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On Essential Self-adjointness, Confining Potentials & the L_p-Hardy Inequality

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Abstract

Let Ω be a domain in $\mathbb{R}^m$ with non-empty boundary and let $H = -\Delta + V$ be a Schrödinger operator defined on $C_0^\infty(\Omega)$ where $V \in L_{\infty,\mathrm{loc}}(\Omega)$. We seek the minimal criteria on the potential V that ensures that H is essentially self-adjoint, i.e. that ensures the closed operator $\bar{H}$ is self-adjoint. Overcoming various technical problems, we extend the results of Nenciu & Nenciu in [1] to more general types of domain, specifically unbounded domains and domains whose boundaries are fractal. As a special case of an abstract condition we show that H is essentially self-adjoint provided that sufficiently close to the boundary

$$V(x) \;\geq\; \frac{1}{d(x)^2}\left[1 - \mu_2(\Omega) - \frac{1}{\ln(\,d(x)^{-1})} - \frac{1}{\ln(\,d(x)^{-1})\,\ln\ln(\,d(x)^{-1})} - \cdots\right], \quad (1)$$

where $d(x) = \mathrm{dist}(x, \partial\Omega)$ and the right hand side of the above inequality contains a finite number of logarithmic terms. The constant $\mu_2(\Omega)$ appearing in (1) is the variational constant associated with the L_2-Hardy inequality and is non-zero if and only if Ω admits the aforementioned inequality. Our results indicate that the existence of an L_2-Hardy inequality, and the specific value of $\mu_2(\Omega)$, depend intimately on the (Hausdorff / Aikawa) dimension of the boundary. In certain cases where Ω is geometrically simple, this constant, as well as the constant '1' appearing in front of each logarithmic term, is shown to be optimal with regards to the essential self-adjointness of H.

Foreword & Acknowledgements

I once read the foreword to someone's PhD thesis that could be paraphrased as follows:

> "From the moment I arrived at the university I knew that I was in the right place. I realized almost immediately that the area of mathematics I was researching was a fruitful one and that I could make a significant contribution to it. Although I felt challenged, I also felt confident that my abilities would allow me to succeed. Pretty soon I began to feel at home amongst the doctors and established professors at the institute..."

For the benefit of any (prospective) PhD student that may read this foreword, I would like to stress that this was **not** my experience. I remember my first meeting with my supervisors Professor Gaven Martin and Professor Boris Pavlov. They informed me that my sole task for the first year of the PhD was to read as much material as possible so that "in twelve months time we can have a meaningful conversation". At the time I remember thinking that they had underestimated me. Now I realize that they had significantly over estimated me. It took two years before I could have a meaningful conversation with them or ask them a question that they did not immediately know the answer to. At times during those first two years it was horrible. I felt like a fraud, completely out of my depth and uncertain as to whether I had the ability to succeed. Had it not been for a combination of bizarre personal circumstance, convoluted rules and regulations concerning my scholarship and sheer geographical distance, I probably would have returned home to Europe. Then after those two years had passed, slowly, things started to come together. In the end nothing of worth comes without struggle.

Looking back at my time in New Zealand, I realize that it is necessary to thank various people. First and foremost, my thanks go to my family - Mum & Dad, Margret & Chris, Vicky & Dave and Tina. Without your constant love, support and sacrifices over the years none of this would have been possible.

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