$ on (M, η), so that L defines a flat Legendre foliation $\mathcal{F}^{\#}$ on $(M, \phi^{\#}, \xi, \eta, g^{\#})$.

Let the subbundle $L^{\#} \subset H$ be given by

$$X_1^{\#} = 2y_1\frac{\partial}{\partial y_1} = y_1 X_1 \qquad X_2^{\#} = 2y_1\frac{\partial}{\partial y_2} = y_1 X_2$$

Let the subbundle $Q^{\#} \subset H$ be given by

$$Y_1^{\#} = \frac{2}{y_1}\left(\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial z}\right) = \frac{2}{y_1} Y_1^{c} \qquad Y_2^{\#} = \frac{2}{y_1}\left(\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial z}\right) = \frac{2}{y_1} Y_2^{c}.$$

Then $\{X_1^{\#}, X_2^{\#}, Y_1^{\#}, Y_2^{\#}, \xi\}$ is an orthonormal frame for TM with respect to the metric $g^{\#}$ defined by

$$g^{\#}|_{L^{\#}} = \frac{1}{(y_1)^2} g^{c}, \quad g^{\#}|_{Q^{\#}} = (y_1)^2 g^{c},$$

$$g^{\#}|_{E^{\#}} = \eta \otimes \eta \text{ and } g^{\#} = 0 \text{ otherwise.}$$

Let $\phi^{\#}$ be the tensor defined by

$$\phi^{\#}X_{\alpha}^{\#} = Y_{\alpha}^{\#}, \qquad \phi^{\#}Y_{\alpha}^{\#} = - X_{\alpha}^{\#}, \qquad \phi^{\#}\xi = 0 \qquad \text{for } \alpha = 1, 2.$$

Then $(\phi^{\#}, \xi, \eta, g^{\#})$ is a contact metric structure on the contact manifold (M, η).

Calculating the Lie brackets we obtain

$$[X_1^{\#}, X_2^{\#}] = 2X_2^{\#} \qquad [Y_1^{\#}, Y_2^{\#}] = 0$$
$$[X_1^{\#}, \xi] = 0 \qquad [X_2^{\#}, \xi] = 0$$
$$[Y_1^{\#}, \xi] = 0 \qquad [Y_2^{\#}, \xi] = 0$$
$$[X_1^{\#}, Y_1^{\#}] = - 2Y_1^{\#} + 2\xi \qquad [X_1^{\#}, Y_2^{\#}] = - 2Y_2^{\#}$$
$$[X_2^{\#}, Y_1^{\#}] = 0 \qquad [X_2^{\#}, Y_2^{\#}] = - 2\xi.$$

Thus $L = \text{span}\{X_1^{\#}, X_2^{\#}\} = \text{span}\{X_1, X_2\}$ defines a flat Legendre foliation $\mathcal{F}^{\#}$ on the contact metric manifold $(M, \phi^{\#}, \xi, \eta, g^{\#})$.

Furthermore by calculation

$$\nabla^{L}_{X_1^{\#}} X_1^{\#} = 2X_1^{\#} \qquad \nabla^{L}_{X_1^{\#}} X_2^{\#} = 2X_2^{\#}$$

$$\nabla^{L}_{X_2^{\#}} X_2^{\#} = 0 \qquad \nabla^{L}_{X_2^{\#}} X_2^{\#} = 0,$$

and

$$\nabla_{X_1^{\#}} X_1^{\#} = 0 \qquad \nabla_{X_1^{\#}} X_2^{\#} = 0$$

$$\nabla_{X_2^{\#}} X_2^{\#} = - 2X_2^{\#} \qquad \nabla_{X_2^{\#}} X_2^{\#} = 2X_1^{\#}.$$

Thus

$$\nabla^{L}_{X^{\#}_{\alpha}} X^{\#}_{\beta} \neq \nabla_{X^{\#}_{\alpha}} X^{\#}_{\beta}.$$

From above we can show that $R^{L} \equiv 0$ but we have

$$\begin{aligned} R(X^{\#}_1, X^{\#}_2)X^{\#}_2 &= \nabla_{X^{\#}_1}\nabla_{X^{\#}_2} X^{\#}_2 - \nabla_{X^{\#}_2}\nabla_{X^{\#}_1} X^{\#}_2 - \nabla_{[X^{\#}_1, X^{\#}_2]}X^{\#}_2 \\ &= \nabla_{X^{\#}_1} 2X^{\#}_1 - \nabla_{2X^{\#}_2} X^{\#}_2 \\ &= 4X^{\#}_1 \neq 0. \end{aligned}$$

Hence the leaves of $\mathcal{F}^{\#}$ are totally geodesic but not flat.

Furthermore by calculation we have

$$h^{\#}W = 0 \qquad \forall\, W \in \Gamma TM.$$

hence $(M, \phi^{\#}, \xi, \eta, g^{\#})$ is a K-contact manifold.

But $(M, \phi^{\#}, \xi, \eta, g^{\#})$ is not a Sasakian manifold as

$$\begin{aligned} N^{1}(X^{\#}_1, X^{\#}_2) &= -[X^{\#}_1, X^{\#}_2] + [\phi^{\#}X^{\#}_1, \phi^{\#}X^{\#}_2] - \phi^{\#}[\phi^{\#}X^{\#}_1, X^{\#}_2] \\ &\qquad - \phi^{\#}[X^{\#}_1, \phi^{\#}X^{\#}_2] \\ &= 4X^{\#}_2 \neq 0. \end{aligned}$$

Example 3.1 (e) shows that the second part of theorem 3.2 can not be generalized to a flat Legendre foliation on an arbitrary contact metric manifold (M, ϕ, ξ, η, g). Therefore the leaves of a flat Legendre foliation have a natural affine structure given by ∇^{L} but they need not be flat.

CHAPTER 4. NON-DEGENERATE LEGENDRE FOLIATIONS

In this chapter we investigate the case where Π is non-degenerate, giving non-degenerate Legendre foliations. The properties of a non-degenerate Legendre foliation will be investigated and examples given.

Among the possible contact metric structures on a contact manifold which carries a non-degenerate Legendre foliation we identify a canonical contact metric structure and show that it is unique. The properties of the canonical contact metric structure and those of the family it belongs to will be investigated. Then the corresponding canonical connection will be defined and its properties given.

4 (a). THE CANONICAL CONTACT METRIC STRUCTURE

Let $\mathcal{F}$ be a Legendre foliation, not necessarily non-degenerate, on a contact manifold (M, η), then the subbundles $H \subset TM$ and $L \subset TM$ are fixed and known. But the subbundle $Q = \phi L \subset TM$ is determined by the contact metric structure (ϕ, ξ, η, g). A different contact metric structure (ϕ', ξ, η, g') on (M, η) may give a subbundle $Q' = \phi' L \subset TM$ different from Q. Hence the Legendre foliation $\mathcal{F}'$ defined by L on the contact metric manifold $(M, \phi', \xi, \eta, g')$ is different from the Legendre foliation $\mathcal{F}$ defined by L on the contact metric manifold (M, ϕ, ξ, η, g). But as Π is not determined by the contact metric structure $\mathcal{F}$ and $\mathcal{F}'$ will be of the same classification.

Alternatively for a contact manifold (M, η) with a completely integrable n-dimensional subbundle $L \subset H$, and a contact metric structure (ϕ, ξ, η, g) L defines a Legendre foliation on the contact metric manifold (M, ϕ, ξ, η, g). Now we fix $g|_L$ and generate a different Legendre foliation defined by L. Let $\{X_\alpha\}$ be a local orthonormal frame of L with respect to $g|_L$ then $\{\phi X_\alpha\}$ is a local orthonormal frame of $Q = \text{span}\{\phi X_\alpha\}$ with respect to $g|_Q$. For some functions c_α on M we define a contact metric structure (ϕ', ξ, η, g') where ϕ' is defined by

(a) $\phi' X_\alpha = \phi X_\alpha + c_\alpha X_\alpha, \qquad \phi'(\phi' X_\alpha) = - X_\alpha, \qquad \phi'\xi = 0,$

and g' is defined by

(b) $g'(\phi' X_\alpha, \phi' X_\beta) = g'(X_\alpha, X_\beta) = g(X_\alpha, X_\beta)$

$g'(X_\alpha, \phi' X_\beta) = g'(X_\alpha, \xi) = g'(\phi' X_\alpha, \xi) = 0.$

Note: $TM = \text{span}\{X_\alpha, \phi' X_\alpha, \xi\} = \text{span}\{X_\alpha, \phi X_\alpha, \xi\}$.

A series of straightforward calculations shows that (ϕ', ξ, η, g') is a contact metric structure. For example for any α, β we have

(i) $\phi'^2(\phi' X_\alpha) = \phi'(\phi'(\phi' X_\alpha)) = - \phi' X_\alpha$ by (a).

(ii)
$$\begin{aligned} d\eta(X_\alpha, \phi' X_\beta) &= d\eta(X_\alpha, \phi X_\beta + c_\beta X_\beta) && \text{by (a)} \\ &= d\eta(X_\alpha, \phi X_\beta) && \text{as } d\eta(X, X') = 0, \forall\, X, X' \in \Gamma L \\ &= - g(X_\alpha, X_\beta) && \text{by 1.2 and 1.3} \\ &= g'(X_\alpha, - X_\beta) && \text{by (b)} \\ &= g'(X_\alpha, \phi'(\phi' X_\beta)) && \text{by (a).} \end{aligned}$$

Note: $Q' \neq Q$ unless $c_\alpha = 0$ for all α as

$$\begin{aligned} Q' &= \text{span}\{\phi' X_\alpha\} \\ &= \text{span}\{\phi X_\alpha + c_\alpha X_\alpha\}. \end{aligned}$$

Hence if $c_\alpha \neq 0$ for some α, L defines a Legendre foliation $\mathcal{F}'$ on $(M, \phi', \xi, \eta, g')$ different from the Legendre foliation $\mathcal{F}$ on (M, ϕ, ξ, η, g), since (M, ϕ, ξ, η, g) and $(M, \phi', \xi, \eta, g')$ are different contact metric manifolds.

In example 3.1 of chapter 3 the contact manifold (M, η) and the subbundle L of TM are fixed and the contact metric structure is changed to give Legendre foliations on different contact metric manifolds. Note that

$$g^c|_L = g'|_L = g''|_L = dy_1 \otimes dy_1 + dy_2 \otimes dy_2.$$

For a non-degenerate Legendre foliation $\mathcal{F}$ on the contact manifold (M, η) $\prod$ is a non-degenerate, symmetric, bilinear form on L so we can use $\prod$ to define a semi-Riemannian metric on L, which can be extended to a contact metric structure on (M, η) using the tensor λ. Furthermore by fixing the semi-Riemannian metric on L we can define a family of contact metric structures. Then from this family we pick out a particular contact metric structure, (ϕ^c, ξ, η, g^c), which we call the canonical contact metric structure of the non-degenerate Legendre foliation.

Lemma 4.1

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on a contact manifold (M, η). Then there exists a contact metric structure (ϕ, ξ, η, g) on (M, η) with

$$g|_L = \frac{1}{4}\prod.$$

Proof

We build up a contact metric structure (ϕ, ξ, η, g) for (M, η) with $g|_L = \frac{1}{4}\prod$. Let $\{X_\alpha\}$, $\alpha = 1, \ldots\ldots, n$, be a local orthonormal frame for L with respect to $\frac{1}{4}\prod$, choose a local frame $\{Y_\alpha\}$ of $H \cap L'$, where

(a) $$\lambda Y_\alpha = X_\alpha$$

λ is defined by 2.14, and $L' = \{W \in TM \mid W \notin L\}$ is the complement of L.

Let $Q = \text{span}\{Y_\alpha\}$ so

$\lambda|_Q: Q \to L$ is an isomorphism.

Let (ϕ, ξ, η, g) be a contact metric structure for (M, η) such that

(b) $$g|_L = \frac{1}{4}\Pi$$

(c) $$Y_\alpha = \phi X_\alpha \qquad \forall\ \alpha.$$

These conditions define the contact metric structure (ϕ, ξ, η, g). From (c) we have for any α that

$$\begin{aligned} \lambda(\phi X_\alpha) &= \lambda Y_\alpha \\ &= X_\alpha \end{aligned} \qquad \text{by (a),}$$

which implies that

$$(\lambda\phi)|_L = \text{identity on L.}$$

Thus

$$\phi|_L = (\lambda|_Q)^{-1}.$$

Also by (c) we have for any α that

$$\begin{aligned} \phi Y_\alpha &= \phi(\phi X_\alpha) \\ &= -X_\alpha && \text{by 1.2} \\ &= -\lambda Y_\alpha && \text{by (a).} \end{aligned}$$

Thus

$$\phi|_Q = -\lambda|_Q.$$

Furthermore for any α, β we have

$$g(\phi X_\alpha, \phi X_\beta) = g(X_\alpha, X_\beta) \qquad \text{by 1.9}$$

$$= \frac{1}{4}\Pi(X_\alpha, X_\beta) \qquad \text{by (b)}$$
$$= \frac{1}{4}\Pi(\lambda Y_\alpha, \lambda Y_\beta) \qquad \text{by (a)}$$
$$= \bar{\Pi}(Y_\alpha, Y_\beta) \qquad \text{by 2.17}$$
$$= \bar{\Pi}(\phi X_\alpha, \phi X_\alpha) \qquad \text{by (c).}$$

Thus

$$g|_Q = \bar{\Pi}|_Q.$$

Hence a contact metric structure (ϕ, ξ, η, g) with $g|_L = \frac{1}{4}\Pi$ can be defined by

(4.2)
$$g|_L = \frac{1}{4}\Pi \quad \text{and} \quad \phi|_L = (\lambda|_Q)^{-1}$$
$$g|_Q = \bar{\Pi}|_Q \qquad \phi|_Q = -\lambda|_Q$$
$$g|_E = \eta \otimes \eta \qquad \phi = 0 \quad \text{otherwise}$$
$$g = 0 \quad \text{otherwise.}$$

□

Note: (ϕ, ξ, η, g) is not a unique contact metric structure on (M, η), for example choose the local frame of $H \cap L'$ given by $\{Y'_\alpha = Y_\alpha + X_\alpha\}$ and define a subbundle of H by $Q' = \text{span}\{Y'_\alpha\}$. Then for any α we have

$$\lambda Y'_\alpha = \lambda(Y_\alpha + X_\alpha)$$
$$= \lambda Y_\alpha \qquad \text{as the kernel of } \lambda \text{ is } L \oplus E$$
$$= X_\alpha.$$

So replacing Q by Q' in 4.2 gives another contact metric structure (ϕ', ξ, η, g') of (M, η).

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on (M, η). The collection of all contact metric manifolds having $g|_L = \frac{1}{4}\Pi$ is called *the canonical family of contact metric structures* for $(M, \mathcal{F})$. Furthermore for any non-degenerate Legendre foliation we

can obtain an invariant formulation for $\bar{\Pi}$ from the coframe, $\{\mathcal{E}^\alpha, \theta^\alpha, \eta\}$, induced by any contact metric structure (ϕ, ξ, η, g) with $g|_L = \frac{1}{4}\Pi$.

Lemma 4.3

For a non-degenerate Legendre foliation $\mathcal{F}$ on the contact manifold (M, η)

$$\bar{\Pi} = k_{\alpha\beta}\, \theta^\alpha \otimes \theta^\beta$$

where $\theta^\alpha = -\frac{1}{2}\, i(X_\alpha)d\eta$, $\{X_\alpha\}$ $\alpha = 1,\ldots\ldots, n$, is a local orthonormal frame for L with respect to $\frac{1}{4}\Pi$, and

$$k = [\, k_{\alpha\beta}\,] = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \text{ and } (p, q) \text{ is the signature of } \Pi.$$

Proof

Let (ϕ, ξ, η, g) be a contact metric structure with $g|_L = \frac{1}{4}\Pi$ given by 4.2 and let $\{\mathcal{E}^\alpha, \theta^\alpha, \eta\}$ be the local coframe induced by it. Hence

$$\begin{aligned} \bar{\Pi}|_Q &= g|_Q && \text{by 4.2} \\ &= k_{\alpha\beta}\, \theta^\alpha \otimes \theta^\beta && \text{by lemma 2.5 (iii).} \end{aligned}$$

Let $\{X_\alpha\}$ $\alpha = 1,\ldots\ldots,n$, be a local orthonormal frame for L with respect to $\frac{1}{4}\Pi$ and $\theta^\alpha = -\frac{1}{2}\, i(X_\alpha)d\eta$. Then θ^α is the local coframe of $Q^* = \text{ann}\{L \oplus E\}$ induced by any contact metric structure (ϕ, ξ, η, g) with $g|_L = \frac{1}{4}\Pi$ hence

$$\bar{\Pi} = k_{\alpha\beta}\, \theta^\alpha \otimes \theta^\beta. \qquad \square$$

There are several equivalent definitions of the canonical family of contact metric structures defined by $g|_L = \frac{1}{4}\Pi$.

Lemma 4.4

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g). Then the following are equivalent

i) $g|_L = \frac{1}{4}\Pi.$

ii) $p_{L^\perp}([\xi, X]) = p_Q([\xi, X]) \qquad \forall X \in \Gamma L$
$= 2\phi X.$

iii) $\phi|_Q = -\lambda|_Q.$

Proof

(i) $\Leftrightarrow$ (ii) follows from 1.12 which shows that the projection

$$p_{L^\perp}([\xi, X]) = p_Q([\xi, X]) \qquad \forall X \in \Gamma L,$$

and 2.11 which states that for any vector fields X, X' tangent to the leaves of $\mathcal{F}$ we have

$$\begin{aligned} \frac{1}{4}\Pi(X, X') &= \frac{1}{2} g([\xi, X], \phi X') \\ &= g(-\frac{1}{2}\phi[\xi, X], X') \qquad \text{by 1.10.} \end{aligned}$$

Hence

$$\begin{aligned} g|_L = \frac{1}{4}\Pi \quad &\Leftrightarrow \quad g(X, X') = g(-\frac{1}{2}\phi[\xi, X], X') \\ &\Leftrightarrow \quad p_L(-\frac{1}{2}\phi[\xi, X]) = X \\ &\Leftrightarrow \quad \phi p_Q([\xi, X]) = -2X \qquad \text{by theorem 2.1} \\ &\Leftrightarrow \quad p_Q([\xi, X]) = 2\phi X \qquad \text{since } X \in \Gamma L. \end{aligned}$$

To prove (ii) $\Leftrightarrow$ (iii) for any $X \in \Gamma L$ use

$$\begin{aligned} -\phi(\phi X) &= X \qquad \text{by 1.2} \\ &= \frac{1}{2}\lambda[\xi, X] \qquad \text{by 2.1} \end{aligned}$$

$$= \frac{1}{2}\lambda(p_Q[\xi, X] + p_L[\xi, X])$$
$$= \frac{1}{2}\lambda(p_Q[\xi, X]) \qquad \text{since } L \subset \text{kernel } \lambda.$$

Hence

$$\text{(ii)} \quad \Leftrightarrow \quad \phi(\phi X) = -\lambda(\phi X)$$
$$\Leftrightarrow \quad \phi|_Q = -\lambda|_Q. \qquad \square$$

For a non-degenerate Legendre foliation, the tensor ϕ from any contact metric structure (ϕ, ξ, η, g) belonging to the canonical family of contact metric structures can be written in terms of the tensor ϕ' from any other contact metric structure (ϕ', ξ, η, g') from this family.

Lemma 4.5

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on the contact metric manifold (M, ϕ, ξ, η, g) with $g|_L = \frac{1}{4}\Pi$. Then if (ϕ', ξ, η, g') is any contact metric structure for (M, η) with $g'|_L = \frac{1}{4}\Pi$ there exist functions $c_{\alpha\beta}$ on M such that

$$\text{(i)} \qquad \phi' X_\alpha = \phi X_\alpha + c_{\alpha\beta} X_\beta$$

where $\{X_\alpha\}$ $\alpha = 1, \ldots\ldots, n$ is a local orthonormal frame for L with respect to $\frac{1}{4}\Pi$.

Furthermore for any α, β

$$\text{(ii)} \qquad c_{\alpha\gamma}\, k_{\gamma\beta} = c_{\beta\gamma}\, k_{\gamma\alpha}.$$

Proof

As $\{X_\alpha, \phi X_\alpha\}$ is a local frame for H and $\phi' X_\alpha \in \Gamma H$, for any α there exist functions $c_{\alpha\beta}, d_{\alpha\beta}$ on M such that

$$\phi' X_\alpha = d_{\alpha\beta}\phi X_\beta + c_{\alpha\beta} X_\beta.$$

So for any α, β we have

$$\begin{aligned} \Pi(\lambda\phi' X_\alpha, X_\beta) &= 4d\eta(\phi' X_\alpha, X_\beta) && \text{by 2.14} \\ &= 4g'(X_\alpha, X_\beta) && \text{by 1.3 and 1.9} \\ &= \Pi(X_\alpha, X_\beta) && \text{as } g'|_L = \tfrac{1}{4}\Pi, \end{aligned}$$

similarly we have for any α, β

$$\Pi(\lambda\phi X_\alpha, X_\beta) = \Pi(X_\alpha, X_\beta).$$

Thus for any α

$$\lambda\phi' X_\alpha = \lambda\phi X_\alpha = X_\alpha \qquad \text{as } \lambda\phi' X_\alpha, \lambda\phi X_\alpha, X_\alpha \in \Gamma L.$$

Using this, for any α we obtain

$$\begin{aligned} X_\alpha &= \lambda\phi' X_\alpha \\ &= \lambda(d_{\alpha\beta}\phi X_\beta + c_{\alpha\beta} X_\beta) \\ &= d_{\alpha\beta}\lambda\phi X_\beta && \text{as the kernel of } \lambda \text{ is } L \oplus E \\ &= d_{\alpha\beta} X_\beta, \end{aligned}$$

which implies that

$$d_{\alpha\beta} = \delta_{\alpha\beta}.$$

Thus for any α we have

$$\phi' X_\alpha = \phi X_\alpha + c_{\alpha\beta} X_\beta.$$

Let $\mathcal{F}'$ be the Legendre foliation defined by L on the contact metric manifold $(M, \phi', \xi, \eta, g')$. Then as (ϕ, ξ, η, g) and (ϕ', ξ, η, g') are the contact metric structures for the Legendre foliations $\mathcal{F}$ and $\mathcal{F}'$ respectively, by 2.2, we have for any α, β that the projections

$$p_E([\phi X_\alpha, \phi X_\beta]) = p_E([\phi' X_\alpha, \phi' X_\beta]) = 0.$$

Thus for any α, β we have

$$\begin{aligned} 0 &= p_E([\phi' X_\alpha, \phi' X_\beta]) \\ &= p_E([\phi X_\alpha + c_{\alpha\gamma} X_\gamma, \phi X_\beta + c_{\beta\delta} X_\delta]) && \text{by (i)} \\ &= p_E(c_{\alpha\gamma}[X_\gamma, \phi X_\beta] + c_{\beta\delta}[\phi X_\alpha, X_\delta]) \\ &\qquad \text{as } [X, X'],\ [\phi X, \phi X'],\ X, X',\ \phi X,\ \phi X' \in \Gamma H,\ \forall\ X, X' \in \Gamma L \\ &= 2c_{\alpha\gamma} k_{\gamma\beta} \xi - 2c_{\beta\delta} k_{\delta\alpha} \xi && \text{by theorem 2.4 (i).} \end{aligned}$$

Which implies

$$c_{\alpha\gamma} k_{\gamma\beta} = c_{\beta\gamma} k_{\gamma\alpha}. \qquad \square$$

We now define a *canonical coframe* for a non-degenerate Legendre foliation and then use it to define the canonical contact metric structure, which belongs to the canonical family of contact metric structures.

Theorem 4.6

For $\mathcal{F}$ a non-degenerate Legendre foliation on a contact manifold (M, η) there exists a canonical set of coframes $\{\mathcal{E}_c^\alpha, \theta_c^\alpha, \eta\}$ on (M, η) such that

(i) $\theta_c^\alpha \in \Gamma Q^*$

(ii) $d\eta = 2\delta_{\alpha\beta} \theta_c^\alpha \wedge \mathcal{E}_c^\beta$

(iii) $\overline{\Pi} = k_{\alpha\beta} \theta_c^\alpha \otimes \theta_c^\beta$

(iv) $\mathcal{L}_\xi \bar{\Pi} = -2\delta_{\alpha\beta}\theta_c^\alpha \otimes \mathcal{E}_c^\beta - 2\delta_{\alpha\beta}\mathcal{E}_c^\alpha \otimes \theta_c^\beta.$

This is theorem 5.4 Pang [2], with some changes of the constants due to our defining the exterior derivative and the interior product with the p + 1 factor included.

From Pang's proof of theorem 5.4, taking into consideration the changes of the constants we get the following relationships.

If $\{X_\alpha\}$ is a local orthonormal frame of L with respect to $\frac{1}{4}\Pi$ then

(4.7) (i) $\theta_c^\alpha = -\frac{1}{2} i(X_\alpha)\, d\eta.$

(ii) $\mathcal{E}_c^\alpha (X_\beta) = \delta_{\alpha\beta}.$

From 1.14 we have $i(\xi)d\eta = 0$ so using theorem 4.6 (ii) we can show that

$$\mathcal{E}_c^\alpha \in H^* = \mathrm{ann}(E).$$

Let $L_c^* = \mathrm{span}\{\mathcal{E}_c^\alpha\}$ then L_c^* is a subbundle of H^* and we get a decomposition of H^* as

$$\begin{aligned} H^* &= L_c^* \oplus Q^* \\ &= \mathrm{span}\{\mathcal{E}_c^\alpha\} \oplus \mathrm{span}\{\theta_c^\alpha\}. \end{aligned}$$

Now we use L_c^* to define Q^c a subbundle of $H \cap L'$ by

(4.8) $$\begin{aligned} Q^c &= \mathrm{ann}(L_c^* \oplus E^*) \\ &= \{Y \in TM \mid \mathcal{E}_c^\alpha(Y) = \eta(Y) = 0\}. \end{aligned}$$

Let $\{Y_\alpha^c\}$ be a local frame of Q^c such that

(4.9) $$\theta_c^\alpha(Y_\beta^c) = k_{\alpha\beta}.$$

We have for any α, β that

$$\begin{aligned}\frac{1}{2}k_{\alpha\beta}\, i(Y_\beta^c)d\eta &= k_{\alpha\beta}\, i(Y_\beta^c)\delta_{\delta\gamma}\theta_c^\delta \wedge \varepsilon_c^\gamma && \text{by theorem 4.6 (ii)}\\ &= k_{\alpha\beta}\delta_{\delta\gamma}\theta_c^\delta(Y_\beta^c)\varepsilon_c^\gamma \\ &= \varepsilon_c^\alpha && \text{by 4.9.}\end{aligned}$$

Thus we have

(4.10) $$\varepsilon_c^\alpha = \frac{1}{2}k_{\alpha\beta}\, i(Y_\beta^c)d\eta.$$

For any α, β we have

$$\begin{aligned}d\eta(Y_\alpha^c, X_\beta) &= 2\delta_{\delta\gamma}\theta_c^\delta \wedge \varepsilon_c^\gamma(Y_\alpha^c, X_\beta) && \text{by theorem 4.6 (ii)}\\ &= \delta_{\delta\gamma}\theta_c^\delta(Y_\alpha^c)\varepsilon_c^\gamma(X_\beta) \\ &= k_{\alpha\beta} && \text{by 4.7 (ii) and 4.9.}\end{aligned}$$

Now we define a tensor field λ^c: $Q^c \to L$ of type (1,1) by

$$\lambda^c Y_\alpha^c = X_\alpha$$

Then

$$\begin{aligned}\Pi(\lambda^c Y_\alpha^c, X_\beta) &= \Pi(X_\alpha, X_\beta) \\ &= 4k_{\alpha\beta} \\ &= 4d\eta(Y_\alpha^c, X_\beta) && \text{from above}\\ &= \Pi((\lambda Y_\alpha^c, X_\beta) && \text{by 2.14,}\end{aligned}$$

which implies that

$$\lambda^c = \lambda|_{Qc}.$$

From 4.7 (ii) it can be easily seen that

$$\frac{1}{4}\Pi = k_{\alpha\beta}\mathcal{E}_c^{\alpha} \otimes \mathcal{E}_c^{\beta}.$$

These properties allow us to define *the canonical contact metric structure of a non-degenerate Legendre foliation.*

Definition 4.11

The canonical contact metric structure for a non-degenerate Legendre foliation is (ϕ^c, ξ, η, g^c) where

(i)
$$g^c|_L = \frac{1}{4}\Pi$$
$$= k_{\alpha\beta}\mathcal{E}_c^{\alpha} \otimes \mathcal{E}_c^{\beta},$$

$$g^c|_{Qc} = \overline{\Pi}|_{Qc}$$
$$= k_{\alpha\beta}\,\theta_c^{\alpha} \otimes \theta_c^{\beta},$$

$$g^c|_E = \eta \otimes \eta,$$

$$g^c = 0 \quad \text{otherwise.}$$

(ii)
$$\phi^c|_L = (\lambda|_{Qc})^{-1},$$

$$\phi^c|_{Qc} = -\lambda|_{Qc},$$

$$\phi^c = 0 \quad \text{otherwise.}$$

Where $\{\mathcal{E}_c^\alpha , \theta_c^\alpha , \eta\}$ is the canonical coframe given by theorem 4.6, Q^c is the subbundle of H given by 4.8 and where

$$k = [\, k_{\alpha\beta} \,] = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \text{ and } (p, q) \text{ is the signature of } \Pi.$$

As

$$\begin{aligned} X_\alpha &= \lambda^c Y_\alpha^c \\ &= \lambda Y_\alpha^c \\ &= -\phi^c Y_\alpha^c \end{aligned}$$ by definition 4.11 (ii),

By 1.2 we have

(4.12) $$Y_\alpha^c = \phi^c X_\alpha.$$

$\{\mathcal{E}_c^\alpha, \theta_c^\alpha, \eta\}$ is the local orthonormal coframe induced by (ϕ^c, ξ, η, g^c) as given by lemma 2.5. From this and definition 4.11 (i) we can see that the canonical contact metric structure is a contact metric structure from the canonical family of contact metric structures whose induced coframe satisfies theorem 4.6. For a non-degenerate Legendre foliation on the contact manifold (M, η) let $(M, \phi^c, \xi, \eta, g^c)$ be *the canonical contact metric manifold*, that is the contact manifold (M, η) with the canonical contact metric structure. In the next theorem we show that the canonical contact metric manifold is unique.

Theorem 4.13

The canonical contact metric structure of a non-degenerate Legendre foliation is unique.

Proof

For any non-degenerate Legendre foliation $\mathcal{F}$ on a contact manifold (M, η) with the canonical contact metric structure (ϕ^c, ξ, η, g^c) let $\{\mathcal{E}_c^\alpha, \theta_c^\alpha, \eta\}$ be the local coframe induced by (ϕ^c, ξ, η, g^c). Suppose (ϕ', ξ, η, g') is another contact metric structure

on (M, η) with $g'|_L = \frac{1}{4}\prod$ whose induced coframe $\{\mathcal{E}^\alpha, \theta^\alpha, \eta\}$ also satisfies theorem 4.6.

Let $\{X_\alpha\}$ $\alpha = 1,.....,n$ be a local orthonormal frame for L with respect to $\frac{1}{4}\prod$. Thus as $g^c|_L = \frac{1}{4}\prod$ by definition 4.11 (i) and $g'|_L = \frac{1}{4}\prod$, $\{X_\alpha\}$ is a local orthonormal frame for L with respect to $g^c|_L$ and $g'|_L$.

We have for any α

$$\theta^\alpha_c = -\frac{1}{2} i(X_\alpha)\, d\eta \qquad \text{by 4.7 (i)}$$
$$= \theta^\alpha \qquad \text{by 2.6 (i).}$$

By lemma 4.5 (i) there exist functions $c_{\alpha\beta}$ on M such that

$$\phi' X_\alpha = \phi^c X_\alpha + c_{\alpha\beta} X_\beta,$$

hence

$$\mathcal{E}^\alpha = \frac{1}{2} k_{\alpha\beta} i(\phi' X_\beta)\, d\eta \qquad \text{by 2.6 (ii)}$$
$$= \frac{1}{2} k_{\alpha\beta} i(\phi^c X_\beta + c_{\beta\gamma} X_\gamma)\, d\eta$$
$$= \frac{1}{2} k_{\alpha\beta} i(\phi^c X_\beta) d\eta + \frac{1}{2} c_{\beta\gamma} k_{\alpha\beta} i(X_\gamma) d\eta$$
$$= \mathcal{E}^\alpha_c - k_{\alpha\beta}\, c_{\beta\gamma} \theta^\gamma_c \qquad \text{by 4.7 (i), 4.10 and 4.12.}$$

As $\{\mathcal{E}^\alpha, \theta^\alpha, \eta\}$ and $\{\mathcal{E}^\alpha_c, \theta^\alpha_c, \eta\}$ both satisfy theorem 4.6 we have

$$\mathcal{L}_\xi \overline{\prod} = -2\delta_{\alpha\beta}\theta^\alpha \otimes \mathcal{E}^\beta - 2\delta_{\alpha\beta}\, \mathcal{E}^\alpha \otimes \theta^\beta \qquad \text{by theorem 4.6 (iv)}$$
$$= -2\delta_{\alpha\beta}\theta^\alpha_c \otimes (\mathcal{E}^\beta_c - k_{\beta\gamma} c_{\gamma\delta} \theta^\delta_c)$$
$$- 2\delta_{\alpha\beta} (\mathcal{E}^\alpha_c - k_{\alpha\gamma} c_{\gamma\delta} \theta^\delta_c) \otimes \theta^\beta_c$$

$$= -2\delta_{\alpha\beta}\theta_c^{\alpha} \otimes \mathcal{E}_c^{\beta} - 2\delta_{\alpha\beta}\,\mathcal{E}_c^{\alpha} \otimes \theta_c^{\beta} + 2\delta_{\alpha\beta}\theta_c^{\alpha} \otimes k_{\beta\gamma}c_{\gamma\delta}\theta_c^{\delta} + 2\delta_{\alpha\beta}k_{\alpha\gamma}c_{\gamma\delta}\theta_c^{\delta} \otimes \theta_c^{\beta}$$

$$= \mathcal{L}_{\xi}\,\bar{\Pi} + 2\delta_{\alpha\beta}\theta_c^{\alpha} \otimes k_{\beta\gamma}c_{\gamma\delta}\theta_c^{\delta} + 2\delta_{\alpha\beta}k_{\alpha\gamma}c_{\gamma\delta}\theta_c^{\delta} \otimes \theta_c^{\beta}$$

by theorem 4.6 (iv).

This implies that

$$2\delta_{\alpha\beta}k_{\beta\gamma}c_{\gamma\delta}\theta_c^{\alpha} \otimes \theta_c^{\delta} = -2\delta_{\alpha\beta}k_{\alpha\gamma}c_{\gamma\delta}\theta_c^{\delta} \otimes \theta_c^{\beta}.$$

Thus for any α, γ, δ we have

$$k_{\alpha\gamma}c_{\gamma\delta} = -k_{\delta\gamma}c_{\gamma\alpha}.$$

Now if we multiply this by $k_{\delta\lambda}$ it enables us to use lemma 4.5 (ii), noting that $k_{\delta\lambda}$ is symmetric in δ, γ. So for any $\alpha, \delta, \gamma, \lambda$, we have

$$k_{\alpha\gamma}c_{\gamma\delta}\,k_{\delta\lambda} = -k_{\delta\gamma}c_{\gamma\alpha}k_{\delta\lambda}$$

$$\Rightarrow \quad k_{\alpha\gamma}c_{\lambda\delta}\,k_{\delta\gamma} = -k_{\delta\gamma}c_{\gamma\alpha}k_{\delta\lambda} \qquad \text{by lemma 4.5 (ii)}$$

$$\Rightarrow \quad k_{\alpha\gamma}k_{\gamma\delta}c_{\lambda\delta} = -k_{\gamma\delta}k_{\delta\lambda}c_{\gamma\alpha}$$

$$\Rightarrow \quad \delta_{\alpha\delta}c_{\lambda\delta} = -\delta_{\gamma\lambda}c_{\gamma\alpha}$$

$$\Rightarrow \quad c_{\lambda\alpha} = -c_{\lambda\alpha}.$$

Thus all α, λ we have

$$c_{\lambda\alpha} = 0,$$

which implies

$$\phi' X_{\alpha} = \phi^{c} X_{\alpha}.$$

Hence we have

$$(\phi', \xi, \eta, g') = (\phi^{c}, \xi, \eta, g^{c}). \qquad \square$$

4 (b). PROPERTIES OF THE CANONICAL CONTACT METRIC STRUCTURE

Lemma 4.14

If $\mathcal{F}$ is a non-degenerate Legendre foliation on the canonical contact metric manifold $(M, \phi^c, \xi, \eta, g^c)$ then

$$h^c: L \to L \qquad \text{and} \qquad h^c: Q^c \to Q^c,$$

which implies

$$A_\xi = 0.$$

Proof

For any $Y, Y' \in \Gamma Q^c$ we have

$$\begin{aligned} 0 &= \mathcal{L}_\xi\, g^c(Y, Y') && \text{by theorem 4.6 (iv)} \\ &= g^c(\nabla_Y \xi, Y') + g^c(\nabla_{Y'} \xi, Y) \\ &= - g^c(\phi^c Y + \phi^c h^c Y, Y') - g^c(\phi^c Y' + \phi^c h^c Y', Y) && \text{by 1.35} \\ &= - 2g^c(\phi^c h^c Y, Y') && \text{by 1.31 and theorem 2.1.} \end{aligned}$$

This implies that

$$h^c: Q^c \to Q^c.$$

As $\phi^c|_H$ is an isomorphism for any $X \in \Gamma L$ there exists a $Y \in \Gamma Q^c$ such that $\phi^c Y = X$. Thus for any $X \in \Gamma L$ we have

$$\begin{aligned} h^c X &= h^c \phi^c Y && \text{where } \phi^c Y = X \\ &= - \phi^c h^c Y && \text{by 1.32.} \end{aligned}$$

Thus by theorem 2.1 as $h^c Y \in \Gamma Q^c$ we have for any $X \in \Gamma L$ that $h^c X \in \Gamma L$, hence

$$h^c: L \to L.$$

Furthermore by lemma 2.9 we have

$$A_\xi = p_L(\phi h^c)$$
$$= 0 \qquad \text{as } h^c: L \to L.$$

¤

The following theorem gives the conditions for a contact metric structure to be the canonical contact metric structure.

Theorem 4.15
For a non-degenerate Legendre foliation, $\mathcal{F}$, on a contact metric manifold (M, ϕ, ξ, η, g), (ϕ, ξ, η, g) is the canonical contact metric structure if and only if

(i) $g|_L = \frac{1}{4}\Pi$

(ii) $h: L \to L.$

Proof
If (ϕ, ξ, η, g) is a canonical contact metric structure then by definition 4.11 (i) $g|_L = \frac{1}{4}\Pi$ and $h: L \to L$ by lemma 4.14.

Suppose

$$g|_L = \frac{1}{4}\Pi \qquad \text{and} \qquad h: L \to L.$$

Let $\{X_\alpha\}$ be a local orthonormal frame of L with respect to g and $\{\mathcal{E}^\alpha, \theta^\alpha, \eta\}$ be the local coframe induced by (ϕ, ξ, η, g). By lemma 2.5 and lemma 4.3 $\{\mathcal{E}^\alpha, \theta^\alpha, \eta\}$

satisfies properties (i), (ii) and (iii) of theorem 4.6 so we must show that it also satisfies property (iv), that is

$$\mathcal{L}_\xi \bar{\Pi} = -2\delta_{\alpha\beta}\theta^\alpha \otimes \mathcal{E}^\beta - 2\delta_{\alpha\beta}\mathcal{E}^\alpha \otimes \theta^\beta.$$

Evaluating the right hand side of this on the local orthonormal frame $\{X_\alpha, \phi X_\alpha, \xi\}$ of TM we obtain

(a) $(-2\delta_{\gamma\delta}\theta^\gamma \otimes \mathcal{E}^\delta - 2\delta_{\gamma\delta}\mathcal{E}^\gamma \otimes \theta^\delta)(X_\alpha, \phi X_\beta) = -2\delta_{\gamma\delta}\,\mathcal{E}^\gamma(X_\alpha)\theta^\delta(\phi X_\beta)$
$= -2k_{\alpha\beta}$ by 4.7 (ii) and 4.9,

(b) $(-2\delta_{\gamma\delta}\theta^\gamma \otimes \mathcal{E}^\delta - 2\delta_{\gamma\delta}\,\mathcal{E}^\gamma \otimes \theta^\delta)(X_\alpha, X_\beta) = 0,$

(c) $(-2\delta_{\gamma\delta}\theta^\gamma \otimes \mathcal{E}^\delta - 2\delta_{\gamma\delta}\,\mathcal{E}^\gamma \otimes \theta^\delta)(\phi X_\alpha, \phi X_\beta) = 0,$

(d) $(-2\delta_{\gamma\delta}\theta^\gamma \otimes \mathcal{E}^\delta - 2\delta_{\gamma\delta}\,\mathcal{E}^\gamma \otimes \theta^\delta)(\xi, W) = 0$ $\quad \forall\, W \in \Gamma TM.$

Now evaluating the left hand side on the local orthonormal frame $\{X_\alpha, \phi X_\alpha, \xi\}$ of TM, and using $g|_Q = \bar{\Pi}|_Q$, we get

$$\begin{aligned}
\mathcal{L}_\xi \bar{\Pi}(X_\alpha, \phi X_\beta) &= \xi\, \bar{\Pi}(X_\alpha, \phi X_\beta) - \bar{\Pi}([\xi, X_\alpha], \phi X_\beta) - \bar{\Pi}(X_\alpha, [\xi, \phi X_\beta]) \\
&= -\bar{\Pi}([\xi, X_\alpha], \phi X_\beta) \\
&= -g|_Q(p_Q[\xi, X_\alpha], \phi X_\beta) \\
&= -2g(\phi X_\alpha, \phi X_\beta) \qquad \text{by lemma 4.4 (ii)} \\
&= -2k_{\alpha\beta} \\
&= (-2\delta_{\gamma\delta}\theta^\gamma \otimes \mathcal{E}^\delta - 2\delta_{\gamma\delta}\mathcal{E}^\gamma \otimes \theta^\delta)(X_\alpha, \phi X_\beta) \qquad \text{by (a).}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_\xi \bar{\Pi}(X_\alpha, X_\beta) &= \xi\, \bar{\Pi}(X_\alpha, X_\beta) - \bar{\Pi}([\xi, X_\alpha], X_\beta) - \bar{\Pi}(X_\alpha, [\xi, X_\beta]) \\
&= 0 \\
&= (-2\delta_{\gamma\delta}\theta^\gamma \otimes \mathcal{E}^\delta - 2\delta_{\gamma\delta}\mathcal{E}^\gamma \otimes \theta^\delta)(X_\alpha, X_\beta) \qquad \text{by (b).}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_\xi \bar{\Pi}(\phi X_\alpha, \phi X_\beta) &= \xi\, \bar{\Pi}(\phi X_\alpha, \phi X_\beta) - \bar{\Pi}([\xi, \phi X_\alpha], \phi X_\beta) \\
&\qquad - \bar{\Pi}(\phi X_\alpha, [\xi, \phi X_\beta]) \\
&= \xi g(\phi X_\alpha, \phi X_\beta) - g([\xi, \phi X_\alpha], \phi X_\beta) - g(\phi X_\alpha, [\xi, \phi X_\beta]) \\
&= g(\nabla_{\phi X_\alpha} \xi, \phi X_\beta) + g(\nabla_{\phi X_\beta} \xi, \phi X_\alpha) \\
&= -2g(\phi h \phi X_\alpha, \phi X_\beta) \qquad \text{by 1.31 and 1.35} \\
&= -2g(h X_\alpha, \phi X_\beta) \qquad \text{by 1.2 and 1.32} \\
&= 0 \qquad \text{as } h: L \to L \\
&= (-2\delta_{\gamma\delta}\theta^\gamma \otimes \mathcal{E}^\delta - 2\delta_{\gamma\delta}\, \mathcal{E}^\gamma \otimes \theta^\delta)(\phi X_\alpha, \phi X_\beta) \qquad \text{by (c).}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_\xi \bar{\Pi}(\xi, W) &= \xi\, \bar{\Pi}(\xi, W) - \bar{\Pi}([\xi, \xi], W) - \bar{\Pi}(\xi, [\xi, W]) \qquad \forall\, W \in \Gamma TM \\
&= 0 \\
&= (-2\delta_{\gamma\delta}\theta^\gamma \otimes \mathcal{E}^\delta - 2\delta_{\gamma\delta}\, \mathcal{E}^\gamma \otimes \theta^\delta)(\xi, W) \qquad \text{by (d).}
\end{aligned}$$

Hence we have

$$\mathcal{L}_\xi \bar{\Pi} = -2\delta_{\alpha\beta}\theta^\alpha \otimes \mathcal{E}^\beta - 2\delta_{\alpha\beta}\mathcal{E}^\alpha \otimes \theta^\beta$$

Thus $\{\mathcal{E}^\alpha, \theta^\alpha, \eta\}$ is the local coframe induced by (ϕ, ξ, η, g) satisfies theorem 4.6 hence it is the canonical contact metric structure. ¤

The examples of chapter 2 illustrate these ideas.

Example 2.1: (a) $\mathcal{F}_F^c$ is a non-degenerate Legendre foliation on $(S_F^*M, \phi^c, \xi, \eta, g^c)$. As $g^c|_L = \frac{1}{4}\Pi$ and $h^c: L \to L$, by theorem 4.15, (ϕ^c, ξ, η, g^c) is the canonical contact metric structure.

(b) $\mathcal{F}_F'$ is a non-degenerate Legendre foliation on $(S_F^*M, \phi', \xi, \eta, g')$. As $g'|_L \neq \frac{1}{4}\Pi$, (ϕ', ξ, η, g') is not from the canonical family of contact metric structures.

For a non-degenerate Legendre foliation we define a tensor, S, of type (1,1) by

(4.16) $S(X) = 0 \qquad \forall\, X \in \Gamma L,$

$$S([\xi, X]) = \frac{1}{2}([\xi, X] - \frac{1}{4}\lambda[\xi, [\xi, X]]) \qquad \forall\, X \in \Gamma L.$$

Thus the kernel of S is L and by using 1.12, 2.14 and lemma 2.16 it can be shown that $S: H \rightarrow H$.

For any $X \in \Gamma L$ we have

(4.17) $$\lambda S([\xi, X]) = \frac{1}{2}(\lambda[\xi, X] - \frac{1}{4}\lambda^2([\xi, [\xi, X]])) \qquad \text{by 4.16}$$
$$= X \qquad \text{as } \lambda^2 = 0 \text{ and by 2.15.}$$

The tensor S can be used to construct the canonical contact metric structure for a non-degenerate Legendre foliation as the following theorem shows.

Theorem 4.18

For a non-degenerate Legendre foliation, $\mathcal{F}$, on a contact manifold (M, η) the canonical transverse bundle Q^c is given by the image of S, that is

$$Q^c = \{S([\xi, X]) \mid X \in \Gamma L\}.$$

Proof

Let $\{X_\alpha\}$ be a local orthonormal frame of L with respect to $\frac{1}{4}\Pi$. Then define vector fields Y_α by

$$Y_\alpha = S([\xi, X_\alpha]).$$

Thus for any α we have

$$\lambda Y_\alpha = X_\alpha \qquad \text{by 4.17.}$$

Let Q be the subbundle of H defined by

$$Q = \text{span}\{Y_\alpha\}.$$

For this Q let (ϕ, ξ, η, g) be the contact metric structure given by 4.2. We will show that this is the canonical contact metric structure for $\mathcal{F}$.

As $g|_L = \frac{1}{4}\Pi$ we have by lemma 4.4 (ii) that for any α

(a) $$p_Q([\xi, X_\alpha]) = 2\phi X_\alpha = 2Y_\alpha.$$

By theorem 4.15 if $h: L \to L$ then (ϕ, ξ, η, g) is the canonical contact metric structure for $\mathcal{F}$.

For any α we have

$$\begin{aligned}
\lambda[\xi, Y_\alpha] &= \lambda[\xi, S([\xi, X_\alpha])] \\
&= \lambda[\xi, \tfrac{1}{2}([\xi, X_\alpha] - \tfrac{1}{4}\lambda[\xi, [\xi, X_\alpha]])] && \text{by 4.16} \\
&= \tfrac{1}{2}\{\lambda[\xi, [\xi, X_\alpha]] - \tfrac{1}{4}\lambda[\xi, \lambda[\xi, [\xi, X_\alpha]]]\} \\
&= \tfrac{1}{2}\{\lambda[\xi, [\xi, X_\alpha]] - \tfrac{1}{2}\lambda[\xi, [\xi, X_\alpha]]\} \\
& && \text{by 2.15 as } \lambda[\xi, [\xi, X]] \in \Gamma L \\
&= \tfrac{1}{4}\{\lambda[\xi, p_L[\xi, X_\alpha]] + \lambda[\xi, p_Q[\xi, X_\alpha]]\} \\
&= \tfrac{1}{2}\{p_L[\xi, X_\alpha] + \lambda[\xi, Y_\alpha]\} && \text{by 2.15 and (a).}
\end{aligned}$$

Thus for any α

(b) $$\lambda[\xi, Y_\alpha] = p_L[\xi, X_\alpha].$$

For any α we have

$$
\begin{aligned}
hY_\alpha &= \frac{1}{2}\{[\xi, \phi Y_\alpha] - \phi[\xi, Y_\alpha]\} && \text{by 1.30}\\
&= \frac{1}{2}\{-[\xi, X_\alpha] - \phi p_L[\xi, Y_\alpha] - \phi p_Q[\xi, Y_\alpha]\} && \text{as } \phi|_Q = -\lambda|_Q \text{ by 4.2}\\
&= \frac{1}{2}\{-[\xi, X_\alpha] - p_Q\phi[\xi, Y_\alpha] + \lambda[\xi, Y_\alpha]\} && \text{as } \phi|_Q = -\lambda|_Q \text{ by 4.2}\\
&= \frac{1}{2}\{-[\xi, X_\alpha] - p_Q\phi[\xi, Y_\alpha] + p_L[\xi, X_\alpha]\} && \text{by (b)}\\
&= \frac{1}{2}\{-p_Q[\xi, X_\alpha] - p_Q\phi[\xi, Y_\alpha]\}\\
&= -Y_\alpha - \frac{1}{2}p_Q\phi[\xi, Y_\alpha] \in \Gamma Q, && \text{by 4.2 and lemma 4.4 (ii).}
\end{aligned}
$$

Thus $h: Q \to Q$ which implies $h: L \to L$ so (ϕ, ξ, η, g) is the canonical contact metric structure for $\mathcal{F}$, and

$$Q^c = \{S([\xi, X]) \mid X \in \Gamma L\}. \qquad \square$$

Lemma 4.19

For $\mathcal{F}$ a non-degenerate Legendre foliation on the canonical contact metric manifold $(M, \phi^c, \xi, \eta, g^c)$ we have the projection

$$p_L([\xi, \phi^c X]) = 2(h^c X - X) \qquad \forall\, X \in \Gamma L.$$

Proof

For any $X \in \Gamma L$ we have

$$
\begin{aligned}
h^c X &= \frac{1}{2}([\xi, \phi^c X] - \phi^c[\xi, X]) && \text{by 1.30}\\
&= \frac{1}{2}(p_L([\xi, \phi^c X]) - p_L(\phi^c[\xi, X])) && \text{as } h^c: L \to L \text{ by lemma 4.15 (ii)}\\
&= \frac{1}{2}(p_L([\xi, \phi^c X]) - \phi^c p_Q([\xi, X])) && \text{by lemma 2.1}\\
&= \frac{1}{2}(p_L([\xi, \phi^c X]) + 2X) && \text{by lemma 4.4 (ii) as } g^c|_L = \frac{1}{4}\Pi.
\end{aligned}
$$

$\square$

Lemma 4.20

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on the canonical contact metric manifold $(M, \phi^c, \xi, \eta, g^c)$, then for any $\alpha = 1,....,n$,

(i) $g^c([X_\alpha, \xi], X_\beta) = - g^c([X_\beta, \xi], X_\alpha)$ thus $g^c([X_\alpha, \xi], X_\alpha) = 0$,

(ii) $g^c([\phi^c X_\alpha, \xi], \phi^c X_\beta) = - g^c([\phi^c X_\beta, \xi], \phi^c X_\alpha)$ thus $g^c([\phi^c X_\alpha, \xi], \phi^c X_\alpha) = 0$,

(iii) Furthermore if $[X_\alpha, \xi] \in \Gamma Q^c$ then for any $X \in \Gamma L$ we have

$$g^c([X_\alpha, \phi^c X_\beta], X) = - g^c([\phi^c X_\alpha, X_\beta], X),$$

where $\{X_\alpha\}$ is a local orthonormal frame of L with respect to g^c.

Proof

By lemma 4.14 h^c: $L \to L$ and h^c: $Q^c \to Q^c$ so for any $X \in \Gamma L$ we have

(a) $\nabla_X \xi = - \phi^c X - \phi^c h^c X \in \Gamma Q^c$ by theorem 2.1 as h^c: $L \to L$

and

(b) $$\begin{aligned} \nabla_{\phi^c X} \xi &= - \phi^c(\phi^c X) - \phi^c h^c \phi^c X \\ &= X - h^c X \in \Gamma L \end{aligned}$$ as h^c: $L \to L$.

(i) For any α we have

$$\begin{aligned} g^c([X_\alpha, \xi], X_\beta) &= - g^c(\nabla_\xi X_\alpha, X_\beta) && \text{by (a)} \\ &= g^c(X_\alpha, \nabla_\xi X_\beta) \\ &= - g^c([X_\beta, \xi], X_\alpha) && \text{by (a).} \end{aligned}$$

(ii) For any α we have

$$g^c([\phi^c X_\alpha, \xi], \phi^c X_\beta) = - g^c(\nabla_\xi \phi^c X_\alpha, \phi^c X_\beta) \qquad \text{by (b)}$$
$$= g^c(\phi^c X_\alpha, \nabla_\xi \phi^c X_\beta)$$
$$= - g^c([\phi^c X_\beta, \xi], \phi^c X_\alpha) \qquad \text{by (b).}$$

(iii) Suppose $[X_\alpha, \xi] \in \Gamma Q^c$ for any α, then for any α, β and any $X \in \Gamma L$ we have

$$g^c([X_\alpha, \phi^c X_\beta], X) = \frac{1}{2} g^c([X_\alpha, [\xi, X_\beta]], X)$$
$$\text{by lemma 4.4 (ii) as } g^c|_L = \frac{1}{4}\Pi$$
$$= \frac{1}{2} g^c([[X_\beta, X_\alpha], \xi] + [[X_\alpha, \xi], X_\beta], X)$$
$$\text{by the Jacobi identity}$$
$$= - g^c(\phi^c[X_\beta, X_\alpha], X) - g^c([\phi^c X_\alpha, X_\beta], X)$$
$$\text{by lemma 4.4 (ii) as } g^c|_L = \frac{1}{4}\Pi$$
$$= - g^c([\phi^c X_\alpha, X_\beta], X) \qquad \text{by lemma 2.1.}$$

¤

4 (c). THE PROPERTIES OF A NON-DEGENERATE LEGENDRE FOLIATION

We will now investigate the properties of a non-degenerate Legendre foliation on a manifold from the canonical family of contact metric manifolds .

Theorem 4.21

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g), then

$$\mathcal{L}_X g(Y, Z) \equiv 0 \qquad \forall\, X \in \Gamma L, \forall\, Y \in \Gamma Q, \forall\, Z \in \Gamma E.$$

if and only if (ϕ, ξ, η, g) is a contact metric structure with $g|_L = \frac{1}{4}\Pi$.

Proof

E is a line bundle so for any $Z \in \Gamma E$ there exists a function f on M such that $Z = f\xi$.

For any $X \in \Gamma L$, $Y \in \Gamma Q$, $Z = f\xi \in \Gamma E$ we have

$$\begin{aligned}
\mathcal{L}_X g(Y, Z) &= Xg(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]) \\
&= - g([X, Y], f\xi) - g(Y, [X, f\xi]) \\
&= - 2fg(\phi X, Y) - fg(Y, [X, \xi]) \qquad \text{by 1.10 and 1.13} \\
&= - f(g(2\phi X + [X, \xi], Y))
\end{aligned}$$

thus the theorem follows by lemma 4.4. ¤

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) from the canonical family of contact metric manifolds and let $\{X_\alpha\}$ be a local orthonormal frame of L with respect to $g|_L = \frac{1}{4}\Pi$ then for any α, β the relative partial connection form of 2.20 (ii) gives

$$\begin{aligned}
\omega^\beta_{n+1}(X_\alpha) &= \theta^\beta([X_\alpha, \xi]) && \text{by 2.20 (ii)} \\
&= g(\phi X_\beta, [X_\alpha, \xi]) && \text{by 2.7} \\
&= - 2g(\phi X_\beta, \phi X_\alpha) && \text{by lemma 4.4 (ii)} \\
&= - 2\theta^\beta(\phi X_\alpha) && \text{by 2.7.}
\end{aligned}$$

Thus for any α, β we have

$$\omega^\beta_{n+1}(X_\alpha) = - 2k_{\alpha\beta}.$$

Substituting this in 2.21 (ii) gives for any α, β the following relationship

$$\begin{aligned}
0 &= (d\omega^\delta_{n+1} + \omega^\delta_\gamma \wedge \omega^\gamma_{n+1})(X_\alpha, X_\beta) \\
&= d\omega^\delta_{n+1}(X_\alpha, X_\beta) + \frac{1}{2}\{\omega^\delta_\gamma(X_\alpha)\omega^\gamma_{n+1}(X_\beta) - \omega^\delta_\gamma(X_\beta)\omega^\gamma_{n+1}(X_\alpha)\} \\
&= \frac{1}{2}\{X_\alpha\omega^\delta_{n+1}(X_\beta) - X_\beta\omega^\delta_{n+1}(X_\alpha) - \omega^\delta_{n+1}([X_\alpha, X_\beta]) \\
&\qquad + \omega^\delta_\gamma(X_\alpha)\omega^\gamma_{n+1}(X_\beta) - \omega^\delta_\gamma(X_\beta)\omega^\gamma_{n+1}(X_\alpha)\}
\end{aligned}$$

$$= \frac{1}{2}\{- \omega^{\delta}_{n+1}([X_\alpha, X_\beta]) + (\omega^{\delta}_{\gamma}(X_\alpha)\omega^{\gamma}_{n+1}(X_\beta) - \omega^{\delta}_{\gamma}(X_\beta)\omega^{\gamma}_{n+1}(X_\alpha))\}$$

$$= \frac{1}{2}\{- \omega^{\delta}_{n+1}([X_\alpha, X_\beta]) - 2\omega^{\delta}_{\gamma}(X_\alpha)k_{\gamma\beta} - \omega^{\delta}_{\gamma}(X_\beta)k_{\gamma\alpha}\}$$

$$= \theta^{\delta}(\phi[X_\alpha, X_\beta]) - \theta^{\delta}([X_\alpha, \phi X_\beta]) + \theta^{\delta}([X_\beta, \phi X_\alpha])$$

by 2.20 (i) and above.

Thus by 2.7 for any α, β, δ we have

$$g(\phi X_\delta, \phi[X_\alpha, X_\beta] - [X_\alpha, \phi X_\beta] - [\phi X_\alpha, X_\beta]) = 0.$$

From which it can be deduced that

(4.22) $$g([X, X'], X'') = g([X, \phi X'] + [\phi X, X'], \phi X'') \quad \forall\ X, X', X'' \in \Gamma L.$$

Furthermore it also implies that

(4.23) $$p_L(\phi^2[X, X'] - \phi[X, \phi X'] - \phi[\phi X, X']) = 0 \qquad \forall\ X, X' \in \Gamma L.$$

From 4.22 and 2.22 we get for any α, β, γ that

(4.24) $$g([X_\alpha, \phi X_\beta], \phi X_\gamma) + g([\phi X_\alpha, X_\beta], \phi X_\gamma) = g(\phi X_\alpha, [X_\beta, \phi X_\gamma]) - g(\phi X_\beta, [X_\alpha, \phi X_\gamma]).$$

Theorem 4.25

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) with $g|_L = \frac{1}{4}\Pi$, then

$$G(X, X', X'') = 2g(\nabla_{\phi X'} \phi X'' + \nabla_{\phi X''} \phi X', X) \qquad \forall\ X, X', X'' \in \Gamma L,$$

where ∇ is the Levi-Civita connection of g.

Proof

Using lemma 4.4 as $g|_L = \frac{1}{4}\Pi$ we can substitute $p_Q([\xi, X]) = 2\phi X$ in 2.18 thus for any $X, X', X'' \in \Gamma L$ we obtain

$$\begin{aligned} G(X, X', X'') &= 2\{ - Xg(\phi X', \phi X'') + g(\phi X'', [X, \phi X']) \\ &\qquad + g(\phi X', [X, \phi X'']) \} \\ &= 2\{ - Xg(\phi X', \phi X'') + g(\phi X'', \nabla_X \phi X') + g(\phi X', \nabla_X \phi X'') \\ &\qquad - g(\phi X'', \nabla_{\phi X'} X) - g(\phi X', \nabla_{\phi X''} X) \} \\ &= 2g(\nabla_{\phi X'} \phi X'' + \nabla_{\phi X''} \phi X', X) \end{aligned}$$

as ∇ is the Levi-Civita connection of g.

¤

Consider a Legendre foliation $\mathcal{F}$ on a contact metric manifold (M, ϕ, ξ, η, g) such that for any tangential vector field, X, the Lie bracket satisfies

$$p_{L^\perp}([\xi, X]) = 2f\phi X$$

for some smooth function $f \neq 0$. Then by lemma 2.13 (ii) $\mathcal{F}$ is a non-degenerate Legendre foliation. Furthermore a contact metric structure from the canonical family can be generated as the next theorem shows.

Theorem 4.26

Note: The summation convention is not used in this theorem and its proof.

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on the contact metric manifold (M, ϕ, ξ, η, g), such that there exist smooth non-zero functions f_α on M which satisfy

$$p_{L^\perp}([\xi, X_\alpha]) = 2f_\alpha \phi X_\alpha$$

where $\{X_\alpha\}$ is a local orthonormal frame of L with respect to g.

Then a contact metric structure, (ϕ', ξ, η, g') with $g'|_L = \frac{1}{4}\Pi$ is given by

i) $g'(X_\alpha, X_\alpha) = f_\alpha g(X_\alpha, X_\alpha)$

$g'(\phi X_\alpha, \phi X_\alpha) = \frac{1}{f_\alpha} g(X_\alpha, X_\alpha)$

$g'(\xi, \xi) = g(\xi, \xi)$

$g' = 0$ otherwise

ii) $\phi' X_\alpha = f_\alpha \phi X_\alpha$

$\phi' \phi X_\alpha = -\frac{1}{f_\alpha} X_\alpha$

$\phi' \xi = 0.$

Proof

Note that when $f_\alpha = 1$ for all α then by lemma 4.4 $g|_L = \frac{1}{4}\Pi$. Suppose $f_\alpha \neq 1$ for some α then we first show that (ϕ', ξ, η, g') as given by (i) and (ii) is a contact metric structure on (M, η). Also note that $\{X_\alpha, \phi X_\alpha, \xi\}$ is a local frame for TM.

By (ii) the definition of ϕ' for any α we have

$$\phi'(\phi' X_\alpha) = f_\alpha \phi'(\phi X_\alpha), \quad \phi'(\phi' \phi X_\alpha) = -\frac{1}{f_\alpha} \phi' X_\alpha, \quad \phi' \xi = 0.$$
$$= -X_\alpha \qquad\qquad = -\phi X_\alpha$$

Thus ϕ' is a tensor of type (1,1) which satisfies 1.2.

It is easily shown that g' is a semi-Riemannian metric with $\{X_\alpha, \phi' X_\alpha, \xi\}$ a local orthogonal frame of TM with respect to g'. For example for any α we have

$$\begin{aligned} g'(\phi' X_\alpha, \phi' X_\alpha) &= g'(f_\alpha \phi X_\alpha, f_\alpha \phi X_\alpha) && \text{by (ii)} \\ &= f_\alpha g(X_\alpha, X_\alpha) && \text{by (i)} \\ &= f_\alpha, \end{aligned}$$

and for any $\alpha \neq \beta$

$$g'(\phi'X_\alpha, \phi'X_\beta) = g'(f_\alpha \phi X_\alpha, f_\beta \phi X_\beta) \qquad \text{by (ii)}$$
$$= 0 \qquad \text{by (i).}$$

Thus for any α, β we have

$$g'(\phi'X_\alpha, \phi'X_\beta) = f_\alpha k_{\alpha\beta}.$$

A series of straight forward calculations shows that (ϕ', ξ, η, g') satisfies 1.3. For example for any α, β we have

$$g'(X_\alpha, \phi'(\phi'X_\beta)) = -\, g'(X_\alpha, X_\beta) \qquad \text{by (ii)}$$
$$= \begin{cases} 0 & \text{if } \alpha \neq \beta \\ -f_\beta & \text{if } \alpha = \beta \end{cases}$$
$$= -f_\beta g(X_\alpha, X_\beta)$$
$$= g(X_\alpha, \phi(f_\beta \phi X_\beta))$$
$$= g(X_\alpha, \phi(\phi'X_\beta)) \qquad \text{by (ii)}$$
$$= d\eta(X_\alpha, \phi'X_\beta) \qquad \text{by 1.3.}$$

Hence (ϕ', ξ, η, g') is a contact metric structure on (M, η) with the property that the projection

$$p_{L^\perp}([\xi, X_\alpha]) = 2f_\alpha \phi X_\alpha$$
$$= 2\phi'X_\alpha \qquad \text{by (ii),}$$

which by lemma 4.4 implies that

$$g'|_L = \frac{1}{4}\Pi.$$

□

This situation is illustrated in example 2.1 of chapter 2.

Example 2.1: (b) $\mathcal{F}'_F$ is a non-degenerate Legendre foliation on $(S^*_F M, \phi', \xi, \eta, g')$ with

$$[\xi, X'_\alpha] = 2\left(\frac{1}{2(y_3)^2}\right)\phi X'_\alpha \qquad \text{for } \alpha = 1, 2$$

where $\{X_\alpha\}$ is an orthonormal frame of L with respect to g'. Thus

$$f_\alpha = \frac{1}{2(y_3)^2} \qquad \text{for } \alpha = 1, 2$$

satisfies the conditions of theorem 4.26. It can be shown that (ϕ^c, ξ, η, g^c) from example 2.1 (a) is the contact metric structure with $g^c|_L = \frac{1}{4}\prod$ given by (i) and (ii) of theorem 4.26.

Corollary 4.27

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on the contact metric manifold (M, ϕ, ξ, η, g), such that there exist smooth non-zero functions f_α on M which satisfy

$$p_{L^\perp}([\xi, X_\alpha]) = 2f_\alpha \phi X_\alpha .$$

where $\{X_\alpha\}$ is a local orthonormal frame of L with respect to g. Then the transverse bundle $L^\perp$ determined by g is also the transverse bundle $L'^\perp$ determined by g' where (ϕ', ξ, η, g') is the contact metric structure with $g'|_L = \frac{1}{4}\prod$ given by theorem 4.26.

Proof

$$\begin{aligned} L^\perp &= \text{span}\{\phi X_\alpha, \xi\} \\ &= \text{span}\{f_\alpha \phi X_\alpha, \xi\} \\ &= \text{span}\{\phi' X_\alpha, \xi\} \qquad \text{by theorem 4.26 (ii)} \\ &= L'^\perp . \end{aligned}$$

□

In the following example we give a contact manifold and introduce two different contact metric structures on it. We then define a non-degenerate Legendre foliation on each of the resultant contact metric manifolds and examine their properties.

Example 4.1: Non-degenerate Legendre foliation

Let

$$M = R^3$$
$$= \{(x, y, z) \mid x, y, z \in R\}$$

and

$$\eta = \frac{1}{2}(\cos z dx + \sin z dy).$$

Then we have

$$d\eta = \frac{1}{2}(\sin z dx - \cos z dy) \wedge dz$$

which by calculation gives

$$\eta \wedge d\eta = -\frac{1}{4}(dx \wedge dy \wedge dz)$$
$$\neq 0 \text{ everywhere on } M.$$

Thus (M, η) is a contact manifold, with characteristic vector field

$$\xi = 2(\cos z \frac{\partial}{\partial x} + \sin z \frac{\partial}{\partial y}).$$

Consider the vector field

$$X = 2\frac{\partial}{\partial z}.$$

This is a global vector field which belongs to H the contact distribution of (M, η).

Let L be the subbundle of H given by

$$L = \text{span}\{X\}.$$

As L is a line bundle, L is a completely integrable subbundle of H. So L defines a Legendre foliation on (M, η). By calculation we get that the Lie bracket

$$[X, \xi] = 4(-\sin z\frac{\partial}{\partial x} + \cos z\frac{\partial}{\partial y}).$$

Using this and 2.11 we get

$$\begin{aligned}\Pi(X, X) &= 2d\eta(X, [X, \xi]) \\ &= 8((\sin z dx - \cos z dy) \wedge dz)(\frac{\partial}{\partial z} - \sin z\frac{\partial}{\partial x} + \cos z\frac{\partial}{\partial y}) \\ &= 4 \neq 0.\end{aligned}$$

Thus Π is non-degenerate, so L defines a non-degenerate Legendre foliation on (M, η). Furthermore $\{X\}$ gives an orthonormal frame of L with respect to $\frac{1}{4}\Pi$. We now give two different contact metric structures on (M, η) each from the canonical family of contact metric structures, and investigate the resultant non-degenerate Legendre foliations defined by L.

(a) We introduce a contact metric structure (ϕ^c, ξ, η, g^c) with $g^c|_L = \frac{1}{4}\Pi$ so that L defines a non-degenerate Legendre foliation $\mathcal{F}^c$ of $(M, \phi^c, \xi, \eta, g^c)$. This contact metric structure is shown to be the canonical one.

Let (ϕ^c, ξ, η, g^c) be the contact metric structure given by

$$g^c = \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\phi^c = \begin{bmatrix} 0 & 0 & \sin z \\ 0 & 0 & -\cos z \\ -\sin z & \cos z & 0 \end{bmatrix}$$

Thus

$$Y = \phi^c X$$
$$= 2(\sin z \frac{\partial}{\partial x} - \cos z \frac{\partial}{\partial y})$$

and $\{X, Y, \xi\}$ is an orthonormal frame of TM with respect to g^c.

Calculating the Lie brackets we get

$$[X, \xi] = -2Y$$
$$[Y, \xi] = 0$$
$$[X, Y] = 2\xi.$$

Thus $L = \text{span}\{X\}$ defines a non-degenerate Legendre foliation $\mathcal{F}^c$ on the contact metric manifold $(M, \phi^c, \xi, \eta, g^c)$ with by lemma 4.4 $g^c|_L = \frac{1}{4}\Pi$.

By calculation we have

$$h^c X = X \quad \text{and} \quad h^c Y = -Y.$$

Hence by theorem 4.15 (ϕ^c, ξ, η, g^c) is the canonical contact metric structure for the non-degenerate Legendre foliation defined by L .

It can be shown that $\mathcal{F}^c$ is totally geodesic.

(b) We now select a different contact metric structure (ϕ', ξ, η, g') on (M, η), so that L defines a non-degenerate Legendre foliation $\mathcal{F}'$ on $(M, \phi', \xi, \eta, g')$.

Let (ϕ', ξ, η, g') be the contact metric structure where ϕ' is given by

$$\phi' = \begin{bmatrix} -\sin^2 z & \sin z \cos z & \sin z \\ \sin z \cos z & -\cos^2 z & -\cos z \\ -\sin z & \cos z & 1 \end{bmatrix},$$

and g' is the semi-Riemannian metric defined by

$$g'(X, X) = g'(\phi'X, \phi'X) = g(\xi, \xi) = 1$$

$$g'(X, \phi'X) = g'(\phi'X, \xi) = g(X, \xi) = 0.$$

Thus

$$\begin{aligned} Y' &= \phi'X \\ &= 2(\sin z \frac{\partial}{\partial x} - \cos z \frac{\partial}{\partial y} + \frac{\partial}{\partial z}) \\ &= Y + X \end{aligned}$$

and $\{X, Y', \xi\}$ is an orthonormal frame of TM with respect to g'.

Calculating the Lie brackets we get

$$[X, \xi] = 2X - 2Y'$$
$$[Y', \xi] = 2X - 2Y'$$
$$[X, Y'] = 2\xi.$$

Thus $L = \text{span}\{X\}$ defines a non-degenerate Legendre foliation $\mathcal{F}'$ on the contact metric manifold $(M, \phi', \xi, \eta, g')$ with by lemma 4.4 $g'|_L = \frac{1}{4}\Pi$.

By calculation we have

$$h'X = 2Y' \qquad \text{and} \qquad h'Y' = 2X.$$

Hence by theorem 4.15 (ϕ', ξ, η, g') is not the canonical contact metric structure for the non-degenerate Legendre foliation defined by L.

Also by calculation we have

$$\nabla_X X = -2\xi \qquad \text{and} \qquad \nabla_{Y'} Y' = 2\xi$$

so the mean curvature vector field of $\mathcal{F}'$ is given by

$$\tau = -2\xi.$$

Hence $\mathcal{F}'$ is not a harmonic Legendre foliation.

4 (d). THE CANONICAL CONNECTION FOR A NON-DEGENERATE LEGENDRE FOLIATION

The canonical contact metric structure of a non-degenerate Legendre foliation induces a canonical connection.

Theorem 4.28 (Theorem 5.10 Pang[2])
For a non-degenerate Legendre foliation $\mathcal{F}$ on a contact manifold (M, η) there exists a unique connection ∇^c on M such that

i) $\nabla^c\eta = \nabla^c(d\eta) = \nabla^c\Pi = \nabla^c\,\overline{\Pi} = \nabla^c(\mathcal{L}_\xi\,\overline{\Pi}) = 0,$

ii) The torsion T^c of ∇^c has vanishing component in $Q \otimes (\wedge^2(Q^* \oplus E^*))$ and in $L \otimes (\wedge^2 L_c^*)$,

where $L_c^* = \text{span}\{\mathcal{E}_c^\alpha\}$
$= \text{ann}(\, Q^c \oplus E \,).$

Furthermore if (ϕ^c, ξ, η, g^c) is the canonical contact metric structure then (i) implies that

$$\nabla^c g^c|_L = \nabla^c g^c|_{Q^c} = \nabla^c(\mathcal{L}_\xi g^c|_{Q^c}) = 0.$$

For $W \in \Gamma TM$, ∇^c is defined by (See Pang [2])

$$\nabla^c_W \omega = \begin{cases} \frac{1}{2} p_{Q^*}(i(W))\left(d\omega + \frac{1}{2}\left(\mathcal{L}_{\omega^*} \overline{\Pi} \right) \right) & \omega \in \Gamma Q^* \\ (\lambda^*)^{-1}\left(\nabla_W \lambda^* \omega \right) & \omega \in \Gamma L_c^* \\ 0 & \omega \in \Gamma E^* \end{cases}$$

where for any $\omega \in \Gamma Q^*$, ω^* is the unique section of Q^c determined by

$$\omega(Y) = \overline{\Pi}(\omega^*, Y) \qquad \forall\, Y \in \Gamma Q^c.$$

With the canonical contact metric structure this becomes

$$\nabla^c_W \omega = \begin{cases} \frac{1}{2} p_{Q^*}(i(W))\left(d\omega + \frac{1}{2}\left(\mathcal{L}_{\omega^*} g^c|_{Q^c} \right) \right) & \omega \in \Gamma Q^* \\ -\, \phi^{c^*}\left(\nabla_W \phi^{c^*} \omega \right) & \omega \in \Gamma L_c^* \\ 0 & \omega \in \Gamma E^* \end{cases}$$

where for any $\omega \in \Gamma Q^*$, ω^* is the unique section of Q^c determined by

$$\omega(Y) = g^c|_{Q^c}(\omega^*, Y) \qquad \forall\, Y \in \Gamma Q^c.$$

Let $\{X_\alpha\}$ be a local orthonormal frame of L with respect to g^c, then $\{Y_\alpha = \phi^c X_\alpha, \xi\}$ is the corresponding local orthonormal frame of $L^\perp$. Evaluating the canonical connection ∇^c on this local orthonormal frame we get the following, where ∇ is the Levi-Civita connection of g^c:

4.29 (i) $$(\nabla^c_{X_\alpha} \theta^\beta_c)(Y_\gamma) = (\nabla_{X_\alpha} \theta^\beta_c)(Y_\gamma) + \frac{1}{2} g^c(X_\alpha, [Y_\beta, Y_\gamma]).$$

(ii) $$(\nabla^c_{Y_\alpha} \theta^\beta_c)(Y_\gamma) = (\nabla_{Y_\alpha} \theta^\beta_c)(Y_\gamma).$$

(iii) $$(\nabla^c_{\xi} \theta^\beta_c)(Y_\gamma) = (\nabla_{\xi} \theta^\beta_c)(Y_\gamma).$$

(iv) $$(\nabla^c_{X_\alpha} \varepsilon^\beta_c)(X_\gamma) = k_{\beta\delta}\{(\nabla_{X_\alpha} \theta^\delta_c)(Y_\gamma) + \frac{1}{2} g^c(X_\alpha, [Y_\delta, Y_\gamma])\}.$$

(v) $$(\nabla^c_{Y_\alpha} \varepsilon^\beta_c)(X_\gamma) = k_{\beta\delta}\{(\nabla_{Y_\alpha} \theta^\delta_c)(Y_\gamma)\}.$$

(vi) $$(\nabla^c_{\xi} \varepsilon^\beta_c)(X_\gamma) = k_{\beta\delta}\{(\nabla_{\xi} \theta^\delta_c)(Y_\gamma)\}.$$

(vii) $$(\nabla^c_W \eta)(Z) = (\nabla_W \eta)(Z) = 0 \qquad \forall\, W \in \Gamma TM, \forall\, Z \in \Gamma E.$$

(viii) $\nabla^c = 0$ otherwise.

A series of straightforward calculations using the definition of ∇^c proves these, we will only prove 4.29 (i) and (iv) the others are proved similarly.

First for any α, β we have

$$\begin{aligned} \theta^\alpha_c(\phi^c X_\beta) &= -\, d\eta(X_\alpha, \phi^c X_\beta) && \text{by 4.7 (i)} \\ &= g^c(\phi^c X_\alpha, \phi^c X_\beta) && \text{by 1.3.} \end{aligned}$$

As $\theta_c^\alpha \in Q^*$ we have $\theta_c^{\alpha *} \in Q^c$, so the above implies that for any α,

$$\begin{aligned}\theta_c^{\alpha *} &= \phi^c X_\alpha \\ &= Y_\alpha.\end{aligned}$$

Using this in the definition of ∇^c, we have any α, β that

$$\begin{aligned}(\nabla^c{}_{X_\alpha}\theta_c^\beta)(Y_\gamma) &= \tfrac{1}{2}(i(X_\alpha)(d\theta_c^\beta + \tfrac{1}{2}(\mathcal{L}_{\theta_c^\beta} * g^c|_{Q^c})(Y_\gamma) \\ &= d\theta_c^\beta(X_\alpha, Y_\gamma) + \tfrac{1}{2}(\mathcal{L}_{Y_\beta} * g^c|_{Q^c})(X_\alpha, Y_\gamma) \\ &= \tfrac{1}{2}\{X_\alpha\theta_c^\beta(Y_\gamma) - Y_\gamma\theta_c^\beta(X_\alpha) - \theta_c^\beta([X_\alpha, Y_\gamma]) \\ &\qquad + Y_\beta g^c|_{Q^c}(X_\alpha, Y_\gamma) - g^c|_{Q^c}([Y_\beta, X_\alpha], Y_\gamma) \\ &\qquad - g^c|_{Q^c}(X_\alpha, [Y_\beta, Y_\gamma])\} \\ &= \tfrac{1}{2}\{X_\alpha g^c(Y_\beta, Y_\gamma) - g^c(Y_\beta, [X_\alpha, Y_\gamma]) \\ &\qquad - g^c([Y_\beta, X_\alpha], Y_\gamma)\} \qquad \text{by 2.7} \\ &= X_\alpha\theta_c^\beta(Y_\gamma) - \theta_c^\beta(\nabla_{X_\alpha} Y_\gamma) + \tfrac{1}{2} g^c(X_\alpha, [Y_\beta, Y_\gamma]) \\ &= (\nabla_{X_\alpha}\theta_c^\beta)(Y_\gamma) + \tfrac{1}{2} g^c(X_\alpha, [Y_\beta, Y_\gamma]).\end{aligned}$$

Thus we have proved 4.29 (i) that for any α, β and γ

$$(\nabla^c{}_{X_\alpha}\theta_c^\beta)(Y_\gamma) = (\nabla_{X_\alpha}\theta_c^\beta)(Y_\gamma) + \tfrac{1}{2} g^c(X_\alpha, [Y_\beta, Y_\gamma]).$$

To show 4.29 (iv) for any α, β we use

$$\begin{aligned}\phi^{c*}\mathcal{E}_c^\alpha(Y_\beta) &= \mathcal{E}_c^\alpha(\phi^c Y_\beta) \\ &= -\mathcal{E}_c^\alpha(X_\beta) \\ &= -\delta_{\alpha\beta} \qquad \text{by 4.7 (ii).}\end{aligned}$$

Hence we have

$$\phi^{c*}\mathcal{E}_c^{\alpha} = -\, k_{\alpha\beta}\theta_c^{\beta} \qquad \text{by 4.9.}$$

Using this in the definition of ∇^c, we have any α, β that

$$\begin{aligned}(\nabla^c{}_{Y_\alpha}\mathcal{E}_c^{\beta})(X_\gamma) &= -\,\phi^{c*}(\nabla^c{}_{Y_\alpha}\phi^{c*}\mathcal{E}_c^{\beta})(X_\gamma)\\ &= k_{\beta\delta}\{(\nabla^c{}_{X_\alpha}\theta_c^{\delta})(Y_\gamma)\}\\ &= k_{\beta\delta}\{(\nabla_{X_\alpha}\theta_c^{\delta})(Y_\gamma) + \tfrac{1}{2}\,g^c(X_\alpha, [Y_\delta, Y_\gamma])\}\end{aligned}$$

by 4.29 (i).

Therefore we have proved 4.29 (iv) that for any α, β and γ

$$(\nabla^c{}_{Y_\alpha}\mathcal{E}_c^{\beta})(X_\gamma) = k_{\beta\delta}\{(\nabla_{X_\alpha}\theta_c^{\delta})(Y_\gamma) + \tfrac{1}{2}\,g^c(X_\alpha, [Y_\delta, Y_\gamma])\}.$$

As the torsion T^c of ∇^c has a $L \otimes (\wedge^2 L_c^*)$ vanishing component and $\nabla^c g^c|_L = 0$, the restriction of ∇^c to the leaves of a non-degenerate Legendre foliation, $\mathcal{F}$, is the Levi-Civita connection, ∇, of g^c restricted to the leaves. Thus for any α, β, γ we have

$$(\nabla^c{}_{X_\alpha}\mathcal{E}_c^{\beta})(X_\gamma) = (\nabla_{X_\alpha}\mathcal{E}_c^{\beta})(X_\gamma).$$

Furthermore by writing this as a difference and expanding using 4.29 (iv) we have

$$\begin{aligned}0 &= (\nabla^c{}_{X_\alpha}\mathcal{E}_c^{\beta})(X_\gamma) - (\nabla_{X_\alpha}\mathcal{E}_c^{\beta})(X_\gamma)\\ &= k_{\beta\delta}\{(\nabla_{X_\alpha}\theta_c^{\delta})(Y_\gamma) + \tfrac{1}{2}\,g^c(X_\alpha, [Y_\delta, Y_\gamma])\} - (\nabla_{X_\alpha}\mathcal{E}_c^{\beta})(X_\gamma)\end{aligned}$$

by 4.29 (iv)

$$\begin{aligned}&= k_{\beta\delta}\{X_\alpha\theta_c^{\delta}(Y_\gamma) - \theta_c^{\delta}(\nabla_{X_\alpha}Y_\gamma) + \tfrac{1}{2}\,g^c(X_\alpha, [Y_\delta, Y_\gamma])\}\\ &\qquad - X_\alpha\mathcal{E}_c^{\beta}(X_\gamma) + \mathcal{E}_c^{\beta}(\nabla_{X_\alpha}X_\gamma)\end{aligned}$$

$$= k_{\beta\delta}\{ X_\alpha g^c(Y_\delta, Y_\gamma) - g^c(Y_\delta, \nabla_{X_\alpha} Y_\gamma) + \frac{1}{2} g^c(X_\alpha, [Y_\delta, Y_\gamma]) - X_\alpha g^c(X_\delta, X_\gamma) + g^c(X_\delta, \nabla_{X_\alpha} X_\gamma)\}$$

by 2.7

$$= k_{\beta\delta}\{g^c(X_\delta, \nabla_{X_\alpha} X_\gamma) - g^c(Y_\delta, \nabla_{X_\alpha} Y_\gamma) + \frac{1}{2} g^c(X_\alpha, [Y_\delta, Y_\gamma])\}$$

by 1.9.

Therefore for any α, γ, δ we have

(4.30) $$g^c(Y_\delta, \nabla_{X_\alpha} Y_\gamma) - g^c(X_\delta, \nabla_{X_\alpha} X_\gamma) = \frac{1}{2} g^c(X_\alpha, [Y_\delta, Y_\gamma]).$$

Let π^α_β be the o(p, q)-valued 1-form such that

$$\nabla^c \theta^\alpha_c = - \pi^\alpha_\beta \otimes \theta^\beta_c \qquad \text{and} \qquad \nabla^c \mathcal{E}^\alpha_c = (\pi^t)^\alpha_\beta \otimes \mathcal{E}^\beta_c$$

for the existence of π^α_β see Pang [2].

The first structure equations of the canonical coframes are given as follows.

Theorem 4.31

The canonical coframes $\{\mathcal{E}^\alpha_c, \theta^\alpha_c, \eta\}$ and π^α_β satisfy the first structure equations

$$\begin{bmatrix} d\mathcal{E}^\alpha_c \\ d\theta^\alpha_c \\ d\eta \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -(\pi^t)^\alpha_\beta & 0 & 0 \\ 0 & \pi^\alpha_\beta & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathcal{E}^\beta_c \\ \theta^\beta_c \\ \eta \end{bmatrix}$$

$$+ \begin{bmatrix} K^\alpha_{\beta\gamma}\theta^\beta_c \wedge \theta^\gamma_c + S^\alpha_\beta \eta \wedge \theta^\beta_c + Q^\alpha_{\beta\gamma}\theta^\beta_c \wedge \mathcal{E}^\gamma_c \\ -2k_{\alpha\beta}\eta \wedge \mathcal{E}^\beta_c + \delta_{\alpha\beta}G_{\sigma\beta\gamma}k_{\sigma\beta}\theta^\beta_c \wedge \mathcal{E}^\gamma_c \\ 2\delta_{\beta\gamma}\theta^\beta_c \wedge \mathcal{E}^\gamma_c \end{bmatrix}$$

where S^α_β, $Q^\alpha_{\beta\gamma} = Q^\beta_{\alpha\gamma}$ and $K^\alpha_{\beta\gamma} = - K^\alpha_{\gamma\beta}$ are functions chosen to satisfy these equations.

This is theorm 5.20 of Pang [2] with a change of constants due to our definition of the exterior derivative with the p+1 factor included.

We use these first structure equations and conditions on the functions $Q^{\alpha}_{\beta\gamma}$ and $K^{\alpha}_{\beta\gamma}$ in chapters five and six.

4 (e). CONTACT METRIC STRUCTURES ON $S^*_F M$

In this section we restrict our attention to non-degenerate Legendre foliations on the unit cotangent bundle $S^*_F M$ and obtain expressions for a contact metric structure (ϕ, ξ, η, g) with $g|_L = \frac{1}{4}\Pi$ in terms of the local coordinates (x_i, y_j) on T^*M.

For a Legendre foliation $\mathcal{T}_F$ on the contact metric manifold $(S^*_F M, \phi, \xi, \eta, g)$, given that $g|_L = g_{ij}\, dy_i \otimes dy_j$, consider the tensor χ of type (1,1) defined by

$$\chi\frac{\partial}{\partial y_i} = 2(-\, g_{ij}\frac{\partial}{\partial x_j} + f_{ik}\frac{\partial}{\partial y_k}) \tag{4.32}$$

where the functions f_{ik} are such that for any point p in $S^*_F M$ the following are satisfied,

(i) $\chi X_p \in L_p^{\perp}$ $\qquad \forall\, X_p \in L_p,$

(ii) $\chi Z_p \in L_p^{\perp}$ $\qquad \forall\, Z_p \in T_p T^*M$ such that $Z_p \notin TS^*_F M.$

From 2.30 we have

$$\begin{aligned} TS^*_F M &= L \oplus \mathrm{span}\{\frac{\partial}{\partial x_i}\} \\ &= \{a_{\alpha i}\frac{\partial}{\partial y_i} \mid F_i a_{\alpha i} = 0\} \oplus \mathrm{span}\{\frac{\partial}{\partial x_i}\}, \end{aligned}$$

which shows that there exist functions f_{ik} that satisfy (i) and (ii).

Lemma 4.33

Let $\mathcal{F}_F$ be a non-degenerate Legendre foliation on the contact metric manifold $(S_F^*M, \phi, \xi, \eta, g)$ with $g|_L = \frac{1}{4}\Pi$ then

$$\chi|_L = \phi|_L.$$

Proof

As $g|_L = \frac{1}{4}\Pi$ by 2.31 we have

$$g|_L = \frac{1}{8}(F^2)_{ij}\, dy_i \otimes dy_j \qquad i, j = 1,.....,n$$
$$= \frac{1}{4}(F_iF_j + FF_{ij})dy_i \otimes dy_j.$$

Let $g_{ij} = \frac{1}{4}(F_iF_j + FF_{ij})$ and $\{X_\alpha = a_{\alpha i}\frac{\partial}{\partial y_i}\}$ be a local orthonormal frame of L with respect to g, then $\{\phi X_\alpha\}$ is a local orthonormal frame of Q with respect to g. By 2.29 the coefficients $a_{\alpha i}$ are restricted so that $F_i a_{\alpha i} = 0$.

First we will show that $\chi X_\alpha \in \Gamma Q$. For any α we have

$$g(\chi X_\alpha, \xi) = \eta(\chi X_\alpha) \qquad \text{by 1.8}$$
$$= \eta(a_{\alpha i}\chi\frac{\partial}{\partial y_i})$$
$$= 2a_{\alpha i}\eta(-g_{ij}\frac{\partial}{\partial x_j} + f_{ik}\frac{\partial}{\partial y_k}) \qquad \text{by 4.32}$$
$$= 2a_{\alpha i}y_l dx_l(-g_{ij}\frac{\partial}{\partial x_j} + f_{ik}\frac{\partial}{\partial y_k})$$
$$= -2a_{\alpha i}y_j g_{ij}$$
$$= -\frac{1}{2}a_{\alpha i}y_j(F_iF_j + FF_{ij}) \qquad \text{as } g_{ij} = \frac{1}{4}(F_iF_j + FF_{ij})$$
$$= -\frac{1}{2}(a_{\alpha i}F_iy_jF_j + a_{\alpha i}Fy_jF_{ij})$$
$$= 0 \qquad \text{as } F_i a_{\alpha i} = 0 \text{ and by 2.28.}$$

Since $\chi X_\alpha \in \Gamma L^\perp$ this implies that $\chi X_\alpha \in \Gamma Q$.

Since $\{X_\alpha\}$ be a local orthonormal frame of L with respect to g we have

(4.34) $$k_{\alpha\beta} = g|_L(X_\alpha, X_\beta)$$
$$= a_{\alpha i}a_{\beta j}g_{ij}.$$

Furthermore the local orthonormal frame $\{\theta^\alpha\}$ of Q^* with respect to g is given by

(4.35) $$\theta^\alpha = -\frac{1}{2}i(X_\alpha)d\eta \qquad \text{by 2.6 (i)}$$
$$= -\frac{1}{2}i(a_{\alpha i}\frac{\partial}{\partial y_i})d\eta$$
$$= -\frac{1}{2}a_{\alpha i}dx_i \qquad \text{by 2.26 (i).}$$

Using 4.34 and 4.35 for any α, β we have that

$$g(\chi X_\alpha, \phi X_\beta) = d\eta(\chi X_\alpha, X_\beta) \qquad \text{by 1.3}$$
$$= \theta^\beta(\chi X_\alpha) \qquad \text{by 2.7}$$
$$= -\frac{1}{2}a_{\beta j}dx_j(a_{\alpha i}\chi\frac{\partial}{\partial y_i}) \qquad \text{by 4.35}$$
$$= a_{\beta j}a_{\alpha i}dx_j\,(g_{ik}\frac{\partial}{\partial x_k} - f_{il}\frac{\partial}{\partial y_l}) \qquad \text{by 4.32}$$
$$= a_{\alpha i}a_{\beta j}g_{ij}$$
$$= k_{\alpha\beta} \qquad \text{by 4.34}$$
$$= g(\phi X_\alpha, \phi X_\beta)$$

as $\{\phi X_\alpha\}$ is a local orthonormal frame of Q with respect to g.

Thus $\chi X_\alpha = \phi X_\alpha$ as $\chi X_\alpha, \phi X_\alpha \in \Gamma Q$. ¤

Theorem 4.36

For a non-degenerate Legendre foliation $\mathcal{F}_F$ on the contact metric manifold $(S_F^*M, \phi, \xi, \eta, g)$ with

$$g|_L = \frac{1}{4}\Pi$$
$$= \frac{1}{4}(F_iF_j + FF_{ij})dy_i \otimes dy_j \qquad \text{by 2.31}$$
$$= \frac{1}{8}(F^2)_{ij}\,dy_i \otimes dy_j \qquad i, j = 1,.....,n,$$

then

$$g|_{L^\perp} = 2((F^2)_{ij})^{-1} dx_i \otimes dx_j$$
$$= (F_iF_j + FF_{ij})^{-1} dx_i \otimes dx_j,$$

and

$$g = 0 \quad \text{otherwise.}$$

Proof

Let $g_{ij} = \frac{1}{4}(F_iF_j + FF_{ij})$ therefore $\frac{1}{4} g^{ij} = (F_iF_j + FF_{ij})^{-1}$ where $(g_{ij})^{-1} = (g^{ij})$, and let $\{X_\alpha = a_{\alpha i} \frac{\partial}{\partial y_i}\}$ be a local orthonormal frame of L with respect to g, then $\{\phi X_\alpha\}$ is a local orthonormal frame of Q with respect to g, by 2.29 the coefficients $a_{\alpha i}$ are restricted so that $F_i a_{\alpha i} = 0$.

We will first show that

$$g|_Q = \frac{1}{4} g^{ij} dx_i \otimes dx_j.$$

Consider a symmetric tensor, $\tilde{h}$ of type (2,0) defined by

(4.37) $$\tilde{h} = \frac{1}{4} g^{ij} dx_i \otimes dx_j.$$

Then

$$\tilde{h}(\phi X_\alpha, \phi X_\beta) = \tilde{h}(\chi X_\alpha, \chi X_\beta) \qquad \text{by lemma 4.33}$$
$$= a_{\alpha i} \tilde{h}(\chi \frac{\partial}{\partial y_i}, \chi \frac{\partial}{\partial y_j}) a_{\beta j}$$
$$= 4 a_{\alpha i} \tilde{h}(-g_{ik} \frac{\partial}{\partial x_k} + f_{il} \frac{\partial}{\partial y_l}, -g_{jm} \frac{\partial}{\partial x_m} + f_{jn} \frac{\partial}{\partial y_n}) a_{\beta j}$$
$$\text{by 4.32}$$
$$= a_{\alpha i} g_{ik} g^{km} g_{jm} a_{\beta j} \qquad \text{by 4.37}$$

$$= a_{\alpha i}\delta_{im}g_{jm}a_{\beta j}$$
$$= a_{\alpha m}g_{jm}a_{\beta j}$$
$$= k_{\alpha\beta} \qquad \text{by 4.34}$$
$$= g(\phi X_\alpha, \phi X_\beta)$$

as $\{\phi X_\alpha\}$ is a local orthonormal frame of Q with respect to g.

Thus we have

$$g|_Q = \tilde{h}|_Q$$
$$= \frac{1}{4} g^{ij}\, dx_i \otimes dx_j.$$

Now consider the local vector field Z on T^*M where

(4.38) $$Z = 2y_i \frac{\partial}{\partial y_i}.$$

Since F is homogenous of degree one we have $F = y_i F_i$ which implies that

$$dF(Z) = F_i dy_i(Z)$$
$$= 2y_i F_i \qquad \text{by 4.38}$$
$$= 2F.$$

At any point p in $S^*_F M$ by the definition of $S^*_F M$ we have F = 1 so

$$dF(Z_p) = 2 \qquad \forall\, p \in S^*_F M.$$

Thus by 2.29 and 2.30 we have

$$Z_p \notin T_p S^*_F M \qquad \forall\, p \in S^*_F M.$$

This shows that Z is a vector field normal to S_F^*M, so at any point p in S_F^*M we have

$$\begin{aligned} T_pT^*M &= T_pS_F^*M \oplus \text{span}\{Z_p\} \\ &= L_p^{\perp} \oplus L_p \oplus \text{span}\{Z_p\}. \end{aligned}$$

Which enables us to define $\tilde{L}$ a subbundle of T^*M by

$$\tilde{L}_p = L_p \oplus \text{span}\{Z_p\} \qquad \forall\, p \in S_F^*M.$$

In the following we restrict F and the vector field Z to S_F^*M, hence F = 1.

We now define a semi-Riemannian metric g' on T^*M such that $g'|_L = g|_L$, $g'|_Q = g|_Q$ and $g'|_E = \tilde{h}|_E$, then we show that $g'|_E = g|_E$, that is g' is an extension to T^*M of the metric g on S_F^*M.

Consider the semi-Riemannian metric on T^*M such that

(4.39) (i) $$g'|_{L^{\perp}} = \frac{1}{4} g^{ij}\, dx_i \otimes dx_j$$

(ii) $$g'|_{\tilde{L}} = g_{ij}\, dy_i \otimes dy_j.$$

$\{X_\alpha, Z\}$ is a local orthonormal frame of $\tilde{L}$ with respect to g' as

(4.40)
$$\begin{aligned} g'|_{\tilde{L}}(Z, Z) &= g'|_{\tilde{L}}(2y_i\frac{\partial}{\partial y_i}, 2y_j\frac{\partial}{\partial y_j}) && \text{by 4.38} \\ &= 4y_iy_jg_{ij} && \text{by 4.39 (ii)} \\ &= y_iF_iy_jF_j + y_jFy_iF_{ij} && \text{as } g_{ij} = \frac{1}{4}(F_iF_j + FF_{ij}) \\ &= 1 && \text{by 2.28.} \end{aligned}$$

Also for any α we have

(4.41) $g'|_{\tilde{L}}(Z, X_\alpha) = g'|_{\tilde{L}}(2y_i\frac{\partial}{\partial y_i}, a_{\alpha j}\frac{\partial}{\partial y_j})$ where $a_{\alpha j}F_j = 0$

$= 2y_i a_{\alpha j} g_{ij}$ by 4.39 (ii)

$= \frac{1}{2}(y_i F_i a_{\alpha j} F_j + a_{\alpha j} F y_i F_{ij})$ as $g_{ij} = \frac{1}{4}(F_i F_j + FF_{ij})$

$= 0$ by 2.29,

and for any α, β

$$g'|_{\tilde{L}}(X_\alpha, X_\beta) = g|_L(X_\alpha, X_\beta)$$
$$= k_{\alpha\beta}$$

as $\{X_\alpha\}$ is a local orthonormal frame of L with respect to g.

We now prove that $g'|_{\tilde{L}}$ and $g'|_{L^\perp}$ are related by the following

(4.42) $g'|_{L^\perp}(\chi X, \chi X') = g'|_{\tilde{L}}(X, X')$ $\forall\ X, X' \in \Gamma\tilde{L}$.

For any i, j we have

$$g'|_{L^\perp}(\chi\frac{\partial}{\partial y_i}, \chi\frac{\partial}{\partial y_j}) = 4g'|_{L^\perp}(-g_{ik}\frac{\partial}{\partial x_k} + f_{il}\frac{\partial}{\partial y_l}, -g_{jm}\frac{\partial}{\partial x_m} + f_{jn}\frac{\partial}{\partial y_n})$$

by 4.32

$= g_{ik}g^{km}g_{jm}$ by 4.39 (i)

$= g_{ij}$

$= g'|_{\tilde{L}}(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j})$ by 4.39 (ii).

Hence as $\tilde{L} = \text{span}\{\frac{\partial}{\partial y_i}\}$ and $\chi: \tilde{L} \to L^\perp$ we have 4.42.

Using 4.42 we now show that $\{\phi X_\alpha, \xi\}$ is a local orthonormal frame of $L^\perp$ with respect to g'.

First we note that

$$(4.43)\qquad \chi Z = 2y_i \chi \frac{\partial}{\partial y_i} \qquad \text{by 4.38}$$

$$= -4y_i g_{ij} \frac{\partial}{\partial x_j} + 4y_i f_{ik} \frac{\partial}{\partial y_k} \qquad \text{by 4.32}$$

$$= -y_i (F_i F_j + F F_{ij}) \frac{\partial}{\partial x_j} + 4y_i f_{ik} \frac{\partial}{\partial y_k} \qquad \text{as } g_{ij} = \frac{1}{4}(F_i F_j + F F_{ij})$$

$$= -F_j \frac{\partial}{\partial x_j} + 4y_i f_{ik} \frac{\partial}{\partial y_k} \qquad \text{by 2.28.}$$

Then we have

$$g'|_{L^\perp}(\xi, \xi) = g'|_{L^\perp}\left(F_i \frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_k}\frac{\partial}{\partial y_k}, F_j \frac{\partial}{\partial x_j} - \frac{\partial F}{\partial x_l}\frac{\partial}{\partial y_l}\right) \qquad \text{by 2.26 (ii)}$$

$$= g'|_{L^\perp}\left(F_i \frac{\partial}{\partial x_i}, F_j \frac{\partial}{\partial x_j}\right) \qquad \text{by 4.39 (i)}$$

$$= g'|_{L^\perp}\left(-F_i \frac{\partial}{\partial x_i} + 4y_i f_{ik} \frac{\partial}{\partial y_k}, -F_j \frac{\partial}{\partial x_j} + 4y_j f_{jl} \frac{\partial}{\partial y_l}\right)$$

by 4.39 (i)

$$= g'|_{L^\perp}(\chi Z, \chi Z) \qquad \text{by 4.43}$$

$$= g'|_{\tilde{L}}(Z, Z) \qquad \text{by 4.42}$$

$$= 1 \qquad \text{by 4.40.}$$

Similarly for any α we have

$$g'|_{L^\perp}(\xi, \phi X_\alpha) = -g'|_{L^\perp}(\chi Z, \phi X_\alpha) \qquad \text{by 2.2 (ii), 4.39 (i) and 4.43}$$

$$= -g'|_{\tilde{L}}(Z, X_\alpha) \qquad \text{by lemma 4.33 and by 4.42}$$

$$= 0 \qquad \text{by 4.41,}$$

and for any α, β we have

$$g'|_{L^\perp}(\phi X_\alpha, \phi X_\beta) = g|_{L^\perp}(\phi X_\alpha, \phi X_\beta)$$

$$= k_{\alpha\beta}$$

as $\{\phi X_\alpha\}$ is a local orthonormal frame of Qwith respect to g.

Thus $g'|_{L^\perp} = g|_{L^\perp}$.

Hence $\{X_\alpha, \phi X_\alpha, \xi\}$ is a local orthonormal frame of $TS^*_F M$ with respect to g' and g, so $g'|_{TS^*_F M} = g$. Thus by 4.39 we have

$$\begin{aligned} g|_{L^\perp} &= \frac{1}{4} g^{ij}\, dx_i \otimes dx_j \\ &= 2((F^2)_{ij})^{-1} dx_i \otimes dx_j \\ &= (F_iF_j + FF_{ij})^{-1} dx_i \otimes dx_j, \end{aligned}$$

where

$$\begin{aligned} g|_L &= \frac{1}{4}\Pi \\ &= \frac{1}{8} (F^2)_{ij}\, dy_i \otimes dy_j \qquad \text{by 2.31} \\ &= \frac{1}{4} (F_iF_j + FF_{ij}) dy_i \otimes dy_j. \end{aligned}$$

□

From lemma 4.3 we have

$$\begin{aligned} \bar{\Pi} &= k_{\alpha\beta}\, \theta^\alpha \otimes \theta^\beta \\ &= \frac{1}{4} k_{\alpha\beta} a_{\alpha i} a_{\beta j} dx_i \otimes dx_j \end{aligned}$$

where $\theta^\alpha = -\frac{1}{2}\, i(X)d\eta$ and $\{X_\alpha = a_{\alpha i} \frac{\partial}{\partial y_i}\}$ is a local orthonormal frame of L with respect to $\frac{1}{4}\Pi$.

Furthermore from theorem 4.36, any contact metric structure (ϕ, ξ, η, g) with $g|_L = \frac{1}{4}\Pi$ will give the same formula for $g|_{L^\perp}$, thus

$$\begin{aligned} \bar{\Pi}|_Q &= g|_Q \\ &= ((F_iF_j + FF_{ij})^{-1} dx_i \otimes dx_j)|_Q \qquad \text{by theorem 4.36.} \end{aligned}$$

This implies that

$$\begin{aligned} \bar{\Pi} &= (F_iF_j + FF_{ij})^{-1} dx_i \otimes dx_j)|_{H \cap L'} \\ &= (2((F^2)_{ij})^{-1} dx_i \otimes dx_j)|_{H \cap L'}. \end{aligned} \tag{4.44}$$

From which we also obtain

$$(F_iF_j + FF_{ij})^{-1} = \frac{1}{4} k_{\alpha\beta} a_{\alpha i} a_{\beta j}. \tag{4.45}$$

CHAPTER 5. CONJUGATE LEGENDRE FOLIATIONS

If for a Legendre foliation $\mathcal{F}$ on a contact metric manifold (M, ϕ, ξ, η, g) the subbundle Q is completely integrable, then Q defines a Legendre foliation $\overline{\mathcal{F}}$ on the contact metric manifold (M, ϕ, ξ, η, g) related to $\mathcal{F}$. This Legendre foliation is called the conjugate Legendre foliation of $\mathcal{F}$. The existence of $\overline{\mathcal{F}}$ depends on the contact metric structure on the contact manifold. In this chapter we define the conjugate Legendre foliation of a Legendre foliation and find conditions for its existence when the Legendre foliation is flat and when it is non-degenerate. Then we give a family of Legendre foliations for which the conjugate always exists.

5 (a). CONJUGATE LEGENDRE FOLIATIONS

On a contact manifold (M, η) of dimension $2n + 1$ a subbundle L of the tangent bundle TM defines a Legendre foliation if and only if

i) L is completely integrable.

ii) L is n-dimensional.

iii) L is a subbundle of the contact distribution H.

Now consider a Legendre foliation, $\mathcal{F}$, on a contact metric manifold (M, ϕ, ξ, η, g). We know that $\phi|_H$ is an isomorphism and $\phi\xi = 0$, so the subbundle $Q = \phi L$ is a n-dimensional subbundle of H. Hence if Q is completely integrable it defines a Legendre foliation $\overline{\mathcal{F}}$ with tangent bundle $\overline{L} = Q$ and with transverse bundle $\overline{L}^{\perp} = L \oplus E$.

We will call $\overline{\mathcal{F}}$ *the conjugate Legendre foliation of* $\mathcal{F}$.

Note: (i) $\mathcal{F}$ and $\overline{\mathcal{F}}$ are complementary foliations when restricted to H.

(ii) $\overline{\mathcal{F}}$ depends on the contact metric structure. In general for a Legendre foliation $\mathcal{F}$ on a contact manifold (M, η) different contact metric structures may give different conjugate Legendre foliations.

(iii) For a Legendre foliation $\mathcal{F}$ the conjugate Legendre foliation need not exist, and if it does it may have a different classification from $\mathcal{F}$.

5 (b). EXISTENCE OF CONJUGATE LEGENDRE FOLIATIONS

We will now find conditions for the existence of the conjugate of a Legendre foliation. First as a line bundle is completely integrable we have the following result.

Theorem 5.1
If $\mathcal{F}$ is a Legendre foliation on a 3-dimensional contact metric manifold (M, ϕ, ξ, η, g) then the conjugate Legendre foliation $\overline{\mathcal{F}}$ of $\mathcal{F}$ exists.

For a Legendre foliation, $\mathcal{F}$, the existence of a flat conjugate Legendre foliation is dependant on the existence of a complementary foliation to $\mathcal{F}$ as the next theorem shows.

Theorem 5.2
For $\mathcal{F}$ a Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) the conjugate Legendre foliation $\overline{\mathcal{F}}$ exists and is flat if and only if there exists a foliation $\tilde{\mathcal{F}}$ complementary to $\mathcal{F}$.

Proof
Suppose there exists a foliation $\tilde{\mathcal{F}}$ complementary to $\mathcal{F}$ then $Q \oplus E$ is completely integrable which, using 1.12, implies that Q is completely integrable so $\overline{\mathcal{F}}$ the

conjugate Legendre foliation of $\mathcal{F}$ exists. $\overline{\mathcal{F}}$ is flat by lemma 3.3 with $\tilde{\mathcal{F}}$ as its associated (n+1)-dimensional foliation.

Suppose $\overline{\mathcal{F}}$ exists and is flat then, by lemma 3.3, there exists an associated (n+1)-dimensional foliation $\tilde{\mathcal{F}}$ defined by $\overline{L} \oplus E = Q \oplus E$. This associated foliation $\tilde{\mathcal{F}}$ is complementary to $\mathcal{F}$. ¤

These theorems are illustrated in the examples of chapter 4.

Example 4.1: (a) $\mathcal{F}$ is a non-degenerate Legendre foliation on a contact metric manifold of dimension 3, thus $\overline{\mathcal{F}}$ the conjugate Legendre foliation of $\mathcal{F}$ exists by theorem 5.1. Furthermore, as $[Y, \xi] = 0$ it is flat. The associated foliation given by theorem 5.2 is $\tilde{\mathcal{F}}$ defined by $\tilde{L} = \text{span}\{Y, \xi\}$. $\tilde{\mathcal{F}}$ is complementary to $\mathcal{F}$.

(b) $\mathcal{F}'$ is a non-degenerate Legendre foliation on a contact metric manifold of dimension 3, thus $\overline{\mathcal{F}}'$ the conjugate Legendre foliation of $\mathcal{F}'$ exists by theorem 5.1. Furthermore, as $[Y', \xi] = 2X - 2Y'$ it is non-degenerate. Thus the subbundle $\text{span}\{Y', \xi\}$ of TM is not completely integrable so doesn't define a foliation of M.

For a non-degenerate Legendre foliation on a manifold from the canonical family of contact metric manifolds we get the following existence theorem for the conjugate Legendre foliation.

Theorem 5.3

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) with $g|_L = \frac{1}{4}\Pi$, then the conjugate Legendre foliation $\overline{\mathcal{F}}$ of $\mathcal{F}$ exists if and only if the projection

$$p_L(N^1(X, X')) = 0 \qquad \forall\, X, X' \in \Gamma L.$$

Proof

For any $X, X' \in \Gamma L$ by 1.18 (i) we have

$$p_L(N^1(X, X')) = p_L(\phi^2[X, X'] - \phi[\phi X, X'] - \phi[X, \phi X'] + [\phi X, \phi X'])$$
$$= p_L([\phi X, \phi X']) \qquad \text{by 4.23,}$$

which with 2.2 implies that Q is completely integrable if and only if

$$p_L(N^1(X, X')) = 0 \qquad \forall\ X, X' \in \Gamma L.$$

□

Therefore if $\mathcal{F}$ is a non-degenerate Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) with $g|_L = \frac{1}{4}\prod$, such that $\phi|_H$ is a complex structure on H then the conjugate Legendre foliation $\overline{\mathcal{F}}$ of $\mathcal{F}$ exists.

Theorem 5.3 is illustrated in the examples of chapter 2.

Example 2.1: (a) $\mathcal{F}_F^c$ is a non-degenerate Legendre foliation on a unit cotangent bundle S_F^*M with the canonical contact metric structure. Looking at the Lie brackets we have for $\alpha = 1, 2$ that

$$[Y_1, Y_2] = 0, \qquad [Y_\alpha, \xi] = 0.$$

Hence $\overline{\mathcal{F}}_F^c$ the conjugate Legendre foliation of $\mathcal{F}_F^c$ exists and is flat. It can be shown that

$$p_L(N_c^1(X, X')) = 0 \qquad \forall\ X, X' \in \Gamma L.$$

(b) $\mathcal{F}_F'$ is a non-degenerate Legendre foliation on a unit cotangent bundle S_F^*M with the contact metric structure (ϕ', ξ, η, g') with $g'|_L \neq \frac{1}{4}\prod$.

Looking at the Lie brackets we have for $\alpha = 1, 2$ that

$$[Y_1', Y_2'] = 0, \qquad [Y_\alpha', \xi] = 0.$$

Hence $\overline{\mathcal{F}}_F'$ the conjugate Legendre foliation of $\mathcal{F}_F'$ exists and is flat. Also from the Lie brackets can be shown that

$$N^1(X_1', X_2') = \frac{4}{y_3} X_2'.$$

Example 2.1 (b) shows that in general theorem 5.3 can not be extended to a non-degenerate Legendre foliation, $\mathcal{F}$, on a contact metric manifold (M, ϕ, ξ, η, g) with $g|_L \neq \frac{1}{4}\Pi$. But when the contact metric structure satisfies the conditions of theorem 4.26, the contact metric structure (ϕ', ξ, η, g') with $g'|_L = \frac{1}{4}\Pi$ generated by theorem 4.26 can be used in theorem 5.3 to get the following theorem.

Theorem 5.4

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) such that there exist smooth non-zero functions f_α on M which satisfy

$$p_{L^\perp}([\xi, X_\alpha]) = 2f_\alpha \phi X_\alpha$$

where $\{X_\alpha\}$ is a local orthonormal frame of L with respect to g.

Then the conjugate Legendre foliation $\overline{\mathcal{F}}$ of $\mathcal{F}$ exists if and only if for any $X, X' \in \Gamma L$ the projection

$$\begin{aligned} p_L(N^1(X, X')) &= p_L(\phi'^2[X, X'] - \phi'[\phi'X, X'] - \phi'[X, \phi'X'] \\ &\qquad + [\phi'X, \phi'X']) \\ &= 0, \end{aligned}$$

where (ϕ', ξ, η, g') is the contact metric structure with $g'|_L = \frac{1}{4}\Pi$ generated by theorem 4.26.

Proof

By corollary 4.27, as Q = Q', Q' is completely integrable if and only if Q is, and by theorem 5.3 Q is completely integrable if and only if

$$p_L(N^1(X, X')) = 0 \qquad \forall\, X, X' \in \Gamma L.$$

□

There are several equivalent conditions for the existence of the conjugate of a non-degenerate Legendre foliation on a manifold from the canonical family of contact metric manifolds.

Lemma 5.5

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) with $g|_L = \frac{1}{4}\Pi$. Then the following are equivalent where ∇ is the Levi-Civita connection of g.

(i) $p_L(N^1(X, X')) = 0 \qquad \forall\, X, X' \in \Gamma L.$

(ii) $p_Q((\nabla_X \phi)X') = 0 \qquad \forall\, X,\ X' \in \Gamma L.$

(iii) $p_L((\nabla_X \phi)Y) = 0 \qquad \forall\, X \in \Gamma L, \forall\, Y \in \Gamma Q.$

(iv) $p_Q((\nabla_Y \phi)Y') = 0 \qquad \forall\, Y, Y' \in \Gamma Q.$

(v) $p_L((\nabla_Y \phi)X) = 0 \qquad \forall\, X \in \Gamma L, \forall\, Y \in \Gamma Q.$

Proof

(i) $\Leftrightarrow$ (v) For any X, X', X'' $\in \Gamma L$ we have

$$g(N^1(X, X'), X'') = -\, g(N^1(X, X'), \phi(\phi X'')) \qquad \text{by 1.2}$$
$$= -\, 2g((\nabla_{\phi X''} \phi)X, X') \qquad \text{by lemma 1.26.}$$

(ii) $\Leftrightarrow$ (iii) For any $X, X' \in \Gamma L$ we have

$$\begin{aligned}
\phi p_Q((\nabla_X \phi)X') &= p_L\phi((\nabla_X \phi)X') && \text{by theorem 2.1}\\
&= p_L(\phi(\nabla_X \phi X' - \phi\nabla_X X')) \\
&= p_L(\phi\nabla_X \phi X' + \nabla_X X') && \text{by 1.2}\\
&= p_L(\phi\nabla_X \phi X' - \nabla_X\phi(\phi X')) && \text{by 1.2}\\
&= - p_L((\nabla_X \phi)\, \phi X').
\end{aligned}$$

(iii) $\Leftrightarrow$ (v) For any $X \in \Gamma L$ and $Y \in \Gamma Q$ we have

$$p_L((\nabla_X \phi)Y) = - p_L((\nabla_{\phi X} \phi)\, \phi Y) \qquad \text{by 1.38.}$$

(ii) $\Leftrightarrow$ (iv) For any $X, X' \in \Gamma L$ we have

$$p_Q((\nabla_X \phi)X') = - p_Q((\nabla_{\phi X} \phi)\, \phi X') \qquad \text{by 1.38.}$$

¤

We also get the following corollary to theorem 5.3

Corollary 5.6

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) with $g|_L = \frac{1}{4}\Pi$. Then if the conjugate Legendre foliation $\overline{\mathcal{F}}$ of $\mathcal{F}$ exists we have

$$p_Q((\nabla_{\phi X'} X) = p_Q((\nabla_{\phi X} X') \qquad \forall\ X, X' \in \Gamma L,$$

where ∇ is the Levi-Civita connection of g.

Therefore the Weingarten map of $\overline{\mathcal{F}}$ satisfies

$$\overline{A}_X \phi X' = \overline{A}_{X'} \phi X \qquad \forall\ X, X' \in \Gamma L.$$

Proof

As Q is completely integrable for any X, X', X'' $\in \Gamma L$ we have

$$\begin{aligned} 0 &= g([\phi X, \phi X'], X'') \\ &= g(\nabla_{\phi X} \phi X' - \nabla_{\phi X'} \phi X, X'') \\ &= - g(\nabla_{\phi X} X' - \nabla_{\phi X'} X, \phi X'') \qquad \text{by 1.10 and lemma 5.5 (v).} \end{aligned}$$

Hence

$$p_Q((\nabla_{\phi X'} X) = p_Q((\nabla_{\phi X} X') \qquad \forall\, X, X' \in \Gamma L,$$

and as $\bar{L} = Q$ we have

$$\bar{A}_X \phi X' = \bar{A}_{X'} \phi X \qquad \forall\, X, X' \in \Gamma L.$$

□

Note that in general for a Legendre foliation on a contact metric manifold

$$A_X \phi X' \neq A_{X'} \phi X$$

for tangential vector fields X and X', see example 3.1 (c).

Using the canonical connection we get the following proposition when the contact metric structure of a non-degenerate Legendre foliation is the canonical contact metric structure.

Proposition 5.7

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on the canonical contact metric manifold $(M, \phi^c, \xi, \eta, g^c)$, then $N_c^1|_H \equiv 0$ if and only if $K^{\alpha}_{\beta\gamma} \equiv 0$, where $K^{\alpha}_{\beta\gamma}$ are the functions given by theorem 4.31.

This is a rewording of proposition 5.42 of Pang [2].

From the structure equations of theorem 4.31 for any α, β, γ we have

$$\begin{aligned} d\mathcal{E}_c^{\alpha}(\phi^c X_\beta, \phi^c X_\gamma) &= (K_{\delta\varepsilon}^{\alpha}\theta_c^{\delta} \wedge \theta_c^{\varepsilon})(\phi^c X_\beta, \phi^c X_\gamma) \\ &= \frac{1}{2}\{K_{\delta\varepsilon}^{\alpha}\theta_c^{\delta}(\phi^c X_\beta)\theta_c^{\varepsilon}(\phi^c X_\gamma) \\ &\qquad - K_{\delta\varepsilon}^{\alpha}\theta_c^{\delta}(\phi^c X_\gamma)\theta_c^{\varepsilon}(\phi^c X_\beta)\} \\ &= \pm\frac{1}{2}(K_{\beta\gamma}^{\alpha} - K_{\gamma\beta}^{\alpha}) \\ &= \pm K_{\beta\gamma}^{\alpha} \qquad \text{as } K_{\beta\gamma}^{\alpha} = -K_{\gamma\beta}^{\alpha}. \end{aligned}$$

Therefore $K_{\beta\gamma}^{\alpha} \equiv 0$ if and only if

$$\begin{aligned} 0 &= d\omega(Y, Y') \qquad \forall\ Y, Y' \in \Gamma Q, \forall\ \omega \in \Gamma L^* \\ &= \frac{1}{2}\omega([Y, Y']). \end{aligned}$$

if and only if the projection

$$p_L([Y, Y']) = 0 \qquad \forall\ Y, Y' \in \Gamma Q.$$

Since by 2.2 the projection

$$p_E([Y, Y']) = 0 \qquad \forall\ Y, Y' \in \Gamma Q,$$

we have that Q is completely integrable if and only if $K_{\beta\gamma}^{\alpha} \equiv 0$. ¤

So when the contact metric structure of a non-degenerate Legendre foliation is the canonical contact metric structure theorem 5.3 can be stated more generally as follows

Theorem 5.8

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on the canonical contact metric manifold $(M, \phi^c, \xi, \eta, g^c)$. Then the conjugate Legendre foliation $\overline{\mathcal{F}}$ of $\mathcal{F}$ exists if and only if

(i) $\qquad N_c^1|_H \equiv 0,$

which is equivalent by lemma 1.26 to the projection

ii) $\qquad p_H((\nabla_U \phi^c)U') = 0 \qquad \forall\ U, U' \in \Gamma H.$

Theorem 5.3 can only be extended to theorem 5.8 when (ϕ, ξ, η, g) with $g|_L = \frac{1}{4}\Pi$ is the canonical contact metric structure, as example 6.1 (d) will show.

We also get the following extension to corollary 5.6.

Corollary 5.9

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on the canonical contact metric manifold $(M, \phi^c, \xi, \eta, g^c)$. Then if the conjugate Legendre foliation $\overline{\mathcal{F}}$ of $\mathcal{F}$ exists the projection

$$p_L((\nabla_{X'} \phi^c X) = p_L((\nabla_X \phi^c X') \qquad \forall\ X, X' \in \Gamma L,$$

where ∇ is the Levi-Civita connection of g.

Therefore the Weingarten map of $\mathcal{F}$ satisfies

$$A_{\phi^c X} X' = A_{\phi^c X'} X \qquad \forall\ X, X' \in \Gamma L.$$

Proof

Suppose the conjugate exists. Then for any $X, X', X'' \in \Gamma L$ as L is completely integrable we have

$$
\begin{aligned}
0 &= g^c([X, X'], \phi^c X'') \\
&= g^c(\nabla_X X' - \nabla_{X'} X, \phi^c X'') \\
&= - g^c(\nabla_X \phi^c X' - \nabla_{X'} \phi^c X, X'') \qquad \text{by 1.10 and theorem 5.8.}
\end{aligned}
$$

Hence

$$
p_L((\nabla_{X'} \phi^c X) = p_L((\nabla_X \phi^c X') \qquad \forall\ X, X' \in \Gamma L,
$$

which implies that

$$
A_{\phi^c X} X' = A_{\phi^c X'} X \qquad \forall\ X, X' \in \Gamma L.
$$

□

When the contact metric structure of a non-degenerate Legendre foliation $\mathcal{F}$ is the canonical contact metric structure and the conjugate Legendre foliation of $\mathcal{F}$ exists, the canonical connection is a restriction of the Levi-Civita connection of g^c, the canonical semi-Riemannian metric, as the next theorem shows.

Theorem 5.10

For a non-degenerate Legendre foliation, $\mathcal{F}$, on the canonical contact metric manifold $(M, \phi^c, \xi, \eta, g^c)$ such that the conjugate Legendre foliation of $\mathcal{F}$ exists the canonical connection ∇^c, where ∇ is the the Levi-Civita connection of g^c,

$$
\nabla^c_W \omega = \begin{cases} p_{L_c^*}\left(\nabla_W \omega\right) & \forall\ W \in \Gamma TM, \text{ if } \omega \in \Gamma L_c^* \\ p_{Q^*}\left(\nabla_W \omega\right) & \forall\ W \in \Gamma TM, \text{ if } \omega \in \Gamma Q^* \\ p_{E^*}\left(\nabla_W \omega\right) & \forall\ W \in \Gamma TM, \text{ if } \omega \in \Gamma E^* \end{cases}
$$

Proof

By 4.29 (vii) and (viii) we have

$$\nabla^c_W \omega = p_{E^*}(\nabla_W \omega) = 0 \qquad \forall\ W \in \Gamma TM, \text{ if } \omega \in \Gamma E^*.$$

As Q is completely integrable from 4.29 (i), (ii), (iii) and (viii) it follows that

$$\nabla^c_W \omega = p_{Q^*}(\nabla_W \omega) \qquad \forall\ W \in \Gamma TM, \text{ if } \omega \in \Gamma Q^*.$$

For any $\alpha, \beta, \delta, \gamma$ and any $W \in \Gamma TM$ we have

$$\begin{aligned}
k_{\beta\delta}(\nabla_W \theta^\delta_c)(\phi^c X_\gamma) &= (\nabla_W k_{\beta\delta}\theta^\delta_c)(\phi^c X_\gamma) \\
&= -(\nabla_W \phi^{c*}\mathcal{E}^\beta_c)(\phi^c X_\gamma) \\
&\qquad \text{as } \phi^{c*}\mathcal{E}^\beta_c = -k_{\beta\delta}\theta^\delta_c \text{ by 4.7 (ii) and 4.9} \\
&= -W\mathcal{E}^\beta_c(\phi^c(\phi^c X_\gamma)) + \mathcal{E}^\beta_c(\phi^c(\nabla_W \phi^c X_\gamma)) \\
&= -W\mathcal{E}^\beta_c(\phi^c(\phi^c X_\gamma)) + \mathcal{E}^\beta_c(\nabla_W \phi^c(\phi^c X_\gamma)) \\
&\qquad \text{by theorem 5.8 (ii)} \\
&= W\mathcal{E}^\beta_c(X_\gamma) - \mathcal{E}^\beta_c(\nabla_W X_\gamma) \\
&= (\nabla_W \mathcal{E}^\beta_c)(X_\beta).
\end{aligned}$$

From this and 4.29 (iv), (v), (vi) and (viii) as Q is completely integrable it follows that

$$\nabla^c_W \omega = p_{L^*_c}(\nabla_W \omega) \qquad \forall\ W \in \Gamma TM, \text{ if } \omega \in \Gamma L^*_c.$$

¤

The following theorem shows that for a flat Legendre foliation there always exists a contact metric structure such that the conjugate Legendre foliation exists.

Theorem 5.11

Let $\mathcal{F}$ be a flat Legendre foliation on the canonical contact metric manifold $(M, \phi^c, \xi, \eta, g^c)$ then the conjugate Legendre foliation, $\overline{\mathcal{F}}$, of $\mathcal{F}$ exists and is flat.

Proof

Take the canonical contact metric structure (ϕ^c, ξ, η, g^c) given by theorem 3.2, then there are local coordinates (x_α, y_α, z) such that

$$X_\alpha = 2\frac{\partial}{\partial y_\alpha}$$
$$Y_\alpha = \phi^c X_\alpha$$
$$= 2\left(\frac{\partial}{\partial x_\alpha} + y_\alpha \frac{\partial}{\partial z}\right)$$
$$\xi = 2\frac{\partial}{\partial z}$$

and $\{X_\alpha\}$ is a local orthonormal frame of L with respect to g^c.

Calculating the Lie brackets we get for all α, β

$$[X_\alpha, X_\beta] = 0$$
$$[Y_\alpha, Y_\beta] = 0$$
$$[X_\alpha, \xi] = 0$$
$$[Y_\alpha, \xi] = 0$$
$$[X_\alpha, Y_\beta] = 2\delta_{\alpha\beta}\xi$$

Hence $\overline{\mathcal{F}}$ the conjugate Legendre foliation of $\mathcal{F}$ exists and by lemma 2.13 (i) it is flat. ¤

The examples of chapter 3 illustrate theorem 5.11

Example 3.1: (a) $\mathcal{F}^c$ is a flat Legendre foliation on the canonical contact metric manifold which is Sasakian. Looking at the Lie brackets we have for $\alpha = 1, 2$ that

$$[Y_1^c, Y_2^c] = 0, \qquad [Y_\alpha^c, \xi] = 0$$

Hence $\overline{\mathcal{F}}^c$ the conjugate Legendre foliation of $\mathcal{F}^c$ exists and is flat.

(b) $\mathcal{F}'$ is a flat Legendre foliation on a contact metric manifold. Looking at the Lie brackets we have that

$$[Y_1', Y_2'] = 2y_2X_1 - 2y_1X_2.$$

Hence the conjugate Legendre foliation of $\mathcal{F}'$ does not exist.

(c) $\mathcal{F}''$ is a flat Legendre foliation on a K-contact manifold. Looking at the Lie brackets we have that

$$[Y_1', Y_2'] = 2y_2X_2 - 2y_1X_1.$$

Hence the conjugate Legendre foliation of $\mathcal{F}''$ does not exist.

(d) $\mathcal{F}$ is a flat Legendre foliation on a Sasakian manifold. Looking at the Lie brackets we have for $\alpha = 1, 2$ that

$$[Y_1, Y_2] = 0, \qquad [Y_\alpha, \xi] = 0$$

Hence $\overline{\mathcal{F}}$ the conjugate Legendre foliation of $\mathcal{F}$ exists and is flat.

Theorem 5.11 shows that for a flat Legendre foliation, $\mathcal{F}$, on the canonical contact metric manifold $(M, \phi^c, \xi, \eta, g^c)$ the conjugate Legendre foliation exists and is flat, but it need not be the only one, for example see example 3.1 (d).

Later a family of flat Legendre foliations will be introduced for which the conjugate exists and is a non-degenerate Legendre foliation.

5 (c). PROPERTIES OF CONJUGATE LEGENDRE FOLIATIONS

The mean curvature vector fields of a Legendre foliation and its conjugate satisfy the following relationship.

Lemma 5.12
Let $\mathcal{F}$ be a Legendre foliation on the contact metric manifold (M, ϕ, ξ, η, g) such that the conjugate exists then

$$g(\tau, \xi) = - g(\bar{\tau}, \xi)$$

where τ is the mean curvature vector field of $\mathcal{F}$ and $\bar{\tau}$ is the mean curvature vector field of $\bar{\mathcal{F}}$ which is given by

$$\bar{\tau} = g(\phi X_\alpha, \phi X_\beta)(p_L(\nabla_{\phi X_\alpha} \phi X_\beta) + p_E(\nabla_{\phi X_\alpha} \phi X_\beta)),$$

where $\{\phi X_\alpha\}$ is a local orthonormal frame of $\bar{L} = Q$ with respect to g.

Proof
Let $\{X_\alpha\}$ be a local orthonormal frame of L Q with respect to g then

$$\begin{aligned} g(\tau, \xi) &= g(X_\alpha, X_\beta) g(\nabla_{X_\alpha} X_\beta, \xi) \\ &= - g(\phi X_\alpha, \phi X_\beta) g(\nabla_{\phi X_\alpha} \phi X_\beta, \xi) \\ &\qquad \text{by 1.9 and theorem 2.4 (iii)} \\ &= - g(\bar{\tau}, \xi). \end{aligned}$$

□

Hence if $\mathcal{F}$ is a non-harmonic foliation such that $g(\tau, \xi) \neq 0$, then $\bar{\mathcal{F}}$ is not harmonic and $g(\bar{\tau}, \xi) \neq 0$.

This is illustrated by example 4.1 (b).

Example 4.1: (b) $\mathcal{F}$ and $\overline{\mathcal{F}}$ exist and are not harmonic also

$$\tau = p_Q(\nabla_X X) + p_E(\nabla_X X) = -2\xi$$

and

$$\overline{\tau} = p_L(\nabla_{\phi X}\phi X) + p_E(\nabla_{\phi X}\phi X) = 2\xi.$$

Thus

$$g(\tau, \xi) = -2 = -g(\overline{\tau}, \xi),$$

which is consistent with lemma 5.12.

5 (d). A FAMILY OF LEGENDRE FOLIATIONS AND THEIR CONJUGATES

Let (M, ϕ, ξ, η, g) be a contact metric manifold such that the curvature of ∇ the Levi-Civita connection of g satisfies

$$R(W, W')\xi = \varpi(g(W', \xi)W - g(W, \xi)W') \qquad \forall\ W, W' \in \Gamma TM,$$

where ϖ is a real number.

Thus for any $U \in \Gamma H$ we have

(5.13) $$R(\xi, U)\xi = -\varpi U.$$

Blair [5] gives the following relationship for any $W \in \Gamma TM$

$$\frac{1}{2}(R(\xi, W)\xi - \phi R(\xi, \phi W)\xi) = h^2W + \phi^2W.$$

By restricting this to $U \in \Gamma H$ and substituting 5.13 into it we get

$$\frac{1}{2}(-\varpi U + \phi\varpi\phi U) = h^2U + \phi^2U \qquad \forall\ U \in \Gamma H$$

therefore

$$h^2U = (\varpi - 1)\phi^2U,$$

thus we obtain

$$(h|_H)^2 = (1 - \varpi).$$

Thus $h|_H$ has eigenvalues $\mu = \sqrt{(1 - \varpi)}$ and $-\mu = -\sqrt{(1 - \varpi)}$, hence if

$\varpi < 1$ $\quad$ $h|_H$ has rank $2n$ and if

$\varpi = 1$ $\quad$ $h \equiv 0$ so (M, ϕ, ξ, η, g) is a K-contact manifold.

The following proposition, which is a rewording of proposition 5.1 of Tanno [15], shows that if $\mathcal{F}$ is a Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) whose curvature satisfies

$$R(W, W')\xi = \varpi(g(W', \xi)W - g(W, \xi)W') \qquad \forall\ W, W' \in \Gamma TM,$$

where $\varpi < 1$, then the conjugate of $\mathcal{F}$ exists.

Proposition 5.14

Let (M, ϕ, ξ, η, g) be a contact metric manifold where for some real number ϖ the curvature of ∇, the Levi-Civita connection of g, satisfies

$$R(W, W')\xi = \varpi(g(W', \xi)W - g(W, \xi)W') \qquad \forall\ W, W' \in \Gamma TM.$$

Then $\varpi \leq 1$, also if $\varpi < 1$ then (M, ϕ, ξ, η, g) admits three mutually orthogonal and completely integrable distributions $D(0)$, $D(\mu)$ and $D(-\mu)$ defined by the eigenspaces of h where $\mu = \sqrt{(1- \varpi)}$.

Therefore $D(\mu)$ and $D(-\mu)$ define a Legendre foliation and its conjugate.

Note: if $\varpi = 0$ then $\mu = 1$ and the conditions of theorem 3.5 are satisfied. Therefore $D(-1)$ defines a flat Legendre foliation and $D(1)$ defines its conjugate Legendre foliation.

CHAPTER 6.SEMI-RIEMANNIAN LEGENDRE FOLIATIONS

Pang defines a Riemannian Legendre foliation as a Legendre foliation that is locally equivalent to one of the form $\mathcal{F}_F$ on $S^*_F\tilde{M}$ where F is the norm induced by a Riemannian metric g on $\tilde{M}$. In this chapter we show that a Riemannian Legendre foliation has a naturally defined contact metric structure (ϕ, ξ, η, g), related to the canonical contact metric structure, where g is a bundle-like metric. Thus we get an alternative definition of a Riemannian Legendre foliation , in terms of this bundle-like metric. We then extend the definition of a bundle-like Riemannian metric to a semi-Riemannian metric and define a semi-Riemannian Legendre foliation.

The properties of semi-Riemannian Legendre foliations are investigated and examples of semi-Riemannian foliations are given.

6 (a). SEMI-RIEMANNIAN LEGENDRE FOLIATIONS

The definition of a bundle-like metric with respect to a foliation can be naturally extended to a bundle-like contact metric structure with respect to a Legendre foliation.

Definition 6.1

Let $\mathcal{F}$ be a Legendre foliation on a contact manifold (M, η), then a contact metric structure (ϕ, ξ, η, g) on (M, η) is bundle-like with respect to $\mathcal{F}$ if

$$\mathcal{L}_X g|_{L^\perp} = 0 \qquad \forall\, X \in \Gamma L$$

where $\mathcal{L}$ is the Lie derivative.

Usually a bundle-like metric g is assumed to be positive definite, and a foliation on a Riemannian manifold (M, g) is a *Riemannian foliation* if g is a bundle-like metric. But we do not make this restriction and define a *semi-Riemannian foliation* as a foliation on the semi-Riemannian manifold (M, g) where g is a bundle-like semi-Riemannian metric, that is for any tangential vector field X the Lie derivative

$$\mathcal{L}_X g|_{L^\perp} = 0.$$

The following theorem shows that a bundle-like contact metric structure belongs to the canonical family of contact metric structures with $g|_L = \frac{1}{4}\prod$.

Theorem 6.2

Let $\mathcal{F}$ be a Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g). If the contact metric structure is bundle-like with respect to $\mathcal{F}$ then

i) $\mathcal{F}$ is a non-degenerate Legendre foliation.

ii) (ϕ, ξ, η, g) is such that $g|_L = \frac{1}{4}\prod$.

Proof

Assume that (ϕ, ξ, η, g) is a bundle-like contact metric structure with respect to $\mathcal{F}$ and let $\{X_\alpha\}$ be a local orthonormal frame of L with respect to g.

i) Suppose $\mathcal{F}$ (and therefore $\prod$) is degenerate, then there exists a X_α such that

$$\prod(X_\alpha, X_\alpha) = 0.$$

For this X_α we have

$$\begin{aligned}\mathcal{L}_{X_\alpha} g(\phi X_\alpha, \xi) &= - g([X_\alpha, \phi X_\alpha], \xi) - g([X_\alpha, \xi], \phi X_\alpha) \\ &= - 2g(X_\alpha, X_\alpha) + \tfrac{1}{2}\prod(X_\alpha, X_\alpha) \quad \text{by 1.13 and 2.11} \\ &= \pm 2 \neq 0.\end{aligned}$$

Hence (ϕ, ξ, η, g) is not a bundle-like contact metric structure, and we have a contradiction. Therefore $\mathcal{F}$ is a non-degenerate Legendre foliation.

ii) As (ϕ, ξ, η, g) is a bundle-like contact metric structure we have

$$\mathcal{L}_X g(Y, Z) = 0 \qquad \forall\, X \in \Gamma L, \forall\, Y \in \Gamma Q, \forall\, Z \in \Gamma E.$$

Thus, as $\mathcal{F}$ is non-degenerate by part (i), theorem 4.21 shows that $g|_L = \frac{1}{4}\Pi$.

□

We call a Legendre foliation, $\mathcal{F}$, on a contact metric manifold (M, ϕ, ξ, η, g) a *semi-Riemannian Legendre foliation* if (ϕ, ξ, η, g) is a bundle-like contact metric structure. Furthermore when the bundle-like contact metric structure is such that g is positive definite we have a *Riemannian Legendre foliation*. For a Riemannian Legendre foliation it follows from theorem 6.2 (ii) that Π is positive definite as g is positive definite. Hence a Riemannian Legendre foliation is a positive definite Legendre foliation.

A semi-Riemannian Legendre foliation is a semi-Riemannian foliation as it has a bundle-like semi-Riemannian metric and a Riemannian Legendre foliation is a Riemannian foliation. We will show later in this chapter that the semi-Riemannian Legendre foliations defined here correspond to the semi-Riemannian Legendre foliations defined by Pang [2]. Furthermore theorem 6.2 shows that a semi-Riemannian Legendre foliation is a foliation on a manifold from the canonical family of contact metric manifolds.

Lemma 6.3

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g), with $g|_L = \frac{1}{4}\Pi$. Then $\mathcal{F}$ is a semi-Riemannian Legendre foliation if and only if the Lie derivative

i) $\mathcal{L}_X g(Y, Y') = 0 \qquad \forall\, X \in \Gamma L, \forall\, Y, Y' \in \Gamma Q,$

which is equivalent to

ii) $g(X, \nabla_Y Y' + \nabla_{Y'} Y) = 0$ $\quad \forall X \in \Gamma L, \forall Y, Y' \in \Gamma Q.$

Proof

(i): For any $Z, Z' \in \Gamma E$ there exist functions f, h such that $Z = f\xi$ and $Z' = h\xi$, then for any $X \in \Gamma L$ we have

$$\begin{aligned} \mathcal{L}_X g(Z, Z') &= \mathcal{L}_X g(f\xi, h\xi) \\ &= Xg(f\xi, h\xi) - g([X, f\xi], h\xi) - g(f\xi, [X, h\xi]) \\ &= X(fh) - (Xf)h - f(Xh) - 2fh(g([X, \xi], \xi)) \\ &= 0 \qquad \text{by 1.12.} \end{aligned}$$

Furthermore as $g|_L = \frac{1}{4}\Pi$ for any $X \in \Gamma L$, $Y \in \Gamma Q$, and $Z \in \Gamma E$ we have

$$\mathcal{L}_X g(Y, Z) = 0 \qquad \text{by theorem 4.21.}$$

These imply that

$$\mathcal{L}_X g|_{L^\perp} = 0 \qquad \forall X \in \Gamma L,$$

if and only if

$$\mathcal{L}_X g(Y, Y') = 0 \qquad \forall X \in \Gamma L, \forall Y, Y' \in \Gamma Q.$$

The equivalence of (i) with (ii) is shown by Tondeur [26]. ¤

Theorem 6.2 shows that a bundle-like contact metric structure belongs to the canonical family of contact metric structures. In the following theorem we show that if one contact metric structure from this family is bundle-like then all contact metric structures in the family are bundle-like.

Theorem 6.4

Let $\mathcal{F}$ be a semi-Riemannian Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) defined by the subbundle $L \subset H$. Then for a contact metric manifold $(M, \phi', \xi, \eta, g')$ with $g'|_L = \frac{1}{4}\Pi$ the Legendre foliation $\mathcal{F}'$ defined by the subbundle $L \subset H$ is semi-Riemannian.

Proof

Let $\{X_\alpha\}$ be a local orthonormal frame of L with respect to $\frac{1}{4}\Pi$. Then $\{\phi X_\alpha\}$ is a local orthonormal frame of $Q = \phi L$ with respect to g and for any α, β, γ we have

a) $$\begin{aligned} \mathcal{L}_{X_\alpha} g(\phi X_\beta, \phi X_\gamma) &= - g([X_\alpha, \phi X_\beta], \phi X_\gamma) - g(\phi X_\beta, [X_\alpha, \phi X_\gamma]) \\ &= - g(\phi[X_\alpha, \phi X_\beta], \phi^2 X_\gamma) - g(\phi^2 X_\beta, \phi[X_\alpha, \phi X_\gamma]) \quad \text{by 1.9} \\ &= - g(\lambda[X_\alpha, \phi X_\beta], X_\gamma) - g(X_\beta, \lambda[X_\alpha, \phi X_\gamma]) \\ &\qquad \text{as } \phi|_Q = - \lambda|_Q \text{ by 4.2} \\ &= - \frac{1}{4} \{\Pi(\lambda[X_\alpha, \phi X_\beta], X_\gamma) + \Pi(X_\beta, \lambda[X_\alpha, \phi X_\gamma])\} \\ &\qquad \text{as } g|_L = \frac{1}{4}\Pi. \end{aligned}$$

For any contact metric structure, (ϕ', ξ, η, g'), with $g'|_L = \frac{1}{4}\Pi$, by lemma 4.5 (i) there exists functions $c_{\alpha\beta}$ on M such that

$$\phi' X_\alpha = \phi X_\alpha + c_{\alpha\beta} X_\beta.$$

As $\{\phi' X_\alpha\}$ is a local orthonormal frame of $Q = \phi' L$ with respect to g', for any α, β, γ we have

$$\begin{aligned} \mathcal{L}_{X_\alpha} g'(\phi' X_\beta, \phi' X_\gamma) &= - g'([X_\alpha, \phi' X_\beta], \phi' X_\gamma) - g'(\phi' X_\beta, [X_\alpha, \phi' X_\gamma]) \\ &= - g'([X_\alpha, \phi X_\beta + c_{\beta\delta} X_\delta], \phi' X_\gamma) \\ &\qquad - g'(\phi' X_\beta, [X_\alpha, \phi X_\gamma + c_{\gamma\varepsilon} X_\varepsilon]) \\ &= - g'([X_\alpha, \phi X_\beta], \phi' X_\gamma) - g'(\phi' X_\beta, [X_\alpha, \phi X_\gamma]) \\ &\qquad \text{as L is completely integrable} \end{aligned}$$

$$= - g'(\phi'[X_\alpha, \phi X_\beta], \phi'^2 X_\gamma) - g'(\phi'^2 X_\beta, \phi'[X_\alpha, \phi X_\gamma])$$
by 1.9

$$= - g'(\lambda[X_\alpha, \phi X_\beta], X_\gamma) - g'(X_\beta, \lambda[X_\alpha, \phi X_\gamma])$$
as $\phi'|_{Q'} = - \lambda|_{Q'}$ by 4.2

$$= - \frac{1}{4} \{\Pi(\lambda[X_\alpha, \phi X_\beta], X_\gamma) - \Pi(X_\beta, \lambda[X_\alpha, \phi X_\gamma])\}$$
as $g'|_L = \frac{1}{4}\Pi$

$$= \mathcal{L}_{X_\alpha} g(\phi X_\beta, \phi X_\gamma)$$ by (a)

$$= 0$$

as (ϕ, ξ, η, g) is a bundle-like contact metric structure.

Thus (ϕ', ξ, η, g') is a bundle-like contact metric structure by lemma 6.3 (i). ¤

A bundle-like contact metric structure can also be defined in terms of the invariant G.

Theorem 6.5

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) with $g|_L = \frac{1}{4}\Pi$. Then $\mathcal{F}$ is a semi-Riemannian Legendre foliation if and only if

$$G \equiv 0.$$

Proof

From theorem 4.25 for any $X, X', X'' \in \Gamma L$, we have

$$G(X, X', X'') = 2g(\nabla_{\phi X'} \phi X'' + \nabla_{\phi X''} \phi X', X).$$

Thus the theorem follows from lemma 6.3 part (ii). ¤

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on a contact manifold (M, η) with $G \equiv 0$, then by lemma 4.1 there exists a contact metric structure (ϕ, ξ, η, g) with $g|_L = \frac{1}{4}\Pi$, hence by theorem 6.5 $\mathcal{F}$ is a semi-Riemannian Legendre foliation on the contact metric manifold (M, ϕ, ξ, η, g). Furthermore, if $\mathcal{F}$ is a Legendre foliation on

a contact metric manifold, (M, ϕ, ξ, η, g), such that $G \equiv 0$, it follows from theorem 6.2 and theorem 6.5 that $\mathcal{F}$ is a semi-Riemannian Legendre foliation if and only if $g|_L = \frac{1}{4}\Pi$.

Pang [2] gives the following definition for a Riemannian Legendre foliation.

Definition
A Legendre foliation, $\mathcal{F}$, on a contact manifold (M, η) is a Riemannian Legendre foliation if $\mathcal{F}$ is locally equivalent to a Legendre foliation of the form $\mathcal{F}_F$ on S_F^*M where F is the norm induced by a Riemannian metric g on $\tilde{M}$.

Furthermore we get the corresponding definition for a semi-Riemannian Legendre foliation.

Definition
A Legendre foliation, $\mathcal{F}$, on a contact manifold (M, η) is a semi-Riemannian Legendre foliation if $\mathcal{F}$ is locally equivalent to a Legendre foliation of the form $\mathcal{F}_F$ on S_F^*M where F is the norm induced by a semi-Riemannian metric g on $\tilde{M}$.

The following proposition is a rewording of Pang's [2] proposition 5.32.

Proposition
A non-degenerate (positive definite) Legendre foliation, $\mathcal{F}$, is a semi-Riemannian (Riemannian) Legendre foliation if and only if $G \equiv 0$.

This proposition and theorem 6.5 imply that if a Legendre foliation $\mathcal{F}$ on a contact manifold (M, η) is a semi-Riemannian Legendre foliation in the sense of Pang, then giving (M, η) its canonical contact metric structure (ϕ^c, ξ, η, g^c) makes $\mathcal{F}$ a semi-Riemannian Legendre foliation according to our definition. Also theorem 6.4 allows us to give (M, η) any contact metric structure from the canonical family of contact metric structures and obtain $\mathcal{F}$ as a semi-Riemannian Legendre foliation under our

definition. Hence if $\mathcal{F}$ is a semi-Riemannian Legendre foliation by Pang's definition we can introduce a bundle-like contact metric structure to make it a semi-Riemannian Legendre foliation by our definition, which shows that our definition of a semi-Riemannian Legendre foliation is consistent with Pang's definition. But note that the condition $G \equiv 0$ is not a sufficient condition for a non-degenerate Legendre foliation on a contact metric manifold to be semi-Riemannian we also need the condition that $g|_L = \frac{1}{4}\Pi$.

For a semi-Riemannian manifold $(\tilde{M}, g)$, let F be the norm induced by the semi-Riemannian metric g on $\tilde{M}$. We will call the unit cotangent bundle given by this F $S_g^*\tilde{M}$ and denote by $\mathcal{F}_g$ the semi-Riemannian Legendre foliation given by the projection

$$p : S_g^*\tilde{M} \to \tilde{M}.$$

Let $\mathcal{F}_g$ be the Legendre foliation on the unit cotangent bundle $S_g^*\tilde{M}$ obtained from the semi-Riemannian manifold $(\tilde{M}, g)$. Then F is the norm induced by g on $\tilde{M}$, so

(a) $$F = \left(g_{ij}(x)y_i y_j\right)^{\frac{1}{2}}$$

where $g(x) = g_{ij}(x)dx_i \otimes dx_j$ is a function of x only.

Thus by 2.31 we have

(b) $$\begin{aligned} \Pi &= F_{ij}dy_i \otimes dy_j && \text{by 2.31} \\ &= \frac{1}{2}(F^2)_{ij}dy_i \otimes dy_j \\ &= (g_{ij}(x))dy_i \otimes dy_j && \text{by (a).} \end{aligned}$$

and by 4.44 we have

$$\begin{aligned}\overline{\Pi} &= (F_iF_j + FF_{ij})^{-1}dx_i \otimes dx_j)|_{H \cap L'} \\ &= 2((F^2)_{ij})^{-1}dx_i \otimes dx_j)|_{H \cap L'} \\ &= (g^{ij}(x)dx_i \otimes dx_j)|_{H \cap L'} \qquad \text{by (a),}\end{aligned}$$

where $(g_{ij}(x))^{-1} = g^{ij}(x)$.

Let (ϕ, ξ, η, g) be a contact metric structure on $S_g^*\tilde{M}$ with

$$\begin{aligned}g|_L &= \frac{1}{4}\Pi \\ &= \frac{1}{4}g_{ij}(x)dy_i \otimes dy_j \qquad \text{by (b).}\end{aligned}$$

Then $\mathcal{F}_g$ on the contact metric manifold $(S_g^*\tilde{M}, \phi, \xi, \eta, g)$ is a semi-Riemannian Legendre foliation and by theorem 4.36 we have

$$\begin{aligned}g|_{L^\perp} &= 2((F^2)_{ij})^{-1}dx_i \otimes dx_j \\ &= g^{ij}(x)dx_i \otimes dx_j \qquad \text{by (a).}\end{aligned}$$

This implies that when g is a Riemannian metric the projection

$$p : S_g^*\tilde{M} \to (\tilde{M}, g^{-1})$$

is a *Riemannian submersion*, (Reinhart [25]), as the tangent map at each point $m \in S_g^*\tilde{M}$

$$p^* : T_m^*S_g^*\tilde{M} \to T_m^*(\tilde{M}, g^{-1})$$

is onto and induces an isometry between $L_m^\perp$ and $T_m^*(\tilde{M}, g^{-1})$. Since any Riemannian Legendre foliation is locally equivalent to one of the form $\mathcal{F}_g$ this shows that any Riemannian Legendre foliation is locally defined by a Riemannian submersion. This

is consistent with Reinhart's [25] definition of a Riemannian foliation as a foliation that is locally defined by Riemannian submersions.

In the following example we give a Legendre foliation on a unit cotangent bundle of R^3 and then introduce 4 different bundle-like contact metric structures. Then we examine the properties of the resultant semi-Riemannian Legendre foliations on the different contact metric manifolds.

Example 6.1: Semi-Riemannian Legendre foliation on a unit cotangent bundle

Let

$$M = R^3$$
$$= \{(x_1, x_2, x_3) \mid x_i \in R\}$$

and the function $F: T^*M \to R$ be defined by

$$F(x, y) = \left((y_1)^2 + (y_2)^2 - (y_3)^2 \right)^{\frac{1}{2}}.$$

So F is the norm of $\tilde{g}$ the semi-Riemannian metric on M defined by

$$\tilde{g} = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 - dx_3 \otimes dx_3.$$

Then

$$S_F^*M = F^{-1}(1)$$
$$= \{(x_1, x_2, x_3, y_1, y_2, y_3) \mid (y_1)^2 + (y_2)^2 - (y_3)^2 = 1\}.$$

Thus by 2.26 (i) and (ii) we have

$$\eta = y_1 dx_1 + y_2 dx_2 + y_3 dx_3$$

$$\xi = y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2} - y_3\frac{\partial}{\partial x_3}.$$

Let $\mathcal{F}_F$ be the Legendre foliation given by the fibres of the projection

$$p: S_F^*M \to M.$$

By 2.31 we have

$$\begin{aligned}\Pi &= \frac{1}{2}(F^2)_{ij}dy_i \otimes dy_j \\ &= dy_1 \otimes dy_1 + dy_2 \otimes dy_2 - dy_3 \otimes dy_3,\end{aligned}$$

and by theorem 4.44

$$\begin{aligned}\overline{\Pi} &= 2((F^2)_{ij})^{-1}dx_i \otimes dx_j \\ &= dx_1 \otimes dx_1 + dx_2 \otimes dx_2 - dx_3 \otimes dx_3.\end{aligned}$$

Π is non-degenerate so $\mathcal{F}_F$ is a non-degenerate Legendre foliation. As F is the norm of a semi-Riemannian metric on M, $\mathcal{F}_F$ on a contact metric manifold $(S_F^*M, \phi, \xi, \eta, g)$, with $g|_L = \frac{1}{4}\Pi$, is a semi-Riemannian Legendre foliation.

The tangent bundle, L, of $\mathcal{F}_F$ is given by

$$L = \text{span}\{X_1, X_2\}$$

where

$$X_1 = 2\mu\left(y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2}\right)$$

$$X_2 = 2\mu\left(y_1y_3\frac{\partial}{\partial y_1} + y_2y_3\frac{\partial}{\partial y_2} + \left((y_1)^2 + (y_2)^2\right)\frac{\partial}{\partial y_3}\right)$$

and

$$\mu = \left((y_1)^2 + (y_2)^2\right)^{-\frac{1}{2}} = \left(1 + (y_3)^2\right)^{-\frac{1}{2}}.$$

Thus

$$\theta_c^1 = \mu(-y_2dx_1 + y_1dx_2)$$

$$\theta_c^2 = \mu(y_1y_3dx_1 + y_2y_3dx_2 + ((y_1)^2 + (y_2)^2)dx_3).$$

As $(y_1)^2 + (y_2)^2 \neq 0$ everywhere on S_F^*M, X_1 and X_2 are well defined, global vector fields on S_F^*M. Furthermore $\{X_1, X_2\}$ is an orthonormal frame of L with respect to $\frac{1}{4}\Pi$.

In (a) to (d) below we give four different contact metric structures for (S_F^*M, η), each from the canonical family of contact metric structures, and investigate the semi-Riemannian Legendre foliations defined by $L = \text{span}\{X_1, X_2\}$.

(a) We introduce a subbundle Q^c of TS_F^*M and a contact metric structure (ϕ^c, ξ, η, g^c) with $g^c|_L = \frac{1}{4}\Pi$, so that L defines a semi-Riemannian Legendre foliation $\mathcal{F}_F^c$ on $(S_F^*M, \phi^c, \xi, \eta, g^c)$. This contact metric structure is shown to be the canonical one.

Let the subbundle $Q^c \subset TS_F^*M$ be given by

$$Q^c = \text{span}\{Y_1^c, Y_2^c\}$$

where

$$Y_1^c = \mu\left(- y_2 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial x_2}\right)$$

$$Y_2^c = \mu\left(- y_1 y_3 \frac{\partial}{\partial x_1} - y_2 y_3 \frac{\partial}{\partial x_2} + \left((y_1)^2 + (y_2)^2\right)\frac{\partial}{\partial x_3}\right).$$

Then $\{Y_1^c, Y_2^c\}$ is an orthonormal frame of Q^c with respect to the metric g^c defined by

$$g^c|_L = \frac{1}{4}\Pi, \quad g^c|_{Q^c} = \bar{\Pi}|_{Q^c}, \quad g^c|_E = \eta \otimes \eta \quad \text{and} \quad g^c = 0 \text{ otherwise.}$$

Let ϕ^c be the tensor defined by

$$\phi^c X_\alpha = - Y_\alpha^c, \qquad \phi^c Y_\alpha^c = X_\alpha, \qquad \phi^c \xi = 0 \qquad \text{for } \alpha = 1, 2.$$

(ϕ^c, ξ, η, g^c) is a contact metric structure on the contact manifold (S_F^*M, η) and $\{X_1, X_2, Y_1^c, Y_2^c, \xi\}$ is an orthonormal frame for TS_F^*M with respect to g^c.

Calculating the Lie brackets we get

$$[X_1, X_2] = 2\mu y_3 X_1 \qquad [Y_1^c, Y_2^c] = 0$$
$$[X_1, \xi] = - 2Y_1^c \qquad [X_2, \xi] = - 2\, Y_2^c$$
$$[Y_1^c, \xi] = 0 \qquad [\, Y_2^c, \xi] = 0$$
$$[X_1, \; Y_1^c] = 2\mu y_3 Y_2^c + 2\xi \qquad [X_1, \; Y_2^c] = 2\mu y_3 Y_1^c$$
$$[X_2, \; Y_1^c] = 0 \qquad [X_2, \; Y_2^c] = - 2\xi.$$

It can be seen from the Lie brackets that $\overline{\mathcal{F}}_F^c$, the conjugate Legendre foliation of $\mathcal{F}_F^c$ exists and is flat.It can also be shown that $\mathcal{F}_F^c$ and $\overline{\mathcal{F}}_F^c$ are both totally geodesic.

By calculation we have for $\alpha = 1, 2$

$$h^c X_\alpha = X_\alpha, \qquad \text{and} \qquad h^c Y_\alpha^c = - Y_\alpha^c.$$

Hence by theorem 4.15 (ϕ^c, ξ, η, g^c) is the canonical contact metric structure for $\mathcal{F}_F^c$.

(b) We now select a different subbundle $Q' \subset TS_F^*M$ and introduce another contact metric structure (ϕ', ξ, η, g') with $g'|_L = \frac{1}{4}\prod$, so that L defines a semi-Riemannian Legendre foliation $\mathcal{F}_F'$ on $(S_F^*M, \phi', \xi, \eta, g')$.

Let the subbundle $Q' \subset TS_F^*M$ be given by

$$Q' = \text{span}\{Y_1', Y_2'\}$$

where

$$\begin{aligned} Y_1' &= \mu\left(- y_2 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial x_2} + 2y_2 \frac{\partial}{\partial y_1} - 2y_1 \frac{\partial}{\partial y_2} \right) \\ &= X_1 + Y_1^c \end{aligned}$$

$$\begin{aligned} Y_2' &= \mu\left(- y_1 y_3 \frac{\partial}{\partial x_1} - y_2 y_3 \frac{\partial}{\partial x_2} + \left((y_1)^2 + (y_2)^2 \right) \frac{\partial}{\partial x_3} \right. \\ &\qquad \left. + 2y_1 y_3 \frac{\partial}{\partial y_1} + 2y_2 y_3 \frac{\partial}{\partial y_2} + 2\left((y_1)^2 + (y_2)^2 \right) \frac{\partial}{\partial y_3} \right) \\ &= X_2 + Y_2^c. \end{aligned}$$

Then $\{Y_1', Y_2'\}$ is an orthonormal frame of Q' with respect to the metric g' defined by

$$g'|_L = \frac{1}{4}\Pi, \quad g'|_{Q'} = \overline{\Pi}|_{Q'}, \quad g'|_E = \eta \otimes \eta \quad \text{and} \quad g' = 0 \text{ otherwise.}$$

Let ϕ' be the tensor defined by

$$\phi' X_\alpha = - Y_\alpha' \qquad \phi' Y_\alpha' = X_\alpha, \qquad \phi'\xi = 0 \qquad \text{for } \alpha = 1, 2.$$

Then (ϕ', ξ, η, g') is a contact metric structure on the contact manifold (S_F^*M, η) and $\{X_1, X_2, Y_1', Y_2', \xi\}$ is an orthonormal frame for TS_F^*M with respect to g'.

Calculating the Lie brackets we get

$$\begin{aligned}
&[X_1, X_2] = 2\mu y_3 X_1 && [Y_1', Y_2'] = 2\mu y_3 Y_1' \\
&[X_1, \xi] = 2X_1 - 2Y_1' && [X_2, \xi] = 2X_2 - 2Y_2' \\
&[Y_1', \xi] = 2X_1 - 2Y_1' && [Y_2', \xi] = 2X_2 - 2Y_2' \\
&[X_1, Y_1'] = - 2\mu y_3 X_2 + 2\mu y_3 Y_2' + 2\xi && [X_1, Y_2'] = 2\mu y_3 Y_1' \\
&[X_2, Y_1',] = - 2\mu y_3 X_1 && [X_2, Y_2'] = - 2\xi.
\end{aligned}$$

It can be seen from the Lie brackets that $\overline{\mathcal{F}}_F'$, the conjugate Legendre foliation of $\mathcal{F}_F'$ exists and is non-degenerate. Furthermore for any $\alpha = 1, 2$ the projection

$$p|_L([Y_\alpha', \xi]) = - 2\phi' Y_\alpha' = 2X_\alpha.$$

Therefore (ϕ', ξ, η, g') belongs to the family of canonical contact metric structures for the conjugate Legendre foliation $\overline{\mathcal{F}}_F'$ which is defined by Q'. It can also be shown that $\overline{\mathcal{F}}_F'$ is a semi-Riemannian Legendre foliation.

By calculation we get

$$\nabla_{X_1} X_1 = 2\mu y_3 X_2 - 2\xi \qquad \nabla_{X_2} X_2 = 2\xi$$

$$\nabla_{Y_1'} Y_1' = 2\mu y_3 Y_2' + 2\xi \qquad \nabla_{Y_2'} Y_2' = -2\xi$$

also for any α, β

$$p_L(\nabla_{X_\alpha} Y_\beta') = 0 \qquad p_L(\nabla_{X_\alpha} \xi) = 2X_\alpha.$$

Thus

$$\tau = -4\xi \qquad \bar{\tau} = 4\xi,$$

therefore $\mathcal{F}_F'$ and $\bar{\mathcal{F}}_F'$ are not harmonic Legendre foliations.

By calculation we have for $\alpha = 1, 2$

$$h'X_\alpha = 2Y_\alpha', \qquad \text{and} \qquad h'Y_\alpha' = 2X_\alpha.$$

Hence by theorem 4.15 (ϕ', ξ, η, g') is not the canonical contact metric structure for either $\mathcal{F}_F'$ or $\bar{\mathcal{F}}_F'$.

(c) We now select another subbundle $Q'' \subset TS_F^*M$ and introduce a different contact metric structure (ϕ'', ξ, η, g'') with $g''|_L = \frac{1}{4}\Pi$, so that L defines a semi-Riemannian Legendre foliation $\mathcal{F}_F''$ on $(S_F^*M, \phi'', \xi, \eta, g'')$.

Let the subbundle $Q'' \subset TS_F^*M$ be given by

$$Q'' = \text{span}\{Y_1'', Y_2''\}$$

where

$$Y_1'' = \mu\Big(- y_2\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial x_2} + 2\big(y_2 + y_1y_3\big)\frac{\partial}{\partial y_1} + 2\big(- y_1 + y_2y_3\big)\frac{\partial}{\partial y_2} + 2\Big(\big(y_1\big)^2 + \big(y_2\big)^2\Big)\frac{\partial}{\partial y_3}\Big)$$
$$= X_1 + X_2 + Y_1^c.$$

$$Y_2'' = \mu\Big(- y_1y_3\frac{\partial}{\partial x_1} - y_2y_3\frac{\partial}{\partial x_2} + \Big(\big(y_1\big)^2 + \big(y_2\big)^2\Big)\frac{\partial}{\partial x_3} - 2y_2\frac{\partial}{\partial y_1} + 2y_1\frac{\partial}{\partial y_2}\Big)$$
$$= - X_1 + Y_2^c.$$

Then $\{Y_1'', Y_2''\}$ is an orthonormal frame of Q" with respect to the metric g" defined by

$$g''|_L = \frac{1}{4}\Pi, \quad g''|_{Q''} = \overline{\Pi}|_{Q''}, \quad g''|_E = \eta \otimes \eta \quad \text{and} \quad g'' = 0 \text{ otherwise.}$$

Let ϕ'' be the tensor defined by

$$\phi''X_\alpha = - Y_\alpha'', \qquad \phi'' Y_\alpha'' = X_\alpha, \qquad \phi''\xi = 0 \qquad \text{for } \alpha = 1, 2.$$

Then (ϕ'', ξ, η, g'') is a contact metric structure on the contact manifold (S_F^*M, η) and $\{X_1, X_2, Y_1'', Y_2'', \xi\}$ is an orthonormal frame for TS_F^*M with respect to g".

Calculating the Lie brackets we get

$$[X_1, X_2] = 2\mu y_3 X_1$$
$$[Y_1'', Y_2''] = 2\mu y_3(X_1 - X_2 + Y_1'' + Y_2'')$$
$$[X_1, \xi] = 2X_1 + 2X_2 - 2Y_1''$$
$$[X_2, \xi] = - 2X_1 - 2Y_2''$$
$$[Y_1'', \xi] = 2X_2 - 2Y_1'' - 2Y_2''$$
$$[Y_2'', \xi] = - 2X_1 - 2X_2 + 2Y_2''$$
$$[X_1, Y_1''] = 2\mu y_3(X_1 + Y_2'') + 2\xi$$
$$[X_1, Y_2''] = 2\mu y_3(- X_1 - X_2 + Y_1'')$$
$$[X_2, Y_1''] = - 2\mu y_3 X_1$$
$$[X_2, Y_2''] = 2\mu y_3 X_1 - 2\xi.$$

It can be seen from the Lie brackets that $\overline{\mathcal{F}}_F''$, the conjugate Legendre foliation of $\mathcal{F}_F''$ does not exist.

By calculation we have

$$\nabla_{X_1} X_1 = 2\mu y_3(X_2 - Y_1'' - Y_2'') - 2\xi \qquad \nabla_{X_2} X_2 = 0$$

$$\nabla_{Y_1''} Y_1'' = 2\mu y_3 Y_2'' + 2\xi \qquad \nabla_{Y_2''} Y_2'' = - 2\mu y_3 Y_1''$$

Thus $\mathcal{F}_F''$ is not a harmonic Legendre foliation.

Furthermore by calculation we have

$$h''X_1 = X_1 - X_2 + 2Y_1'' + 2Y_2''$$
$$h''X_2 = X_1 + 2X_2 - 2Y_1''$$
$$h''Y_1'' = 2X_1 + 2X_2 - Y_1'' + Y_2''$$
$$h''Y_2'' = - 2X_1 - Y_1'' + 2Y_2''.$$

Hence by theorem 4.15 (ϕ'', ξ, η, g'') is not the canonical contact metric structure.

(d) We select a another subbundle $Q \subset TS_F^*M$ and introduce a different contact metric structure (ϕ, ξ, η, g) with $g|_L = \frac{1}{4}\Pi$, so that L defines a semi-Riemannian Legendre foliation $\mathcal{F}_F$ on $(S_F^*M, \phi'', \xi, \eta, g'')$.

Let the subbundle $Q \subset TS_F^*M$ be given by

$$Q = \text{span}\{Y_1, Y_2\}$$

where

$$\begin{aligned} Y_1 &= \mu\Big(-y_2\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial x_2} + 2\big(y_2 + y_1y_3\big)\frac{\partial}{\partial y_1} \\ &\qquad + 2\big(-y_1 + y_2y_3\big)\frac{\partial}{\partial y_2} - 2\Big(\big(y_1\big)^2 + \big(y_2\big)^2\Big)\frac{\partial}{\partial y_3}\Big) \\ &= X_1 + X_2 + Y_1^c. \end{aligned}$$

$$\begin{aligned} Y_2 &= \mu\Big(-y_1y_3\frac{\partial}{\partial x_1} - y_2y_3\frac{\partial}{\partial x_2} + \Big(\big(y_1\big)^2 + \big(y_2\big)^2\Big)\frac{\partial}{\partial x_3} \\ &\qquad - 2\big(y_2 + y_1y_3\big)\frac{\partial}{\partial y_1} - 2\big(-y_1 + y_2y_3\big)\frac{\partial}{\partial y_2} \\ &\qquad - 2\Big(\big(y_1\big)^2 + \big(y_2\big)^2\Big)\frac{\partial}{\partial y_3}\Big) \\ &= -X_1 - X_2 + Y_2^c. \end{aligned}$$

Then $\{Y_1, Y_2\}$ is an orthonormal frame of Q with respect to the metric g defined by

$$g|_L = \frac{1}{4}\Pi, \quad g|_Q = \overline{\Pi}|_Q, \quad g|_E = \eta \otimes \eta \quad \text{and} \quad g = 0 \text{ otherwise.}$$

Let ϕ be the tensor defined by

$$\phi X_\alpha = - Y_\alpha, \qquad \phi Y_\alpha = X_\alpha, \qquad \phi \xi = 0 \quad \text{for } \alpha = 1, 2.$$

Then (ϕ, ξ, η, g) is a contact metric structure on the contact manifold (S_F^*M, η) and $\{X_1, X_2, Y_1, Y_2, \xi\}$ is an orthonormal frame for TS_F^*M with respect to g.

Calculating the Lie brackets we get

$$[X_1, X_2] = 2\mu y_3 X_1$$
$$[Y_1, Y_2] = 2\mu y_3 (Y_1 + Y_2)$$
$$[X_1, \xi] = 2X_1 + 2X_2 - 2Y_1$$
$$[X_2, \xi] = - 2X_1 - 2X_2 - 2Y_2$$
$$[Y_1, \xi] = - 2Y_1 - 2Y_2$$
$$[Y_2, \xi] = 2Y_1 + 2Y_2$$
$$[X_1, Y_1] = 2\mu y_3 (2X_1 + X_2 + Y_2) + 2\xi$$
$$[X_2, Y_1] = - 2\mu y_3 X_1$$
$$[X_1, Y_2] = 2\mu y_3 (- 2X_1 - X_2 + Y_1)$$
$$[X_2, Y_2] = 2\mu y_3 X_1 - 2\xi.$$

It can be seen from the Lie brackets that $\overline{\mathcal{F}}_F$, the conjugate Legendre foliation of $\mathcal{F}_F$ exists and is flat.

By calculation we have

$$\nabla_{X_1} X_1 = 2\mu y_3 (X_2 - 2Y_1 - 2Y_2) - 2\xi \qquad \nabla_{X_2} X_2 = - 2\xi$$

$$\nabla_{Y_1} Y_1 = 2\mu y_3 Y_2 + 2\xi \qquad \nabla_{Y_2} Y_2 = - 2\mu y_3 Y_1 + 2\xi$$

which shows that $\mathcal{F}_F$ is not a harmonic Legendre foliation and $\overline{\mathcal{F}}_F$ is a harmonic but not a totally geodesic Legendre foliation.

Furthermore we have

$$\nabla_{X_1} Y_2 = 2\mu y_3(-2X_1 - X_2 + Y_1) \qquad \nabla_{Y_2} X_1 = 0$$

$$\nabla_{X_2} Y_1 = -2\mu y_3 X_1 \qquad \nabla_{Y_1} X_2 = 0.$$

Thus

$$A_{Y_2} X_1 = 2\mu y_3(2X_1 + X_2) \qquad \overline{A}_{X_2} Y_1 = 0$$

$$A_{Y_1} X_2 = 2\mu y_3 X_1 \qquad \overline{A}_{X_1} Y_2 = 0.$$

Hence

$$A_{Y_2} X_1 \neq A_{Y_1} X_2.$$

But as the conjugate exists by corollary 5.6 we have for any α, β that

$$\overline{A}_{X_\alpha} Y_\beta = \overline{A}_{X_\beta} Y_\alpha.$$

By calculation we have

$$hX_1 = X_1 + 2Y_1 + 2Y_2 \qquad hX_2 = X_2 - 2Y_1 - 2Y_2$$
$$hY_1 = 2X_1 + 2X_2 - Y_1 \qquad hY_2 = -2X_1 - 2X_2 + Y_2.$$

Hence by theorem 4.15 (ϕ, ξ, η, g) is not the canonical contact metric structure.

From the Lie brackets

$$N^1(X_1, X_2) = -[X_1, X_2] + [\phi X_1, \phi X_2] - \phi[\phi X_1, X_2] - \phi[X_1, \phi X_2]$$
$$= 4\mu y_3(Y_1 + Y_2).$$

Thus (ϕ, ξ, η, g) satisfies theorem 5.3 but not theorem 5.8. As (ϕ, ξ, η, g) is not the canonical contact metric structure this shows that theorem 5.3 can only be extended to theorem 5.8 when (ϕ, ξ, η, g) with $g|_L = \frac{1}{4}\Pi$, is the canonical contact metric structure.

6 (b). THE PROPERTIES OF A SEMI-RIEMANNIAN LEGENDRE FOLIATION

We will now investigate the properties of semi-Riemannian Legendre foliations.

Lemma 6.6

A Legendre foliation $\mathcal{F}$ on a 3-dimensional contact metric manifold (M, ϕ, ξ, η, g) is a semi-Riemannian Legendre foliation if and only if

(i) $\mathcal{F}$ is non-degenerate.

(ii) $g|_L = \frac{1}{4}\Pi$

(iii) $g([X, \phi X], \phi X) = 0$

where $\{X\}$ is a local orthonormal frame of L with respect to g.

Proof

(i): For any Y', Y'' $\in \Gamma Q$ and any X' $\in \Gamma L$ there exist functions f, h and k such that Y' = fϕX, Y'' = hϕX and X' = kX, for this X', Y', Y'' we have

$$\begin{aligned}\mathcal{L}_{X'}g(Y',\ Y'') &= \mathcal{L}_{X'}g(f\phi X,\ h\phi X)\\ &= X'g(f\phi X,\ h\phi X) - g([X',\ f\phi X],\ h\phi X)\\ &\qquad - g(f\phi X,\ [X',\ h\phi X])\\ &= X'(fh) - (X'f)h - f(X'h) - 2fh(g([X',\ \phi X],\ \phi X))\\ &= -\,2fhk(g([X,\ \phi X],\ \phi X)).\end{aligned}$$

Then the theorem follows from theorem 6.2 and lemma 6.3 (i). ¤

The bundle-like contact metric structure of a semi-Riemannian Legendre foliation allows us to say the following about the conjugate Legendre foliation.

Theorem 6.7

Let $\mathcal{F}$ be a semi-Riemannian Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g),

i) Then $\overline{\mathcal{F}}$ the conjugate Legendre foliation of $\mathcal{F}$ exists if and only if the projection

$$p|_L(\nabla_Y Y') = 0 \qquad \forall\ Y,\ Y' \in \Gamma Q.$$

ii) Suppose the conjugate Legendre foliation, $\overline{\mathcal{F}}$, of $\mathcal{F}$ exists. Then $\overline{\mathcal{F}}$ is totally geodesic if and only if (ϕ, ξ, η, g) is the canonical contact metric structure.

Proof

As $\mathcal{F}$ is a semi-Riemannian Legendre foliation by theorem 6.2 (ii) we have $g|_L = \frac{1}{4}\Pi$.

i) For any $X \in \Gamma L$ and any $Y, Y' \in \Gamma Q$ as $\mathcal{F}$ is a semi-Riemannian Legendre foliation we have

$$\begin{aligned} 0 &= \mathcal{L}_X g(Y, Y') \\ &= g(\nabla_Y X, Y') + g(\nabla_{Y'} X, Y) \\ &= g([Y', Y], X) - 2g(\nabla_{Y'} Y, X), \end{aligned}$$

which implies that

$$g([Y', Y], X) = 2g(\nabla_{Y'} Y, X) \qquad \forall\, X \in \Gamma L, \forall\, Y, Y' \in \Gamma Q.$$

Part (i) of the theorem follows from this.

ii) If the conjugate Legendre foliation of $\mathcal{F}$ exists then from part (i) we have that the projection

$$p|_L(\nabla_Y Y') = 0 \qquad \forall\, Y, Y' \in \Gamma Q.$$

Also for any $Y, Y' \in \Gamma Q$ we have

$$\begin{aligned} g(\nabla_{Y'} Y, \xi) &= - g(Y, \nabla_{Y'} \xi) \\ &= g(Y, \phi h Y' + \phi Y') && \text{by 1.35} \\ &= g(Y, \phi h Y') && \text{by 2.1,} \end{aligned}$$

which implies that the projection

$$p|_E(\nabla_Y Y') = 0 \qquad \forall\, Y, Y' \in \Gamma Q,$$

if and only if h: Q $\rightarrow$ Q. From theorem 4.15 this is so if and only if (ϕ, ξ, η, g) is the canonical contact metric structure. ¤

These results are illustrated in the following examples.

Example 4.1: (a) $\mathcal{F}$ is a Riemannian Legendre foliation by lemma 6.6, as the Lie bracket $[X, Y] = 2\xi$, and $\overline{\mathcal{F}}$ the conjugate of $\mathcal{F}$ exists by theorem 5.1. $\overline{\mathcal{F}}$ can be shown to be totally geodesic which is consistent with theorem 6.7 (ii) as (ϕ^c, ξ, η, g^c) is the canonical contact metric structure.

(b) $\mathcal{F}'$ is a Riemannian Legendre foliation by lemma 6.6, as the Lie bracket $[X, Y'] = 2\xi$, and $\overline{\mathcal{F}}'$ the conjugate of $\mathcal{F}'$ exists by theorem 5.1. $\overline{\mathcal{F}}'$ is not totally geodesic as

$$\nabla_{Y'}Y' = 2\xi.$$

This is consistent with theorem 6.7 (ii) as (ϕ', ξ, η, g') is not the canonical contact metric structure.

Example 6.1: (a) $\overline{\mathcal{F}}_F^c$, the conjugate Legendre foliation of $\mathcal{F}_F^c$ exists and is totally geodesic. Also (ϕ^c, ξ, η, g^c) is the canonical contact metric structure.

(b) $\overline{\mathcal{F}}_F'$ the conjugate Legendre foliation of $\mathcal{F}_F'$ exists and is a semi-Riemannian Legendre foliation. $\mathcal{F}_F'$ and $\overline{\mathcal{F}}_F'$ are both harmonic but not totally geodesic and (ϕ', ξ, η, g') is not the canonical contact metric structure for either $\mathcal{F}_F'$ or $\overline{\mathcal{F}}_F'$.

(d) $\overline{\mathcal{F}}_F$, the conjugate Legendre foliation of $\mathcal{F}_F$ exists and is not totally geodesic. Also (ϕ', ξ, η, g') is not the canonical contact metric structure.

Corollary 6.8

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on the canonical contact metric manifold $(M, \phi^c, \xi, \eta, g^c)$ such that the conjugate exists. Then $\mathcal{F}$ is a semi-Riemannian Legendre foliation if and only if $\overline{\mathcal{F}}$ is totally geodesic.

Proof

Suppose $\mathcal{F}$ is a semi-Riemannian Legendre foliation then as (ϕ^c, ξ, η, g^c) is the canonical contact metric structure $\overline{\mathcal{F}}$ is totally geodesic by theorem 6.7.

Suppose $\overline{\mathcal{F}}$ is totally geodesic then by theorem 4.25 for any $X, X', X'' \in \Gamma L$, we have

$$G(X, X', X'') = 2g(\nabla_{\phi X'} \phi X'' + \nabla_{\phi X''} \phi X', X)$$
$$= 0.$$

Hence $\mathcal{F}$ is semi-Riemannian by theorem 6.5. ¤

Corollary 6.8 is illustrated by the following example from chapter 2.

Example 2.1: (a) $\mathcal{F}^c_F$ is a non-degenerate Legendre foliation on the canonical contact metric manifold $(S^*_F M, \phi^c, \xi, \eta, g^c)$ and $\overline{\mathcal{F}}_F$ the conjugate Legendre foliation of $\mathcal{F}^c_F$ exists. But $\overline{\mathcal{F}}^c_F$ is not totally geodesic as

$$\nabla_{Y_1} Y_1 = -3\sqrt{2}X_1.$$

This is consistent with corollary 6.8 as $\mathcal{F}_F^c$ is not a semi-Riemannian Legendre foliation .

The bundle-like contact metric structure of a semi-Riemannian Legendre foliation allows us to say the following about the partial connection on L.

Lemma 6.9
If $\mathcal{F}$ is a semi-Riemannian Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g), then the partial connection on L

$$\nabla^L_X X' = p_L(\phi[\phi X', X]) \qquad \forall\ X, X' \in \Gamma L.$$

Proof
Let $\{X_\alpha\}$ be a local orthonormal frame of L with respect to g and ∇ be the Levi-Civita connection of g. $\mathcal{F}$ is a semi-Riemannian Legendre foliation so (ϕ, ξ, η, g) is a bundle-like contact metric structure hence for any α, β we have

$$\begin{aligned} g(\nabla^L_{X_\alpha} X_\beta, X_\gamma) &= -\, g([X_\alpha, \phi X_\gamma], \phi X_\beta) && \text{by 2.7 and 2.23} \\ &= -\, g(\nabla_{X_\alpha} \phi X_\gamma - \nabla_{\phi X_\gamma} X_\alpha, \phi X_\beta) \\ &= -\, g(\nabla_{X_\alpha} \phi X_\gamma, \phi X_\beta) - g(\nabla_{\phi X_\beta} X_\alpha, \phi X_\gamma) \\ & && \text{by lemma 6.3 (ii)} \\ &= g([X_\alpha, \phi X_\beta], \phi X_\gamma) \\ &= g(\phi[\phi X_\beta, X_\alpha], X_\gamma) && \text{by 1.10.} \end{aligned}$$

This implies that

$$\nabla^L_X X' = p_L(\phi[\phi X', X]) \qquad \forall\ X, X' \in \Gamma L.$$

¤

Therefore if (ϕ, ξ, η, g) is any bundle-like contact metric structure on (M, η) and (ϕ^c, ξ, η, g^c) the corresponding canonical contact metric structure we have

$$p_L(\phi[\phi X', X]) = \nabla^L_X X' \\ = p_L(\phi^c[\phi^c X', X]).$$

A semi-Riemannian Legendre foliation is non-degenerate, hence for a semi-Riemannian Legendre foliation on a contact manifold of dimension ≥ 5 we have by theorem 2.25 that $R^L \neq 0$. Which implies that for some α, β the projection

$$p_L(\phi[\phi X_\beta, X_\alpha]) = \nabla^L_{X_\alpha} X_\beta \neq 0.$$

Hence for some α, β by 2.1 the projection

$$p_Q([X_\alpha, \phi X_\beta]) \neq 0.$$

Furthermore when the conjugate exists we get the following lemma.

Lemma 6.10

Let $\mathcal{F}$ be a semi-Riemannian Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) such that the conjugate Legendre foliation exists. Then the partial connection on L

$$\nabla^L_X X' = p_L(\nabla_X X') \qquad \forall\, X, X' \in \Gamma L$$

where ∇ is the Levi-Civita connection of g.

Proof

Let $\{X_\alpha\}$ be a local orthonormal frame of L with respect to g.

As the conjugate exists we have

$$\begin{aligned}
\nabla^L_{X_\alpha} X_\beta &= p_L(\phi[\phi X_\beta, X_\alpha]) && \text{by lemma 6.9} \\
&= p_L(\phi\nabla_{\phi X_\beta} X_\alpha - \phi\nabla_{X_\alpha}\phi X_\beta) \\
&= p_L(\nabla_{\phi X_\beta}\phi X_\alpha + \nabla_{X_\alpha} X_\beta) && \text{by lemma 5.5 (iii) and (v)} \\
&= p_L(\nabla_{X_\alpha} X_\beta) && \text{by theorem 6.7 (i).}
\end{aligned}$$

□

Thus by Lemma 6.9 we have

$$p_L(\nabla_X X') = p_L(\phi[\phi X', X]) \qquad \forall\ X, X' \in \Gamma L.$$

For a semi-Riemannian Legendre foliation on the contact metric manifold, (M, ϕ, ξ, η, g), with the property that the conjugate Legendre foliation exists, lemma 6.10 shows that the curvature of the Levi-Civita connection when restricted to the leaves, is the curvature of the partial connection ∇^L thus by 2.24

$$R(X, X')X'' = \prod(X', X'')X - \prod(X, X'')X' \qquad \forall\ X, X', X'' \in \Gamma L,$$

where R is the curvature tensor field of g.

Theorem 6.11

Let $\mathcal{F}$ be a semi-Riemannian Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) such that the conjugate Legendre foliation exists then the leaves of $\mathcal{F}$ have constant sectional curvature 4.

Proof

Let $\{X_\alpha\}$ be a local orthonormal frame of L with respect to g. Then for any α, β we have

$$K(X_\alpha, X_\beta) = \frac{g(R(X_\alpha, X_\beta)X_\beta, X_\alpha)}{g(X_\alpha, X_\alpha)g(X_\beta, X_\beta)}$$

$$= \frac{g(\Pi(X_\beta, X_\beta)X_\alpha - \Pi(X_\alpha, X_\beta)X_\beta, X_\alpha)}{g(X_\alpha, X_\alpha)g(X_\beta, X_\beta)}$$

$$= \frac{\Pi(X_\beta, X_\beta)g(X_\alpha, X_\alpha)}{g(X_\alpha, X_\alpha)g(X_\beta, X_\beta)}$$

$$= \frac{4g(X_\beta, X_\beta)g(X_\alpha, X_\alpha)}{g(X_\alpha, X_\alpha)g(X_\beta, X_\beta)} \qquad \text{as } g|_L = \frac{1}{4}\Pi \text{ by theorem 6.2 (ii)}$$

$$= 4. \qquad \square$$

This theorem can be extended to semi-Riemannian Legendre foliations where the conjugate does not exist as follows.

Corollary 6.12

Let $\mathcal{F}$ be a semi-Riemannian Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) defined by the subbundle $L \subset H$. If there exists a semi-Riemannian Legendre foliation $\mathcal{F}'$ on a contact metric manifold $(M, \phi', \xi, \eta, g')$ defined by the subbundle $L \subset H$, such that the conjugate Legendre foliation of $\mathcal{F}'$ exists, then the leaves of $\mathcal{F}$ have constant sectional curvature 4.

Proof

As $g|_L = g'|_L = \frac{1}{4}\Pi$ by 6.2 (ii) we have for any $X, X' \in \Gamma L$ the projections

$$p|_L(\nabla_X X') = p|_L(\nabla'_X X')$$

where ∇ and ∇' are the Levi-Civita connection of g and g' respectively.

Thus as the leaves of $\mathcal{F}'$ have constant sectional curvature 4, by theorem 6.11, so do the leaves of $\mathcal{F}$. $\square$

Proposition 5.23 of Pang states that if $G \equiv 0$ then $Q^{\alpha}_{\beta\gamma} = 0$, where $Q^{\alpha}_{\beta\gamma}$ is given by theorem 4.31. Hence for a semi-Riemannian Legendre foliation on the canonical contact metric manifold $(M, \phi^c, \xi, \eta, g^c)$ theorem 4.31 is reduced to,

Theorem 6.13

For a semi-Riemannian Legendre on the canonical contact metric manifold, $(M, \phi^c, \xi, \eta, g^c)$. The canonical coframes $\{\varepsilon^{\alpha}_c , \theta^{\alpha}_c, \eta\}$ and π^{α}_{β} satisfy the first structure equations

$$\begin{bmatrix} d\varepsilon^{\alpha}_c \\ d\theta^{\alpha}_c \\ d\eta \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -(\pi^t)^{\alpha}_{\beta} & 0 & 0 \\ 0 & \pi^{\alpha}_{\beta} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} K^{\alpha}_{\beta\gamma}\theta^{\beta}_c \wedge \theta^{\gamma}_c + S^{\alpha}_{\beta}\eta \wedge \theta^{\beta}_c \\ -2k_{\alpha\beta}\eta \wedge \varepsilon^{\beta}_c \\ 2\delta_{\beta\gamma}\theta^{\beta}_c \wedge \varepsilon^{\gamma}_c \end{bmatrix}$$

where S^{α}_{β} and $K^{\alpha}_{\beta\gamma} = -K^{\alpha}_{\gamma\beta}$ are functions chosen to satisfy these equations.

We will now give an example of a Riemannian Legendre foliation on a unit cotangent bundle, give it a canonical contact metric structure and calculate its first structure equations.

Example 6.2: Semi-Riemannian Legendre foliation on a unit cotangent bundle

Let

$$M = R^2$$
$$= \{(x_1, x_2) \mid x_1 > 0\}$$

and the function F: $T^*M \to R$ be defined by

$$F(x, y) = \frac{1}{x_1}\left((y_1)^2 + (y_2)^2\right)^{\frac{1}{2}}$$

Thus F is the norm of $\tilde{g}$ the semi-Riemannian metric on M defined by

$$\tilde{g} = \frac{1}{x_1^2} dx_1 \otimes dx_1 + \frac{1}{x_1^2} dx_2 \otimes dx_2.$$

Then

$$\begin{aligned} S_F^*M &= F^{-1}(1) \\ &= \{(x_1, x_2, y_1, y_2) \mid (y_1)^2 + (y_2)^2 = (x_1)^2\}. \end{aligned}$$

Therefore by 2.26 (i) and (ii) we have

$$\eta = y_1 dx_1 + y_2 dx_2$$

$$\xi = \frac{1}{x_1^2}\left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial y_1}\right)$$

Let $\mathcal{F}_F$ be the Legendre foliation given by the fibres of the projection

$$p: S_F^*M \to M.$$

By 2.31 we have

$$\begin{aligned} \Pi &= \frac{1}{2}(F^2)_{ij} dy_i \otimes dy_j \\ &= \frac{1}{(x_1)^2}(dy_1 \otimes dy_1 + dy_2 \otimes dy_2), \end{aligned}$$

and by theorem 4.44

$$\begin{aligned}\bar{\Pi} &= 2((F^2)_{ij})^{-1}dx_i \otimes dx_j \\ &= (x_1)^2(dx_1 \otimes dx_1 + dx_2 \otimes dx_2).\end{aligned}$$

Π is non-degenerate so $\mathcal{F}_F$ is a non-degenerate Legendre foliation. As F is the norm of a semi-Riemannian metric on M, the foliation $\mathcal{F}_F$ on the contact metric manifold $(S_F^*M, \phi, \xi, \eta, g)$, with $g|_L = \frac{1}{4}\Pi$, is a semi-Riemannian Legendre foliation.

The tangent bundle, L, of $\mathcal{F}_F$ is given by

$$L = \text{span}\{X\}$$

where

$$X = 2\left(y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2}\right)$$

As $(y_1)^2 + (y_2)^2 = (x_1)^2$ and $x_1 > 0$, X is a global vector field. Furthermore $\{X\}$ is an orthonormal frame of L with respect to $\frac{1}{4}\Pi$.

Thus by 2.6 (i) we have

$$\theta = -y_2dx_1 + y_1dx_2.$$

We now introduce a subbundle Q^c of TS_F^*M and a contact metric structure (ϕ^c, ξ, η, g^c) with $g^c|_L = \frac{1}{4}\Pi$, so that L defines a semi-Riemannian Legendre foliation $\mathcal{F}_F^c$ of $(S_F^*M, \phi^c, \xi, \eta, g^c)$. This contact metric structure is shown to be the canonical one.

Let the subbundle $Q^c \subset TS_F^*M$ be given by

$$Q^c = \text{span}\{Y\}$$

where

$$Y = \frac{1}{(x_1)^2}\left(- y_2\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial y_2}\right).$$

Then $\{Y\}$ is an orthonormal frame of Q^c with respect to the metric g^c defined by

$$g^c|_L = \frac{1}{4}\Pi, \quad g^c|_{Q^c} = \overline{\Pi}|_{Q^c}, \quad g^c|_E = \eta \otimes \eta \quad \text{and} \quad g^c = 0 \text{ otherwise.}$$

Let ϕ^c be the tensor defined by

$$\phi^c X = - Y, \qquad \phi^c Y = X, \qquad \phi^c \xi = 0.$$

(ϕ^c, ξ, η, g^c) is a contact metric structure on the contact manifold (S_F^*M, η) and $\{X, Y, \xi\}$ is an orthonormal frame for TS_F^*M with respect to g^c.

Furthermore by 2.6 (ii) we have

$$\varepsilon = \frac{1}{2(x_1)^2}(- x_1dx_2 + y_2dy_1 - y_1dy_2).$$

Calculating the Lie brackets we get

$$[X, \xi] = - 2Y$$
$$[Y, \xi] = \frac{1}{2(x_1)^4} X$$
$$[X, Y] = 2\xi$$

It can be seen from the Lie brackets that $\overline{\mathcal{F}}_F^c$, the conjugate Legendre foliation of $\mathcal{F}_F^c$ exists and is non-degenerate. Furthermore as the projection

$$p_L([Y, \xi]) = \frac{1}{2(x_1)^4} X \neq - 2\phi Y$$

(ϕ^c, ξ, η, g^c) does not belong to the canonical family of contact metric structures for the conjugate Legendre foliation $\overline{\mathcal{F}}_F^c$ defined by Q^c. It can also be shown that $\mathcal{F}_F^c$ and $\overline{\mathcal{F}}_F^c$ are both totally geodesic.

By calculation we have

$$h^c X = \frac{(4(x_1)^4 - 1)}{4(x_1)^4} X.$$

Hence by theorem 4.15 (ϕ^c, ξ, η, g^c) is the canonical contact metric structure of $\mathcal{F}_F^c$.

We will now calculate the first structure equations of $\mathcal{F}_F^c$. By calculation using $x_1 dx_1 = y_1 dy_1 + y_2 dy_2$ it can be shown that

$$\begin{aligned} d\mathcal{E} &= \frac{1}{2(x_1)^3} (x_1 dx_1 \wedge dx_2 - 2y_2 dx_1 \wedge dy_1 + 2y_1 dx_1 \wedge dy_2 \\ &\qquad - 2x_1 dy_1 \wedge dy_2) \\ &= \frac{1}{2(x_1)^4} (\eta \wedge \theta). \end{aligned}$$

$$\begin{aligned} d\theta &= dx_1 \wedge dy_2 - dx_2 \wedge dy_1 \\ &= - 2\eta \wedge \mathcal{E}. \end{aligned}$$

$$\begin{aligned} d\eta &= - dx_1 \wedge dy_1 - dx_2 \wedge dy_2 \\ &= - 2\theta \wedge \mathcal{E}. \end{aligned}$$

Therefore we have

$$\begin{bmatrix} d\varepsilon \\ d\theta \\ d\eta \end{bmatrix} = \begin{bmatrix} \frac{1}{2(x_1)^4}\eta \wedge \theta \\ -2\eta \wedge \varepsilon \\ 2\theta \wedge \varepsilon \end{bmatrix}$$

and it follows from theorem 6.13 that $\pi = K = 0$ and $S = \frac{1}{2(x_1)^4}$.

(M, g) is the Poincaré half-plane, which has constant curvature of -1. This is a counter example to corollary 5.40 of Pang [2] as $S = \frac{1}{2(x_1)^4} \neq -1$.

6 (c). A FAMILY OF SEMI-RIEMANNIAN LEGENDRE FOLIATIONS

This section is an adaptation of the work of Blair [5] to Legendre foliations.

If we have a contact metric manifold which satisfies proposition 5.14 with $\varpi = 0$, then ± 1 are the non-zero eigenvalues of h and

$$R(W, W')\xi = 0 \qquad \forall\ W, W' \in \Gamma TM.$$

In which case proposition 5.14 can be extended to

Theorem 6.14

Let (M, ϕ, ξ, η, g) be a contact metric manifold such that

$$R(W, W')\xi = 0 \qquad \forall\ W, W' \in \Gamma TM$$

then D(1) defines a semi-Riemannian Legendre foliation and D(-1) defines its conjugate which is flat and totally geodesic. Furthermore (ϕ, ξ, η, g) is the canonical contact metric structure of the Legendre foliation defined by D(1).

Proof

From proposition 5.14 D(1) and D(-1) define a Legendre foliation and its conjugate. Also D(-1) defines a flat Legendre foliation by theorem 3.5. Let $\mathcal{F}$ be the Legendre foliation defined by D(1). Then $\mathcal{F}$ has tangent bundle $L = D(1)$ and transverse bundle $Q \oplus E = D(-1) \oplus E$. The conjugate Legendre foliation $\overline{\mathcal{F}}$ of $\mathcal{F}$ is the flat Legendre foliation defined by D(-1). $\overline{\mathcal{F}}$ has transverse bundle $\overline{Q} \oplus E = L \oplus E = D(1) \oplus E$ and tangent bundle $\overline{L} = Q = D(-1)$.

We now show that (ϕ, ξ, η, g) is the canonical contact metric structure $\mathcal{F}$.

For any $X \in \Gamma D(1) = \Gamma L$ we have

(a) $hX = X$

which implies by 1.35 that

$$\nabla_X \xi = - \phi X - \phi hX$$
$$= - 2\phi X.$$

Furthermore for any $Y \in \Gamma D(-1) = \Gamma Q$ we have

(b) $hY = - Y$

which implies by 1.35 that

$$\nabla_Y \xi = - \phi Y - \phi hY$$
$$= 0.$$

As $\bar{\mathcal{F}}$ is a flat Legendre foliation, we have by lemma 2.13 (i) that ξ is a foliate vector field with respect $\bar{\mathcal{F}}$, that is

$$p_{\bar{Q}}([\phi X, \xi]) = p_L([\phi X, \xi]) = 0 \qquad \forall\, \phi X \in \Gamma \bar{L} = \Gamma Q.$$

From this we obtain

(c)
$$\begin{aligned} 0 &= p_L([\phi X, \xi]) \\ &= p_L(\nabla_{\phi X}\, \xi - \nabla_{\xi}\, \phi X) \\ &= p_L(-\nabla_{\xi}\, \phi X) \qquad \text{by (b), as } \phi X \in \Gamma Q = \Gamma D(-1). \end{aligned}$$

Thus we have

$$\begin{aligned} p_L(\phi[X, \xi]) &= p_L(\phi \nabla_X\, \xi - \phi \nabla_{\xi}\, X) \\ &= p_L(-2\phi^2 X - \nabla_{\xi}\, \phi X) \qquad \text{by 1.27 and (a)} \\ &= 2X \qquad \text{by (c),} \end{aligned}$$

therefore

$$p_Q([X, \xi]) = -2\phi X \qquad \forall\, X \in \Gamma D(1) = \Gamma L.$$

This implies by lemma 4.4 that $g|_L = \frac{1}{4}\Pi$. Furthermore from (a) we have that $h: L \to L$ so (ϕ, ξ, η, g) satisfies the conditions of theorem 4.15 thus (ϕ, ξ, η, g) is the canonical contact metric structure for $\mathcal{F}$.

Now we show that D(1) defines a semi-Riemannian Legendre foliation, $\mathcal{F}$.

Let $\{X_\alpha\}$ be a local orthonormal frame of $L = D(1)$ with respect to g, then for any α, β we have

$$\begin{aligned} 0 &= R(X_\alpha, \phi X_\beta)\xi \\ &= \nabla_{X_\alpha}\nabla_{\phi X_\beta}\xi - \nabla_{\phi X_\beta}\nabla_{X_\alpha}\xi - \nabla_{[X_\alpha,\phi X_\beta]}\xi \\ &= 2(\nabla_{\phi X_\beta}\phi X_\alpha) + 2\phi p_L([X_\alpha, \phi X_\beta]) \end{aligned}$$

by (a), (b) 1.28 and theorem 2.4 (i).

Therefore for any α, β we have

(d) $$\nabla_{\phi X_\beta}\phi X_\alpha = - p_Q\phi([X_\alpha, \phi X_\beta]).$$

From which for any α, β, γ we have by theorem 4.25 that

$$\begin{aligned} G(X_\alpha, X_\beta, X_\gamma) &= 2g(\nabla_{\phi X_\beta}\phi X_\gamma + \nabla_{\phi X_\gamma}\phi X_\beta, X_\alpha) \\ &= 0 \end{aligned}$$ by (d).

Thus by theorem 6.5 the Legendre foliation $\mathcal{F}$ defined by D(1) is a semi-Riemannian Legendre foliation. Furthermore as (ϕ, ξ, η, g) is the canonical contact metric structure of $\mathcal{F}$ and D(-1) defines the conjugate Legendre foliation $\overline{\mathcal{F}}$ of $\mathcal{F}$, also $\overline{\mathcal{F}}$ is totally geodesic by theorem 6.7 (ii). ¤

CHAPTER 7. SPECIAL CASES

7 (a). LEGENDRE FOLIATIONS ON K-CONTACT AND SASAKIAN MANIFOLDS

In this section we will investigate the properties of Legendre foliations on K-contact manifolds and on Sasakian manifolds.

A contact metric manifold (M, ϕ, ξ, η, g) is a K-contact manifold if ξ is a Killing vector field with respect to g which by lemma 1.20 implies that

$$h = \frac{1}{2} N^3 \equiv 0.$$

Lemma 7.1
Let (M, ϕ, ξ, η, g) be a K-contact manifold then

(i) $\nabla_W \xi = - \phi W \qquad \forall\, W \in \Gamma TM$

(ii) $[\xi, \phi W] = \phi[\xi, W] \qquad \forall\, W \in \Gamma TM.$

Proof

(i) For any $W \in \Gamma TM$ we have

$$\begin{aligned} \nabla_W \xi &= - \phi W - \phi h W && \text{by 1.35} \\ &= - \phi W && \text{as } h \equiv 0. \end{aligned}$$

(ii) For any $W \in \Gamma TM$ we have

$$\begin{aligned} 0 &= 2hW \\ &= [\xi, \phi W] - \phi[\xi, W] && \text{by 1.30.} \end{aligned}$$

¤

Lemma 7.2

Let $\mathcal{F}$ be a Legendre foliation on a K-contact manifold (M, ϕ, ξ, η, g) then the projections of the Levi-Civita connection of g satisfy

(i) $\quad p_E(\nabla_{X_\alpha} \phi X_\beta) = k_{\alpha\beta}\xi, \qquad p_E(\nabla_{\phi X_\alpha} X_\beta) = - k_{\alpha\beta}\xi.$

(ii) $\quad p_E(\nabla_{X_\alpha} X_\beta) = p_E(\nabla_{\phi X_\alpha} \phi X_\beta) = 0.$

Where $\{X_\alpha, \phi X_\alpha, \xi\}$ is a local orthonormal frame for TM with respect to g.

Proof

For any α, β we have

$$\begin{aligned} g(\nabla_{X_\alpha} \phi X_\beta, \xi) &= - g(\phi X_\beta, \nabla_{X_\alpha} \xi) \\ &= g(\phi X_\beta, \phi X_\alpha) \qquad \text{by lemma 7.1 (i)} \\ &= k_{\alpha\beta}. \end{aligned}$$

The rest are proved similarly. ¤

The contact metric manifold (M, ϕ, ξ, η, g) is a Sasakian manifold if $N^1 \equiv 0$. A Sasakian manifold is also a K-contact manifold as $N^1 \equiv 0$ implies $N^3 \equiv 0$ by lemma 1.19.

Lemma 7.3

Let (M, ϕ, ξ, η, g) be a Sasakian manifold then the Levi-Civita connection has the additional projection property

$$p_H((\nabla_U \phi)U') = 0 \qquad \forall\ U, U' \in \Gamma H.$$

Proof

As $N^1 \equiv 0$ for any U, U', U'' $\in \Gamma H$ we have

$$\begin{aligned} 0 &= g(N^1(U', U''), \phi U) \\ &= 2g((\nabla_U \phi)U', U'') \qquad \text{by lemma 1.26.} \end{aligned}$$

¤

For a Legendre foliation on a K-contact manifold we show that the second fundamental form is a section of the subbundle Q.

Lemma 7.4

Let $\mathcal{F}$ be a Legendre foliation on a K- contact manifold (M, ϕ, ξ, η, g) then

(i) $A_\xi \equiv 0.$

(ii) $B(X, X') \in \Gamma Q \qquad \forall\ X, X' \in \Gamma L,$

which implies that

$$\tau = g(X_\alpha, X_\beta)B(X_\alpha, X_\beta) \in \Gamma Q$$

where $\{X_\alpha\}$ is a local orthonormal frame of L with respect to g.

Proof

(i) $A_\xi = p_L(\phi h)$ by lemma 2.9

$\equiv 0$ since h = 0.

(ii) For any tangential vector fields X, X' we have

$$\begin{aligned} B(X, X') &= p_{L^\perp}(\nabla_X X') \\ &= p_Q(\nabla_X X') \qquad \text{since } p_E(\nabla_X X') = 0 \text{ by lemma 7.2 (ii).} \end{aligned}$$

Hence $B(X, X') \in \Gamma Q$. ¤

From lemma 7.4 we get the following theorem

Theorem 7.5
Let $\mathcal{F}$ be a Legendre foliation on a K-contact manifold (M, ϕ, ξ, η, g) then

(i) $\mathcal{F}$ is totally geodesic if and only if the second fundamental form B of $\mathcal{F}$ is parallel in the transverse bundle,

(ii) $\mathcal{F}$ is harmonic if and only if the mean curvature vector field τ of $\mathcal{F}$ is parallel in the transverse bundle.

Proof

(i) Suppose B is parallel in the transverse bundle, that is

$$p_{L^\perp}(\nabla_X B(X', X'')) = 0 \qquad \forall\, X, X', X'' \in \Gamma L.$$

Then for any vectors $X, X', X'' \in \Gamma L$, we have

$$\begin{aligned} 0 &= g(\nabla_X B(X', X''), \xi) \\ &= - g(B(X', X''), \nabla_X \xi) \\ &= g(B(X', X''), \phi X) \qquad \text{by lemma 7.1 (i).} \end{aligned}$$

As $B(X', X'') \in \Gamma Q$ by lemma 7.4 (ii) we have $g(B(X', X''), W) = 0$ for any vector field W on M, that is $B \equiv 0$, thus $\mathcal{F}$ is a totally geodesic foliation.

Suppose $\mathcal{F}$ is totally geodesic that is $B \equiv 0$ hence B is parallel in the transverse bundle.

(ii) Suppose τ is parallel in the transverse bundle that is

$$p_{L^\perp}(\nabla_X \tau) = 0 \qquad \forall\, X \in \Gamma L.$$

Then for any vector $X \in \Gamma L$, we have

$$\begin{aligned} 0 &= g(\nabla_X \tau, \xi) \\ &= - g(\tau, \nabla_X \xi) \\ &= g(\tau, \phi X) \qquad \text{by lemma 7.1 (i).} \end{aligned}$$

As $\tau \in \Gamma Q$ by lemma 7.4 (ii) we have $g(\tau, W) = 0$ for any vector field W on M, that is $\tau \equiv 0$, hence $\mathcal{F}$ is a harmonic foliation.

Suppose $\mathcal{F}$ is harmonic that is $\tau \equiv 0$ which implies that τ is parallel in the transverse bundle. ¤

Lemma 7.6

Let $\mathcal{F}$ be a Legendre foliation on a Sasakian manifold (M, ϕ, ξ, η, g) then

$$A_{\phi X} X' = A_{\phi X'} X \qquad \forall\ X, X' \in \Gamma L,$$

which implies that

$$g(B(X', X''), \phi X) = g(B(X, X''), \phi X') \qquad \forall\ X, X', X'' \in \Gamma L.$$

Proof

For any tangential vector fields X, X', X'' we have

$$\begin{aligned} 0 &= g([X, X'], \phi X'') && \text{as L is completely integrable} \\ &= - g(\phi \nabla_X X' - \phi \nabla_{X'} X, X'') && \text{by 1.10} \\ &= - g(\nabla_X \phi X' - \nabla_{X'} \phi X, X'') && \text{by lemma 7.3} \\ &= g(A_{\phi X} X' - A_{\phi X'} X, X''). && ¤ \end{aligned}$$

The next theorem proves that if the conjugate of a Legendre foliation, $\mathcal{F}$, on a K-contact manifold exists it is of the same classification as $\mathcal{F}$.

Theorem 7.7

Let $\mathcal{F}$ be a Legendre foliation on a K-contact manifold (M, ϕ, ξ, η, g) such that the conjugate Legendre foliation of $\mathcal{F}$ exists, then the conjugate is of the same classification as $\mathcal{F}$.

Proof

Let $\{X_\alpha, \phi X_\alpha, \xi\}$ be a local orthonormal frame for TM with respect to g, then for any α the projection

$$\begin{aligned} p_L([\xi, \phi X_\alpha]) &= p_L(\phi[\xi, X_\alpha]) && \text{by lemma 7.1 (ii)} \\ &= \phi p_Q([\xi, X_\alpha]) && \text{by theorem 2.1.} \end{aligned}$$

As $\phi|_H$ is an isomorphism the theorem follows by lemma 2.13. □

The following examples from chapter 3 illustrate these ideas.

Example 3.1: (a) $\mathcal{F}^c$ is a flat Legendre foliation on the canonical Sasakian manifold for which the conjugate Legendre foliation exists and is flat by theorem 5.11.

(c) $\mathcal{F}''$ is a flat Legendre foliation on a K-contact manifold that is not a Sasakian manifold. The conjugate Legendre foliation of $\mathcal{F}''$ of does not exist.

(d) $\mathcal{F}$ is a flat Legendre foliation on a Sasakian manifold which is not the canonical Sasakian manifold. The conjugate Legendre foliation of $\mathcal{F}$ exists and is flat.

(e) $\mathcal{F}^\#$ is a flat Legendre foliation on a K-contact manifold that is not a Sasakian manifold. The conjugate Legendre foliation of $\mathcal{F}^\#$ exists and is flat.

We now investigate the properties of non-degenerate Legendre foliations on K-contact manifolds.

Theorem 7.8

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on a K-contact manifold (M, ϕ, ξ, η, g). Then (ϕ, ξ, η, g) is the canonical contact metric structure if and only if $g|_L = \frac{1}{4}\Pi$.

Proof

As $h \equiv 0$, the conditions of theorem 4.15 are satisfied if and only if $g|_L = \frac{1}{4}\Pi$. ¤

Lemma 7.9

If $\mathcal{F}$ is a non-degenerate Legendre foliation on a K-contact manifold $(M, \phi^c, \xi, \eta, g^c)$ where (ϕ^c, ξ, η, g^c) is the canonical contact metric structure then the projection

(i) $\quad p_L([\xi, \phi^c X]) = -2X.$

Furthermore for any $X \in \Gamma L$ we have

(ii) $\quad p_L(\nabla_\xi \phi^c X) = (\phi^c)^2 X = -X \qquad$ and $\qquad p_Q(\nabla_\xi X) = \phi^c X.$

Proof

(i) Let $\{X_\alpha, \phi^c X_\alpha, \xi\}$ be a local orthonormal frame for TM with respect to g^c, then using lemma 7.1 (ii) for any α we obtain

$$\begin{aligned} p_L([\xi, \phi^c X_\alpha]) &= p_L \phi^c([\xi, X_\alpha]) \\ &= \phi^c p_Q([\xi, X_\alpha]) && \text{by lemma 2.1} \\ &= \phi^c(2\phi^c X_\alpha) && \text{by lemma 4.4 (ii)} \\ &= -2X_\alpha. \end{aligned}$$

(ii) From part (i) for any $X \in \Gamma L$ we have

$$\begin{aligned} 2X &= - p_L([\xi, \phi^c X]) \\ &= - p_L(\nabla_\xi \phi^c X - \nabla_{\phi^c X} \xi) \\ &= - p_L(\nabla_\xi \phi^c X) + X \qquad \text{by lemma 7.1 (i).} \end{aligned}$$

This implies that

$$p_L(\nabla_\xi \phi^c X) = - X$$

and

$$\begin{aligned} p_Q(\nabla_\xi X) &= - p_Q(\nabla_\xi \phi^c(\phi^c X)) \\ &= - \phi^c p_L(\nabla_\xi \phi^c X) \qquad \text{by 1.27} \\ &= \phi^c X. \qquad \square \end{aligned}$$

When $\mathcal{F}$ is a non-degenerate Legendre foliation and the K-contact metric structure is the canonical contact metric structure we have the following existence theorem for the conjugate.

Theorem 7.10

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on a K-contact manifold (M, ϕ, ξ, η, g) with $g|_L = \frac{1}{4}\Pi$, then $\overline{\mathcal{F}}$ the conjugate Legendre foliation of $\mathcal{F}$ exists if and only if (M, ϕ, ξ, η, g) is a Sasakian manifold.

Proof

(ϕ, ξ, η, g) is the canonical contact metric structure by theorem 7.8 so

$\overline{\mathcal{F}}$ exists if and only if $N_c^1|_H \equiv 0$ by theorem 5.8

if and only if $\phi|_H$ is a complex structure on H by theorem 1.24

if and only if (M, ϕ, ξ, η, g) is a Sasakian manifold by corollary 1.25.

$\square$

For a Legendre foliation, $\mathcal{F}$, on a K-contact manifold we have shown in theorem 7.7, that if the conjugate Legendre foliation of $\mathcal{F}$ exists, it is of the same classification as $\mathcal{F}$. Example 6.1 (b) shows that we can have a Legendre foliation with a conjugate of the same classification on a contact metric manifold which is not a K-contact manifold. Thus this property is not restricted to K-contact manifolds. But for a non-degenerate Legendre foliation on the canonical contact metric manifold we get the following theorem.

Theorem 7.11

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on the canonical contact metric manifold $(M, \phi^c, \xi, \eta, g^c)$ such that $\overline{\mathcal{F}}$ the conjugate Legendre foliation of $\mathcal{F}$ exists. Then $(M, \phi^c, \xi, \eta, g^c)$ is a Sasakian manifold if and only if $\overline{\mathcal{F}}$ is non-degenerate and (ϕ^c, ξ, η, g^c) is the canonical contact metric structure of $\overline{\mathcal{F}}$.

Proof

Suppose $\overline{\mathcal{F}}$ is non-degenerate and (ϕ^c, ξ, η, g^c) is the canonical contact metric structure of $\overline{\mathcal{F}}$. Then for any $X \in \Gamma L$ we have the projection

$$\begin{aligned} p_L([\xi, \phi^c X]) &= p_{\overline{Q}}([\xi, \phi^c X]) \\ &= 2\phi^c(\phi^c X) \qquad \text{by lemma 4.4 (ii)} \\ &= -2X. \end{aligned}$$

By lemma 4.19 this implies that $h^c \equiv 0$, that is $(M, \phi^c, \xi, \eta, g^c)$ is a K-contact manifold. Thus as the conjugate exists, $(M, \phi^c, \xi, \eta, g^c)$ is a Sasakian manifold by theorem 7.10.

Suppose $(M, \phi^c, \xi, \eta, g^c)$ is a Sasakian manifold then $\overline{\mathcal{F}}$ is non-degenerate by theorem 7.7. Furthermore we have $h^c \equiv 0$ and for any $X \in \Gamma L$ the projection

$$p_L([\xi, \phi^c X]) = 2\phi^c(\phi^c X) \qquad \text{by lemma 7.9 (i).}$$

Hence by lemma 4.4 the conditions of theorem 4.15 are satisfied for $\overline{\mathcal{F}}$ so (ϕ^c, ξ, η, g^c) is the canonical contact metric structure of $\overline{\mathcal{F}}$. ¤

For a Sasakian manifold corollary 6.8 can be extended to

Theorem 7.12
Let $\mathcal{F}$ be a non-degenerate Legendre foliation on a Sasakian manifold $(M, \phi^c, \xi, \eta, g^c)$ then $\mathcal{F}$ ($\overline{\mathcal{F}}$) is a semi-Riemannian Legendre foliation if and only if $\overline{\mathcal{F}}$ ($\mathcal{F}$) is a totally geodesic Legendre foliation.

Proof
The conjugate Legendre foliation exists by theorem 7.10 as $g^c|_L = \frac{1}{4}\Pi$. Also by theorem 7.11 (ϕ^c, ξ, η, g^c) is the canonical contact metric structure of $\mathcal{F}$ and $\overline{\mathcal{F}}$. Thus the theorem follows by corollary 6.8. ◻

We now give an example of a non-degenerate Legendre foliation on a Sasakian manifold, which illustrates these theorems.

Example 7.1: Non-degenerate Legendre foliation on a Sasakian manifold
Let

$$M = S^3$$
$$= \{(x_1, x_2, y_1, y_2) \mid (x_1)^2 + (x_2)^2 + (y_1)^2 + (y_2)^2 = 1\}$$

and

$$\eta = y_1dx_1 + y_2dx_2 - x_1dy_1 - x_2dy_2.$$

Then we have

$$d\eta = 2(dy_1 \wedge dx_1 + dy_2 \wedge dx_2)$$

which by calculation gives

$$\eta \wedge d\eta = 2(y_2dx_1 \wedge dx_2 \wedge dy_1 - y_1dx_1 \wedge dx_2 \wedge dy_2 + x_2dx_1 \wedge dy_1 \wedge dy_2 - x_1dx_2 \wedge dy_1 \wedge dy_2)$$
$$\neq 0 \text{ everywhere on } M.$$

Thus (M, η) is a contact manifold. Furthermore $(M, \phi^c, \xi, \eta, g^c)$ a Sasakian manifold where (ϕ^c, ξ, η, g^c) is the Sasakian structure given by

$$g^c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\phi^c = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\xi = y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial y_1} - x_2 \frac{\partial}{\partial y_2}.$$

Consider the vector field

$$X = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} - y_2 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial y_2}.$$

Then

$$\begin{aligned} Y &= \phi^c X \\ &= y_2 \frac{\partial}{\partial x_1} - y_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial y_1} - x_1 \frac{\partial}{\partial y_2}. \end{aligned}$$

These are global vector fields which belong to H the contact distribution of $(M, \phi^c, \xi, \eta, g^c)$.

Calculating the Lie brackets we get

$$[X, \xi] = -2Y = -2\phi^c X$$
$$[Y, \xi] = 2X = -2\phi^c Y$$
$$[X, Y] = 2\xi.$$

Thus L = span{X} defines a non-degenerate Legendre foliation $\mathcal{F}$ and Q = span {Y} defines a non-degenerate Legendre foliation $\overline{\mathcal{F}}$, the conjugate of $\mathcal{F}$. The Lie brackets show that (ϕ^c, ξ, η, g^c) is the canonical contact metric structure for $\mathcal{F}$ and $\overline{\mathcal{F}}$. As $[X, Y] = 2\xi$, lemma 6.6 (iii) shows that $\mathcal{F}$ and $\overline{\mathcal{F}}$ are both semi-Riemannian Legendre foliations, it can also be shown that they are both totally geodesic.

7 (b). TOTALLY UMBILIC AND TOTALLY GEODESIC LEGENDRE FOLIATIONS

A foliation $\mathcal{F}$ on a semi-Riemannian manifold (M, g) is *totally umbilic* if there exists a one form, a, on $L^{\perp}$ such that

$$A_V = a(V)I \qquad \forall\, V \in \Gamma L^{\perp}.$$

The leaves of a totally umbilic foliation are totally umbilic submanifolds of (M, g).

Remember that a foliation $\mathcal{F}$ on a semi-Riemannian manifold (M, g) is totally geodesic if its second fundamental form vanishes identically, that is,

$$B \equiv 0 \quad \text{which is equivalent to } A \equiv 0.$$

Hence a totally geodesic foliation is also a totally umbilic foliation.

Note that a foliation which is harmonic and totally umbilic is also totally geodesic.

Lemma 7.13

Let $\mathcal{F}$ be a totally umbilic Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) then for any α, β the projection

$$p_{L^{\perp}}(\nabla_{X_\alpha} X_\beta) = g(X_\alpha, X_\beta)(k_{\gamma\delta}a(\phi X_\delta)\phi X_\gamma + a(\xi)\xi),$$

where $A_{\phi X_\beta} = a(\phi X_\beta)I$, $A_\xi = a(\xi)I$, and $\{X_\alpha\}$ is a local orthonormal frame of L with respect to g.

Proof

For any α, β, γ we have

$$\begin{aligned} a(\phi X_\gamma)g(X_\alpha, X_\beta) &= g(a(\phi X_\gamma)X_\alpha, X_\beta) \\ &= g(A_{\phi X_\gamma} X_\alpha, X_\beta) \\ &= - g(\nabla_{X_\alpha} \phi X_\gamma, X_\beta) \\ &= g(\phi X_\gamma, \nabla_{X_\alpha} X_\beta), \end{aligned}$$

which implies that

$$p_Q(\nabla_{X_\alpha} X_\beta) = g(X_\alpha, X_\beta)k_{\gamma\delta}a(\phi X_\delta)\phi X_\gamma.$$

Furthermore for any α, β we have

$$\begin{aligned} a(\xi)g(X_\alpha, X_\beta) &= g(a(\xi)X_\alpha, X_\beta) \\ &= g(A_\xi X_\alpha, X_\beta) \\ &= - g(\nabla_{X_\alpha} \xi, X_\beta) \\ &= g(\xi, \nabla_{X_\alpha} X_\beta), \end{aligned}$$

which implies that

$$p_E(\nabla_{X_\alpha} X_\beta) = g(X_\alpha, X_\beta)a(\xi)\xi. \qquad \square$$

Lemma 7.13 implies that for a totally umbilic Legendre foliation $\mathcal{F}$ the projection

$$p_{L^\perp}(\nabla_{X_\alpha} X_\beta) = 0 \qquad \forall\ \alpha \neq \beta.$$

Furthermore the mean curvature vector field of $\mathcal{F}$ is given by

$$\begin{aligned}\tau &= g(X_\alpha, X_\beta)p_{L^\perp}(\nabla_{X_\alpha} X_\beta) \\ &= g(X_\alpha, X_\beta)g(X_\alpha, X_\beta)(k_{\gamma\delta}a(\phi X_\delta)\phi X_\gamma + a(\xi)\xi) \\ &= n(k_{\gamma\delta}a(\phi X_\delta)\phi X_\gamma + a(\xi)\xi).\end{aligned}$$

Theorem 7.14

Let $\mathcal{F}$ be a Legendre foliation on a Sasakian manifold (M, ϕ, ξ, η, g) then $\mathcal{F}$ is a totally umbilic Legendre foliation if and only if $\mathcal{F}$ is totally geodesic.

Proof

We have already noted that a totally geodesic foliation is totally umbilic.

Suppose $\mathcal{F}$ is a totally umbilic Legendre foliation. Let $\{X_\alpha\}$ be a local orthonormal frame of L with respect to g, then as (M, ϕ, ξ, η, g) is a Sasakian manifold by lemma 7.6 for any α, β we have

$$\begin{aligned}0 &= -\, g(A_{\phi X_\beta} X_\alpha - A_{\phi X_\alpha} X_\beta, X_\beta) \\ &= g(\nabla_{X_\alpha} \phi X_\beta - \nabla_{X_\beta} \phi X_\alpha, X_\beta) \\ &= -\, g(\phi X_\beta, \nabla_{X_\alpha} X_\beta) + g(\phi X_\alpha, \nabla_{X_\beta} X_\beta).\end{aligned}$$

For $\alpha \neq \beta$ this implies that

$$\begin{aligned}g(\phi X_\alpha, \nabla_{X_\beta} X_\beta) &= g(\phi X_\beta, \nabla_{X_\alpha} X_\beta) \\ &= 0 \qquad \text{by lemma 7.13.}\end{aligned}$$

But for any α we have

$$\begin{aligned}g(\phi X_\alpha, \nabla_{X_\beta} X_\beta) &= -\, g(\nabla_{X_\beta} \phi X_\alpha, X_\beta) \\ &= g(A_{\phi X_\alpha} X_\beta, X_\beta) \\ &= a(\phi X_\alpha)g(X_\beta, X_\beta) \qquad \text{as } \mathcal{F} \text{ is totally umbilic.}\end{aligned}$$

This implies for all α that

$$a(\phi X_\alpha) = 0.$$

Hence $\mathcal{F}$ is harmonic as

$$\tau = p_Q(\nabla_{X_\alpha} X_\alpha) \qquad \text{by corollary 2.10. as } h \equiv 0$$
$$= 0 \qquad \text{by lemma 7.13, as } a(\phi X_\alpha) = 0 \text{ for all } \alpha.$$

Thus $\mathcal{F}$ is totally geodesic as it is totally umbilic and harmonic. ¤

Theorem 7.15
Let $\mathcal{F}$ be a non-degenerate Legendre foliation on the canonical contact metric manifold $(M, \phi^c, \xi, \eta, g^c)$, such that $\overline{\mathcal{F}}$ the conjugate Legendre foliation of $\mathcal{F}$ exists. Then $\mathcal{F}$ is a totally umbilic Legendre foliation if and only if $\mathcal{F}$ is totally geodesic.

Proof
From corollary 5.9 as the conjugate exists we have

$$A_{\phi^c X} X' = A_{\phi^c X'} X \qquad \forall\ X, X' \in \Gamma L,$$

and as (ϕ^c, ξ, η, g^c) is the canonical contact metric structure, we have $h^c : L \to L$, from theorem 4.15.

Using these two conditions the proof is the same as for theorem 7.14. ¤

An example of a Legendre foliation that is totally umbilic but not totally geodesic is given in chapter 6.

Example 6.1 (b): $\mathcal{F}'_F$ is a semi-Riemannian Legendre foliation on a contact metric manifold (M, ϕ', ξ, η, g') which is not the canonical contact metric manifold. By calculation for any α, β we have

$$A_{Y'_\beta} X_\alpha = - p_L(\nabla_{X_\alpha} Y'_\beta) = 0$$

$$A_\xi X_\alpha = - p_L(\nabla_{X_\alpha} \xi) = -2X_\alpha.$$

This implies that $\mathcal{F}'_F$ is a totally umbilic Legendre foliation with $a(Y'_\alpha) = 0$ and $a(\xi) = - 2$. Furthermore

$$\begin{aligned} \tau &= n(k_{\gamma\delta} a(\phi X_\delta)\phi X_\gamma + a(\xi)\xi). \\ &= 2(- 2\xi) \\ &= - 4\xi. \end{aligned}$$

Hence $\mathcal{F}'_F$ is not a harmonic Legendre foliation.

7 (c). ISOPARAMETRIC AND HARMONIC LEGENDRE FOLIATIONS

A foliation $\mathcal{F}$ on a semi-Riemannian manifold (M, g) is isoparametric if the mean curvature form κ is *parallel along the leaves*, that is

$$\mathcal{L}_X \kappa = 0 \qquad \forall\, X \in \Gamma L$$

where $\kappa \in \Gamma L^{\perp *}$ is the one form given by

$$\kappa(Y) = g(\tau, V) \qquad \forall\, V \in \Gamma L^{\perp}.$$

The leaves of an isoparametric foliation are isoparametric submanifolds of (M, g).

Remember a foliation $\mathcal{F}$ on a semi-Riemannian manifold (M, g) is harmonic if the mean curvature form κ vanishes that is

$$\kappa \equiv 0 \text{ which is equivalent to } \tau \equiv 0.$$

Hence a harmonic foliation is also an isoparametric foliation.

Lemma 7.16

Let $\mathcal{F}$ be an isoparametric Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) such that $h: L \to L$ then

$$g(\tau, [X, \xi]) = 0 \qquad \forall\, X \in \Gamma L.$$

Proof

As $\mathcal{F}$ is a isoparametric Legendre foliation for any $X \in \Gamma L$ we have

$$\begin{aligned} 0 &= \mathcal{L}_X \kappa(\xi) \\ &= X\kappa(\xi) - \kappa([X, \xi]) \\ &= Xg(\tau, \xi) - g(\tau, [X, \xi]) \\ &= -\, g(\tau, [X, \xi]) \qquad \text{as } \tau \in \Gamma Q, \text{ by corollary 2.10.} \end{aligned}$$

□

Theorem 7.17

If $\mathcal{F}$ is a non-degenerate Legendre foliation on the canonical contact metric manifold $(M, \phi^c, \xi, \eta, g^c)$. Then $\mathcal{F}$ is an isoparametric Legendre foliation if and only if $\mathcal{F}$ is a harmonic Legendre foliation.

Proof

We have already noted that a harmonic foliation is isoparametric.

Suppose $\mathcal{F}$ is an isoparametric Legendre foliation. Let $\{X_\alpha\}$ be a local orthonormal frame of L with respect to g^c, then h^c: L $\rightarrow$ L by theorem 4.15, thus for any α we have

$$0 = g^c(\tau, [X_\alpha, \xi]) \qquad \text{by lemma 7.16}$$
$$= -2g^c(\tau, \phi^c X_\alpha) \qquad \text{by lemma 4.4 (ii), as } g|_L = \frac{1}{4}\Pi.$$

As $\tau \in \Gamma Q$ by corollary 2.10, we have $g^c(\tau, W) = 0$ for any vector field W on M, that is $\tau \equiv 0$, so $\mathcal{F}$ is a harmonic Legendre foliation. ¤

Corollary 7.18
If $\mathcal{F}$ is a non-degenerate Legendre foliation on the canonical contact metric manifold $(M, \phi^c, \xi, \eta, g^c)$. Then $\mathcal{F}$ is a totally umbilic and isoparametric Legendre foliation if and only if $\mathcal{F}$ is a totally geodesic Legendre foliation.

Proof
If $\mathcal{F}$ is a totally geodesic Legendre foliation then by definition $\mathcal{F}$ is totally umbilic and isoparametric.

Suppose $\mathcal{F}$ is a totally umbilic and isoparametric Legendre foliation then by Theorem 7.17 $\mathcal{F}$ is harmonic so as $\mathcal{F}$ is also totally umbilic $\mathcal{F}$ is a totally geodesic Legendre foliation. ¤

Theorem 7.19
If $\mathcal{F}$ is a non-degenerate Legendre foliation on a K-contact manifold (M, ϕ, ξ, η, g). Then $\mathcal{F}$ is an isoparametric Legendre foliation if and only if $\mathcal{F}$ is a harmonic Legendre foliation.

Proof
We have already noted that a harmonic foliation is isoparametric.

Suppose $\mathcal{F}$ is a isoparametric Legendre foliation. Let $\{X_\alpha\}$ be a local orthonormal frame of L with respect to Π, then by theorem 2.1 $Q = \text{span}\{\phi X_\alpha\}$, note $\{X_\alpha, \phi X_\alpha\}$ need not be a local orthonormal frame of H with respect to g.

As $h: L \to L$, $\tau \in \Gamma Q$ by corollary 2.10, so there exists functions a_α such that

$$\tau = a_\alpha \phi X_\alpha.$$

For any β we have

$$\begin{aligned} 0 &= g(\tau, [X_\beta, \xi]) && \text{by lemma 7.16} \\ &= g(a_\alpha \phi X_\alpha, [X_\beta, \xi]) \\ &= a_\alpha g(\phi X_\alpha, [X_\beta, \xi]) \\ &= -\frac{1}{2} a_\alpha \Pi(X_\alpha, X_\beta) && \text{by 2.11} \\ &= -\frac{1}{2} a_\alpha k_{\alpha\beta} \\ &= \pm \frac{1}{2} a_\beta, \end{aligned}$$

which implies that $\tau \equiv 0$, so $\mathcal{F}$ is a harmonic Legendre foliation. ¤

Corollary 7.20

Let $\mathcal{F}$ be a non-degenerate Legendre foliation on a K-contact manifold (M, ϕ, ξ, η, g). Then $\mathcal{F}$ is totally umbilic and isoparametric if and only if $\mathcal{F}$ is a totally geodesic Legendre foliation.

Proof

If $\mathcal{F}$ is a totally geodesic Legendre foliation then by definition $\mathcal{F}$ is totally umbilic and isoparametric.

Suppose $\mathcal{F}$ is a totally umbilic and isoparametric Legendre foliation then by Theorem 7.19 $\mathcal{F}$ is harmonic so as $\mathcal{F}$ is also totally umbilic $\mathcal{F}$ is a totally geodesic Legendre foliation. ¤

Theorem 7.21
Let $\mathcal{F}$ be a semi-Riemannian Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) then $\mathcal{F}$ is an isoparametric Legendre foliation if and only if $\mathcal{F}$ is a harmonic foliation.

Proof
As g is a bundle-like metric for any $X \in \Gamma L$ and $V \in \Gamma L^{\perp}$ we have

$$\begin{aligned} 0 &= \mathcal{L}_X g(\tau, V) \\ &= Xg(\tau, V) - g([X, \tau], V) - g(\tau, [X, V]) \\ &= X\kappa(V) - \kappa([X, V]) - g([X, \tau], V) \\ &= \mathcal{L}_X \kappa(V) - g([X, \tau], V). \end{aligned}$$

This implies that

$$\mathcal{L}_X \kappa(V) = g([X, \tau], V).$$

Thus $\mathcal{L}_X \kappa \equiv 0$ if and only if τ is a foliate vector field.

Suppose $\mathcal{F}$ is an isoparametric Legendre foliation which is not a harmonic foliation, then $\tau \neq 0$ is a foliate vector field. This implies by theorem 2.8 that $\tau \in \Gamma E$. Thus there exists a non-zero function f such that $\tau = f\xi$, as τ is a foliate vector field it can be shown that ξ is also a foliate vector field. This by lemma 2.13 (i) implies that $\mathcal{F}$ is a flat Legendre foliation, but as (ϕ, ξ, η, g) is a bundle-like contact metric structure $\mathcal{F}$ is a non-degenerate Legendre foliation by theorem 6.2 (i), which is a contradiction.

Thus $\mathcal{F}$ is an isoparametric Legendre foliation if and only if $\mathcal{F}$ is a harmonic foliation.

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CHAPTER 8 .
COMPLEX LEGENDRE FOLIATIONS

In this chapter we extend our work on Legendre foliations on contact metric manifolds to complex Legendre foliations on complex contact metric manifolds.

A complex contact manifold is a complex manifold of odd complex dimension which carries a complex contact form. The definition of a complex contact manifold is given and then we define a complex contact metric structure on a complex manifold.

We then define a complex Legendre foliation on a complex contact manifold and investigate its properties especially where they differ from the properties of a Legendre foliation on contact a manifold.

8 (a). COMPLEX CONTACT MANIFOLDS

A complex manifold M of complex dimension 2n + 1 is a *complex contact manifold* if it carries a global holomorphic 1-form η such that

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M.

The *complex contact form* η determines a holomorphic global vector field ξ, the *complex characteristic vector field* defined by Lin [30],

$$\eta(\xi) = 1.$$

The complex contact distribution H of a complex contact manifold is defined by

$$\begin{aligned} H &= \mathrm{ann}(\eta) \\ &= \{W \in TM \mid \eta(W) = 0\}, \end{aligned}$$

and a complex line bundle E is defined by

$$E = \text{span}(\xi).$$

Let J be the natural almost complex structure on the complex contact manifold (M, η), and T^cM be the complexified tangent bundle of M.

Define global vector fields by

$$U = \xi + \bar{\xi} \qquad V = -i(\xi - \bar{\xi}),$$

and one-forms by

$$u = \frac{1}{2}(\eta + \bar{\eta}) \qquad v = \frac{1}{2}i(\eta - \bar{\eta}).$$

Therefore we have

(8.1) (i) $V = -JU$ $\qquad v = u \circ J,$

(ii) $U = JV$ $\qquad u = -v \circ J,$

and furthermore

(8.2) $u(U) = v(V) = 1 \qquad u(V) = v(U) = 0.$

The complexified contact distribution H^c of a complex contact manifold is defined by

$$\begin{aligned} H^c &= \text{ann}(u, v) \\ &= \{W \in T^cM \mid u(W) = v(W) = 0\}, \end{aligned}$$

and the complexified line bundle E^c is defined by

$$E^c = \text{span}(U, V).$$

For a complex manifold M with almost complex structure J, g is a *Hermitian metric* on M if g is a Riemannian metric which for any vector fields W and W' on M satisfies

$$g(JW, JW') = g(W, W'),$$

as $J^2 = -I$ this implies that

$$g(W, JW') = -g(JW, W') \qquad \forall\, W, W' \in \Gamma T^cM.$$

A complex contact manifold (M, η) has a *complex contact metric structure* (G, ξ, η, g) where g is a Hermitian metric and G is a tensor of type (1,1) such that, (see Ishihara and Konishi [28])

(8.3) (i) $\quad G^2 = -I + u \otimes U + v \otimes V.$

(ii) $$g(W, GW') = du(W, W') + \frac{1}{2} g(\nabla_{W'} U, V)v(W) - \frac{1}{2} g(\nabla_W U, V)v(W')$$
$$\forall\, W, W' \in \Gamma T^cM.$$

(iii) $\quad GJ = -JG.$

Note that G is the negative of the tensor defined by Ishihara and Konishi [28]. This change in sign gives a direct correspondence between property 8.3 (ii) and property 1.2 of a contact metric structure. Furthermore as g is a Hermitian metric g is positive definite.

In this chapter (M, η) is used to denote a complex contact manifold when the complex contact metric structure is not specified while (M, G, ξ, η, g) is used to denote the complex contact manifold (M, η) with the complex contact metric structure (G, ξ, η, g), (M, G, ξ, η, g) is called a *complex contact metric manifold.*

From the properties of 8.3 it can be shown that for any $W, W' \in \Gamma T^c M$,

(8.4) (i) $GU = GV = 0.$

(ii) $u \circ G = v \circ G = 0.$

(iii) $\text{rank } G = 4n.$

(iv) $g(U, U) = g(V, V) = 1$ and $g(U, V) = 0.$

(v) $g(W, U) = u(W)$ and $g(W, V) = v(W).$

(vi) $g(GW, W') = - g(W, GW').$

(vii) $g(GW, GW') = g(W, W') - u(W)u(W') - v(W)v(W').$

We now define a tensor K of type (1,1) by

$$K = GJ.$$

Then if (G, ξ, η, g) is a complex contact metric structure on a complex contact manifold (M, η), for any $W, W' \in \Gamma T^c M$ we have,

(8.5) (i) $K^2 = - I + u \otimes U + v \otimes V.$

(ii) $$g(W, KW') = dv(W, W') - \frac{1}{2} g(\nabla_{W'} U, V)u(W) + \frac{1}{2} g(\nabla_W U, V)u(W').$$

From the properties of 8.3, 8.4, 8.5 and $J^2 = - I$, it can be shown that for any $W, W' \in \Gamma T^c M$ we have,

(8.6) (i) $KJ = - JK = - G.$

(ii) $KG = -GK = J + u \otimes V - v \otimes U$.

(iii) $KU = KV = 0$.

(iv) $u \circ K = v \circ K = 0$.

(v) rank $K = 4n$.

(vi) $g(KW, W') = - g(W, KW')$.

(vii) $g(KW, KW') = g(W, W') - u(W)u(W') - v(W)v(W')$.

(viii) $g(GW, KW') = g(W, JW') + u(W)v(W') - v(W)u(W')$.

From 8.3 (i), 8.4 (i) & (iii), 8.5 (i) and 8.6 (iii) & (v) we can show that $G|_{H^c}$ and $K|_{H^c}$ are isomorphisms with

$$(G|_{H^c})^{-1} = - G|_{H^c} \qquad \text{and} \qquad (K|_{H^c})^{-1} = - K|_{H^c}.$$

For any $W \in \Gamma H^c$ we have

$$\begin{aligned} g([W, U], U) &= u([W, U]) && \text{by 8.4 (v)} \\ &= - 2du(W, U) && \text{as } W \in \Gamma H^c \\ &= - 2g(W, GU) + g(\nabla_U U, V)v(W) - g(\nabla_W U, V)v(U) \\ & && \text{by 8.3 (ii)} \\ &= 0 && \text{by 8.2, 8.4 (i) as } W \in \Gamma H^c. \end{aligned}$$

Similarly for any $W \in \Gamma H^c$ we have

$$g([W, V], V) = 0.$$

Furthermore for any $W \in \Gamma H^c$ we have

$$\begin{aligned} g([W, V], U) &= u([W, V]) && \text{by 8.4 (v)} \\ &= -2du(W, V) && \text{as } W \in \Gamma H^c \\ &= -2g(W, GV) + g(\nabla_V U, V)v(W) - g(\nabla_W U, V)v(V) && \text{by 8.3 (ii)} \\ &= -g(\nabla_W U, V) && \text{by 8.2, 8.4 (i) and (iv) as } W \in \Gamma H^c. \end{aligned}$$

Similarly for any $W \in \Gamma H^c$ we have

$$g([W, U], V) = g(\nabla_W U, V).$$

These relationships imply that

$$g([W, V], U) = -g([W, U], V).$$

Therefore we have

(8.7) (i) $g([W, U], U) = u([W, U]) = 0$ $\quad \forall\ W \in \Gamma H^c.$

(ii) $g([W, V], V) = v([W, V]) = 0$ $\quad \forall\ W \in \Gamma H^c.$

(iii) $g([W, V], U) = u([W, V]) = -g(\nabla_W U, V)$ $\quad \forall\ W \in \Gamma H^c.$

(iv) $g([W, U], V) = v([W, U]) = g(\nabla_W U, V)$ $\quad \forall\ W \in \Gamma H^c.$

(v) $g([W, V], U) = -g([W, U], V)$ $\quad \forall\ W \in \Gamma H^c,$

which implies that

$$u([W, V]) = -v([W, U]).$$

For any $W, W' \in \Gamma H^c$ we have

$$\begin{aligned} g([W, GW'], U) &= u([W, GW']) && \text{by 8.4 (v)} \\ &= -2du(W, GW') && \text{as } W, W' \in \Gamma H^c \\ &= -2g(W, G^2W') && \text{by 8.3 (ii)} \\ &= 2g(W, W') && \text{by 8.3 (i).} \end{aligned}$$

Similarly for any $W, W' \in \Gamma H^c$ we have

$$g([W, KW'], V) = 2g(W, W').$$

Furthermore for any $W, W' \in \Gamma H^c$ we have

$$\begin{aligned} g([W, GW'], V) &= v([W, GW']) && \text{by 8.4 (v)} \\ &= -2dv(W, GW') && \text{as } W, W' \in \Gamma H^c \\ &= -2g(W, KGW') && \text{by 8.5 (ii)} \\ &= -2g(W, JW') && \text{by 8.6 (ii).} \end{aligned}$$

Similarly for any $W, W' \in \Gamma H^c$ we have

$$g([W, KW'], U) = 2g(W, JW').$$

Therefore we have

(8.8) (i) $g([W, GW'], U) = 2g(W, W')$ $\forall\ W, W' \in \Gamma H^c$.

(ii) $g([W, KW'], V) = 2g(W, W')$ $\forall\ W, W' \in \Gamma H^c$.

(iii) $g([W, GW'], V) = -2g(W, JW')$ $\forall\ W, W' \in \Gamma H^c$.

(iv) $g([W, KW'], U) = 2g(W, JW')$ $\forall\ W, W' \in \Gamma H^c$.

Theorem 8.9

For a complex contact metric manifold (M, G, ξ, η, g), U and V are analytic vector fields, that is the Lie derivatives

$$\mathcal{L}_U J = \mathcal{L}_V J = 0$$

which is equivalent to

$$[U, JW] = J[U,W] \quad \text{and} \quad [V, JW] = J[V, W] \quad \forall\, W \in \Gamma T^c M.$$

Proof

Yano and Kon [16], page 117-118, give an isomorphism θ

$$\theta : X \to \frac{1}{2}(X - iJX)$$

from the Lie algebra of analytic vector fields to the Lie algebra of holomorphic vector fields.

Using the inverse of this isomorphism, as

$$\xi = \frac{1}{2}(U - iJU)$$

is a holomorphic vector field, we obtain that U is an analytic vector field.

Similarly as ξ is a holomorphic vector field so is $-i\xi$ and we have

$$-i\xi = \frac{1}{2}(V - iJV).$$

Hence by the inverse of θ we obtain that V is also an analytic vector field. ¤

From theorem 8.9 we obtain that the Lie bracket

$$\begin{aligned} [U, V] &= - [U, JU] && \text{by 8.1 (i)} \\ &= J[U, U] && \text{by theorem 8.9} \\ &= 0. \end{aligned}$$

A *holomorphic foliation* is a foliation on a complex manifold (M, g), where g is a Hermitian metric, with an almost complex structure J which satisfies, Kamber and Tondeur [20].

(i) $J : L^{\perp} \to L^{\perp}$.

(ii) $g((\mathcal{L}_X J)W, W') = 0 \qquad \forall\, X \in \Gamma L, \forall\, W, W' \in \Gamma L^{\perp},$

which is equivalent to the projection

$$p_{L^{\perp}}([X, JW]) = p_{L^{\perp}}(J[X, W]) \qquad \forall\, X \in \Gamma L, \forall\, W, \in \Gamma L^{\perp}.$$

As $[U, V] = 0$, E^c is a completely integrable distribution on T^cM. Hence E^c defines a foliation on M of real dimension 2. Since U and V are analytic vector fields this is a holomorphic foliation.

Lemma 8.10

Let (M, G, ξ, η, g) be a complex contact metric manifold then

(i) $\nabla_V U = \nabla_U V = \nabla_U U = \nabla_V V = 0,$

(ii) $du(U, V) = dv(U, V) = 0,$

where ∇ is the Levi-Civita connection of g.

Proof

(i) For any $W \in \Gamma H^c$ we have

$$g(\nabla_W U, V) = g([W, U], V) \qquad \text{by 8.7 (iv)}$$
$$= g(\nabla_W U, V) + g(W, \nabla_U V).$$

Thus

$$g(\nabla_U V, W) = 0 \qquad \forall\, W \in \Gamma H^c.$$

Also, as $[U, V] = 0$ we have

$$0 = g([U, V], U)$$
$$= g(\nabla_U V, U) - g(\nabla_V U, U)$$
$$= g(\nabla_U V, U) \qquad \text{as } \{U, V\} \text{ is an orthonormal frame of } E^c.$$

Thus

$$g(\nabla_U V, U) = 0.$$

Furthermore, as $\{U, V\}$ is an orthonormal frame of E^c we have

$$g(\nabla_U V, V) = 0.$$

Hence

$$\nabla_U V = 0.$$

The others are proved similarly.

(ii) We have

$$2du(U, V) = 2g(U, GV) - g(\nabla_V U, V)v(U) + g(\nabla_U U, V)v(V)$$

by 8.3 (ii)

$$= g(\nabla_U U, V)v(V) \qquad \text{by 8.2 and 8.4 (i)}$$

$$= 0 \qquad \text{by part (i).}$$

$dv(U, V) = 0$, is proved similarly. □

8 (b). COMPLEX LEGENDRE FOLIATIONS

Consider a complex contact manifold (M, η) of complex dimension $2n + 1$. A *complex Legendre foliation*, $\mathcal{F}$, of (M, η) is a holomorphic foliation of M by integral submanifolds of η, of complex dimension n.

Since the leaves of $\mathcal{F}$ are integral submanifolds of η, we have $L^c \subset H^c$, where L^c is the complexified tangent bundle of $\mathcal{F}$. Let (G, ξ, η, g) be a complex contact metric structure on (M, η) then the complexified transverse bundle of $\mathcal{F}$ is defined by

$$L^{c\perp} = \{ Y \in T^cM \mid g(Y, X) = 0, \forall X \in L^c \}.$$

Define a subbundle Q^c of H^c by

$$Q^c = H^c \cap L^{c\perp}.$$

Then the foliation gives the following natural orthogonal decomposition of the complexified tangent bundle of (M, η)

$$T^cM = L^c \oplus Q^c \oplus E^c$$

where $L^{c\perp} = Q^c \oplus E^c$ and $H^c = L^c \oplus Q^c$.

As $\mathcal{F}$ is a holomorphic foliation we have the following relationships, see Kamber and Tondeur [20],

(8.11) (i) $J: L^{c\perp} \to L^{c\perp}$ and $J: L^{c} \to L^{c}$.

(ii) $g(\mathcal{L}_X J)\, W, W') = 0$ $\forall\, W, W' \in \Gamma L^{c\perp}, \forall\, X \in \Gamma L^{c}$,

which is equivalent to the projection

$$p_{L^{c}\perp}([X, JW]) = p_{L^{c}\perp}(J[X, W]) \qquad \forall\, X \in \Gamma L^{c}, \forall\, W \in \Gamma L^{c\perp}.$$

Hence by 8.11 (i) the leaves of a complex Legendre foliation on a complex contact metric manifold (M, G, ξ, η, g) are invariant submanifolds with respect to the complex structure J.

If W = U is substituted in 8.11 (ii) for any $X \in \Gamma L^{c}$ we have the projection

(8.12) (i) $p_{L^{c}\perp}(J[U, X]) = p_{L^{c}\perp}([JU, X])$
$= - p_{L^{c}\perp}([V, X])$ by 8.1 (i).

Also with W = V in 8.11 (ii) for any $X \in \Gamma L^{c}$ we have the projection

(8.12) (ii) $p_{L^{c}\perp}(J[V, X]) = p_{L^{c}\perp}([JV, X])$
$= p_{L^{c}\perp}([U,X])$ by 8.1 (ii).

Theorem 8.13
If $\mathcal{F}$ is a complex Legendre foliation on the complex contact metric manifold (M, G, ξ, η, g) then

(i) $u(X) = v(X) = 0$ $\forall\, X \in L^{c}$.

(ii) $GL^c = Q^c$ and $GQ^c = L^c$,

where

$GL^c = \{GX \mid X \in L^c\}$ and $GQ^c = \{GY \mid Y \in Q^c\}$.

(iii) $KL^c = Q^c$ and $KQ^c = L^c$,

where

$KL^c = \{KX \mid X \in L^c\}$ and $KQ^c = \{KY \mid Y \in Q^c\}$.

Proof

i) As L^c is a subbundle of H^c for any $X \in \Gamma L^c$ we have

$$u(X) = v(X) = 0.$$

(ii) & (iii)

For any $X, X' \in \Gamma L^c$ we have

$$\begin{aligned} 2g(X, GX') &= 2du(X, X') && \text{by 8.3 (ii) and part (i)} \\ &= -u([X, X']) && \text{as } L^c \subset H^c \\ &= 0 && \text{by part (i), as } L^c \text{ is completely integrable.} \end{aligned}$$

Thus $GX' \in L^{c\perp}$ but by 8.4 (ii) $GX' \in H^c$, so

$$GX \in Q^c \qquad \forall\, X \in L^c.$$

Similarly

$$KX \in Q^c \qquad \forall\, X \in L^c.$$

As $G|_{H^c}$ and $K|_{H^c}$ are isomorphisms the theorem follows. □

Thus the leaves of a complex Legendre foliation on a complex contact metric manifold (M, G, ξ, η, g) are anti-invariant submanifolds with respect to the complex tensors G and K.

Lemma 8.14

Let $\mathcal{F}$ be a complex Legendre foliation on the complex contact metric manifold (M, G, ξ, η, g) if

$\{X_\alpha, JX_\alpha\}$, $\alpha = 1,.....,n$ is a local orthonormal frame for L^c with respect to g then

$\{Y_\alpha, JY_\alpha\}$, $\alpha = 1,.....,n$ is a local orthonormal frame for Q^c with respect to g, where $\{Y_\alpha = GX_\alpha, JY_\alpha = JGX_\alpha = - GJX_\alpha = - KX_\alpha\}$.

Furthermore

(i) $g([X_\alpha, GX_\beta], U) = 2\delta_{\alpha\beta}$
$g([X_\alpha, GX_\beta], V) = 0$
$g([JX_\alpha, GJX_\beta], U) = g([JX_\alpha, KX_\beta], U) = 2\delta_{\alpha\beta}$
$g([JX_\alpha, GJX_\beta], V) = g([JX_\alpha, KX_\beta], V) = 0$
$g([JX_\alpha, GX_\beta], U) = 0.$

(ii) $g([X_\alpha, KX_\beta], V) = 2\delta_{\alpha\beta}$
$g([X_\alpha, KX_\beta], U) = 0$
$g([JX_\alpha, KJX_\beta], V) = - g([JX_\alpha, GX_\beta], V) = 2\delta_{\alpha\beta}$
$g([JX_\alpha, KJX_\beta], U) = - g([JX_\alpha, GX_\beta], U) = 0$
$g([JX_\alpha, KX_\beta], V) = 0.$

(iii) $\mathcal{L}_X g(Z, Z') = 0$ $\quad \forall X \in L^c, \forall Z, Z' \in E^c.$

Proof

Using 8.4 (vii), 8.6 (vii) & (viii) and theorem 8.13 it can be easily shown that $\{Y_\alpha, JY_\alpha\}$, $\alpha = 1,.....,n$ is a local orthonormal frame for Q^c with respect to g.

(i) & (ii)

For any α we have

$$g([X_\alpha, GX_\beta], U) = 2g(X_\alpha, X_\beta) \qquad \text{by 8.8 (i)}$$
$$= 2\delta_{\alpha\beta}.$$

The others are proved similarly.

(iii) For any $X \in L^c$ we have

$$\begin{aligned} \mathcal{L}_X g(U, V) &= - g([X, U], V) - g([X, V], U) \\ &= - g([X, JV], V) - g([X, V], JV) && \text{by 8.1 (ii)} \\ &= - g([X, JV], V) + g(J[X, V], V) && \text{since g is Hermitian} \\ &= - g((\mathcal{L}_X J)V, V) \\ &= 0 && \text{by 8.11 (ii).} \end{aligned}$$

□

8 (c). THE INVARIANTS Π, AND $\overline{\Pi}$ ON A COMPLEX LEGENDRE FOLIATION

We introduce for a complex Legendre foliation on a complex contact manifold (M, η) a symmetric, bilinear form Π on L^c defined by

$$\Pi(X, X') = - ((\mathcal{L}_X \mathcal{L}_{X'} u)(U)) + u([X', V])v([X, U]) \qquad \forall\ X, X' \in \Gamma L^c.$$

By definition Π is an invariant of a complex Legendre foliation, so that its value does not depend on the complex contact metric structure on (M, η). This definition corresponds to the definition for the invariant Π in chapter 2 (c) for a Legendre foliation, with an extra term $u([X', V])v([X, U])$. We see in the following lemma that this term enables us to find an expression for Π in terms of any complex contact metric structure on (M, η) that corresponds to 2.11.

Lemma 8.15

For a complex Legendre foliation the invariant Π satisfies the relationship

$$\Pi(X, X') = 2g([U, X], GX') \qquad \forall\ X, X' \in \Gamma L^c$$
$$= 2g([V, X], KX') \qquad \forall\ X, X' \in \Gamma L^c.$$

For any complex contact metric structure (G, ξ, η, g) on (M, η).

Proof

For any $X, X' \in \Gamma L^c$, we have

$$-(\mathcal{L}_X\mathcal{L}_{X'}u)(U) = -\{X(X'u(U) - u([X', U])) - X'u([X, U]) + u([X', [X, U]])\}$$
$$= -u([X', [X,U]]) \qquad \text{by 8.2 and 8.7 (i)}$$
$$= 2du(X', [X, U]) \qquad \text{by 8.7 (i) as } X, X' \in \Gamma L^c$$
$$= 2g(X', G[X, U]) + g(\nabla_{X'} U, V)v([X, U]) \qquad \text{by 8.3 (ii) as } X' \in \Gamma L^c$$
$$= 2g(X', G[X, U]) - u([X', V])v([X, U]) \qquad \text{by 8.7 (ii).}$$

Hence for any $X, X' \in \Gamma L^c$ we have

$$\Pi(X, X') = -((\mathcal{L}_X\mathcal{L}_{X'}u)(U)) + u([X', V])v([X, U])$$
$$= 2g(X', G[X, U])$$
$$= 2g([U, X], GX') \qquad \text{by 8.4 (vi)}$$
$$= 2g(J[V, X], GX') \qquad \text{by 8.12 (ii) as } GX' \in \Gamma Q^c$$
$$= 2g([V, X], GJX') \qquad \text{by 8.3 (iii) since g is Hermitian}$$
$$= 2g([V, X], KX'). \qquad \square$$

Furthermore for any $X, X' \in \Gamma L^c$ we have

$$-(\mathcal{L}_X\mathcal{L}_{X'}v)(V) = -\{X(X'v(V) - v([X', V])) - X'v([X, V]) + v([X', [X, V]])\}$$
$$= -v([X', [X, V]]) \qquad \text{by 8.2 and 8.7 (ii)}$$

$$= 2dv(X', [X, V]) \qquad \text{by 8.7 (ii) as } X, X' \in \Gamma L^c$$
$$= 2g(X', K[X, V]) + g(\nabla_{X'} V, U)u([X, V])$$
$$\text{by 8.5 (ii) as } X' \in \Gamma L^c$$
$$= 2g(X', K[X, V]) - v([X', U])u([X, V]) \qquad \text{by 8.7 (iv).}$$

Thus for any $X, X' \in \Gamma L^c$ we have

$$- (\mathcal{L}_X \mathcal{L}_{X'} v)(V) + v([X', U])u([X, V])$$
$$= 2g(X', K[X, V])$$
$$= 2g([V, X], KX') \qquad \text{by 8.6 (vi)}$$
$$= \Pi(X, X') \qquad \text{by lemma 8.15.}$$

Therefore Π can be equivalently defined by

$$\Pi(X, X') = - (\mathcal{L}_X \mathcal{L}_{X'} v)(V) - g(\nabla_{X'} V, U)u([X,V])$$
$$\forall\ X, X' \in \Gamma L^c.$$

We now use Π to classify complex Legendre foliations.

Definition 8.16

Let $\mathcal{F}$ be a complex Legendre foliation on the complex contact metric manifold (M, G, ξ, η, g) then

(i) $\mathcal{F}$ is a flat Legendre foliation if $\Pi \equiv 0$.

(ii) $\mathcal{F}$ is a non-degenerate (degenerate) Legendre foliation if Π is non-degenerate (degenerate).

Note that $\prod$ can not be positive definite as for any $X, X' \in \Gamma L^c$

$$\begin{aligned} \prod(X, X') &= 2g([U, X], GX') && \text{by lemma 8.15} \\ &= 2g(J[U,X], JGX') && \text{since g is Hermitian} \\ &= -2g([U, JX], GJX') && \text{by 8.3 (iii) and theorem 8.9} \\ &= -\prod(JX, JX') && \text{by lemma 8.15.} \end{aligned}$$

This shows that $\prod$ cannot be regarded as a Hermitian metric on L^c so $\prod$ cannot be used to define a canonical complex metric structure.

Lemma 8.17

Let $\mathcal{F}$ be a complex Legendre foliation on the complex contact metric manifold (M, G, ξ, η, g) then

(i) $\mathcal{F}$ is a flat complex Legendre foliation

if and only if $p_{Q^c}([U, X]) = 0 \quad \forall X \in \Gamma L^c$,

if and only if $p_{Q^c}([V, X]) = 0 \quad \forall X \in \Gamma L^c$.

(ii) $\mathcal{F}$ is a non-degenerate complex Legendre foliation

if and only if $p_{Q^c}([U, X]) \neq 0 \quad \forall X \in \Gamma L^c$,

if and only if $p_{Q^c}([V, X]) \neq 0 \quad \forall X \in \Gamma L^c$.

These follow readily using Lemma 8.15.

Theorem 8.18

Let $\mathcal{F}$ be a complex Legendre foliation on the complex contact metric manifold (M, G, ξ, η, g) then

$$\mathcal{L}_X g(Y, Z) \neq 0 \qquad \text{for some } X \in \Gamma L^c, Y \in \Gamma Q^c, Z \in \Gamma E^c.$$

Proof

Let $\{X_\alpha, JX_\alpha, GX_\alpha, GJX_\alpha, U, V\}$ be a local orthonormal frame for T^cM with respect to g.

Case I: $\mathcal{F}$ is degenerate

By lemma 8.17 (ii) there exists an α such that the projection

$$p_{\mathcal{Q}^c}([X_\alpha, U]) = 0$$

For this α we have

$$\begin{aligned} \mathcal{L}_{X_\alpha} g(GX_\alpha, U) &= - g([X_\alpha, GX_\alpha], U) - g([X_\alpha, U], GX_\alpha) \\ &= - 2 \neq 0 \qquad \text{by lemma 8.14 (i).} \end{aligned}$$

Case II: $\mathcal{F}$ is non-degenerate

Suppose

$$\mathcal{L}_X g(Y, Z) = 0 \qquad \forall\, X \in \Gamma L^c, \forall\, Y \in \Gamma Q^c, \forall\, Z \in \Gamma E^c.$$

Then for any α, β we have

$$\begin{aligned} 0 &= \mathcal{L}_{X_\alpha} g(GX_\beta, U) \\ &= - g([X_\alpha, GX_\beta], U) - g([X_\alpha, U], GX_\beta) \\ &= - 2\delta_{\alpha\beta} - g([X_\alpha, U], GX_\beta) \qquad \text{by lemma 8.14 (i).} \end{aligned}$$

Thus for any α, β we have

(a) $$g([X_\alpha, U], GX_\beta) = - 2\delta_{\alpha\beta}.$$

For any α we also have

$$\begin{aligned} 0 &= \mathcal{L}_{JX_\alpha} g(KX_\beta, U) \\ &= - g([JX_\alpha, KX_\beta], U) - g([JX_\alpha, U], KX_\beta) \\ &= - 2\delta_{\alpha\beta} - g([JX_\alpha, U], KX_\beta) \qquad \text{by lemma 8.14(i).} \end{aligned}$$

Hence for any α, β we have

$$g([JX_\alpha, U], KX_\beta) = -2\delta_{\alpha\beta}.$$

But also for any α, β we have that

$$\begin{aligned} g([JX_\alpha, U], KX_\beta) &= g(J[X_\alpha, U], GJX_\beta) && \text{by theorem 8.9} \\ &= -\, g([X_\alpha, U], GX_\beta) && \text{by 8.3 (iii) since g is Hermitian} \\ &= 2\delta_{\alpha\beta} && \text{by (a),} \end{aligned}$$

which is a contradiction. ¤

The following corollary is a direct result of theorem 8.18.

Corollary 8.19

For a complex Legendre foliation $\mathcal{F}$ on the complex contact manifold (M, η) there is no complex contact metric structure (M, G, ξ, η, g) such that g is bundle-like. That is semi-Riemannian complex Legendre foliations do not exist.

8 (d). CONJUGATE COMPLEX LEGENDRE FOLIATIONS

On a complex contact metric manifold (M, G, ξ, η, g) a subbundle, L^c, of the complexified tangent bundle, T^cM, defines a complex Legendre foliation if,

(i) L^c is completely integrable.

(ii) L^c is of real dimension 2n.

(iii) L^c is a subbundle of the complexified contact distribution H^c.

(iv) L^c satisfies the holomorphic condition, that is

$$g((\mathcal{L}_X J)\, W, W') = 0 \qquad \forall\, W, W' \in \Gamma L^{c\perp},\ \forall\, X \in \Gamma L^c.$$

For a complex Legendre foliation, $\mathcal{F}$, on a complex contact metric manifold (M, G, ξ, η, g), as $G|_{H^c}$ is an isomorphism and $GU = GV = 0$, the subbundle $Q^c = GL^c = \{GX \mid X \in L^c\}$ is a real $2n$-dimensional subbundle of H^c. Suppose Q^c is completely integrable then Q^c defines a foliation whose leaves are real $2n$-dimensional integral submanifolds of η. But this foliation will only be a complex Legendre foliation if Q^c also satisfies the holomorphic condition. So we get the following definition of the conjugate complex Legendre foliation.

Definition 8.20
Given a complex Legendre foliation $\mathcal{F}$ on a complex contact metric manifold (M, G, ξ, η, g). If Q^c defines a complex Legendre foliation $\overline{\mathcal{F}}$ on (M, G, ξ, η, g) then $\overline{\mathcal{F}}$ is the *conjugate complex Legendre foliation* of $\mathcal{F}$.

Lemma 8.21
If $\mathcal{F}$ is a complex Legendre foliation on a complex contact metric manifold (M, G, ξ, η, g) then the conjugate complex Legendre foliation $\overline{\mathcal{F}}$ exists if and only if

i) Q^c is completely integrable.

ii) $g((\mathcal{L}_Y J)W, W') = 0 \qquad \forall\, W, W' \in \Gamma(L^c \oplus E^c),\ \forall\, Y \in \Gamma Q^c.$

Note Q^c could also be defined as $Q^c = KL^c = \{KX \mid X \in L^c\}$.

We give an example of a non-degenerate complex Legendre foliation whose conjugate exists and is flat.

Example 8.1 : Non-degenerate complex Legendre foliation

Let

$$M = \mathbf{C}^3 - \{(z_1, z_2, z_3) \mid \cos z_3 \cos \bar{z}_3 + \sin z_3 \sin \bar{z}_3 = 0\}\}$$
$$= \{(z_1, z_2, z_3) \mid z_i \in \mathbf{C},\ \cos z_3 \cos \bar{z}_3 + \sin z_3 \sin \bar{z}_3 \neq 0\}$$

and

$$\eta = \frac{1}{2}(\cos z_3 dz_1 + \sin z_3 dz_2).$$

Then we have

$$d\eta = \frac{1}{2}(-\sin z_3 dz_3 \wedge dz_1 + \cos z_3 dz_3 \wedge dz_2)$$

which by calculation gives

$$\eta \wedge d\eta = -\frac{1}{4}(dz_1 \wedge dz_2 \wedge dz_3)$$
$$\neq 0 \qquad \text{everywhere on M.}$$

Hence (M, η) is a complex contact manifold, with complex characteristic vector field given by

$$\xi = 2\left(\cos z_3 \frac{\partial}{\partial z_1} + \sin z_3 \frac{\partial}{\partial z_2}\right)$$

Thus

$$U = 2\left(\cos z_3 \frac{\partial}{\partial z_1} + \sin z_3 \frac{\partial}{\partial z_2} + \cos \bar{z}_3 \frac{\partial}{\partial \bar{z}_1} + \sin \bar{z}_3 \frac{\partial}{\partial \bar{z}_2}\right)$$

$$V = -2i\left(\cos z_3 \frac{\partial}{\partial z_1} + \sin z_3 \frac{\partial}{\partial z_2} - \cos \bar{z}_3 \frac{\partial}{\partial \bar{z}_1} - \sin \bar{z}_3 \frac{\partial}{\partial \bar{z}_2}\right)$$

Let (G, ξ, η, g) be the complex contact metric structure of (M, η) given by

$$g = \frac{1}{8\mu}\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \sin z_3 \\ 0 & 0 & 0 & 0 & 0 & -\cos z_3 \\ 0 & 0 & 0 & -\sin \bar{z}_3 & \cos \bar{z}_3 & 0 \\ 0 & 0 & \sin \bar{z}_3 & 0 & 0 & 0 \\ 0 & 0 & -\cos \bar{z}_3 & 0 & 0 & 0 \\ -\sin z_3 & \cos z_3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Where $\mu = \cos z_3 \cos \bar{z}_3 + \sin z_3 \sin \bar{z}_3$.

Consider the vector fields

$$X = 2\left(\frac{\partial}{\partial z_3} + \frac{\partial}{\partial \bar{z}_3}\right)$$

$$JX = 2i\left(\frac{\partial}{\partial z_3} - \frac{\partial}{\partial \bar{z}_3}\right).$$

Then we obtain the vector fields

$$Y = GX$$

$$= 2\left(\sin z_3 \frac{\partial}{\partial z_1} - \cos z_3 \frac{\partial}{\partial z_2} + \sin \bar{z}_3 \frac{\partial}{\partial \bar{z}_1} - \cos \bar{z}_3 \frac{\partial}{\partial \bar{z}_2}\right)$$

$$JY = -GJX$$

$$= 2i\left(\sin z_3 \frac{\partial}{\partial z_1} - \cos z_3 \frac{\partial}{\partial z_2} - \sin \bar{z}_3 \frac{\partial}{\partial \bar{z}_1} + \cos \bar{z}_3 \frac{\partial}{\partial \bar{z}_2}\right).$$

It can be shown that {X, JX, Y, JY} is a orthonormal frame of H^c with respect to g.

By calculation we have

GU = 0	KU =0
GV = 0	KV =0
GX = Y	KX = - JY
GJX = - JY	KJX = - Y
GY= - X	KY = JX
GJY = JX	KJY = X.

Calculating the Lie brackets we get

[X, U] = - 2Y	[X, V] = 2JY
[JX, U] = - 2JY	[JX, V] = - 2Y
[Y, U] = 0	[Y, V] = 0
[JY, U] = 0	[JY, V] = 0
[X, Y] = 2U	[X, JY] = - 2V
[JX, Y] = - 2V	[JX, JY] = - 2U
[X, JX] = 0	[Y, JY] = 0
[U, V] = 0.	

It is clear that [W, W'] = 0 for all W, W' $\in$ span{X, JX} so this is a completely integrable submanifold of real dimension 2. Also X and JX belong to H^c and the holomorphic condition 8.11 (ii) can be verified from the Lie brackets. Thus L^c = span{X, JX} defines a complex Legendre foliation, $\mathcal{F}$, with Q^c = span{Y, JY}. Similarly Q^c defines $\overline{\mathcal{F}}$ the conjugate complex Legendre foliation of $\mathcal{F}$.

From the Lie brackets it can be seen that $\mathcal{F}$ is a non-degenerate complex Legendre foliation while $\overline{\mathcal{F}}$ is a flat complex Legendre foliation.

8 (e). KAHLER CONTACT MANIFOLDS

A Hermitian manifold (M, J, g) is a Kahler manifold if and only if the Levi-Civita connection of g satisfies $\nabla J = 0$.

By theorem 3.2 page 128 Yano and Kon [16] the condition $\nabla J = 0$ is equivalent to the almost complex structure having no torsion and the fundamental 2-form Φ being closed, that is, $N = 0$, and $d\Phi = 0$.

Suppose $\mathcal{F}$ is a complex Legendre foliation on a complex contact metric manifold (M, G, ξ, η, g) such that the Hermitian manifold (M, J, g) is Kahlerian, then the leaves of $\mathcal{F}$ are Kahlerian submanifolds of real dimension 2n. Yano and Kon [16] proposition 1.1 page 181, gives the following theorem

Theorem 8.22
If $\mathcal{F}$ is a complex Legendre foliation on a complex contact metric manifold (M, G, ξ, η, g) such that the Hermitian manifold (M, J, g) is Kahlerian, then $\mathcal{F}$ is harmonic.

From the work of Vaisman [31] we get the following theorem when a complex contact metric manifold (M, G, ξ, η, g) is also a Kahler manifold.

Lemma 8.23
Let $\mathcal{F}$ be a complex Legendre foliation on the complex contact metric manifold (M, G, ξ, η, g) such that (M, J, g) is a Kahler manifold.Then

(i) $\mathcal{L}_X g(W, JW') = g([W, W'], JX) \qquad \forall\ W, W' \in \Gamma L^{c\perp}, \forall\ X \in \Gamma L^{c}.$

Furthermore if $\overline{\mathcal{F}}$ the conjugate complex Legendre foliation of $\mathcal{F}$ exists then

(ii) $$\mathcal{L}_Y g(W, JW') = g([W, W'], JY)$$

$$\forall\ W, W' \in \Gamma(L^c \oplus E^c),\ \forall\ Y \in \Gamma Q^c.$$

Proof

Let Φ be the fundamental 2-form of (M, J, g) which is defined by

$$\Phi(W, W') = g(W, JW') \qquad \forall\ W, W' \in \Gamma T^c M.$$

Note that for a Kahler manifold the fundamental 2-form is closed, that is $d\Phi = 0$.

(i) As (M, J, g) is a Kahler manifold for any $X \in \Gamma L^c$ and any $W, W' \in \Gamma L^{c\perp}$ we have

$$\begin{aligned} 0 &= 3d\Phi(W, W', X) \\ &= Wg(W', JX) + W'g(X, JW) + Xg(W, JW') - g([W, W'], JX) \\ &\qquad - g([W', X], JW) - g([X, W], JW') \\ &= Xg(W, JW') - g([X, W], JW') - g(J[X, W'], W) \\ &\qquad - g([W, W'], JX) \\ &= Xg(W, JW') - g([X, W], JW') - g([X, JW'], W) \\ &\qquad - g([W, W'], JX) \\ &\qquad \text{by 8.11 (ii)} \\ &= \mathcal{L}_X g(W, JW') - g([W, W'], JX), \end{aligned}$$

which implies that

$$\mathcal{L}_X g(W, JW') = g([W, W'], JX) \qquad \forall\ W, W' \in \Gamma L^{c\perp},\ \forall\ X \in \Gamma L^c.$$

(ii) Suppose $\overline{\mathcal{F}}$ the conjugate complex Legendre foliation of $\mathcal{F}$ exists then as (M, J, g) is a Kahler manifold for any $W, W' \in \Gamma(L^c \oplus E^c)$ and for any

$Y \in \Gamma Q^c$ we have $0 = 3d\Phi(W, W', Y)$. Hence similar to the proof of (i) using lemma 8.21 (ii) we obtain

$$\mathcal{L}_Y g(W, JW') = g([W, W'], JY)$$

$$\forall\ W, W' \in \Gamma(L^c \oplus E^c),\ \forall\ Y \in \Gamma Q^c.$$

□

Theorem 8.24

Let $\mathcal{F}$ be a complex Legendre foliation on a complex contact metric manifold (M, G, ξ, η, g) such that (M, J, g) is a Kahler manifold and $\overline{\mathcal{F}}$ the conjugate complex Legendre foliation of $\mathcal{F}$ exists then

(i) $\mathcal{F}$ is a non-degenerate complex Legendre foliation.

(ii) $\overline{\mathcal{F}}$ is a non-degenerate complex Legendre foliation.

Proof

Let $\{X_\alpha, JX_\alpha, GX_\alpha, GJX_\alpha, U, V\}$ be a local orthonormal frame for T^cM with respect to g.

(i) Suppose $\mathcal{F}$ is degenerate then by lemma 8.17 (ii) there exists an α such that

$$p_{Q^c}([X_\alpha, U]) = 0,$$

and using theorem 8.9 as $J: Q^c \to Q^c$ this implies that

$$p_{Q^c}(J[X_\alpha, U]) = p_{Q^c}([JX_\alpha, U]) = 0.$$

For this α, and any β we have

$$\begin{aligned} 0 &= g([X_\alpha, U], GX_\beta) \\ &= -\, g([X_\alpha, U], J(JGX_\beta)) && \text{since } J^2 = -I \\ &= \mathcal{L}_{GJX_\beta} g(X_\alpha, JU) && \text{by 8.3 (iii) and lemma 8.23 (ii)} \end{aligned}$$

$$= - g([GJX_\beta, X_\alpha], JU) - g([GJX_\beta, JU], X_\alpha)$$
$$= - 2\delta_{\alpha\beta} - g([GJX_\beta, JU], X_\alpha) \qquad \text{by 8.1 (i) and lemma 8.14 (ii)}$$
$$= - 2\delta_{\alpha\beta} + g([GJX_\beta, U], JX_\alpha)$$
$$\text{by lemma 8.21 (ii) as g is Hermitian,}$$

which implies that

(a) $\quad g([GJX_\beta, U], JX_\alpha) = 2\delta_{\alpha\beta}.$

Furthermore for this α, and any β we have

$$0 = g([JX_\alpha, U], GJX_\beta)$$
$$= - g([JX_\alpha, U], JGX_\beta) \qquad \text{by 8.3 (iii)}$$
$$= - \mathcal{L}_{GX_\beta} g(JX_\alpha, JU) \qquad \text{by lemma 8.23 (ii)}$$
$$= g([GX_\beta, JX_\alpha], JU) + g([GX_\beta, JU], JX_\alpha)$$
$$= - 2\delta_{\alpha\beta} + g([GX_\beta, JU], JX_\alpha) \qquad \text{by 8.1 (i) and lemma 8.14 (ii)}$$
$$= - 2\delta_{\alpha\beta} + g([GX_\beta, U], X_\alpha) \qquad \text{by lemma 8.21 (ii) since } J^2 = - I,$$

which implies that

(b) $\quad g([GX_\beta, U], X_\alpha) = 2\delta_{\alpha\beta}.$

But we also have

$$g([GJX_\beta, U], JX_\alpha) = g(J[JGX_\beta, U], X_\alpha)$$
$$\text{by 8.3 (iii) since g is Hermitian}$$
$$= - g([GX_\beta, U], X_\alpha) \qquad \text{by theorem 8.9}$$
$$= - 2\delta_{\alpha\beta} \qquad \text{by (b),}$$

which contradicts (a) hence $\mathcal{F}$ can not be degenerate.

(ii) Similar to (i). ¤

From theorem 8.24 we get the following corollary.

Corollary 8.25
Let $\mathcal{F}$ be a complex Legendre foliation on a complex contact metric manifold (M, G, ξ, η, g) such that $\overline{\mathcal{F}}$ the conjugate complex Legendre foliation of $\mathcal{F}$ exists. If $\mathcal{F}$ or $\overline{\mathcal{F}}$ is a degenerate complex Legendre foliation then (M, J, g) is not a Kahler manifold.

This corollary is illustrated by

Example 8.1: $\mathcal{F}$ is a non-degenerate complex Legendre foliation on the complex contact metric manifold (M, G, ξ, η, g). By calculation we obtain

$$
\begin{aligned}
&3d\Phi(JY, V, JX) \\
&\quad = JY\Phi(V, JX) + V\Phi(JX, JY) + JX\Phi(JY, V) \\
&\qquad - \Phi([JY, V], JX) - \Phi([V, JX], JY) - \Phi([JX, JY], V) \\
&\quad = g([JY, V], X) + g([V, JX], Y) - g([JX, JY], JV) \\
&\quad = 2g(Y, Y) + 2g(U, U) \\
&\quad = 4 \neq 0.
\end{aligned}
$$

This implies that (M, J, g) is not a Kahler manifold, which is consistent with corollary 8.25 as $\overline{\mathcal{F}}$ the conjugate of $\mathcal{F}$ exists but is a flat complex Legendre foliation.

8 (f). COMPLEX K-CONTACT MANIFOLDS

Definition 8.26
A complex contact metric manifold (M, G, ξ, η, g) is a complex K-contact manifold if ξ is a Killing vector field with respect to g.

Theorem 8.27

A complex contact metric manifold (M, G, ξ, η, g) is a complex K-contact manifold if and only if U and V are Killing vector fields with respect to g.

Proof

As $\xi = \frac{1}{2}(U + iV)$ we have,

$$\begin{aligned} \mathcal{L}_{2\xi} g &= \mathcal{L}_{U + iV} g \\ &= \mathcal{L}_U g + i\mathcal{L}_V g. \end{aligned}$$

Thus

$$\mathcal{L}_\xi g = 0 \qquad \text{if and only if} \qquad \mathcal{L}_U g = 0 \qquad \text{and} \qquad \mathcal{L}_V g = 0. \qquad \square$$

Lemma 8.28

For a complex K-contact manifold (M, G, ξ, η, g)

(i) $g(W, GW') = du(W, W')$ $\quad \forall\ W, W' \in \Gamma T^c M,$

$g(W, HW') = dv(W, W')$ $\quad \forall\ W, W' \in \Gamma T^c M.$

(ii) $\nabla_W U = - GW$ $\quad \forall\ W \in \Gamma T^c M,$

$\nabla_W V = - KW$ $\quad \forall\ W \in \Gamma T^c M.$

iii) $\mathcal{L}_U G = \mathcal{L}_V K = 0.$

Proof

(i) As U is a Killing vector field for any $W \in \Gamma T^cM$ we have

$$\begin{aligned} 0 &= \mathcal{L}_U g(V, W) \\ &= g(\nabla_V U, W) + g(V, \nabla_W U) \\ &= g(V, \nabla_W U) \qquad \text{by 8.10 (i).} \end{aligned}$$

Thus by 8.3 (ii) and 8.5 (ii) we have

$$g(W, GW') = du(W, W') \qquad \forall\ W, W' \in \Gamma T^cM$$

and

$$g(W, KW') = dv(W, W') \qquad \forall\ W, W' \in \Gamma T^cM.$$

(ii) From part (i) for any $W, W' \in \Gamma T^cM$ we have

$$\begin{aligned} 2g(W, GW') &= 2du(W, W') \\ &= Wu(W') - W'u(W) - u([W, W']) \\ &= Wg(U, W') - W'g(U, W) - g(U, [W, W']) \\ &\qquad \text{by 8.4 (v)} \\ &= g(\nabla_W U, W') - g(\nabla_{W'} U, W) \\ &= -2g(W, \nabla_{W'} U) \qquad \text{as U is a Killing vector field.} \end{aligned}$$

Hence

$$\nabla_W U = -GW \qquad \forall\ W \in \Gamma T^cM$$

and similarly

$$\nabla_W V = -KW \qquad \forall\ W \in \Gamma T^cM.$$

(iii) For any $W \in \Gamma T^cM$ we have

$$\begin{aligned} 0 &= 2g(W, GU) && \text{by 8.4 (i)} \\ &= 2du(W, U) && \text{by part (i)} \\ &= Wu(U) - Uu(W) - u([W, U]) \\ &= - \{Uu(W) - u([U, W])\} \\ &= - \{(\mathcal{L}_U u)(W)\}. \end{aligned}$$

Thus

$$\mathcal{L}_U u = 0.$$

Furthermore since for any $W \in \Gamma T^cM$

$$\begin{aligned} du(W, U) &= g(W, GU) && \text{by part (i)} \\ &= 0 && \text{by 8.4 (i),} \end{aligned}$$

we have

$$\begin{aligned} \mathcal{L}_U du &= i(U)d(du) + d(i(U)du) \\ &= 0. \end{aligned}$$

Thus for any $W, W' \in \Gamma T^cM$ we obtain

$$\begin{aligned} 0 &= (\mathcal{L}_U du)(W, W') \\ &= Udu(W, W') - du([U, W], W') - du(W, [U, W']) \\ &= Ug(W, GW') - g([U, W], GW') - g(W, G[U, W']) && \text{by part (i)} \\ &= \mathcal{L}_U g(W, GW') + g(W, (\mathcal{L}_U G)W') \\ &= g(W, (\mathcal{L}_U G)W') && \text{as U is a Killing vector field.} \end{aligned}$$

Hence

$$\mathcal{L}_U G = 0,$$

and similarly

$$\mathcal{L}_V K = 0.$$ □

Theorem 8.29

Let $\mathcal{F}$ be a complex Legendre foliation on a complex K-contact manifold (M, G, ξ, η, g) such that the conjugate complex Legendre foliation of $\mathcal{F}$ exists, then the conjugate is of the same classification as $\mathcal{F}$.

Proof

Let $\{X_\alpha, JX_\alpha, GX_\alpha, GJX_\alpha, U, V\}$ be a local orthonormal frame for T^cM with respect to g.

Then from lemma 8.27 (iii) for any α we have

$$[U, GX_\alpha] = G[U, X_\alpha], \qquad [U, GJX_\alpha] = G[U, JX_\alpha]$$

and

$$[V, KX_\alpha] = K[V, X_\alpha], \qquad [V, KJX_\alpha] = K[V, JX_\alpha].$$

Thus theorem follows by lemma 8.17. □

Example 8.2 : Flat complex Legendre foliation on a complex K-contact manifold

Let

$$M = \mathbf{C}^3$$
$$= \{(z_1, z_2, z_3) \mid z_i \in \mathbf{C}\}$$

and

$$\eta = \frac{1}{2}(dz_1 + z_2 dz_3).$$

Then we have

$$d\eta = \frac{1}{2}(dz_2 \wedge dz_3)$$

which by calculation gives

$$\eta \wedge d\eta = \frac{1}{4}(dz_1 \wedge dz_2 \wedge dz_3)$$
$$\neq 0 \qquad \text{everywhere on M.}$$

Hence (M, η) is a complex contact manifold, with the complex characteristic vector field given by

$$\xi = 2\left(\frac{\partial}{\partial z_1}\right)$$

Thus we obtain the global vector fields

$$U = 2\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial \bar{z}_1}\right)$$

$$V = -2i\left(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial \bar{z}_1}\right)$$

Let (G, ξ, η, g) be the complex contact metric structure on (M, η) given by

$$g = \frac{1}{8}\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & \bar{z}_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z_2 & 0 & 1+z_2\bar{z}_2 \\ 1 & 0 & z_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \bar{z}_2 & 0 & 1+z_2\bar{z}_2 & 0 & 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & \bar{z}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Consider the vector fields

$$X = 2\left(\frac{\partial}{\partial z_2} + \frac{\partial}{\partial \bar{z}_2}\right)$$

$$JX = 2i\left(\frac{\partial}{\partial z_2} - \frac{\partial}{\partial \bar{z}_2}\right).$$

Then we obtain the vector fields

$$\begin{aligned} Y &= GX \\ &= 2\left(z_2\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_3} + \bar{z}_2\frac{\partial}{\partial \bar{z}_1} - \frac{\partial}{\partial \bar{z}_3}\right) \end{aligned}$$

$$\begin{aligned} JY &= JGX \\ &= 2i\left(z_2\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_3} - \bar{z}_2\frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial \bar{z}_3}\right). \end{aligned}$$

Note that X, JX, Y, JY, are global vector fields and that {X, JX, Y, JY} is an orthonormal frame of H^c with respect to g.

By calculation we have

$$GU = 0 \qquad KU = 0$$
$$GV = 0 \qquad KV = 0$$
$$GX = Y \qquad KX = -JY$$

$GJX = - JY$ $\qquad$ $KJX = - Y$

$GY = - X$ $\qquad$ $KY = JX$

$GJY = JX$ $\qquad$ $KJY = X$

Calculating the Lie brackets we get

$[X, U] = 0$ $\qquad$ $[X, V] = 0$

$[JX, U] = 0$ $\qquad$ $[JX, V] = 0$

$[Y, U] = 0$ $\qquad$ $[Y, V] = 0$

$[JY, U] = 0$ $\qquad$ $[JY, V] = 0$

$[X, Y] = 2U$ $\qquad$ $[X, JY] = - 2V$

$[JX, Y] = - 2V$ $\qquad$ $[JX, JY] = - 2U$

$[X, JX] = 0$ $\qquad$ $[Y, JY] = 0$

$[U, V] = 0.$

As

$[W, U] = 0$ $\qquad$ $\forall\ W \in \Gamma T^cM$ such that $g(W, W) = 1.$

We have

$$\mathcal{L}_U g(W, W') = 0$$

$\forall\ W, W' \in \Gamma T^cM$ such that $g(W, W) = g(W', W') = 1.$

This implies that U is a Killing vector field with respect to g. Similarly the Lie brackets show that V is a Killing vector field with respect to g. Hence by theorem 8.26 (M, G, ξ, η, g) is a complex K-contact manifold.

It is clear that $[W, W'] = 0$ for all $W, W' \in \text{span}\{X, JX\}$ so this is a completely integrable submanifold of real dimension 2. Also X and JX belong to H^c and the

holomorphic condition 8.11 (ii) can be verified from the Lie brackets. Thus $L^c = \text{span}\{X, JX\}$ defines a complex Legendre foliation, $\mathcal{F}$, with $Q^c = \text{span}\{Y, JY\}$. Similarly Q^c defines $\overline{\mathcal{F}}$ the conjugate complex Legendre foliation of $\mathcal{F}$.

From the Lie brackets it can be seen that $\mathcal{F}$ and $\overline{\mathcal{F}}$ are flat complex Legendre foliations.

Also by calculation we obtain

$$
\begin{aligned}
3d\Phi(JY, V, JX) &= JY\Phi(V, JX) + V\Phi(JX, JY) + JX\Phi(JY, V) \\
&\quad - \Phi([JY, V], JX) - \Phi([V, JX], JY) - \Phi([JX, JY], V) \\
&= g([JY, V], X) + g([V, JX], Y) - g([JX, JY], JV) \\
&= 2g(U, U) \\
&= 2 \neq 0.
\end{aligned}
$$

This implies that (M, J, g) is not a Kahler manifold which is consistent with corollary 8.25 as $\mathcal{F}$ and $\overline{\mathcal{F}}$ are flat complex Legendre foliations.

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