DYNAMICS AND NUMERICS OF GENERALISED EULER EQUATIONS

by

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Abstract

This thesis is concerned with the well-posedness, dynamical properties and numerical treatment of the generalised Euler equations on the Bott-Virasoro group with respect to the general H^k metric , $k \geq 2$.

The term "generalised Euler equations" is used to describe geodesic equations on Lie groups, which unifies many differential equations and has found many applications in such as hydrodynamics, medical imaging in the computational anatomy, and many other fields. The generalised Euler equations on the Bott-Virasoro group for $k = 0, 1$ are well-known and intensively studied—the Korteweg-de Vries equation for $k = 0$ and the Camassa-Holm equation for $k = 1$. Unlike these, the equations for $k \geq 2$, which we call the modified Camassa-Holm (mCH) equation, is not known to be integrable. This distinction motivates the study of the mCH equation.

In this thesis, we derive the mCH equation and establish the short time existence of solutions, the well-posedness of the mCH equation, long time existence, the existence of the weak solutions, both on the circle S and $\mathbb{R}$, and three conservation laws, show some quite interesting properties, for example, they do not lead to the blowup in finite time, unlike the Camassa-Holm equation.

We then consider two numerical methods for the modified Camassa-Holm equation: the particle method and the box scheme. We prove the convergence result of the particle method. The numerical simulations indicate another interesting phenomenon: although mCH does not admit blowup in finite time, it admits solutions that *blow up* (which means their maximum value becomes infinity) at *infinite time*, which we call *weak blowup*. We study this novel phenomenon using the method of matched asymptotic expansion. A whole family of self-consistent blowup profiles is obtained. We propose a mechanism by which the actual profile is selected that is consistent with the simulations, but the mechanism is only partly supported by the analysis.

We study the four particle systems for the mCH equation finding numerical evidence both for the non-integrability of the mCH equations and for the existence of the fourth integral. We also study the higher dimensional case and obtain the short time existence and well-posedness for the generalised Euler equation in the two dimension case.

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Xingyou Philip Zhang, July 11, 2008

Contents

List of Spaces

Here is the list of various spaces in Chapter 1 − Chapter 7.

The notation	Its meaning
$B(X,Y)$	The space of all bounded linear operators from X to Y
$C([a,b],X)$	The set of all continuous functions from $[a,b]$ to X
$C^\alpha(\Omega)$	Hölder spaces defined in Section 2.1
$C^{1,c}(\Omega)$	The set of all continuous functions with compact supports in Ω
	and continuous first order derivatives in Ω
$\mathrm{Diff}(S)$	The set of all diffeomorphisms from S to S preserving the orientation
$\mathcal{D}^s(S)$	The set of all H^s diffeomorphisms on S
$\widehat{\mathcal{D}}(S)$	Bott-Virasoro group defined in Section 3.1
$G(X,1,\beta)$	The set of quasi-m-accretive operators in X defined in Section 2.1.2
$GL(V)$	The set of all invertible linear operators from V to V
$H^s(\mathbb{R}^n)$	$W^{s,2}(\mathbb{R}^n)$
$H^s(S)$	The s-th order Sobolev space $W^{s,2}(S)$
$H^\infty(S)$	$\bigcap_{s=1}^{\infty} H^s(S)$
$L^p(\Omega)$	The set of all measurable functions u with $\int_\Omega \lvert u\rvert^p \mathrm{d}x < \infty$
$L^\infty(\Omega)$	The set of essentially bounded measurable functions on Ω
$\mathbb{R}^1$	The standard one dimensional Euclidean space
$\mathbb{R}^n$	The standard n dimensional Euclidean space
S	The unit circle $\mathbb{R}^1/2\pi\mathbb{Z}$
$SO(n)$	The group of special orthogonal transforms in $\mathbb{R}^{n+1}$
$W^{k,p}(\Omega)$	Sobolev spaces defined in Section 2.1

Chapter 1

Introduction

However sublime are the researches on fluids which we owe to Messrs Bernoulli, Clairaut and d'Alembert, they flow so naturally from my two general formulae that one cannot sufficiently admire this accord of their profound meditations with the simplicity of the principles from which I have drawn my two equations...

——L. Euler, 1752.

The Euler fluid equation for the perfect (ie, inviscid incompressible) fluid has been one of the most attractive PDEs in the mathematical physics since it appeared about 250 years ago. It can be derived from the conservation of the mass and momentum of the fluid. However, V.I. Arnold [3] proposed in 1966 a completely different perspective which says that the Euler fluid equation can be viewed as the geodesic equation on some diffeomorphism groups with respect to some invariant metric. Arnold's approach has laid a theoretical foundation to exploit a simple construction in Lie group to unify various dynamical systems in mathematical physics. Now the term "generalised Euler equations" (or Euler-Poincaré equations) are used to describe the general geodesic equations on Lie groups.

The Korteweg-de Vries (KdV) equation and the Camassa-Holm (CH) equation are two examples of generalised Euler equations. They can be viewed as the Euler equations on the Virasoro group $\widehat{\mathcal{D}}^s(S)$ (or on the diffeomorphism group $\mathrm{Diff}(S)$ for the limiting case) with respect to the L^2 metric (ie H^0 metric) and H^1 metric respectively on its Lie algebra [64]. These equations read

$$m_t + um_x + 2u_x m = a\partial_x^3 u$$

with $m = u$ for the KdV equation and $m = (1 - \partial_x^2)u$ for the Camassa-Holm equation.

My thesis mainly concerns the analytical and dynamical properties of their generalised version (which we call the modified Camassa-Holm equation)

$$m_t + um_x + 2u_x m = a\partial_x^3 u \quad \text{with} \quad m = (1 - \partial_x^2 + \cdots + (-1)^k \partial_x^{2k})u, \quad k \in \mathbb{N}, \tag{1.1}$$

with the following motivations:

- Mathematically, KdV and Camassa-Holm have some significant differences in dynamics. For example, the Camassa-Holm equation leads to blowup in finite time for some initial values and admits smooth solutions for some other initials while for KdV [111] we have the global wellposedness for all smooth enough initial values. So it is natural to ask how the dynamics of the generalised Euler equations depends on the metric on the Lie algebra? Or more specifically, how do the solutions of the generalised Euler equations on $\widehat{\mathcal{D}}^s(S)$ corresponding to the H^k metric, with $k \neq 0, 1$, on its Lie algebra behave dynamically? We know that the KdV and Camassa-Holm equations are integrable systems, then how about the general H^k metric case?

- Just as the Euler fluid equation has the point vortex solutions, the KdV equation has the so-called soliton solutions, while CH admits the peakon solutions. All these solutions are the so-called "particle solutions". The peakons in the Camassa-Holm equation corresponds to the Dirac δ function in the momentum, just the same as the vortex in the fluid dynamics. The study of peakons (and solitons) are closely related to the integrability, what can we say about the δ solutions of the generalised Euler equations for the H^k metric while we do not know they are integrable or not?

- D. Holm et al [53] discussed the applications of the generalised Euler equations in the computational anatomy and mentioned that a smoother kernel than the inverse of $I - \triangle$ is used there. Mathematically, this means that we need to consider the dynamics of the generalised Euler equations of H^k metric other than H^1 as in the Camassa-Holm equation.

1.1 Euler Fluid Equations: A Brief History

Leonhard Euler, arguably one of the three greatest mathematicians in the history of human beings, published a number of major pieces of work [35] through the 1750s setting up the main formulae for the study of fluid dynamics: the continuity equation, the Laplace velocity potential equation, and the Euler equations for the motion of an inviscid compressible fluid. After Euler's work, Cauchy [20] described in 1823 the conservation of mass and angular momentum by PDEs. Material symmetry and frame invariance were used by Cauchy [20] and Poisson [95] to reduce the constitutive equations. The dissipative effects of internal frictional forces were modeled mathematically by Navier [90], Poisson [95] etc.

In modern fluid dynamics, the Euler equations are in general used to term the equations that govern the motion of an incompressible, inviscid fluid:

$$\begin{cases} u_t + u \cdot \nabla u &= -\nabla p \quad \text{in } \mathbb{R}^2 \text{ or } \mathbb{R}^3 \\ \\ \mathrm{div} u &= 0 \quad \text{in } \mathbb{R}^2 \text{ or } \mathbb{R}^3. \end{cases} \tag{1.2}$$

where p stands for the pressure, u the velocity. They correspond to the Navier-Stokes equations with zero viscosity, although they are usually written in the form of conservation laws and emphasize the fact that they directly represent conservation of mass and momentum.

For 250 years, Euler's equations have formed an essential part of the bedrock of our understanding of fluid flow. They are still one of the most fascinating PDEs. In the memorial issue of Bulletin of AMS, no.4, Vol. 44, 2007, devoted to the 300th anniversary of Euler's birth, two of the six surveys are about the Euler fluid equations.

1.2 Arnold's Viewpoint

Arnold [3] in 1966 introduced a completely new viewpoint on the Euler equations: they are equivalent to the equation of geodesics on a diffeomorphism group with an invariant metric! His method uses one simple construction in Lie group to give a unified approach to a great variety of differential dynamical systems, from the simple (Euler) equation of a rotating top (corresponding to the group $SO(3)$ with the metric $\langle \omega, I\omega \rangle$, here $I = \mathrm{diag}(I_1, I_2, I_3)$) to the (Euler) hydrodynamics equations (corresponding to the group $\mathrm{SDiff}(M)$ with the metric L^2). This led to a totally new stage of development of the Euler equations.

Ebin and Marsden [34] in 1970 used this viewpoint to prove the well-posedness for the Euler equations and Navier-Stokes equations of an incompressible fluid on a (possibly with boundary) Riemannian manifold (corresponding the group SDiff(M) with the L^2 metric), which has not been bettered until nowadays.

A curious application [5, 63] of this theory is an explanation of why long-term dynamical weather forecasts are not reliable: Arnold's explicit estimates related to curvatures of diffeomorphism groups show that the earth's weather is essentially unpredictable after two weeks as the error in the initial condition grows by a factor of 10^5 for that period, that is, one loses 5 digits of accuracy. Another application [5, 63] is related to the Sakharov-Zeldovich problem on whether a neutron star can extinguish by "reshaping" and turning to radiation the excessive magnetic energy.

Now the equations of geodesics on Lie groups are called generalized Euler equations (or Euler-Poincaré equations), which include the Euler equations in fluid dynamics and many interesting partial differential equations in mathematical physics and other areas, for example:

- Landau-Lifshitz equations of micromagnetics
- Template matching equations used in image processing.

With this term, the well-known Camassa-Holm equation can be also categorized as an Euler equation.

1.2.1 Shallow Water Equations

In the study of shallow water waves, Camassa and Holm [18] derived in 1993 the following partial differential equation

$$(I - \partial_x^2)u_t + 2\partial_x u \cdot (I - \partial_x^2)u + u \cdot (\partial_x - \partial_x^3)u = 0, \tag{1.3}$$

by an asymptotic expansion directly in the Hamiltonian and intensively studied its properties: its complete integrability, its bi-Hamiltonian structure, infinite conservation laws and the existence of peaked soliton solutions. For these reasons, this PDE is called the Camassa-Holm equation and often considered as one of the most fascinating PDEs in mathematical physics. Since the birth of this equation, many people have contributed to the well-posedness study on the whole real line $\mathbb{R}$ or on the unit circle S: to mention a few, Arnold and Khesin [5], Constantin and McKean [25], McKean [84], and the references therein. Local well-posedness for (1.3) was discussed by Constantin [27], Constantin and Escher [28] for the initial data in $H^s(\mathbb{S})$ with $s \geq 4$ and $s \geq 3$ respectively, and by Misiołek [88] with $s > 3/2$. Local well-posedness in the non-periodic case was proved for the initial data in $H^s(\mathbb{R})$

with $s > 3/2$ by Li and Olver [70] and Rodriguez-Blanco [97]. Classical solutions can become singular in finite time if the initial momentum $(I - \partial_x^2)u$ changes sign. It is worthwhile to mention that Xin and Zhang [108] proved the global existence of the *weak solution* in the *energy space* $H^1(\mathbb{R})$ without any sign conditions on the initial value, and the uniqueness of this *weak solution* is obtained under some restrictions on the solution [109].

Khesin and Misiołek [64] proved that the Camassa-Holm equation is the equation of the geodesic flow associated to $H^1(\mathbb{S})$ metric on the diffeo-group $\mathrm{Diff}(\mathbb{S})$ of the circle, which is the Euler-Poincaré equation by using the Lagrangian associated with the H^1 metric for the fluid velocity, ie, the Lagrangian as a function of the fluid velocity which is given by the quadratic form,

$$l(u) = \frac{1}{2}\int (u^2 + u_x^2)\mathrm{d}x.$$

1.2.2 Abstract Euler-Poincaré Equations

Let G be a Lie group and $\mathfrak{g}$ its associated Lie algebra (identified with the tangent space to G at the identity element), with Lie bracket denoted by $[\xi, \eta]$ for $\xi, \eta \in \mathfrak{g}$. Let $l : \mathfrak{g} \mapsto \mathbb{R}$ be a given Lagrangian and $L : TG \mapsto \mathbb{R}$ the right invariant Lagrangian on G obtained by translating l from the identity element to other points of G via the right action of G on TG. A basic result of Euler-Poincaré theory [50, 52] is that the Euler-Lagrange equations for L on G are equivalent to the (right) Euler-Poincaré equations for l on $\mathfrak{g}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta l}{\delta \xi} = -\mathrm{ad}_\xi^* \frac{\delta l}{\delta \xi}.$$

Here $\mathrm{ad}_\xi : \mathfrak{g} \mapsto \mathfrak{g}$ is the adjoint operator of the linear map given by the Lie bracket $\eta \mapsto [\xi, \eta]$, $\mathrm{ad}_\xi^* : \mathfrak{g}^* \mapsto \mathfrak{g}^*$ is given by $(\mathrm{ad}_\xi^*(\mu), \eta) = (\mu, [\xi, \eta])$, where $(\cdot, \cdot)$ is the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. The Euler-Poincaré equation can also be written in the variational form $\delta \int l \mathrm{d}t = 0$ for all the variations of the form $\delta\xi = \dot{\eta} - [\xi, \eta]$ for some curve η in $\mathfrak{g}$ that vanishes at the endpoints.

If the reduced Legendre transformation $\xi \mapsto \mu = \frac{\delta l}{\delta \xi}$ is invertible, then the Euler-Poincaré equations are equivalent to the (right) Lie-Poisson equations [51]:

$$\dot{\mu} = -\mathrm{ad}^*_{\frac{\delta h}{\delta \mu}} \mu,$$

where the reduced Hamiltonian is given by $h(\mu) = (\mu, \xi) - l(\xi)$. These equations are equivalent to Hamiltonian equations on T^*G relative to the Hamiltonian $H : T^*G \mapsto \mathbb{R}$, obtained by right translating h from the identity element to other points via the right action of G on T^*G.

1.3 Particle solutions

The Euler fluid equation has a striking feature: it admits the particle solutions. The following explanation is quite basic and can be found in any good mathematical textbook on fluid dynamics (e.g. [23]).

Taking the curl on the first equation in (1.2) gives the differential equation on the vortex $\omega \equiv \nabla \times u$

$$\frac{D\omega}{Dt} - (\omega \cdot \nabla)u = 0 \tag{1.4}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla$, from which the stream function $\psi(x,t)$ can be defined by $-\triangle\psi = \omega$, so we have the relation between u and ω

$$u(x,t) = K\omega \tag{1.5}$$

for some convolution operator K depending on the dimension of the space under consideration. We have

$$\omega(\phi(x,t),t) = \nabla\phi(x,t) \cdot \omega(x,0), \tag{1.6}$$

where $\phi(x,t)$ is the flow map

$$\frac{\partial \phi}{\partial t} = u(\phi(x,t),t),\ \phi(x,0) = x. \tag{1.7}$$

Now imagine the vorticity in a fluid is concentrated in N vortices

$$\omega = \sum_{j=1}^{N} \Gamma_j \delta(x - x_j),$$

then the stream function is (here we take $x = (x^1, x^2) \in \mathbb{R}^2$ as an example):

$$\psi(x) = -\int \omega(x')G(x - x')\mathrm{d}x' = -\frac{1}{2\pi}\sum_{j=1}^{N}\Gamma_j \log||x - x_j||, \tag{1.8}$$

where $G(x) = \frac{1}{2\pi}\log||x||$ is the Green's function of the Laplacian operator $\triangle$ in $\mathbb{R}^2$. The velocity field generated by these N point vortices is

$$u(x,t) = (\partial_{x^2}\psi, -\partial_{x^1}\psi) = \left[-\sum_{j=1}^{N}\frac{\Gamma_j}{2\pi}\frac{x^2 - x_j^2}{r_j^2},\ \sum_{j=1}^{N}\frac{\Gamma_j}{2\pi}\frac{x^1 - x_j^1}{r_j^2}\right], \tag{1.9}$$

where $r_j = ||x - x_j||$. Then the equation of motion that the points $x_j = (x_j^1(t), x_j^2(t)),\ j = 1, 2, \cdots, N$ satisfies is

$$\frac{\mathrm{d}x_j^1}{\mathrm{d}t} = -\frac{1}{2\pi}\sum_{i\neq j}\frac{\Gamma_i(x_j^2 - x_i^2)}{r_{ij}^2} \quad \text{and} \quad \frac{\mathrm{d}x_j^2}{\mathrm{d}t} = \frac{1}{2\pi}\sum_{i\neq j}\frac{\Gamma_i(x_j^1 - x_i^1)}{r_{ij}^2} \tag{1.10}$$

where $r_{ij} = ||x_i - x_j||$. This is a Hamiltonian system with the Hamiltonian

$$H = -\frac{1}{4\pi}\sum_{i\neq j}\Gamma_i\Gamma_j \log ||x_j - x_i||.$$

Through the above process, we reduce the Euler fluid equation into a 2N ODE system, this is quite interesting and leads directly to the particle method in the numerical simulation we will mention in the next section. What makes it more attractive is that not only does the Euler fluid equation have the particle solutions as above, the other Euler equations also have this property.

The Euler fluid equation has two forms: one is (1.2), the other is (1.4) with (1.5). Let us look at the generalised Euler equations, take the CH equation

$$m_t + um_x + 2um_x = a\partial_x^3 u \qquad \text{with} \ \ m = (1 - \partial_x^2)u$$

as an example. This form of the CH equation corresponds to (1.4) with (1.5), and if we express m in terms of u, then we get a differential equations in u, which corresponds to the Euler fluid equation (1.2). The so-called "peakon solutions " in the CH equation is exactly the counterpart of "point vortices" solutions in the Euler fluid dynamics. There is a natural parallel relation between the Euler fluid equation and the other generalised Euler equations. Now that the point vortices are interesting and important in the study of fluid dynamics, we naturally want to find the role of the Dirac δ solution in the generalised Euler equation and how the smooth solutions of the equation tend to it? This motivates us to the asymptotic study which we do in Chapter 5.

Another thing worth mentioning: up to now, all the studies on the solitons or the peakons are related to the complete integrability, which is a very beautiful and wonderful but very special property that the KdV equation, the CH equations and some other special PDEs have. But for the mCH equations we study in this thesis, we do not know if they are completely integrable (actually, we tend to believe they are not, supported by our numerical study), however, they admit the Dirac δ solutions (which we call "soliton solutions" too). Can we find a mechanism other than the integrability that generates the solitons?

1.4 Numerical Approaches

Many attempts to simulating numerically the Camassa-Holm equation can be found in the literature. We just mention a few of them.

1.4.1 Particle Method

In recent years, particle methods have become one of the most useful and widespread tools for approximating solutions of partial differential equations in a variety of fields. In these methods, a solution of a given equation is represented by a collection of particles, located in points q_i and carrying weights p_i. Equations of evolution in time are then written to describe the dynamics of the location of the particles and their weights. Due to the Lagrangian nature of the method, small scales that might develop in a solution can be easily described with a relatively small number of particles. This property and the fact that they are mesh-free made particle methods so attractive in a variety of problems with extremely large deformation, moving boundaries or discontinuities, such as hydrodynamics, electrodynamics and molecular dynamics etc.

In the numerical study of shallow water equation, Camassa, Huang and Lee [17] proposed an algorithm corresponding to a completely integrable particle lattice, which has some analogies with the vortex methods for the two dimensional Euler equations. Actually, the particles they used correspond to a solution of the following form to the CH equation:

$$u(x,t) = \sum_{i=1}^{N} p_i(t) e^{-|x-q_i(t)|}.$$

They proved the convergence of the method and furthermore introduced a fast summation algorithm to evaluate the integrals of the particle method so that the computational cost can be reduced from $O(N^2)$ to $O(N)$, where N is the number of particles.

1.4.2 Box Scheme

The box scheme in numerically solving PDEs is related to Preissmann [96]. Zhao and Qin [110] and Ascher and McLachlan [6] developed it. Ascher and McLachlan [6] compared the box schemes with other numerical methods when applied to the KdV equation and proved that the box scheme preserves unconditionally the dispersion relation, which accounts for the very good robustness and stability. We know that the box scheme is a multi-symplectic scheme for the KdV equation(see next subsection for this concept).

1.4.3 Multi-symplectic Methods

One of the very active fields in numerical differential equations in the recent two decades is the structure-preserving algorithms. Among them the so-called symplectic method is the fastest growing area. We now have some excellent monographs on Geometric Numerical Integration, e.g., [49],[68] etc and the review [86].

For Hamiltonian PDE, we have the so-called multi-symplectic algorithm, which extends the symplectic algorithm for ODEs to PDEs and preserves the symplectic structure in both the time direction and the space direction.

There are two approaches to multi-symplectic algorithms: T. J. Bridges and S. Reich [13, 14] proposed the Hamiltonian PDEs into the forms of first order partial differential equations

$$\mathbf{K}z_t + \mathbf{L}z_x = \nabla_z S(z), \tag{1.11}$$

for some skew symmetric matrices $\mathbf{K}, \mathbf{L}$, and S is a smooth function, and then introduced some numerical schemes preserving the symplecity in both x and t directions. Wonderful reviews on this approach and its recent developments include [15], [55],[89] and Brett Ryland's thesis [99].

The other approach, proposed by Marsden et al., is from the variational principle [76, 78], where they showed the existence and preservation of the fundamental multi-symplectic structures for Hamiltonian PDEs and can be obtained directly from the variational principle by using the multi-symplectic geometry.

The Camassa-Holm equation is rich in geometric structures. Kouranbaeva and Shkoller [66] studied the second order multi-symplectic field theory and showed that the multi-symplectic structure can be obtained from the variation of the action functional, which generalised the theory of [76] from the first order field theory to the second order field theory. They applied their abstract formulation to the Camassa-Holm equation to get a multi-symplectic algorithm. Recently, Cohen, Owren and Raymaud [24] proposed two different multi-symplectic formulations for the Camassa-Holm equation and proved that the Euler box scheme preserve the multi-symplecity, and one of their methods behaves very well in simulating the head-on collision of solitons.

1.5 Applications

The theory on generalised Euler equations unifies various differential equations arisen from mathematical physics, so it has naturally found many applications in the field of mathematical physics, [5, 21, 52, 54] etc. At the same time, it has many significant applications in other fields such as image processing [53, 85] and so on. In [85], the authors first studied the singular solutions of the general Euler equations, and their connection with the vortex sheet solutions of the incompressible Euler equations, and then analysed the stability of straight and circular sheets, and studied the stability for various metrics, and for various directions of the momentum vectors on the sheet.

D. Holm et al. [53] found some applications of EPDiff (ie the generalised Euler equation) in the Computational Anatomy (CA) which was pioneered by Grenander [46] through the notion of *deformable templates*. Roughly speaking, a deformable template is an "object, or examplar" $I_0 \subset \mathbb{R}^2$ or $\mathbb{R}^3$ on which a group $\mathcal{G}$ acts and generates a set of new objects through the orbit $\mathcal{G}I_0$. The authors of [53] drew parallels between the *template matching* in CA and the fluid flows in hydrodynamics. They pointed out that, the Green kernels for the operators used to defined the quadratic cost or effort function in CA are typically smoother than the inverse of the Helmholtz operator $1 - \triangle$ used in the H^1 model in hydrodynamics.

1.6 Thesis Preview

With the motivations mentioned in the beginning of this chapter, we are concerned with the following topics in this thesis:

- the derivation of the Euler equation with respect to the general $H^k(S)$ metric, $k \geq 2$;
- the well-posedness of the derived equation;
- some numerical approaches to this equation;
- some asymptotic analysis for the equation;
- the four particle system of the H^2 metric; and
- some local well-posedness for higher dimensional case.

More specifically, Chapter 2 is concerned with the preliminary mathematics which provides us with some functional, geometrical and numerical tools for later development.

Chapter 3 is about the derivation of the generalised Euler equation on Diff(S) with respect to the general H^k metric and its well-posedness. Section 3.1 is about the derivation of the mCH equation; Section 3.2 is about the local well-posedness while Section 3.3 about the global well-posedness; Section 3.4 is concerned with the global weak solution of the mCH. Then we discuss some generalisations in Sections 3.5 and 3.6. In Section 3.7, we discuss the existence of the conjugate points of the geodesic curve starting from constant solutions of the mCH equation.

Chapter 4 presents the numerical study on the equations. Here we consider the limiting case, ie, the case of $a = 0$ just for simplifying the presentation. We propose two schemes: particle method (for the circle case) and the box scheme (for the whole real line $\mathbb{R}^1$ case). The first corresponds to the important properties of mCH equations: they admit the particle solutions and the Hamiltonian structure. The latter scheme is known to be multi-symplectic when applied to the KdV equation and some other Hamiltonian PDEs, but we don't know whether it is so when applied to mCH. However, it gives a very stable simulation.

Chapter 5 studies the blowup profile by asymptotic analysis. In section 5.1, we derive some asymptotic PDE by the method of asymptotic expansion, and then discuss the stability of the stationary solutions. Then in Section 5.3, we show some numerical simulations that suggest the blowup profile may wander within the one parameter family of steady solutions, which we can not explain why.

We use the particle method to study a four particle system corresponding the mCH in Chapter 6. We know that the mCH has the following conserved quantities: $\int m, \int mu$ (and the third one $\int |m|^{\frac{1}{2}}$ for the limiting case). We want to know if there are some other conserved quantities? From our study of the four particle system, it is expected that there is another conserved quantity because the numerical simulation on the Lyapunov exponents strongly suggests that at most one of the Lyapunov exponents is positive which means there should be another conserved quantity!

Higher dimensional case is studied in Chapter 7, where we have obtained some local existence for the two dimensional case by the regularisation method.

Chapter 8 outlines some future work on this equation.

At last, there are two appendices, we prove some very nice analytic properties for the Green functions for the operator $1-\partial_x^2+\partial_x^4$ in the first appendix; and then some material on the multi-symplectic formulation for mCH, which is obtained via the multi-symplectic geometry approach.

Let us conclude this chapter with a table which describes the similarities and differences between the Camassa-Holm equation and the modified Camassa-Holm equations.

The Camassa-Holm equation vs the modified Camassa-Holm equation for $k \geq 2$

	The Camassa-Holm equation	The Modified Camassa-Holm equation
Completely integrable	✓	? but we tend to believe it is not
Local well-posedness	✓ (Kato theory)	✓ Kato theory but an extra effort needed to get the nice properties of the Green functions
Finite time blowup	✓ for some initial values	never
Global well-posedness	✓ for + or − initial values	always ✓ (the proof is surprisingly simple)
Number of conserved quantities	infinite conserved quantities because of the integrability	two obvious conserved, and our study suggests another conserved integral
Particle methods	✓	✓
Box scheme	✓ and multi-symplectic	✓ but we don't know if it is multi-symplectic
Conjugate Points to constant solutions	✓	✓ but we need to solve a higher order linear PDE
Asymptotic profile	only numerical simulation	numerical results and partial analysis results

Chapter 2

Preliminary Tools

子曰：“工欲善其事，必先利其器……”
——《论语》，约公元前480年。

The mechanic, who wishes to do his work well, must first sharpen his tools...

—Confucius, The Confucian Analects, $\sim$480 BC.

A long time ago, when younger and rasher mathematicians, we both momentarily harboured the ambition that one day, older and wiser, we might write a multivolume treatise titled "On the Mathematical Foundations of Numerical Analysis". Then it dawned that such a creation already exists: it is called "a mathematics library".

—B. Baxter and A. Iserles [8], 2003.

In this chapter, we collect some basic concepts and theorems from partial differential equations, Lie groups and Riemannian geometry, mainly for getting acquaintance with the basic notations, ideas and terminology.

2.1 PDE Basics

We will present some fundamental concepts in PDE and Kato theory on quasilinear evolutionary equations. The material can be found, e.g. in [37], [61] and [102] etc.

2.1.1 Sobolev Spaces

Definitions

Fix $1 \leq p \leq \infty$, k a non-negative integer and $\Omega \subset \mathbb{R}^n$ a domain.

Definition 2.1 *The* Sobolev space $W^{k,p}(\Omega)$ *consists of all locally summable functions* $u : \Omega \mapsto \mathbb{R}$ *such that for each (non-negative integer-components) multi-index* $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ *with* $|\alpha| \equiv \sum_{i=1}^n \alpha_i \leq k$*, the weak derivatives* $D^\alpha u$ *exists and* $D^\alpha u \in L^p(\Omega)$*. When* $p = 2$*, we usually write* $H^k(\Omega) = W^{k,2}(\Omega)$.

With the standard norms

$$||u||_{W^{k,p}(\Omega)} \equiv \begin{cases} \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p \mathrm{d}x \right)^{\frac{1}{p}} & (1 \leq p \leq \infty) \\ \sum_{|\alpha| \leq k} \mathrm{esssup} |D^\alpha u| & (p = \infty), \end{cases} \tag{2.1}$$

it is easy to check that $W^{k,p}(\Omega)$ are Banach spaces and $H^k(\Omega)$ are Hilbert spaces if we identify $f = g$ as an element of $W^{k,p}(\Omega)$ provided $f(x) = g(x)$ for almost all $x \in \Omega$.

When $\Omega = \mathbb{R}^n$ or $\mathbb{T}^n (= \mathbb{R}^n/\mathbb{Z}^n)$ and $p = 2$, we have another equivalent approach to Sobolev spaces:

$$u \in H^k(\mathbb{R}^n) \text{ or } H^k(\mathbb{T}^n) \Longleftrightarrow \text{ both } u \text{ and } \langle \xi \rangle^k \hat{u} \in L^2,$$

where $\langle \xi \rangle \equiv (1 + |\xi|^2)^{\frac{1}{2}}$ and $\hat{u}$ is the Fourier transform of u if $\Omega = \mathbb{R}^n$ and $\hat{u}$ is the corresponding Fourier coefficients of u if $\Omega = \mathbb{T}^n$. This approach can be easily extended from integer k to general real $s \in \mathbb{R}$:

$$H^s(\mathbb{R}^n) \equiv \left\{ u \in \mathcal{S}'(\mathbb{R}^n) : \text{ both } u \text{ and } \langle \xi \rangle^s \hat{u} \in L^2(\mathbb{R}^n) \right\},$$

where the Schwarz space $\mathcal{S}'$ stands for the tempered growth distributions in $\mathbb{R}^n$.

We can introduce the Sobolev spaces with negative index by the dual space, ie, for non-negative real $s \geq 0$, we define

$$W^{-s,p}(\mathbb{R}^n) \equiv \left(W^{s,p'}(\mathbb{R}^n) \right)^*,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. For $p = 2$, we have

$$H^{-s}(\mathbb{R}^n) \equiv (H^s(\mathbb{R}^n))^*.$$

Basic Properties of Sobolev Spaces

Theorem 2.2 *Let Ω be bounded, and suppose that $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$, then there exist functions $u_m \in C^\infty(\Omega) \bigcap W^{k,p}(\Omega)$ such that*

$$\lim_{m\to\infty} ||u_m - u||_{W^{k,p}(\Omega)} = 0.$$

This is Theorem 2 in Section 5.3.2 of [37]. We denote $W_0^{k,p}(\Omega)$ the closure in $W^{k,p}(\Omega)$ of the set $C_0^\infty(\Omega)$ which consists of the smooth functions with compact support in Ω.

If $\Omega = S$ is the unit circle, then we can define the Sobolev space $W^{k,p}(S)$ as the completion of the smooth function space $C^\infty(S)$ with respect to the corresponding norm given by (2.1).

For Sobolev spaces, the most important theorem is the following embedding theorem:

Theorem 2.3 *(Embedding Theorem)*

(1) If $s > \frac{n}{2} + k$ for an integer $k \geq 0$, then $H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$.

(2) If $s > \frac{n}{2} + \alpha$ for a real $0 \leq \alpha \leq 1$, then $H^s(\mathbb{R}^n) \subset C^\alpha(\mathbb{R}^n)$.

(3) If Ω is a bounded domain with locally Lipschitz boundary, then $W^{k,p}(\Omega)$ can be compactly embedded into $C^{\lambda,\alpha}(\Omega)$ if $k - \frac{n}{p} > \lambda + \alpha$ for some integer $\lambda \geq 0$ and $0 \leq \alpha < 1$. $W^{k,p}(\Omega)$ can be compactly embedded into $L^q(\Omega)$ if $k - \frac{n}{p} > -\frac{n}{q}$.

(4) $W^{1,2}(\mathbb{R}^1) \subset L^\infty(\mathbb{R}^1)$ and $W^{1,1}(\mathbb{R}^1) \subset L^\infty(\mathbb{R}^1)$ for the whole real line $\mathbb{R}$.

Here, the Hölder spaces $C^\alpha(\Omega)$ and $C^{\lambda,\alpha}(\Omega)$ are defined by

$$C^\alpha(\Omega) \equiv \left\{ u \in C(\Omega) : \ ||u||_{C^\alpha} = \sup_{x\neq y\in\Omega} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < +\infty \right\}$$

$$C^{\lambda,\alpha}(\Omega) \equiv \left\{ u \in C(\Omega) : \ D^\lambda u \in C^\alpha(\Omega) \right\}.$$

Proof The proof of the first three items can be found in, eg, [37], Theorem 6 in Section 5.6 of [37] and Theorem 1 in Section 5.7 there. The last item can be proved very easily as follows: For any $u \in W^{1,2}(\mathbb{R}^1)$, the Cauchy inequality gives us

$$2u(x)^2 = \int_{-\infty}^{x} 2uu_x \mathrm{d}x - \int_{x}^{+\infty} 2uu_x \mathrm{d}x \leq ||u||^2_{W^{1,2}(\mathbb{R}^1)},$$

so

$$||u||_{L^\infty}^2 \le \frac{1}{2}||u||_{W^{1,2}(\mathbb{R}^1)}^2.$$

If $u \in W^{1,1}(\mathbb{R}^1)$, then

$$|u(x)| = \left|\int_{-\infty}^{x} u_x \mathrm{d}x\right| \le \int_{-\infty}^{x} |u_x|\mathrm{d}x,$$

so

$$||u||_{L^\infty} \le ||u||_{W^{1,1}(\mathbb{R}^1)}.$$

From the embedding theorem, we know that if $u \in H^s(\mathbb{R}^n)$ for $s > \frac{n}{2}$, then u is bounded and continuous, whose dual proposition tells us that

$$\text{The Dirac delta function } \delta \in H^{-\frac{n}{2}-\varepsilon}(\mathbb{R}^n), \quad \text{for all } \ \varepsilon > 0. \tag{2.2}$$

Properties of $H^s(S)$

We are mainly concerned with, in the later chapters, the Sobolev spaces $H^s(S)$ which consists of the periodic one-variable Sobolev functions. We need some estimates on the product of two Sobolev functions and on the commutators which we collect here without detail proof.

Lemma 2.4 *[62] For the 1D periodic Sobolev functions on $S = [0, 2\pi]$:*

(1) $H^s(S)$ is a Banach algebra for $s > 1/2$, and

$$||uv||_{H^s} \le ||u||_{H^s}||v||_{H^s} \quad \textit{for} \ \ u, v \in H^s. \tag{2.3}$$

(2) For any $s > \frac{3}{2}$ and $u \in H^s(S)$ we have

$$\langle u, uu_x\rangle_{H^s} \le C(s)||u_x||_{L^\infty}||u||_{H^s}^2, \tag{2.4}$$

here $\langle\cdot,\cdot\rangle_s$ means the standard inner product in $H^s(S)$.

If we denote $\Lambda_{2k} \equiv (1 - \partial_x^2 + \cdots + (-1)^k\partial_x^{2k})^{\frac{1}{2k}}$ for integer $k \ge 1$, then $\Lambda_{2k}^{2k} : H^r(S) \mapsto H^{r-2k}(S)$ is an invertible mapping, and $\partial_x\Lambda_{2k} = \Lambda_{2k}\partial_x$. We denote Λ_{2k} by Λ if no confusion occurs. Moreover, if we denote $[\Lambda, f]g = \Lambda(fg) - f\Lambda g$, then from [62] [103], we have

Lemma 2.5 *If $s > 0$ and $1 < p < \infty$,*

(1) $W^{s,p}(S) \bigcap L^\infty(S)$ is an algebra, and

$$||uv||_{W^{s,p}} \le C(||u||_{L^\infty}||v||_{W^{s,p}} + ||u||_{W^{s,p}}||v||_{L^\infty}). \tag{2.5}$$

(2)

$$||[\Lambda^s, u]v||_{L^p} \leq C(p,s)(||u_x||_{L^\infty}||\Lambda^{s-1}v||_{L^p} + ||\Lambda^s u||_{L^p}||v||_{L^\infty}). \quad (2.6)$$

(3)

$$||[D^\alpha, u]v||_{L^2} \leq C(||u||_{H^k}||v||_{L^\infty} + ||u_x||_{L^\infty}||v||_{H^{k-1}}) \quad (2.7)$$

for all $\alpha \leq k$ with k positive integer.

Kato and Ponce [62] proved this lemma for $k = 1$ and the same method yields the result for the general case.

Lemma 2.6 *On the norm estimates of product of two functions on S, we have*

(1) [61] *Let s, t be real numbers such that $-s < t \leq s$, then*

$$C||f||_{H^s}||g||_{H^t} \geq \begin{cases} ||fg||_{H^t} & \text{if} \quad s > 1/2, \\ ||fg||_{H^{s+t-1/2}} & \text{if} \quad s < 1/2, \end{cases} \quad (2.8)$$

where C is a positive constant depending on s, t.

(2) [62] *For $s \geq 0$, we have*

$$||fg||_{H^s} \leq C\left(||f||_{L^\infty}||g||_{H^s} + ||f||_{H^s}||g||_{L^\infty}\right). \quad (2.9)$$

(3) For $s \geq 0$, we have

$$||fg||_{H^s} \leq C(||f||_{W^{s,\infty}}||g||_{L^2} + ||g||_{H^s}||f||_{L^\infty}). \quad (2.10)$$

Proof of (3) The inequality (2.9) is Lemma X4 from [62] whose proof is based on a lemma due to R. Coifman and Y. Meyer. Actually, the same ideas with obvious modifications yield a proof of (2.10).

Another formal approach suggested by T. Tao [101] (page 338) is that we can heuristically think

$$\langle\Lambda\rangle^s(fg) \backsim g(\langle\Lambda\rangle^s f) + f(\langle\Lambda\rangle^s g), \quad (2.11)$$

then take L^2 norm on both sides and use the Hölder inequality to get the required inequality (2.10).

Lemma 2.7 *For any two functions f, g defined on S, we have*

(1) $||fg||_{H^t} \leq C||f||_{H^t}||g||_{H^t}$ *for* $t > \frac{1}{2}$;

(2) $||fg||_{H^t} \leq C||f||_{L^\infty}||g||_{H^t}$ *for* $t \leq 0$;

(3) $||fg||_{H^t} \leq C||f||_{H^{t+1/2}}||g||_{H^{t+1/2}}$ *for* $0 < t \leq \frac{1}{2}$.

(4) $||fg||_{H^t} \leq C(||f||_{L^\infty}||g||_{H^t} + ||g||_{L^\infty}||f||_{H^t})$ *for* $t > 0$.

Proof (1) is the consequence of the fact that H^t is a Banach algebra for $t > \frac{1}{2}$.
(2) For $t \leq 0$, and any $h \in H^{-t}(S)$, we have

$$\begin{aligned} \left|\int_S fgh\mathrm{d}x\right| &\leq |f|_{L^\infty}\int |gh|\,\mathrm{d}x \\ &\leq |f|_{L^\infty}||g||_{H^t}||h||_{H^{-t}}, \end{aligned} \tag{2.12}$$

from which (2) follows.
(3) The inequality follows from (1) and the fact $||fg||_{H^t} \leq ||fg||_{H^{t+1/2}}$.
(4) The inequality is the Lemma X4 in [62], see also Lemma 2.6.
The similar proof of Lemmas A.2 and A.3 in [61] will give

Lemma 2.8 *If* $s > 1/2 + 1$, *then*

$$||[\Lambda_4^s, f]\Lambda_4^{1-s}|| \leq C||f'||_{H^{s-1}}, \tag{2.13}$$

where $||\cdot||$ *on the left denotes the operator norm in* $L^2(S)$.

Lemma 2.9 *Let* $u \in H^s$ *for some* $s > 3/2$. *Then*

$$||\Lambda^{-r_1}[\Lambda^{r_1+r_2+1}, u]\Lambda^{-r_2}||_{B(L^2)} \leq C||u'||_{H^{s-1}}, \quad |r_1|, |r_2| \leq s-1. \tag{2.14}$$

Kato [61] proved this lemma for $\Lambda = (1-\partial_x^2)^{\frac{1}{2}}$, but a obvious modification yields the same results for the general $\Lambda = (1-\partial_x^2 + \cdots + (-1)^k\partial_x^{2k})^{\frac{1}{2k}}$ and we omit the details.

2.1.2 Kato Theory

Our local well-posedness is based on the Kato Theory [61] which sets up the abstract theorems on the quasilinear evolutionary equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(t,u)u = f(t,u) \in X, \quad t \geq 0, \\ u(0) = u_0 \in Y, \end{cases} \tag{2.15}$$

where $A(t,u)$ is a linear operator depending on the unknown u and X,Y are two Banach spaces satisfying some conditions. The equations we are concerned with in this thesis can be put into the form of

$$(C) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + A(u)u = f(u) \in \ X, \quad t \geq 0, \\ u(0) = u_0 \in Y. \end{cases} \tag{2.16}$$

That means, the operators A and f do not depend on t explicitly.

Some notations: $\mathcal{B}(X,Y)$ denotes the space of all bounded linear operators from a Banach space X to a Banach space Y ($\mathcal{B}(X)$ if $X = Y$); $\partial = \partial_x = \frac{\partial}{\partial x}$; $\Lambda_2^s = (I - \partial_x^2)^{s/2}, \Lambda_4^s = (I - \partial_x^2 + \partial_x^4)^{s/4}$, $s \in \mathbb{R}$; $\langle\ ,\ \rangle_s$ for the inner product on H^s; $H^\infty = \bigcap_{s\geq 0} H^s$; $[A,B] = AB - BA$ denotes the commutator of the linear operators A and B.

Some assumptions on (C):

(X) X and Y are Hilbert spaces where $Y \subset X$ is dense and the inclusion continuous, and there is an isomorphism S from Y to X such that $||\omega||_Y = ||S\omega||_X$ for all $\omega \in Y$.

(A_1) Let $W \subset Y$ be an open ball centered at 0. The linear operator $A(u)$ belongs to $G(X,1,\beta)$ where β is a real number, where $G(X,1,\beta)$ is defined as the set of linear operators $A(u)$ satisfying:

1. $\langle A\omega, \omega\rangle_X \geq -\beta||\omega||_X^2, \quad \forall\ \omega \in \mathcal{D}(A)$, the domain of A.
2. $(A + \lambda)$ is onto for some (all) $\lambda > \beta$.

(A_2) The map

$$\omega \in W \mapsto B(\omega) = [S, A(\omega)]S^{-1} = SAS^{-1} - A \in \mathcal{B}(X) \tag{2.17}$$

is uniformly bounded and Lipschitz continuous, i.e., there exist constants $\lambda_1, \mu_1 > 0$ such that

$$||B(\omega)||_{\mathcal{B}(X)} \leq \lambda_1, \quad ||B(\omega) - B(\nu)||_{\mathcal{B}(X)} \leq \mu_1||\omega - \nu||_Y$$

for all $\omega, \nu \in W$.

(A_3) $Y \subseteq \displaystyle\bigcap_{\omega\in W} \mathcal{D}(A(\omega))$, so that $A(\omega)|_Y \in \mathcal{B}(Y,X)$ by the Closed Graph Theorem. Moreover, there exists $\mu_2 > 0$ such that, for all $\omega, \nu \in W$, we have

$$||A(\omega) - A(\nu)||_{\mathcal{B}(Y,X)} \leq \mu_2||\omega - \nu||_X.$$

(f_1) $f : W \to Y$ is bounded and there exist a constant $\mu_3 > 0$ such that

$$||f(\omega) - f(\nu)||_X \leq \mu_3 ||\omega - \nu||_X, \quad \forall\ \omega, \nu \in W,$$

$$||f(\omega) - f(\nu)||_Y \leq \mu_3 ||\omega - \nu||_Y, \quad \forall\ \omega, \nu \in W,$$

Theorem 2.10 (Kato [61]) *Under assumptions above on (C) with $u_0 \in W$, there exists a $T > 0$ such that there exits a unique solution $u \in C([0,T], Y) \cap C^1([0,T], X)$ to (C). Moreover, the map $u_0 \in Y \to u \in C([0,T], Y)$ is continuous in the following sense: suppose*

$$\lim_{n\to\infty} ||A_n(\omega) - A_\infty(\omega)||_{\mathcal{B}(Y,X)} = 0, \quad \lim_{n\to\infty} ||B_n(\omega) - B_\infty(\omega)||_{\mathcal{B}(X)} = 0,$$

$$\lim_{n\to\infty} ||f_n(\omega) - f_\infty(\omega)||_Y = 0, \quad \lim_{n\to\infty} ||u_{0,n}(\omega) - u_{0,\infty}(\omega)||_Y = 0,$$

and consider the Cauchy problems

$$(C_n) \quad \begin{cases} \frac{\partial u_n}{\partial t} + A_n(u_n)u_n = f_n(u_n) \in \ X, \quad t \geq 0, \\ \\ u_n(0) = u_{0,n} \in W, \qquad\qquad n \in \mathbb{Z} \cup \{\infty\}. \end{cases} \tag{2.18}$$

Suppose the assumptions above hold also for (C_n) with the same X, Y, S, W and the constants β, λ_i, μ_i are independent of n. Let T_n be the time of existence of u_n. Then all u_n, with n large enough, can be extended to $[0, T_\infty]$ and

$$\lim_{n\to\infty} ||u_n(t) - u_\infty(t)||_{C([0,T_\infty];Y)} = 0.$$

The proof of Theorem 2.1 can be found in [61].

2.2 Riemannian Geometry

We will state some basic concepts in Riemannian geometry which we will use in the subsequent chapters.

Definition 2.11 *Let M be a differentiable manifold modelled on $\mathbb{R}^m$, and if there is a positive definite mapping*

$$g : TM \times TM \mapsto \mathbb{R}^1$$

which is called the Riemannian metric, then (M, g) is called a **Riemannian manifold**.

The fundamental theorem of Riemannian geometry states that on any Riemannian manifold there is a unique *torsion-free* metric connection, called the *Levi-Civita connection* of the given metric. Here a metric (or Riemannian) connection is a connection which preserves the metric tensor.

More precisely:

Let (M, g) be a Riemannian manifold (or pseudo-Riemannian manifold), then there is a unique connection ∇ which satisfies the following conditions:

(1) for any vector fields X, Y, Z on M, we have,

$$\partial_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

where $\partial_X g(Y, Z)$ denotes the derivative of the function $g(Y, Z)$ along the vector field X.

(2) For any vector fields X, Y on M, we have

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

where $[X, Y]$ denotes the Lie brackets for vector fields X, Y.

(The first condition expresses the fact that the connection is compatible with the Riemannian metric, so that the metric tensor is preserved by parallel transport, while the second condition expresses the fact that the torsion of the connection is zero.)

Definition 2.12 *A parametrised curve* $\gamma : I \mapsto M$ *is a* geodesic at $t_0 \in I$ *if* $\frac{D}{\mathrm{d}t}(\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}) \equiv \nabla_{\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}} \frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = 0$ *at the point* t_0*; if* γ *is a geodesic at all* $t \in I$*, we say* γ *is a* geodesic curve.

Definition 2.13 *The* curvature R *of a Riemannian manifold* M *is a correspondence that associates to every pair of vector fields* $X, Y \in TM$ *a mapping* $R(X, Y) : TM \mapsto TM$ *given by*

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, \quad Z \in TM,$$

where ∇ *is the Levi-Civita connnection.*

Now we can introduce *Jacobi fields* along a geodesic γ which, in some sense, describe the spreading rate of the nearby geodesics close to γ.

Definition 2.14 *Let* $\gamma(t)$ *be a geodesic curve on* M*, and if a vector field* $J(t)$ *on* M *satisfies the* Jacobi equation

$$\frac{D^2}{\mathrm{d}t^2} J(t) + R(\gamma'(t), J(t))\gamma'(t) = 0,$$

then $J(t)$ *is called a* Jacobi field *along* $\gamma(t)$.

Clearly, $\gamma'(t)$, $t\gamma'(t)$ are Jacobi fields along $\gamma(t)$ because $R(\gamma'(t),\gamma'(t))\gamma'(t) = 0$. So, one normally considers only the Jacobi fields along γ that are normal to γ'.

Definition 2.15 *Let $\gamma : [0,a] \mapsto M$ be a geodesic. The point $\gamma(t_0)$, with some $t_0 \in (0,a]$, is said to be conjugate to $\gamma(0)$ along γ if there exists a Jacobi field $J(t)$ along γ, not identically zero, with $J(0) = J(t_0) = 0$.*

For example, on the two dimensional sphere S^2, any semicircle jointing the north and south poles is a geodesics, and the two antipoles are conjugate points each other. Intuitively, the existence of the conjugate point means that the nearby geodesic curves γ_1, generated by perturbating a geodesics γ along $J(t)$, will comes back to $\gamma(t)$ at some time.

We conclude this section with the introduction of Lie derivative.

Definition 2.16 *Let X be a smooth vector field on M, and T a differentiable tensor field of rank (p,q) on M, then the* Lie derivative of T *with respect to X is defined at the point $p \in M$ by*

$$(\pounds_X T)_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\phi_t)^* T_{\phi_t(p)}$$

where ϕ_t (or ϕ) is the one parameter flow generated by X and ϕ^* is the pull-back of ϕ.

For example, $\pounds_X f = Xf$ is the directional derivative of f along the directional field X for any smooth function $f : M \mapsto \mathbb{R}^1$. One can easily check that, $\pounds_X Y = [X,Y]$ for vector fields X, Y on M. We have also the Cartan's Magic Formula:

$$\pounds_X \omega = \mathrm{d}\imath_X \omega + \imath_X(\mathrm{d}\omega)$$

for any form ω on M, where d is the exterior differential operator and $\imath_X \omega$ the contraction of ω and X.

2.3 Lie Groups

Lie groups play a fundamentally important role in the study of differential equations with some (continuous) symmetry [92]. We pick up some basic concepts and results on Lie groups in this section, mainly from [5].

2.3.1 Lie Group and Adjoint Representation

Definition 2.17 *A group G is called a* Lie group *if G has a smooth structure and the two group operations are smooth. Its tangent space $\mathfrak{g}$ at the identity e, as a linear space, is called the* vector space of the Lie algebra *of G.*

For example, all rotations of a rigid body about the origin is the Lie group SO(3).

We know a coordinate change C in $\mathbb{R}^4$ leads to the similar transform which sends a matrix $B \in \mathrm{SO}(3)$ to the matrix $CBC^{-1} \in \mathrm{SO}(3)$. A similar structure exists for a general Lie group G.

Definition 2.18 *Given a Lie group G and $g \in G$, the map $A_g : G \mapsto G$ defined by*

$$A_g : h \mapsto ghg^{-1}$$

is called an inner automorphism *of G.*

Obviously, $A_g(e) = geg^{-1} = e$, so the derivative of the mapping A_g maps $\mathfrak{g}$ to $\mathfrak{g}$.

Definition 2.19 *The derivative* Ad_g *of A_g at the identity e:*

$$\mathrm{Ad}_g : \mathfrak{g} \mapsto \mathfrak{g}, \;\; \mathrm{Ad}_g a = (A_{g*}|_e)a, \quad a \in \mathfrak{g} = T_eG$$

is called the group adjoint operator *of G. Here $F_*|x : T_xM \mapsto T_{F(x)}M$ is the derivative (or push-forward) of a mapping $F : M \mapsto M$ at x.*

If we denote $GL(\mathfrak{g})$ the space consisting of the invertible linear operators from $\mathfrak{g}$ to $\mathfrak{g}$, then the mapping $\mathrm{Ad} : g \mapsto \mathrm{Ad}_g \in GL(\mathfrak{g})$ can be thought as a mapping from $G \mapsto GL(\mathfrak{g})$. So we can introduce

Definition 2.20 *Suppose $g(t)$ is a curve in a Lie group G, $g(0) = e, \dot{g}(0) = \xi \in \mathfrak{g}$, then the differential* ad *of the mapping* Ad *at the identity e*

$$\mathrm{ad} = \mathrm{Ad}_*|_e : \mathfrak{g} \mapsto GL(\mathfrak{g}), \quad \mathrm{ad}_\xi = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{g(t)}$$

is called the adjoint representation of the Lie algebra.

For example, for the special orthogonal group $G = SO(n)$ in $\mathbb{R}^n$, we have

$$\mathrm{ad}_\xi \omega = [\xi, \omega] = \xi\omega - \omega\xi$$

is the commutator of skew symmetric matrices $\xi, \omega \in \mathfrak{so}(n)$.

Definition 2.21 *The* commutator *in the Lie algebra* $\mathfrak{g}$ *is defined as the operation* $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$ *that associates to a pair of vectors* $\xi,\ \eta \in \mathfrak{g}$ *the vector* $\mathrm{ad}_\xi \eta$*, ie,*

$$[\xi, \eta] = \mathrm{ad}_\xi \eta.$$

The tangent space $\mathfrak{g}$ *with the operation* $[\ ,\ ]$ *is called the* Lie algebra of the Lie group G*.*

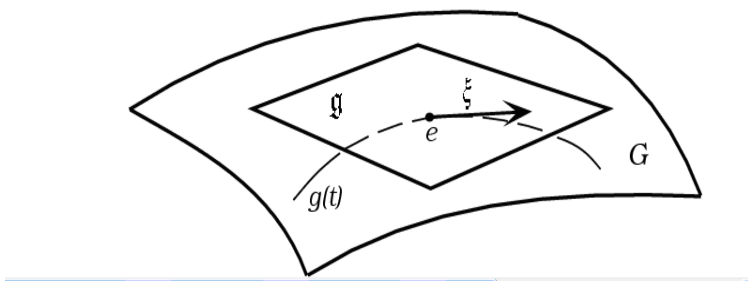

Figure 2.3.1: The vector ξ in the Lie algebra $\mathfrak{g}$

It is not difficult to verify that the operation $[\ ,\ ]$ is bilinear, skew symmetric, and satisfies the Jacobi identity, ie, for any $\xi, \eta, \omega \in \mathfrak{g}$, we have:

$$[\lambda\xi + \nu\omega, \eta] = \lambda[\xi, \eta] + \nu[\omega, \eta] \quad \lambda, \nu \in \mathbb{R} \text{ or } \mathbb{C};$$

$$[\xi, \eta] = -[\eta, \xi];$$

$$[[\xi, \eta], \omega] + [[\eta, \omega], \xi] + [[\omega, \xi], \eta] = 0.$$

Definition 2.22 *For a vector* $\xi \in \mathfrak{g}$*, the set* $\mathrm{Orbit}(\xi) = \{\mathrm{Ad}_g \xi :\ g \in G\}$ *is called the* adjoint (group) orbit of ξ*.*

One can easily find that the vectors $\mathrm{ad}_v u,\ v \in \mathfrak{g}$ form the tangent space to the adjoint orbit of the point $u \in \mathfrak{g}$.

For example, the adjoint orbit of a matrix, regarded as an element of the Lie algebra of all complex matrices, is the set of matrices with the same Jordan normal form.

2.3.2 Co-adjoint Representation of a Lie Group

When working with the Eulerian hydrodynamics, we are not dealing with the Lie algebra $\mathfrak{g}$ and its adjoint representation, but with its dual space $\mathfrak{g}^*$ and the co-adjoint representation.

Definition 2.23 **(1)** *Denote* $G, \mathfrak{g}, \mathfrak{g}^*$ *as above, then for every* $g \in G, \xi \in \mathfrak{g}^*, \omega \in \mathfrak{g}$, *we can define a mapping*

$$\mathrm{Ad}_g^* : \mathfrak{g}^* \mapsto \mathfrak{g}^*$$

by

$$(\mathrm{Ad}_g^* \xi)(\omega) = \xi(\mathrm{Ad}_g \omega).$$

The corresponding map $\mathrm{Ad}^* : G \mapsto GL(\mathfrak{g}^*)$ *is called the* co-adjoint representation *of the Lie group* G*.*

(2) *For* $\xi \in \mathfrak{g}^*$*, the set* $\mathrm{Orbit}(\xi) \equiv \{\mathrm{Ad}_g^* \xi : \ g \in G\}$ *is called the* co-adjoint orbit *of* ξ*.*

(3) *The* co-adjoint representation *of* $v \in \mathfrak{g}$ *is the operator* $\mathrm{ad}_v^* : \mathfrak{g}^* \mapsto \mathfrak{g}^*$ *defined by*

$$\mathrm{ad}_v^* \omega(u) = \omega(\mathrm{ad}_v u) = \omega([v, u])$$

for $u, v \in \mathfrak{g}, \omega \in \mathfrak{g}^*$.

Clearly, the co-adjoint representation $\mathrm{ad}_v^* \omega$ for $v \in \mathfrak{g}$, form the tangent space to the co-adjoint orbit of the point ω.

2.3.3 Invariant Metrics of Lie Groups

A Riemannian metric $\langle \cdot, \cdot \rangle$ on a Lie group G is *left-(right-)invariant* if it is conserved under any left (or right) translation L_f (or R_f), which means that for any $\xi, \eta \in \mathfrak{g}$ and $g \in G$, we have

$$\langle L_{g*} \xi, L_{g*} \eta \rangle_g = \langle \xi, \eta \rangle.$$

So we can uniquely define the left-(right-)invariant metric on the whole group G by translating from the Lie algebra $\mathfrak{g}$, ie, once we know a quadratic form on $\mathfrak{g}$, we can define a invariant metric on G.

Let $A : \mathfrak{g} \mapsto \mathfrak{g}^*$ be a symmetric definite operator that defines the inner product

$$\langle \xi, \eta \rangle = (A\xi, \eta) = (A\eta, \xi)$$

for any $\xi, \eta \in \mathfrak{g}$. Here the round brackets stand for the standard pairing of the elements of $\mathfrak{g}$ and $\mathfrak{g}^*$. The operator A is called the *inertia operator*. For any $g \in G$, A induces an operator $A_g : T_gG \mapsto T_g^*G$ by

$$A_g\xi = L_g^{*-1} A L_{g^{-1}*}\xi, \qquad \text{for } \xi \in T_gG.$$

We have the very illustrative commutative diagram Figure 2.3.2 from [5]:

Any tangent vector $\xi \in T_gG$ can be translated by $R_{g^{-1}}$ or $L_{g^{-1}}$ to the Lie algebra $\mathfrak{g}$ and we obtain two different vectors in $\mathfrak{g}$.

Definition 2.24 *For $\xi \in T_gG$, we call the two vectors*

$$\omega_c = L_{g^{-1}*}\xi \in \mathfrak{g}, \ \omega_s = R_{g^{-1}*}\xi \in \mathfrak{g}$$

the angular velocity *and spatial angular velocity respectively, here the subscript c in ω_c stands for "corps"=body. Clearly, $\omega_s = \mathrm{Ad}_g\omega_c$.*

Similarly, for the vector $m = A_g\xi \in T_g^*G$, we have two different vectors in $\mathfrak{g}^*$:

$$m_c = L_g^* m \in \mathfrak{g}^*, \qquad m_s = R_g^* m \in \mathfrak{g}^*$$

which are called *angular momentum relative to the body* and the *angular momentum relative to the space*. We note that $m_c = \mathrm{Ad}_g^* m_s$. It is easy to check that, for a motion $g(t) \in G$, the energy $E \equiv \frac{1}{2}\langle \dot{g}, \dot{g}\rangle$ can be expressed as

$$E = \frac{1}{2}\langle \dot{g}, \dot{g}\rangle = \frac{1}{2}\langle \omega_c, \omega_c\rangle = \frac{1}{2}(A\omega_c, \omega_c) = \frac{1}{2}(m_c, \omega_c) = \frac{1}{2}(m, \dot{g}).$$

Euler proved for $G = \mathrm{SO}(3)$ that any geodesic curve $g(t) \in G$ must satisfy

$$\frac{\mathrm{d}m_s}{\mathrm{d}t} = 0, \quad \text{or equivalently} \quad \frac{\mathrm{d}m_c}{\mathrm{d}t} = \mathrm{ad}_{\omega_c}^* m_c.$$

Arnold [3, 5] pointed out that the proofs are almost literally extendable to the general case. That is, we have the abstract Euler Theorem (or general Euler equations):

Theorem 2.25 *Let G be a Lie group with a left-invariant metric, and $\mathfrak{g}^*$ its Lie algebra, then any geodesic curve $g(t)$ in G must satisfy*

$$\frac{\mathrm{d}m_s}{\mathrm{d}t} = 0, \ \text{or equivalently} \quad \frac{\mathrm{d}m_c}{\mathrm{d}t} = \mathrm{ad}_{\omega_c}^* m_c. \tag{2.19}$$

For a Lie group with a right-invariant metric, we have

Theorem 2.26 *Let G be a Lie group with a right-invariant metric, and $\mathfrak{g}^*$ its Lie algebra, then any geodesic curve $g(t)$ in G must satisfy*

$$\frac{\mathrm{d}m_s}{\mathrm{d}t} = 0, \quad \text{or equivalently} \quad \frac{\mathrm{d}m_c}{\mathrm{d}t} = -\mathrm{ad}_{\omega_c}^* m_c. \tag{2.20}$$

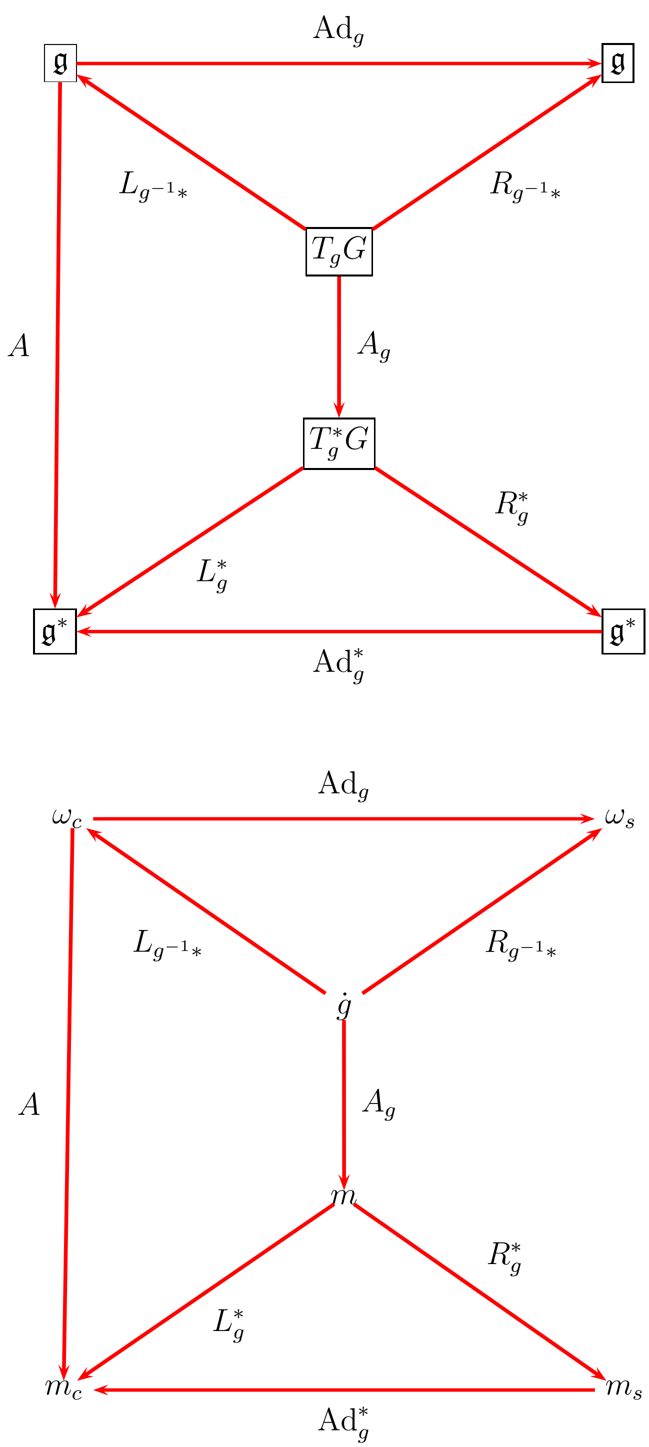

Figure 2.3.2: Diagram of the operators in $\mathfrak{g}$ and $\mathfrak{g}^*$

2.3.4 Applications to Hydrodynamics

Let M be a domain in $\mathbb{R}^n$, and $\mathrm{Diff}(M)$ denotes the set of C^∞-diffeomorphisms on M, and $\mathrm{Diff}_{\mathrm{vol}}(M)$ for the volume-preserving C^∞-diffeomorphisms on M. Then both $\mathrm{Diff}(M)$ and $\mathrm{Diff}_{\mathrm{vol}}(M)$ are Lie groups if we take the composition of two diffeomorphisms as the group multiplication operation. $\mathrm{Diff}(M)$ and $\mathrm{Diff}_{\mathrm{vol}}(M)$ are the configuration manifolds of compressible and incompressible fluid flows respectively.

Remark If one is concerned with the C^r-diffeomorphisms on M, then neither $\mathrm{Diff}^r_{\mathrm{vol}}(M)$ nor $\mathrm{Diff}^r(M)$ is a Lie group although they are both continuous groups. The reason is that the left translation L_f defined by $L_f(g) \equiv f \circ g$ is only continuous but not smooth. The detailed explanations can be found, eg, in [34].

Chapter 3

Well-posedness

庄子曰："人皆知有用之用，而莫知无用之用也。"
——《人间世》，约公元前300年。

All men know the utility of useful things; but they do not know the utility of futility.

—Zhuang Zi, $\sim$300 BC.

In this chapter, we will first derive the generalised Euler equation with respect to the H^k, $k \geq 2$ metric on the Bott-Virasoro group $\widehat{\mathcal{D}}(S)$:

$$m_t + 2u_x m + u m_x + a\partial_x^3 u = 0 \qquad \text{on } S, \quad \text{with } m = \Lambda_{2k}^{2k} u, \tag{3.1}$$

where the operator $\Lambda_{2k}^s = (1 - \partial_x^2 + \cdots + (-1)^k \partial_x^{2k})^{\frac{s}{2k}}$, k is a positive integer and $a \in \mathbb{R}$ (here we use this cumbersome notation just to indicate that Λ_{2k}^s is a s-th order pseudo-differential operator), then study its well-posedness and other analytical properties of its solution.

3.1 Derivation of the Equations

3.1.1 Bott-Virasoro Group

Let $\mathcal{D}^s(S)$ be the group of orientation preserving Sobolev H^s diffeomorphisms of the unit circle S, and let $\mathrm{Vect}^s(S) = T_e\mathcal{D}^s(S)$ be the corresponding Lie algebra. We assume s to be large enough so that all our formal calculations can be rigorously justified.

The Bott-Virasoro group $\widehat{\mathcal{D}}^s(S)$ is the non-trivial central extension of $\mathcal{D}^s(S)$ which is defined as follows: the set

$$\widehat{\mathcal{D}}^s(S) \equiv \mathcal{D}^s(S) \times \mathbb{R}^1$$

and the group operation is defined by Bott [12]

$$\hat{\eta} \circ \hat{\xi} = \left(\eta \circ \xi, \alpha + \beta + \int_S \log \partial_x (\eta \circ \xi) \mathrm{d} \log \partial_x \xi \right), \tag{3.2}$$

where $\hat{\eta} = (\eta, \alpha), \hat{\xi} = (\xi, \beta)$ with $\eta, \xi \in \mathcal{D}^s(S)$ and $\alpha, \beta \in \mathbb{R}$.

The corresponding Virasoro algebra $\widehat{\mathrm{Vect}}(S)$ is the tangent space of $\widehat{\mathcal{D}}^s(S)$ at the identity which is the non trivial extension of $\mathrm{Vect}(S)$, the tangent space of $\mathcal{D}^s(S)$ at the identity of $\mathcal{D}^s(S)$. The commutator in the Lie algebra is given by [5]

$$[\widehat{V}, \widehat{W}] \equiv -\left((v\partial_x w - w\partial_x v) \frac{\partial}{\partial x}, c(v, w) \right), \tag{3.3}$$

where $c(v, w) \equiv \int_S v \partial_x^3 w \mathrm{d}x$, $\widehat{V} = \left(v \frac{\partial}{\partial x}, a \right)$, $\widehat{W} = \left(w \frac{\partial}{\partial x}, b \right)$ with $a, b \in \mathbb{R}$ and $v \frac{\partial}{\partial x}, w \frac{\partial}{\partial x} \in T_e \mathcal{D}^s(S)$.

3.1.2 Derivation of the Equations

In order to derive the equation (3.1), according to Theorem 2.26, the key point we need to find is the formula of ad^*.

Let $\widehat{U} = (u \frac{\partial}{\partial x}, a), \widehat{V} = (v \frac{\partial}{\partial x}, b), \widehat{W} = (w \frac{\partial}{\partial x}, c) \in \widehat{\mathrm{Vect}}^s(S)$, and define the H^k inner product on $\widehat{\mathrm{Vect}}^s(S)$ by

$$(\widehat{U}, \widehat{V})_{H^k} = \int_S (uv + u_x v_x + \cdots + \partial_x^k u \partial_x^k v) \mathrm{d}x + ab, \tag{3.4}$$

then we find $\mathrm{ad}^*_{\widehat{U}}$ by direct calculations

$$\begin{aligned} (\mathrm{ad}^*_{\widehat{U}} \widehat{V}, \widehat{W})_{H^k} &= (\widehat{V}, \mathrm{ad}_{\widehat{U}} \widehat{W})_{H^k} = (\widehat{V}, [\widehat{U}, \widehat{W}])_{H^k} \\ &= (v, u_x w - u w_x)_{H^k} - b \cdot c(u, w) \\ &= (\Lambda_{2k}^{2k} v, u_x w - u w_x)_{L^2} - b \cdot c(u, w) \\ &= (g + b\partial_x^3 u, w)_{L^2} = (\Lambda_{2k}^{-2k}(g + b\partial_x^3 u), w)_{H^k}, \end{aligned} \tag{3.5}$$

where $g = 2u_x \Lambda_{2k}^{2k} v + u \Lambda_{2k}^{2k} v_x$. So

$$\mathrm{ad}^*_{\widehat{U}} \widehat{V} = \left(\Lambda_{2k}^{-2k} (2u_x \Lambda_{2k}^{2k} v + u \Lambda_{2k}^{2k} v_x + b\partial_x^3 u) \frac{\partial}{\partial x}, 0 \right). \tag{3.6}$$

The group $\widehat{\mathcal{D}}^s(S)$ is a right-invariant group, so by Theorem 2.26, the generalized Euler equation $\frac{\mathrm{d}}{\mathrm{dt}}\widehat{U} = -\mathrm{ad}^*_{\widehat{U}}\widehat{U}$ on the Virasoro group gives us

$$\frac{\mathrm{d}\Lambda^{2k}_{2k}u}{\mathrm{d}t} = -(2u_x\Lambda^{2k}_{2k}u + u\Lambda^{2k}_{2k}u_x + a\partial^3_x u), \qquad \frac{\mathrm{d}a}{\mathrm{d}t} = 0, \tag{3.7}$$

which is (3.1) for $m = \Lambda^{2k}_{2k}u$.

We can put the equation (3.7) in the Hamiltonian form:

$$m_t = -(m\partial_x + \partial_x m + a\partial_x^3)\frac{\delta H}{\delta m}, \quad \text{with } \; H = \frac{1}{2}\int um\mathrm{d}x. \tag{3.8}$$

The KdV equation and the CH equation can be also put into this form (3.8) but with $m = u$ and $m = (1-\partial_x^2)u$ respectively. On the other hand, we know that the KdV equation can be expressed as

$$m_t = -\partial_x\frac{\delta H_1}{\delta m}, \quad \text{with } \; m = u, \; H_1 = \frac{1}{2}\int\left(\frac{1}{3}u^3 - au_x^2\right)\mathrm{d}x, \tag{3.9}$$

and the Camassa-Holm equation

$$m_t = -\partial_x(1-\partial_x^2)\frac{\delta H_1}{\delta m}, \quad \text{with } \; H_1 = \frac{1}{2}\int\left(u(u^2+u_x^2) - au_x^2\right)\mathrm{d}x. \tag{3.10}$$

These equations give the second Hamiltonian structure for the KdV and CH equations respectively, where the term “bi-Hamiltonian structure” in some literature comes from, and the bi-Hamiltonian structure leads to the integrability of the equations and gives infinite conserved quantities for KdV and CH. Another interesting point is that they yield a constant Poisson structure $\mathcal{K} = \partial_x, \; \partial_x(1-\partial_{xx})$ for KdV and CH, which we do not know if it is true also for the general $k > 1$ case.

3.2 Local Well-posedness

We will establish the well-posedness of (3.1) for $k = 2$ and the similar result holds for the general case $k \geq 2$.

Theorem 3.1 *Let $k = 2, \; u_0 \in H^s(S), \; s > 2k - \frac{1}{2} = 7/2$. Then, there exist a $T > 0$ depending on $||u_0||_s$, and a unique solution u satisfying (3.1) in the distribution sense such that*

$$u \in C([0,T], H^s(S)) \cap C^1([0,T], H^{s-1}(S)).$$

Moreover, the map $u_0 \in H^s \mapsto u \in C([0,T], H^s(S))$ is continuous.

We can rewrite (3.1) for $k = 2$ in two ways:

$$\begin{cases} m_t = -um_x - 2mu_x - a\partial_x^3 u, & x \in S,\ t \in \mathbb{R}, \\ m(x,0) = m_0(x) = \Lambda_4^4 u_0(x), \end{cases} \tag{3.11}$$

where $m = \Lambda_4^4 u = (I - \partial_x^2 + \partial_x^4)u$, $\Lambda_4^s = (I - \partial_x^2 + \partial_x^4)^{\frac{s}{4}}$. Or

$$\begin{cases} u_t = -uu_x - \partial_x \Lambda_4^{-4}(u^2 + \frac{1}{2}u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x\partial_x^3 u) - a\partial_x^3\Lambda_4^{-4}u, & x \in S, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{3.12}$$

If we denote $A(u) = u\partial_x$, $f \equiv -\partial_x\Lambda_4^{-4}(u^2 + \frac{1}{2}u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x\partial_x^3 u) - a\partial_x^3\Lambda_4^{-4}u$, then (3.12) has the form of (2.16):

$$(C) \qquad \begin{cases} \frac{\partial u}{\partial t} + A(u)u = f(u) \in X, \quad t \geq 0, \\ u(0) = u_0 \in Y. \end{cases} \tag{3.13}$$

In order to use the Kato's theory on quasilinear evolutionary equations to prove this theorem, we need to verify that the conditions in Kato's Theorem are satisfied, ie, we need the following lemmas:

Lemma 3.2 *The operator $A(u) = u\partial_x$, with $u \in H^s$, $s > \frac{3}{2}$ belongs to $G(H^{s-1}, 1, \beta)$ for some $\beta > 0$.*

Proof According to the definition of $G(H^{s-1}, 1, \beta)$ in the assumption (A_1) in Section 2.1.2 we need to verify two conditions:
(1)

$$\langle A(u)\phi, \phi\rangle_{H^{s-1}} \geq -\beta||\phi||^2_{H^{s-1}}, \tag{3.14}$$

(2) $A(u) + \lambda$ is onto H^{s-1} for for some $\lambda > \beta$.

We use Λ for Λ_4 in this proof just for simplifying the notations.

$$\begin{aligned} \langle u\partial_x\phi, \phi\rangle_{H^{s-1}} &= \langle \Lambda^{s-1}(u\partial_x\phi), \Lambda^{s-1}\phi\rangle_{L^2} \qquad \text{for } \phi \in H^{s-1}(S) \\ &= \langle [\Lambda^{s-1}, u]\partial_x\phi + u\partial_x\Lambda^{s-1}\phi, \Lambda^{s-1}\phi\rangle_{L^2} \\ &= \langle [\Lambda^{s-1}, u]\partial_x\phi, \Lambda^{s-1}\phi\rangle_{L^2} - \tfrac{1}{2}\langle u_x, (\Lambda^{s-1}\phi)^2\rangle_{L^2} \end{aligned} \tag{3.15}$$

The first term can be estimated as follows:

$$\begin{aligned}
& \langle[\Lambda^{s-1}, u]\partial_x\phi, \Lambda^{s-1}\phi\rangle_{L^2} \\
\leq\ & ||\Lambda^{s-1}, u]\partial_x\phi||_{L^2}||\Lambda^{s-1}\phi||_{L^2} \\
\leq\ & C(||u_x||_{L^\infty}||\Lambda^{s-2}\partial_x\phi||_{L^2} + ||\Lambda^{s-1}u||_{L^2}||\partial_x\phi||_{L^\infty})||\Lambda^{s-1}\phi||_{L^2} \qquad (3.16) \\
& \qquad \text{(by Lemma 2.5)} \\
\leq\ & C||u||_{H^s}||\Lambda^{s-1}\phi||^2_{L^2}.
\end{aligned}$$

It is easier to get the estimate for the second term:

$$\begin{aligned}
\langle u_x, (\Lambda^{s-1}\phi)^2\rangle_{L^2} &\leq C||u_x||_{L^\infty}||\Lambda^{s-1}\phi||^2_{L^2} \\
&\leq C||u||_{H^s}||\Lambda^{s-1}\phi||^2_{L^2}.
\end{aligned} \qquad (3.17)$$

Now we are going to verify the second condition: $A(u) + \lambda$ is onto H^{s-1} for some $\lambda > \beta$. Clearly, $A(u) + \lambda$ is a closed operator, so we need only to prove that $A(u) + \lambda$ has dense range for $\lambda > \beta$.

Suppose $\psi \in H^{s-1}(S)$ such that

$$\langle(A(u)+\lambda)\phi, \psi\rangle_{H^{s-1}} = 0 \quad \text{for any } \phi \in \mathcal{D}(A(u)) = \{\phi \in H^{s-1} : u\partial_x\phi \in H^{s-1}\}, \qquad (3.18)$$

then we have in the distributional sense

$$-\partial_x(u\Lambda^{2s-2}\psi) + \lambda\Lambda^{2s-2}\psi = 0,$$

which yields

$$\langle\Lambda^{2s-2}\psi, u\psi_x\rangle_{(H^{1-s},H^{s-1})} + \lambda\langle\Lambda^{2s-2}\psi, \psi\rangle_{(H^{1-s},H^{s-1})} = 0, \qquad (3.19)$$

here (H^{1-s}, H^{s-1}) stands for the natural pairing between $H^{1-s} = (H^{s-1}(S))^*$ and H^{s-1} and $\Lambda^{2s-2} : H^{s-1}(S) \mapsto H^{1-s}(S)$ is an isomorphism. On the other hand, by the inequality (3.14), we have

$$\langle\Lambda^{2s-2}\psi, u\psi_x\rangle_{(H^{1-s},H^{s-1})} = \langle\psi, u\psi_x\rangle_{H^{s-1}} \geq -\beta||\psi||^2_{H^{s-1}}, \qquad (3.20)$$

so the above two equations and the condition $\lambda > \beta$ imply that $\psi \equiv 0$. This means, $A(u) + \lambda$ has dense range for $\lambda > \beta$.

Lemma 3.3 $B(u) = [\Lambda_4^1, u\partial_x]\Lambda_4^{-1} \in \mathcal{B}(H^{s-1})$ *for* $u \in H^s$, $s > 3/2$.

Proof Obviously, $[\Lambda, u\partial_x]\Lambda^{-1}w = [\Lambda, u]\Lambda^{-1}w_x$, so we have

$$\begin{aligned} ||B(u)w||_{H^{s-1}} &= ||\Lambda^{s-1}[\Lambda, u]\Lambda^{-1}w_x|| \\ &= ||\Lambda^{s-1}[\Lambda, u]\Lambda^{1-s}\Lambda^{(s-2)}w_x|| \\ &\leq C||u_x||_{H^{s-1}}||\Lambda^{(s-2)}w_x||_{L^2} \\ &\leq C||u||_{H^s}||w||_{H^{s-1}}. \end{aligned} \tag{3.21}$$

In the second-to-last estimate, we have used Lemma 2.9.

From the proof, we have

$$||(B(u) - B(v))w||_{H^{s-1}} \leq C||w||_{H^{s-1}}||u - v||_{H^s} \quad \text{for } u, v \in H^s. \tag{3.22}$$

Lemma 3.4 *For* $u \in H^s(S)$ *with* $s > 3/2$,
(a) $H^s \subset \mathcal{D}(u\partial_x) = \{f \in H^{s-1} : u\partial_x f \in H^{s-1}\}$, $s > 3/2$.

(b) $u\partial_x \in \mathcal{B}(H^s, H^{s-1})$, $s > 3/2$.

(c) $||u\partial_x - v\partial_x||_{\mathcal{B}(H^s,H^{s-1})} \leq C||u - v||_{s-1}$.

Proof For $f \in H^{s-1}$ we have

$$\begin{aligned} ||u\partial_x f||_{H^{s-1}} &\leq C||u||_{H^{s-1}}||\partial_x f||_{H^{s-1}} \quad \text{because } H^{s-1} \text{ is a Banach algebra} \\ &\leq C||u||_{H^{s-1}}||f||_{H^s} \end{aligned} \tag{3.23}$$

which proves the first two parts. The third part follows directly from the above inequality.

Lemma 3.5 *Let* $f(u) = -\partial_x\Lambda_4^{-4}(u^2 + \frac{1}{2}u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x\partial_x^3 u) - a\partial_x^3\Lambda_4^{-4}u$, $s > 7/2$, *then for any* u, v *with* $||u||_{H^s}, ||v||_{H^s} \leq C$, *we have*
(a) $||f(u) - f(v)||_{H^{s-1}} \leq C||u - v||_{H^{s-1}}$.

(b) $||f(u) - f(v)||_{H^s} \leq C||u - v||_{H^s}$.

Proof In the proof, we use $||\cdot||_s$ to stand for $||\cdot||_{H^s}$.
(a) We need only to verify that

$$||\partial_x \Lambda_4^{-4}(\frac{7}{2}u_{xx}^2 + 3u_x\partial_x^3 u - \frac{7}{2}v_{xx}^2 - 3v_x\partial_x^3 u)||_{s-1} \le C||u-v||_{s-1},$$

for the corresponding inequality for the other three terms is easy to verify.

$$\begin{aligned}
&||\partial_x \Lambda_4^{-4}(u_{xx}^2 - v_{xx}^2)||_{s-1} \le C||\partial_x^2(u+v)\partial_x^2(u-v)||_{s-4} \\
&\le \max\{||\partial_x^2(u+v)||_{s-4}, ||\partial_x^2(u+v)||_{L^\infty}, ||\partial_x^2(u+v)||_{s-7/2}\} \\
&\cdot \max\{||\partial_x^2(u-v)||_{s-4}, ||\partial_x^2(u-v)||_{s-7/2}\} \text{ (by Lemma 2.7)} \\
&\le C||u-v||_{s-3/2} \le C||u-v||_{s-1}.
\end{aligned}$$

$$||\partial_x \Lambda_4^{-4}(u_x\partial_x^3 u - v_x\partial_x^3 v)||_{s-1} \le C||u_x\partial_x^3 u - v_x\partial_x^3 v||_{s-4} \tag{3.24}$$
$$= C||u_x\partial_x^3 u - u_x\partial_x^3 v + u_x\partial_x^3 v - v_x\partial_x^3 v||_{s-4} \tag{3.25}$$
$$\le C||u_x(\partial_x^3 u - \partial_x^3 v)||_{s-4} + ||(u_x - v_x)\partial_x^3 v||_{s-4}. \tag{3.26}$$

We estimate these two terms separately. If $s-4 > \frac{1}{2}$ or $-\frac{1}{2} < s-4 \le 0$, we can easily get from Lemma 2.7 in Chapter 2 that

$$||u_x(\partial_x^3 u - \partial_x^3 v)||_{s-4} \le C||\partial_x^3(u-v)||_{s-4} \le C||u-v||_{s-1}. \tag{3.27}$$

If $0 < s-4 \le \frac{1}{2}$, we have to use Lemma 2.6 in Chapter 2, to get

$$\begin{aligned}
||u_x\partial_x^3(u-v)||_{H^{s-4}} &\le C(||u_x||_{L^\infty}||\partial_x^3(u-v)||_{H^{s-4}} + ||u_x||_{W^{s-4,\infty}}||\partial_x^3(u-v)||_{L^2}) \\
&\le C(||u||_{W^{1,\infty}}||u-v||_{H^{s-1}} + ||u||_{W^{s-3,\infty}}||u-v||_{H^3}) \\
&\le C||u||_{H^s}||u-v||_{H^{s-1}} \quad \text{because } s-4>0.
\end{aligned} \tag{3.28}$$

Similarly, we can estimate the other term in (3.26). Here we just write out the formula for the case $0 < s-4 \le \frac{1}{2}$.

$$\begin{aligned}
||(u_x - v_x)\partial_x^3 v||_{H^{s-4}} &\le C(||u_x - v_x||_{s-4}||\partial_x^3 v||_{L^\infty} + ||\partial_x^3 v||_{s-4}||u_x - v_x||_{L^\infty} \\
&\le C||v||_{H^s}||u-v||_{H^{s-1}}.
\end{aligned} \tag{3.29}$$

Adding up all the above estimates yields

$$||f(u) - f(v)||_{H^{s-1}} \le C||u-v||_{H^{s-1}}. \tag{3.30}$$

(b) Similar situation as in **(a)**.

$$\begin{aligned} & ||\partial_x \Lambda_4^{-4}(u_{xx}^2 - v_{xx}^2)||_s \leq C||\partial_x^2(u+v)\partial_x^2(u-v)||_{s-3} \\ \leq \; & C||\partial_x^2(u+v)||_{s-3}||\partial_x^2(u-v)||_{s-3} \leq C||u+v||_{s-1}||u-v||_{s-1} \\ \leq \; & C||u-v||_s, \end{aligned}$$

$$\begin{aligned} & ||\partial_x \Lambda_4^{-4}(u_x\partial_x^3 u - v_x\partial_x^3 v)||_s \\ \leq \; & C||u_x\partial_x^3 u - v_x\partial_x^3 v||_{s-3} = C||u_x\partial_x^3 u - u_x\partial_x^3 v + u_x\partial_x^3 v - v_x\partial_x^3 v||_{s-3} \\ \leq \; & C||u||_{s-2}||u-v||_s + C||v||_s||u_x - v_x||_{s-3} \\ \leq \; & C||u-v||_s, \end{aligned}$$

here we have used the fact that H^s is a Banach algebra for $s > 1/2$.

Proof of Theorem 3.1 Now Theorem 3.1 is just a direct consequence of Kato's Theorem with $Y = H^s(S), X = H^{s-1}(S)$ and the above Lemmas.

Theorem 3.6 *If the Theorem 3.1 yields the maximal time interval of existence is* $[0, T)$, *then we have* $T = +\infty$ *or*

$$\lim_{t\to T^-} ||u(t)||_{H^s} = +\infty \quad \text{if} \quad T < \infty. \tag{3.31}$$

Proof From Theorem 3.1, we have $T = +\infty$ or

$$\lim_{t\to T^-} (||u(t)||_{H^s} + ||u_t(t)||_{H^{s-1}}) = +\infty \quad \text{if} \quad T < \infty. \tag{3.32}$$

On the other hand, we have from the proof of Theorem 3.1 and the equation (3.12) that

$$||u_t(t)||_{H^{s-1}} \leq C||u(t)||_{H^{s-1}}||u(t)||_{H^s} \leq C||u(t)||_{H^s}^2, \tag{3.33}$$

which yields what we want.

3.2.1 Conservation Laws

Based on the local well-posedness, some conservation laws can be established. In this subsection, we assume that the solutions are smooth enough that all the calculations can be done rigorously.

Theorem 3.7 *Let* $u(x,t)$ *be the solution to (3.1) with* $u_0 \in H^\infty$, *and* $m_0 = (1 - \partial_x^2 + \partial_x^4)u_0$,*, then in the time interval of existence of u, we have the following conserved quantities:*

$$I_1 = \int m = \int u, \tag{3.34}$$

$$I_2 = \int um = \int (u^2 + u_x^2 + u_{xx}^2). \tag{3.35}$$

Proof Integrating directly the equation (3.1), we have the first conserved quantity. We can exploit the equations (3.11)(3.12) to verify that

$$\frac{\mathrm{d}I_2}{\mathrm{d}t} = \int u_t m \mathrm{d}x + \int u m_t \mathrm{d}x = 0.$$

Geometrically, the fact I_2 is conserved just means that velocity vector of the geodesic curve has a constant length along the geodesics.

3.3 Global Well-posedness

The Camassa-Holm equation (1.3) can reach a singularity in a finite time if $m_0 = (I - \partial_x^2)u_0$ changes sign. However, this can not happen for the modified equation (3.1) by our next theorem.

Theorem 3.8 *Suppose $k \geq 2$ in (3.1). If the initial value $m(0,x) \in L^2(S)$, then $m(t,x) \in L^2(S)$ for any finite time $t > 0$, and there exists a constant C_0 depending only on the norm of initial values u such that*

$$||m||_{L^2} \leq e^{C_0 t}||m_0||_{L^2}. \tag{3.36}$$

Proof of Theorem 3.8 We prove the Theorem 3.8 for sufficiently smooth function m and the general case $m_0 \in L^2$ follows by a standard density argument. Multiply (3.1) by m and integrate over S, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||m||_{L^2}^2 + 2\int u_x m^2 + \int u m m_x = a\int m\partial_x^3 u, \tag{3.37}$$

Clearly, $\int m\partial_x^3 u = \int \partial_x^3 u \Lambda_{2k}^{2k} u = 0$. So

$$\frac{\mathrm{d}}{\mathrm{d}t}||m||_{L^2}^2 = -3\int m^2 u_x, \tag{3.38}$$

from which

$$\frac{\mathrm{d}}{\mathrm{d}t}||m||_{L^2}^2 \leq 3|u_x|_{L^\infty}||m||_{L^2}^2. \tag{3.39}$$

On the other hand, $I_2 = \int_S u m \mathrm{d}x$ is a conserved quantity for (3.1), ie

$$\sum_{l=0}^{k} ||\partial_x^l u(t,x)||_{L^2}^2 = \sum_{l=0}^{k} ||\partial_x^l u(0,x)||_{L^2}^2. \tag{3.40}$$

So from the Sobolev embedding theorem and $k \geq 2$ we have

$$|u_x|_{L^\infty} \leq C||u_{xx}||_{L^2} \leq C_0, \tag{3.41}$$

where C_0 is a constant depending only on the initial condition. The Gronwall inequality and (3.39) yield

$$||m||_{L^2} \leq e^{C_0 t}||m_0||_{L^2}. \tag{3.42}$$

For the limiting mCH, ie, $a = 0$, we have even better results:

Theorem 3.9 *If $m_0 \in L^p(S)$, where $2 \leq p < +\infty$, then there exists a unique global solution $m(t, x) \in L^p(S)$ to (3.1) with $a = 0$ and $k = 2$ such that $m \in L^p(S)$ and*

$$||m(\cdot, t)||_{L^p(S)} \leq e^{Ct}||m_0||_{L^p(S)}, \tag{3.43}$$

where C is a constant independent of p. Moreover, if $m_0 \in W^{1,p}(S)$, then $m(\cdot, t) \in W^{1,p}(S)$ and there exist constants C_1, C_2 independent of p such that

$$||m(\cdot, t)||_{W^{1,p}(S)} \leq e^{C_1 e^{C_2 t}}||m_0||_{W^{1,p}(S)} \tag{3.44}$$

Proof of Theorem 3.9: We prove the conclusions for the smooth enough data and the general case can be reached by standard approximations. Multiplying (3.11) by $|m|^{p-1}\text{sgn}m$, we have

$$\frac{1}{p}\frac{\mathrm{d}|m|^p}{\mathrm{d}t} + \frac{1}{p}\left(|m|^p u\right)_x + (2 - \frac{1}{p})|m|^p u_x = 0, \tag{3.45}$$

integrating this equation, we have

$$\begin{aligned} \frac{1}{p}\frac{\mathrm{d}||m||^p_{L^p}}{\mathrm{d}t} &= -(2 - \frac{1}{p})\int |m|^p u_x \\ &\leq (2 - \frac{1}{p})||u_x||_{L^\infty}||m||^p_{L^p}. \end{aligned} \tag{3.46}$$

Combining with the embedding $||u_x||_{L^\infty} \leq C'||u_{xx}||_{L^2} \leq C$, we can easily see that

$$||m(\cdot, t)||_{L^p} \leq e^{Ct}||m_0||_{L^p}. \tag{3.47}$$

Taking the derivative with respect to x in (3.11), we have got

$$\frac{\mathrm{d}m_x}{\mathrm{d}t} + 3m_x u_x + 2m u_{xx} + m_{xx} u = 0. \tag{3.48}$$

Multiplying this equation by $|m_x|^{p-1}\mathrm{sgn} m_x$,

$$\frac{1}{p}\frac{\mathrm{d}|m_x|^p}{\mathrm{d}t}+\frac{1}{p}(|m_x|^p u)_x+(3-\frac{1}{p})|m_x|^p u_x+2mu_{xx}|m_x|^{p-1}\mathrm{sgn} m_x=0. \quad (3.49)$$

Now integrating over S gives

$$\frac{1}{p}\frac{\mathrm{d}||m_x||_{L^p}^p}{\mathrm{d}t}+(3-\frac{1}{p})\int|m_x|^p u_x+2\int mu_{xx}|m_x|^{p-1}\mathrm{sgn} m_x=0. \quad (3.50)$$

So, if $m_0\in W^{1,p}(S),\ p\geq 1$, then $m_0\in L^\infty(S)$, and (3.47) yields that, for some constants $C_1,C_2>0$,

$$||m(\cdot,t)||_{L^\infty(S)}\leq C_1e^{C_2t}. \quad (3.51)$$

At the same time

$$||u_x||_{L^\infty}\leq C.$$

Moreover, the embedding theorems and (3.47) tell us that

$$||u_{xx}||_{L^\infty}\leq C||\partial_x^3u||_{L^2}\leq C||\partial_x^4u||_{L^2}\leq C(||m||_{L^2}+||u-u_{xx}||_{L^2})\leq e^{Ct}. \quad (3.52)$$

Now it is easy to derive from (3.50)∼(3.52) that there exist constants C_1,C_2 independent of p such that

$$||m_x||_{L^p}\leq e^{C_1e^{C_2t}}||m_{0x}||_{L^p}. \quad (3.53)$$

3.3.1 Extra Properties for $a=0$

Lemma 3.10 *Let $u(x,t)$ be the solution to (3.1) with $a=0$, $u_0\in H^\infty$, and suppose that $m_0=(1-\partial_x^2+\partial_x^4)u_0\geq 0$ (or ≤ 0), then $m=(1-\partial_x^2+\partial_x^4)u\geq 0$ (respectively ≤ 0), moreover, if $m\geq 0$, then*

$$\int_S m^{1/2}\mathrm{d}x=\int_S m_0^{1/2}\mathrm{d}x.$$

Proof The proof of Lemma 3.3 in [28] applies here with little change, but we include it here for completeness.

Let $\varepsilon>0,\Omega\subset\mathbb{R}^2$ be a bounded domain, $v\in H^1(\Omega)$. Then from [44], we know that $\sqrt{\varepsilon+v_+},\sqrt{\varepsilon+v_-}\in H^1(\Omega)$, with

$$\nabla\sqrt{\varepsilon+v_+}=\frac{\nabla v}{2\sqrt{\varepsilon+v_+}}\chi[v>0],\quad \nabla\sqrt{\varepsilon+v_-}=\frac{\nabla v}{2\sqrt{\varepsilon+v_-}}\chi[v<0],$$

where χ stands for the characteristic function. Now if we take $\Omega = [0, 2\pi] \times [0, t]$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_S \sqrt{\varepsilon + m_+} &= \frac{1}{2}\int_S \frac{m_t}{\sqrt{\varepsilon + m_+}}\chi[m>0] \\ &= -\int_S \frac{mu_x}{\sqrt{\varepsilon + m_+}}\chi[m>0] - \frac{1}{2}\int_S \frac{m_x u}{\sqrt{\varepsilon + m_+}}\chi[m>0] \\ &= -\int_S \sqrt{\varepsilon + m_+}u_x\chi[m>0] + \varepsilon\int_S \frac{u_x}{\sqrt{\varepsilon + m_+}}\chi[m>0] \\ &\quad -\frac{1}{2}\int_S \frac{m_x u}{\sqrt{\varepsilon + m_+}}\chi[m>0].\end{aligned} \tag{3.54}$$

Integrating the first integral by parts ,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \sqrt{\varepsilon + m_+} = \varepsilon\int_S \frac{u_x}{\sqrt{\varepsilon + m_+}}\chi[m>0] + R(t,\varepsilon), \tag{3.55}$$

where $R(t,\varepsilon) = \sum_{x_s\in A}\sigma(x_s)\sqrt{\varepsilon}u(x_s)$ with $A \equiv \{x \in S : m(x) = 0\}$ and $\sigma(x_s) = 1$ or -1 depending on x_s is the left or right end point of the composing intervals of $\{x \in S : m(x) > 0\}$. Whatever the value of $R(t,\varepsilon)$ is, we always have

$$|R(t,\varepsilon)| \le \sqrt{\varepsilon}\int_S |u_x|, \tag{3.56}$$

and so

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_S \sqrt{\varepsilon + m_+}\right| &= \left|\varepsilon\int_S \frac{u_x}{\sqrt{\varepsilon + m_+}}\chi[m>0] + R(t,\varepsilon)\right| \\ &\le 2\sqrt{\varepsilon}\int_S |u_x| \le \sqrt{\varepsilon}\left(1 + \int_S (u^2 + u_x^2 + u_{xx}^2)\right).\end{aligned} \tag{3.57}$$

So from the obvious fact that $\int_S (u^2 + u_x^2 + u_{xx}^2)$ is conserved, we have

$$\left|\int_S \sqrt{\varepsilon + m_+} - \int_S \sqrt{\varepsilon + m_{0+}}\right| \le \sqrt{\varepsilon}\left(1 + \int (u_0^2 + u_{0x}^2 + u_{0xx}^2)\right)t. \tag{3.58}$$

Letting $\varepsilon \to 0$, we get

$$\int_S \sqrt{m_+} = \int_S \sqrt{m_{0+}}. \tag{3.59}$$

Similarly we have the conservation for $\int_S \sqrt{m_-}$, from which the lemma follows.

Remark

(a) From the proof, we can find that the essential part in the proof is the equation $m_t = -2um_x - u_x m$ and the conservation of $\int(u^2 + u_x^2 + u_{xx}^2)$ (which is $\int um$). The exact relation between m and u does not really matter as long as $\int |u_x|$ can be controlled by $\int um$.

(b) If $m_0(x)$ changes sign on S, we have

$$\int_S |m|^{\frac{1}{2}}\mathrm{d}x = \int_S |m_0|^{\frac{1}{2}}\mathrm{d}x. \tag{3.60}$$

In fact, the limiting mCH has a very nice property: the zero points of m evolve along the characteristics, this can easily be seen from $m_t + um_x = 0$ at the points where $m = 0$. So on each subinterval of S where m does not change sign, the integral $\int |m|^{\frac{1}{2}}\mathrm{d}x$ over the subinterval is conserved. There is another point of view: formally the limiting mCH can be put in the form of

$$\frac{\mathrm{d}}{\mathrm{d}t}|m|^{\frac{1}{2}} + (|m|^{\frac{1}{2}}u)_x = 0. \tag{3.61}$$

This means the so-called Casimir functional is conserved:

$$\int |m|^{\frac{1}{2}}\mathrm{d}x = \int |m_0|^{\frac{1}{2}}\mathrm{d}x. \tag{3.62}$$

Lemma 3.11 *Let $u_0 \in H^s(S)$, $s > 7/2$ and $m_0 = (1 - \partial_x^2 + \partial_x^4)u_0 \geq 0$(or ≤ 0), then $\exists K > 0$ such that $||u_{xxx}||_{L^\infty} \leq K$.*

Proof At first, we assume that $u_0 \in H^\infty$, u solves (3.1), then it is easy to show that $||u||_{L^2}^2 + ||u_x||_{L^2}^2 + ||u_{xx}||_{L^2}^2$ is conserved as long as u exists as a solution to (3.1). From Lemma 3.10, we have $m = \Lambda_4^4 u \geq 0$ (or ≤ 0). Let $x_0 \in S$ satisfy $u_{xxx}(x_0) = 0$, then $\forall y \in S$, we have

$$\begin{aligned} u_{xxx}(y) &= \int_{x_0}^{y} \partial_x^4 u\mathrm{d}x = \int_{x_0}^{y}(u - \partial_x^2 u + \partial_x^4 u)\mathrm{d}x - \int_{x_0}^{y}(u - \partial_x^2 u)\mathrm{d}x \\ &\leq \int_S m\mathrm{d}x + ||u||_{L^1} + ||u_{xx}||_{L^1} = \int_S m_0\mathrm{d}x + ||u||_{L^1} + ||u_{xx}||_{L^1} \\ &\leq \int_S m_0\mathrm{d}x + C||u||_{L^2} + C||u_{xx}||_{L^2} \leq K, \end{aligned} \tag{3.63}$$

where K depends on m_0 and $||u_0||_{H^2}$ and C is a constant independent of u. Similarly, we have (here we use x_0 and $x_0 + 2\pi$ to stand for the same point on the circle S assuming the circumference of S is 1 with $x_0 \leq y \leq x_0 + 2\pi$)

$$-u_{xxx}(y) = \int_y^{x_0+2\pi} \partial_x^4 u\mathrm{d}x \leq K.$$

So far we have proved the Lemma for $u_0 \in H^\infty$. A standard approximation can give the proof for $u_0 \in H^s(S),\ s > 7/2$.

Theorem 3.12 *Suppose $k = 2,\ u_0 \in H^s(S),\ s > 7/2$, then Equation (3.1) with $a = 0$ admits a unique solution in $C([0, +\infty), H^s(S)) \bigcap C^1([0, +\infty), H^{s-1}(S))$ if the initial momentum $m_0 \geq 0$.*

This theorem holds also valid for $k \geq 2$ as long as $u_0 \in H^s(S)$ with $s > 2k - \frac{1}{2}$. In order to prove Theorem 3.12, we need the following lemma:

Lemma 3.13 *Assume the conditions in Theorem 3.12 hold, then $||u(t)||_{H^s}$ is finite for any $0 < t < \infty$.*

Proof Apply Λ_4^s to $u_t = -uu_x - f(u)$, where $f(u) = \partial_x \Lambda_4^{-4}(u^2 + \frac{1}{2}u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x\partial_x^3 u)$, and multiply by $\Lambda_4^s u$ and then integrate over S, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}||u||_s^2 = -2\langle u, uu_x\rangle_s - 2\langle u, f(u)\rangle_s \tag{3.64}$$

By Lemma 2.5, we have

$$|\langle u, uu_x\rangle_s| \leq C_s||u_x||_{L^\infty}||u||_s^2. \tag{3.65}$$

The Cauchy inequality gives

$$|\langle u, f(u)\rangle_s| \leq ||u||_s||f(u)||_s, \tag{3.66}$$

and

$$\begin{aligned}
||f(u)||_s &\leq C||u^2 + \tfrac{1}{2}u_x^2 - \tfrac{7}{2}u_{xx}^2 - 3u_x\partial_x^3 u||_{H^{s-3}} \\
&\leq C(||u^2||_{s-3} + ||u_x^2||_{s-3} + ||u_{xx}^2||_{s-3} + ||u_x\partial_x^3 u||_{s-3} + ||u||_{H^s}) \\
&\leq C(||u||_{L^\infty}||u||_{s-3} + ||u_x||_{L^\infty}||u_x||_{s-3} + ||u_{xx}||_{L^\infty}||u_{xx}||_{s-3} \\
&\qquad + ||u_x||_{L^\infty}||\partial_x^3 u||_{s-3} + ||\partial_x^3 u||_{L^\infty}||u_x||_{s-3} + ||u||_{H^s}) \\
&\leq C||u||_s,
\end{aligned} \tag{3.67}$$

where we used Lemma 2.5 and Lemma 3.11. So we have

$$\frac{\mathrm{d}}{\mathrm{d}t}||u||_s^2 \leq C||u||_s^2, \tag{3.68}$$

and so the Gronwall's inequality completes the proof of Lemma.

Proof of Theorem 3.12 Theorem 3.12 is a direct consequence of Theorem 3.6 and Lemma 3.13 above.

3.4 Weak Solutions for $a = 0$

What we have obtained is the well-posednss for u in $H^s(S)$ with $s > 2k - \frac{1}{2}$, this excludes the δ-momentum solutions. But we know that the δ-momentum solutions play a very important role in the study of Euler equations. So we want to enlarge a little bit the solution space to include the δ-momentum solutions.

Equation (3.12) with $a = 0$ can be rewritten as

$$\begin{cases} u_t + F(u)_x = 0, \quad x \in S, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{3.69}$$

where

$$\begin{aligned} F(u) &= \frac{1}{2}u^2 + \Lambda_4^{-4}(u^2 + \frac{1}{2}u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x\partial_x^3 u) \\ &= \frac{1}{2}u^2 + \Lambda_4^{-4}(u^2 + \frac{1}{2}u_x^2 - \frac{1}{2}u_{xx}^2) - 3(\partial_x\Lambda_4^{-4})(u_x u_{xx}). \end{aligned}$$

Definition 3.14 *Let $u_0 \in H^2(S)$. A function $u : [0, +\infty) \times S \to \mathbb{R}$ is called a global weak solution to (3.69) if $u \in C([0,\infty); H^2)$ and $\forall\ T > 0$, we have*

$$\int_0^T\int_S (u\varphi_t + F(u)\varphi_x)\mathrm{d}x\mathrm{d}t + \int_S u_0(x)\varphi(0,x)\mathrm{d}x = 0, \quad \forall\ \varphi \in C^{1,c}([0,T) \times S), \tag{3.70}$$

where $C^{1,c}([0,T)\times S)$ is the set of all first order smooth function with compact support in $[0,T) \times S$.

Theorem 3.15 *Let $u_0 \in H^2(S)$, and $m_0 = (I - \partial_x^2 + \partial_x^4)u_0$ is a positive Radon measure on S. Then there exists a unique global* weak *solution $u \in C([0,\infty); H^2(S))$ of (3.1) and such that $m = \Lambda_4^4 u$ is a positive Radon measure on S whose total variation on S is uniformly bounded for $t \geq 0$. Moreover we have*

$$\int_S u\mathrm{d}x = \int_S u_0\mathrm{d}x, \qquad \int_S (u^2 + u_x^2 + u_{xx}^2)\mathrm{d}x = \int_S (u_0^2 + u_{0x}^2 + u_{0xx}^2)\mathrm{d}x, \tag{3.71}$$

Proof of Theorem 3.15 Let $\theta \equiv ||m_0||_{\mathcal{M}} = ||u_0 - \partial_x^2 u_0 + \partial_x^4 u_0||_{\mathcal{M}}$ be the variation of the Radon measure m_0, then by Lemma 5.2 in [28], there exist positive functions $m_0^n \in C^\infty(S)$ such that $||m_0^n||_{L^1} \leq C$ for a constant

C independent of n, and $m_0^n \to m_0$ in $\mathcal{D}'(S)$. If we denote $u_0^n = \Lambda_4^{-4} m_0^n$, then $m_0^n = u_0^n - \partial_x^2 u_0^n + \partial_x^4 u_0^n$,

$$u_0^n \to u_0 \text{ in } H^2(S),$$

and

$$\begin{aligned} ||u_0^n||_{H^2}^2 &= \int_S |u_0^n|^2 + |u_{0x}^n|^2 + |u_{0xx}^n|^2 \mathrm{d}x = |\int_S m_0^n \cdot u_0^n \mathrm{d}x| \\ &\leq ||m_0^n||_{L^1} ||u_0^n||_{L^\infty} \leq C ||m_0^n||_{L^1} ||u_0^n||_{H^1}, \end{aligned} \tag{3.72}$$

which implies that

$$||u_0^n||_{H^2}^2 = \int_S |u_0^n|^2 + |u_{0x}^n|^2 + |u_{0xx}^n|^2 \mathrm{d}x \leq C ||m_0^n||_{L^1}^2 \leq C\theta^2. \tag{3.73}$$

Then by applying Theorems 3.1 and 3.8 with the smooth initial value $u_0^n(x)$, there exists a unique solution to (3.69) $u^n \in C([0,\infty); H^s) \cap C^1([0,\infty); H^{s-1})$. From Lemma 3.10, we know that $m^n(t,x) > 0$ if we denote $m^n = u^n - \partial_x^2 u^n + \partial_x^4 u^n$, so

$$||u^n(t)||_{H^2} = ||u_0^n||_{H^2} \leq C \quad \text{and} \quad ||m^n(t)||_{L^1} = ||m_0^n(t)||_{L^1} \leq C,$$

where C is a constant independent of n. Hence

$$||\partial_x^4 u^n||_{L^1} \leq ||u^n||_{L^1} + ||\partial_x^2 u^n||_{L^1} + ||m^n(t)||_{L^1} \leq C$$

and

$$||\partial_x^3 u^n||_{L^\infty} \leq C, \qquad \text{with } C \text{ independent of } n.$$

So $\{u^n(t)\}$ is a compact set in $H^2(S)$ for any $t \geq 0$.

On the other hand, $||\frac{\mathrm{d}u^n}{\mathrm{d}t}||_{H^2} = ||F(u^n)_x||_{H^2}$ can be estimated as follows:

$$\begin{aligned} ||[(u^n)^2]_x||_{H^2} &= 2||u^n u_x^n||_{H^2} \leq C(||u^n u_x^n||_{L^2} + ||u_{xx}^n u_x^n||_{L^2} + ||u^n u_{xxx}^n||_{L^2}) \\ &\leq C||\partial_x^3 u^n||_{L^2} \leq C||\partial_x^3 u^n||_{L^\infty} \leq C, \end{aligned} \tag{3.74}$$

$$\begin{aligned} &||\partial_x \Lambda_4^{-4}(v^2 + \tfrac{1}{2}v_x^2 - \tfrac{7}{2}v_{xx}^2 - 3v_x\partial_x^3 v)||_{H^2} \\ &\leq C||v^2 + \tfrac{1}{2}v_x^2 - \tfrac{7}{2}v_{xx}^2 - 3v_x\partial_x^3 v||_{H^{-1}} \\ &\leq C \quad \text{if } v = u^n. \end{aligned} \tag{3.75}$$

So $||\frac{\mathrm{d}u^n}{\mathrm{d}t}||_{H^2} = ||F(u^n)_x||_{H^2} \leq C$ with C independent of t and n. So the family $u^n(t)$, as functions from $[0,+\infty)$ to $H^2(S)$, is equi-continuous. Therefore Arzelà-Ascoli theorem ([98] or see the " Arzelà-Ascoli Theorem" in the Wikipedia) tells us that $\{u^n(t)\}_{n\geq 1} \subset C([0,T];H^2)$ is a compact subset for any $T > 0$. So we can extract a subsequence u^{n_k} and there is a function $u \in C([0,\infty);H^2)$ such that

$$u^{n_k} \to u \quad \text{in } C([0,\infty);H^2),$$

with

$$||u(t) - u(s)||_{H^2} \leq C|t-s|, \quad \forall\, t,s \geq 0.$$

From $\lim\limits_{n\to\infty} u^n(t) = u(t)$ in $H^2(S)$ for any $t \geq 0$, we have

$$u(0) = u_0.$$

Taking $n_k \to \infty$ in

$$\int_0^T\!\!\int_S (u^{n_k}\varphi_t + F(u^{n_k})\varphi_x)\mathrm{d}x\mathrm{d}t + \int_S u_0^{n_k}(x)\varphi(0,x)\mathrm{d}x = 0, \quad \forall\, \varphi \in C^{1,c}([0,T)\times S) \tag{3.76}$$

yields that $u \in C([0,\infty);H^2)$ is the weak solution to (3.69).

From the proof above, we can easily get the conserved quantities and that the total variation $||m(t,\cdot)||_{\mathcal{M}}$ of the limit measure m satisfies

$$||m(t,\cdot)||_{\mathcal{M}} \leq ||m^n(t)||_{L^1} = ||m_0^n(t)||_{L^1} \leq C.$$

Uniqueness: Now we are proving the uniqueness of the solution. Here we just sketch the proof, and a rigorous argument can be realised by a standard mollification method. Let $G(x)$ be the Green's function for the operator $\Lambda_4^4 = I - \partial_x^2 + \partial_x^4$ acting on $H^\infty(S)$, then from

$$(I - \partial_x^2 + \partial_x^4)G(x) = \delta(x) = \sum_{n=-\infty}^{\infty} e^{nix}, \tag{3.77}$$

we have

$$\begin{aligned} G(x) &= \sum_{n=-\infty}^{\infty} \frac{1}{1+n^2+n^4} e^{inx} \\ &= 1 + 2\sum_{n=1}^{\infty} \frac{1}{1+n^2+n^4}\cos(nx) \quad x \in S. \end{aligned} \tag{3.78}$$

Obviously, for any $0 \leq \varepsilon < 1$, $G(x) \in C^{2+\varepsilon}(S)$. Moreover, from the Appendix A of this thesis, we can even have that $\partial_x^3 G \in L^\infty(S)$.

Suppose $u, v \in C([0,\infty); H^2)$ are two solutions of (3.12), ie, they both solve the equation

$$\begin{cases} u_t = -uu_x - \partial_x \Lambda_4^{-4}(u^2 + \frac{1}{2}u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x\partial_x^3 u), \quad x \in S, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{3.79}$$

Or equivalently,

$$\begin{cases} u_t = -uu_x - G_x * (u^2 + \frac{1}{2}u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x\partial_x^3 u), \quad x \in S, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{3.80}$$

here $*$ stands for the convolution. Denote

$$M \equiv \sup_{t\geq 0}\{||\Lambda_4^4 u||_{\mathcal{M}} + ||\Lambda_4^4 v||_{\mathcal{M}}\} < \infty, \tag{3.81}$$

then for all $t, x \in \mathbb{R}_+ \times S$, we have

$$\begin{aligned} ||u(x,t)||_{L^\infty} &= ||G * m||_{L^\infty} \leq ||G||_{L^\infty}||m||_{\mathcal{M}} \leq CM, \\ ||u_x(x,t)||_{L^\infty} &= ||G_x * m||_{L^\infty} \leq CM, \\ ||u_{xx}(x,t)||_{L^\infty} &= ||G_{xx} * m||_{L^\infty} \leq CM, \\ ||u_{xxx}(x,t)||_{L^\infty} &= ||G_{xxx} * m||_{L^\infty} \leq CM, \end{aligned} \tag{3.82}$$

and same estimates hold true for v as well.

Let $w = u - v$ and $A(u) = u^2 + \frac{1}{2}u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x\partial_x^3 u$, then

$$\begin{cases} w_t = -uw_x - wv_x - G_x * (A(u) - A(v)), \quad x \in S,\ t > 0, \\ w|_{t=0} = 0. \quad x \in S \end{cases} \tag{3.83}$$

so

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_S |w|\mathrm{d}x = \int_S w_t \mathrm{sgn} w \\ = &\int -uw_x\mathrm{sgn}w - wv_x\mathrm{sgn}w - G_x * (A(u) - A(v))\mathrm{sgn}w. \end{aligned} \tag{3.84}$$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_S |w_x|\mathrm{d}x = \int_S w_{xt}\mathrm{sgn}w_x \\ = &\int -[w_x(u_x+v_x) + uw_{xx} + wv_{xx}]\mathrm{sgn}w_x - G_{xx} * (A(u)-A(v))\mathrm{sgn}w_x. \end{aligned} \tag{3.85}$$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_S |w_{xx}|\mathrm{d}x = \int_S w_{xxt}\mathrm{sgn}w_{xx} \\ = &-\int [w_{xx}(2u_x+v_x) + w_x(2v_{xx}+u_{xx}) + u\partial_x^3 w + w\partial_x^3 v]\mathrm{sgn}w_{xx} \\ &-\int G_{xxx} * (A(u)-A(v))\mathrm{sgn}w_{xx}. \end{aligned} \tag{3.86}$$

Using the estimates (3.82) for u, v, we have

$$\begin{aligned} \left|\int -uw_x\mathrm{sgn}w - wv_x\mathrm{sgn}w\right| &\leq CM\left(\int |w| + |w_x|\right); \\ \left|\int [w_x(u_x+v_x) + uw_{xx} + wv_{xx}]\mathrm{sgn}w_x\right| &\leq CM\left(\int |w| + |w_x| + |w_{xx}|\right); \\ \left|\int [w_{xx}(2u_x+v_x) + w_x(2v_{xx}+u_{xx})]\mathrm{sgn}w_{xx}\right| &\leq CM\left(\int |w_x| + |w_{xx}|\right). \end{aligned} \tag{3.87}$$

On the other hand, $A(u)-A(v) = w(u+v) + \frac{1}{2}w_x(u_x+v_x) - \frac{7}{2}w_{xx}(u_{xx} + v_{xx}) - 3u_x\partial_x^3 w - 3w_x\partial_x^3 v$, and integration by parts gives us

$$\begin{aligned} G_x * (u_x\partial_x^3 w) &= G_{xx} * (u_x w_{xx}) - G_x * (u_{xx}w_{xx}); \\ G_{xx} * (u_x\partial_x^3 w) &= G_{xxx} * (u_x w_{xx}) - G_{xx} * (u_{xx}w_{xx}) \end{aligned} \tag{3.88}$$

which enables us to estimate

$$\begin{aligned} \left|\int G_x * (A(u)-A(v))\mathrm{sgn}w\right| &\leq CM\left(\int |w| + |w_x| + |w_{xx}|\right); \\ \left|\int G_{xx} * (A(u)-A(v))\mathrm{sgn}w_x\right| &\leq CM\left(\int |w| + |w_x| + |w_{xx}|\right). \end{aligned} \tag{3.89}$$

The other terms in (3.86) can be estimated as follows:

$$\int u\partial_x^3 w\mathrm{sgn}w_{xx} = \int u\frac{\mathrm{d}}{\mathrm{d}x}|w_{xx}|\mathrm{d}x = -\int |w_{xx}|u_x\mathrm{d}x, \tag{3.90}$$

so

$$\left|\int u\partial_x^3 w \mathrm{sgn} w_{xx}\right| \leq CM \int |w_{xx}|. \tag{3.91}$$

It is easy to see

$$\left|\int w\partial_x^3 v \mathrm{sgn} w_{xx}\right| \leq C||\partial_x^3 v||_{L^\infty} \int |w| \leq CM \int |w|. \tag{3.92}$$

In order to estimate $\int G_{xxx} * (A(u) - A(v))\mathrm{sgn} w_{xx}$, we need only estimate $\int G_{xxx} * (u_x \partial_x^3 w)$ because the other terms can be estimated in the same way as the above terms. Again, the integration by parts yields

$$\begin{aligned} G_{xxx} * (u_x \partial_x^3 w) &= G_{xxxx} * (u_x w_{xx}) - G_{xxx} * (u_{xx} w_{xx}) \\ &= G_{xx} * (u_x w_{xx}) - G * (u_x w_{xx}) + u_x w_{xx} - G_{xxx} * (u_{xx} w_{xx}) \end{aligned} \tag{3.93}$$

here, we have used the definition of G, which gives us

$$G_{xxxx} * f - G_{xx} * f + G * f = f.$$

Now it is clear that

$$\left|\int G_{xxx} * (u_x \partial_x^3 w)\right| \leq CM \int |w_{xx}|. \tag{3.94}$$

Taking all the above estimates in account, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_S (|w| + |w_x| + |w_{xx}|)\mathrm{d}x \leq CM \int_S (|w| + |w_x| + |w_{xx}|)\mathrm{d}x, \tag{3.95}$$

and so the Gronwall's inequality yields $w \equiv 0$. That completes the proof of Theorem 3.15.

3.5 The Whole Real Line Case

We have discussed the well-posedness of equation (3.1) in the periodic case. Actually, some of the above results hold true with $\Lambda = (1-\partial_x^2)^k$ on the whole real line case:

$$m_t + 2u_x m + u m_x = a\partial_x^3 u \qquad \text{in } \mathbb{R}^1, \quad \text{with } m = (1 - \partial_x^2)^k u. \tag{3.96}$$

More specifically, the local well-posedness Theorem 3.1 holds true if $u_0 \in L^1(\mathbb{R}^1) \cap H^s(\mathbb{R}^1)$, combining our arguments here and those estimates established for $(1-\partial_x^2)$ in [97]. Theorem 3.8 with $m_0 \in L^2(\mathbb{R}^1)$ and $u_0 \in H^s(\mathbb{R}^1) \cap L^1(\mathbb{R}^1)$ holds true. Using Lemma 3.16 we are going to prove, we can prove that Theorem 3.12 holds true for (3.96) with $a = 0$ if we suppose $m_0 \geq 0$, $u_0 \in L^1(\mathbb{R}^1) \cap H^s(\mathbb{R}^1)$ with some $s > 2k - \frac{1}{2}$. If $a = 0$ in (3.96), Theorem 3.15 holds true for $u_0 \in H^s(\mathbb{R}^1) \cap L^(\mathbb{R}^1)$ with $m_0 = (1-\partial_x^2)^2 u_0$ a positive Radon measure.

In fact, the only things we need to check are

(a) $G(x) \geq 0$ ($x \in \mathbb{R}^1$) for the fundamental solution $G(x)$ of the operator $(1-\partial_x^2)^k$ on $\mathbb{R}^1$;

(b) $||\partial_x^3 G||_{L^\infty} < \infty$;

(c) the proof of Lemma 3.11;

The items (a) and (b) will be proved in the Appendix A, and here we just prove a lemma analogous to Lemma 3.11 (take $k = 2$ as an example).

Lemma 3.16 *Let* $a = 0$, $u_0 \in H^s(\mathbb{R}^1)$, $s > 7/2$, $m_0 = (1-\partial_x^2)^2 u_0 \geq 0$*(or* ≤ 0*) smooth enough and* $u_0 \in L^1(\mathbb{R}^1)$, *then* $\exists K > 0$ *such that* $||u_{xxx}||_{L^\infty} \leq K$.

Proof From the assumption $m_0 = (1-\partial_x^2)^2 u_0 \geq 0$, we can prove that $m(x,t) \geq 0$ for any $t \geq 0$ using the argument in Lemma 3.10, so we have $u = G * m \geq 0$ because $G(x) > 0$. From (3.96), we have

$$||u(t,\cdot)||_{L^1(\mathbb{R}^1)} = ||m(t,\cdot)||_{L^1(\mathbb{R}^1)} = ||m_0(t,\cdot)||_{L^1(\mathbb{R}^1)}, \tag{3.97}$$

and the conservation law

$$\int um = \int_{\mathbb{R}^1} (u^2 + 2u_x^2 + u_{xx}^2)\mathrm{d}x = \int_{\mathbb{R}^1} (u_0^2 + 2u_{0x}^2 + u_{0xx}^2)\mathrm{d}x, \tag{3.98}$$

which implies

$$||u_x||_{L^\infty} \leq C \tag{3.99}$$

by the Sobolev embedding theorem.

On the other hand, neither m nor u changes sign, so we have

$$0 \leq \int^x m\mathrm{d}x = \int_{-\infty}^x (u - 2\partial_x^2 u + \partial_x^4 u)\mathrm{d}x \leq ||u||_{L^1} - 2u_x + \partial_x^3 u \tag{3.100}$$

$$||u||_{L^1} = ||m||_{L^1} \geq \int_{-\infty}^x (u - 2\partial_x^2 u + \partial_x^4 u)\mathrm{d}x \geq -2u_x + \partial_x^3 u, \tag{3.101}$$

which implies

$$||2u_x - \partial_x^3 u||_{L^\infty} \leq ||u||_{L^1}. \tag{3.102}$$

So combining the equations (3.99)(3.102), we have

$$||\partial_x^3 u||_{L^\infty} \leq C \tag{3.103}$$

with a constant C depending only on the L^1 norm and H^2 norm of the initial u_0.

3.6 Remarks on the Generalisations

In fact, for the circle S case, the arguments in this chapter up to now can be easily extended to the general H^k metric case ($k \geq 1$) with some obvious modifications.

Going further, from the derivation of the equation, we can find that the specific form of the inertia operator Λ does not really matter. We can take, for example, $\Lambda \equiv (1 - \partial_x^2)^k$ with some integer $k \geq 1$ (that is what we have done in the whole real line case), or even more generally, $\Lambda = (1 - \partial_x^2)^{\frac{r}{2}}$ for any real number $r \geq 0$, and derive the corresponding generalised Euler equation with respect to the H^r metric. From the arguments above, for the limiting case $a = 0$, we know that as long as $r > \frac{3}{2}$, there will be no finite time blowup phenomena.

But for the whole line $\mathbb{R}$ case, it seems that we need to choose $\Lambda = (1-\partial_x^2)^k$ in order to guarantee that the corresponding Green's function is positive. We have to impose another assumption $m_0 \in L^1(\mathbb{R}^1)$ and some extra assumptions on u_0 as we did in the previous section.

3.7 Conjugate Points and Beyond

In this section, we turn back to the geometrical aspect of the mCH. Now that mCH is the geodesic equation and its solution the geodesic curve, it is natural to study the geometry of $\widehat{\mathcal{D}}(S)$ around the geodesic curve. Here, we are going to exploit the sectional curvature etc to investigate the existence of conjugate points.

Theorem 3.17 *The geodesic in $\widehat{\mathcal{D}}(S)$ with initial conditions $\hat{\eta}(0) = (e, 0)$ and $\dot{\hat{\eta}}(0) = (v_0 \frac{\partial}{\partial x}, b)$, where v_0, b are constants, contains points conjugate to $\hat{\eta}(0)$ along $\hat{\eta}$.*

Proof For the right-invariant vector fields $\widehat{U} = \left(u\frac{\partial}{\partial x}, a\right), \widehat{V} = \left(v\frac{\partial}{\partial x}, b\right)$, the covariant derivative $\nabla_{\widehat{U}}\widehat{V}$ can be obtained from the formula ([22])

$$\begin{aligned}
&2\nabla_{\widehat{U}}\widehat{V} \\
&= [\widehat{U}, \widehat{V}] - \mathrm{ad}^*_{\widehat{U}}\widehat{V} - \mathrm{ad}^*_{\widehat{V}}\widehat{U} \\
&= [\widehat{U}, \widehat{V}] - \left((\Lambda_{2k}^{-2k}(2u_x\Lambda_{2k}^{2k}v + u\Lambda_{2k}^{2k}v_x + 2v_x\Lambda_{2k}^{2k}u + v\Lambda_{2k}^{2k}u_x + b\partial_x^3 u + a\partial_x^3 v)\tfrac{\partial}{\partial x}, 0\right) \\
&= \left((u_xv - uv_x - \Lambda_{2k}^{-2k}(2u_x\Lambda_{2k}^{2k}v + u\Lambda_{2k}^{2k}v_x + 2v_x\Lambda_{2k}^{2k}u + v\Lambda_{2k}^{2k}u_x + b\partial_x^3 u + a\partial_x^3 v)\tfrac{\partial}{\partial x},\right. \\
&\left. c(u,v)\right),
\end{aligned} \tag{3.104}$$

ie

$$\nabla_{\widehat{U}}\widehat{U} = -\left((\Lambda_{2k}^{-2k}(2u_x\Lambda_{2k}^{2k}u + u\Lambda_{2k}^{2k}u_x + a\partial_x^3 u)\frac{\partial}{\partial x}, 0)\right).$$

On the other hand,

$$[[\widehat{U}, \widehat{V}], \widehat{V}] = \left(((u_xv - uv_x)_x v - (u_xv - uv_x)v_x)\frac{\partial}{\partial x}, -c(u_xv - uv_x, v)\right),$$

so by the formula ([22])

$$R(\widehat{U}, \widehat{V})\widehat{V} = \nabla_{\widehat{U}}\nabla_{\widehat{V}}\widehat{V} - \nabla_{\widehat{V}}\nabla_{\widehat{U}}\widehat{V} - \nabla_{[\widehat{U},\widehat{V}]}\widehat{V} \tag{3.105}$$

$$\begin{aligned}
R(\widehat{U}, \widehat{V}) &= (R(\widehat{U}, \widehat{V})\widehat{V}, \widehat{U})_{H^k} = \tfrac{1}{4}||\mathrm{ad}^*_{\widehat{U}}\widehat{V} + \mathrm{ad}^*_{\widehat{V}}\widehat{U}||^2_{H^k} - \left(\mathrm{ad}^*_{\widehat{U}}\widehat{U},\ \mathrm{ad}^*_{\widehat{V}}\widehat{V}\right)_{H^k} \\
&\quad -\tfrac{3}{4}||[\widehat{U}, \widehat{V}]||^2_{H^k} - \tfrac{1}{2}\left([[\widehat{U}, \widehat{V}], \widehat{V}], \widehat{U}\right)_{H^k} - \tfrac{1}{2}\left([[\widehat{V}, \widehat{U}], \widehat{U}], \widehat{V}\right)_{H^k},
\end{aligned} \tag{3.106}$$

we can get the Riemaniann curvature $R(\widehat{U}, \widehat{V})\widehat{V}$ and the sectional curvature $R(\widehat{U}, \widehat{V}) \equiv (R(\widehat{U}, \widehat{V})\widehat{V}, \widehat{U})_{H^k}$ although the calculation is lengthy and messy. However, if $\widehat{V} = (v_0\frac{\partial}{\partial x}, b)$ is a constant vector field, then the calculation is much simpler:

$$\nabla_{\widehat{U}}\widehat{V} = -\left((v_0\Lambda_{2k}^{-2k}u_x + \frac{1}{2}b\Lambda_{2k}^{-2k}\partial_x^3 u)\frac{\partial}{\partial x}, 0\right), \tag{3.107}$$

$$\nabla_{\widehat{V}}\widehat{U} = -\left((v_0 u_x + v_0\Lambda_{2k}^{-2k}u_x + \frac{1}{2}b\Lambda_{2k}^{-2k}\partial_x^3 u)\frac{\partial}{\partial x}, 0\right), \tag{3.108}$$

$$\nabla_{[\widehat{U},\widehat{V}]}\widehat{V} = -\left((v_0^2\Lambda_{2k}^{-2k}u_{xx} + \frac{1}{2}bv_0\Lambda_{2k}^{-2k}\partial_x^3 u)\frac{\partial}{\partial x}, 0\right), \tag{3.109}$$

$$\begin{aligned}\nabla_{\widehat{V}}\nabla_{\widehat{U}}\widehat{V} = \Big((v_0^2\Lambda_{2k}^{-2k}u_{xx} &+ \frac{1}{2}b\Lambda_{2k}^{-2k}\partial_x^4 u + v_0^2\Lambda_{2k}^{-4k}u_{xx} \\ &+ bv\Lambda_{2k}^{-4k}\partial_x^4 u + \frac{1}{4}\Lambda_{2k}^{-4k}\partial_x^6 u)\frac{\partial}{\partial x}, 0\Big),\end{aligned} \tag{3.110}$$

so the Riemannian curvature

$$\begin{aligned} R(\widehat{U},\widehat{V})\widehat{V} &= \nabla_{\widehat{U}}\nabla_{\widehat{V}}\widehat{V} - \nabla_{\widehat{V}}\nabla_{\widehat{U}}\widehat{V} - \nabla_{[\widehat{U},\widehat{V}]}\widehat{V} \\ &= \left((-\frac{1}{4}b^2\Lambda_{2k}^{-4k}\partial_x^6 u - v_0^2\Lambda_{2k}^{-4k}u_{xx} - bv_0\Lambda_{2k}^{-4k}\partial_x^4 u)\frac{\partial}{\partial x}, 0\right). \end{aligned} \tag{3.111}$$

and the sectional curvature

$$\begin{aligned} R(\widehat{U},\widehat{V}) &= (R(\widehat{U},\widehat{V})\widehat{V},\widehat{U})_{H^k} \\ &= \frac{1}{4}b^2\int_S \partial_x^3 u\Lambda_{2k}^{-2k}\partial_x^3 u + v_0^2\int u_x\Lambda_{2k}^{-2k}u_x - bv_0\int u_{xx}\Lambda_{2k}^{-2k}u_{xx} \\ &= \frac{1}{4}\int_S \left(-b\Lambda_{2k}^{-k}\partial_x^3 u + 2v_0\Lambda_{2k}^{-k}u_x\right)^2 \mathrm{d}x \geq 0. \end{aligned} \tag{3.112}$$

Let $\widehat{\eta}(t)$ be the geodesic with the initial condition $\dot{\widehat{\eta}}(t) = \widehat{V}$, and $\widehat{W}(t)$ be an arbitrary vector along $\widehat{\eta}(t)$ and

$$(w(t,x)\frac{\partial}{\partial x}, s(t)) \equiv \mathrm{d}_{\widehat{\eta}(t)}R_{\widehat{\eta}^{-1}(t)}\widehat{W}(t),$$

where R_g denote the right multiplication by g on the Virasoro group.

$$\begin{aligned} &\mathrm{d}_{\widehat{\eta}_t}R_{\widehat{\eta}_t^{-1}}\left(R(\widehat{W}(t),\dot{\widehat{\eta}}(t))\dot{\widehat{\eta}}(t)\right) \\ = \;& R\left(\left(w(t,x)\frac{\partial}{\partial x}, s(t)\right), \left(v_0\frac{\partial}{\partial x}, b\right)\right)\left(v_0\frac{\partial}{\partial x}, b\right) \\ = \;& -\frac{1}{4}\left(\left(b^2\Lambda_{2k}^{-4k}\partial_x^6 w + 4v_0^2\Lambda_{2k}^{-4k}w_{xx} + bv_0\Lambda_{2k}^{-4k}\partial_x^4 w\right)\frac{\partial}{\partial x}, 0\right). \end{aligned} \tag{3.113}$$

$$
\begin{aligned}
& \mathrm{d}_{\widehat{\eta}_t} R_{\widehat{\eta}_t^{-1}} \left(\nabla_{\dot{\widehat{\eta}}_t} \nabla_{\dot{\widehat{\eta}}_t} \widehat{W}(t) \right) \\
= \;& \nabla_{(v_0 \frac{\partial}{\partial x}, b)} \nabla_{(v_0 \frac{\partial}{\partial x}, b)} \left(w(t,x) \frac{\partial}{\partial x}, s(t) \right) \\
= \;& \nabla_{(v_0 \frac{\partial}{\partial x}, b)} \left([w_t - (v_0 w_x + v_0 \Lambda_{2k}^{-2k} \partial_x w + \frac{1}{2} \Lambda_{2k}^{-2k} \partial_x^3 w)] \frac{\partial}{\partial x}, s'(t) \right) \\
= \;& \left(H(t,x) \frac{\partial}{\partial x}, s''(t) \right),
\end{aligned}
\tag{3.114}
$$

where

$$
\begin{aligned}
H(t,x) = \;& \partial_x^2 w - 2v_0 w_{tx} + v_0^2 w_{xx} - 2v_0 \Lambda_{2k}^{-2k} w_{xt} - b\Lambda_{2k}^{-2k} \partial_t \partial_x^3 w + 2v_0^2 \Lambda_{2k}^{-2k} w_{xx} \\
& + b v_0 \Lambda_{2k}^{-2k} \partial_x^4 w + v_0^2 \Lambda_{2k}^{-4k} w_{xx} + b v_0 \Lambda_{2k}^{-4k} \partial_x^4 w + \frac{1}{4} b^2 \Lambda_{2k}^{-4k} \partial_x^6 w.
\end{aligned}
\tag{3.115}
$$

Then the Jacobi equation along $\widehat{\eta}(t)$

$$
\nabla_{\dot{\widehat{\eta}}(t)} \nabla_{\dot{\widehat{\eta}}(t)} \widehat{W}(t) + R(\widehat{W}(t), \dot{\widehat{\eta}}(t)) \dot{\widehat{\eta}}(t) = 0 \tag{3.116}
$$

reads $s''(t) = 0$ and

$$
\begin{aligned}
& \frac{\partial^2 w}{\partial t^2} - 2v_0 \frac{\partial^2 w}{\partial t \partial x} + v_0^2 \frac{\partial^2 w}{\partial x^2} + 2v_0^2 \Lambda_{2k}^{-2k} \frac{\partial^2 w}{\partial x^2} - 2v_0 \Lambda_{2k}^{-2k} \frac{\partial^2 w}{\partial t \partial x} \\
& \qquad - b\Lambda_{2k}^{-2k} \frac{\partial^4 w}{\partial t \partial x^3} + b v_0 \Lambda_{2k}^{-2k} \frac{\partial^4 w}{\partial x^4} = 0
\end{aligned}
\tag{3.117}
$$

that is

$$
\left(\frac{\partial}{\partial t} - v_0 \frac{\partial}{\partial x} \right)^2 w - 2v_0 \Lambda_{2k}^{-2k} \left(\frac{\partial}{\partial t} - v_0 \frac{\partial}{\partial x} \right) w_x - b\Lambda_{2k}^{-2k} \left(\frac{\partial}{\partial t} - v_0 \frac{\partial}{\partial x} \right) \partial_x^3 w = 0. \tag{3.118}
$$

For any integer $n \geq 1$, if we denote $k(n) = (1 + n^2 + n^4 + \cdots + n^{2k})^{-1}$,

$$
\mu = n v_0 k(n) - \frac{1}{2} b k(n) n^3 \quad \text{and} \quad \lambda = n v_0 + n v_0 k(n) - \frac{1}{2} b k(n) n^3,
$$

then a direct calculation tells us that

$$
w(t,x) = \sin(\mu t) \sin(nx + \lambda t), \qquad s(t) \equiv 0,
$$

is a non-trivial solution to the Jacobi equation (3.117). Clearly, $\widehat{W}$ is always perpendicular to $\dot{\widehat{\eta}}(t)$, so it is a Jacobi field along $\widehat{\eta}(t)$. If we take

$$t = \frac{2\pi j}{\mu} \quad \text{for } j = 0, \pm 1, \pm 2, \cdots$$

we get the points conjugate to $\widehat{\eta}(0)$, which completes the proof of Theorem 3.17.

Remark From the theorem 3.17, we can obtain that the constant solutions are stable. However, if we are concerned only with the stability of the constant solutions, we can use the energy method to give a very simple proof.

Theorem 3.18 *Any constant solution* (v_0, b) *is nonlinearly stable.*

Proof Let $m = m_0 = v_0$ be the constant solution, we can introduce the functional

$$H_1(m) = \frac{1}{2}\int_S um\mathrm{d}x - \int_S v_0 m\mathrm{d}x, \tag{3.119}$$

then it is easy to check that

$$\left.\frac{\delta H_1}{\delta m}\right|_{m_0} = 0, \quad \left.\frac{\delta^2 H_1}{\delta m^2}\right|_{m_0} = \Lambda_{2k}^{-2k} > 0, \tag{3.120}$$

which yields that m_0 is a local strict minimum point of H_1.

3.8 Conclusions

We have studied various analytical properties of the one dimensional mCH. We have first derived the one dimensional mCH according to Arnold's viewpoint in Section 3.1, then have exploited the Kato theory to establish its local well-posedness in $H^s(S)$ with $s > 2k - \frac{1}{2}$ in Section 3.2. After that, we have proved that $||m||_{L^2}$ is always finite in finite time t if the initial momentum m_0 belongs to $L^2(S)$, which means that the solutions of the mCH will not blow-up in finite time if the initial value is smooth enough. This is totally different from the Camassa-Holm equation which may admit some finite time blow-up solution even for some very smooth initial values. Then we have studied the extra properties for the limiting mCH, ie, the case of $a = 0$. In this case, we have proved that the mCH admits a unique solution $u \in C([0, +\infty), H^s(S)) \cap C^1([0, +\infty), H^{s-1}(S))$ if the initial momentum does not change sign (here m_0 may not be in $L^2(S)$). In Section 3.4, we have introduced the notion of weak solution which includes the δ-momentum solutions, and then proved its well-posedness for the limiting mCH with the initial momentum being positive Radon measures, ie, Theorem 3.15, by an approximation process. I guess the assumption $a = 0$ is only of technique significance and the similar results hold true also for the general case $a \neq 0$ although I have not yet found a proof. The difficulty here is that for the case $a \neq 0$, we can not have $\int |m| = \int |m_0|$ from the conservation of $\int m$. Then I have made some remarks on the generalisations of the previous results to the whole line $\mathbb{R}^1$ case under some mild extra assumptions, and to different inertia operators in Sections 3.5, 3.6 respectively. Here in order to get the required positivity of the Green's function G, we have to switch to the inertia operator $(1 - \partial_x^2)^k$. Then, we have looked at the geometry of $\widehat{\mathcal{D}}(S)$ around the geodesic curve and proved that the geodesic in $\widehat{\mathcal{D}}(S)$ has conjugate points to the starting point.

Chapter 4

Numerics

"A major task of mathematics is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both."

—E. T. Bell, *Men of Mathematics,* 1965.

In this chapter, we will consider the numerical methods for the limiting case of mCH for $k = 2$, ie,

$$\begin{cases} m_t = -um_x - 2mu_x \\ m(x,0) = m_0(x). \end{cases} \tag{4.1}$$

Why do we consider the case $a = 0$? This is because in the study of the CH equation, the limiting case is the most interesting case. Moreover, our study on weak solutions in Chapter 3 is restricted to the limiting case. So we focus here on the limiting case $a = 0$ for the mCH.

We will consider (4.1) on the circle S in Section 4.1 and on the whole real line $\mathbb{R}^1$ in Section 4.2.

4.1 Particle methods

The point vortex algorithm is one of the most efficient methods in the study of ideal hydrodynamics. Similarly, we can introduce the particle method in the computation of mCH. The basic idea is as follows: from Theorem 3.15, we know (4.1) has solutions supported at points on S via the following sum over Dirac delta measures,

$$m(t,x) = \sum_{i=1}^{N} p_i(t)\delta(x - q_i(t)), \tag{4.2}$$

and the velocity

$$u(t,x) = G * m = \sum_{i=1}^{N} p_i(t) G(x - q_i(t)) \tag{4.3}$$

is the superposition of the velocity of each "soliton" supported at $q_i(t)$, where G is the fundamental solution of the inertia operator Λ. Plugging (4.2)(4.3) into (4.1), we have got an ODE of 2N variables:

$$\begin{cases} \dot{q}_i = h \sum_{j=1}^{N} G(q_i - q_j) p_j, & i = 1, 2, \cdots, N, \\ \dot{p}_i = -h p_i \sum_{j=1}^{N} G'(q_i - q_j) p_j, & i = 1, 2, \cdots, N. \end{cases} \tag{4.4}$$

Camassa et al. [17] proved that the particle method is convergent for the Camassa-Holm equation, their proof used the complete integrability of the equation although then mentioned that the use of integrability is perhaps "overkill". We can see that the reduction of the PDE (4.1) to the ODE (4.4) is not related to the integrability. We will show that it applies to the modified CH equation too. Our proof is similar to those of [17] except that we do not use the integrability of its discretization to prove the global existence of the corresponding ODEs.

Let $G(x)$ be the Green's function for the operator $\Lambda_4^4 = I - \partial_x^2 + \partial_x^4$ acting on $H^\infty(S)$, then from

$$(I - \partial_x^2 + \partial_x^4) G(x) = \delta(x) = \sum_{n=-\infty}^{\infty} e^{nix}, \tag{4.5}$$

we have

$$G(x) = \sum_{n=-\infty}^{\infty} \frac{1}{1 + n^2 + n^4} e^{inx} = 1 + 2 \sum_{n=1}^{\infty} \frac{1}{1 + n^2 + n^4} \cos(nx) \quad x \in S. \tag{4.6}$$

Obviously, for any $0 \leq \varepsilon < 1$, $G(x) \in C^{2+\varepsilon}(S)$.

Now that

$$u(x,t) = \int_0^{2\pi} G(x-y)m(y,t)\mathrm{d}y \tag{4.7}$$

and if $m_0 \geq c > 0$, then from Lemma 3.10 we have $m(x,t) \geq 0$ for any $t > 0$, so (4.1) can be rewritten as

$$(m^{1/2})_t = -(um^{1/2})_x \ . \tag{4.8}$$

Let us introduce an auxiliary function

$$w(x,t) = \int_0^{x} m(y,t)^{1/2}\mathrm{d}y, \tag{4.9}$$

then $w_{xt} + (uw_x)_x = 0, \ \forall \ x \in S$. So there exists a function $g(t)$ such that

$$w_t + uw_x = g(t) \qquad \forall \ x \in S. \tag{4.10}$$

Introducing characteristic curves

$$x = q(\xi,t), \qquad q(\xi,0) = \xi, \tag{4.11}$$

then Equation (4.10) reads as

$$\dot{x} = \dot{q} = u(q,t), \quad \dot{w} = g, \tag{4.12}$$

where $\dot{f}$ denotes the total derivative

$$\dot{f} \equiv \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) f.$$

From (4.12), we have

$$w(q(\xi,t),t) = \int_0^t g(s)\mathrm{d}s + w(\xi,0), \qquad \text{and so} \qquad \frac{\mathrm{d}w}{\mathrm{d}\xi} = \frac{\mathrm{d}w_0}{\mathrm{d}\xi}, \tag{4.13}$$

where $w_0(\xi) \equiv w(\xi,0)$ and $\frac{\mathrm{d}w}{\mathrm{d}\xi}$ is uniquely determined by m. Combining (4.7) with the first equation of (4.12) gives

$$u(q(\xi,t),t) = \dot{q}(\xi,t) = \int_0^{2\pi} G(q(\xi,t)-q(\eta,t))m(q(\eta,t),t)\frac{\partial q(\eta,t)}{\partial \eta}\mathrm{d}\eta. \tag{4.14}$$

From (4.9) and (4.13) we have

$$m(q(\xi,t),t) = \left(\frac{\frac{\mathrm{d}w}{\mathrm{d}\xi}}{\frac{\partial q(\xi,t)}{\partial \xi}}\right)^2 = \left(\frac{\frac{\mathrm{d}w_0}{\mathrm{d}\xi}}{\frac{\partial q(\xi,t)}{\partial \xi}}\right)^2. \tag{4.15}$$

Introducing an auxiliary function

$$p(\xi,t) = m(q(\xi,t),t)\frac{\partial q(\xi,t)}{\partial \xi} = \frac{(w_0'(\xi))^2}{\frac{\partial q(\xi,t)}{\partial \xi}}, \tag{4.16}$$

then we have

$$\dot{p}(\xi,t) = -p(\xi,t)\int_0^{2\pi} G'(q(\xi,t)-q(\eta,t))p(\eta,t)\mathrm{d}\eta, \tag{4.17}$$

and (4.14) becomes

$$\dot{q}(\xi,t) = \int_0^{2\pi} G(q(\xi,t)-q(\eta,t))p(\eta,t)\mathrm{d}\eta. \tag{4.18}$$

The solution to (4.17)(4.18) with the initial conditions $q(\xi,0)=\xi,\ p(\xi,0)=(w_0'(\xi))^2$ determines the characteristic curves $x=q(\xi,t)$. On the other hand, (4.17)(4.18) is a Hamiltonian system with

$$H = \frac{1}{2}\int_{S\times S} G(q(\xi,t)-q(\eta,t))p(\xi,t)p(\eta,t)\mathrm{d}\xi\mathrm{d}\eta.$$

Integrating directly (4.17) yields that

$$P \equiv \int_0^{2\pi} p(\xi,t)\mathrm{d}\xi$$

is independent of time t because $G'(x)$ is symmetric with respect to $x=\pi$.

From (4.17) we have $|\dot{p}(\xi,t)/p(\xi,t)| \le c_1 P$, where $c_1 = |G'(x)|_{L^\infty}$. So

$$p(\xi,0)e^{-c_1Pt} \le p(\xi,t) \le p(\xi,0)e^{c_1Pt}, \qquad \forall \xi\in S \tag{4.19}$$

In order to approximate the Hamiltonian equations (4.17)(4.18):

$$\begin{cases} \dot{q}(\xi,t) &= \displaystyle\int_0^{2\pi} G(q(\xi,t)-q(\eta,t))p(\eta,t)\mathrm{d}\eta, \\ \dot{p}(\xi,t) &= \displaystyle -p(\xi,t)\int_0^{2\pi} G'(q(\xi,t)-q(\eta,t))p(\eta,t)\mathrm{d}\eta, \end{cases} \tag{4.20}$$

we can use the so-called *particle method*, which takes

$$q_i(t) \equiv q_i(\xi_i,t), \qquad p_i(t) \equiv p_i(\xi_i,t), \quad i\in\mathbb{N}$$

as position coordinates and momenta, and if, for example, q and p are evaluated at points $\xi_i = ih,\ i = 1, 2, \cdots, N$, obtain the (finite dimensional) discretised version of (4.20):

$$\begin{cases} \dot{q}_i = h\sum_{j=1}^{N} G(q_i - q_j)p_j, & i = 1, 2, \cdots, N, \\ \dot{p}_i = -hp_i\sum_{j=1}^{N} G'(q_i - q_j)p_j, & i = 1, 2, \cdots, N. \end{cases} \tag{4.21}$$

Compared with other classical numerical methods for PDE, one of the main advantages of the particle method is that it preserves the Hamiltonian structure of (4.20) and so we can use the geometric integration [49] to simulate it numerically.

Proposition 4.1 *For any $l_p > 0$, the right hand side of (4.21) is Lipschitz continuous on $(q, p) \in D$, where $D \subset \mathbb{R}^{2N}$ is the set of points $(q, p) = (q_1, q_2, \cdots, q_N; p_1, p_2, \cdots, p_N)$:*

$$0 \le q_i \le 2\pi, \quad i = 1, 2, \cdots, N, \quad \max_i |p_i| < l_p < \infty.$$

So the system of ODEs (4.21) admits a unique local solution.

Proof We demonstrate only for the first equation of (4.21) and omit the second one for it is analogous. Let $(p, q),\ (\tilde{p}, \tilde{q}) \in D \subset \mathbb{R}^{2N},\ c_0 =$

$\max_{x\in S}|G(x)|,\ c_1=\max_{x\in S}|G'(x)|$, then

$$
\begin{aligned}
&\left|h\sum_{j=1}^N G(q_i-q_j)p_j - h\sum_{j=1}^N G(\tilde{q}_i-\tilde{q}_j)\tilde{p}_j\right| \\
\leq\ & \left|h\sum_{j=1}^N G(q_i-q_j)p_j - h\sum_{j=1}^N G(q_i-q_j)\tilde{p}_j\right. \\
&\left. +h\sum_{j=1}^N G(q_i-q_j)\tilde{p}_j - h\sum_{j=1}^N G(\tilde{q}_i-\tilde{q}_j)\tilde{p}_j\right| \\
\leq\ & \left|h\sum_{j=1}^N G(q_i-q_j)p_j - h\sum_{j=1}^N G(q_i-q_j)\tilde{p}_j\right| \\
&+\left|h\sum_{j=1}^N G(q_i-q_j)\tilde{p}_j - h\sum_{j=1}^N G(\tilde{q}_i-\tilde{q}_j)\tilde{p}_j\right| \\
\leq\ & c_0h\sum_{j=1}^N|p_j-\tilde{p}_j| + h\sum_{j=1}^N|G(q_i-q_j)-G(\tilde{q}_i-\tilde{q}_j)|\,|\tilde{p}_j| \\
\leq\ & c_0h\sum_{j=1}^N|p_j-\tilde{p}_j| + c_1h\max_j|\tilde{p}_j|\left(N|q_i-\tilde{q}_i|+\sum_{j=1}^N|q_j-\tilde{q}_j|\right)
\end{aligned}
$$

So if we denote $||v||=\displaystyle\sum_{i=1}^N|v_i|$ for $v\in\mathbb{R}^N,\ L=\max\{c_0h, c_1hl_pN\}$, then we have the wanted estimate:

$$\left|h\sum_{j=1}^N G(q_i-q_j)p_j - h\sum_{j=1}^N G(\tilde{q}_i-\tilde{q}_j)\tilde{p}_j\right| \leq L(||p-\tilde{p}||+||q-\tilde{q}||). \tag{4.22}$$

For the global in time existence we have

Proposition 4.2 *If the initial momenta are positive, $p_i\geq\varepsilon>0,\ i=1,2,\cdots,N$, then the solution to (4.21) exists uniquely for all times.*

Proof From the Hamiltonian structure of (4.21), it is not difficult to prove that $P=h\sum_{i=1}^N p_i$ is independent of time t. From the second equation of (4.21), we have $|\frac{\dot{p}_i}{p_i}|\leq c_1P$, where $c_1=\max|G'(x)|$, so

$$p_i(0)e^{-c_1Pt}\leq p_i(t)\leq p_i(0)e^{c_1Pt}. \tag{4.23}$$

If the initial momenta are positive, then $p_i(t)$ are positive and bounded for all times $t<+\infty$. So the global existence follows.

Now we will prove that the solution of (4.21) converges to that of (4.20) as $h \to 0$.

Let $q(\xi, t), p(\xi, t)$ be the solution to (4.20) with the initial data

$$q(\xi, 0) = \xi, \quad p(\xi, 0) = p^0(\xi), \tag{4.24}$$

while $\tilde{q}(t),\ \tilde{p}(t)$ stand for the solution to (4.21) with

$$\tilde{q}_i(0) = q(\xi_i, 0) = \xi_i, \quad \tilde{p}_i(0) = p^0(\xi_i). \tag{4.25}$$

$q_i(t) = q(\xi_i, t),\ p_i(t) = p(\xi_i, t)$ denotes the PDE solution evaluated at the grid points, $\phi_i = q_i - \tilde{q}_i, \psi_i = p_i - \tilde{p}_i$ and $||\phi|| = h \sum_{i=1}^N |\phi_i|,\ ||\psi|| = h \sum_{i=1}^N |\psi_i|$ denote the l_1 norm. From (4.19)(4.23) we easily know that for any $T > 0$, there exists a constant $P_T < \infty$, independent of h, such that

$$\max\{\tilde{p}_i(t) : 1 \le i \le N; 0 \le t \le T\}, \max\{p(\xi, t) : \xi \in S; 0 \le t \le T\} \le P_T$$

for h small enough (or equivalently, for N large enough).

Theorem 4.3 *Consider (4.20) with (4.24) and (4.21) with (4.25). If $p^0(\xi) > 0,\ \xi \in S$, smooth enough, then for any finite time $T > 0$, there exists a grid length h such that*

$$||\phi(t)|| + \frac{1}{P_T}||\psi(t)|| \le \frac{Ch^2}{P_T}(e^{C'P_T t} - 1) \tag{4.26}$$

for $0 \le t \le T$, where $C,\ C'$ are constants independent of T and h.

Proof Because the two equations have the same initial values at the grid points, so

$$\begin{aligned} |\tilde{q}_i(t) - q(\xi_i, t)| \ \le\ & \int_0^t \Bigg| h \sum_{j=1}^N G(q_i(s) - q_j(s)) p_j(s) \\ & - \int_S G(q(\xi_i, s) - q(\eta, s)) p(\eta, s) \mathrm{d}\eta \Bigg| \mathrm{d}s \\ + \ & h \int_0^t \Bigg| \sum_{j=1}^N G(\tilde{q}_i(s) - \tilde{q}_j(s)) \tilde{p}_j(s) - \sum_{j=1}^N G(q_i(s) - q_j(s)) p_j(s) \Bigg| \mathrm{d}s. \end{aligned} \tag{4.27}$$

The first integral of the right hand side is controlled above by $Ch^2 t$ because the Riemannian sum $h \sum_{j=1}^N G(q_i(s) - q_j(s)) p_j(s)$ is the composite trapezoidal approximation of the integral $\int_S G(q(\xi_i, s) - q(\eta, s))p(\eta, s)\mathrm{d}\eta$. The second one is estimated as follows (letting $c_0 = \max|G(x)|,\ c_1 = \max|G'(x)|$.):

$$\begin{aligned}
& h\left|\sum_{j=1}^N G(\tilde{q}_i(s) - \tilde{q}_j(s))\tilde{p}_j(s) - \sum_{j=1}^N G(q_i(s) - q_j(s))p_j(s)\right| \\
\leq\ & h\left|\sum_{j=1}^N G(\tilde{q}_i(s) - \tilde{q}_j(s))\tilde{p}_j(s) - \sum_{j=1}^N G(\tilde{q}_i(s) - \tilde{q}_j(s))p_j(s)\right| \\
& + h\left|\sum_{j=1}^N G(\tilde{q}_i(s) - \tilde{q}_j(s))p_j(s) - \sum_{j=1}^N G(q_i(s) - q_j(s))p_j(s)\right| \\
\leq\ & c_0 h \sum_{j=1}^N |p_j(s) - \tilde{p}_j(s)| + c_1 h \sum_{j=1}^N \left(|q_i(s) - \tilde{q}_i(s)| + |q_j(s) - \tilde{q}_j(s)|\right) p_j(s) \\
\leq\ & c_0 h \sum_{j=1}^N |p_j(s) - \tilde{p}_j(s)| + c_1 P_T h \sum_{j=1}^N \left(|q_i(s) - \tilde{q}_i(s)| + |q_j(s) - \tilde{q}_j(s)|\right),
\end{aligned}$$

so

$$|\phi_i| \leq \int_0^t \left(c_0 h \sum_{j=1}^N |\psi_j| + c_1 P_T h \sum_{j=1}^N (|\phi_i| + |\phi_j|) \right) \mathrm{d}s + Ch^2 t, \tag{4.28}$$

and hence

$$||\phi(t)|| \leq \int_0^t \left(c_0||\psi|| + 2c_1 P_T ||\phi||\right) \mathrm{d}s + Ch^2 t. \tag{4.29}$$

Similarly, we have (letting $c_2 = \max|G''(x)|$)

$$\begin{aligned}
|\psi_i(t)| \quad \leq \quad & \int_0^t (2c_1 P_T h |p_i(s) - \tilde{p}_i(s)| + c_2 p_i P_T h \sum_{j=1}^N (|q_i(s) - \tilde{q}_i(s)| \\
& + |q_j(s) - \tilde{q}_j(s)|))\mathrm{d}s + Ch^2 t,
\end{aligned} \tag{4.30}$$

and hence

$$||\psi(t)|| \leq \int_0^t \left(2c_1 h P_T ||\psi(s)|| + 2c_2 {P_T}^2 ||\phi(s)||\right) \mathrm{d}s + Ch^2 t, \tag{4.31}$$

from which we have

$$\begin{aligned}
||\phi(t)|| + \frac{1}{P_T}||\psi(t)|| \quad \leq \quad & \int_0^t \left(2(c_1 + c_2)P_T||\phi|| + (c_0 + 2c_1 h)||\psi||\right) \mathrm{d}s + Ch^2 t \\
= \quad & 2(c_1 + c_2)P_T \int_0^t (||\phi|| + \frac{1}{P_T}||\psi||)\mathrm{d}s \\
& + (2c_1 h + c_0 - 2c_1 - 2c_2) \int_0^t ||\psi||\mathrm{d}s + Ch^2 t \\
\leq \quad & 2(c_1 + c_2)P_T \int_0^t (||\phi|| + \frac{1}{P_T}||\psi||)\mathrm{d}s + Ch^2 t,
\end{aligned} \tag{4.32}$$

as long as $h < 1 + \frac{2c_2 - c_0}{2c_1}$! (It's easy to verify that $1 + \frac{2c_2 - c_0}{2c_1}$ is indeed a positive number.) Now (4.32) and Gronwall inequality yield

$$||\phi(t)|| + \frac{1}{P_T}||\psi(t)|| \le \frac{Ch^2}{2(c_1 + c_2)P_T}(e^{2(c_1+c_2)P_T t} - 1) \tag{4.33}$$

for $0 < t < T$.

Remark The convergence proof of the particle method here is similar to that in [17]. But for the CH equation, in order to establish the global well-posedness of the reduced particle system, Camassa et al. [17] used the complete integrability, although they mentioned that this property *may be overkill*. But for the mCH, we know that the solution will not blowup in finite time and so the global existence of the corresponding ODE system follows without the use of complete integrability.

Remark Similar to the vortex method in hydrodynamics [9, 10], the particle algorithm has several distinctive features:

- the interactions of the "solitons" mimic the physical mechanisms described by the original PDE;
- there are no inherent errors which behave like the numerical viscosity of conventional Eulerian difference methods. Such diffusive errors can overtake the effects of physical viscosity in high Reynolds number flow simulation.
- One can use the geometric numerical integrators [49] to solve the resulted ODE to preserve the geometric structure that the ODE has.

4.2 Box Scheme

In this section, we propose a so-called box scheme to solve the equation:

$$\left\{\begin{array}{rcl} m_t & = & -um_x - 2mu_x \quad \text{on } \mathbb{R}^1, \\ m(x,0) & = & m_0(x). \end{array}\right. \tag{4.34}$$

The box scheme dated back to Preissmann [96] and was mathematically developed by Zhao and Qin [110]. It is a nondissipative scheme and extensively used in the computational fluid dynamics. Ascher and McLachlan [6] have compared the box scheme and other geometric integrators for the KdV equation and have analysed the dispersion relation to give an explanation of the stability of the box scheme when applied to hyperbolic systems.

Introduce the finite difference operators D_x, D_t and the mean operator M:

$$D_x m_i^n = \frac{m_{i+1}^n - m_i^n}{\triangle x}, \quad D_t m_i^n = \frac{m_i^{n+1} - m_i^n}{\triangle t}$$

$$M_x m_i^n = \frac{m_{i+1}^n + m_i^n}{2}, \quad M_t m_i^n = \frac{m_i^{n+1} + m_i^n}{2},$$

then the box scheme for (4.34) reads

$$D_t M_x m + M_t M_x u \cdot D_x M_t m + 2 D_x M_t u \cdot M_t M_x m = 0$$

$$\begin{aligned} \tfrac{1}{2\triangle t}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} m \;&=\; -\tfrac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} u \cdot \tfrac{1}{2\triangle x}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} m \\ &\quad -\; \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} m \cdot \tfrac{1}{2\triangle x}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} u. \end{aligned} \tag{4.35}$$

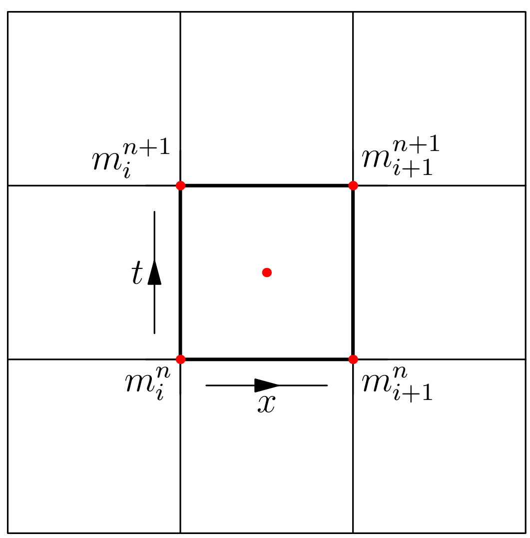

Figure 4.2.1: The box scheme for (4.34)

This is an implicit scheme and we can solve the obtained algebraic equations by Newton iteration method. The result for the Gaussian initial value is shown in the Figure 4.2.2. Here the equation (4.1) is solved by the box scheme with a moving frame, ie, we use $u - \max_x(u)$ as our velocity function u in the simulation and concentrate on region where the blob is mainly supported. The region is $[0, L] = [0, 16]$, with $n = 400$ grid points, dt $= 0.01$, and the initial value is $m_0(x) = e^{-|x-8|^2}$.

The graph indicates that the Gaussian initial value does evolves to a Dirac δ function as $t \to +\infty$ although we know that the mCH equation has no finite time blowup solution. This directly leads to the study we do in Chapter 5.

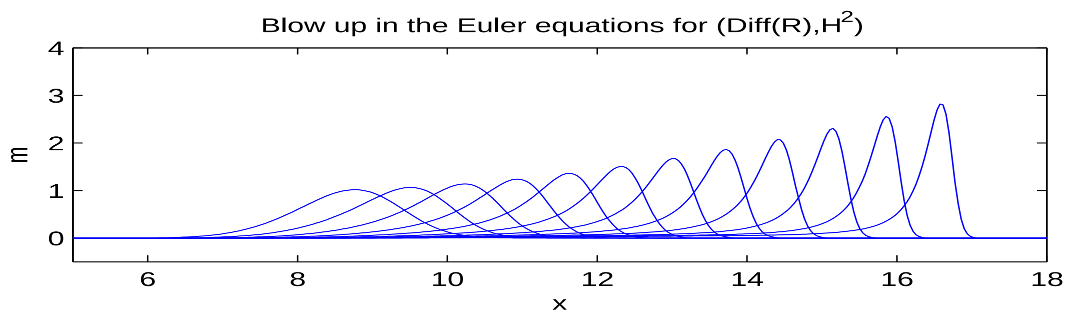

Figure 4.2.2: The evolution of Gaussian initial value $m_0(x) = e^{-|x-8|^2}$.

Remark Zhao and Qin [110] proved that the box scheme is symplectic and multi-symplectic for KdV equation, this is because KdV has another Hamiltonian (Poisson) structure ∂_x with constant coefficient. D. Cohen et al. [24] proposed a box scheme for the Camassa-Holm equation which they proved is multi-symplectic, this is also because the CH equation has a constant Hamiltonian (Poisson) structure $\partial_x(1 - \partial_x^2)$ (see the last part of the section 3.1.2). But for the general mCH, we have only one Hamiltonian structure $m\partial_x + \partial_x m$, which leads to no easy way to get a Hamiltonian discretization in space. Actually, we can propose a multi-symplectic formalization for the mCH equation in the Appendix B via the multi-symplectic geometry approach, but it is in the Lagrangian coordinates and is not of much practical use.

4.3 Conclusions

We have studied the numerical aspect of the limiting mCH, ie, $a = 0$, on the circle S and on the whole real line $\mathbb{R}^1$ respectively in this chapter. We have proved the convergence of the so-called particle methods in Section 4.1, which is quite similar to that in [17]. But here in our case, the global existence of the corresponding ODEs is very easy due to the nice property of the mCH with $k \geq 2$, and we do not need to use the integrability which the mCH probably does not have. Then we have proposed a box-scheme for (4.1). This scheme is locally a second order method because it is a symmetric method, and it is not too expensive to implement (Newton's iteration will solve the resulting algebraic equations). The simulation we have done gives a reliable result in a not-very-long time. However, the nonlinearity, especially the nonlocal dependence of u on m, make it very hard to rigorously prove the convergence of the box-scheme.

Chapter 5

Asymptotics

[Paradoxes of the infinite arise] only when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited.

— Galileo Galilei

5.1 Introduction

In this chapter, we are going to study the asymptotics for the limiting mCH equation on $\mathbb{R}^1$, that is,

$$\begin{cases} m_t &= -um_x - 2mu_x \quad \text{on } \mathbb{R}^1, \\ m(x,0) &= m_0(x). \end{cases} \tag{5.1}$$

where $(1-\partial_x^2)^k u = m, \quad k \geq 2$. As we mentioned before, this is a generalised Euler equation on Diff($\mathbb{R}$) with respect to H^k metric. We have proved in Chapter 3 that the solution of this equation does not blowup in any finite time, unlike the Camassa-Holm equation. However, our numerical simulation shows that the solution m does tend to some *weak soliton-like* solution as the time $t \to +\infty$ for some initial values (see Figure 4.2.2 in Chapter 4 showing that the initial Gaussian value evolves with linear growth in height towards a δ like function).

This is sort of similar to the phenomenon in Camassa-Holm equation (see [51], where they depicted the solutions of the CH equation on the circle with some Gaussian initial value that evolves into an ordered soliton-train as the time t increases). We have not yet found any study on this phenomenon in the existing literature.

First, we are going to use the asymptotic expansion to get an ODE which the asymptotic profile satisfies, then use the so-called *matched asymptotic expansion method* [105] to analyse the asymptotic behaviour for the generalised Euler equation (5.1) as t goes to infinity.

The method of matched asymptotic expansions is quite powerful dealing with two scale problems. Roughly speaking, it approximates the problem in two separate coordinates, one in the fine coordinate and the other in the coarse coordinate, and matches the boundary conditions using the matching principle.

In general, if we know that an evolutionary PDE has a solution $v(x,t)$ which blows up at time T and we are to study the (asymptotic) self-similar blowup profile f for the solution $v(x,t)$ of the PDE, we need to choose an appropriate "similarity variable ξ", which may depends on x and t, and a scaled factor $\phi(t)$ (may depending on T) according to the nature of PDE such that when plugging the solution $v(x,t) = \phi(t)f(\xi)$ into the PDE, we can obtained a differential equation of f as t goes to the blowup time T. The travelling wave solutions of PDEs are closely related to the self-similar solutions which Barenblatt [7] discussed intensively for partial differential equations. Here by the term "travelling wave" we mean the solutions of the form $f(x - ct)$. If one is looking for travelling wave solutions $u(t,x) = f(x - ct)$, then he can regard $\xi = x - ct$ as a new variable and plug this ansatz $u(t,x) = f(x - ct)$ back into the PDE to get an ODE in ξ.

For some PDEs, the resulted differential equation on f has a unique (stable) solution, e.g. [80]-[83] for the generalised KdV equations and [106] for travelling wave solutions of parabolic systems, and we can use various tools to prove that the solution of the PDE tends to the unique steady solution in some sense.

But for the mCH equation, we will find in this chapter that the situation is quite different.

5.2 Asymptotic PDE

We can observe from Figure 4.2.2 that the solution is getting taller and thinner as the time t increases and the bump moves to the right at an almost constant speed. This observation is helpful although it is not necessary to our analysis. In order to study how the solutions are approaching the blowup profile, we consider the travelling wave solutions of the form

$$m(t,x) = \phi(t)f(\phi(t) \cdot (x - ct))$$

with a scaling factor $\phi(t)$ (we take this form to guarantee $\int m = \int f$). We will first plug this ansatz into the equation (5.1) to find the right choice of $\phi(t)$, and then get the differential equation that f satisfies. If G denotes the Green's function for the operator $(1-\partial_x^2)^k$, then for very large $\phi = \phi(t)$, we can approximate u by

$$\begin{aligned} u(y) &= \int G(y-x)m(x)\mathrm{d}x \\ &= \int G(y-x)\phi f(\phi(x-ct))\mathrm{d}x \\ &= \int G(y-\xi/\phi - ct)f(\xi)\mathrm{d}\xi \qquad \xi = \phi(t)\cdot(x-ct) \\ &= \int G(\frac{\eta-\xi}{\phi})f(\xi)\mathrm{d}\xi \qquad \eta = \phi(t)\cdot(y-ct) \\ &= \int \big(G_0 + \frac{(\eta-\xi)^2}{2\phi^2}G''(0)\big)f(\xi)\mathrm{d}\xi + O(\frac{1}{\phi^3}) \quad \text{with } G_0 = G(0) \\ &= G_0 f_0 + \frac{G''(0)}{2\phi^2}f_2 + \frac{G''(0)}{2\phi^2}f_0\eta^2 - \frac{\eta}{\phi^2}G''(0)f_1 + O(\frac{1}{\phi^3}), \end{aligned} \tag{5.2}$$

where $f_i = \displaystyle\int \xi^i f(\xi)\mathrm{d}\xi$. Here we have used the fact $G'(0) = 0$. From (5.2), we have for very large ϕ that

$$\begin{aligned} u_y &= \textstyle\int G_y'(y-x)m(x)\mathrm{d}x \\ &= \int \frac{G''(0)(\eta-\xi)}{\phi}f(\xi)\mathrm{d}\xi + O(\frac{1}{\phi^2}) \\ &= G''(0)(\eta f_0 - f_1)/\phi + O(\frac{1}{\phi^2}) \end{aligned} \tag{5.3}$$

Substituting (5.2)(5.3) into (5.1), we get a differential equation:

$$\begin{aligned} \phi f'\cdot(\phi'\eta/\phi - c\phi) + \phi' f + 2G''(0)(\eta f_0 - f_1)f & \quad + \\ + \left(G_0 f_0 + \frac{G''(0)}{2\phi^2}f_2 + \frac{G''(0)}{2\phi^2}\eta^2 f_0 - \frac{G''(0)}{\phi^2}\eta f_1\right)\phi^2 f' &= 0. \end{aligned} \tag{5.4}$$

ie

$$\eta f'\phi' - cf'\phi^2 + f\phi' + G_0 f_0 f'\phi^2 + 2G''(0)(\eta f_0 - f_1)f + \frac{G''(0)}{2}\big(f_2 + \eta^2 f_0 - 2\eta f_1\big)f' = 0. \tag{5.5}$$

In order to make this equation balance in ϕ, we have to assume $\phi' \sim 1$ or $\phi' \sim \phi^2$. We will choose which is suitable to our goal.

- If $\phi' \sim \phi^2$, take $\phi' = \phi^2$ as an example, then $\phi(t)$ will become infinity at some finite time, which contradicts with our result in Chapter 3. Moreover, if $\phi' = \phi^2$, then the leading term in (5.5) will lead to

$$\eta f' - cf' + f + G_0 f_0 f' = 0$$

  which has only unbounded solutions. This is not of interest to us because we are looking for some smooth profile. This rules out the choice $\phi' \sim \phi^2$.

- If $\phi' \sim 1$, take $\phi' = 1$ for example, we have $\phi(t) = t$ and we can get the case in which we are interested.

Now if we take $\phi(t) = t$. Then we match the coefficients of t^i

$$\begin{aligned} t^0: \quad & [2G''(0)(\eta f_0 - f_1) + 1]\, f \\ & + \left(\frac{G''(0)}{2} f_2 + \frac{G''(0)}{2} \eta^2 f_0 - G''(0)\eta f_1 + \eta \right) f' = 0, \\ t^2 = \phi^2: \quad & -cf' + G_0 f_0 f' = 0. \end{aligned}$$

The leading order term $t^2(-cf' + G_0 f_0 f')$ is the limiting (δ function) part of the motion. If we divide the equation (5.4) by t^2 and let $t \to \infty$, then we get $-cf' + G_0 f_0 f' = 0$, which means the leading order term determines the speed c of the soliton: $c = G_0 f_0$. If we denote $U(\eta) = \frac{1}{2}G''(0) f_0 \eta^2 - G''(0) f_1 \eta + \frac{1}{2}G''(0) f_2$, then the ODE for the coefficients in t^0 takes the form

$$(\eta f)_\eta + f_\eta U + 2U_\eta f = 0. \tag{5.6}$$

This equation is nonlinear and nonlocal (U depends on the integrals of f), but fortunately it can be explicitly solved as follows:

Multiplying the LHS of the ODE in t^0 by η and then integrating it, we find that $f_1 = 0$. Because $G''(0) < 0$, we assume $f_0 G''(0) = -1$, $f_2 G''(0) = -1 - a^{-2}$, then we have

$$(-2\eta + 1) f + \frac{1}{2}(2\eta - \eta^2 - 1 - a^{-2}) \cdot f' = 0, \tag{5.7}$$

so

$$\frac{\mathrm{d}f}{f} = \frac{-2(2\eta - 1)\mathrm{d}\eta}{(\eta - 1)^2 + a^{-2}}$$

then

$$-\frac{\mathrm{d}f}{f} = \frac{2+4(\eta-1)}{(\eta-1)^2+a^{-2}}\mathrm{d}\eta = \frac{2}{(\eta-1)^2+a^{-2}}\mathrm{d}\eta + \frac{4(\eta-1)}{(\eta-1)^2+a^{-2}}\mathrm{d}\eta.$$

From which we have

$$f(a,\eta) = C(a)\cdot\frac{e^{-2a[\arctan(a(\eta-1))+\frac{\pi}{2}]}}{((\eta-1)^2+a^{-2})^2}. \tag{5.8}$$

We can find, with the help of the software Mathematica, that

$$\int_{-\infty}^{+\infty} f\mathrm{d}\eta = \frac{C(a)a^2}{4}\frac{1-e^{-2a\pi}}{a^2+1}, \qquad \int_{-\infty}^{+\infty} \eta^2 f\mathrm{d}\eta = \frac{C(a)}{4}(1-e^{-2a\pi}).$$

If we take

$$C(a) = -\frac{1}{G''(0)}\frac{a^2+1}{a^2}\cdot\frac{4}{1-e^{-2a\pi}}$$

then f satisfies the conditions $f_0G''(0) = -1,\ f_2G''(0) = -1-a^{-2}$. The limit of (5.8) is

$$\lim_{a\to+\infty} f(a,\eta) = \tilde{f} = \begin{cases} 4(\eta-1)^{-4}e^{\frac{2}{\eta-1}} & \text{if } \eta-1<0, \\ 0, & \text{if } \eta-1>0. \end{cases} \tag{5.9}$$

Here in (5.8), we have taken an integral constant $C(a)e^{-a\pi}$ in order to guarantee the limit function $\tilde{f}\in L^1(\mathbb{R})$. It's easy to verify that $\tilde{f}$ is a solution to (5.6) too, and we call this solution *the limit steady solution*. The steady solutions are depicted in Figure 5.2.1

Remark There are two magic features of the equation (5.6).

- The equation (5.6) is a nonlinear differential equation, so in general we can not multiply or divide the solution by a constant to get a solution, which means the assumption $f_0G''(0) = -1$ is not very plausible, because when we assume $f_0G''(0) = -1$ and $f_2G''(0) = -1-a^{-2}$, then the equation becomes a linear equation and we can solve it without any difficulty. But the check with Mathematica shows that the two a-prior assumptions on f_0 and f_2 does not lead to contradiction. Perhaps, we can think of it this way: because f_0 could be any number, so we take $f_0G''(0) = -1$ and think f_2 as a parameter (where the parameter a comes from).

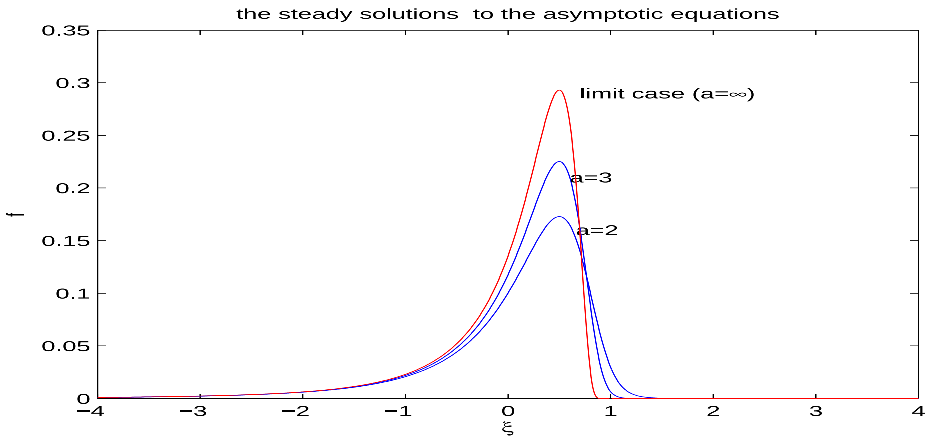

Figure 5.2.1: The steady solutions to the asymptotic equations

- The most interesting point here is that there is a family of self-similar blowup profiles, unlike most familiar cases where just a unique profile exists [80]-[83], [106]. This is the first equation with this kind of property we have ever seen.

If we take $m(x,t) = tf(t(x-ct),\tau)$, with a time scale factor $\tau = g(t)$, then we can do the same things as before

$$\begin{aligned} & tf_\xi \cdot (\eta/t - ct) + f + tf_\tau g'(t) + 2G''(0)(\eta f_0 - f_1)f \\ & + \left(G_0 f_0 + \frac{G''(0)}{2t^2} f_2 + \frac{G''(0)}{2t^2}\eta^2 f_0 - \frac{G''(0)}{t^2}\eta f_1\right) t^2 f_\xi \quad = \quad 0. \end{aligned} \tag{5.10}$$

This means that if we want to take f_τ into the equation, we have to take $g'(t) = \frac{1}{t}$, that is $g(t) = \ln t$. Then we have got the following asymptotic slow time PDE (the self-similar case) for which the equation (5.7) is the stationary equation:

$$f_\tau + (\xi f)_\xi + Uf_\xi + 2U_\xi f = 0, \tag{5.11}$$

here $U = \frac{G''(0)}{2}(\xi^2 f_0 - 2\xi f_1 + f_2)$. It is easy to verify that f_0 is independent of t, which we can assume satisfies $f_0 G''(0) = -1$ in order to simplify the calculation.

The numerical simulation strongly suggests that the steady solutions (5.8) and (5.9) are stable for the asymptotic equations (5.11) and are stable asymptotical solutions to the equation (5.1)(see Figures 5.4.1— 5.4.3) (here we use the term "stable asymptotical solutions" to mean the solutions of (5.1) tends to the family of "steady solutions" (5.8) and (5.9), but we have to point out that (5.8) and (5.9) are not the steady solutions of (5.1), although they are the steady solutions of (5.11)). We will give some trials to studying this stability in the next section.

Remark Our analysis applies to all H^k metric with $k \geq 2$, but not directly to H^1 metric which corresponds to the Camassa-Holm equation because in this case $G'(0)$, $G''(0)$ do not exist at all. We will have a look at the Camassa-Holm equation in the last section of this chapter.

5.3 Are Steady Solutions Stable?

In order to study the stability of the steady solution f, f as in (5.8)(5.9), we linearise the equation (5.11) around this steady f by a small perturbation $f + \varepsilon h(\tau, \xi)$ and get the linearised equation on h

$$h_\tau + [(U + \xi)h]_\xi + U_\xi h + (Wf)_\xi + W_\xi f = 0, \tag{5.12}$$

where $U = \frac{G''(0)}{2}(f_0\xi^2 + f_2)$, $W = \frac{G''(0)}{2}(h_0\xi^2 - 2h_1\xi + h_2)$ and $h_i = \int \xi^i h(\xi) \mathrm{d}\xi$. One can easily find that

$$\frac{\mathrm{d}h_0}{\mathrm{d}\tau} = 0, \quad \frac{\mathrm{d}h_1}{\mathrm{d}\tau} = h_1, \quad \frac{\mathrm{d}h_2}{\mathrm{d}\tau} + \frac{G''(0)}{2}(h_0 f + f_0 h)\xi^4 \Big|_{-\infty}^{+\infty} = 2h_2. \tag{5.13}$$

In order to study the stability of f, the natural idea is to prove that $h(\tau, \xi)$ stays small in some sense as $\tau \to +\infty$ for small initial value $h(0, \xi)$.

We are trying to study the stability of the steady solutions by characteristic methods. One of the difficulties is that one can easily see that some characteristics of (5.12) can not reach the ξ axis(ie, $\tau = 0$) (see Figure 5.3.3 and Figure 5.3.5 below) and we need some upstream boundary conditions $h(\tau, \xi_0)$ for some large ξ_0.

5.3.1 Upstream Boundary Conditions for (5.12)

In the previous section, we let $\xi = t(x - ct)$, with $c = G(0)f_0$, as a new variable, which means we concentrate on the region where the bump is supported and which is getting smaller and smaller as t increases. Now we are going to obtain some upstream boundary conditions for the equations (5.11) and (5.12), that is the boundary condition for ξ very large, that is where the so-called matched asymptotic expansion method comes in.

The technique of matched asymptotic expansion concerns different differential equations in two regions known as the "fine scale" (or inner) region and the "coarse scale" (or outer) region, matching the boundary condition on the "common" boundary (see [7] and [105] for example). The multi-scale asymptotic expansion is very subtle. It appears like "what you get depends on what you want". The key point here is the scale under which the problem is considered, or in other words, what limit process is of interest to us. Specific to our problem here, we can think the region A with the variable $\xi = t(x - ct)$ as the inner region (with $\xi = O(1)$) and think the region out of A as an outer region with another variable much coarse, for example, $z = x - ct = O(1)$. Now we need the upper stream boundary condition for $f(\xi, \tau)$ or $h(\xi, \tau)$, we can think this as the boundary condition at $z \to 0^+$ of the outer region equation as $t \to +\infty$ (see Figure 5.3.1).

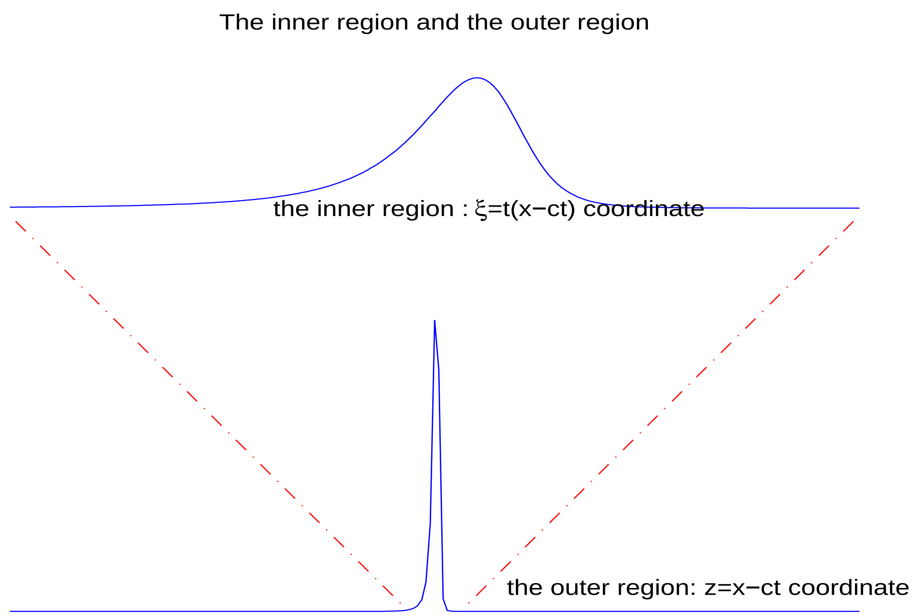

Figure 5.3.1: The illustration of the inner and outer regions. Take $t = 5$ for example, and there are two points P_1, P_2, if the distance of P_1, P_2 in the ξ coordinate is $\mathrm{d}_\xi(P_1, P_2) = 10$, then the distance of these two points in the z coordinate is $\mathrm{d}_z(P_1, P_2) = 10/5 = 2$. Here we plot a function $f(\xi)$ on the top and the function $m(t, z) = tf(\xi)$ on the bottom

Now we let $z = x - ct$ as a new variable, and have another approximation for $u(x)$ in (5.1) (when t is very large, while we think $z = x - ct = O(1)$, because now we are in the z coordinate):

$$
\begin{aligned}
u(y) &= \int G(y-x)m(x)\mathrm{d}x \\
&= \int G(y-ct-\frac{\xi}{t})f(\xi)\mathrm{d}\xi, \\
&\qquad \text{if } m(x)=tf(t(x-ct)),\ \xi=t(x-ct),\ c=G_0f_0, \\
&\approx G(y-ct)f_0-\frac{1}{t}\cdots
\end{aligned}
\tag{5.14}
$$

so in a moving frame (which means we can let $z=x-ct$ denote the relative position of the point x with respect to the point where the Dirac δ function is supported, and consider m as a function of $z:\ m(t,x)=n(t,z)=n(t,x-ct)$), then $m(0,x)=n(0,z)=n(0,x)$ and $m_t=n_t-cn_z$. If we denote $v(z)=(G(z)-G_0)f_0$, then $v(z)=u(x)-c,\ v_z=u_x$ and we can approximate the equation (5.1) (in the region where t is large enough and $z=x-ct=O(1)$) by

$$
n_t+vn_z+2v_zn=0. \tag{5.15}
$$

If we take $f_0=1$ just for simplifying the notations, then $v(z)=(G(z)-G_0)f_0=G(z)-G_0$, and this linear equation can be *explicitly* solved by the characteristic method:

$$
\begin{aligned}
n(t,z) &= n(0,F(-t,z))\left(\frac{G_0-G(F(-t,z))}{G_0-G(z)}\right)^2 \\
&= m(0,F(-t,z))\left(\frac{G_0-G(F(-t,z))}{G_0-G(z)}\right)^2 \\
&= m_0(F(-t,z))\left(\frac{G_0-G(F(-t,z))}{G_0-G(z)}\right)^2,
\end{aligned}
\tag{5.16}
$$

where $z=F(t,z_0)$ is defined by the characteristics curve

$$
\int_a^z \frac{1}{G(x')-G_0}\mathrm{d}x' = t+\int_a^{z_0}\frac{1}{G(x')-G_0}\mathrm{d}x' \tag{5.17}
$$

where

$$
a=\begin{cases} 1 & \text{if} \quad z_0>0, \\ -1 & \text{if} \quad z_0<0. \end{cases}
$$

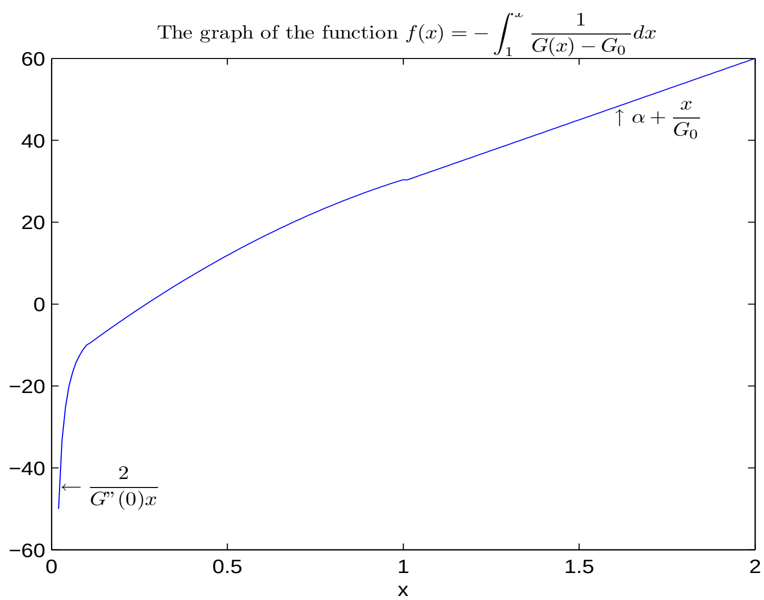

Figure 5.3.2: Asymptotic for the function $f(z) \equiv -\int_1^z \frac{1}{G(x')-G_0}\mathrm{d}x'$

$z = F(t, z_0)$ means the curve starting from z_0 at $t = 0$ arrives at z at the time t, so it is clear that $z_0 = F(-t, z)$.

Now we are going to use the limit of $m(t, z)$ as $t \to +\infty$ and $z \to 0^+$ as the boundary condition of the outer region with (z, t) coordinates and then transfer this limit to the upstream boundary condition of the inner region with (ξ, τ) coordinates.

In order to do that, we need to find the asymptotic expression of $m(t, z)$ as $z > 0$ very small and t large enough, ie, we are interested in the case $z \to 0^+$, and $t \to \infty$, such that $\xi = zt = t(x - ct) > 0$ very large. We are concerned with how fast the initial value $m_0(z_0)$ decreases as $z_0 \to +\infty$ can guarantee the upstream boundary conditions of $f(\xi, \tau)$ tends to 0 as $\tau \to +\infty$, so we consider the situation where $z_0 \to +\infty,\ z \to 0^+,\ t \to +\infty$. In this case, taking into account that $G(z) = G_0 + \frac{1}{2}G''(0)z^2 + o(z^3)$ for $z > 0$ small enough and $G(z) \to 0$ as $z \to +\infty$, we have

$$-\int_1^z \frac{1}{G(x')-G_0}\mathrm{d}x' \approx \frac{2}{G''(0)z} \quad \text{is negatively large as } z > 0 \ \text{ small enough}$$

$$-\int_1^{z_0} \frac{1}{G(x')-G_0}\mathrm{d}x' \approx \left(\alpha + \frac{z_0}{G_0}\right) \quad \text{is positively large as } z_0 \text{ large enough,}$$

(see Figure 5.3.2). On the other hand, from (5.17), we have

$$-\int_1^{z_0} \frac{1}{G(x') - G_0} \mathrm{d}x' \approx \frac{1}{z}\left(tz + \frac{2}{G''(0)}\right),$$

and so $z_0 \approx \frac{G_0}{z}(tz + \frac{2}{G''(0)})$, and

$$\begin{aligned} m(t,x) = n(t,z) &= m_0(z_0)\left(\frac{G_0 - G(F(-t,z))}{G_0 - G(z)}\right)^2 \\ &\approx \frac{G_0^2}{(G_0 - G(z))^2} m_0\left(\frac{G_0}{z}(tz + \frac{2}{G''(0)})\right). \end{aligned} \tag{5.18}$$

This approximation holds true for t large enough and $z > 0$ small enough. Now we will match this $m(t, z)$ to the upstream boundary condition of $f(\xi, \tau)$. Notice $G(z) - G(0) = \frac{G''(0)}{2} z^2 + o(z^3)$ as $z \to 0^+$, so the estimate (5.18), together with $m = tf(\xi, \tau), \tau = \ln t, \xi = zt$, yields that for large ξ and τ,

$$f(\xi, \tau) = e^{3\tau}\xi^{-4}(\frac{2G_0}{G''(0)})^2 m_0\left(G_0 e^{\tau} + 2\frac{G_0 e^{\tau}}{G''(0)\xi}\right). \tag{5.19}$$

This can give a $f(\xi, \tau)$ such that $\lim\limits_{\tau\to\infty} f(\xi, \tau) = 0$ as long as the initial $m_0(x) = o(x^{-3})$ as $x \to +\infty$. We call this $f(\xi, \tau)$ for large $\xi = \xi_0$ *the upstream boundary condition* for the *inner problem* (5.11). So the perturbation $h(\tau, \xi_0)$ can be assumed to be $h(\tau, \xi) = O(1) \cdot f(\xi_0, \tau)$ which we call *the upstream boundary condition for the inner problem (5.12)*. The above process is the so-called *the matched asymptotic expansion* method.

5.3.2 Around the General Steady Solution (5.8)

We first consider the general steady solution (5.8) case. In this case, the problem is that the characteristics of (5.12) are like arctan(ξ) (see Figure 5.3.3) and the initial value of $h(\tau, \xi)$ is not the initial $h(0, \xi)$, and instead, is determined by some boundary conditions $h(\tau_0, \xi_0)$ for some large ξ_0 and $\tau_0 > 0$.

In fact, the characteristic is determined by

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = U + \xi, \tag{5.20}$$

and if we put $G''(0)f_0 = -1, G''(0)f_2 = -1 - a^{-2}$, then this equation reads

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = -\frac{1}{2}\left[(\xi - 1)^2 + a^{-2}\right], \tag{5.21}$$

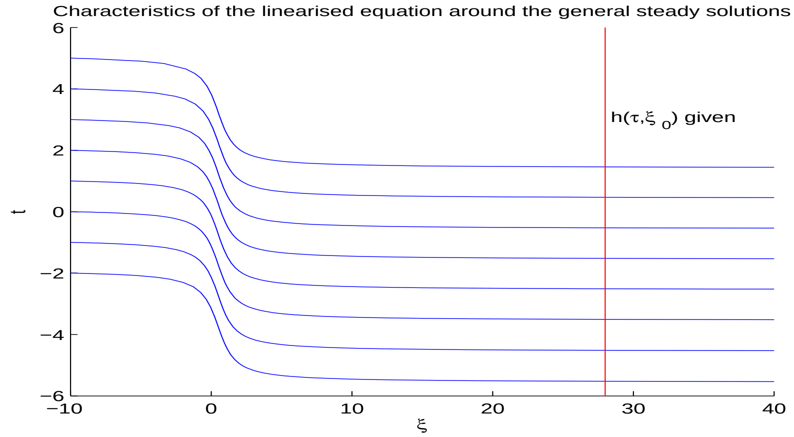

Figure 5.3.3: Characteristics of the linearised equations around the general steady solutions

and the characteristic curves are

$$\arctan a(\xi - 1) = -\frac{1}{2a}\tau + K, \tag{5.22}$$

where K is determined by the upstream boundary, ie,

$$K = \frac{1}{2a}\tau_0 + \arctan(a(\xi_0 - 1)). \tag{5.23}$$

Along the characteristics, the equation (5.12) reads

$$\frac{\mathrm{d}h}{\mathrm{d}\tau} + [1 + 2U_\xi]h + (Wf)_\xi + W_\xi f = 0. \tag{5.24}$$

For any $\xi_1 > 0$ fixed and any τ_1 large, there is a unique characteristic connecting (ξ_1, τ_1) and (ξ_0, τ_0) for some τ_0, and $|\tau_1 - \tau_0| < a\pi$, where a is the parameter from $G''(0)f_2 = -1 - a^{-2}$. Let $A = \max\{|1 - 2\xi| : \xi \text{ between } \xi_1 \text{ and } \xi_0\}$, A is large but fixed once ξ_0, ξ_1 fixed, then from

$$h(\tau_1, \xi_1) = e^{-\int_{\tau_0}^{\tau_1}(1-2\xi)} \left[h(\tau_0, \xi_0) + \int_{\tau_0}^{\tau_1} (-2W_\xi f - Wf_\xi) e^{\int(1-2\xi)} \right], \tag{5.25}$$

if $h(\tau_0, \xi_0) \to 0$ as $\tau_0 \to +\infty$ and W, W_ξ small(which means from (5.13) that h_0, h_2 small and $h_1 = 0$), we have

$$\limsup_{\tau_1 \to \infty} h(\tau_1, \xi_1) \text{ is small.} \tag{5.26}$$

If we take $h_0 = h_1 = 0$ and $G''(0)f_0 = -1$, then (5.13) reads

$$\frac{\mathrm{d}h_2}{\mathrm{d}\tau} - \frac{1}{2}c(\tau) + \frac{1}{2}h\xi^4|^{-\infty} = 2h_2, \tag{5.27}$$

where $c(\tau) = \lim_{\xi\to+\infty} h\xi^4$ and $\lim_{\tau\to\infty} c(\tau) = 0$. From (5.25) we have

$$h(\tau_1,\xi_1) = e^{-\int_{\tau_0}^{\tau_1}(1-2\xi)}\left[h(\tau_0,\xi_0) - \frac{G''(0)}{2}\int_{\tau_0}^{\tau_1} f_\xi h_2 e^{\int_{\tau_0}^{\tau_1}(1-2\xi)}\right] \equiv I + II. \tag{5.28}$$

But along the characteristics, we have

$$e^{-\int_{\tau_0}^{\tau_1}(1-2\xi)} = e^{\tau_1-\tau_0}\left[\frac{1+a^2(\xi_0-1)^2}{1+a^2(\xi_1-1)^2}\right]^2, \tag{5.29}$$

and so

$$\begin{aligned} I &= e^{-\int_{\tau_0}^{\tau_1}(1-2\xi)}h(\tau_0,\xi_0) \\ &= e^{\tau_1-\tau_0}\left[\frac{1+a^2(\xi_0-1)^2}{1+a^2(\xi_1-1)^2}\right]^2 h(\tau_0,\xi_0) \\ &= e^{\tau_1-\tau_0}\left[\frac{1}{1+a^2(\xi_1-1)^2}\right]^2 c(\tau_0)a^4 \end{aligned} \tag{5.30}$$

because $h(\tau_0,\xi_0) \approx c(\tau_0)\xi_0^4$. If we let τ_1 fixed, then $\tau_0 = \tau_0(\xi_1)$ depends on ξ_1.

$$\lim_{\xi_1\to-\infty} I\cdot\xi_1^4 = \lim_{\xi_1\to-\infty} e^{\tau_1-\tau_0}c(\tau_0) = e^{2a\pi}c(\tau_1 - 2a\pi) \tag{5.31}$$

for $\tau_0 = \tau_1 - 2a\arctan(a(\xi_0-1)) + 2a\arctan(a(\xi_1-1))$. On the other hand,

$$f_\xi = \frac{-4\xi}{a^{-2}+(\xi-1)^2}f, \tag{5.32}$$

so

$$II = 2G''(0)e^{\tau_1-\tau_0}\frac{1}{[a^{-2}+(\xi_1-1)^2]^2}\int_{\tau_0}^{\tau_1} h_2 f\xi e^{\tau_0-\tau}(a^{-2}+(\xi-1)^2)\mathrm{d}\tau. \tag{5.33}$$

Plugging the formula of f and the characteristic equation (5.22), then a direct calculation yields

$$\begin{aligned}
&\lim_{\xi_1\to-\infty} II\cdot\xi_1^4\\
=\;& 2G''(0)e^{\tau_1-a\pi}\lim_{\xi_1\to-\infty}\int_{\tau_0}^{\tau_1} h_2 e^{-2aK}\frac{\frac{1}{a}\tan(K-\frac{1}{2a}\tau)+1}{a^{-2}\tan^2(K-\frac{1}{2a}\tau)+a^{-2}}\mathrm{d}\tau\\
=\;& 2aG''(0)e^{\tau_1-a\pi}\lim_{\xi_1\to-\infty}\int_{\tau_0}^{\tau_1} h_2 e^{-2aK}\left[a\cos^2(K-\frac{1}{2a}\tau)\right.\\
&\left.+\cos(K-\frac{1}{2a}\tau)\sin(K-\frac{1}{2a}\tau)\right]\mathrm{d}\tau\\
=\;& 2aG''(0)\int_{\tau_1-2a\pi}^{\tau_1}\left[\frac{ah_2}{2}\left\{1+\cos\left(\frac{\tau_1-\tau}{a}-\pi\right)\right]\right.\\
&\left.+\frac{1}{2}h_2\sin\left(\frac{\tau_1-\tau}{a}-\pi\right)\right\}\mathrm{d}\tau,
\end{aligned}\tag{5.34}$$

because

$$\lim_{\xi_1\to-\infty}K=\lim_{\xi_1\to-\infty}\frac{1}{2a}\tau_0+\arctan(a\xi_0-a)=\frac{1}{2a}\tau_1-\pi+\frac{\pi}{2}=\frac{1}{2a}\tau_1-\frac{\pi}{2}.\tag{5.35}$$

So

$$\begin{aligned}
&\lim_{\xi_1\to-\infty} II\cdot\xi_1^4\\
=\;& 2aG''(0)\int_{\tau_1-2a\pi}^{\tau_1}\left[\frac{ah_2(\tau)}{2}\left(1-\cos\left(\frac{\tau_1-\tau}{a}\right)\right)\right.\\
&\left.-\frac{1}{2}h_2(\tau)\sin\left(\frac{\tau_1-\tau}{a}\right)\right]\mathrm{d}\tau,
\end{aligned}\tag{5.36}$$

so h_2 satisfies at $\tau=\tau_1$

$$\begin{aligned}
&\frac{\mathrm{d}h_2}{\mathrm{d}\tau}-\frac{1}{2}c(\tau_1)+\frac{1}{2}c(\tau_1-2a\pi)e^{2a\pi}-2h_2\\
=\;& -\frac{1}{2}aG''(0)\int_{\tau_1-2a\pi}^{\tau_1}\left[a\left(1-\cos\left(\frac{\tau_1-\tau}{a}\right)\right)-\sin\left(\frac{\tau_1-\tau}{a}\right)\right]h_2\mathrm{d}\tau.
\end{aligned}\tag{5.37}$$

If we can obtain from this differential integral equation that $h_2(\tau)$ stays small for all large time τ, then it is easy to see from (5.25)

$$\lim_{\tau\to+\infty}h(\tau,\xi)=0.\tag{5.38}$$

But the problem here is that we can not have the smallness of h_2 from (5.37)! In fact, we can numerically solve the equation (5.37) and the result is shown in the following Figure 5.3.4, which indicates that $h_2(\tau)$ increases exponentially as a function of τ. (In this figure, we take $c(\tau) \equiv 0,\ a = 1, G''(0) = -1$ and $h_2(\tau) = -0.1$ for $\tau \leq 0$.) However, this example does not contradict with (5.38) although we can not derive any definite result from it.

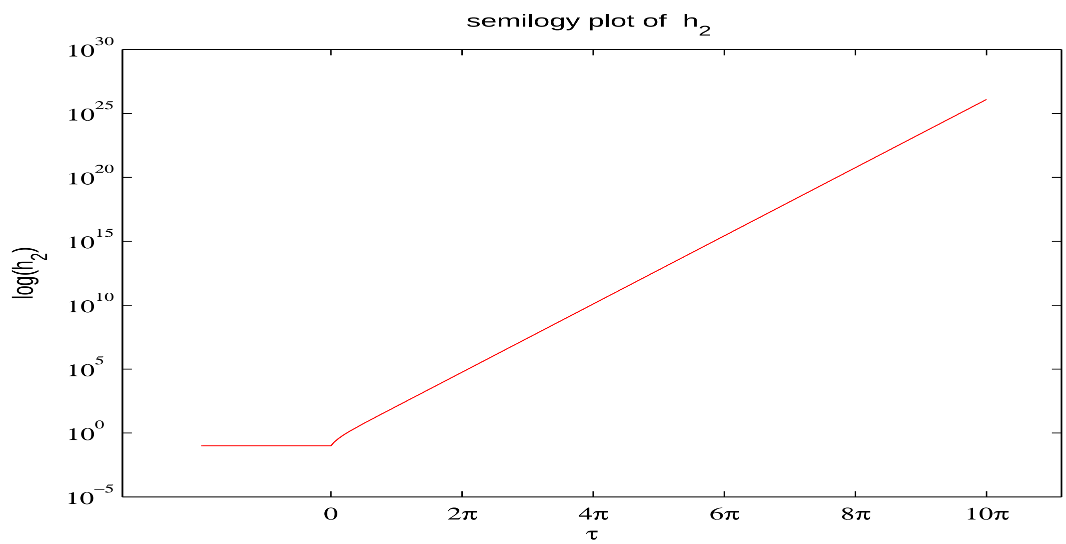

Figure 5.3.4: Plot of $h_2(\tau)$

5.3.3 Around the Limit Steady Solution (5.9)

At this stage, the characteristic equation of (5.12) is

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = -\frac{1}{2}(\xi - 1)^2 \tag{5.39}$$

and the characteristics from $\xi|_{\tau=\tau_0} = \xi_0$ are

$$(\xi - 1)^{-1} = \frac{1}{2}(\tau - \tau_0) + (\xi_0 - 1)^{-1}$$

whose diagrams are shown in Figure 5.3.5. Clearly, $\tau - \tau_0 < (\xi - 1)^{-1}$ for $\xi > 1$.

Along each characteristic curve, the linearised equation (5.12) becomes

$$\frac{\mathrm{d}h}{\mathrm{d}\tau} + (1 + 2U_\xi)h + (Wf)_\xi + W_\xi f = 0, \tag{5.40}$$

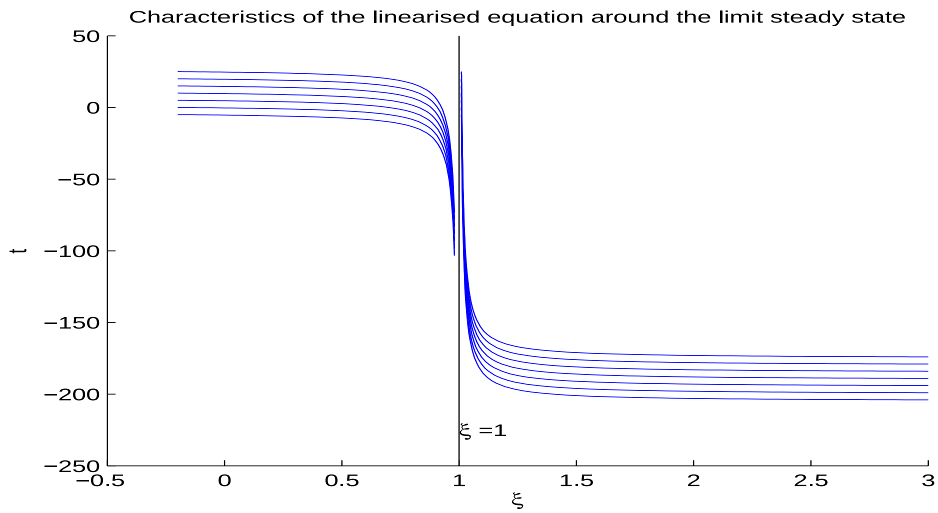

Figure 5.3.5: Characteristics of the linearised equations around the limit steady solution

ie

$$\frac{\mathrm{d}h}{\mathrm{d}\tau} + (1-2\xi)h + (Wf)_\xi + W_\xi f = 0. \tag{5.41}$$

Case $\xi > 1$.
For any $\xi > 1$, we have $f \equiv 0$, so this equation reads

$$\frac{\mathrm{d}h}{\mathrm{d}\tau} = (2\xi - 1)h \tag{5.42}$$

whose solution can be found explicitly

$$h(\tau, \xi) = e^{\tau-\tau_0}\left(\frac{\xi_0 - 1}{\xi - 1}\right)^4 h(\tau_0, \xi_0). \tag{5.43}$$

If $\xi > 1$ is fixed, then $\tau_0 \to +\infty$ as $\tau \to +\infty$ because $\tau - \tau_0 < (\xi - 1)^{-1}$, so $e^{\tau-\tau_0}$ is bounded, from which we have for fast decaying $h(\tau, \xi_0)$

$$\lim_{\tau\to\infty} h(\tau, \xi) = 0 \quad \text{ for any } \xi > 1.$$

Case $\xi < 1$.
If $\xi < 1$, then for any $\tau > 0$ there is a characteristic curve connecting (ξ, τ) and $(\xi_0, 0)$ for some unique $\xi_0 < 1$. Along each such curve, the equation (5.12) reads

$$\frac{\mathrm{d}h}{\mathrm{d}\tau} + (1-2\xi)h = -(Wf)_\xi - W_\xi f. \tag{5.44}$$

Denote $l(\xi) = -(Wf)_\xi - W_\xi f$, then $l(\xi) = o(1) \cdot (\xi - 1)^{-4} e^{\frac{2}{\xi-1}} (\xi - 1)^{-2}$ if $W,\ W_\xi = o(1)$. Along the characteristic curve, we have

$$e^{-\int_0^\tau (1-2\xi)} = e^{\frac{2}{\xi-1} - \frac{2}{\xi_0 - 1}} \left(\frac{\xi_0 - 1}{\xi - 1} \right)^4 \tag{5.45}$$

and the solution is

$$h(\tau, \xi) = e^{-\int_0^\tau (1-2\xi)} \left[h(0, \xi_0) + \int_0^\tau l(\xi(s)) e^{\int_0^s (1-2\xi)} \mathrm{d}s \right]. \tag{5.46}$$

the first part of the solution

$$e^{\frac{2}{\xi-1} - \frac{2}{\xi_0-1}} \left(\frac{\xi_0 - 1}{\xi - 1} \right)^4 h(0, \xi_0) \tag{5.47}$$

can be small if $h(0, \xi_0) = o(1) \cdot e^{\frac{2}{\xi_0 - 1}} (\xi_0 - 1)^{-4} = o(1) \cdot \tilde{f}(\xi_0)$.

But the other part

$$e^{-\int_0^\tau (1-2\xi)} \int_0^\tau l(\xi(s)) e^{\int_0^s (1-2\xi)} \mathrm{d}s = e^{\frac{2}{\xi-1}} \left(\frac{1}{\xi - 1} \right)^4 \cdot o(1) \cdot \int_0^\tau (\xi(s) - 1)^{-2} \mathrm{d}s \tag{5.48}$$

is not small! This means that we can not use the standard approach to establish the linear stability of the limit steady solution.

5.4 The Family of Steady Solutions

Then we go further with the numerical simulation on the stability of steady solutions. We found that the real profile seems to wander within the family of steady solutions, as shown in the Figures 5.4.1– 5.4.3. In these figures, the differential equation (5.1) is solved by the box scheme with a moving frame, ie, we use $u - \max_x(u)$ as our velocity function u in the simulation and concentrate on region where the blob is mainly supported. The region is $[0, L] = [0, 12]$, with $n = 400$ grid points, $\mathrm{dt} = 0.01$, and the initial value is $m_0(x) = e^{-|x|^2}$. The solid line of the top plot of each figure stands for the real solution $m(x, t)$, and then use the nonlinear least square method to solve a parameter optimization problem to find the closest profile (hence the shape parameter a) from the family

$$a_3 e^{a_4} \frac{\exp(-2a_2 \arctan(a_2 a_3 (x - a_1)))}{((a_3(x - a_1))^2 + a_2^{-2})^2}, \tag{5.49}$$

with four parameters $a_i,\ i = 1, \cdots, 4$: a_1 is the maximum point, a_2 is the shape parameter a, a_3 is the spatial scaling factor and a_4 the height scaling factor. The solid line in the figures is the numerical solution of the equation (5.1) and the dotted line correspond to the asymptotic profile selected from (eq:ft) that best fits the solution. We can see from the Figure 4.2.2 and the Figures 5.4.1– 5.4.3 that the Gaussian initial value evolves very quickly to the family of steady solution profiles, then it does not stay at any steady solution, instead, it wanders in this family of steady solutions (see the first plot of the evolution of a in Figure 5.4.4). It is so strange that we have not ever seen the similar phenomenon before.

Here are some excuses why we failed to rigorously solve this problem:

- Analytically, both the original PDE (5.1) and its slow time asymptotic PDE (5.11) are nonlinear and nonlocal, and are not real hyperbolic equations. Normally, when talking about the stability of a steady solution, we need some nice properties such as the dissipation or some enough number of conserved quantities to guarantee the solutions of the PDE tends to the steady solution in some sense, of which the PDEs (5.1)(5.11) lack.

- Numerically, we can not reliably solve the asymptotic PDE (5.11) because of the coefficients in (5.11) depends on $f_i,\ i = 0, 1, 2$. When we solve numerically (5.11), after some time, the solution $f(\eta, \tau)$ behaves like (5.8) or (5.9), which means

$$f \sim \eta^{-4} \qquad \text{for} \quad |\eta| \text{ large}$$

  which incurs an $O(\frac{1}{L})$ error when we replace $f_2 = \int_{-\infty}^{\infty} \xi^2 f(\xi) \mathrm{d}\xi$ with the evaluation of $f_2 = \int_{-L}^{L} \xi^2 f(\xi) \mathrm{d}\xi$ in the simulation. This error will take over the true solution, so we are not able to check the stability numerically.

- We can not solve the mCH reliably for long times because of the weak blowup. If we solve directly the original equation (5.1), then the solution will be getting larger and larger as the time increases due to the weak blowup.

- If we use a moving frame in the simulation, then we concentrate on the very narrow region where the bump is supported, which is good. But a moving frame means that we are not capturing the correct upstream boundary conditions. It seems that it would need the adaptive grid method used by Budd [16] which we could not get working.

- The reason that the standard characteristic method does not work is probably that the characteristic curves depend on the solutions, which means that, as the solutions of (5.1) tend to (5.8)(5.9), the characteristic curves themselves change!

From the numerical simulations, we make the following conjectures:

- The one parameter family of the steady solutions is exponentially stable, every initial value not in but close to this set will tend to it very quickly and then wanders along this set.
- If the initial value $m(x, 0)$ is zero for all large $x > 0$, then the solution will tend to the limit steady solution (5.9).
- If $m(x, 0)$ is zero for all large $x > 0$ and there is a small perturbation $h(x_0, t)$ at some large x_0 with $h(x_0, t) \to 0$ as $t \to \infty$, then the solution will track this perturbation and tends to the limit steady solution (5.9) as $t \to \infty$.

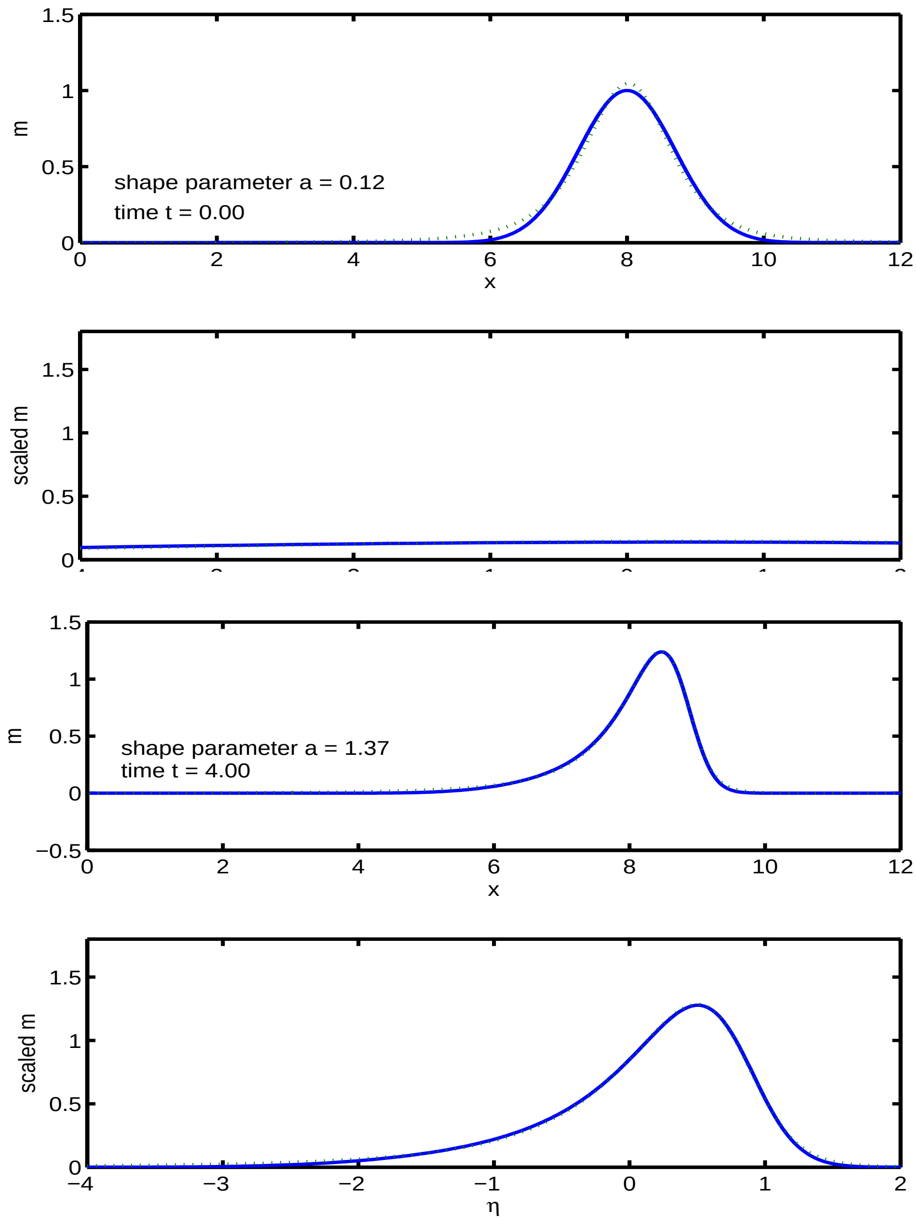

Figure 5.4.1: Fit of the solution to the true profiles: the equation is solved by the box scheme with a moving frame. Starting off from the Gaussian initial value, the solution is soon almost indistinguishable from the true profile.

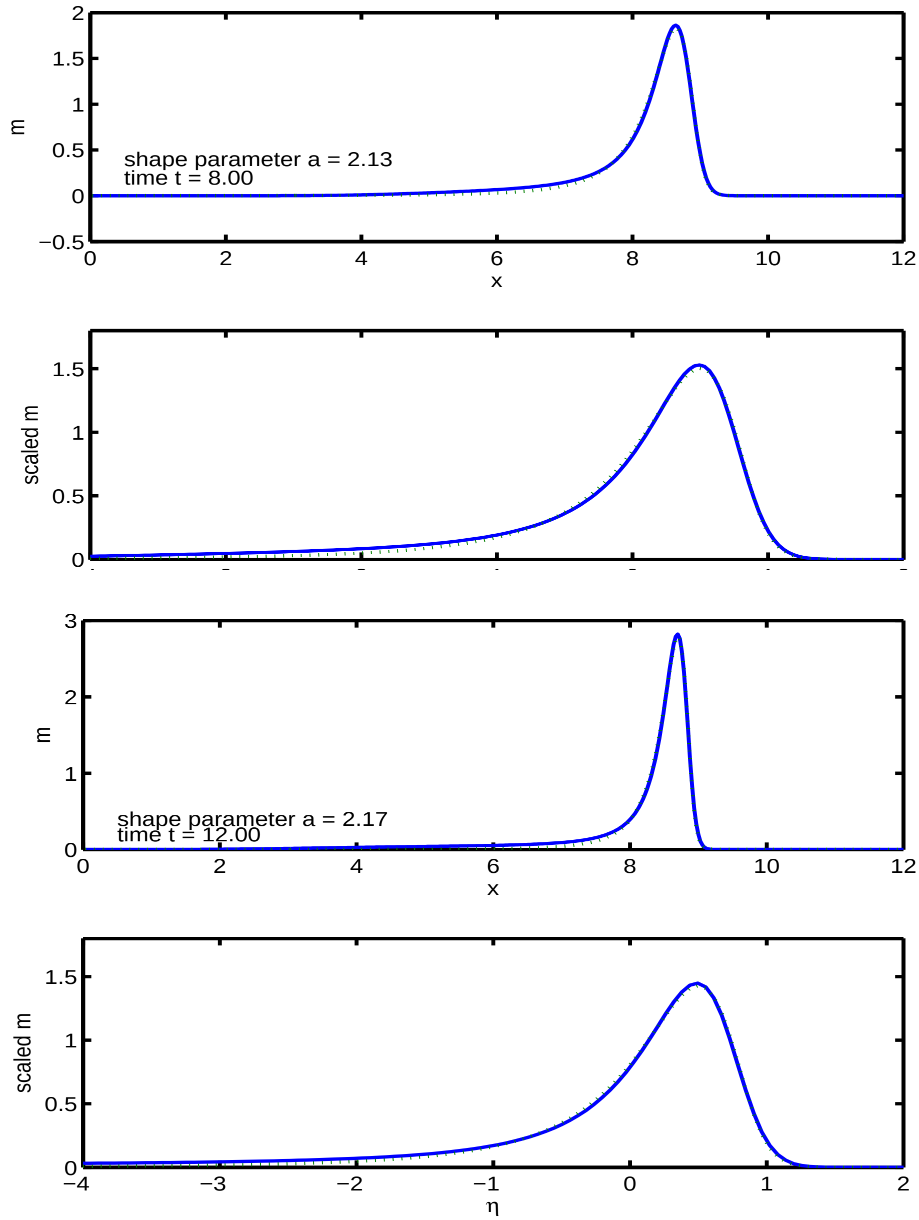

Figure 5.4.2: Fit of the solution to the true profiles

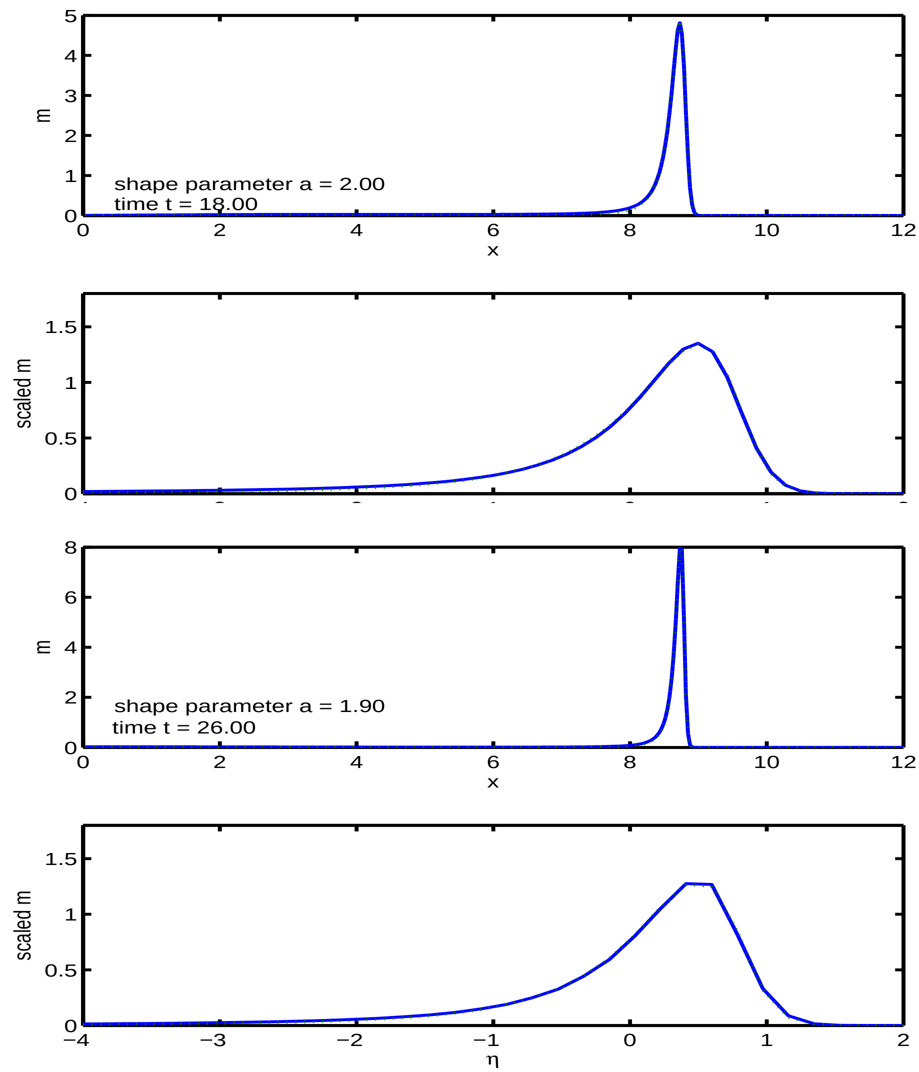

Figure 5.4.3: Fit of the solution to the true profiles: the solution is getting concentrated on a narrower region as t increases, and the graph starts to wiggle as can be seen from the plot of the scaled m.

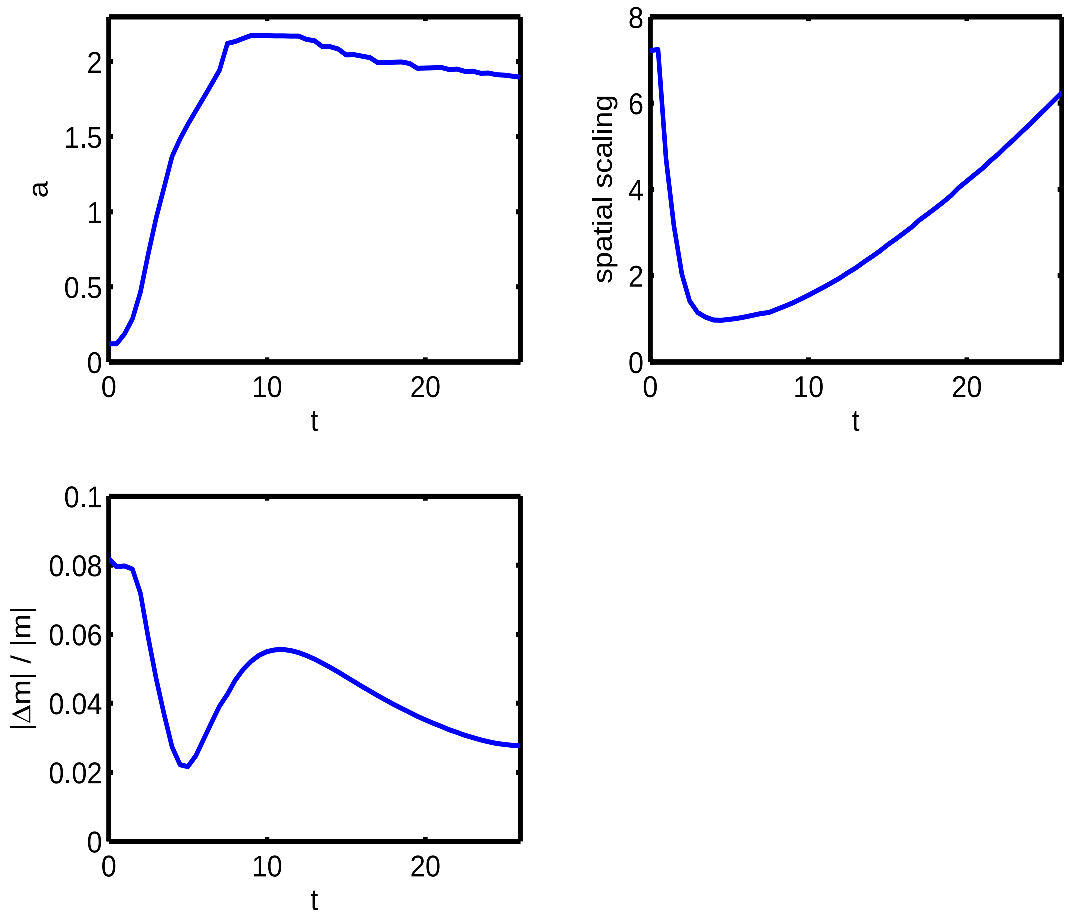

Figure 5.4.4: The evolution of the parameters: the first plot is on the shape parameter a; the second is on the spatial scaling factor and the last one on the error of the fit. The spatial scaling factor increases linearly after $t \geq 10$, as we expect, and the error decreases steadily after $t \geq 10$.

5.5 Remarks on the Camassa-Holm Equation

For the H^1 metric case, ie the Camassa-Holm equation, $G(x) = \frac{1}{2}e^{-|x|}$, if we let $m(t,x) = \phi(t)f(\phi(t)(x-ct))$ and plug this ansatz into the Camassa-Holm equation, then similarly as we did in section 5.2, we can obtain $\phi(t) = e^t$ from the balance of ϕ in the resulted equation. In fact, if we suppose $m(t,x) = \phi(t)f(\phi(t)(x-ct))$, we know that $G'(0)$ does not exist in this case, but we still have

$$\begin{aligned} u(y) &= \int G(y-x)m(x)\mathrm{d}x \\ &= \left(\int_{\xi<\eta} + \int_{\xi>\eta}\right) G(\frac{\eta-\xi}{\phi})f(\xi)\mathrm{d}\xi \\ &= G_0 f_0 + \frac{1}{\phi(t)}\left[G'_+(\eta f_0^- - f_1^-) + G'_-(\eta f_0^+ - f_1^+)\right] \\ &\quad + \frac{G''(0)}{2\phi(t)^2}\left[\eta^2 f_0 - 2\eta f_1 + f_2\right] + O(\frac{1}{\phi(t)^3}), \end{aligned} \tag{5.50}$$

where f_i as before, and $G'_\pm = \lim_{\xi\to\pm 0} G'(\xi) = \mp\frac{1}{2}$, $f_i^+ = \int_{\xi>\eta} \xi^i f(\xi)\mathrm{d}\xi$, $f_i^- = \int_{\xi<\eta} \xi^i f(\xi)\mathrm{d}\xi$ and we denote $G''(0) = G''_+(0) = G''_-(0) = \frac{1}{2}$ although $G''(0)$ does not exist actually. Similarly,

$$u_y(y) = G'_+ f_0^- + G'_- f_0^+ + \frac{G''(0)}{\phi(t)}\left[\eta f_0 - f_1\right] + O(\frac{1}{\phi(t)^2}). \tag{5.51}$$

On the other hand,

$$m_t = \phi' \cdot (f + \eta f_\eta) - \phi^2 \cdot c f_\eta, \quad m_y = \phi^2 \cdot f_\eta. \tag{5.52}$$

Plugging all these terms into (5.1), and noticing that $G'_+ = -G'_-$, we have

$$\begin{aligned} &\phi' \cdot (\eta f)_\eta - \phi^2 c f_\eta + \phi^2 f_\eta G_0 f_0 \\ &+\phi f_\eta G'_+[\eta(f_0^- - f_0^+) + (f_1^+ - f_1^-)] + 2\phi f G'_+(f_0^- - f_0^+) \\ &+\frac{1}{2}G''(0) f_\eta[\eta^2 f_0 - 2\eta f_1 + f_2] + 2fG''(0)(f_0\eta - f_1) = 0. \end{aligned} \tag{5.53}$$

In order to balance in ϕ in this equation, we may have three choices: $\phi' \sim \phi^2$; $\phi' \sim 1$ or $\phi' \sim \phi$. We can discuss them separately as we did for the H^2

metric case and obtain that only $\phi' \sim \phi$ is of interest to us. So we take $\phi(t) = e^t$ and suppose

$$m = e^t f(e^t(x - ct)), \tag{5.54}$$

then, similarly to the previous derivation, we have the asymptotic steady equation

$$(\eta f)_\eta + W f_\eta + 2W_\eta f \quad = \quad 0, \tag{5.55}$$

where

$$W(\eta) = \left[G'_+ \cdot (\eta f_0^- - f_1^-) + G'_- \cdot (\eta f_0^+ - f_1^+)\right] = \frac{1}{2}[\eta(f_0^+ - f_0^-) - (f_1^+ - f_1^-)]. \tag{5.56}$$

Unlike in the H^2 metric case, this equation is really nonlinear in f because W depends on f in a sort of complicated way. We are not so lucky any more as with the equation (5.6) and can not find the solution explicitly at the moment. Hopefully, we will work on this equation in the future.

5.6 Conclusions

In this chapter, we have studied the asymptotic behaviour of the mCH (5.1) on the whole real line $\mathbb{R}^1$ by the asymptotic expansion and the so-called asymptotic matching method. After a short introduction, we have used the asymptotic expansion in Section 5.2 to derive the ODE (5.6), which the asymptotic steady solutions should satisfy, and the slow-time evolutionary PDE (5.11). The ODE (5.6) admits a family of solutions (5.8),(5.9). When linearising the slow-time PDE (5.11) around the steady solutions (5.8)(5.9), we have found that the characteristics do not intersect with the x-axis, which means we have to assign some upstream boundary conditions for the linearised equations. So we have approximated the equation (5.1) in another (coarser) scale to get (5.15). Matching the solution of (5.15) to the upstream boundary condition of (5.15), we have shown that, if $m_0(x) = o(x^{-3})$ as $x \to +\infty$, then the upstream boundary condition $f(\xi, \tau) \to 0$ as $\tau \to +\infty$. However, we have not yet rigorously proved that the asymptotic steady solutions are stable despite all the efforts we have made here. In Section 5.4, we have tried to find numerically the best fit profiles of the solutions of (5.1), which shows that the true profiles seem wandering within the family of steady solutions. We have listed some reasons why we have not yet completely solved this problem. After that, we have tried to apply the same method to the CH equation in Section 5.5, which suggests that a more involved calculation will be needed.

Chapter 6

Four Particle Systems

In this chapter, we will study the mCH in a very specific form. That is, we will consider the Hamiltonian ODE system

$$\begin{cases} \dot{q}_i = & h \displaystyle\sum_{j=1}^{4} G(q_i - q_j) p_j \\ \dot{p}_i = & -h p_i \displaystyle\sum_{j=1}^{4} G'(q_i - q_j) p_j \end{cases} \tag{6.1}$$

where $h = \pi/2$, $G(x)$ is the Green function corresponding to the H^2 metric on $S^1 = [0, 2\pi]$.

6.1 Motivation

Why do we study the four particle system? The answer is that we know that the KdV equation and the CH equation are completely integrable but the general mCH equation with $k > 1$ is very likely not, at the same time we have two conserved quantities: $\int m$ and $\int mu$ for the general mCH equation and an extra $\int |m|^{\frac{1}{2}}$ for the limiting mCH, so it is natural to ask if there is another conserved quantity? The possible approaches to the question include (i) constructing a conserved quantity explicitly and/or (ii) studying the Lyapunov exponents of the corresponding four particle system.

For the particle systems, the conserved quantity $\int |m|^{\frac{1}{2}} \mathrm{d}x$ becomes zero (because the particle systems correspond to the evolution of the sum of some δ momentum). If the four particle system has only one positive Lyapunov exponent, then it is expected that the systems have some other integral than $\int m$ and $\int mu$ in general. We will use the Lyapunov exponents method to show that this ODE system is likely to have another conserved quantity than the obvious conserved quantities corresponding to $\int m$ and $\int mu$. At the same time, a positive Lyapunov exponent means the four particle system is not integrable and hence the mCH equation is very unlikely to be integrable.

6.2 Lyapunov Exponents

Lyapunov exponents (or characteristic numbers) were first introduced by Lyapunov [73] in 1892 to study the stability of nonstationary solutions of ODEs and many papers and books (see, for example Nemyskii et al. [91], E. Ott [93] and the reference therein) are devoted to it.

Let X be a differentiable manifold with a Riemannian metric, then an ODE $\dot{x} = f(t, x)$ on X defines a (flow) mapping $F(t, \cdot) : X \mapsto X$ for any $t > 0$. The Lyapunov exponents are introduced to describe the long-time dependence of the solution on the initial perturbation. More precisely, if $V \in T_{x_0}M$ for some $x_0 \in M$, $\mathrm{d}F(t, x)$ stands for the differential of F w.r.t. x, then the Lyapunov is defined by

$$\lambda(x_0, V) \equiv \lim_{t\to\infty} \frac{1}{t} \log ||\mathrm{d}F(t, x_0)(V)||. \tag{6.2}$$

6.3 How to Compute Them?

Specifically to an ODE in $\mathbb{R}^n$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x), \qquad x(0) = x_0 \in \mathbb{R}^n, \tag{6.3}$$

its linearised equation reads

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = A(t)y(t), \quad y(t) \in \mathbb{R}^n, \tag{6.4}$$

whose fundamental solution matrix $Y(t)$ satisfies

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = A(t)Y(t), \quad Y(0) = Y_0 \in \mathbb{R}^{n\times n} \text{ is orthogonal.} \tag{6.5}$$

If $\{\mathbf{p}_i\}$ is an orthonormal basis of $\mathbb{R}^n$, then

$$\lambda_i = \limsup_{t\to\infty} \frac{1}{t} \log(||Y(t)\mathbf{p}_i||) \quad i = 1, 2, \cdots, n, \tag{6.6}$$

are well-defined. When the sum $\sum_{i=1}^{n} \lambda_i$ is minimised, the orthonormal basis $\{\mathbf{p}_i\}$ is called **normal** and the λ_i are the so-called **Lyapunov exponents**. It is clear from above that the concept of Lyapunov exponents is a sort of generalisation of the (real part of) the eigenvalues for the constant coefficient matrix $A(t) \equiv A$ in the asymptotic stability analysis of $\dot{y} = A(t)y$, so one can expect it will play a very important role in the study of asymptotic behaviour of ODEs. That is why so many existing literatures are devoted to it.

Then how to compute them?

It is easy to get that for any normal basis we have

$$\sum_{i=1}^{n} \lambda_i \geq \limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{trace}(A(s))\mathrm{d}s = \limsup_{t\to\infty} \frac{1}{t} \log|\mathrm{det}Y(t)|. \tag{6.7}$$

If this inequality becomes an equality for some normal basis, then the linear system is called **regular**. One can find that the Lyapunov exponents are invariant if we change $Y(t)$ to another matrix $X(t) = T(t)Y(t)$ with $T(t)$, $T^{-1}(t)$ are uniformly bounded. Perron and Diliberto (see [30]) show that for bounded continuous $A(t)$, there is an orthogonal $Q(t)$ such that $X(t) = Q^T(t)Y(t)$ satisfies

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = \tilde{A}(t)X(t), \tag{6.8}$$

where $\tilde{A}(t)$ is an upper triangular matrix. The reason we transform the coefficient matrix $A(t)$ to an upper triangular matrix $\tilde{A}(t)$ becomes clear when we look at the following theorem[73]:

Theorem 6.1 *If $A(t) \in \mathbb{R}^{n\times n}(t)$ is an upper triangular matrix with all entries continuous and bounded, then the equation (6.1) is regular if and only if the following limits exist*

$$\mu_i = \lim_{t\to\infty} \frac{1}{t}\int_0^t A_{ii}(s)\mathrm{d}s, \quad i = 1, 2, \cdots, n, \tag{6.9}$$

and in this case, $\lambda_i = \mu_i,\ i = 1, 2, \cdots, n$.

The key point in computing Lyapunov exponents is the continuous QR decomposition of $Y(t)$,

$$Y(t) = Q(t)R(t), \tag{6.10}$$

where $Q(t)$ is orthogonal and $R(t)$ is upper triangular with positive diagonal entries $R_{ii},\ i = 1, 2, \cdots, n$. From the orthogonality of $Q(t)$, we have

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log||Y(t)\mathbf{p}_i|| = \lim_{t\to\infty} \frac{1}{t} \log||R(t)\mathbf{p}_i||. \tag{6.11}$$

So by Theorem 6.1,

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log |R_{ii}(t)|, \quad 1 \geq i \geq n. \tag{6.12}$$

From this, G. Benettini et al. [11] proposed the now very popular discrete QR decomposition method in computing the Lyapunov exponents (L. Dieci et al. [31] proved the convergence of this algorithm):

Discrete QR method: The point here is to QR decompose $Y(t)$ indirectly at $t_0 < t_1 < \cdots < t_j < \cdots$. More precisely, let

$$Y_0 \equiv Q_0 = I,$$

and for $j = 0, 1, \cdots$, one solves

$$\dot{Z}_j = AZ_j,\ Z_j(t_j) = Q_j, \quad t_j \leq t \leq t_{j+1},$$

and then QR factorise $Z_j(t_{j+1})$

$$Z_j(t_{j+1}) = Q_{j+1}R_{j+1},$$

with R_{j+1} having positive diagonal entries. Since $Q_0 = I$ and

$$\dot{Y} = AY, \quad Y(0) = I,$$

so if we denote $Y_j = Y(t_j)$,, then we have $\dot{Y} = AY, \quad Y(t_j) = Y_j$ and

$$\begin{aligned} Y_{j+1} &= Z_j(t_{j+1})Q_j^T Y_j = Q_{j+1}R_{j+1}Q_j^T Y_j = \cdots \\ &= Q_{j+1}R_{j+1}R_j \cdots R_1 Q_0 = Q_{j+1} \prod_{k=j+1}^{1} R_k. \end{aligned}$$

The Lyapunov exponents can thus be obtained by

$$\lambda_i = \lim_{j\to\infty} \frac{1}{t_j} \log ||(R_j)_{ii} \cdots (R_1)_{ii}|| = \lim_{j\to\infty} \frac{1}{t_j} \sum_{k=1}^{j} \log ||(R_k)_{ii}||. \tag{6.13}$$

6.4 Four Particle Systems

In order to study the Lyapunov exponents of (6.1) by the method of Dieci et al. [31], we need to linearise the equation (6.1). The linearised equations are

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = JY, \tag{6.14}$$

where J is the matrix

$$J = \frac{\pi}{2}\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{6.15}$$

and

$$A = \begin{bmatrix} \sum_{j\neq 1} G'(q_1-q_j)p_j & -G'(q_1-q_2)p_2 & -G'(q_1-q_3)p_3 & -G'(q_1-q_4)p_4 \\ G'(q_1-q_2)p_1 & \sum_{j\neq 2} G'(q_2-q_j)p_j & -G'(q_2-q_3)p_3 & -G'(q_2-q_4)p_4 \\ G'(q_1-q_3)p_1 & G'(q_2-q_3)p_2 & \sum_{j\neq 3} G'(q_3-q_j)p_j & -G'(q_3-q_4)p_4 \\ G'(q_1-q_4)p_1 & G'(q_2-q_4)p_2 & G'(q_3-q_4)p_3 & \sum_{j\neq 4} G'(q_4-q_j)p_j \end{bmatrix} = -D^T \tag{6.16}$$

$$B = \begin{bmatrix} G(0) & G(q_1-q_2) & G(q_1-q_3) & G(q_1-q_4) \\ G(q_1-q_2) & G(0) & G(q_2-q_3) & G(q_2-q_4) \\ G(q_1-q_3) & G(q_2-q_3) & G(0) & G(q_3-q_4) \\ G(q_1-q_4) & G(q_2-q_4) & G(q_3-q_4) & G(0) \end{bmatrix} \tag{6.17}$$

$$C = \begin{bmatrix} -p_1\sum_{j\neq 1} G''(q_1-q_j)p_j & p_1p_2G''(q_1-q_2) & p_1p_3G''(q_1-q_3) & p_1p_4G''(q_1-q_4) \\ p_1p_2G''(q_1-q_2) & -p_2\sum_{j\neq 2} G''(q_2-q_j)p_j & p_2p_3G''(q_2-q_3) & p_2p_4G''(q_2-q_4) \\ p_1p_3G''(q_1-q_3) & p_2p_3G''(q_2-q_3) & -p_3\sum_{j\neq 3} G''(q_3-q_j)p_j & p_3p_4G''(q_3-q_4) \\ p_1p_4G''(q_1-q_4) & p_2p_4G''(q_2-q_4) & p_3p_4G''(q_3-q_4) & -p_4\sum_{j\neq 4} G''(q_4-q_j)p_j \end{bmatrix} \tag{6.18}$$

We can form the matrix J as follows: first construct the Hessian matrix H'' of the Hamiltonian functional

$$H = \frac{1}{2}\sum_{i,j=1}^{4} p_ip_jG(q_i-q_j), \tag{6.19}$$

and then get J by

$$J = \begin{bmatrix} O & I \\ -I & O \end{bmatrix} \times H''. \tag{6.20}$$

We try different initial momentum values at four equi-spaced points (ie, with $q_k(0) = \frac{k\pi}{2}$ for $k = 0, 1, 2, 3$). The numerical results are shown in the following figures. The bottom plot is a zoom-in of the top one in each figure.

T=1500 in all the simulations. There do exist some cases for which the Lyapunov exponents are convergent to zeros, but some other figures indicate that the situations seem different. The Lyapunov exponent is notorious at its convergence rate when computed, however, from these figures, the following assertion is quite certain: at most one of the Lyapunov exponents is positive as the time $t \to \infty$!

The simulation suggests that although the four particle systems are not integrable, there should exist another conserved quantity, but we have not yet found any good candidate.

6.5 Conclusions

We have considered a four particle system corresponding to the limiting mCH on S with $m = (1 - \partial_x^2 + \partial_x^4)u$. We have used the method from [31] to compute the Lyapunov exponents of the system of four particles initially equi-distributed on the circle S. The numerical result suggests that the four particle system is very likely non-integrable and at the same time it seems to have another conserved quantity other than $\int m$ and $\int mu$. A possible candidate for the third integral is $\int u(u^2 + u_x^2 + u_{xx}^2)$ (because we know that $\int u(u^2 + u_x^2)$ is the third integral of the CH equation), and actually I once believed I found a "proof" of that, but the numerical check disconfirmed that and then I found that the "proof" was wrong!

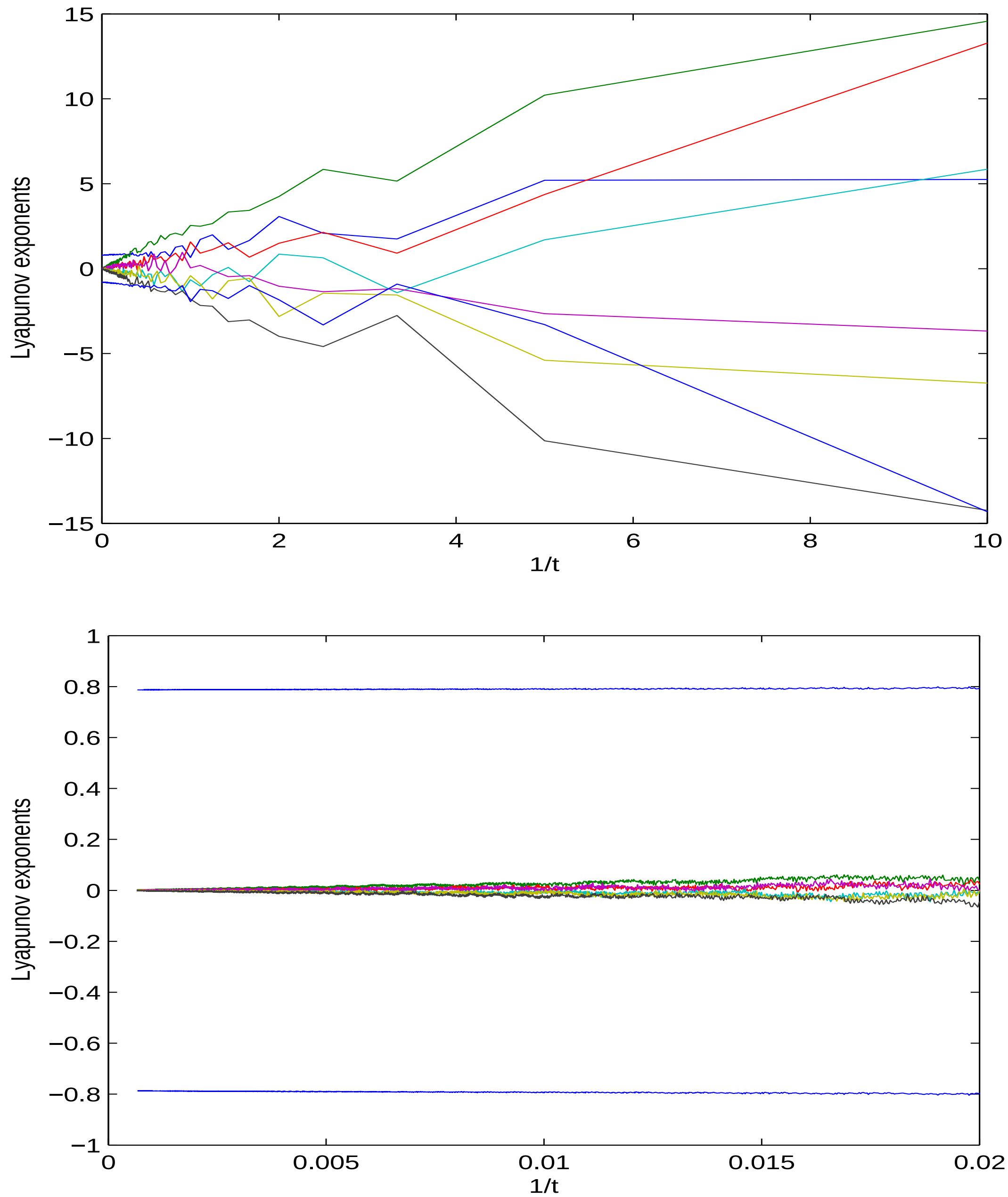

Figure 6.4.1: Lyapunov exponents for $p = [1, 5, 10, 4]'$: one has a positive limit, six exponents go to zero as $t \to \infty$.

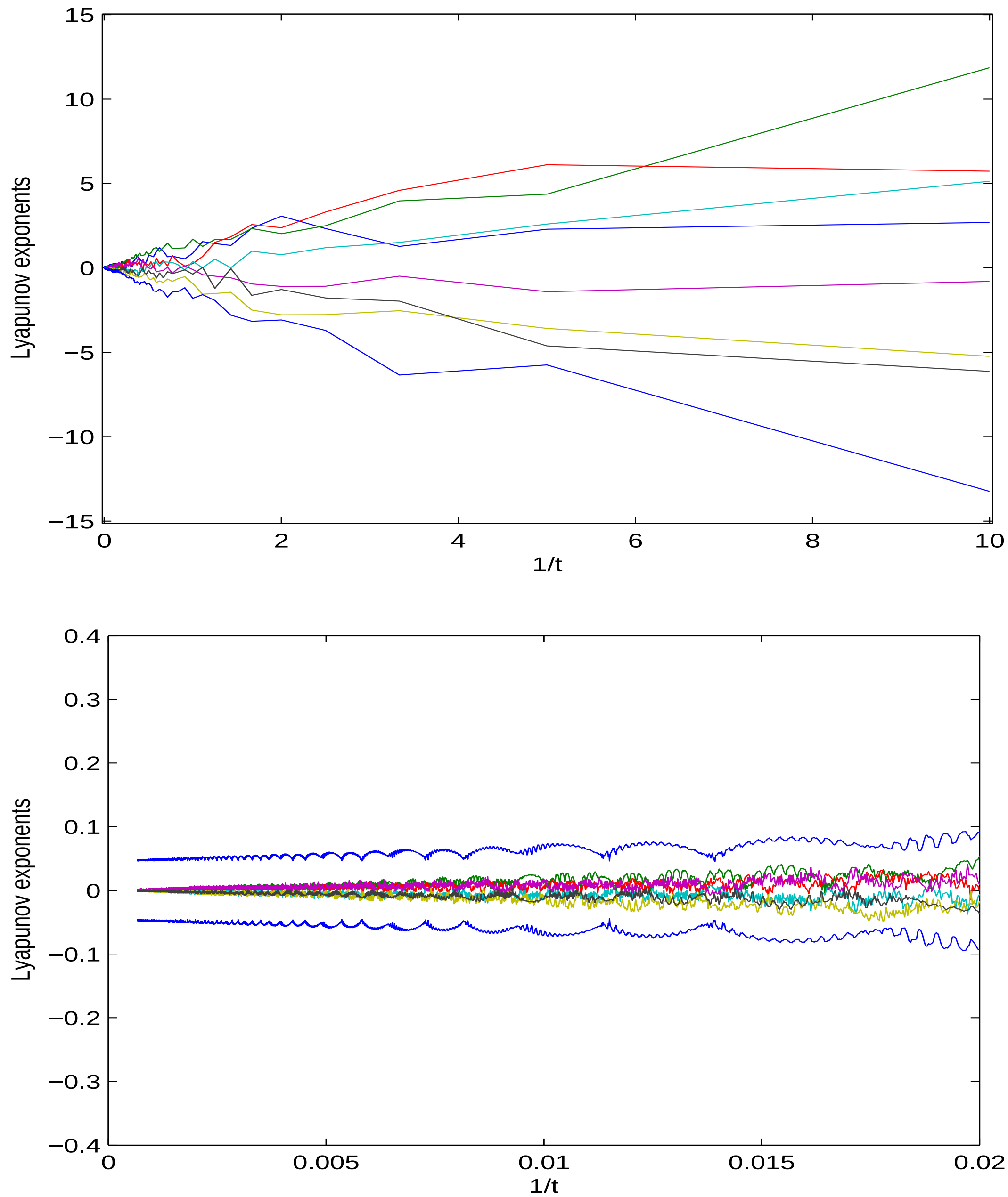

Figure 6.4.2: Lyapunov exponents for $p = [1, 5, 1, 10]'$: one has a positive limit, six exponents go to zero

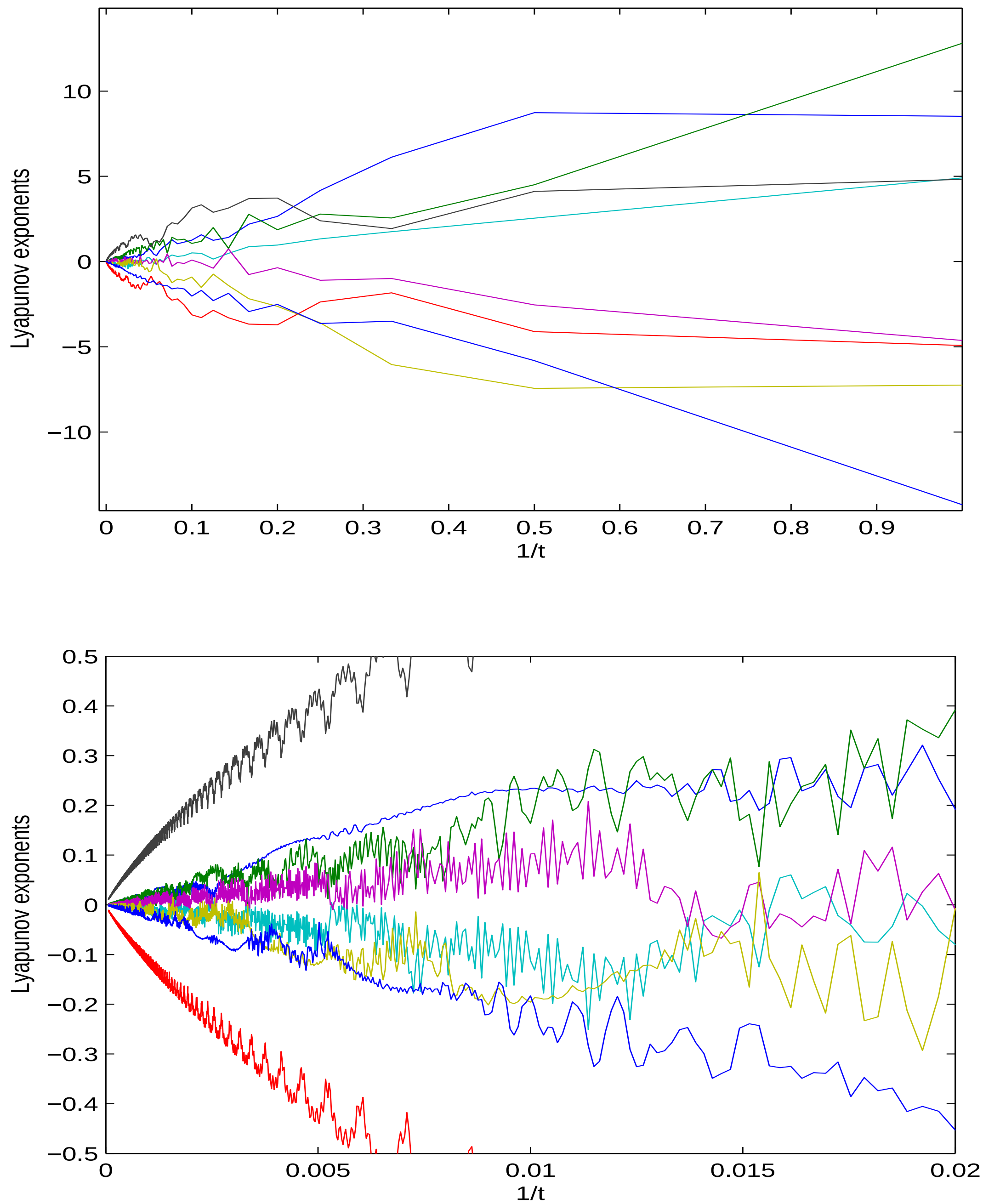

Figure 6.4.3: Lyapunov exponents for $p = [2, 10, 0, 5]'$: $p_3(0) = 0$ implies $p_3(t) \equiv 0$ giving another conserved quantity. All exponents go to 0.

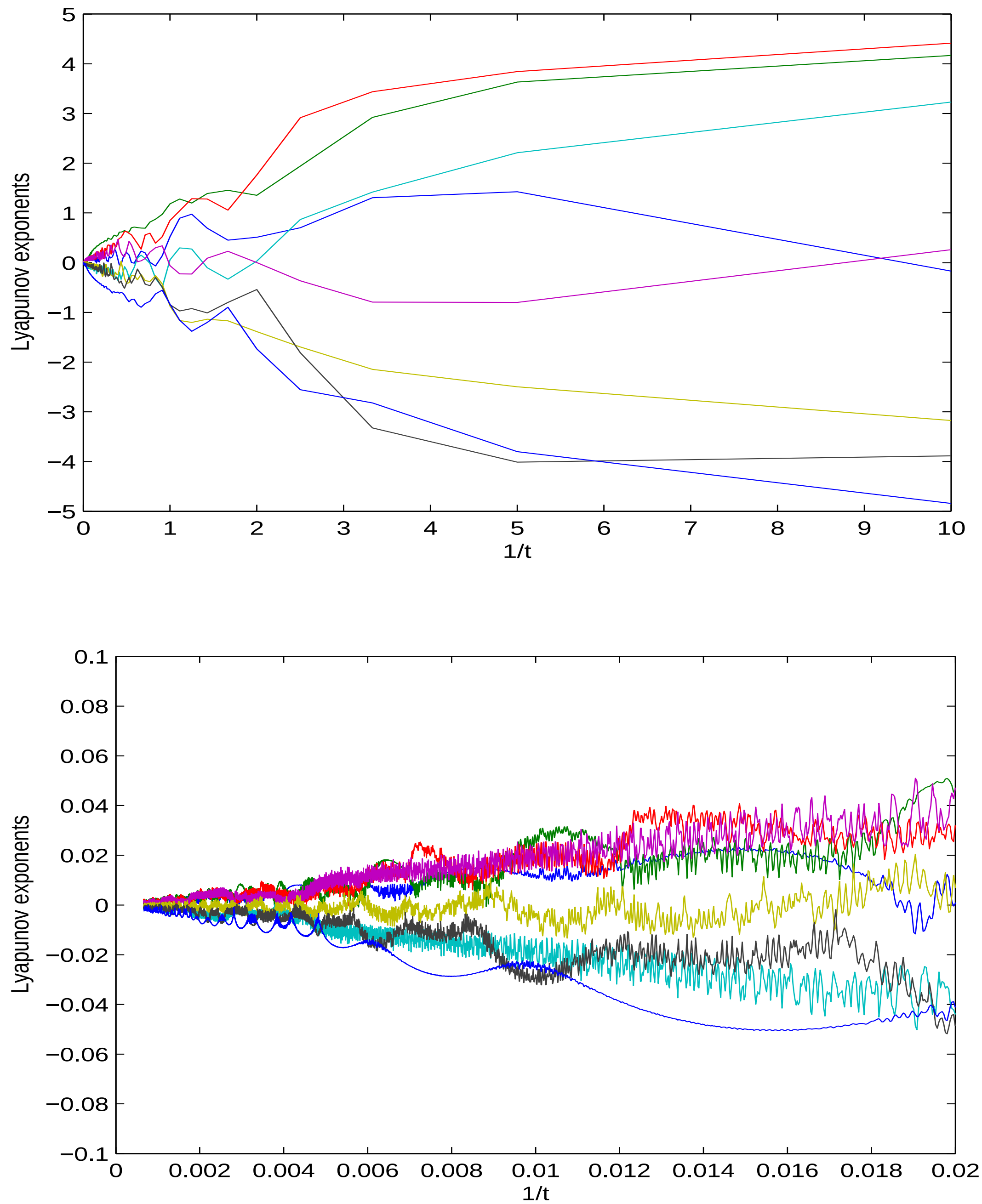

Figure 6.4.4: Lyapunov exponents for $p = [1, 2, 3, 4]'$: lower $\dfrac{H}{||p||^2}$ means more likely to have quasiperiodic motion (ie, $\lambda = 0$).

Chapter 7

Higher Dimensional Case

7.1 Introduction

Up to now, all we have studied is about the one dimensional generalised Euler equations. But as we mentioned in section 1.5 of Chapter 1, the higher dimensional generalised Euler equations can find many applications in Computational Anatomy [51, 53]. So we turn to the two dimensional case.

In this chapter we will study the so-called *two dimensional Camassa-Holm equation*, i.e., the Euler equation for $G = \mathrm{Diff}(\mathbb{R}^2)$ with $H^1(\mathbb{R}^n)$ metric

$$\frac{\partial}{\partial t}\mathbf{m} + \mathbf{u}\cdot\nabla\mathbf{m} + \nabla\mathbf{u}^T\cdot\mathbf{m} + \mathbf{m}(\mathrm{div}\mathbf{u}) = 0. \tag{7.1}$$

It was derived in [52] as a higher dimensional generalization of the 1d Camassa-Holm equation, where the momentum density $\mathbf{m}(x,t) = (I-\Delta)\mathbf{u}(x,t)$ is a function from $\mathbb{R}^2 \times \mathbb{R} \mapsto \mathbb{R}^2$. Holm and Marsden [51] mentioned its well-posedness in $H^s(\mathbb{R}^2)$ can be established by extending the arguments in [34], which are mainly the geometric approach, and will be given in another publication (but I have not yet found any). Here we will prove the local existence of classical solutions by an analysis approach.

7.2 Local Well-posedness

We will use some regularization method to establish the well-posedness. First, we will give some properties of the mollifiers. Given any radial function

$$\rho(|x|) \in C_0^\infty(\mathbb{R}^2), \quad \rho \geq 0, \quad \int_{\mathbb{R}^2} \rho = 1,$$

and define the mollification

$$(\mathcal{J}_\varepsilon v)(x) = \varepsilon^{-2} \int_{\mathbb{R}^2} \rho(\frac{x-y}{\varepsilon}) v(y) \mathrm{d}y, \quad \varepsilon > 0.$$

Lemma 7.1 *([74]) The mollifier $\mathcal{J}_\varepsilon$ defined above has the properties:*

(i) $\forall\ u \in L^p(\mathbb{R}^2),\ 1 < p < \infty,\ \mathcal{J}_\varepsilon u$ *is a* C^∞ *function.*

(ii) *For all* $v \in C^0(\mathbb{R}^2)$, *we have* $\mathcal{J}_\varepsilon v \to v$ *uniformly on any compact set* Ω *in* $\mathbb{R}^2$ *and*

$$|\mathcal{J}_\varepsilon v|_{L^\infty} \leq |v|_{L^\infty}.$$

(iii) $D^\alpha \mathcal{J}_\varepsilon v = \mathcal{J}_\varepsilon D^\alpha v, \quad \forall\ |\alpha| \leq m, \quad v \in H^m(\mathbb{R}^2).$

(iv) *For all* $u \in L^p(\mathbb{R}^2),\ v \in L^q(\mathbb{R}^2),\ \frac{1}{p} + \frac{1}{q} = 1$, *we have*

$$\int_{\mathbb{R}^2} (\mathcal{J}_\varepsilon u) v \mathrm{d}x = \int_{\mathbb{R}^2} (\mathcal{J}_\varepsilon v) u \mathrm{d}x.$$

(v) *For all* $v \in H^s(\mathbb{R}^2)$, $\mathcal{J}_\varepsilon v$ *converges to* v *in* H^s *and the rate of the convergence in the* H^{s-1} *norm is linear in* ε,

$$\lim_{\varepsilon \to 0} ||\mathcal{J}_\varepsilon v - v||_s = 0,$$

$$||\mathcal{J}_\varepsilon v - v||_{s-1} \leq C\varepsilon ||v||_s.$$

(vi) *For all* $v \in H^m(\mathbb{R}^2),\ k \in \mathbb{Z}^+ \cup \{0\}$, *and* $\varepsilon > 0$, *we have*

$$||\mathcal{J}_\varepsilon v||_{m+k} \leq \frac{c_{mk}}{\varepsilon^k} ||v||_m,$$

$$|\mathcal{J}_\varepsilon D^k v|_{L^\infty} \leq \frac{c_k}{\varepsilon^{1+k}} ||v||_0.$$

Now we approximate the 2d CH equation (7.1) by the following smoothed equations

$$\frac{\partial}{\partial t}\mathbf{m}^\varepsilon + \mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon \mathbf{u}^\varepsilon \cdot \nabla \mathcal{J}_\varepsilon \mathbf{m}^\varepsilon) + \mathcal{J}_\varepsilon((\nabla \mathcal{J}_\varepsilon \mathbf{u}^\varepsilon)^T \cdot \mathcal{J}_\varepsilon \mathbf{m}^\varepsilon) + \mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon \mathbf{m}^\varepsilon (\mathrm{div} \mathcal{J}_\varepsilon \mathbf{u}^\varepsilon)) = 0, \tag{7.2}$$

$$\mathbf{m}^\varepsilon = (I - \triangle)\mathbf{u}^\varepsilon, \tag{7.3}$$

with $\mathbf{m}^\varepsilon(0) = \mathbf{m}_0(x)$ given.

Let

$$\begin{aligned} F(\mathbf{m}^\varepsilon) &= -\mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon\mathbf{u}^\varepsilon\cdot\nabla\mathcal{J}_\varepsilon\mathbf{m}^\varepsilon) - \mathcal{J}_\varepsilon((\nabla\mathcal{J}_\varepsilon\mathbf{u}^\varepsilon)^T\cdot\mathcal{J}_\varepsilon\mathbf{m}^\varepsilon) - \mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon\mathbf{m}^\varepsilon(\mathrm{div}\mathcal{J}_\varepsilon\mathbf{u}^\varepsilon)) \\ &\equiv F_1(\mathbf{m}^\varepsilon) + F_2(\mathbf{m}^\varepsilon) + F_3(\mathbf{m}^\varepsilon). \end{aligned} \tag{7.4}$$

Then for any $m \in \mathbb{Z}^+ \cup \{0\}$, we have

$$\begin{aligned} & ||F_1(\mathbf{m}_1^\varepsilon) - F_1(\mathbf{m}_2^\varepsilon)||_m \\ =\; & ||\mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon\mathbf{u}_1^\varepsilon\cdot\nabla\mathcal{J}_\varepsilon\mathbf{m}_1^\varepsilon) - \mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon\mathbf{u}_2^\varepsilon\cdot\nabla\mathcal{J}_\varepsilon\mathbf{m}_2^\varepsilon)||_m \\ \le\; & ||\mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon\mathbf{u}_1^\varepsilon\cdot\nabla\mathcal{J}_\varepsilon(\mathbf{m}_1^\varepsilon - \mathbf{m}_2^\varepsilon))||_m + ||\mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon(\mathbf{u}_1^\varepsilon - \mathbf{u}_2^\varepsilon)\cdot\nabla\mathcal{J}_\varepsilon\mathbf{m}_2^\varepsilon)||_m \\ \equiv\; & G_1 + G_2, \end{aligned}$$

and

$$\begin{aligned} G_1 &\le |\mathcal{J}_\varepsilon\mathbf{u}_1^\varepsilon|_{L^\infty}||D^{m+1}\mathcal{J}_\varepsilon(\mathbf{m}_1^\varepsilon - \mathbf{m}_2^\varepsilon)||_0 + ||D^m\mathcal{J}_\varepsilon\mathbf{u}_1^\varepsilon||_0|\nabla\mathcal{J}_\varepsilon(\mathbf{m}_1^\varepsilon - \mathbf{m}_2^\varepsilon)|_{L^\infty} \\ &\le \tfrac{1}{\varepsilon}|\mathcal{J}_\varepsilon\mathbf{u}_1^\varepsilon|_{L^\infty}||\mathbf{m}_1^\varepsilon - \mathbf{m}_2^\varepsilon||_m + \tfrac{C}{\varepsilon^2}||D^m\mathcal{J}_\varepsilon\mathbf{u}_1^\varepsilon||_0||\mathbf{m}_1 - \mathbf{m}_2||_0 \\ &\le \tfrac{C}{\varepsilon^2}||\mathbf{u}_1^\varepsilon||_0||\mathbf{m}_1^\varepsilon - \mathbf{m}_2^\varepsilon||_m + \tfrac{C}{\varepsilon^2}||\mathbf{u}_1^\varepsilon||_m||\mathbf{m}_1 - \mathbf{m}_2||_0 \\ &\le \tfrac{C(||\mathbf{m}_1^\varepsilon||_m)}{\varepsilon^2}||\mathbf{m}_1^\varepsilon - \mathbf{m}_2^\varepsilon||_m. \end{aligned}$$

Similarly, we have

$$G_2 \le \frac{C}{\varepsilon^2}||\mathbf{m}_1^\varepsilon - \mathbf{m}_2^\varepsilon||_m. \tag{7.5}$$

So we have

$$||F_1(\mathbf{m}_1^\varepsilon) - F_1(\mathbf{m}_2^\varepsilon)||_m \le \frac{C}{\varepsilon^2}||\mathbf{m}_1^\varepsilon - \mathbf{m}_2^\varepsilon||_m, \tag{7.6}$$

where C depends on $||\mathbf{m}_1^\varepsilon||_m$ and $||\mathbf{m}_2^\varepsilon||_m$.

Analogously, we have

$$||F_i(\mathbf{m}_1^\varepsilon) - F_i(\mathbf{m}_2^\varepsilon)||_m \le \frac{C}{\varepsilon^2}||\mathbf{m}_1^\varepsilon - \mathbf{m}_2^\varepsilon||_m, \quad i = 2, 3. \tag{7.7}$$

So, if we define an open subset O of $H^m(\mathbb{R}^2)$ by $O \equiv \{\mathbf{m} \in H^m(\mathbb{R}^2) : ||\mathbf{m}||_m < M\ \}$ for some fix $M > 0$, then from the Picard Theorem, there exists a $T = T_\varepsilon > 0$ depending only on M and ε such that the smoothed equations (7.2) admit a unique solution in $C^1((-T,T);O)$, ie a unique solution in $C^1((-T,T);H^m(\mathbb{R}^2))$.

We take the derivative $D^\alpha,\ |\alpha| \le m,\ m \ge 2$ of the equation (7.2) and then inner product with $D^\alpha \mathbf{m}^\varepsilon$, use the fact

$$\langle \mathcal{J}_\varepsilon \mathbf{u}^\varepsilon \cdot \nabla D^\alpha \mathcal{J}_\varepsilon \mathbf{m}^\varepsilon, D^\alpha \mathcal{J}_\varepsilon \mathbf{m}^\varepsilon \rangle_0 = -\frac{1}{2}\langle \mathrm{div} \mathcal{J}_\varepsilon \mathbf{u}^\varepsilon D^\alpha \mathcal{J}_\varepsilon \mathbf{m}^\varepsilon, D^\alpha \mathcal{J}_\varepsilon \mathbf{m}^\varepsilon \rangle_0, \quad (7.8)$$

we have (omitting the superscript ε and $\mathcal{J}_\varepsilon$)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||D^\alpha \mathbf{m}||^2_{L^2} = -\left\langle D^\alpha(\mathbf{u}\nabla \mathbf{m}) + D^\alpha(\nabla \mathbf{u}^T \cdot \mathbf{m}) + D^\alpha(\mathbf{m}\mathrm{div}\mathbf{u}), D^\alpha \mathbf{m}\right\rangle_0, \quad (7.9)$$

We estimate the right side terms one by one.

$$\left\langle D^\alpha(\mathbf{u}\nabla\mathbf{m}), D^\alpha\mathbf{m}\right\rangle_0 \ = \ \left\langle \nabla\mathbf{m}D^\alpha\mathbf{u} + \mathbf{u}\nabla D^\alpha\mathbf{m} + \text{ other terms } Q, D^\alpha\mathbf{m}\right\rangle_0,$$

where Q is some intermediate terms between $\nabla\mathbf{m}D^\alpha\mathbf{u}$ and $\mathbf{u}\nabla D^\alpha\mathbf{m}$. We can estimate

$$\left\langle \nabla\mathbf{m}D^\alpha\mathbf{u}, D^\alpha\mathbf{m}\right\rangle_0 \le |D^\alpha\mathbf{u}|_{L^\infty}||\mathbf{m}||^2_m, \quad (7.10)$$

and from (7.8) we have

$$|\langle \mathbf{u}\nabla D^\alpha\mathbf{m}, D^\alpha\mathbf{m}\rangle_0| \le |\mathrm{div}\mathbf{u}|_{L^\infty}||\mathbf{m}||^2_m. \quad (7.11)$$

Similarly we can estimate $\langle Q, D^\alpha\mathbf{m}\rangle$. The point here is that we need to cancel the terms involving the $m+1$-th order derivative of $\mathbf{m}$. If there is no such higher order terms, then the estimate is straightforward. That is the estimate for the first term in (7.9). We can obtain the similar estimates for the other two terms in (7.9). Summing the resulting inequalities, we have

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}||\mathbf{m}||^2_m \ &\le \ C(m)\sum_{|\alpha|\le m}\langle D^\alpha\mathbf{u}\nabla\mathbf{m} + D^\alpha\nabla\mathbf{u}\cdot\mathbf{m} + \\ &\qquad \nabla\mathbf{u}D^\alpha\mathbf{m} + D^\alpha\mathbf{m}\mathrm{div}\mathbf{u} + \mathbf{m}\mathrm{div}D^\alpha\mathbf{u}, D^\alpha\mathbf{m}\rangle_0 \end{aligned} \quad (7.12)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}||\mathbf{m}||^2_m \le C(m)(|\nabla\mathbf{u}|_{L^\infty}||\mathbf{m}||^2_m + |D^m\mathbf{u}|_{L^\infty}||\mathbf{m}||^2_m + |\mathbf{m}|_{L^\infty}||\mathbf{u}||_{m+1}||\mathbf{m}||_m). \quad (7.13)$$

So Sobolev embedding theorems give us

$$\frac{\mathrm{d}}{\mathrm{d}t}||\mathbf{m}||_m \le C||\mathbf{m}||^2_m \quad (7.14)$$

and integrating yields

$$||\mathbf{m}(\cdot,t)||_m \leq \frac{||\mathbf{m}_0||_m}{1-Ct||\mathbf{m}_0||_m}, \tag{7.15}$$

which means that for $0 < t < T^* = \frac{1}{C||\mathbf{m}_0||_m}$, $||\mathbf{m}^\varepsilon(\cdot,t)||_m$ has an upper bound independent of ε.

For any $\varepsilon > 0$, the equation (7.2) has a unique solution on $[0, T_\varepsilon)$, and by the Continuation of an Autonomous ODE on a Banach Space ([74], page 103) we can extend the existence time interval to $[0, \tilde{T}_\varepsilon)$ such that $\tilde{T}_\varepsilon = \infty$ or

$$\lim_{t\to\tilde{T}_\varepsilon} ||\mathbf{m}^\varepsilon(\cdot,t)||_m = \infty.$$

On the other hand, from the inequality (7.15), we find that for $0 < t < T^*$, $||\mathbf{m}^\varepsilon(\cdot,t)||_m$ has an upper bound independent of ε. So we have

$$[0, T^*) \subset \bigcap_{\varepsilon>0} [0, \tilde{T}_\varepsilon). \tag{7.16}$$

Proposition 7.2 *If $\{\mathbf{m}^\varepsilon(x,t)\}$ is a bounded set in $H^m(\mathbb{R}^2)$ for all $t \in [0,T]$ and some $m \geq 2$, then there exist a constant $C = C(||\mathbf{m}_0||_m, T)$ and a constant $0 < \delta < 1$ such that for all $\varepsilon, \varepsilon' > 0$, we have*

$$\sup_{0\leq t<T} ||\mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}||_0 \leq C \max(\varepsilon, \varepsilon')^{1-\delta}. \tag{7.17}$$

Proof From the equation (7.2) we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}||_0^2 &= \langle F(\mathbf{m}^\varepsilon) - F(\mathbf{m}^{\varepsilon'}), \mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}\rangle_0 \\ &= \sum_{i=1}^{3} \langle F_i(\mathbf{m}^\varepsilon) - F_i(\mathbf{m}^{\varepsilon'}), \mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}\rangle_0. \end{aligned} \tag{7.18}$$

Let us take the second term as an example, and the other two terms are proved analogously.

$$\langle F_2(\mathbf{m}^\varepsilon) - F_2(\mathbf{m}^{\varepsilon'}), \mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}\rangle_0 = -\langle (\nabla\mathcal{J}_\varepsilon\mathbf{u}^\varepsilon)^T \cdot \mathcal{J}_\varepsilon\mathbf{m}^\varepsilon - (\nabla\mathcal{J}_{\varepsilon'}\mathbf{u}^{\varepsilon'})^T \cdot \mathcal{J}_{\varepsilon'}\mathbf{m}^{\varepsilon'}, \mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}\rangle_0$$

$$\begin{aligned} &= -\Big\langle (\nabla\mathcal{J}_\varepsilon\mathbf{u}^\varepsilon)^T \cdot \mathcal{J}_\varepsilon(\mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}) + (\nabla\mathcal{J}_\varepsilon\mathbf{u}^\varepsilon)^T \cdot (\mathcal{J}_\varepsilon - \mathcal{J}_{\varepsilon'})\mathbf{m}^{\varepsilon'} + \\ &\quad (\nabla\mathcal{J}_\varepsilon(\mathbf{u}^\varepsilon - \mathbf{u}^{\varepsilon'})^T \cdot \mathcal{J}_{\varepsilon'}\mathbf{m}^{\varepsilon'} + (\nabla(\mathcal{J}_\varepsilon - \mathcal{J}_{\varepsilon'})\mathbf{u}^{\varepsilon'})^T \cdot \mathcal{J}_{\varepsilon'}\mathbf{m}^{\varepsilon'},\ \ \mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}\Big\rangle_0 \\ &= T_1 + T_2 + T_3 + T_4. \end{aligned}$$

$$
\begin{aligned}
|T_1| &= \langle (\nabla \mathcal{J}_\varepsilon \mathbf{u}^\varepsilon)^T \cdot \mathcal{J}_\varepsilon(\mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}),\ \mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'} \rangle_0 \\
&\le C||\mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}||_0^2 ||\nabla \mathbf{u}^\varepsilon||_{L^\infty} \le C||\mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}||_0^2 ||\mathbf{m}^\varepsilon||_2.
\end{aligned}
$$

Similarly,

$$|T_3| \le C||\mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}||_0 ||\mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}||_2 ||\mathbf{m}^{\varepsilon'}||_0.$$

From Lemma 7.1, we have

$$|T_2| \le C \max(\varepsilon, \varepsilon')||\mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}||_0 ||\mathbf{m}^\varepsilon||_2 ||\mathbf{m}^{\varepsilon'}||_1,$$

For some $0 < \delta < 1$, we have

$$
\begin{aligned}
|T_4| &\le C||\mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}||_0 ||\mathbf{m}^{\varepsilon'}||_0 ||\nabla(\mathcal{J}_\varepsilon - \mathcal{J}_{\varepsilon'})\mathbf{u}^{\varepsilon'}||_{L^\infty} \\
&\le C||\mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}||_0 ||\mathbf{m}^{\varepsilon'}||_0 ||\nabla(\mathcal{J}_\varepsilon - \mathcal{J}_{\varepsilon'})\mathbf{u}^{\varepsilon'}||_{H^{1+\delta}} \\
&\le C||\mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}||_0 ||\mathbf{m}^{\varepsilon'}||_0 ||(\mathcal{J}_\varepsilon - \mathcal{J}_{\varepsilon'})\mathbf{m}^{\varepsilon'}||_{H^{1+\delta}} \\
&\le C \max(\varepsilon, \varepsilon')^{1-\delta} ||\mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}||_0 ||\mathbf{m}^{\varepsilon'}||_0 ||\mathbf{m}^{\varepsilon'}||_2,
\end{aligned}
$$

here we have used the following inequality which can be easily proved:

$$||\mathcal{J}_\varepsilon v - v||_{s-\delta} \le C\varepsilon^\delta ||v||_{H^s}, \text{ for all } v \in H^s(\mathbb{R}^2).$$

So if $||\mathbf{m}^\varepsilon(x,t)||_m \le M$ for some constant $M > 0$ and for all $0 < t < T$, then the above inequalities yield

$$\frac{\mathrm{d}}{\mathrm{d}t}||\mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}||_0 \le C(\max(\varepsilon, \varepsilon') + \max(\varepsilon, \varepsilon')^{1-\delta} + ||\mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}||_0), \tag{7.19}$$

and Gronwall inequality tells us

$$\sup_{0 \le t \le T} ||\mathbf{m}^\varepsilon - \mathbf{m}^{\varepsilon'}||_0 \le C \max(\varepsilon, \varepsilon')^{1-\delta} \tag{7.20}$$

because $\mathbf{m}^\varepsilon$ have the same initial value.

We have just proved that the existence of an $\mathbf{m}$ such that

$$\sup_{0 \le t \le T} ||\mathbf{m}^\varepsilon - \mathbf{m}||_0 \le C\varepsilon^{1-\delta}. \tag{7.21}$$

Moreover, from the Interpolation of Sobolev Spaces:

$$||v||_{s'} \le C_s ||v||_0^{1-s'/s} ||v||_s^{s'/s}, \quad \text{for } 0 < s' < s,$$

we have for $0 < r < m$

$$\sup_{0 \le t \le T} ||\mathbf{m}^{\varepsilon} - \mathbf{m}||_r \le C(||\mathbf{m}_0||, T)\varepsilon^{(1-\delta)(1-r/m)} \tag{7.22}$$

Hence we have the strong convergence in $C([0,T]; H^r(\mathbb{R}^2))$. If $0 < 2 < r < m$, this implies strong convergence in $C([0,T]; C^1(\mathbb{R}^2))$. In the equation (7.2), the last three terms of (7.2) converge in $C(0,T; C(\mathbb{R}^2))$, so does the first term of it, ie, $\mathbf{m}_t^{\varepsilon}$, which means that,if $\mathbf{m}_0 \in H^m(\mathbb{R}^2)$ with $m > 2$, then the equation (7.1) has a unique classical solution for $0 < t < T$.

Chapter 8

Future Work

路漫漫其修远兮，吾将上下而求索。
——屈原，《离骚》，约公元前290年。

The way ahead is long; I see no ending, yet high and low, I'll search with my will unbending.

—Qu Yuan, *Li Sao*, ~290 BC.

8.1 Stability of the Asymptotic Solutions

Our numerical simulation shows that the solution of the equation

$$m_t + um_x + 2u_x m = 0, \tag{8.1}$$

where $(1 - \partial_x^2)^k u = m, \quad k = 2$, with the Gaussian initial values, tends to the asymptotic solution

$$f(a, \eta) = C(a) \frac{e^{-2a[\arctan(a(\eta-1)) + \frac{\pi}{2}]}}{((\eta - 1)^2 + a^{-2})^2}, \tag{8.2}$$

with some constant $C(a)$, and

$$\lim_{a \to +\infty} f(a, \eta) = \tilde{f} = \begin{cases} 4(\eta - 1)^{-4} e^{\frac{2}{\eta - 1}} & \text{if } \eta - 1 < 0, \\ 0, & \text{if } \eta - 1 > 0. \end{cases} \tag{8.3}$$

But we have not yet found the rigorous explanation of this phenomena. In Chapter 5, we tried to use the characteristics method to understand how this *weak blowup* forms, but have not solved this completely.

There are lots of literature devoted to the stability of steady solutions of some evolutionary equations. For example, the blowup rate and profile for the nonlinear heat equation

$$u_t = \triangle u + u^p \quad \text{in} \quad \mathbb{R}^N$$

with some restrictions on p, has been determined by Giga and Kohn [43] and Fermanian-Kammerer et al. [38]. However, the existence of Lyapunov functions plays a fundamental role in their stability study of the blowup profile. Another approach is the so-called *Evans function method* introduced by J. Evans [36], whose application in stability of travelling wave solutions of the dissipative systems is summarised by T. Kapitula [60].

The Evans function method also finds some successful applications to Hamiltonian PDEs. R. Pego and M. Weinstein [94] used the Evans function to discuss the instability of solitary waves. There are many other approaches as well to the stability problem for the Hamiltonian PDEs. For example, Grillakis et al. [48] established some sharp conditions for the stability and instability of solitary waves of some Hamiltonian PDEs. A. Constantin and W. Strauss [26], J. Lenells [69] exploited the first three conserved quantities to establish the stability of peakons for Camassa-Holm equations on the whole $\mathbb{R}^1$ and periodic case respectively. Another important contribution to the stability of the blowup profile is made by Y. Martel and F. Merle [80]– [83], where they introduced some new tools to study the stability of the blowup profile of the generalised KdV equation

$$u_t + (u_{xx} + u^p)_x = 0, \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}.$$

But all the approaches above cannot directly apply to our case. The equation (8.1) is indeed a Hamiltonian PDE, but the functions in (8.2) and (8.3) do not correspond to the travelling wave solutions of (8.1), ie, $tf(t(x - ct))$ is not a solution of (8.1), it is only an *asymptotic solution*! If we consider the asymptotic slow time PDE (5.11):

$$f_\tau + (\xi f)_\xi + U f_\xi + 2U_\xi f = 0, \tag{8.4}$$

here $U = \frac{G''(0)}{2}(\xi^2 f_0 - 2\xi f_1 + f_2)$. Then these f's are steady solutions of (8.4), but now the PDE (8.4) is no longer a Hamiltonian PDE, and the fact that the coefficients in (8.4) depend on the integrals of f makes things even more difficult.

8.2 Positivity of Solutions

For the periodic Camassa-Holm equation, it is well-known that the positivity of the initial momentum implies the global well-posedness and the positivity of the momentum for any time $t > 0$. The proof depends on the complete integrability [25] [65]. Our numerical simulation supports that the equation (8.1) has this property as well, which is consistent with the geometric observation that the generalised Euler equation preserves the co-adjoint orbits, whereas we do not know whether it is an integrable system or not. This means that the preservation of the positivity of momentum may not relate to the complete integrability.

8.3 Higher Dimensional Case

In Chapter 7 we discussed the local well-posedness of the Camassa-Holm equation, but we do not know anything about the global well-posedness and other properties. For the torus $\mathbb{T}^2$ case, we conjecture that if all components of the initial momentum have no simultaneous zero point (ie, the momentum vector is never zero), then the solution should exist globally. But we have no idea how to prove this.

Another question is to study the dynamics of the higher-dimensional generalised Euler equations and how they depend on the choice of the metric.

Appendix A

Properties of the Green's Function

Lemma A.1 *Let*

$$G(x) = 1 + 2\sum_{n=1}^{\infty} \frac{1}{1+n^2+n^4}\cos(nx) \quad x \in S = [0, 2\pi],$$

then $\partial_x^3 G(x) \in L^\infty(S)$.

Proof We know from Abel-Dirichlet criterion [107] that
(or see http://en.wikibooks.org/wiki/Real_Analysis/Series)

$$\sum_{n=1}^{\infty} \frac{n^3}{1+n^2+n^4}\sin(nx)$$

converges for any $x \in S$, and uniformly converges in any $[\alpha, \beta] \subset (0, 2\pi)$ if $0 < \alpha < \beta < 2\pi$. That means the Fourier series

$$2\sum_{n=1}^{\infty} \frac{n^3}{1+n^2+n^4}\sin(nx)$$

converges to $\partial_x^3 G$:

$$\partial_x^3 G(x) = 2\sum_{n=1}^{\infty} \frac{n^3}{1+n^2+n^4}\sin(nx). \tag{A.1}$$

On the other hand, we know

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi}{2}(1 - \frac{x}{\pi}) \quad \text{for} \quad 0 < x < 2\pi, \tag{A.2}$$

and if we denote $g(x)$ for this function, then

$$|g(x) - \partial_x^3 G(x)| = \left| \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{n^3}{1+n^2+n^4} \right) \sin(nx) \right| \tag{A.3}$$

which converges uniformly to a bounded function on S. So we have

$$\partial_x^3 G \in L^\infty(S).$$

Now we are going to discuss some properties of the Green's function of the operator Λ_4^4 acting on $H^\infty(\mathbb{R}^1)$, that is,

$$\Lambda_4^4 G(x) = (I - \partial_x^2)^2 G(x) = \delta \quad \text{ on } \mathbb{R}^1. \tag{A.4}$$

The Green's function $G(x)$ can be expressed (up to a multiplicative constant) via Fourier transform

$$\begin{aligned} G(x) &= \int_{\mathbb{R}^1} \frac{1}{1+2|\xi|^2+|\xi|^4} \exp(ix\xi) \mathrm{d}\xi \\ &= \int_{\mathbb{R}^1} \frac{1}{1+2|\xi|^2+|\xi|^4} \cos(x\xi) \mathrm{d}\xi. \end{aligned} \tag{A.5}$$

Clearly, $G(x)$ is uniformly continuous on $\mathbb{R}^1$ and $G \in C^{2+\varepsilon}(\mathbb{R}^1)$ for any $0 \leq \varepsilon < 1$.

Lemma A.2 *Let $G(x)$ be the Green's function as above, then*

$$||\partial_x^3 G||_{L^\infty(\mathbb{R}^1)} < \infty. \tag{A.6}$$

Proof The idea is similar to that in the proof of Lemma A.1.

$$G(x) = \int_{\mathbb{R}^1} \frac{1}{1+2|\xi|^2+|\xi|^4} \cos(x\xi) \mathrm{d}\xi, \tag{A.7}$$

Abel-Dirichlet criterion for the improper integral tells us that

$$\int_{\mathbb{R}^1} \frac{\xi^3}{1+2|\xi|^2+|\xi|^4} \sin(x\xi) \mathrm{d}\xi$$

converges on $\mathbb{R}^1$ and uniformly converges on any bounded interval away from $x = 0$, which means

$$\partial_x^3 G(x) = -\int_{\mathbb{R}^1} \frac{\xi^3}{1+2|\xi|^2+|\xi|^4} \sin(x\xi) \mathrm{d}\xi. \tag{A.8}$$

Now let us introduce a bounded function

$$g(x) = \begin{cases} e^x, & x < 0 \\ -e^{-x}, & x > 0 \end{cases} \tag{A.9}$$

then it is easy to find the Fourier transform of g is $2i\frac{\xi}{1+|\xi|^2}$, which means

$$\begin{aligned} g(x) &= \int_{\mathbb{R}^1} 2i\frac{\xi}{1+\xi^2}\exp ix\xi \mathrm{d}\xi \\ &= -2\int_{\mathbb{R}^1} \frac{\xi}{1+\xi^2}\sin x\xi \mathrm{d}\xi. \end{aligned} \tag{A.10}$$

On the other hand,

$$\left|\partial_x^3 G(x) - \frac{1}{2}g(x)\right| = \left|\int_{\mathbb{R}^1}\left(\frac{\xi}{1+\xi^2} - \frac{\xi^3}{1+2|\xi|^2+|\xi|^4}\right)\sin x\xi \mathrm{d}\xi\right| \tag{A.11}$$

is clearly convergent to a C^1 function on $\mathbb{R}^1$ and so is bounded on any bounded interval. Moreover, the Riemann-Lebesgue Lemma tells us that

$$\lim_{|x|\to\infty}\left|\partial_x^3 G(x) - \frac{1}{2}g(x)\right| = 0, \tag{A.12}$$

so

$$|\partial_x^3 G|_{L^\infty(\mathbb{R}^1)} < \infty.$$

Lemma A.3 *The Green's function of* $\Lambda = (1-\partial_x^2)^2$ *on* $\mathbb{R}^1$ *is*

$$G(x) = \frac{1}{4}(|x|e^{-|x|} + e^{-|x|}), \tag{A.13}$$

so it is positive on $\mathbb{R}^1$.

Proof We use the method of undetermined coefficients to solve the equation

$$(1-\partial_x^2)^2 G(x) = \delta.$$

First, find the eigenvalues of this operator, $\lambda = \pm 1$ (double eigenvalues), so the solutions have the form of $h(x) = ae^x + bxe^x + ce^{-x} + dxe^{-x}$. Then we impose the requirements

$$h(0_+) = h(0_-); \qquad h'(0_+) = 0 = h'(0_-);$$

$$h''(0_+) = h''(0_-); \quad h'''(0_-) = -\tfrac{1}{2} = -h'''(0_+)$$

to find that the solution is $G(x) = \frac{1}{4}e^{-|x|}(1+|x|)$.

The arguments here apply to the Green's function of the general initial operator and we have

Lemma A.4 *Let $G(x)$ be the Green's function of the operator $\Lambda_{2k}^{2k} = (1-\partial_x^2)^k$ with $k \geq 1$, then $G(x) \geq 0$ and*

$$|\partial_x^{2k-1} G|_{L^\infty} < \infty. \tag{A.14}$$

Proof The idea is same as before and we can find the Green function in this case is

$$G(x) = \frac{1}{2k} e^{-|x|}(1 + |x| + \cdots + |x|^{k-1}).$$

Appendix B

Multi-symplectic Formulation

In this appendix, we will establish a multi-symplectic formulation for the mCH equation via the multi-symplectic geometry. [45] provides the details of the foundation of multi-symplectic geometry.

B.1 Multi-symplectic Geometry

We recall some aspects of the so-called multi-symplectic geometry. Let X be an orientable $n+1$ dimensional manifold (which in our case is $S \times \mathbb{R}^1$) and let π_{XY} be a fiber bundle over X, which we call the *covariant configuration bundle*. The space of sections of π_{XY} will be denoted by $C^\infty(\pi_{XY})$ (which in our case is the space of smooth functions on $S \times \mathbb{R}^1$.)

Definition B.1 The first jet bundle $J^1(Y)$ *of* Y *is the affine bundle over* Y *whose fiber over* $y \in Y_x$ *consists of those linear mappings* $\gamma : T_xX \mapsto T_yY$ *satisfying*

$$T\pi_{XY} \circ \gamma = \textit{identity on } T_xX. \tag{B.1}$$

The map γ corresponds to $T_x\phi$ for a local section ϕ because we can easily find that

$$\pi_{XY}(\phi(x)) = x \quad \forall x \in X \Longrightarrow T\pi_{XY} \circ T_x\phi(x)(v) = v \quad \forall v \in T_xX. \tag{B.2}$$

If X has local coordinates $x^\nu, \nu = 1, 2, \cdots, n$, adapted coordinates on Y are $y^A,\ A = 1, 2, \cdots, N$, along the fiber Y_x. Then Coordinates (x^ν, y^A) induce coordinates y^A_ν on the fibers of $J^1(Y)$. The map $x \mapsto T_x\phi$ defines a section of $J^1(Y)$ regarded as a bundle over X, and we denote this section by $j^1(\phi)$ and call it *the first jet of* ϕ or the first prolongation of ϕ. In coordinates, $j^1(\phi)$ is given by

$$x^\nu \mapsto (x^\nu, \phi^A(x^\nu), \partial_\mu\phi^A(x^\nu)),$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$. A section of the bundle $J^1(Y) \mapsto X$ which is the first jet of a section of π_{XY} is said to be *holonomic*. We can define the higher order jet bundles by induction: $J^2(Y) \equiv J^1(J^1(Y))$, $J^k(Y) \equiv J^1(J^{k-1}(Y))$ for any integer $k \geq 2$. Introduce the k-th jet prolongation $j^k(\phi) \equiv j^1(j^{k-1}(\phi))$ of a section $\phi : X \mapsto Y$. In coordinates, $j^2(\phi)$ is given by

$$x^\nu \mapsto (x^\nu, \phi^A(x^\nu), \partial_\mu \phi^A(x^\nu), \partial_{\nu_1}\partial_{\nu_2}\phi^A(x^\nu)).$$

A section ρ of $J^k(Y) \mapsto X$ is said to be k-holonomic if $\rho = j^k(\pi_{Y,J^k(Y)} \circ \rho)$.

If, for example, $X = S \times \mathbb{R}^1$, $Y = X \times \mathbb{R}^1$, $\phi \in C^\infty(\pi_{XY})$ means that $\phi = \phi(x,t)$ is a smooth function, and $j^1(\phi)$ means a map

$$(x,t) \mapsto (x, t, \phi(x,t), \phi_x(x,t), \phi_t(x,t)), \tag{B.3}$$

and $j^2(\phi)$ means a map

$$(x,t) \mapsto (x, t, \phi(x,t), \phi_x(x,t), \phi_t(x,t), \phi_{xx}(x,t), \phi_{xt}(x,t), \phi_{tx}(x,t), \phi_{tt}(x,t)). \tag{B.4}$$

We use $j^k(\phi)$ for these maps and their values $j^k(\phi)(x,t)$ if no confusions occur.

Define the set

$$\mathcal{C}^\infty \equiv \Big\{\phi : X \mapsto Y \Big| \ \forall x \in X, \text{ there is a smooth open manifold } U \subset X \text{ with}$$
$$\text{smooth closed boundary, such that } \pi_{XY} \circ \phi|_U : U \mapsto X \text{ is an embedding}\Big\}.$$

For each $\phi \in \mathcal{C}^\infty$ set $\phi_X \equiv \pi_{XY} \circ \phi$ and $U_X \equiv \phi_X(U)$ so that $\phi_X : U \mapsto U_X$ is a diffeomorphism. Let $\mathcal{C}$ be the closure of $\mathcal{C}^\infty$ in some Hilbert norm. Denote $\mathcal{C}_k \equiv \{j^k(\phi \circ \phi_X^{-1}) | \ \phi \in \mathcal{C}\}$.

The tangent space to the manifold $\mathcal{C}$ at a point $\phi \in \mathcal{C}$ is

$$\Big\{V \in \mathcal{C}^\infty(X, TY) | \text{ locally } \pi_{Y,TY} \circ V = \phi \text{ and } V_X \equiv T\pi_{XY} \circ V \circ \phi_X^{-1}$$
$$\text{is a vector field on } U_X\Big\},$$

which means that it consists of vector fields V such that the diagram in the Figure B.1.1 commutes.

We introduce

$$G \equiv \{\eta_Y : Y \mapsto Y | \ \eta_Y \text{ covers a diffeomorphism } \eta_X : X \mapsto X,$$
$$\eta_Y \ \text{ is a } \pi_{XY} \text{ bundle automorphism}\}.$$

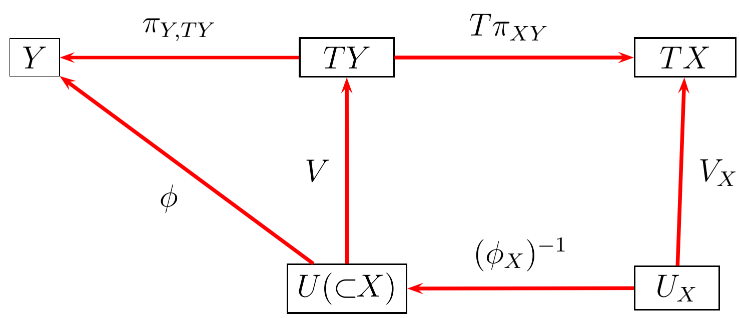

Figure B.1.1: The tangent vector $V \in T_\phi \mathcal{C}$

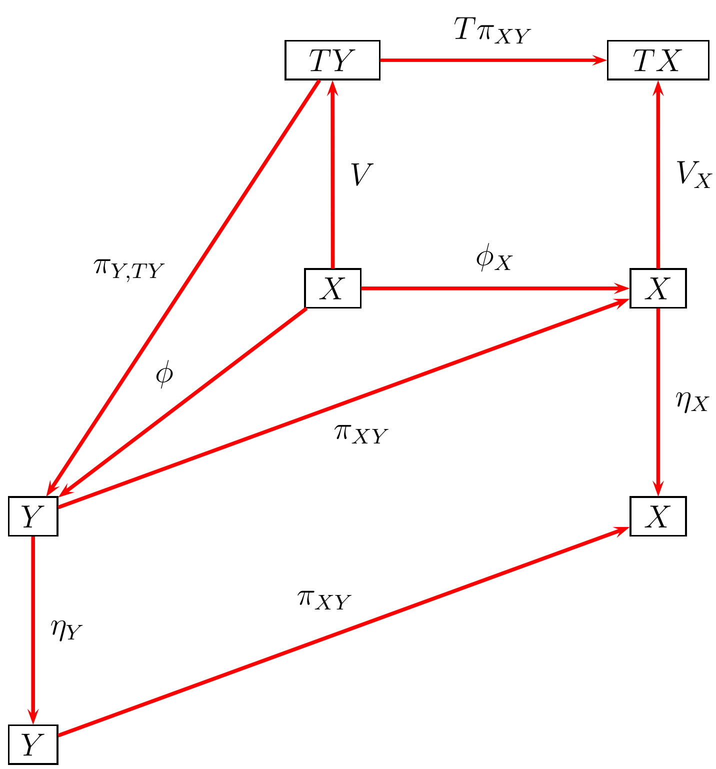

Figure B.1.2: Various spaces and mappings

We can summarise the various sets and mappings in the Figure B.1.2.

We need introduce the vertical sub-bundle VY of Y: it is defined as a sub-bundle whose fiber over $y \in Y_x \equiv \pi_{XY}^{-1}(x)$ is

$$V_yY \equiv \{v \in T_yY : \ T\pi_{XY} \cdot v = 0\}. \tag{B.5}$$

For $\gamma \in J_y^1(Y)$, we have the splitting

$$T_yY = \text{image of } \gamma \oplus V_yY. \tag{B.6}$$

We will have some intuitive ideas about these concepts through the following example:

Example B.2 If $X = \mathbb{R}^1, Y = X \times \mathbb{R}^1 = \mathbb{R}^2$, then a smooth section f of $Y \mapsto X$ means a smooth function $f : \mathbb{R}^1 \mapsto \mathbb{R}^1$. In other words, we have a smooth mapping

$$f : x \in X = \mathbb{R}^1 \mapsto (x, f(x)) \in Y = \mathbb{R}^2$$

whose composition with π_{XY} is $f_X = \pi_{XY} \circ f$ =identity on $\mathbb{R}^1$. If denote Γ the graph of f: $\Gamma = \{(x, y) : y = f(x), \ x \in \mathbb{R}^1\}$, and that $V = (V_1, V_2) \in T_p\mathbb{R}^2$ for some $p = (x, f(x)) \in \Gamma$, then the first prolongation $\gamma = j^1(f)$ of f is a mapping

$$j^1(f) : (x, V_1) \in T_x\mathbb{R}^1 \mapsto (x, f(x), V_1, f'(x)V_1) \in T_p\mathbb{R}^2,$$

or in other words,

$$\gamma \cdot V_1 = (V_1, f'(x)V_2) = V_1\frac{\partial}{\partial x} + f'(x)V_1\frac{\partial}{\partial y} \in T_p\mathbb{R}^2.$$

So

$$V^v = V - \gamma \cdot V_1 = (V_2 - f'(x)V_1)\frac{\partial}{\partial y} \in T_p\mathbb{R}^2$$

is a vector in the vertical sub-bundle of Y.

If we consider a mapping $\phi \in \mathcal{C}$ instead of the section f above, then locally, we have

$$\phi : \ x \mapsto (\phi_X(x), f(x)) \in \mathbb{R}^2$$

with a diffeomorphism ϕ_X and a smooth function f, which means we have a smooth section $f(\phi_X^{-1}(x))$ of $Y \mapsto U_X$. In other words, $\phi \circ (\phi_X)^{-1} : x \mapsto (x, f \circ \phi_X^{-1}(x))$ is a smooth section of $Y \mapsto U_X$, and we have the similar results for this section $\phi \circ (\phi_X)^{-1}$.

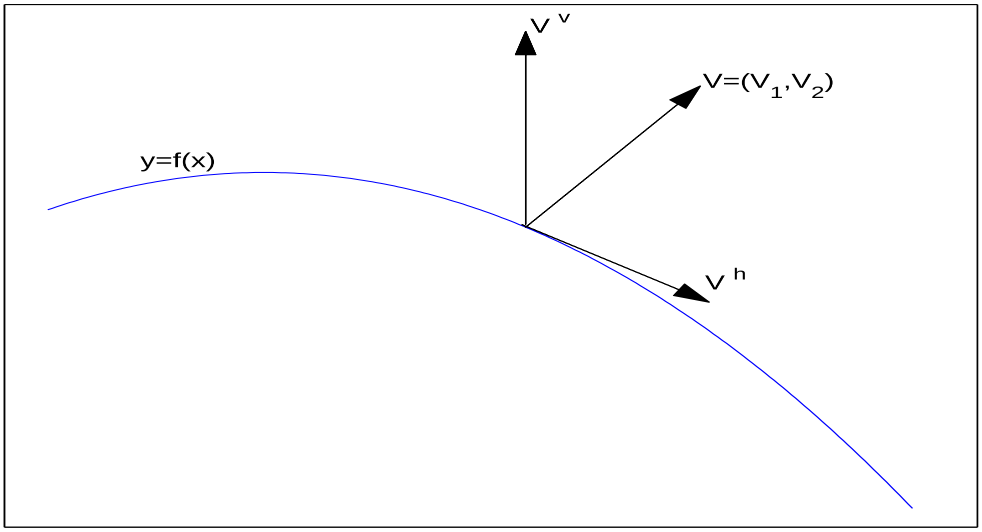

Figure B.1.3: The vertical part V^v of a vector V

Without loss of generality, from now on, we take an open bounded domain $U \subset X$ and consider $U_X = \phi_X(U)$ only.

Definition B.3 *The* **action functional** $\mathcal{S} : \mathcal{C} \mapsto \mathbb{R}$ *is given by*

$$\mathcal{S}(\phi) = \int_{U_X} \mathcal{L}(j^k(\phi \circ \phi_X^{-1})), \quad \phi \in \mathcal{C} \tag{B.7}$$

for some Lagrangian density $\mathcal{L} : J^k(Y) \mapsto \Lambda^{n+1}(X)$, *and* $\phi \in \mathcal{C}$ *is called* **an extremum** *of* $\mathcal{S}$ *if*

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \mathcal{S}(\eta_Y^\lambda \circ \phi) = 0$$

for all smooth paths $\lambda \mapsto \eta_Y^\lambda$ *in* G, *where* η_Y^λ *covers a diffeomorphism* η_X^λ *and* $\eta_Y^0 = id_Y$.

For any $\phi^\lambda \in \mathcal{C}$ such that $\phi^0 = \phi$, and $\frac{\mathrm{d}\phi^\lambda}{\mathrm{d}\lambda}|_{\lambda=0} = V$, there is $\eta_Y^\lambda : Y \mapsto Y$ such that $\phi^\lambda = \eta_Y^\lambda \circ \phi$. We can see that $\eta_Y^\lambda \circ (\phi \circ \phi_X^{-1}) \circ (\eta_X^\lambda)^{-1}$ is a section of $Y \mapsto X$, and it maps $U_X^\lambda \equiv \eta_X^\lambda \circ \phi_X(U)$ to $\phi^\lambda(U)$.

We are going to introduce the prolongations of automorphisms η_Y of Y and of elements $V \in T_\phi \mathcal{C}$.

Definition B.4 *The* **first prolongation** $j^1(\eta_Y) : J^1(Y) \mapsto J^1(Y)$ *of an automorphism* η_Y *of* $Y \mapsto X$ *is defined by (as shown in Figure B.1.4):*

$$j^1(\eta_Y)(\gamma) = T\eta_Y \circ \gamma \circ T\eta_X^{-1}.$$

$$\begin{tikzcd} \boxed{Y} \arrow[r, "\eta_Y"] \arrow[d, "\pi_{XY}"] & \boxed{Y} \arrow[d, "\pi_{XY}"] \\ \boxed{X} \arrow[r, "\eta_X"] & \boxed{X} \end{tikzcd} \qquad \begin{tikzcd} \boxed{TY} \arrow[r, "T\eta_Y"] & \boxed{TY} \\ \boxed{TX} \arrow[r, "T\eta_X"] \arrow[u, "\gamma"] & \boxed{TX} \arrow[u, "j^1(\eta_Y)(\gamma)"'] \end{tikzcd}$$

Figure B.1.4: The prolongation of η_Y

For a vector $V \in T_\phi \mathcal{C}$, let η_Y^λ be the flow of a vector field v on Y with $v \circ \phi = V$. Then the **first prolongation** $j^1(V)$ *of V is a vector field on $J^1(Y)$ defined by*

$$j^1(V) = \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} j^1(\eta_Y^\lambda).$$

We can define the k**-th prolongations of an automorphism** η_Y **and a vector** $V \in T_\phi \mathcal{C}$ *for all $k \geq 1$ by induction.*

Definition B.5 The variational derivative of $f : J^k(Y) \mapsto \mathbb{R}$ *is the function on $J^{2k}(Y)$ defined by*

$$\frac{\delta f}{\delta y^A} \equiv \sum_{s=0}^{k} (-1)^s D_{\mu_1} D_{\mu_2} \cdots D_{\mu_s} \left(\frac{\partial f}{\partial y^A_{\mu_1 \cdots \mu_s}} \right), \tag{B.8}$$

where $D_\mu(f) \equiv \frac{\partial f}{\partial x_\mu} + \frac{\partial f}{\partial y^A} y^A_\mu + \cdots + \frac{\partial f}{\partial y^A_{\mu_1 \cdots \mu_k}} y^A_{\mu_1 \cdots \mu_k \mu}$.

Definition B.6 *The* **first dual jet bundle** $J^1(Y)^*$ *is the vector bundle over Y whose fiber at $y \in Y_x$ is the set of affine maps from $J^1(Y)_y$ to $\Lambda^{n+1}(X)_x$, the bundle of $n+1$-forms on X. A smooth section of $J^1(Y)^*$ is therefore an affine bundle map of $J^1(Y)$ to $\Lambda^{n+1}(X)$ covering π_{XY}.*

Fiber coordinates on $J^1(Y)^*$ are (p, p^ν_A), which correspond to the affine map given in coordinates by

$$v^A_\nu \mapsto (p + v^A_\nu p^\nu_A)\mathrm{d}^{n+1}x, \tag{B.9}$$

where $\mathrm{d}^{n+1}x = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \cdots \wedge \mathrm{d}x^n \wedge \mathrm{d}x^0$.

Analogous to the canonical one- and two-forms on a cotangent bundle, there exist canonical $(n+1)$- and $(n+2)$-forms on the jet bundle $J^1(Y)^*$. In coordinates, with $\mathrm{d}^n x_\nu \equiv \partial_\nu \dashv \mathrm{d}^{n+1}x$ standing for the interior product of $\frac{\partial}{\partial x_\nu}$ and $\mathrm{d}^{n+1}x$, these forms are given by

$$\theta = p_A^\nu \mathrm{d}y^A \wedge \mathrm{d}^n x_\nu + p\mathrm{d}^{n+1}x, \quad \Omega = -\mathrm{d}y^A \wedge \mathrm{d}p_A^\nu \wedge \mathrm{d}^n x_\nu + \mathrm{d}p \wedge \mathrm{d}^{n+1}x. \tag{B.10}$$

Similarly, we can define the k**-th dual jet bundle** $J^k(Y)^*$ and the canonical one- and two-forms on it.

A Lagrangian density $\mathcal{L} : J^k(Y) \mapsto \Lambda^{n+1}(X)$ is a fiber preserving map:

$$\mathcal{L}(j^k(\phi)) = L(j^k(\phi))\mathrm{d}x^1 \wedge \mathrm{d}x^2 \cdots \wedge \mathrm{d}x^n \wedge \mathrm{d}x^0. \tag{B.11}$$

For k-th order Lagrangian field theory, the fundamental geometric structure is the Cartan form θ_L, which is defined as the pullback of the canonical $n+1$-form θ on $J^k(Y)^*$ by $(\mathbb{F}\mathcal{L})^*$:

$$\theta_L \equiv (\mathbb{F}\mathcal{L})^*\theta, \tag{B.12}$$

where the fiber derivative $\mathbb{F}\mathcal{L} : J^k(Y) \mapsto J^k(Y)^*$, expressed intrinsically as the first order vertical Taylor approximation to $\mathcal{L}$, is defined by

$$\mathbb{F}\mathcal{L}(\gamma) \cdot \gamma' = \mathcal{L}(\gamma) + \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathcal{L}(\gamma + \varepsilon(\gamma' - \gamma)), \tag{B.13}$$

where $\gamma, \gamma' \in J^k(Y)$.

Now denote $\omega = \mathrm{d}^{n+1}x,\ \omega_\nu = \frac{\partial}{\partial x_\nu} \dashv \omega$.

Theorem B.7 *Given a smooth Lagrangian density $\mathcal{L} : J^k(Y) \mapsto \Lambda^{n+1}(X)$, there exist a unique $\Psi \in \Lambda^{n+2}(J^{2k}(Y))$ given by*

$$\Psi = \frac{\delta L}{\delta y^A}\mathrm{d}y^A \wedge \omega,$$

a unique map $\mathcal{D}_{EL}\mathcal{L} \in C^\infty(\mathcal{C}_{2k}, T^*\mathcal{C} \otimes \Lambda^{n+1}(X))$ given by

$$\mathcal{D}_{EL}\mathcal{L}(\phi) \cdot V = j^{2k}(\phi \circ \phi_X^{-1})^* \left(\frac{\delta L}{\delta y^A} \mathbf{i}_V (\mathrm{d}y^A \wedge \omega) \right), \tag{B.14}$$

and a unique $n+1$ form $\theta_L \in \Lambda^{n+1}(J^{2k-1}(Y))$ given by

$$\begin{aligned} \theta_L &= \sum_{s=0}^{k} p_A^{\mu_1\mu_2\cdots\mu_s} \mathrm{d}y^A_{\mu_1\cdots\mu_{s-1}} \wedge \omega_{\mu_s} \\ &= p\omega + p_A^\mu \mathrm{d}y^A \wedge \omega_\mu + p_A^{\mu_1\mu_2}\mathrm{d}y^A_{\mu_1} \wedge \omega_{\mu_2} + \cdots + p_A^{\mu_1\mu_2\cdots\mu_k}\mathrm{d}y^A_{\mu_1\cdots\mu_{k-1}} \wedge \omega_{\mu_k} \end{aligned} \tag{B.15}$$

such that for any $V \in T_\phi \mathcal{C}$ *and any open subset* U_X *of* X *with* $\overline{U}_X \bigcap \partial X = \varnothing$, *we have*

$$\mathrm{d}\mathcal{S}_\phi \cdot V = \int_{U_X} \mathcal{D}_{EL}\mathcal{L}(\phi) \cdot V + \int_{\partial U_X} j^{2k-1}(\phi \circ \phi_X^{-1})^*[j^{2k-1}(V) \dashv \theta_L], \quad \text{(B.16)}$$

where $\mathbf{i}_V$ *or* $V \dashv$ *denotes the interior product with* V, *and*

$$\begin{aligned}
p \;\; = \;\; & L - \frac{\partial L}{\partial y^A_{\mu_1}} y^A_{\mu_1} + D_{\mu_2}\left(\frac{\partial L}{\partial y^A_{\mu_1\mu_2}}\right) y^A_{\mu_1} - D_{\mu_2} D_{\mu_3}\left(\frac{\partial L}{\partial y^A_{\mu_1\mu_2\mu_3}}\right) y^A_{\mu_1} + \cdots \\
& \quad + (-1)^k D_{\mu_2} \cdots D_{\mu_k}\left(\frac{\partial L}{\partial y^A_{\mu_1\cdots\mu_k}}\right) y^A_{\mu_1} \\
& - \frac{\partial L}{\partial y_{\mu_1\mu_2}} y^A_{\mu_1\mu_2} + D_{\mu_3}\left(\frac{\partial L}{\partial y^A_{\mu_1\mu_2\mu_3}}\right) y^A_{\mu_1\mu_2} + \cdots \\
& + (-1)^{k-1} D_{\mu_3} \cdots D_{\mu_k}\left(\frac{\partial L}{\partial y^A_{\mu_1\cdots\mu_k}}\right) y^A_{\mu_1\mu_2} + \cdots - \frac{\partial L}{\partial y^A_{\mu_1\cdots\mu_k}} y^A_{\mu_1\cdots\mu_k}, \\
p^\mu_A \;\; = \;\; & \frac{\partial L}{\partial y^A_\mu} - D_{\mu_2}\left(\frac{\partial L}{\partial y^A_{\mu\mu_2}}\right) + D_{\mu_1} D_{\mu_2}\left(\frac{\partial L}{\partial y^A_{\mu\mu_1\mu_2}}\right) + \cdots \\
& + (-1)^{k-1} D_{\mu_2} \cdots D_{\mu_k}\left(\frac{\partial L}{\partial y^A_{\mu\mu_2\cdots\mu_k}}\right) \\
p^{\mu_1\mu_2}_A \;\; = \;\; & \frac{\partial L}{\partial y^A_{\mu_1\mu_2}} - D_{\mu_3}\left(\frac{\partial L}{\partial y^A_{\mu_1\mu_2\mu_3}}\right) + D_{\mu_3} D_{\mu_4}\left(\frac{\partial L}{\partial y^A_{\mu_1\mu_2\mu_3\mu_4}}\right) + \cdots \\
& + (-1)^k D_{\mu_3} \cdots D_{\mu_k}\left(\frac{\partial L}{\partial y^A_{\mu_1\mu_2\cdots\mu_k}}\right), \\
& \cdots\cdots \\
p^{\mu_1\mu_2\cdots\mu_k}_A \;\; = \;\; & \frac{\partial L}{\partial y^A_{\mu_1\mu_2\cdots\mu_k}}.
\end{aligned}$$

Moreover, the θ_L *agrees with the* $n+1$ *form introduced by (B.12), and* $\Omega_L = \mathrm{d}\theta_L$ *is the multi-symplectic form on* $J^{2k-1}(Y)$. *Furthermore,*

$$\mathcal{D}_{EL}\mathcal{L}(\phi) \cdot V = j^{2k-1}(\phi \circ \phi_X^{-1})^*[j^{2k-1}(V) \dashv \Omega_L] \quad \text{in } U_X$$

and the variational principle (B.14) gives, on the interior of the domain, the Euler-Lagrange equation

$$\begin{aligned}
& \frac{\partial L(j^k(\phi \circ \phi_X^{-1}))}{\partial y^A} - \frac{\partial}{\partial x^{\mu_1}}\left(\frac{\partial L(j^k(\phi \circ \phi_X^{-1}))}{\partial y^A_{\mu_1}}\right) + \cdots + \\
& \cdots + (-1)^k \frac{\partial^k}{\partial x^{\mu_1} \cdots \partial x^{\mu_k}}\left(\frac{\partial L(j^k(\phi \circ \phi_X^{-1}))}{\partial y^A_{\mu_1\cdots\partial\mu_k}}\right) = 0.
\end{aligned} \quad \text{(B.17)}$$

Proof J. Marsden et al [76] and S. Kouranbaeva et al [66] have proved the result for $k = 1$ and $k = 2$ respectively, and their ideas work for the general $k \geq 1$ case although the calculation is quite involved. Here we omit the details but only mention the main ideas as follows:

For any $\phi^\lambda \in \mathcal{C}$ such that $\phi^0 = \phi$, and $\frac{\mathrm{d}\phi^\lambda}{\mathrm{d}\lambda}|_{\lambda=0} = V$, there is $\eta_Y^\lambda : Y \mapsto Y$ such that $\phi^\lambda = \eta_Y^\lambda \circ \phi$. We can see that $\eta_Y^\lambda \circ (\phi \circ \phi_X^{-1}) \circ (\eta_X^\lambda)^{-1}$ is a section of $Y \mapsto X$, and it maps $U_X^\lambda \equiv \eta_X^\lambda \circ \phi_X(U)$ to $\phi^\lambda(U)$ and $\pi_{XY} \circ \phi^\lambda \circ \phi_X^{-1} = \eta_X^\lambda$, $V_X = \frac{\mathrm{d}\eta_X^\lambda}{\mathrm{d}\lambda}|_{\lambda=0}$.

So we have

$$
\begin{aligned}
\mathrm{d}\mathcal{S}_\phi \cdot V &= \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \int_{U_X^\lambda} \mathcal{L}(j^k(\phi^\lambda \circ (\phi_X^\lambda)^{-1})) \\
&= \int_{U_X} \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \mathcal{L}(j^k(\phi^\lambda \circ (\phi_X^\lambda)^{-1})) + \\
&\quad \int_{U_X} \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} (\eta_X^\lambda)^* \mathcal{L}(j^k(\phi \circ (\phi_X)^{-1})) \\
&\triangleq I + II.
\end{aligned}
\tag{B.18}
$$

Then use the Cartan's formula to evaluate the second part

$$
\begin{aligned}
II &= \int_{U_X} \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} (\eta_X^\lambda)^* \mathcal{L}(j^k(\phi \circ (\phi_X)^{-1})) \\
&= \int_{U_X} \pounds_{V_X} \mathcal{L}(j^k(\phi \circ (\phi_X)^{-1})) \\
&= \int_{U_X} \pounds_{V_X}(L\omega) = \int_{U_X} \mathrm{d}\ \imath_{V_X}(L\omega) + \ \imath_{V_X}\mathrm{d}(L\omega) \\
&= \int_{\partial U_X} L\ \imath_{V_X}\omega = \int_{\partial U_X} L V^\mu \omega_\mu.
\end{aligned}
$$

The first part

$$
\begin{aligned}
I &= \int_{U_X} \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \mathcal{L}(j^k(\phi^\lambda \circ (\phi_X^\lambda)^{-1})) \\
&= \int_{U_X} \left[\frac{\partial L}{\partial y^A}(j^k(\phi \circ \phi_X^{-1}))(V^v)^A + \frac{\partial L}{\partial y^A_{\mu_1}}(j^k(\phi \circ \phi_X^{-1}))(V^v)^A_{\mu_1} + \right. \\
&\qquad \left. \cdots + \frac{\partial L}{\partial y^A_{\mu_1 \cdots \mu_k}}(j^k(\phi \circ \phi_X^{-1}))(V^v)^A_{\mu_1 \cdots \mu_k} \right] \omega,
\end{aligned}
$$

here we have used that $\frac{\mathrm{d}}{\mathrm{d}\lambda}\phi^\lambda\big|_{\lambda=0} = V$, $\phi_X^\lambda = \pi_{XY}\circ\phi^\lambda = \pi_{XY}\circ(\eta_Y^\lambda\circ\phi) = \eta_X^\lambda\circ\phi_X$, so $(\phi_X^\lambda)^{-1} = \phi_X^{-1}\circ(\eta_X^\lambda)^{-1}$ and so

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0}\phi^\lambda\circ(\phi_X^\lambda)^{-1} &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0}\phi^\lambda\circ(\phi_X)^{-1} + T\phi\cdot\frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0}(\phi_X^\lambda)^{-1} \\
&= V\circ\phi_X^{-1} + T(\phi\circ\phi_X^{-1})(-V_X) \\
&= V\circ\phi_X^{-1} - T(\phi\circ\phi_X^{-1})V_X = V^v.
\end{aligned}$$

Then use integration by parts to directly check that (B.18) gives the interior integral and boundary integral parts in the Theorem.

We can summarise the analogy between the classical mechanics and field theory in the following table:

Analogy between classical mechanics and field theory

	Classical Mechanics	k-th Field Theory
Configuration Space	Q	the fiber bundle $Y\mapsto X$
Phase Space	TQ	k-th jet bundle $J^k(Y)$
1-form / $(n+1)$-form	$\theta_L = (\mathbb{F}L)^*\theta$ for an $L: TQ\mapsto\mathbb{R}$	$\theta_L = (\mathbb{F}\mathcal{L})^*\theta$ for a covariant $\mathcal{L}: J^k(Y)\mapsto\Lambda^{n+1}(X)$.
2-form /$(n+2)$-form	symplecity: $\omega_L = \mathrm{d}\theta_L$	multi-symplecity: $\Omega_L = \mathrm{d}\theta_L$

We call the critical points of the action functional $\mathcal{S}$ the solutions of the Euler-Lagrange equation, and introduce the notations:

Definition B.8

$$\mathcal{P}\equiv\{\phi\in\mathcal{C}|\ j^{2k-1}(\phi\circ\phi_X^{-1})^*\ \imath_W\Omega_L = 0 \quad \textit{for all vector fields } W \textit{ on } J^{2k-1}(Y)\} \tag{B.19}$$

denotes the solution space of the Euler-Lagrange equation.

For any $\phi\in\mathcal{P}$, let

$$\begin{aligned}\mathcal{F}\equiv\big\{V\in T_\phi\mathcal{C}|\ & j^{2k-1}(\phi\circ\phi_X^{-1})^*\pounds_{j^{2k-1}(V)}[W\lrcorner\,\Omega_L] = 0 \\ & \textit{for all vector fields } W \quad \textit{on } J^{2k-1}(Y)\big\}\end{aligned} \tag{B.20}$$

be the set of the solutions of the first variation equations of the Euler-Lagrange equations, where $\pounds_V$ stands for the Lie derivative along V.

Then we have the following result whose proof is similar to that in [66].

Theorem B.9 (Multi-symplectic form formula) *If $\phi \in \mathcal{P}$, then for all V and W in $\mathcal{F}$, we have*

$$\int_{\partial U_X} j^{2k-1}(\phi \circ \phi_X^{-1})^*[j^{2k-1}(V) \dashv j^{2k-1}(W) \dashv \Omega_L] = 0. \tag{B.21}$$

B.2 Formulation for mCH

We adopt the Lagrangian approach to mCH which leads naturally to the multi-symplectic formulation. For the modified Camassa-Holm equation with H^2 metric:

$$m_t + 2u_y m + u m_y = 0 \quad \text{for } y \in S, \quad m = \Lambda_4^4 u. \tag{B.22}$$

This equation is the generalised Euler equation for the reduced Lagrangian:

$$l(u) \equiv \frac{1}{2}\int (u^2 + u_y^2 + u_{yy}^2)\mathrm{d}y. \tag{B.23}$$

On the other hand, we can express it in the Lagrangian variable $\eta(t,x)$ which is the solution of

$$\begin{cases} \dfrac{\partial}{\partial t}\eta(t,x) &= & u(t,\eta(t,x)), \\ \eta(0,x) &= & x. \end{cases} \tag{B.24}$$

Now let $X = S \times \mathbb{R}, Y = X \times \mathbb{R} = S \times \mathbb{R} \times \mathbb{R}$ with coordinates $(x^1, x^0) = (x,t)$ for X and $(x^1, x^0, y) = (x,t,y)$ for Y. A smooth section ϕ of $Y \mapsto X$ is a mapping $(x,t) \mapsto (x,t,\eta(t,x))$, where $\eta(t,x)$ is the solution of (B.24). The material or Lagrangian velocity $\frac{\partial}{\partial t}\eta(t,x)$ is an element of $T_{\phi(t,x)}Y = T_{(t,x,y)}Y$, where $y = \eta(t,x)$.

With (B.24) and $u_y = \eta_{tx}/\eta_x, u_{yy} = \frac{\eta_{txx}\eta_x - \eta_{tx}\eta_{xx}}{\eta_x^3}$ the Lagrangian representation for the action may be expressed as

$$\mathcal{S}(\phi) = \frac{1}{2}\int_{S\times[0,T]} (\eta_x \eta_t^2 + \eta_{tx}^2/\eta_x + (\eta_x \eta_{txx} - \eta_{tx}\eta_{xx})^2/\eta_x^5)\mathrm{d}x\mathrm{d}t. \tag{B.25}$$

The third jet bundle $J^3(Y)$ is a 17 dimensional manifold and three-holonomic sections of $J^3(Y) \mapsto X$ have local coordinates

$$j^3(\phi) = (t, x, \eta, \eta_t, \eta_x, \eta_{tt}, \eta_{tx}, \eta_{xt}, \eta_{xx}, \eta_{ttt}, \eta_{ttx}, \eta_{txx}, \eta_{txt}, \eta_{xtt}, \eta_{xtx}, \eta_{xxt}, \eta_{xxx}), \tag{B.26}$$

where for smooth section, $\eta_{tx} = \eta_{xt}, \eta_{txx} = \eta_{xtx} = \eta_{xxt}, \cdots$. The Lagrangian density along the third jet of a section ϕ can be given by

$$\mathcal{L}(j^3(\phi)) = \frac{1}{2}\left(\eta_x\eta_t^2 + \frac{\eta_{tx}^2}{\eta_x} + \frac{(\eta_x\eta_{txx} - \eta_{tx}\eta_{xx})^2}{\eta_x^5}\right)\mathrm{d}x \wedge \mathrm{d}t. \tag{B.27}$$

Because the Lagrangian density depends only on $\eta_t, \eta_x, \eta_{tx}, \eta_{xx}, \eta_{txx}$, so the Cartan form $\theta_L = (\mathbb{F}L)^*\theta$ can be found by direct calculation or from Theorem B.7 (from now on we change the notion η to y):

$$\theta_L = T_1\mathrm{d}x \wedge \mathrm{d}t + T_2 + T_3 + T_4 \tag{B.28}$$

where

$$\begin{aligned} T_1 \;=\;& L - \frac{\partial L}{\partial y_t}y_t - \frac{\partial L}{\partial y_x}y_x + D_x\left(\frac{\partial L}{\partial y_{tx}}\right)y_t + D_x\left(\frac{\partial L}{\partial y_{xx}}\right)y_x \\ & - \frac{\partial L}{\partial y_{tx}}y_{tx} - \frac{\partial L}{\partial y_{xx}}y_{xx} - D_{xx}\left(\frac{\partial L}{\partial y_{txx}}\right)y_t + D_x\left(\frac{\partial L}{\partial y_{txx}}\right)y_{tx} - \frac{\partial L}{\partial y_{txx}}y_{txx}. \end{aligned}$$

$$T_2 = p^1\mathrm{d}y \wedge \mathrm{d}t - p^0\mathrm{d}y \wedge \mathrm{d}x,$$
$$T_3 = p^{01}\mathrm{d}y_t \wedge \mathrm{d}t + p^{11}\mathrm{d}y_x \wedge \mathrm{d}t,$$
$$T_4 = p^{011}\mathrm{d}y_{tx} \wedge \mathrm{d}t,$$

and

$$p^0 = p^t = \frac{\partial L}{\partial y_t} - D_x\left(\frac{\partial L}{\partial y_{tx}}\right) + D_{xx}\left(\frac{\partial L}{\partial y_{txx}}\right),$$
$$p^1 = p^x = \frac{\partial L}{\partial y_x} - D_x\left(\frac{\partial L}{\partial y_{xx}}\right),$$
$$p^{01} = p^{tx} = \frac{\partial L}{\partial y_{tx}} - D_x\left(\frac{\partial L}{\partial y_{txx}}\right),$$
$$p^{11} = p^{xx} = \frac{\partial L}{\partial y_{xx}},$$
$$p^{011} = p^{txx} = \frac{\partial L}{\partial y_{txx}},$$
$$p^{10} = p^{00} = p^{000} = p^{110} = p^{010} = p^{001} = p^{100} = p^{111} = p^{101} = 0.$$

We can think p^0, p^1 as the temporal and spatial conjugate momentum of the field component y, and p^{01}, p^{11} as the temporal and spatial conjugate momentum of the field component y_x...

The corresponding 3-form $\Omega_L = \mathrm{d}\theta_L$ is

$$\Omega_L = \mathrm{d}T_1 \wedge \mathrm{d}x \wedge \mathrm{d}t + \mathrm{d}p^1 \wedge \mathrm{d}y \wedge \mathrm{d}t - \mathrm{d}p^0 \wedge \mathrm{d}y \wedge \mathrm{d}x + \mathrm{d}p^{01} \wedge \mathrm{d}y_t \wedge \mathrm{d}t + \mathrm{d}p^{11} \wedge \mathrm{d}y_x \wedge \mathrm{d}t + \mathrm{d}p^{011} \wedge \mathrm{d}y_{tx} \wedge \mathrm{d}t \tag{B.29}$$

and if we introduce a transform

$$\mathbb{F}L: \ (y, y_x, y_t, y_{xx}, y_{xt}, y_{tx}, y_{tt}, y_{xxx}, \cdots) \mapsto (y, y_x, y_t, y_{tx}, p^1, p^0, p^{01}, p^{11}, p^{011})$$

from the space of the vertical sections of $J^5(Y) \mapsto X$ into the phase space $\mathcal{M} = \mathbb{R}^9$ modeled over $X = \mathbb{R}^2$, then we have the following

Proposition B.10 *For any π_{XY} vertical vectors V, W in $\mathcal{F}$ defined in the formula (B.20), we have*

$$\frac{\partial}{\partial x}\omega^1(T\mathbb{F}L \cdot j^5(V), T\mathbb{F}L \cdot j^5(W)) + \frac{\partial}{\partial t}\omega^0(T\mathbb{F}L \cdot j^5(V), T\mathbb{F}L \cdot j^5(W)) = 0. \tag{B.30}$$

Moreover, the mCH equation (B.22) is equivalent to the Hamiltonian system of equations on the multi-symplectic structure

$$B_1 Z_x + B_0 Z_t = \nabla H, \tag{B.31}$$

with the Hamiltonian defined by

$$H = L - p^x y_x - p^t y_t - p^{01} y_{tx} - p^{11} y_{xx} - p^{011} y_{txx}, \tag{B.32}$$

where ω^i, B_i are defined as follows: $\omega^i(u,v) = v^T B_i u$ for any $u, v \in \mathbb{R}^9$, $i = 0, 1$ and

$$B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$
B_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

Proof The idea is the same as that in the [66]. The only difference consists of the calculations: for any π_{XY} vertical vectors V, W in $\mathcal{F}$ defined in the formula (B.20), we have

$$
\begin{aligned}
j^5(W) \lrcorner j^5(V) \lrcorner \Omega_L \;=\; & \left(V^{p^1}W^{y} - W^{p^1}V^{y} + V^{p^{01}}W^{y_t} - W^{p^{01}}V^{y_t}\right)\mathrm{d}t \\
& + \left(V^{p^{011}}W^{y_{tx}} - W^{p^{011}}V^{y_{tx}} + V^{p^{11}}W^{y_x} - W^{p^{11}}V^{y_x}\right)\mathrm{d}t \\
& - \left(V^{p^0}W^{y} - W^{p^0}V^{y}\right)\mathrm{d}x.
\end{aligned} \tag{B.33}
$$

Then the fact that the multi-symplectic formula in Theorem (B.9) holds for arbitrary ∂U_x and Stokes theorem imply

$$
\frac{\partial}{\partial x}\omega^1(Z_t, Z_x) + \frac{\partial}{\partial t}\omega^0(Z_t, Z_x) = 0 \tag{B.34}
$$

where ω^i and B_i are defined in the Proposition.

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