Estimating and Evaluating the Archimedean-Copula-Based Models in Financial Risk Management

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To my family

Abstract

Copula is used to model multivariate data, as it accounts for the dependence structure and provides a flexible representation of the multivariate distribution. Recently a large number of Archimedean copulas have been proposed to deal with various dependence aspects in financial risk management, which invokes several new questions in some important yet under-researched areas. These questions, therefore, need further investigation.

This dissertation comprises three essays and probes into three untouched questions all involving the Archimedean-copula-based models. The first essay studies whether the Archimedean-copula-based portfolio value-at-risk (PVaR) model outperforms the Gaussian-copula-based PVaR model in out-of-sample forecasting. My empirical findings in this essay show that the Archimedean-copula-based PVaR model, especially the Clayton copula-based model, has better forecasting performance than the Gaussian-copula-based PVaR model in most cases in both the in-sample and out-of-sample periods. In addition, the data snooping problem (i.e., model risk) associated with the copula-based PVaR model is also explored.

The second essay examines the question of how to evaluate the non-Gaussian multivariate density forecasts. In this essay, I propose a test procedure, by using the likelihood ratio test based on the Kullback-Leibler information criterion, to evaluate the Archimedean-copula-based multivariate density forecasts, and apply the procedure to foreign exchange markets. The test procedure is not only conducive to fully ranking competing sophisticated models with the non-Gaussian-distributed multivariate densities, but also allows for model misspecification in both marginal and copula functions under the null and the alternative hypothesis.

The third essay focuses on this question: Will the PVaR estimation be improved if the Archimedean copula model takes into account conditional asymmetric tail dependence and time-varying investors' heterogeneous beliefs? I use the conditional skewed-t distribution (as the marginal function) to represent time-varying investors' heterogeneous beliefs, and employ three two-parameter Archimedean copulas to investigate dynamic asymmetric tail dependence between two of three Asian developed futures markets. My results provide strong evidence that such conditional copula models can improve the PVaR estimation and so a greater amount of diversification benefits can be reaped at a higher confidence level.

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Finally, as always, my greatest debts of gratitude are due to my family – my father Genlin Xu, my mother Mei Wang, and my wife Meili Sun– for their love and unfailing support. I would therefore like to dedicate this dissertation to them.

However, none of the above individuals hold responsibility for the opinions expressed, or for any remaining errors and omissions, which are my own.

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Chapter 1 Introduction

This dissertation investigates the estimation and evaluation of the Archimedean-copula-based models for portfolio value-at-risk, multivariate density forecasts and conditional asymmetric tail dependence.

1.1 Motivation

Traditional mean-variance-based portfolio theory assumes that risk factors are normally (elliptically) distributed. Under this assumption, the standard deviation of a portfolio is used as a proxy of portfolio risk, and linear correlation as a measure of dependence between returns of risky assets constituting a portfolio. However, traditional portfolio theory often does not comport with abundant empirical evidence that portfolio returns are asymmetrically distributed with heavy tails.[1] Linear correlation has also been found to be questionable in explaining financial market contagion.[2] The stylised fact that the multivariate distribution of financial time series is non-normal has catalysed methodological changes in financial modeling, such as the use of copula models.

A copula is a multivariate distribution function that connects marginal distributions of variables. It contains all the information about the dependence structure

[1] A number of studies have addressed some of the non-normality characteristics of portfolio returns. For instance, Arditti and Levy (1975) and Kraus and Litzenberger (1976) model portfolio returns by incorporating the effect of skewness on valuation. Lai (1991) and Chunhachinda et al (1997) analyse the problem of portfolio selection taking into account the skewness of returns. Harvey and Siddique (2000) present an asset pricing model that includes conditional skewness. In addition, Fang and Lai (1997) find that investors are compensated for by higher expected returns for bearing systematic cokurtosis risk.

[2] For example, Erb et al (1994), King et al (1994), De Santis and Gerard (1997), Longin and Solnik (1995, 2001), Ang and Bekaert (2002) and, Ang and Chen (2002) uncovered that correlations between international equity markets is higher during bear markets than during bull markets. These empirical results indicate the necessity to go beyond the linear approach to address the existence of contagion.

of the involved variables. In the last decade, copula models, especially the so-called Archimedean copulas, have become increasingly popular in financial applications, such as *measuring portfolio's market risk* (Cherubini and Luciano (2001), Glasserman *et al* (2002), Embrechts *et al* (2003), Ané and Kharoubi (2003), Malevergne and Sornette (2004, 2006), Dowd (2005a), Junker and May (2005), and Rosenberg and Schuermann (2006)), *pricing multivariate contingent claims* (Rosenberg (2003) and Bennett and Kennedy (2004)), and *modeling extreme market comovement* (Patton (2006a), Rodriguez (2007), and Bartram *et al* (2007)). This is because Archimedean copulas are a powerful tool to model an array of real-world multivariate distributions that exhibit various asymmetries of dependence.

Because of the increasing popularity of Archimedean copulas, this dissertation contributes by further investigating their applications in some important yet under-researched areas. Specifically, I probe into three questions as follows.

First, does the Archimedean copula-based portfolio VaR (PVaR) model outperform the Gaussian copula-based PVaR model in out-of-sample forecasting? One of the main tasks of financial risk management is to evaluate and improve the performance of risk measurement models. PVaR is a widely-used risk measure for a portfolio. It is defined as the loss in the portfolio's market value over a given time horizon that is exceeded with a probability 1- ξ where ξ is a given confidence level. Early studies on PVaR were based mainly on the extreme value method.[3] However, one drawback of the extreme value method is that it utilizes the information in the tail region, while missing the information contained in such non-normality properties as co-skewness and nonlinear dependence which the entire portfolio return distribution contains. The Archimedean

[3] See, for example, Jansen *et al* (2000), Consigli (2002), and Frey and McNeil (2002).

copula-based PVaR approach can overcome this problem. Thus, not only the use of, but also the study on, this approach, has received growing interest. Many studies have found that in-sample fitting of the Archimedean copula-based PVaR model outperforms that of the Gaussian copula-based PVaR model (See, for example, Ané and Kharoubi (2003) and Junker and May (2005)). However, that a model best fits historical data does not necessarily mean that it will also provide the best VaR forecasts. Since risk managers are concerned with the future losses of portfolio returns, out-of-sample forecasting evaluation of the two PVaR models should be of more relevance than in-sample one.

Second, how can we evaluate the non-Gaussian multivariate density forecast? Multivariate density forecast is important for market risk measurement, especially when the future portfolio return distributions are non-Gaussian. It is well known that most financial time series have persistent dependence in conditional variance and higher-order conditional moments. A financial risk manager thus needs to go beyond conditional mean and variance and to get a complete picture of financial return dynamics via density forecast. A density forecast is a prediction of the future probability distribution of a random variable. Recently, the need to consider the full predictive density has made *density forecast evaluation* an increasingly important tool in assessing the performance of sophisticated risk measurement models. Density forecast evaluation is to see whether density forecasts are correct. Thus, how well a constructed forecasting model, especially its density component such as the (portfolio) VaR measure and the (multivariate) option pricing), performs can be judged by evaluating whether its density forecasts are correct. In the past few years, several methods for the univariate density forecast evaluation have been proposed, including Diebold et al's (1998) graphical method, Berkowitz's (2001) likelihood ratio (LR) test method, Hong and Li's (2005) and Egorov et al's (2006) nonparametric method, and Corradi and Swanson's (2005, 2006a and 2006b) conditional Komogorov test method. For the multivariate density forecast evaluation, there are very few methods proposed (e.g., Diebold *et al* (1999) and Clements and Smith (2002)). However, these methods are all confined within the normality framework, and so are not of much use in assessing whether a non-Gaussian multivariate density-based model, such as the Archimedean copula-based model, is adequate. To fill this gap, I propose a test procedure which can fully take account of the non-Gaussian property for evaluating the multivariate density forecast.

Third, will the PVaR estimation be improved if the Archimedean copular model takes into account conditional asymmetric tail dependence and time-varying investors' heterogeneous beliefs as represented by the conditional skewed-t distribution? When hedging dependent risk, portfolio managers should care not only about movements of individual markets, but also about comovements among them. Following market integration and financial liberalization in the last two decades, asymmetric dependence among international equity markets has become increasingly significant: Negative shocks originating in one country are more likely to spread to other countries than positive shocks. This asymmetry suggests that downside dependent risk deserves particular attention of portfolio managers. Moreover, a number of studies reveal that investors possess "loss aversion", which refers to the tendency for investors strongly to prefer avoiding losses than acquiring gains.[4] This phenomenon, termed as investors' heterogeneity, can be modeled by either preferences (utility functions) or beliefs

[4] See Kahneman and Tversky (1973) and Benartzi and Thaler (1995).

(probability distributions).[5] Levy (2007) shows that investors' heterogeneity plays a key role in determining the possibility of financial disturbances. In their differences-of-opinion theory, Hong and Stein (2003) demonstrate that there are large negative price changes during market declines, and the negative-skewed returns are closely associated with investors' heterogeneity.[6] Meanwhile, they further point out that large negative price changes are a contagious marketwide phenomenon, that is, a highly correlated drop in the prices of an entire class of stocks. Hence, for international diversification and optimal assets allocation, it is necessary to consider a PVaR model that fully takes account of the conditional lower tail dependent risk along with the factor of time-varying investors' heterogeneity.

Throughout this dissertation, I also consider "model risk" which is the risk arising from the use of an inadequate/mis-specified model. For copula model estimation, there exist three methods: the full parametric method (e.g. Patton (2006a), Jondeau and Rockinger (2006), and Rodriguez (2007)), the semiparametric method (e.g. Ané and Kharoubi (2003), Chen and Fan (2006a, 2006b), and Chen *et al* (2006)), and the nonparametric method (e.g. Fermanian and Scaillet (2003)). The semiparametric and nonparametric methods mainly rely on the kernel density estimate. Although the kernel

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[5] For instance, the asymmetric preference methods are employed in Krause and Litzenberger (1976) and Conine and Tamarkin (1981), and the dynamic heterogeneous beliefs method is used by Hueng and McDonald (2005).

[6] Hong and Stein's (2003) model assumes that difference of opinion (a private signal about a stock's terminal payoff) exist among two groups, bullish and bearish groups, of investors, and that these investors face short-sale constraints. Furthermore, each of the two groups only pays attention to their own signals. When the difference between opinions is large, it is more likely that bearish investors do not initially participate in the market, and their information is not fully incorporated into prices, because of their short-sale constraints. If the market receives positive news, bullish investors' information is still revealed in prices, while bearish investors' information remains hidden. On the other hand, if the market receives negative news and the previous bullish investors have changed their opinions and bailed out of the market, those previous bearish investors may become the "support buyers" and hence reveal more of their information. Thus, accumulated hidden information tends to come out when the market is falling. That is, given high investor heterogeneity, volatility is higher when returns are low. This explains why returns are negatively skewed.

density estimate may be used to overcome the distribution uncertainty problem, it nevertheless has the bandwidth selection problem (Wand and Jones (1995)). In addition, using the semiparametric and nonparametric methods to estimate the PVaR measure makes it hard to calculate the quantile of the multivariate distribution. In view of these, I employ the full parametric method in the estimation of all copula functions throughout this dissertation. The full parametric method is implemented via two-stage maximum likelihood estimation (MLE) proposed by Joe (1997, 2005).

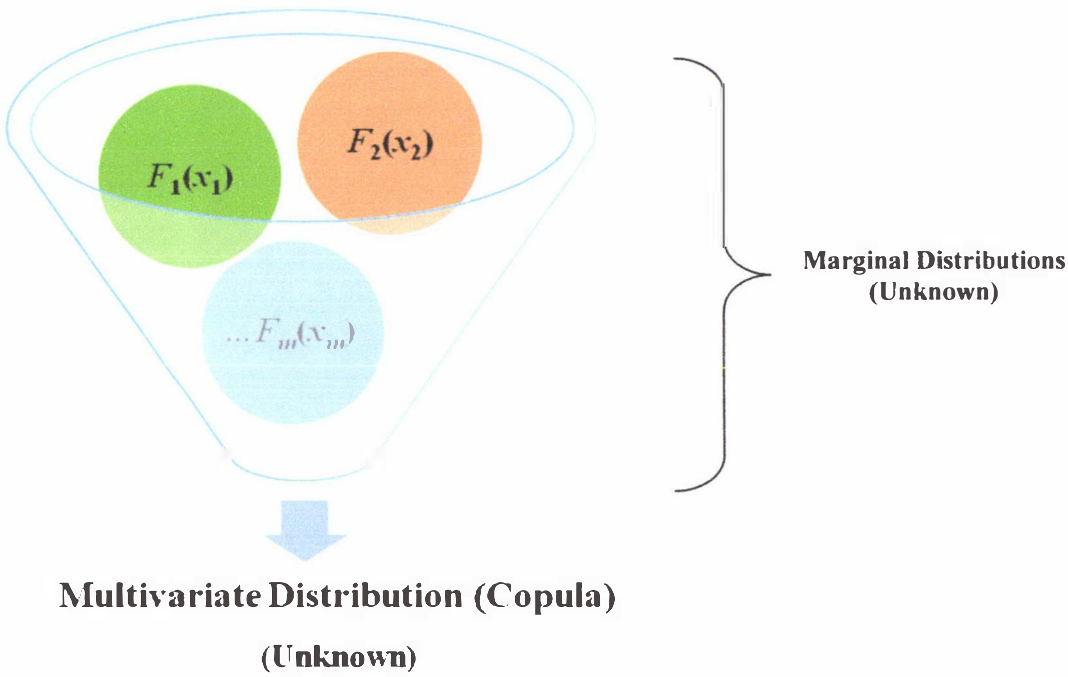

Figure 1.1

Note: This figure plots the relationship between marginal distributions and the multivariate distribution (copula). $F_1(x_1)$, $F_2(x_2)$, and $F_m(x_m)$ indicate the marginal distributions.

It is well known that the copula is a flexible tool to model dependent data, as it enables one to separate the dependence properties of the data from their marginal properties and to construct multivariate models with marginal distributions of an arbitrary form. Therefore, as shown in Figure 1.1, the difficulty with the full parametric method is that both the true marginal and the true multivariate distributions are unknown. As far as modeling the marginal distribution is concerned, Rosenberg and Schuermann (2006) and Rodriguez (2007) use the standard normal distribution, Patton (2006a) and Bartram *et al* (2007) use the Student's *t* distribution, and Patton (2004) suggests using Hansen's (1994) skewed-*t* distribution. Jondeau and Rockinger (2006) consider all the three distributions, and employ a goodness-of-fit test to select and retain the "best" one while dropping out the other two for the marginals, when moving on to the estimation of the copula parameters. In other words, a copula model is deemed to be correctly specified once the best marginals have been determined. However, Jondeau and Rockinger's (2006) approach is, in come cases, not very useful for making model-selection decisions. This is because the selected marginal based on the goodness-of-fit tests is not necessarily the "best" one in the sense that it *will* lead to a correctly-specified copula. Also, the test statistics for different distributions can be all significant, or all insignificant, or mixed. Based on these considerations, the criterion I used is the forecasting performance of a candidate copula model, bearing in mind that a copula model that has the "best" marginals does not necessarily perform better in forecasting than another that has the "second or third best" marginals. Thus, my approach to judging on whether a copula is correctly specified is different from those of the above cited studies. I retain all the three aforementioned marginal restrictions at the second stage of the two-stage MLE of all candidate copulas. The advantage of my approach is twofold. First, my approach allows the two-stage MLE to be performed in the presence of "misspecification" of both the marginal and the copula models. Then all copula-based model performances will finally be examined by the bootstrap-based test (to be introduced later on). Second, with my approach one can investigate how sensitive each candidate copula model's forecasting performance is to different marginals chosen as the possible data generating processes.

However, estimating different models by repeatedly using the same dataset can cause the so-called "data snooping" bias. As White (2000) noted, "[w]hen such data reuse occurs, there is always the possibility that any satisfactory results obtained may simply be due to chance rather than to any merit inherent in the method yielding the results." Data snooping is a serious issue when a set of competing models are considered. Suppose we have several PVaR estimates yielded by different copula-based models with the same data set, and we choose one estimate whose underlying model is however misspecified. When a model is misspecified, its PVaR estimates will be either too low or too high, in relation to the PVaR estimates of the correctly-specified model. A PVaR estimated too low could result in a financial institution having insufficient capital to cover future losses and hence having possibly financial distress and even company failure. The 1998 failure of Long-Term Capital Management (LTCM) is a good example[7]. On the other hand, an

[7] LTCM was a hedge fund founded in 1994. Its board of directors once included Myron Scholes and Robert C. Merton, the two academic heavyweights in finance who shared the 1997 Nobel Memorial Prize in Economics. The core strategy of LTCM was "relative-value" or "convergence-arbitrage" trades, trying to take advantage of small differences in prices among closely related securities. Initially, enormous success with annualized returns of over 40% in its first year made the new venture eminently profitable. Capital grew from $1 billion to more than $7 billion by 1997. This led LTCM to undertake trading strategies outsides their expertise by switching to extremely large positions in areas such as merger arbitrage and index options. The downfall of the fund started in May and June 1998 when net returns fell 6.42% and 10.14% respectively, reducing its capital by $461 million. Meanwhile, LTCM failed to diversify the systemic risk, and relied on recent history to estimate risk while ignoring non-normal risk factors caused by rare but extreme events. The unexpected Russian government bond default on 17 August 1998 caused widespread shock waves across markets as investors panicked. The collapse of LTCM was sudden. By 1 August 1998, the firm's equity stood at $4.1 billion, by 1 September 1998 it dropped to $2.3 billion, and over the next 21 days the fund's equity sharply slipped to $600 million: 85% of the equity was wiped out in 54 days. For more details, see Jorion (2000), Dunbar (2000), and MacKenzie (2006).

excessive high PVaR estimate could result in a financial institution tying up too much of its capital in an unprofitable fashion.

Data snooping is not easy to detect. White (2000) proposes a "reality check" test with a stationary bootstrap method to correct for the data snooping bias. The application of the reality check test in finance can be found in Sullivan et al (1999, 2001, 2003), González-Rivera et al (2004), and Bao et al (2007) among others. However, White's reality check test is conservative when a poor model is included in the set of competing models (see Hansen (2005) and Hansen and Lunde (2005)). As a modified version of the reality check test, Hansen's (2005) superior predictive ability (SPA) test is more powerful and robust to the inclusion of poor and irrelevant alternatives. In view of this, I have implemented the SPA test in each research project.

1.2 Outline of the Dissertation

The main body of this dissertation includes three essays, each related to one of the three questions detailed above.

Chapter 2 briefly introduces the concepts of copula models including Archimedean copulas and elliptical copulas. Also introduced in this chapter are the concepts of rank correlation and tail dependence.

Chapter 3 contains Essay One and focuses on the first research question. The title of the essay is "Evaluating the out-of-sample forecasting performances of the Archimedean-copula-based portfolio VaR models". In this essay, I fit the PVaR models to three international equity indexes (FTSE 100, Nikkei 225, and S&P 500), the models being based on three Archimedean copulas (the Clayton, the Gumbel, and the BB1 copulas) and one elliptical copula (Gaussian copula). The four copulas represent four possibilities regarding to tail dependence: Lower tail dependence only, upper tail dependence only, asymmetry between lower and upper tail dependence, and no lower nor upper tail dependence. For comparisons, I choose three different marginal distributions (the Gaussian, the Student's *t*, and Hansen's (1994) skewed-*t* distribution) respectively as the possible underlying data generating processes for each copula. This is to investigate the sensitiveness of the copula-based PVaR measure to the choice of the marginals. For model evaluation, I split whole sample into two subsets, the former being for in-sample model estimation while the latter for out-of-sample forecasting. To take account of the data snooping bias, I conduct Hansen's (2005) SPA test based on the quantile loss function with stationary bootstrap to evaluate the in- and the out-of-sample performances of the copula-based PVaR models. I show that, compared with the traditional PVaR models based on the full Gaussian copula (the Gaussian copula with the Gaussian marginals), the Archimedean-copula-based PVaR models, especially the Clayton copula-based model, have better forecasting performances at both the lower and higher confidence levels in most cases.

Chapter 4 contains Essay Two and deals with the second research question. The title of the essay is "A test for evaluating the Archimedean-copula-based multivariate density forecasts in foreign exchange markets". In this essay, I extend Vuong's (1989) likelihood ratio test based on the Kullback-Leibler information criterion (KLIC) loss function to evaluate the density forecasting performances of the Archimedean copula-based multivariate density models for foreign exchange markets. For multiple comparisons, I consider five Archimedean copulas (the Clayton, the Frank, the Gumbel, the BB1, and the BB7 copula) as well as two elliptical copulas (the Gaussian, and the *t* copula) for the test. Again, I use three different marginal distributions (the standard normal, the Student's *t*, and Hansen's skewed-*t* distribution) respectively as the possible underlying data generating processes for each copula. In order to avoid model misspecification (data snooping), I implement Hansen's (2005) SPA test with stationary bootstrap to evaluate models' density forecasting performances. Based on the bootstrapped *p*-values, I find strong evidence that the multivariate densities modeled by the Archimedean copulas outperform those modeled by the Gaussian copulas in both in- and out-of-sample periods. Previous studies of the multivariate density forecast evaluation assumed that the multivariate densities are normal, and so their results cannot be applied to non-normal multivariate densities which are often the case in the real world. My study embraces both normal and non-normal multivariate densities, and proposes for the first time a test procedure applicable in such a general circumstance for density forecast evaluation. Moreover, my proposed test method is straightforward to use by financial risk managers.

Chapter 5 contains Essay Three and explores the third research question. The title of the essay is "Estimating dynamic asymmetric tail dependences with time-varying investors' heterogeneous beliefs in Asian developed futures markets". In this essay, I employ three two-parameter Archimedean copulas (the BB1, the BB4, and the BB7 copula) to investigate the dynamic asymmetric tail dependence between two of three index futures returns of Hang Seng, Nikkei 225, and MSCI SIN during the post-crisis period. I first model the marginals by the conditional skewed-t distribution and find that higher moments of each filtered index futures return series are time dependent. This indicates that investors' risk preferences are time-varying. I next extend a class of two-parameter copulas by incorporating time-varying tail dependences to capture their dynamic asymmetries. The estimation results provide strong evidence in support of time-varying asymmetric dependence across all Asian developed futures markets. Moreover, I implement the multivariate density forecast evaluation test developed in Chapter 4 to judge on the fits of the specified models. I find that the conditional BB7 copula for the Hang Seng-MSCI SIN pair and the conditional BB1 copula for the Nikkei 225-MSCI SIN pair outperform the simple symmetric Gaussian copula. Based on the model evaluation results, I select the best-fit models for estimating PVaRs and diversification benefits at both the lower and higher confidence levels. Since the selected models are found to provide a higher degree of diversification benefits at a high confidence level, I take this to imply that, by taking into account conditional asymmetric tail dependence and time-varying investors' risk preferences, Archimedean copulas can improve the PVaR estimation relative to the Gaussian copula. These results should send a useful message to financial risk managers.

Finally, Chapter 6 summaries the key findings of the three essays and gives the outlook for future study.

Chapter 2 Copula, rank correlation, and tail dependence

This chapter briefly introduces copula models including elliptical and Archimedean copulas, as well as rank correlation and tail dependence. These concepts will be used in the research reported later on.

2. 1 Definition of Copula

A copula function $C(u_1, u_2, \ldots, u_m)$ is a multivariate distribution function for an m-dimensional vector $(Y_1, Y_2, \ldots, Y_m)$ with support in $[0, 1]^m$:

$$C\ (u_1, u_2, \ldots, u_m) = \Pr(Y_1 \leq y_1, Y_2 \leq y_2, \ldots, Y_m \leq y_m) \qquad (2.1)$$

The formal definition of copula can be found in Nelsen (1999, p39):

Definition 1 An m-dimensional copula is a function C with the following properties:

1. Domain $C = \mathbf{I}^m = [0, 1]^m$ where $\mathbf{I}$ is unit interval [0, 1].
2. C is grounded and m-increasing.
3. C has margins C_m which satisfy $C_m(u) = u$ if all coordinates of u are 1 except u.

An important theorem due to Sklar (1959) offers an easy way to analyze the dependence structure of a multivariate distribution:

Theorem 1 (Sklar's theorem) Let F be an m-dimensional distribution function with continuous marginals $F_1, F_2,...F_m$. Then F has a unique copula for all y:

$$F(y_1, y_2,..., y_m) = C[F_1(y_1), F_2(y_2),..., F_m(y_m)] \tag{2.2}$$

If each F_i (i = 1, 2,..., m) and C are differentiable, then the joint density $f(y_1, y_2,..., y_m)$ is:

$$f(y_1, y_2,..., y_m) = f_1(y_1) \times f_2(y_2) \times ... \times f_m(y_m) \times c[F_1(y_1), F_2(y_2),..., F_m(y_m)] \tag{2.3}$$

where $f_i(y_i)$ is the density corresponding to F_i (y_i), and

$$\begin{aligned} c[F_1(y_1), F_2(y_2),..., F_m(y_m)] &= \frac{\partial^m C[F_1(y_1), F_2(y_2),..., F_m(y_m)]}{\partial F_1(y_1) \partial F_2(y_2)...\partial F_m(y_m)} \\ &= \frac{f(y_1, y_2,..., y_m)}{f_1(y_1) \times f_2(y_2) \times ... \times f_m(y_m)} \end{aligned} \tag{2.4}$$

is the density of a copula.

Throughout this dissertation, I study two groups of copulas. One is the elliptical copula, and the other is the Archimedean copula.

2.2 Elliptical Copulas

The elliptical copula contains a class of elliptical distributions where random variables are linearly correlated via Pearson's (linear) correlation coefficient ρ_P. In a bivariate case, for example,

$$\rho_P = \mathrm{Cov}(Y_1, Y_2) / \sqrt{\mathrm{Var}(Y_1) \cdot \mathrm{Var}(Y_2)} \tag{2.5}$$

The formal definition of the elliptical distribution can be found in Bradley and Taqqu (2003) and Embrechts *et al* (2003). The elliptical distributions have tractable properties and can be used for modeling symmetric multivariate extremes. An early example of applying the elliptical distribution to portfolio theory is Owen and Rabinovitch (1983).

In empirical studies, two most popular elliptical copulas are the Gaussian and the *t* copula. Their formulas and densities are presented in panel B of Table 2.1. For graphical illustration, Figure 2.1 shows the contours, and the three-dimensional plots, of the bivariate densities based on the Gaussian and the *t* copula respectively. One can see from the figure that both copulas have elliptically-shaped contours which are radially symmetric. However, the ellipses of the *t* copula become "sharper" at the southwest and northeast ends than the ellipses of the Gaussian copula, which has implications for tail dependence (to be introduced later on). It is well known that a full Gaussian copula, i.e., a Gaussian copula constructed by Gaussian (normal) marginals, is another way to describe Markowitz's mean-variance portfolio theory.

Table 2.1: Bivariate Copulas

Model	$C(u,\upsilon)$	$c(u,\upsilon)$	Parameter	Tail Dependence	
				τ^{L}	τ^{U}
		Panel A. Archimedean Copula			
Clayton	$(u^{-\alpha}+\upsilon^{-\alpha}-1)^{-1/\alpha}$	$(1+\alpha)(u\upsilon)^{-(\alpha+1)}(u^{-\alpha}+\upsilon^{-\alpha}-1)^{-(2+1/\alpha)}$	$\alpha\in[-1,\infty)$ $\alpha\neq 0$	$2^{-1/\alpha}$	—
Frank	$-\frac{1}{\alpha}\ln\left[1+\frac{(e^{-\alpha u}-1)(e^{-\alpha\upsilon}-1)}{e^{-\alpha}-1}\right]$	$\frac{\alpha(1-e^{-\alpha})e^{-\alpha(u+\upsilon)}}{[(1-e^{-\alpha})-(1-e^{-\alpha u})(1-e^{-\alpha\upsilon})]^2}$	$\alpha\in(-\infty,\infty)$ $\alpha\neq 0$	—	—
Gumbel	$\exp\left\{-\left[(-\ln u)^{\alpha}+(-\ln\upsilon)^{\alpha}\right]^{1/\alpha}\right\}$	$\left\{\frac{C_{Gumbel}\cdot(\log u\cdot\log\upsilon)^{\alpha-1}}{u\upsilon[(-\log u)^{\alpha}+(-\log\upsilon)^{\alpha}]^{2-1/\alpha}}\right\}$ $\times\{[(-\log u)^{\alpha}+(-\log\upsilon)^{\alpha}]^{1/\alpha}+\alpha-1\}$	$\alpha\in[1,\infty)$	—	$2\text{-}2^{1/\alpha}$
BB1	$\{1+[(u^{-\alpha}-1)^{\beta}+(\upsilon^{-\alpha}-1)^{\beta}]^{1/\beta}\}^{-1/\alpha}$	*	$\alpha\in(0,\infty)$ $\beta\in[1,\infty)$	$2^{-1/(\alpha\beta)}$	$2\text{-}2^{1/\beta}$
BB4	$\{u^{-\alpha}+\upsilon^{-\alpha}-1-[(u^{-\alpha}-1)^{-\beta}+(\upsilon^{-\alpha}-1)^{-\beta}]^{-1/\beta}\}^{-(1+1/\alpha)}$	*	$\alpha\in[0,\infty)$ $\beta\in(0,\infty)$	$(2-2^{-1/\beta})^{-1/\alpha}$	$2^{-1/\beta}$
BB7	$1-\{1-[(1-\bar{u}^{\alpha})^{-\beta}+(1-\bar{\upsilon}^{\alpha})^{-\beta}-1]^{-1/\beta}\}^{1/\alpha}$ $\bar{u}=1-u,\ \bar{\upsilon}=1-\upsilon$	*	$\alpha\in[1,\infty)$ $\beta\in(0,\infty)$	$2^{-1/\beta}$	$2\text{-}2^{1/\alpha}$

Panel B. Elliptical Copula

Gaussian	$\Phi_{\rho}(\Phi^{-1}(u),\Phi^{-1}(\upsilon))$	$\frac{1}{\sqrt{\det(\Sigma)}}\times \exp\left\{-\frac{(\Phi^{-1}(u),\Phi^{-1}(\upsilon))'(\Sigma^{-1}-I)(\Phi^{-1}(u),\Phi^{-1}(\upsilon))}{2}\right\}$ where $\Sigma=\begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}$	$\rho\in[-1,1]$	—	—
t	$T_{\rho,\delta}(T_{\delta}^{-1}(u),T_{\delta}^{-1}(\upsilon))$	$\frac{\Gamma[(\delta+2)/2]\cdot\Gamma(\delta/2)}{\sqrt{\det(\Sigma)}[\Gamma((\delta+1)/2)]^{2}}\left(1+\frac{\mathbf{x}'\Sigma^{-1}\mathbf{x}}{\delta}\right)^{-(\delta+2)/2}\times\left\{\left(1+\frac{x_1^2}{\delta}\right)\cdot\left(1+\frac{x_2^2}{\delta}\right)\right\}^{(\delta+1)/2}$ where $\Sigma=\begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}, \mathbf{x}=(x_1,x_2),$ $x_1=T_{\delta}^{-1}(u), x_2=T_{\delta}^{-1}(\upsilon)$	$\rho\in[-1,1]$ $\delta\in(2,\infty)$	†	†

Note: * The density functions of the BB1 and BB7 copulas are too long to present in this table.

† Lower and upper tail dependence of the t copula are calculated as $\tau^{L}=\tau^{U}=2\times T_{\delta+1}\left(-\sqrt{\frac{(\delta+1)(1-\rho)}{(1+\rho)}}\right)$.

For the Gaussian copula, Φ denotes the standard normal distribution. For the t copula, $T_{\delta+1}$ denotes the Student's t distribution with $\delta+1$ degrees of freedom.

For more details see Joe (1997) and Nelsen (1999).

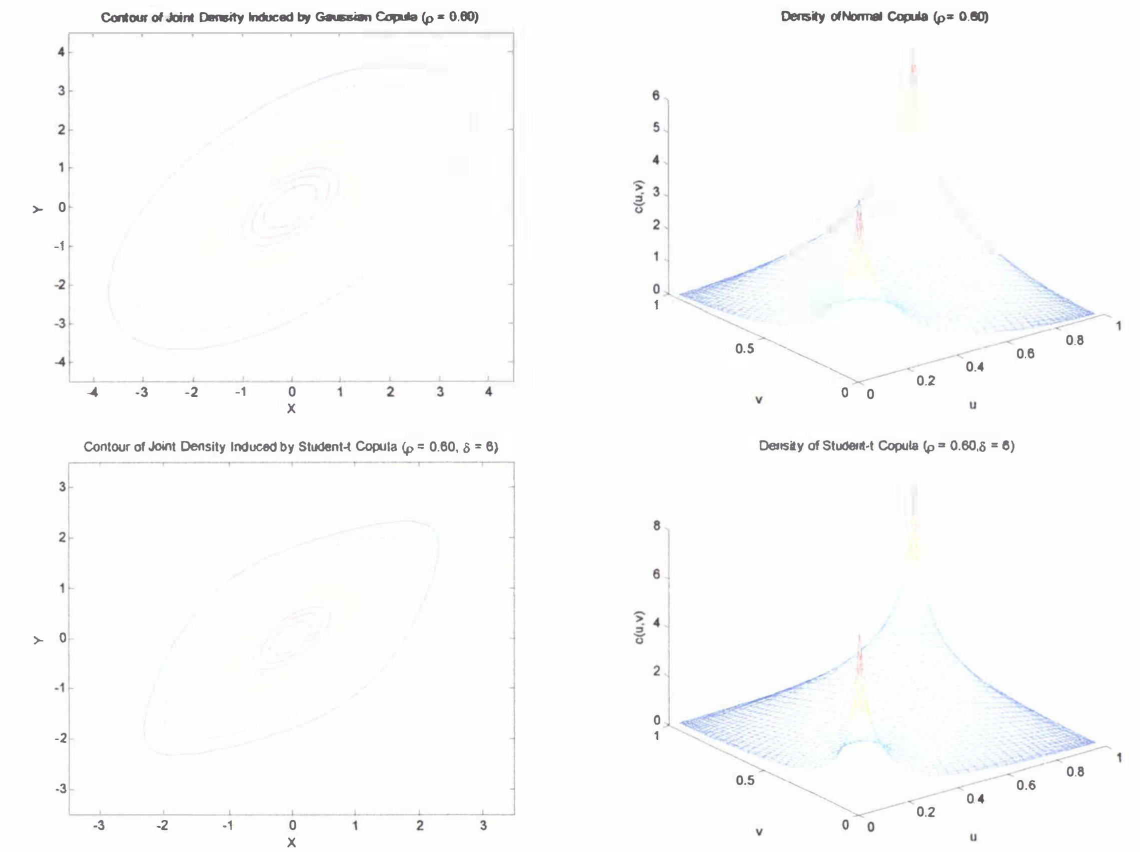

Figure 2.1

Note: The left column shows the contours of the bivariate densities based on the Gaussian and t copulas respectively. The right column shows the three-dimensional plots of densities of these two copulas respectively.

2.3 Rank Correlations

Within Markowitz's framework, multivariate returns are assumed to be normally distributed. For a multivariate normal distribution, linear correlation is an adequate means of describing the dependence between variables involved. However, when the joint distribution of two or more returns is not normal (especially not elliptical), linear correlation may become a very misleading measure of dependence. Embrechts *et al* (2000, 2002) demonstrate that Pearson's ρ_P is a deficient measure for financial risk management due to the non-normality of financial data. Poon *et al* (2004) also find that Pearson's ρ_P is a poor measure for asset pricing as far as tail dependence between the multivariate returns is concerned. To overcome the shortcoming of linear correlation, Embrechts *et al* (2002) propose the use of one of the following two rank correlations, Spearman's ρ_S or Kendall's τ_K, in measuring correlation between non-elliptically distributed variables.

Let $F_1(y_1) = \int f(y_1, y_2)dy_2 = u$ and $F_2(y_2) = \int f(y_1, y_2)dy_1 = \upsilon$ be the two marginal distributions. The two rank correlations are given by:

$$\begin{aligned} \rho_S &= 12\int_0^1\int_0^1 C(u,\upsilon)dud\upsilon - 3 \\ \tau_K &= 4\int_0^1\int_0^1 C(u,\upsilon)dC(u,\upsilon) - 1 \end{aligned} \qquad (2.6)$$

Figure 2.2 displays linear correlation and the relationship between each of the two rank correlations and linear correlation. The differences between linear and rank correlations stem from the differences between the elliptical and non-elliptical multivariate distributions. A class of copulas, known as Archimedaen copulas, can be used to characterize some of the non-elliptical bivariate distributions, to which I now turn.

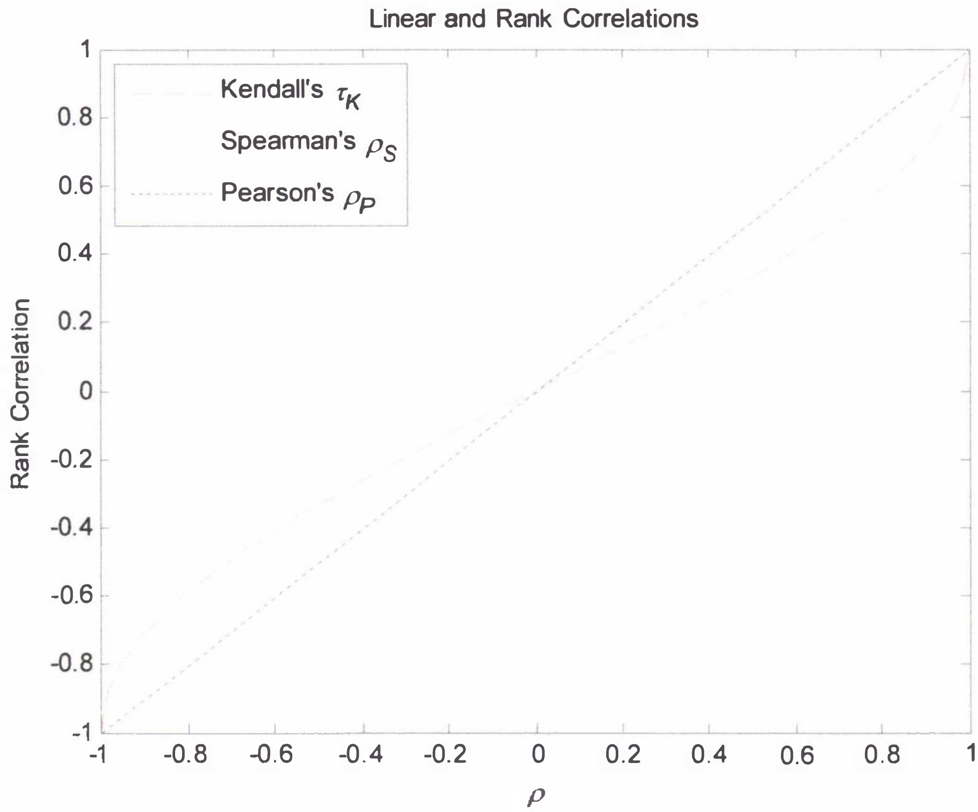

Figure 2.2

Note: The dotted line represents linear correlation, i.e., Pearson's ρ_P. The solid line shows the relationship between Spearman's ρ_S and Pearson's ρ_P. The dashed line shows the relationship between Kendall's τ_K and Pearson's ρ_P.

2.4 Archimedean Copulas

One of the advantages of Archimedean copulas is that they enable one to model the dependence structure beyond linear correlation, and so provide a high degree of flexibility in analyzing market comovements. The Archimedean copula has an important component known as generator φ:

Definition 2 (Nelsen (1999), p90) Let φ be a continuous, strictly decreasing function from [0, 1] to [0, ∞] such that $\varphi(1) = 0$. The pseudo-inverse of φ is the function $\varphi^{[-1]}$: $[0, \infty] \dashrightarrow [0, 1]$ given by

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t), & 0 \le t \le \varphi(0), \\ 0, & \varphi(0) \le t \le \infty. \end{cases} \qquad (2.7)$$

Note that $\varphi^{[-1]}$ is continuous and decreasing on [0, ∞], and strictly decreasing on [0, $\varphi(0)$]. Moreover, $\varphi^{[-1]}(\varphi(u)) = u$ on [0, 1], and

$$\varphi[\varphi^{[-1]}(t)] = \begin{cases} t, & 0 \le t \le \varphi(0), \\ \varphi(0), & \varphi(0) \le t \le \infty. \end{cases} \qquad (2.8)$$

Finally, if $\varphi(0) = \infty$, then $\varphi^{[-1]} = \varphi^{-1}$.

The following theorem defines the Archimedean copula:

Theorem 2 (Nelsen (1999), p90) Let φ be a continuous, strictly decreasing function from [0, 1] to [0, ∞] such that $\varphi(1) = 0$, and let $\varphi^{[-1]}$ be the pseudo-inverse of φ. Let C be the function from $[0, 1]^2$ to [0, 1] given by

$$C(u,\upsilon) = \varphi^{-1}[\varphi(u) + \varphi(\upsilon)] \quad \forall u,\upsilon \in [0, 1] \qquad (2.9)$$

Then C is an Archimedean copula if and only if φ is convex.

In this study, I employ three popular one-parameter Archimedean copulas: the Clayton, the Frank and the Gumbel copula; and three two-parameter Archimedean copulas: the BB1, the BB4 and the BB7 copula. The closed forms of these copula functions are displayed in panel A of Table 2.1. The contours of the bivariate densities based on these six Archimedean copulas and the three-dimensional plots of densities of these six copulas are exhibited in Figures 2.3 and 2.4. The differences across the six copulas can be clearly seen from the figures, and will be discussed after the concept of tail dependence is introduced in the next section.

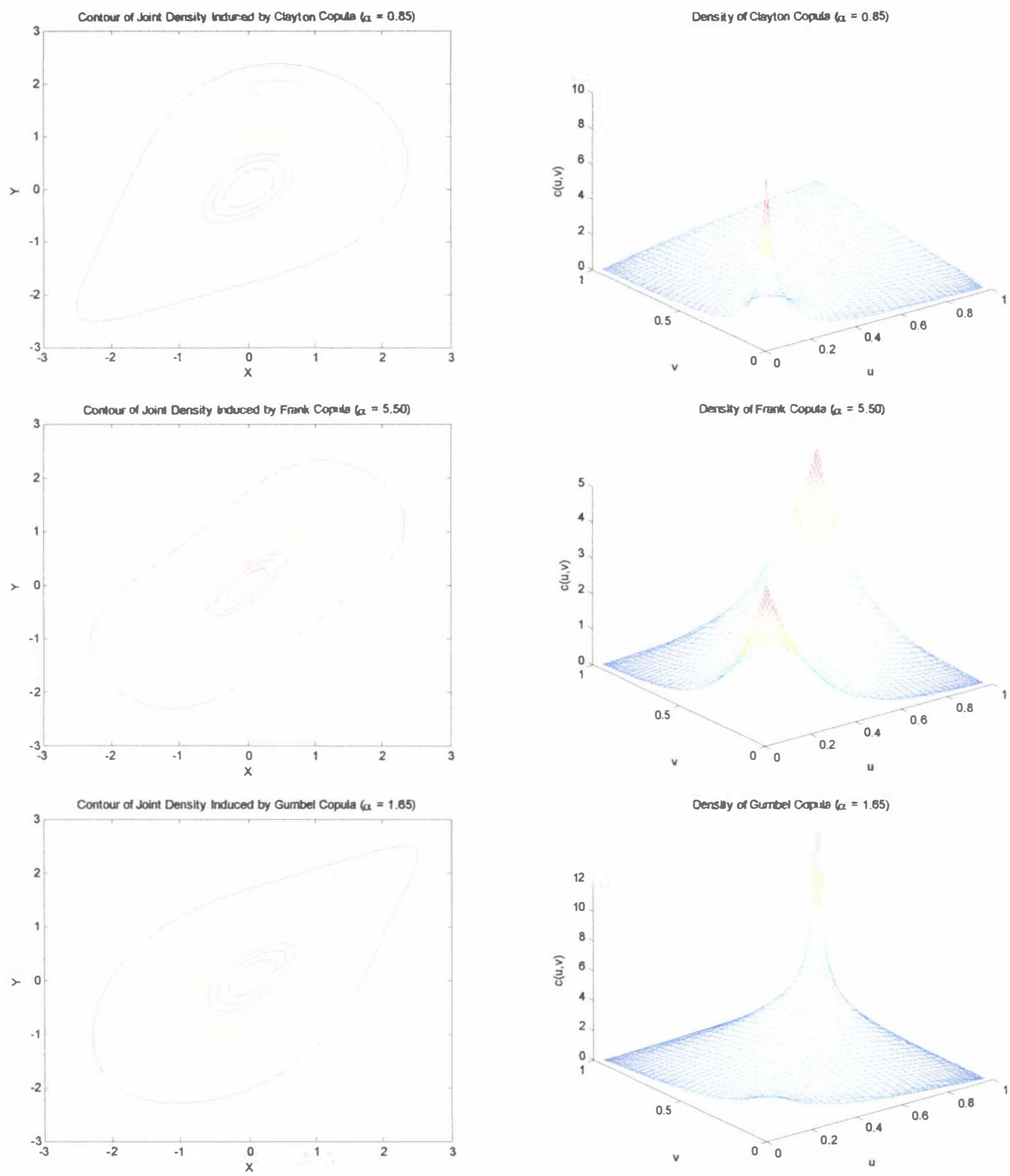

Figure 2.3

Note: The left column shows the contours of the bivariate densities based on the Clayton, Frank and Gumbel copulas respectively. The right column shows the three-dimensional plots of densities of these three copulas respectively.

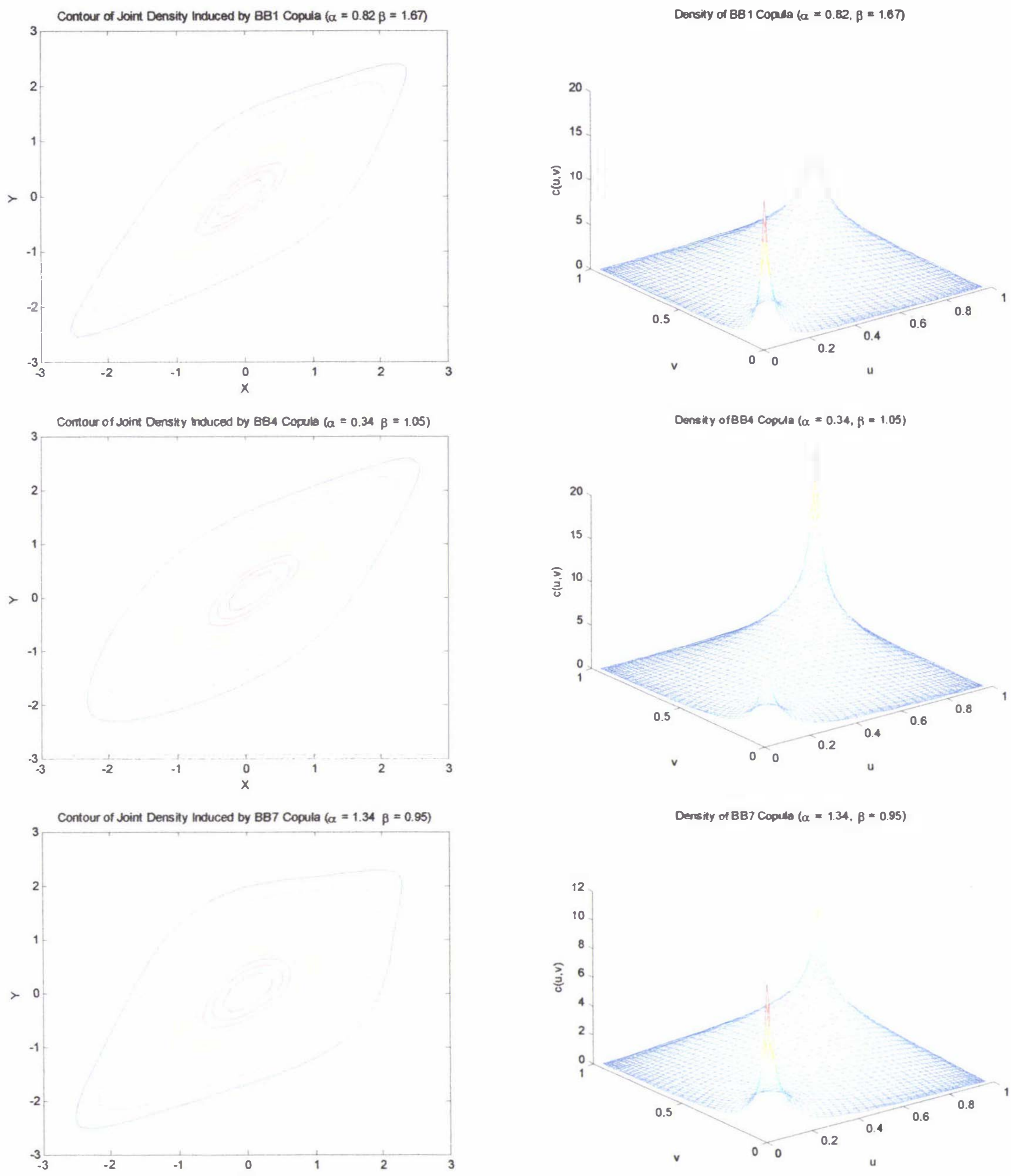

Figure 2.4

Note: The left column shows the contours of the bivariate densities based on the BB1, BB4 and BB7 copulas respectively. The right column shows the three-dimensional plots of densities of these three copulas respectively.

2.5 Tail Dependence

Studying tail dependence is an important application of the Archimedean copula. In finance, tail dependence can be understood as the probability of an extremely large negative (positive) return on one asset given that the other asset has yielded an extremely large negative (positive) return. Suppose the return of asset 1 is represented by Y_1 with a distribution $F_1(\cdot)$, and the return of asset 2 by Y_2 with a distribution $F_2(\cdot)$. Upper tail dependence is then defined as:

$$\tau^U = \lim_{u \to 1} \Pr(Y_1 > F_1^{-1}(u) \mid Y_2 > F_2^{-1}(u))$$

$$= \frac{1 - \Pr\{Y_1 \le F_1^{-1}(u)\} - \Pr\{Y_2 \le F_2^{-1}(u)\} + \Pr\{Y_1 \le F_1^{-1}(u), Y_2 \le F_2^{-1}(u)\}}{1 - \Pr\{Y_2 \le F_2^{-1}(u)\}} \qquad (2.10)$$

Upper tail dependence can also be expressed using the copula function as follows:

$$\tau^U = \lim_{u \to 1} \frac{1 - 2u + C(u,u)}{1 - u} \qquad (2.11)$$

Similarly, lower tail dependence is defined as:

$$\tau^L = \lim_{u \to 0} \Pr\{Y_1 \le F_1^{-1}(u) \mid Y_2 \le F_2^{-1}(u)\} = \lim_{u \to 0} \frac{\Pr\{Y_1 \le F_1^{-1}(u), Y_2 \le F_2^{-1}(u)\}}{\Pr\{Y_2 \le F_2^{-1}(u)\}}$$

$$= \lim_{u \to 0} \frac{C(u,u)}{u} \qquad (2.12)$$

Now recall from the left column of Figure 2.1 the different elliptical shapes of the Gaussian and t copulas. This difference actually reflects the fact that the Gaussian copula does not have upper and lower tail dependence, while the t copula has and tail dependence is symmetric between the upper and lower tails (For more details see Panel B of Table 2.1).

Regarding Archimedean copulas, the left column of Figure 2.3 shows that, except for the Frank copula, the one-parameter copulas can only delineate either lower or upper tail dependence, not both. However, correlation between financial markets may be asymmetric due to investors distinguishing between upside and downside risks. In this case, asymmetric tail dependence may result. The two-parameter copulas, as shown in the left column of Figure 2.4, can capture such asymmetry between upper and lower tail dependence.

Chapter 3 Evaluating the out-of-sample forecasting performances of the Archimedean-copula-based portfolio VaR models

3.1. Introduction and Literature Review

The main purpose of the research reported in this chapter is to investigate the relative performances of the bivariate Archimedean-copula- and Gaussian-copula-based models as outlined in Chapter 2 in out-of-sample forecasting for portfolio Value-at-Risk (hereafter PVaR).

VaR has become one of the most popular risk measures in quantifying downside market risk. For a given probability of, say, 1%, the VaR of a portfolio is defined as the 1% quantile of the portfolio's return distribution such that we can be 99% certain the portfolio's returns will not be less than the VaR over a given time horizon. PVaR has been regarded as an alternative tool to the traditional mean-variance method for optimal portfolio selection and asset allocation. Unlike the traditional mean-variance framework which focuses on the standard deviation (dispersion) around the mean of portfolio returns, the PVaR, by measuring extreme losses in the lower distributional tail,[8] concerns the mean, dispersion and higher order moments of the portfolio return distribution.

Recently, a number of empirical studies show that portfolio returns have fatter distributional tails than the normal distribution, especially during market downturns. See, for example, Chunhachinda et al (1997), Harvey and Siddique (2000), Longin and Solnik

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[8] See Duffie and Pan (1997) and Glasserman et al (2001).

(2001), Ang and Chen (2002), Prakash *et al* (2003), and Sun and Yan (2003). To allow for such tail fatness, several researchers suggest using the extreme value (EV) method to estimate the PVaR. See, for example, Jansen *et al* (2000), Consigli (2002), and Frey and McNeil (2002). Focusing on tail fatness and hence extreme values in the tail region, the EV method ignores the central part and possible skewness of a return distribution when estimating the quantile as a key component of PVaR. However, the central part and skewness, as well as possible nonlinear relations between assets in a portfolio, contain useful information. By modeling the entire marginal and joint distributions, the copula-based PVaR estimation method enables one to utilize the information omitted by the EV method in estimating the quantile, and hence has received increasing attention by researchers. Applications of this method can be found in Cherubini and Luciano (2001), Glasserman et al (2002), Embrechts et al (2003), Ané and Kharoubi (2003), Malevergne and Sornette (2004, 2006), Dowd (2005a), Junker and May (2005), and Rosenberg and Schuermann (2006).

As discussed in Chapter 2, a copula is a multivariate distribution function that connects marginal distributions of individual variables. Where marginal distribution functions are given but the multivariate distribution is unknown, copulas can be used in search for an appropriate form of the multivariate distribution. Therefore, the main advantage of the copula method is that it can flexibly model the unknown multivariate distribution through testing several candidate copula functions regardless of whatever marginal distributions are involved. This then gives rise to the need to compare the performances of alternative copulas, and a great deal of research effort has been devoted to this issue in the literature.

Studies such as Ané and Kharoubi (2003) and Junker and May (2005) have found that the in-sample fitting performance of the Archimedean copula-based PVaR model is better than the Gaussian copula-based PVaR model. This seems to be unsurprising, since the Archimedean copula is able to capture the non-normality dependence structure of assets in a portfolio, such as nonlinear association, co-skewness and co-kurtosis. However, out-of-sample forecasting performance should be more important and interesting than in-sample forecasting performance, since risk managers are more concerned with the *future* losses of portfolio returns. In view of these, I ask whether the Archimedean-copula-based PVaR model also outperforms the Gaussian-copula-based PVaR model in making out-of-sample forecasts.

My research focus on out-of-sample evaluation is motivated by the following considerations. First, just because a model fits historical data does not necessarily mean that it will also provide good forecasts into the future periods. Therefore, in-sample analysis should be accompanied with out-of-sample evaluation, especially when comparing competing models. Second, an accurate estimate of a risk measure is a key input for achieving risk diversification and effective capital allocation. If the PVaR is set too low, this would imply that a financial firm does not have sufficient capital to cover future losses, leading to financial distress and possibly company failure. On the other hand, if the PVaR is set too high, the firm would tie up too much of its capital which is unprofitable. So, as future losses are uncertain, a portfolio manager cares about the accuracy of risk estimation, which requires the use of an adequate model. Out-of-sample forecast performance is often deemed to be the good standard of model evaluation. Third, a risk manager may tend to search for a "best" model from a large set of candidate models using the same dataset. The involved model uncertainty problem will likely lead to the so-called "data-snooping" bias (see White (2000)). Out-of-sample evaluation is an effective way to redress the data-snooping bias.

I evaluate the out-of-sample forecast performances of the Archimedean copula-based PVaR models using Hansen's (2005) superior predictive ability (SPA) test. The SPA test is an extension of White's (2000) reality check test, and deals with the data snooping bias resulting from a given set of data being reused for statistical inference or model selection. White (2000) notes, "when such data reuse occurs, there is always the possibility that any satisfactory results obtained may simply be due to chance rather than to any merit inherent in the method yielding the results." To alleviate the data snooping bias, White (2000) develops a reality check test with the stationary bootstrap method for model evaluation. However, Hansen (2005) and Hansen and Lunde (2005) demonstrate that White's reality check test may be manipulated by including poor and irrelevant alternatives in the set of competing models. Compared with the reality check test, the SPA test is more powerful and less sensitive to the inclusion of poor and irrelevant alternatives.

I apply the SPA test in assessing four bivariate copula-based PVaR models' performances, using data on three international equity indexes (FTSE 100, Nikkei 225, and S&P 500) which have been frequently studied in the literature. The four copulas include the Clayton copula, the Gumbel copula, the BB1 copula and the Gaussian copula. The first three belong to the Archimedean copula family and the last to the elliptical copula family. The reason of choosing these four copulas is that they represent four possible market comovements. Figure 3.1 presents the scatter plots of simulated bivariate random variables based on these four copulas, with each variable being a standard normal variate. Figure 3.2 shows the corresponding efficient frontiers of trade-off between these four copula-based PVaRs and expected returns at the 99% confidence level.

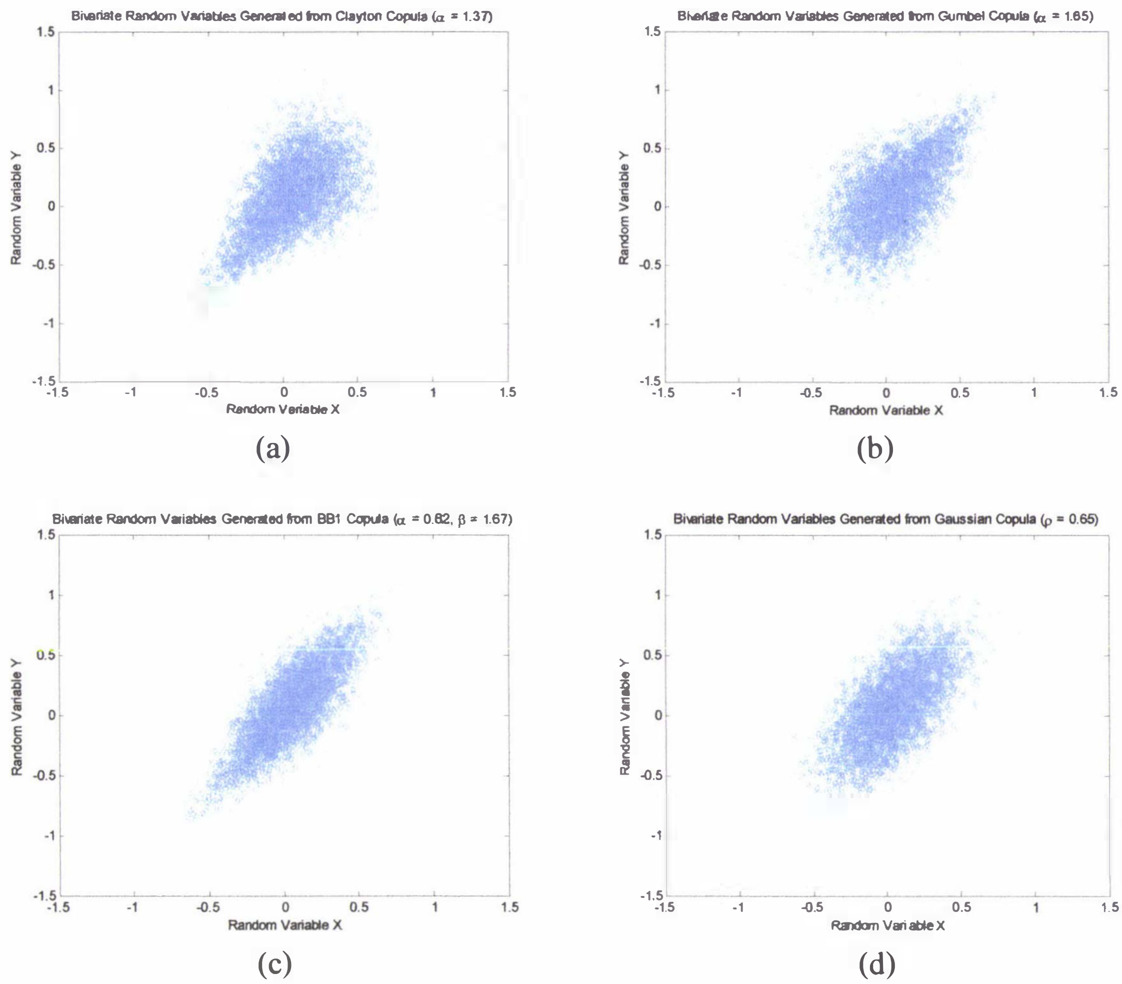


Figure 3.1

Note: This figure shows the scatter plots of 5000 simulated realizations from four different bivariate copula functions with standard normal marginals. (a) Clayton copula with α = 1.37. (b) Gumbel copula with α = 1.65. (c) BB1 copula with α = 0.82 and β = 1.67. (d) Gaussian copula with ρ = 0.65.

As shown in Figure 3.1, the Cla ton copula captures lower tail dependence or extreme downside comovements only of two markets. The Gumbel copula exhibits an opposite picture to the Clayton copula. The BB1 copula characterizes asymmetry between lower and upper tail dependence, reflecting that extreme downside market comovements are different from extreme upside ones. Unlike Archimedean copulas, the Gaussian copula is elliptically shaped without lower and upper tail dependence, implying no joint extreme events in two markets.

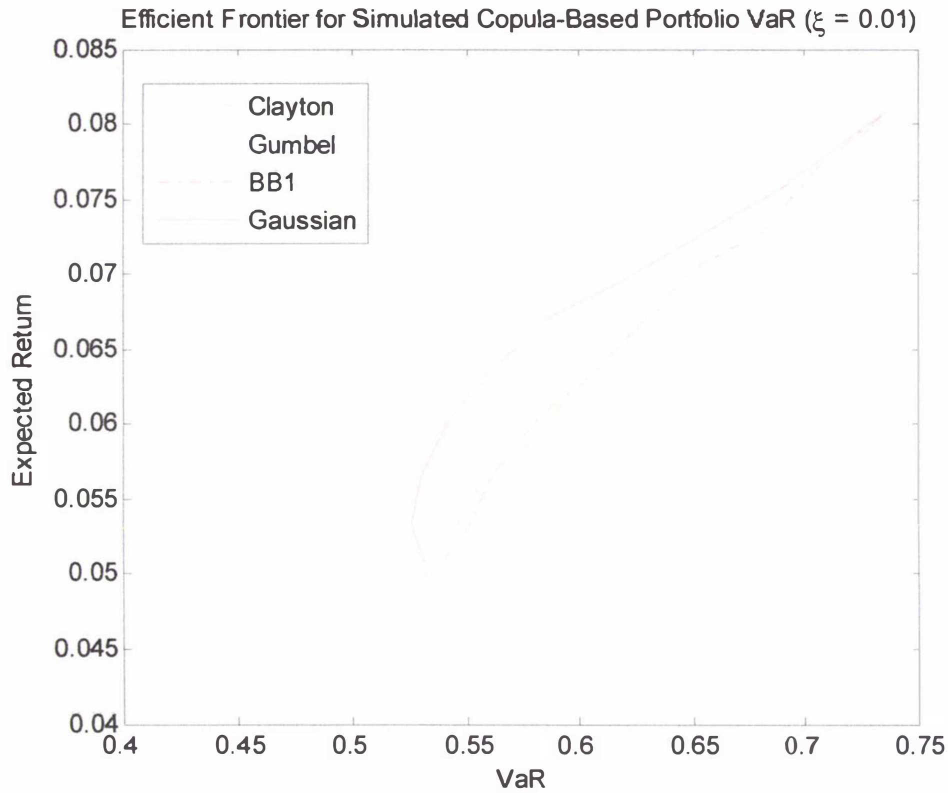


Figure 3.2

Note: In this figure, the copula-based PVaRs measured at the $1-\xi = 99\%$ confidence level based on 500,000 realizations where the parameter for each copula is the same as given in Figure 1. Each portfolio contains two risky assets whose returns follow the normal distribution $N(\mu_i, \sigma_i^2)$ $i = 1, 2$ where $\mu_1 = 0.04$, $\sigma_1 = 0.25$ and $\mu_2 = 0.08$, $\sigma_2 = 0.35$.

There are three methods to estimate the copula: the full parametric method (See, e.g., Patton (2006a), Jondeau and Rockinger (2006), and Rodriguez (2007)), the semiparametric method (See, e.g., Ané and Kharoubi (2003), Chen and Fan (2006a, 2006b), and Chen *et al* (2006)), and the nonparametric method (See, e.g., Fermanian and Scaillet (2003)). The first two methods involve a two-stage estimation procedure proposed by Joe (1997, 2005). Since I am interested in estimating the in-sample and the out-of-sample PVaR, the full parametric method is employed.

In previous studies which apply the full parametric method, researchers first determine marginals and then determine copulas with the marginals determined in the first stage, using various statistical tests. This approach to achieving correct specification of the copula model including its marginals may not necessarily ensure that the selected model will have superior forecasting ability. There is a possibility that the discarded/unconsidered distributions for marginals connected by an "inferior" copula (deemed to be so in the second stage) may actually have a better forecasting performance than the "superior" copula (deemed to be so in the second stage) which joins the retained distributions for marginals from the first stage. To allow for this possibility, I adopt a different approach to judging on correct model specification: The yardstick used is that a correctly specified model should have a superior out-of-sample forecasting ability, and statistical tests are selected to serve the purpose of assessing forecasting performance. This approach means that I try all possible combinations of different marginal distributions and different copulas in the estimation and testing process. Specifically, I choose the Gaussian, the Student-*t*, and Hansen's (1994) skewed-*t* distribution as the three possible forms of marginals, and join the maginals in each of these three forms by each of the four copulas discussed earlier. This way, I can reduce/eliminate the potential bias in the results of model misspecification testing due to insufficient possibilities allowed for.

The remainder of this chapter proceeds as follows. Section 3.2 presents the details of how to calculate the copula-based PVaR. Section 3.3 outlines the SPA test. The empirical results are provided in Section 3.4 and conclusions are given in Section 3.5.

3.2 Copula-Based Portfolio VaR

3.2.1 Portfolio VaR

Consider a portfolio containing two or more risky assets without short-sales constraints. Portfolio returns are computed as:

$$R_t = \mathbf{Y} \cdot \mathbf{W}' \quad (t = 1, 2, \ldots, n) \tag{3.1}$$

where $\mathbf{Y} = (y_{1,t}, y_{2,t}, \ldots, y_{m,t})$ denotes a vector of risky asset returns, $\mathbf{W} = (w_1, w_2, \ldots, w_m)$ denotes a vector of portfolio weights subject to $\sum_{i=1}^{m} w_i = 1$, and n is the sample size.

Let $\xi \in (0, 1)$ be a given probability level, ψ be a certain minimal expected portfolio return, and F_{R_t} be a cumulative distribution function (*c.d.f.*) of portfolio returns R_t. The PVaR is the $\xi \times 100\%$ percentile (e.g., 5% or 1%) of the distribution:

$$\mathrm{VaR}_{\xi}(R_t) = -\inf\{\psi \mid \mathbf{P}_{\{R_t \le \psi\}} \ge \xi\} = -F_{R_t}^{-1}(\xi) \tag{3.2}$$

where $\mathbf{P}$ denotes the probability of R_t being smaller than or equal to ψ.

Alternatively, according to Bradley and Taqqu (2003), with μ_{R_t} and σ_{R_t} being the portfolio expected return and the portfolio variance respectively, the PVaR can be written as:

$$\mathrm{VaR}_{\xi}(R_t) = \mu_{R_t} + \sigma_{R_t} q_{\xi} \tag{3.3}$$

where q_ξ is the ξ quantile of $\widetilde{R}_t = (R_t - \mu_{R_t}) / \sigma_{R_t}$ (the standardized R_t): $q_\xi = -F_{\widetilde{R}_t}^{-1}(\xi)$, and hence termed "standardized quantile" (Rosenberg and Schuermann (2006)). Note that the PVaR estimated using Equation (3.3) is static rather than dynamic, since here I focus on how different copula models chosen will affect the PVaR estimates given μ_{R_t} and σ_{R_t} which are assumed to be constant. Meanwhile, Equation (3.3) also indicates that the standardized quantile q_ξ is a key component of PVaR. So, the next section introduces how to estimate q_ξ using the copula method.

3.2.2 Estimating the standardized quantile by copula method

3.2.2.1 Copula functions

Recall from Chapter 2 that an m-dimensional copula function $C(u_1, u_2, ..., u_m)$ is defined as a distribution function on $[0,1]^m$ with standard uniform marginal distributions $u_i \equiv F_{i,t}(y_{i,t})$ ($i = 1, 2, \ldots, m$). Redefine F as an m-dimensional distribution function with continuous marginals $F_{1,t}, F_{2,t}, ..., F_{m,t}$. It has a unique copula for all $y_{i,t}$ (Sklar (1959)):

$F(y_{1,t}, y_{2,t}, ..., y_{m,t}) = C[F_{1,t}(y_{1,t}), F_{2,t}(y_{2,t}), ..., F_{m,t}(y_{m,t})]$.

In this study, I consider three bivariate Archimedean copulas (whose general definition was given in Chapter 2) and a bivariate Gaussian copula for estimating the standardized quantile. Let $u = F_{1,t}(y_{1,t})$ and $\upsilon = F_{2,t}(y_{2,t})$ be two marginal distributions. The four candidate copulas have their functional forms as follows:

The Clayton copula:

$$C_{Clayton}(u, \upsilon; \alpha) = (u^{-\alpha} + \upsilon^{-\alpha} - 1)^{-1/\alpha} \qquad \alpha \in [-1, \infty)\backslash\{0\} \qquad (3.4)$$

The Gumbel copula:

$$C_{Gumbel}(u,\upsilon;\alpha) = \exp\left\{-\left[(-\ln u)^{\alpha} + (-\ln \upsilon)^{\alpha}\right]^{1/\alpha}\right\} \qquad \alpha \in [1, \infty) \qquad (3.5)$$

The BB1 copula:

$$C_{BB1}(u,\upsilon;\alpha,\beta) = \{1+[(u^{-\alpha}-1)^{\beta}+(\upsilon^{-\alpha}-1)^{\beta}]^{1/\beta}\}^{-1/\alpha} \qquad \alpha \in (0, \infty),\ \beta \in [1, \infty) \qquad (3.6)$$

The Gaussian copula:

$$C_{Gaussian}(u,\upsilon;\rho) = \int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(\upsilon)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{\phi^2 - 2\rho\phi\varphi + \varphi^2}{2(1-\rho^2)}\right\} d\phi d\varphi \qquad (3.7)$$

where $\rho \in [-1, 1]$ is the linear correlation coefficient, $\Phi^{-1}(.)$ is the inverse of the univariate standard normal distribution function, $\phi = \Phi^{-1}(u)$, and $\varphi = \Phi^{-1}(\upsilon)$.

The parameters α, β, and ρ in Equations (3.4) to (3.7) are estimated via the two-stage full parametric maximum likelihood (ML) estimator. In the first stage, I estimate the marginal distribution model specified as the threshold GARCH (1, 1) (TGARCH) process with respectively the following three possible distributions of the standardized (volatility-filtered) returns: the Gaussian, the Student-t, and Hansen's (1994) skewed-t distribution. The TGARCH (1, 1) model is included in the following system:

$$\begin{aligned}
Y_t &= \mu + e_t \\
e_t &= \sqrt{h_t} \cdot z_t \\
e_t^{+} &= \max(e_t, 0) \qquad\qquad (3.8) \\
e_t^{-} &= \max(-e_t, 0) \\
h_t &= a_0 + b_0(e_{t-1}^{+})^2 + c_0(e_{t-1}^{-})^2 + d_0 h_{t-1}
\end{aligned}$$

where Y_t denotes the return series of each price index, μ is the unconditional mean, and $z_t \sim N(0, 1)$, or $z_t \sim$ Student-t (η), or $z_t \sim$ skewed-t (λ, η), denotes the standardized returns.

The GARCH model combined with the normal or the Student-t distribution has been widely applied in the literature, but the application of the GARCH model with Hansen's (1994) skewed-t distribution is relatively young (See, for example, Jondeau and Rockinger (2003), and Hueng and McDonald (2005)). So, it deserves some space to discuss the density of the skewed-t distribution defined as follows:

$$f_{skt}(z_t) = \begin{cases} B \cdot C\left(1 + \dfrac{1}{\eta - 2}\left(\dfrac{B \cdot z_t + A}{1 - \lambda}\right)^2\right)^{-(\eta+1)/2} & \text{if } z_t < -A/B \\ B \cdot C\left(1 + \dfrac{1}{\eta - 2}\left(\dfrac{B \cdot z_t + A}{1 + \lambda}\right)^2\right)^{-(\eta+1)/2} & \text{if } z_t \geq -A/B \end{cases} \tag{3.9}$$

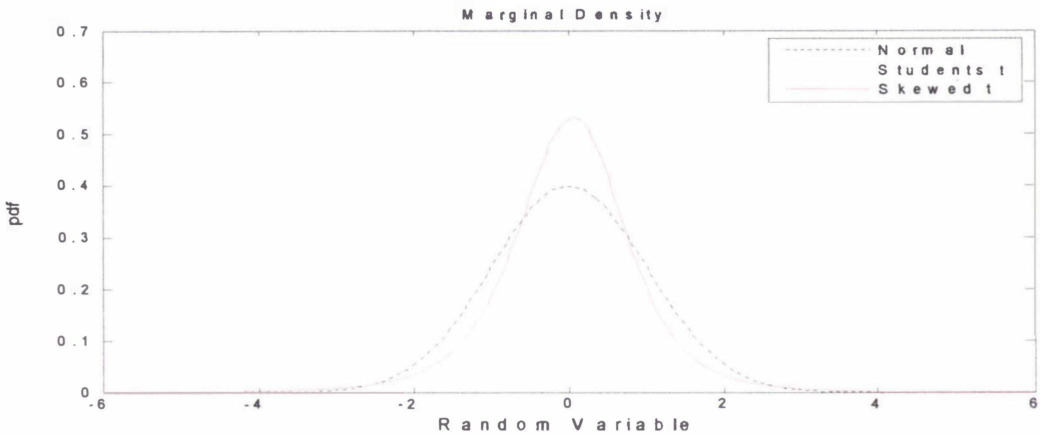
where $A \equiv 4\lambda C \dfrac{\eta - 2}{\eta - 1}$, $B^2 \equiv 1 + 3\lambda^2 - A^2$, $C \equiv \dfrac{\Gamma((\eta+1)/2)}{\sqrt{\pi(\eta-2)}\Gamma(\eta/2)}$. The parameter λ introduces asymmetry to the standard Student-t distribution: If $\lambda = 0$, then $A = 0$ and $B = 1$, giving rise to the standard Student-t distribution which is symmetric. If $-1 < \lambda < 0$ ($0 < \lambda < 1$), the shape of $f_{skt}(z_t)$ is negatively (positively) skewed. The parameter η (>2) measures the degree of freedom. A finite η means positive excess kurtosis, and the smaller η is, the fatter will be the distribution tails than the normal distribution. When η becomes infinite, the normal distribution results.

According to Jondeau and Rockinger (2003), the *c.d.f* of the skewed-t distribution is:

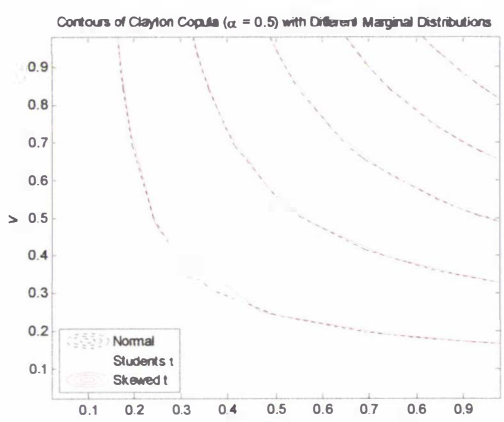
$$F_{skt}(z_t) = \begin{cases} (1-\lambda) \cdot F_{St,\eta} \cdot \left(\dfrac{B \cdot z_t + A}{1 - \lambda}\sqrt{\dfrac{\eta}{\eta - 2}}\right) & \text{if } z_t < -A/B \\ (1-\lambda) \cdot F_{St,\eta} \cdot \left(\dfrac{B \cdot z_t + A}{1 + \lambda}\sqrt{\dfrac{\eta}{\eta - 2}}\right) - \lambda & \text{if } z_t \geq -A/B \end{cases} \tag{3.10}$$

where $F_{St,\eta}$ is the *c.d.f* of the standard Student-t distribution with η degrees of freedom.

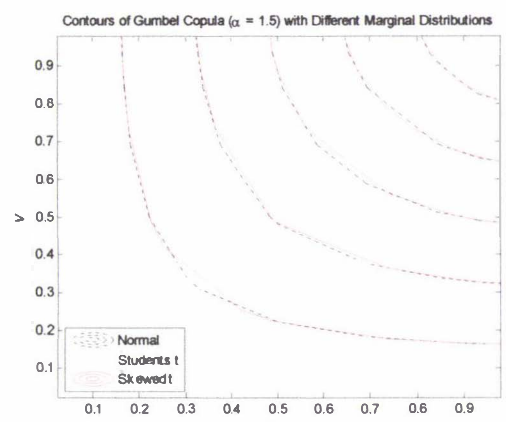
Figure 3.3 displays the density plots of the standard normal, the Student-t and Hansen's skewed-t distribution, and the contours of the four copulas each with respectively the three distributions as marginals.



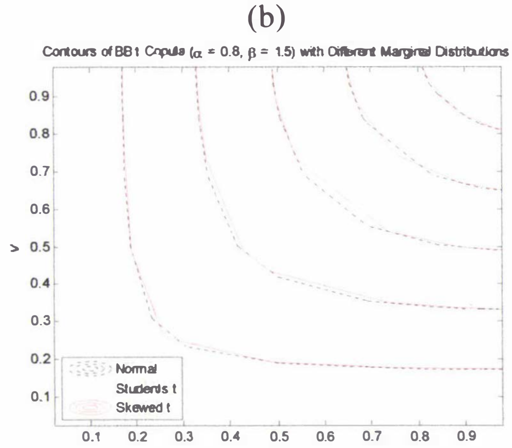
(a)



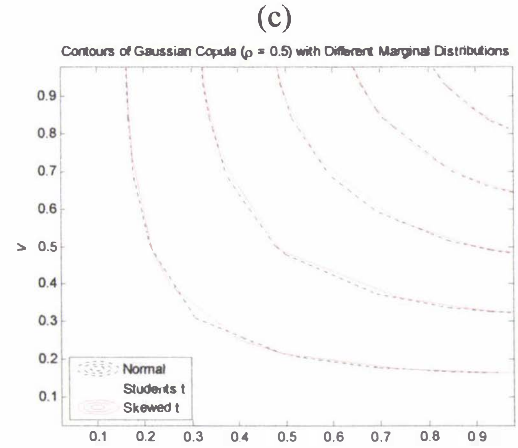
(b)



(c)



(d)



(e)

Figure 3.3

Note: This figure shows the density plots of three different marginal distributions and the contours of four different copulas each being constructed by every one of the three different marginal distributions. (a) Densities of the standard normal ($\mu = 0, \sigma^2 = 1$), the Student-t (η = 4), and Hansen's (1994) skewed-t (η = 4, λ = -0.05) distribution. (b) Clayton copula with α = 0.50. (c) Gumbel copula with α = 1.50. (d) BB1 copula with α = 0.80 and β = 1.50. (e) Gaussian copula with ρ = 0.50.

With the marginal distributions $u = F_{1,t}(z_{1,t})$ and $\upsilon = F_{2,t}(z_{2,t})$ given, I further estimate the bivariate copula model $C(u,\upsilon;\theta)$ in the second stage, before deriving the standardized quantile q_{ξ} from the copula model.

3.2.2.2 Standardized quantile estimation

There are three ways in the literature to derive q_{ξ} from the copula model: the Monte Carlo simulation method suggested by Ané and Kharoubi (2003), Geman and Kharoubi (2004), Dowd (2005a), and Junker and May (2005), the discrete-sum approximation method proposed by Dowd (2005b), and the transformation method utilized by Malevergne and Sornette (2006). The Monte Carlo method is computationally demanding when the number of competing models is large, and the transformation method is practically intractable. Compared with them, Dowd's (2005b) discrete-sum approximation method not only makes it easy to plug in any copula functions, but also reduces computational burden. Thus, I employ Dowd's method in this research. The details of the method are provided in Appendix 3.1.

3.3 Hansen's (2005) SPA Test

This section outlines Hansen's (2005) SPA test, and I use the same notation as Hansen's. The purpose of the test is to help compare various PVaR models' out-of-sample forecasting ability as well as their in-sample fitting. Below I present the statistical loss function for the SPA test. The statistical loss function is a quantile loss function suggested by Koenker and Bassett (1978), and has been used by Bertail *et al* (2004), Engle and Manganelli (2004), González-Rivera *et al* (2004) and Bao *et al* (2007) in empirical studies.

3.3.1 Loss function

Let us split a sample of n observations into two parts: an in-sample (training) part of size T and an out-of-sample (predicting) part of size P, so that $n = T + P$. The predictive quantile loss function (Koenker and Bassett (1978)) at the ξ level for a PVaR model is:

$$L_{\xi} = \mathbf{E}[(\xi - \mathbf{1}_{\{R_{t+1} < \mathrm{VaR}_{\xi,t+1}\}}) \cdot (R_{t+1} - \mathrm{VaR}_{\xi,t+1})] \qquad (t = T, \ldots, n) \qquad (3.11)$$

where $\mathbf{E}[\cdot]$ is the expectation operator. Now let $d_{k,t} \equiv L_{0,t,\xi} - L_{k,t,\xi}$ be the performance measure of the candidate model k ($k = 1, \ldots, g$) relative to the benchmark model at time t, and stack these performance measures in a vector to obtain $\mathbf{d}_t = (d_{1,t}, d_{2,t}, \ldots, d_{g,t})'$. Then the mean value of $d_{k,t}$ is $\bar{d}_k = P^{-1}\sum_{t=T}^{n} d_{k,t}$, the mean value of $\mathbf{d}_t$ is $\bar{\mathbf{d}} \equiv P^{-1}\sum_{t=T}^{n} \mathbf{d}_t$, and the vector of the expected excess performance is $\boldsymbol{\mu} \equiv \mathbf{E}(\mathbf{d}_t)$. My interest is in testing the null hypothesis that all models are no better than a benchmark,[9] that is,

$$H_0: \max_{k=1,\ldots,g} \boldsymbol{\mu} \le \mathbf{0}. \qquad (3.12)$$

3.3.2 SPA test

Based on the regularity conditions given by Theorem 4.1 in West (1996), the test ensures $P^{1/2}(\bar{\mathbf{d}} - \boldsymbol{\mu}) \xrightarrow{d} N(\mathbf{0}, \boldsymbol{\Omega})$, where $\boldsymbol{\Omega}$ is an asymptotic $g \times g$ covariance matrix. Then, the test statistic of the superior predictive ability proposed by Hansen (2005) reads:

[9] This hypothesis is also known as the multiple hypothesis. See Lee and Saltoglu (2002) and Gonzalez-Rivera et al (2004).

$$T^{SPA} \equiv \max\left[\max_{k=1,\dots,g} \frac{P^{1/2}\bar{d}_k}{\hat{\omega}_k}, 0\right] \qquad (3.13)$$

where the consistent estimator $\hat{\omega}_k$ is defined as $\hat{\omega}_k \equiv \sqrt{\mathrm{var}(P^{1/2}\bar{d}_k)}$. The distribution of T^{SPA} is unknown, but Hansen's Corollary 3 shows that it can be approximated under the null hypothesis by the empirical distribution obtained using the so-called stationary bootstrap method (Politis and Romano (1994)). The details of the stationary bootstrap method are provided in Appendix C of Sullivan *et al* (1999). Now the bootstrapped statistic for the SPA test reads:

$$T_b^{SPA*} = \max\left[0, \max_{k=1,\dots,g} \frac{P^{1/2}\bar{Z}^*_{k,b}}{\hat{\omega}^*_k}\right] \qquad (3.14)$$

where $\bar{Z}^*_{k,b} = P^{-1}\sum_{t=T}^{n} Z^*_{k,b,t}$ (b = 1, 2,..., B is the number of resamples), and $\bar{Z}^*_{k,b,t} = d^*_{k,b,t} - g_c(\bar{d}_k)$ with $d^*_{k,b,t}$ being the bootstrapped performance of $d_{k,t}$ of model k and $g_c(\bar{d}_k) = \bar{d}_k \cdot \mathbf{1}_{\{\bar{d}_k \geq -\sqrt{(\hat{\omega}_k^{*2}/P)2\log\log P}\}}$.

The bootstrapped estimator $\hat{\omega}_k^{*2}$ in function $g_c(\cdot)$ is calculated as $\hat{\omega}_k^{*2} \equiv \hat{\gamma}_{0,k} + 2\sum_{i=1}^{P-1}\kappa(P,i)\hat{\gamma}_{i,k}$ where $\hat{\gamma}_{i,k} \equiv P^{-1}\sum_{j=1}^{P-i}(d_{k,j} - \bar{d}_k)(d_{k,j+i} - \bar{d}_k)$ (i = 0, 1,..., P-1) are the empirical covariances, and $\kappa(P,i) \equiv \frac{P-i}{P}(1-q)^i + \frac{i}{P}(1-q)^{P-i}$ are the kernel weights. Note that the bootstrap smoothing parameter $q = l^{-1}$ is determined by the *random* block size l of the stationary bootstrap (conditional on the sample), rather than the deterministic block length. Previous empirical studies on the stationary bootstrap, such as González-Rivera et al (2004) and Bao and Lee (2006), suggest the block size to be 4, i.e. $q = 4^{-1} = 0.25$. This suggestion, however, is only suitable to the block bootstrap and would lead the resampled series to be nonstationary.[10] Therefore, to ensure the stationarity of the resampled series, I employ Politis and White's (2004) method to estimate the random block size.

Now the bootstrap p-value, to be used in identifying the superior predictive model, is given by:

$$\hat{p}_{SPA} \equiv \sum_{b=1}^{B} \frac{\mathbf{1}_{\{T_b^{SPA*} > T^{SPA}\}}}{B} \quad (3.15)$$

The null hypothesis that the benchmark PVaR model is superior to other PVaR models is rejected by small p-values[11].

3.4 Empirical Results

3.4.1 Data

To gauge the relative performances of the four copulas in out-of-sample forecasting for PVaR, I use three daily stock price indexes FTSE100, Nikkei 225 and S&P 500 which represent the international equity markets among the best known. The sample period spans from 07 December, 1987 to 29 June, 2006 (totaling 4844 observations) for each price index, and the data were sourced from Datastream.

I am particularly interested in investigating the forecasting ability of the bivariate Archimedean copula-based PVaR model. To this end, I split the whole sample into an in-

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[10] As discussed in Awartani and Corradi (2005) and Corradi and Swanson (2006b, 2006c), one important feature of the stationary bootstrap is that the resampled series (conditional on the sample) is stationary, while a series resampled from the block bootstrap is nonstationary, even if the original sample is strictly stationary.

[11] The p-value computed for the benchmark model in White's (2000) reality test and Hansen's (2005) SPA test must be compared with other p-values computed when the remaining models are given a chance to be the benchmark model. The statistical inference to be made here is not in a conventional sense that the p-value indicates a significance level compared with, say, 5% or 1%.

sample part (or training set) for obtaining the coefficients, and an out-of-sample part (or test set), as follows:

$$t = \underbrace{07\,\text{Dec}\,1987,......,16\,\text{Oct}\,1997,}_{\text{In}-\text{Sample}(2574\,\text{observations})} \quad \underbrace{17\,\text{Oct}\,1997,......,29\,\text{Jun}\,2006}_{\text{Out}-\text{of}-\text{Sample}\,(2270\,\text{observations})}$$

With the latter set of data, I plug in the coefficients, which are estimated by the *fixed* scheme[12] and obtained from the training set, to see how well they perform with the new data set (test set). Note that my divide (17 Oct, 1997) between the in-sample and out-of-sample periods follows Forbes and Rigobon (2002). Both of the two subsample periods witnessed some extreme episodes. During the in-sample period, there were the Mexican peso crisis and the early episode of the Asian financial crisis as two well-known examples. The out-of-sample period embraces the late episode of the Asian financial crisis, the collapse of Long Term Capital Management, and the dot.com crash etc. I expect to see that the Archimedean copula-based PVaR models are superior to the Gaussian copula-based PVaR model in both in- and out-of-sample periods[13].

The daily returns of each price index are calculated as $Y_t = 100 \cdot \log(X_t / X_{t-1})$ where X_t is the price index. The usual descriptive statistics of the three index returns for the whole, in-sample and out-of-sample periods are set out in Table 3.1. The three return series have common characteristics in the higher-order moments: All of them seem to be nonnormally distributed with heavy tails for both the whole and subsample periods. The LM statistics of Engle (1982) with up to 10 lags indicate the presence of an ARCH effect

[12] There are three different schemes to estimate in-sample coefficients, namely, recursive, rolling and fixed schemes. More discussions on these schemes can be found on page 819 of West and McCracken (1998). In addition, the fixed scheme is also called the once-and-for-all division.

[13] One may wonder whether the final evaluation results are sensitive to the choice of the split point for the two subsamples. To address this concern, I have tried extending in-sample observations by 200 while reducing out-of-sample observations by 200. The evaluation results (not reported in this dissertation but are available upon request) are similar to the ones reported in this chapter, leading to qualitatively the same conclusions. This suggests the robustness of the reported results.

in each series. Table 3.2 reports the linear and rank correlation coefficients between these return series in the three periods. Interestingly, the linear and rank correlations for the out-of-sample period are higher than for the whole and in-sample periods. Furthermore, the highly significant rank correlation estimates reveal that the joint distributions of two return series are not elliptical. Figure 3.4-3.6 present the plots of the three index return pairs and the corresponding scatter plots for the in-sample and out-of-sample periods. A common observation is that the scattered points for the out-of-sample period are more disperse than for the in-sample period, indicating greater extreme gains and losses due to financial instability.

Table 3.1: Summary Statistics of Price Index Returns

	Whole Sample (07 December 1987 – 29 June 2006)			In-Sample (07 December 1987 – 16 October 1997)			Out-of-Sample (17 October 1997 – 29 June 2006)		
	FTSE100	Nikkei225	S&P500	FTSE100	Nikkei225	S&P500	FTSE100	Nikkei225	S&P500
Observations	4844	4844	4844	2574	2574	2574	2270	2270	2270
Maximum	5.9026	12.4303	5.5733	5.4396	12.4303	3.6673	5.9026	7.6605	5.5733
Minimum	-5.8853	-7.2340	-7.1128	-4.1399	-6.8267	-7.0082	-5.8853	-7.2340	-7.1128
Mean	0.0266	-0.0083	0.0354	0.0465	-0.0095	0.0556	0.0040	-0.0070	0.0127
Std.	0.9896	1.3801	0.9876	0.7923	1.3153	0.7977	1.1736	1.4503	1.1658
Skewness	-0.1382	0.1598	-0.2122	0.0563	0.3910	-0.5108	-0.1651	-0.0372	-0.0587
Kurtosis	6.2500	6.9236	7.5738	5.2203	9.2009	8.8111	5.4981	5.0624	6.1413
Engle (10)	889.3137*	317.1759*	408.6297*	77.8200*	192.1592*	64.7449*	452.3209*	155.8132*	231.2863*

Note: The daily percentage price index returns are calculated as $100\times\log(X_t/X_{t-1})$. Engle (10) represents the LM test statistic of Engle (1982) using 10 lags for the presence of ARCH effects. * indicates significance at the 5% level. The critical value of the LM statistic at 5% is 18.307.

Table 3.2: Linear and Nonlinear Correlations

	Whole Sample (07 December 1987 – 29 June 2006)			In-Sample (07 December 1987 – 16 October 1997)			Out-of-Sample (17 October 1997 – 29 June 2006)		
	FTSE100	Nikkei225	S&P500	FTSE100	Nikkei225	S&P500	FTSE100	Nikkei225	S&P500
				Panel A. Pearson's ρ_P					
FTSE 100	—	**0.2365**	**0.3990**	—	**0.2309**	**0.3447**	—	**0.2447**	**0.4271**
Nikkei 225		—	**0.1121**		—	**0.1058**		—	**0.1185**
S&P 500			—			—			—
				Panel B. Spearman's ρ_S					
FTSE 100	—	**0.2155**	**0.3598**	—	**0.1986**	**0.3069**	—	**0.2332**	**0.4068**
Nikkei 225		—	**0.1207**		—	**0.1017**		—	**0.1378**
S&P 500			—			—			—
				Panel C. Kendall's τ_K					
FTSE 100	—	**0.1473**	**0.2512**	—	**0.1346**	**0.2100**	—	**0.1597**	**0.2888**
Nikkei 225		—	**0.0817**		—	**0.0686**		—	**0.0932**
S&P 500			—			—			—

Note: The linear correlation coefficient ρ_P is calculated as $\rho_P = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$. The rank correlation coefficients ρ_S and τ_K are calculated as

$\rho_S = \frac{12}{n(n^2-1)} \sum_{i=1}^{n} \left(rank(x_i) - \frac{n+1}{2} \right) \left(rank(y_i) - \frac{n+1}{2} \right)$ and $\tau_K = \binom{n}{2}^{-1} \sum_{1 \le i \le j \le n} sign[(x_i - x_j)(y_i - y_j)]$ respectively.

Figures in bold type indicate significance at the 5% level.

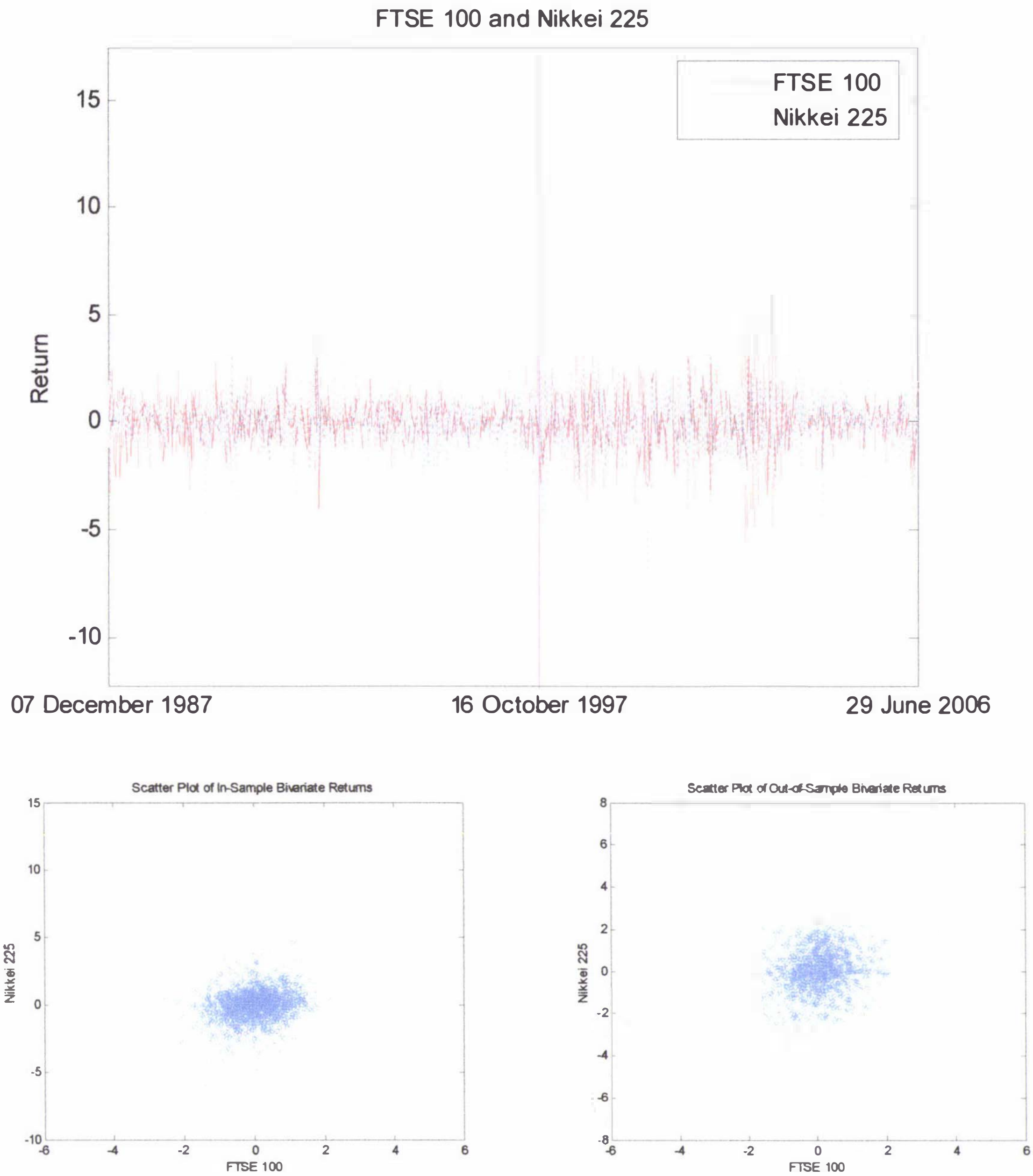

Figure 3.4

Note: This figure shows the time series plot of two index returns from 07 December 1987 to 29 June 2006, and the corresponding scatter plots of in- and out-of-sample periods for the FTSE 100-vs-Nikkei 225 pair.

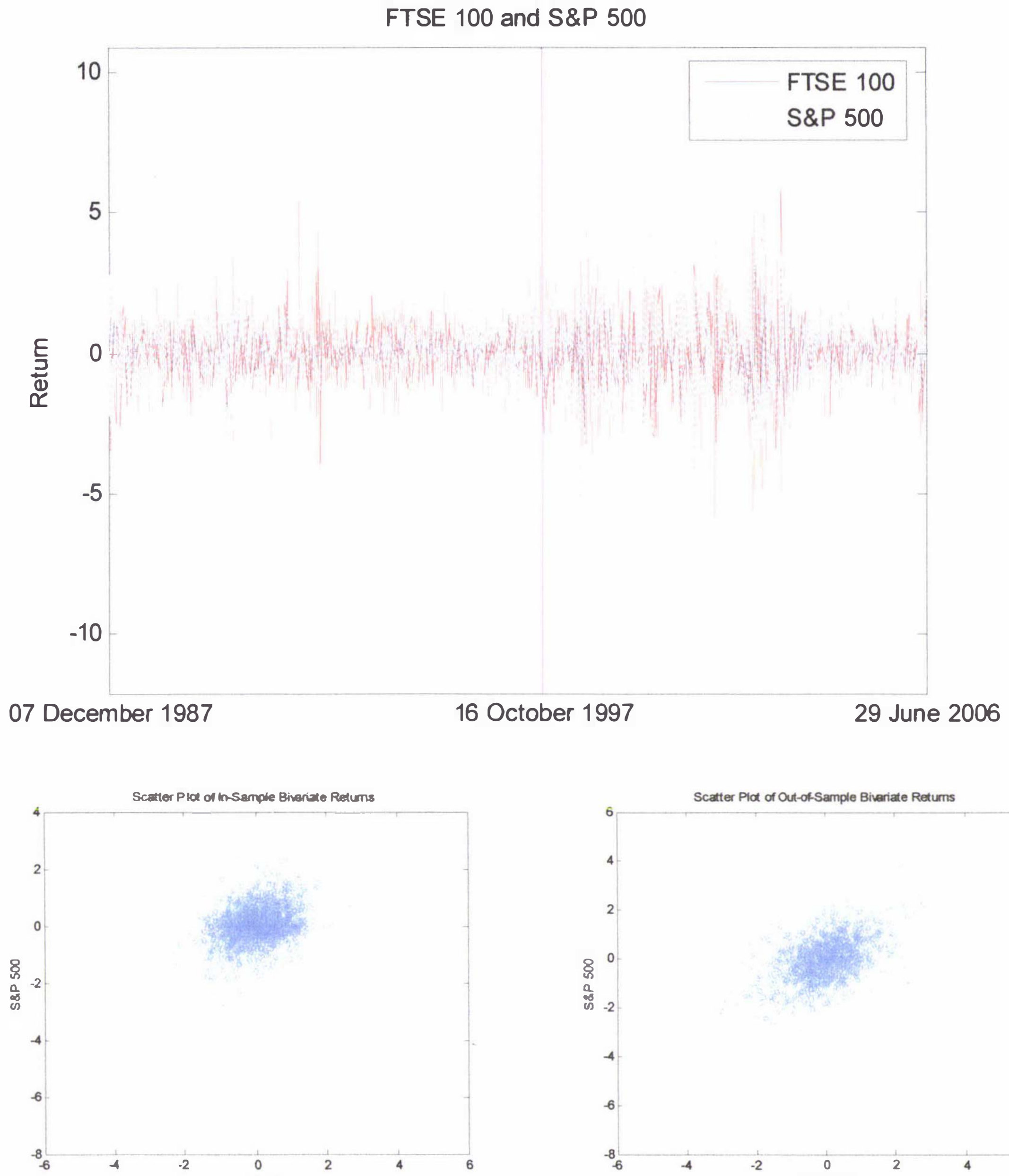

Figure 3.5

Note: This figure shows the time series plot of two index returns from 07 December 1987 to 29 June 2006, and the corresponding scatter plots of in- and out-of-sample periods for the FTSE 100-vs-S&P 500 pair.

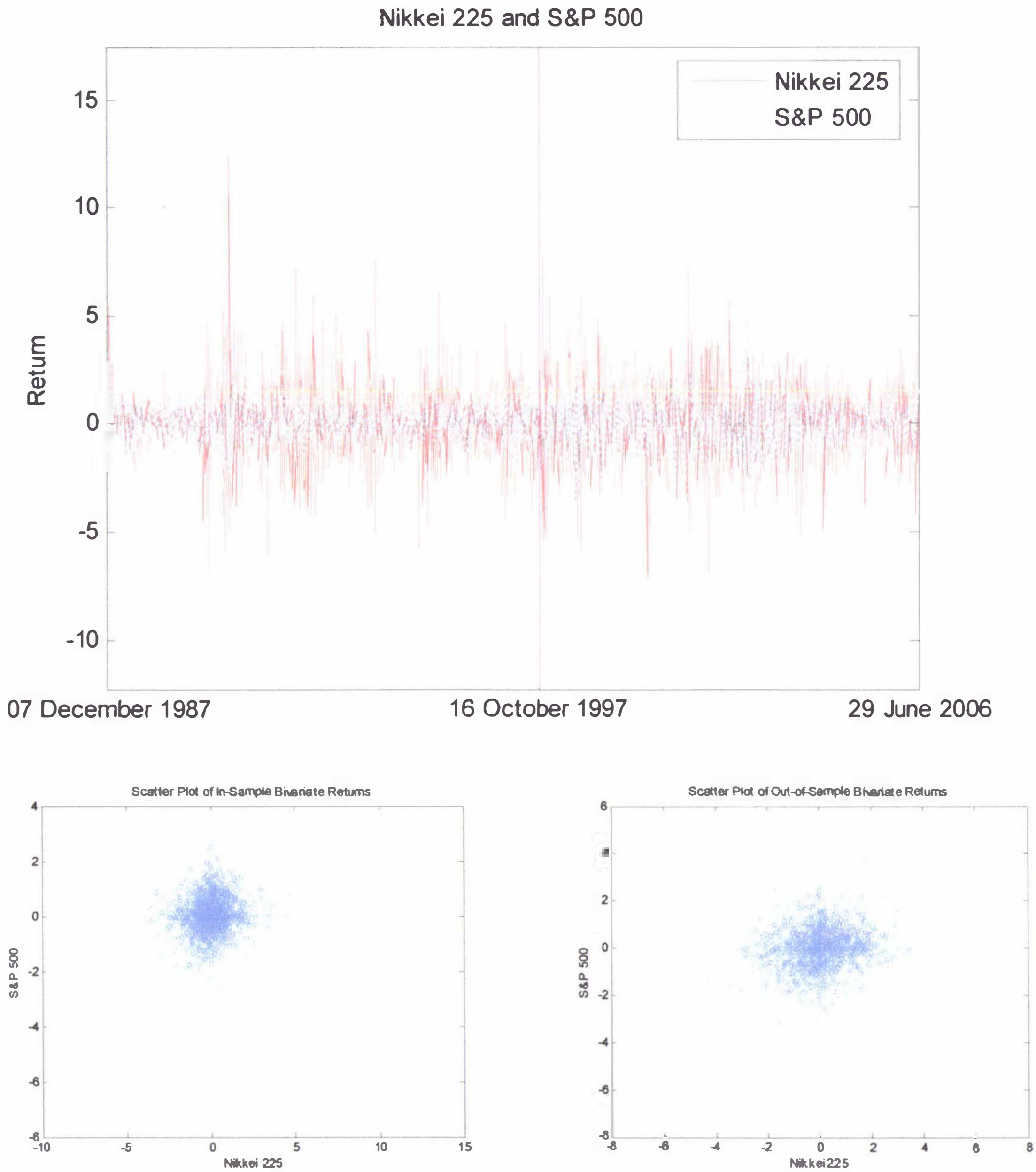

Figure 3.6

Note: This figure shows the time series plot of two index returns from 07 December 1987 to 29 June 2006, and the corresponding scatter plots of in- and out-of-sample periods for the Nikkei 225-vs-S&P 500 pair.

3.4.2 In-sample model estimation

The two-stage full parametric ML estimation method is used to estimate model parameters. Table 3.3 pertains to the first stage estimation. It reports the in-sample estimation results of the TGARCH (1, 1) model with three univariate distributions for each index return series. All the estimated parameters are significant at the 5% level, except for c_0 in the standard normal and the Student-t distribution for the S&P 500 returns, and λ in the skewed-t distributions for the FTSE 100 and the S&P 500 returns. Variance-stationarity is ensured, as the estimated parameters all satisfy the condition $(b_0 + c_0)/2 + d_0 < 1$. In addition, the values of η in both the Student-t and the skewed-t distributions are estimated to be low for the Nikkei 225 and the S&P 500 returns, demonstrating that these two series are distributed with heavy tails. For all the three return series, the skewness parameter λ in the skewed-t distribution is negative, revealing that large negative returns tend to occur more often than large positive ones in the in-sample period.

To make sure the marginal model fittings are reliable, I also employ a goodness-of-fit test suggested by Patton (2006a, 2006b) and Jondeau and Rockinger (2006). The goodness-of-fit test includes two parts. The first part is based on the idea of Diebold et al (1998) who argue that if a marginal distribution is correctly specified, the marginal u_i (i = 1, 2, 3 indicating the normal, student-t and skewed-t distributions respectively) should be *iid* Uniform(0, 1). Hence, I examine the serial correlation of the kth centered moments of $(u_i - \bar{u}_i)^k$ for k = 1,..., 4 by regressing itself on 20 own lags. The LM test statistic is given by $(n-20)\cdot R^2$, where n and R^2 are the (in-)sample size and the coefficient of determination of the regression respectively, and is distributed as a χ^2_{20} under the null. The second part is to test the null hypothesis that u_i is Uniform(0, 1) via Komogorov-Smirnov (henceforth K-S) test. In this case, I follow Jondeau and Rockinger (2006) and cut the empirical and theoretical distributions into 20 bins. Accordingly, a 20-bin K-S test formed on the in-sample 2574 observations should have, on average, 2574 / 20 = 128.7 ≈ 129 observations in each bin. Therefore the 20-bin K-S statistic (denoted as K-S(20)) is obtained from $\mathrm{K-S}(20) = \sum_{j=1}^{20} \max | u_{Emp,ijt} - u_{ijt} |$ where u_{Emp} indicates empirical marginal distribution, and t = 1, 2,..., 129. Under the null, the K-S(20) statistic is also distributed as χ^2_{20}. Table 3.4 presents the p-values of the goodness-of-fit tests. As shown in this table, except for the third moment in Student's t distribution for the FTSE 100 series and in the normal and skewed-t distributions for the S&P 500 series, all three different marginal distribution models for each series pass the test at the 5% level. These results not only indicate that these marginal distributions are generally adequately modeled, but also further confirm that there are three possible distributions for each series.

In the second stage, I undertake the in-sample estimation of copula parameters. Table 3.5 sets out the results of twelve competing copula models for each of the three pairs: FTSE 100 vs Nikkei 225, FTSE 100 vs S&P 500, and Nikkei 225 vs S&P 500. One can see that all the estimated copula parameters are significant at the 5% level. In addition, Table 3.5 also reports the estimates of tail dependence for Archimedean copulas (the definition of tail dependence is given in Chapter 2). To interpret the tail dependence results, take the BB1 copula with the normal marginals as an example. The lower tail dependence parameter κ^L for the FTSE 100-vs-Nikkei 255 pair is estimated to be 0.9071, meaning that given FTSE 100 having a price drop below a certain value, the probability of Nikkei 225 having a price drop below a corresponding value is about 90.71%. The upper tail dependence parameter κ^U for the FTSE 100-vs-Nikkei 255 pair is estimated to be 0.0672, meaning that given FTSE 100 having a price jump above a certain value, the probability of Nikkei 225 having a price jump above a corresponding value is about 6.72%. Interestingly, all the three copula-marginal complexes of model (i.e., BB1-normal, BB1-Student-*t* and BB1-skewed-*t*) yield evidence for all the three pairs of index returns that lower tail dependence is stronger than upper tail dependence: $\kappa^L > \kappa^U$. In other words, the three international equity markets are more likely to fall together than to rise together during the in-sample period.

Table 3.3: In-Sample ML Estimation of the TGARCH (1,1) Model with Different Distributions of Standardized Returns

Parameter	FTSE 100			Nikkei 225			S&P 500		
	Normal	Student-t	Skewed-t	Normal	Student-t	Skewed-t	Normal	Student-t	Skewed-t
a_0	**0.0163**	**0.0126**	**0.0125**	**0.0160**	**0.0160**	**0.0152**	**0.0032**	**0.0039**	**0.0039**
	(0.0045)	(0.0048)	(0.0025)	(0.0021)	(0.0036)	(0.0031)	(0.0007)	(0.0016)	(0.0012)
b_0	**0.0372**	**0.0246**	**0.0247**	**0.0122**	**0.0128**	**0.0129**	**0.0230**	**0.0222**	**0.0221**
	(0.0083)	(0.0085)	(0.0070)	(0.0047)	(0.0081)	(0.0065)	(0.0031)	(0.0085)	(0.0065)
c_0	**0.0278**	**0.0310**	**0.0553**	**0.1331**	**0.1350**	**0.1487**	0.0040	0.0180	**0.0400**
	(0.0101)	(0.0121)	(0.0079)	(0.0090)	(0.0180)	(0.0126)	(0.0046)	(0.0109)	(0.0063)
d_0	**0.9228**	**0.9389**	**0.9392**	**0.9168**	**0.9154**	**0.9158**	**0.9700**	**0.9633**	**0.9633**
	(0.0138)	(0.0145)	(0.0014)	(0.0055)	(0.0097)	(0.0028)	(0.0028)	(0.0071)	(0.0002)
η		**10.8920**	**10.8970**		**6.0014**	**5.9512**		**4.7314**	**4.7380**
		(1.6688)	(2.0303)		(0.6547)	(0.7548)		(0.4615)	(0.5514)
λ			-0.0092			**-0.0442**			-0.0011
			(0.0198)			(0.0194)			(0.0143)
Loglik	-2973.3	-2945.8	-2945.7	-3918.0	-3838.8	-3836.2	-2920.1	-2794.6	-2794.6

Note: This table reports the in-sample maximum likelihood estimation results for the marginal models. The in-sample period is from 07 December 1987 to 16 October 1997. The specified models for estimation are: $R_t = \mu + e_t, e_t = \sqrt{h_t} \cdot z_t, e_t^+ = \max(e_t, 0), e_t^- = \max(-e_t, 0), h_t = a_0 + b_0(e_{t-1}^+)^2 + c_0(e_{t-1}^-)^2 + d_0 h_{t-1}$, where $z_t \sim N$ (0, 1), or $z_t \sim$ *Student-t* (η), or $z_t \sim Skewed-t(\eta, \lambda \mid \mathcal{F}_{t-1})$. Loglike denotes the log likelihood function. Standard errors are given in parentheses. Figures in bold indicate significance at the 5% level.

Table 3.4: Goodness-of-Fit Test Statistics for Different Distributional Restrictions

	FTSE 100	Nikkei 225	S&P 500
Panel A: Normal distribution			
1st Moment LM Test	0.38	0.58	0.06
2nd Moment LM Test	0.15	0.10	0.05
3rd Moment LM Test	0.05	0.24	0.04*
4th Moment LM Test	0.13	0.27	0.12
K-S(20)	0.65	0.63	0.63
Panel B: Student-t distribution			
1st Moment LM Test	0.38	0.56	0.07
2nd Moment LM Test	0.17	0.10	0.07
3rd Moment LM Test	0.04*	0.22	0.05
4th Moment LM Test	0.12	0.33	0.16
K-S(20)	0.66	0.64	0.65
Panel C: Skewed-t distribution			
1st Moment LM Test	0.40	0.64	0.06
2nd Moment LM Test	0.18	0.05	0.08
3rd Moment LM Test	0.05	0.27	0.04*
4th Moment LM Test	0.12	0.21	0.07
K-S(20)	0.65	0.63	0.64

Note: This table reports p-values of the goodness-of-fit test statistics for the normal, the Student-t, and the skewed-t distributional restrictions of the general univariate model. The test contains two parts. The first part of the test which is labeled as kth Moment LM Test (k = 1,..., 4) in each panel evaluates whether the kth centered moments of the transformed residuals u_i are serially correlated. I regress $(u_i - \bar{u}_i)^k$ on 20 own lags. Under the null of no autocorrelation of the residuals, the statistics $(n\text{-}20)\cdot R^2$ is distributed as a χ^2_{20} where n and R^2 are the sample size and the coefficient of determination of the regression. The second part of the test which is labeled as K-S(20) in each panel employs Kolmogorov-Smirnov test within each bin to evaluate whether u_i are Uniform(0, 1), where the total number of the bin is 20. Under the null, the statistics is also distributed as a χ^2_{20}. *indicates significance at the 5% level.

Table 3.5: In-Sample ML Estimation of Copulas with Different Marginals

Copula	Marginals	Index Return Pair								
		FTSE 100 vs Nikkei 225			FTSE 100 vs S&P 500			Nikkei 225 vs S&P 500		
		$\{\theta\}$	κ^L	κ^U	$\{\theta\}$	κ^L	κ^U	$\{\theta\}$	κ^L	κ^U
Clayton	Normal	**0.2022**	0.0325	—	**0.3009**	0.0999	—	**0.0686**	0.0000	—
		(0.0247)			(0.0265)			(0.0184)		
	Student-t	**0.3380**	0.1286	—	**0.5847**	0.3056	—	**0.2365**	0.0534	—
		(0.0343)			(0.0375)			(0.0374)		
	Skewed-t	**0.2432**	0.0579	—	**0.4161**	0.1890	—	**0.1252**	0.0040	—
		(0.0269)			(0.0292)			(0.0242)		
Gumbel	Normal	**1.1135**	—	0.1364	**1.2424**	—	0.2530	**1.0428**	—	0.0561
		(0.0158)			(0.0188)			(0.0134)		
	Student-t	**1.1910**	—	0.2104	**1.3608**	—	0.3358	**1.1333**	—	0.1566
		(0.0203)			(0.0236)			(0.0214)		
	Skewed-t	**1.1246**	—	0.1478	**1.2429**	—	0.2534	**1.0552**	—	0.0712
		(0.0154)			(0.0179)			(0.0133)		
BB1	Normal	**0.1479**	0.9071	0.0672	**0.1461**	0.9163	0.1805	**0.0515**	0.9658	0.0324
		(0.0267)			(0.0139)			(0.0145)		
		1.0519			**1.1580**			**1.0242**		
		(0.0155)			(0.0160)			(0.0123)		
	Student-t	**0.1993**	0.8826	0.1293	**0.2893**	0.8482	0.2332	**0.1383**	0.9153	0.1024
		(0.0420)			(0.0459)			(0.0446)		
		1.1067			**1.2179**			**1.0820**		
		(0.0243)			(0.0295)			(0.0254)		
	Skewed-t	**0.1665**	0.8968	0.0763	**0.2455**	0.8601	0.1520	**0.0982**	0.9356	0.0303
		(0.0316)			(0.0350)			(0.0280)		
		1.0594			**1.1287**			**1.0226**		
		(0.0167)			(0.0209)			(0.0136)		

Gaussian	Normal	**0.2188**	—	—	**0.3423**	—	—	**0.1051**	—	—
		(0.0183)			(0.0164)			(0.0193)		
	Student-*t*	**0.2789**	—	—	**0.4445**	—	—	**0.1892**	—	—
		(0.0217)			(0.0171)			(0.0282)		
	Skewed-*t*	**0.2092**	—	—	**0.3429**	—	—	**0.1080**	—	—
		(0.0184)			(0.0164)			(0.0193)		

Note: This table reports the in-sample two-stage maximum likelihood estimation results of copula functions for each pair of index returns. The specified models for estimation are presented in Section 3.2.2.1. Standard errors are given in parentheses. {θ} indicates a set of the estimated parameters. κ^L and κ^U indicate lower and upper tail dependence. Figures in bold indicate significance at the 5% level.

3.4.3 Copula-based PVaR estimation

This subsection computes the one-day copula-based PVaRs. In order to focus on the performance of the copula-based PVaR model, I let a portfolio consist of two indexes with a fixed weight 0.5 assigned to each index. This gives me three equally-weighted portfolios corresponding to three pairs of indexes. For ease of exposition, I denote each copula-marginal complex of model by $C_{\text{"name of copula"-"name of marginal"}}$. For example, $C_{Clayton-Normal}$ denotes the Clayton copula constructed by the normal marginal distributions. I then use the discrete-sum approximation method proposed by Dowd (2005b) to estimate the standardized quantile and compute the PVaR according to Equation (3.3).

The PVaR estimates for the three portfolios at the 95% and 99% confidence levels are provided in Table 3.6 for the in-sample period and in Table 3.7 for the out-of-sample period. Taking a look at these two tables, we can see that copulas with the Student-*t* distribution as marginals yield lower PVaRs than the same copulas but with the normal or the skewed-*t* distribution as marginals. In particular, the model $C_{Clayton-Student-t}$ yields the lowest PVaRs at both the 95% and 99% confidence levels for all the three portfolios. Using the decision rule that favors the minimum-PVaR, one would be tempted to vote for $C_{Clayton-Student-t}$ as the best and use the lowest PVaRs generated by the model for risk management purposes. However, due to potential data snooping involved, the PVaR estimates yielded by $C_{Clayton-Student-t}$ could be spurious or overstated. Thus, it is necessary to check on the possible spurious predictive ability of the model caused by data snooping.

Table 3.6: In-Sample Copula-Based PVaR Estimates ($\times 10^{-2}$)

Copula	Marginal	Equally Weighted Portfolios					
		FTSE 100 vs. Nikkei 225		FTSE 100 vs. S&P 500		Nikkei 225 vs. S&P 500	
		95%	99%	95%	99%	95%	99%
Clayton	Normal	-2.1246	-3.0992	-1.6673	-2.4645	-1.9049	-2.7321
	Student-*t*	-2.5385	-4.0218	-2.1040	-3.4580	-2.5033	-4.0756
	Skewed-*t*	-2.1275	-3.3573	-1.6678	-2.7177	-1.9005	-3.0470
Gumbel	Normal	-2.0366	-2.8569	-1.6157	-2.2701	-1.8769	-2.6447
	Student-*t*	-2.4067	-3.5933	-1.9765	-3.0369	-2.4069	-3.7171
	Skewed-*t*	-2.0275	-3.0374	-1.5746	-2.3964	-1.8419	-2.8553
BB1	Normal	-2.1220	-3.0648	-1.6648	-2.4034	-1.9111	-2.7269
	Student-*t*	-2.5164	-3.9117	-2.0724	-3.3325	-2.4930	-3.9827
	Skewed-*t*	-2.1162	-3.2943	-1.6516	-2.6378	-1.8979	-3.0225
Gaussian	Normal	-2.1295	-2.9977	-1.6923	-2.3947	-1.9295	-2.7231
	Student-*t*	-2.4977	-3.7680	-2.0712	-3.2298	-2.4618	-3.8354
	Skewed-*t*	-2.0997	-3.1720	-1.6464	-2.5409	-1.8830	-2.9336

Note: This table reports the in-sample copula-based PVaR estimates using Dowd's (2005b) discrete-sum approximation method.

Table 3.7: Out-of-Sample Copula-Based PVaR Estimates ($\times 10^{-2}$)

Copula	Marginal	Equally Weighted Portfolios					
		FTSE 100 vs. Nikkei 225		FTSE 100 vs. S&P 500		Nikkei 225 vs. S&P 500	
		95%	99%	95%	99%	95%	99%
Clayton	Normal	-2.6432	-3.8443	-2.5963	-3.8046	-2.3524	-3.3631
	Student-*t*	-3.1533	-4.9816	-3.2583	-5.2105	-3.0836	-5.0044
	Skewed-*t*	-2.6467	-4.1627	-2.5969	-4.1884	-2.3471	-3.7477
Gumbel	Normal	-2.5347	-3.5457	-2.5180	-3.5100	-2.3182	-3.2563
	Student-*t*	-2.9909	-4.4535	-3.0649	-4.6720	-2.9657	-4.5664
	Skewed-*t*	-2.5235	-3.7683	-2.4558	-3.7013	-2.2755	-3.5136
BB1	Normal	-2.6399	-3.8022	-2.5923	-3.7119	-2.3600	-3.3567
	Student-*t*	-3.1262	-4.8460	-3.2102	-5.0203	-3.0709	-4.8908
	Skewed-*t*	-2.6328	-4.0851	-2.5724	-4.0673	-2.3440	-3.7179
Gaussian	Normal	-2.6492	-3.7193	-2.6329	-3.6987	-2.3825	-3.3521
	Student-*t*	-3.1030	-4.6689	-3.2085	-4.9647	-3.0328	-4.7109
	Skewed-*t*	-2.6124	-3.9342	-2.5645	-3.9205	-2.3256	-3.6091

Note: This table reports the out-of-sample copula-based PVaR estimates using Dowd's (2005b) discrete-sum approximation method.

3.4.4 In-sample fitting performance

This subsection performs Hansen's (2005) SPA test to evaluate in-sample fitting for the copula-based PVaR models.[14] Tables 3.8 reports the estimated p-values of the SPA test for the in-sample period.

The procedure of the SPA test is as follows. Given the twelve models available, I first take each competing model as a benchmark model with which the remaining eleven models are to be compared. I then use the quantile loss function with stationary bootstrapping to make multiple model comparisons. The null hypothesis to test is that the best of the remaining eleven models is no better than the benchmark. The bootstrapped p-values of the SPA test statistic are computed with 1000 resamples and the random smoothing parameter q is calculated following Politis and White's (2004) suggestion. The decision rule for determining the "best" model is based on the largest p-value.

To illustrate, take from Table 3.8 the portfolio of the FTSE 100-vs-Nikkei 225 pair (at the 95% confidence level) for example. I first let $C_{Clayton-Normal}$ be the benchmark model, and then compare its in-sample PVaR fitting with that of each of the remaining eleven models $C_{Clayton-Student-t}$, $C_{Clayton-Skewed-t}$, ..., $C_{Gaussian-Student-t}$, $C_{Gaussian-Skewed-t}$. The comparisons are made by looking at the differences in the quantile loss functions described by Equation (3.11). After stationary bootstrapping, the p-value for the comparisons with $C_{Clayton-Normal}$ being the benchmark model is 0.2840. Next, I let $C_{Clayton-Student-t}$ be the benchmark model with which the other eleven models are compared in terms of in-sample PVaR fitting. This case yields a p-value of 0.1010. And so on and so forth. In the end, I obtained twelve p-values. Of these p-values, the largest one

[14] Although White's (2000) reality check test and Hansen's (2005) SPA test are designed for the out-of-sample applications, the in-sample applications of these two tests can also be found in many empirical studies including, for instance, Sullivan et al (1999), Hsu and Kuan (2005) and Hsu and Hsu (2006).

(0.4980) is associated with $C_{Gumbel-Normal}$, and so one cannot reject the null that the other eleven models is no better than $C_{Gumbel-Normal}$. This allows me to take $C_{Gumbel-Normal}$ as the model with the best in-sample PVaR fitting. By the same token, the same portfolio (comprising FTSE 100 and Nikkei 225) finds, at the higher confidence level of 99%, that the PVaR models constructed by $C_{BB1-Skewed-t}$ has the best in-sample PVaR fitting, since the corresponding p-value of 0.7750 is the largest one.

Now let us turn to the in-sample results in Table 3.8 for other two portfolios: one comprising FTSE 100 and S&P 500, and the other Nikkei 225 and S&P 500. An observation is that for the portfolio of FTSE 100 + S&P 500 and at the 95% confidence level, not only the best in-sample PVaR fitting performance belongs to $C_{Clayton-Student-t}$, but also all the other eleven models demonstrate the "poorest" performance as their p-values are all 0. However, moving to the 99% level, we can see that my in-sample performance investigation ranks No. 1 the Clayton copula with the normal rather than the Student-t marginals, whose p-value of 0.8520 is the largest one. Regarding the portfolio of Nikkei 225 + S&P 500, Table 3.8 shows that $C_{BB1-Normal}$ has the best in-sample performance at the 95% confidence level, while $C_{Gumbel-Normal}$ at the 99% level.

One summary result can be derived from the above discussions on Table 3.8: $C_{Clayton-Student-t}$ is the "best" only in one case but the "worst" in almost all other cases. This is in striking contrast with the result, if based on the "minimum-PVaR" rule, that $C_{Clayton-Student-t}$ is the "best" for *all* cases as shown by Table 3.6. Such a contrast arises from the presence of the data snooping problem. If a risk manager ignores the problem and simply chooses $C_{Clayton-Student-t}$ and its PVaR estimates as the optimal ground on which to make risk management decisions, the consequence would be a significant waste of resources. To illustrate, let us go back to Table 3.6 and consider the portfolio of FTSE 100 + Nikkei 225 for the 99% confidence level. When ignoring the data snooping bias, the PVaR estimate to use is 4.0218×10^{-2} based on $C_{Clayton-Student-t}$. When correcting for the data snooping bias, the PVaR estimate to use should be 3.2943×10^{-2} based on $C_{BB1-Skewed-t}$ (which is the most preferred according to Table 3.8). In the former case, a risk manager would hold, unprofitably, an excess amount of capital equal to $(4.0218 \times 10^{-2} - 3.2943 \times 10^{-2}) \times \$10^{8} = \$727{,}500$ for a portfolio of 100 million dollars. This indicates the importance of correcting for the data snooping bias to obtain risk estimates which are neither overstated nor understated.

Table 3.8: Bootstrap p-Values of the In-Sample SPA Test

Copula	Marginal	Equally Weighted Portfolios		
		FTSE 100 vs. Nikkei 225	FTSE 100 vs. S&P 500	Nikkei 225 vs. S&P 500
		95% Confidence Level		
Clayton	Normal	0.2840	0.0000	0.7400
	Student-t	0.1010	**0.0570**	0.0000
	Skewed-t	0.3220	0.0000	0.6890
Gumbel	Normal	**0.4980**	0.0000	0.3520
	Student-t	0.0000	0.0000	0.0000
	Skewed-t	0.0000	0.0000	0.0000
BB1	Normal	0.2970	0.0000	**0.7780**
	Student-t	0.1540	0.0000	0.0000
	Skewed-t	0.2350	0.0000	0.6180
Gaussian	Normal	0.3080	0.0000	0.6550
	Student-t	0.1320	0.0000	0.0000
	Skewed-t	0.2490	0.0000	0.5620
		99% Confidence Level		
Clayton	Normal	0.6620	**0.8520**	0.9310
	Student-t	0.0000	0.0000	0.0000
	Skewed-t	0.4760	0.7860	0.4140
Gumbel	Normal	0.0000	0.0000	**0.9960**
	Student-t	0.1260	0.2560	0.0440
	Skewed-t	0.5940	0.6960	0.3570
BB1	Normal	0.7180	0.8380	0.9430
	Student-t	0.0000	0.0000	0.0000
	Skewed-t	**0.7750**	0.8110	0.1900
Gaussian	Normal	0.4630	0.4580	0.9280
	Student-t	0.0000	0.0130	0.0000
	Skewed-t	0.7120	0.8400	0.0680

Note: This table reports the p-values of the in-sample Hansen's (2005) SPA test with stationary bootstrapping. The number of resample is 1000. The sample block length for the stationary bootstrap is selected following Politis and White (2004). The p-values in bold indicate the most preferred model.

3.4.5 Out-of-sample forecasting performance

The analysis in the preceding section shows that the in-sample fittings of the Archimedean-copula-based PVaR models generally are superior to those of the Gaussian-copula-based PVaR models at both the 95% and 99% confidence levels. A natural question is whether this conclusion will also hold for PVaR forecasting in the out-of-sample period. Table 3.9 provides the out-of-sample evaluation results which I discuss in what follows.

For the portfolio of the FTSE 100-vs-Nikkei 225 pair at the 95% level, the full Gaussian copula $C_{Gaussian-Normal}$ now is ranked the best PVaR forecasting model, as its p-value of 0.6220 is the highest. In other words, other models either understate or overstate the risk. However, for the same portfolio at the 99% confidence level, the model $C_{Clayton-Normal}$ has the best forecasting performance, with the highest p-value of 0.8070. This implies that $C_{Clayton-Normal}$ can, whereas $C_{Gaussian-Normal}$ cannot, provide the most adequate PVaR forecast when the equity markets in U.K. and Japan experience *extreme* downside comovements.

Pertaining to the portfolio of the FTSE 100-vs-S&P 500 pair, the out-of-sample results of the copula-based PVaR models are similar to their in-sample results at both the 95% and 99% confidence levels. The SPA test picks the Clayton copula as the best, although its marginals (the skewed-t distribution) for the out-of-sample period are not the same as those (the Student-t distribution) for the in-sample period. This suggests that the Clayton copula-based PVaR model not only has an excellent in-sample fitting ability but also has an excellent out-of-sample forecasting ability, as far as the joint extreme losses in the U.K. and U.S equity markets are concerned.

The SPA statistics for the portfolio of the Nikkei 225-vs-S&P 500 pair show that at the 95% confidence level, $C_{BB1-Normal}$ beats all other models in terms of the out-of-sample forecasting performance, consistent with the in-sample fitting evaluation result. However, at the 99% level, the champion of out-of-sample contention goes to the model $C_{Gaussian-Skewed-t}$ (not to $C_{Gumbel-Normal}$ as in in-sample contention). One possible explanation is that dependence in the extreme left tails between the two markets is so low that using Archimedean copulas would overestimate the PVaR relative to using the Gaussian copula.[15] This proposed explanation seems to be supported by Table 3.2 which shows that the rank correlation coefficients ρ_S and τ_K for the Nikkei 225-vs-S&P 500 pair are all much smaller relative to those for the other two pairs.

The evaluation of out-of-sample forecasting performance yields the following main results. Firstly, the out-of-sample forecasting performances of the Archimedean copula-based PVaR models are, in most cases, superior to those of the Gaussian-copula-based PVaR models. This is consistent with the in-sample fitting result. The reason is easy to see: Archimedean copulas can capture joint extreme events while the Gaussian copula cannot.

Secondly, a model which has the best in-sample fitting performance does not necessarily have the best out-of-sample forecasting performance, and vice versa. This suggests that it is inappropriate to base risk-management decisions only on in-sample analysis, and that it is insufficient to evaluate a copular-based PVaR model only in terms of its in-sample fitting.

Thirdly, it seems that any type of copula with any type of marginals will have a chance to become the best in terms of out-of-sample forecasting (as well as in-sample fitting) performance. Thus, in determining the "best" copula-based PVaR model, all possible copula-marginal

[15] When left tail dependence is very low, say, zero, the probability that both asset returns have extreme downside movements is zero. This would mean that when one asset return falls, the other will not. The portfolio of the two assets will have a PVaR which should be most accurately estimated using the Gaussian copula. But if we use an Archimedean copula that characterizes left tail dependence, we are actually forcing left tail dependence to be non zero. Non-zero left tail dependence would mean that the two asset returns have a certain chance to fall together. A portfolio of such two assets should demonstrate a higher PVaR than the portfolio whose constituent assets have no left tail dependence. But this higher PVaR is generated by wrongly using the Archimedean copula, and so is spurious. In this sense, we can say that Archimedean copula may over-estimate PVaR when let tail dependence is low.

complexes should be considered, and a test should be conducted for the copula and its marginals simultaneously. Testing to select marginals first and then testing to select the copula with the marginals selected would lead to what I term as the "preselection omission" problem: It would greatly reduce the number of candidate copula-marginal complexes from which the best is to be identified.

Finally, the model $C_{Clayton-Student-t}$ is inferior for out-of-sample PVaR forecasting in most cases according to Table 3.9. This is largely consistent with the summary result for in-sample analysis of model-fitting performances. This further confirms the presence of the data-snooping bias which one cannot ignored.

Table 3.9: Bootstrap p-Values of the Out-of-Sample SPA Test

Copula	Marginal	Equally Weighted Portfolios: FTSE 100 vs. Nikkei 225	FTSE 100 vs. S&P 500	Nikkei 225 vs. S&P 500
		95% Confidence Level		
Clayton	Normal	0.5240	0.6010	0.8600
	Student-t	0.4370	0.0000	0.0940
	Skewed-t	0.5840	**0.6840**	0.7890
Gumbel	Normal	0.4760	0.3480	0.3850
	Student-t	0.0000	0.0000	0.0000
	Skewed-t	0.0000	0.0000	0.0000
BB1	Normal	0.4810	0.5650	**0.9120**
	Student-t	0.3750	0.0000	0.0900
	Skewed-t	0.3830	0.5250	0.7360
Gaussian	Normal	**0.6220**	0.4800	0.8810
	Student-t	0.3590	0.0000	0.0530
	Skewed-t	0.3090	0.4330	0.6260
		99% Confidence Level		
Clayton	Normal	**0.8070**	0.2210	0.8240
	Student-t	0.0000	**1.0000**	0.0000
	Skewed-t	0.7250	0.0000	0.8950
Gumbel	Normal	0.0000	0.9930	0.0000
	Student-t	0.1000	0.0000	0.0570
	Skewed-t	0.6750	0.4960	0.8840
BB1	Normal	0.7860	0.4850	0.7230
	Student-t	0.0000	0.8500	0.0000
	Skewed-t	0.7170	0.0000	0.8940
Gaussian	Normal	0.3770	0.6570	0.4500
	Student-t	0.0000	0.0000	0.0000
	Skewed-t	0.7870	0.0150	**0.9130**

Note: This table reports the p-values of the out-of-sample Hansen's (2005) SPA test with stationary bootstrapping. The number of resample is 1000. The sample block length for the stationary bootstrap is selected following Politis and White (2004). The p-values in bold indicate the most preferred model.

3.5 Summary and Conclusions

This chapter reports my research work which investigates the performances of the Archimedean-copula-based PVaR models relative to that of Gaussian copula-based PVaR model. I am particularly interested to see whether the former models have a superior out-of-sample forecasting ability over the latter. This question has received little attention in the literature where previous studies have been focused on these models' in-sample fitting performances. Exploring the research question, I employ Hansen's SPA test and Dawd's (2005b) discrete-sum approximation method, and adopt a multiple model comparison approach. The copula models studied include the Clayton, the Gumbel, the BB1 and the Gaussian copula, and the marginal functions are selected from the normal, the Student-t and the skewed-t distribution. In empirical investigation, I utilize data on three equity indexes (FTSE 100, Nikkei 225 and S&P 500).

My study has reached four main conclusions. First, in most cases, the Archimedean-copula-based PVaR model, especially the Clayton-copula-based model, have better forecasting performance than the Gaussian-copula-based PVaR model, for both in-sample fitting and out-of-sample forecasting. Second, the discrepancy between the in-sample and out-of-sample evaluation results of the copula-based PVaR models suggests that in-sample analysis should be supplemented by out-of-sample evaluation. Third, it is circumspect to check on the specifications of copula and marginal functions simultaneously rather than separately, as otherwise some potential good candidate models would be omitted. Fourth, the data-snooping check is important in determining an appropriate model to use, and simply relying on the minimum value of PVaR for model selection may be a hazard to risk management.

Several other issues deserve further investigation and may include the following. In this study, I use equally weighted portfolios for examining the copula-based PVaR models. Since changing portfolio weights will affect the PVaR of a portfolio, it would also be interesting to evaluate the performances of the copula-based PVaR models over a set of portfolio weights, for the purpose of optimal asset allocation. Another research direction in evaluating the copula-based PVaR models may be to consider *conditional* higher-order moments and conditional copulas. The former allows higher-order moments of marginal distributions to be time-varying (See, for example, Jondeau and Rockinger (2003)), and the latter allows the dependency parameters in a copula to be time-varying (See, for example, Patton (2006a), and Jondeau and Rockinger (2006)). Incorporating conditional higher-order moments and conditional copulas may further improve the accuracy of the PVaR estimates/forecasts.

Appendix 3.1

Dowd's (2005b) Discrete-Sum Approximation Method

The details of Dowd's (2005b) discrete-sum approximation method are as follows. Based on a bivariate copula function $F(z_1, z_2) = C(F_1(Z_1 \le z_1),\ F_2(Z_2 \le z_2))$ where Z_1 and Z_2 are two vectors of random variables, the probability of the worst case that the portfolio profit or loss $w \cdot Z_1 + (1-w) \cdot Z_2$ exceeds the VaR at $1-\xi$ confidence level can be expressed as

$$\mathbf{P}_{\{w \cdot Z_1 + (1-w) \cdot Z_2 \le \mathrm{VaR}_{\xi}\}} = \int_0^1 \partial_1 C(F_1(z), F_2(\mathrm{VaR}_{\xi} - z)) F_1'(z) dz \qquad \text{(A1)}$$

where ∂_1 refers to the partial derivative with respect to the first argument in the copula function.

Now denote $\varsigma = F_1(z)$, $d\varsigma = F_1'(z)dz$ and $z = F_1^{-1}(\varsigma)$ where $\varsigma \in (0,\ 1)$ is taken in a specific increment size $\Delta\varsigma$ (e.g. 0.001). Then the right-hand side of Equation (A1) can be easily approximated as

$$\mathbf{P}_{\{w \cdot Z_1 + (1-w) \cdot Z_2 \le \mathrm{VaR}_{\xi}\}} = \int_0^1 \partial_1 C(\varsigma, F_2(\mathrm{VaR}_{\xi} - F_1^{-1}(\varsigma))) d\varsigma$$

$$\approx \sum_{\varsigma \in (0,1)} [C(\varsigma, F_2(\mathrm{VaR}_{\xi} - F_1^{-1}(\varsigma))) - C((\varsigma - \Delta\varsigma), F_2(\mathrm{VaR}_{\xi} - F_1^{-1}(\varsigma)))]\Delta\varsigma \qquad \text{(A2)}$$

Then the portfolio VaR can be worked out by a suitable root finding algorithm, such as bisection algorithm.

Chapter 4 A test for evaluating the Archimedean-copula-based multivariate density forecasts in foreign exchange markets

4.1 Introduction

While Chapter 3 was devoted to PVaR forecast evaluation, this chapter turns to multivariate density forecast[16] evaluation. Specifically, I propose a full ranking statistical test to evaluate non-Gaussian multivariate density forecasts via the Kullback-Leibler information criterion (KLIC).

4.1.1 Motivations

The motivation of this study is threefold. First, pricing contingent claims on multiple assets and estimating portfolio VaR are two issues of great concern to financial practitioners and academic researchers. These two issues require an understanding of the joint distribution of returns on several assets in a portfolio.[17] It has been widely reported that this joint distribution is often non-Gaussian, implying that lower-order moments, such as covariance, carry very limited information about the full dependence structure of asset returns. In other words, forecasts of lower-order moments can only provide partial and hence inaccurate information about the future comovements of these returns. Therefore, forecasting the non-Gaussian multivariate density is

[16] The density forecast is also called the predictive density. In this essay, the two terms are used interchangeably.
[17] The theoretical and empirical studies of the pricing contingent claims on multiple assets related to the Archimedean-copula-based multivariate density can be found in Cherubini and Luciano (2002), Rosenberg (2003), Cherubini et al (2004), and van den Goorbergh et al (2005), among others.

called for. This need catalyzes the need of having a test for evaluating non-Gaussian multivariate density forecasts.

Second, as will reviewed later on, the existing studies on density forecast evaluation have left some voids for further research to fill: They only focused on the Gaussian multivariate density and they did not investigate higher-order moments of a multivariate density. My work thus attempts to contribute in these regards.

Third, in practice, many risk and portfolio managers are often interested in how a (parametric) model is compared with another, and thus need a statistical means to that end. It is therefore also the motivation of this study to provide financial practitioners with a testing procedure for comparing alternative models in terms of non-Gaussian multivariate density forecasting performance.

4.1.2 Literature review

Density forecast evaluation has been receiving increasing academic attention. A comprehensive survey can be found in Tay and Wallis (2000) and Corradi and Swanson (2006c). The main body of this literature is devoted to univariate density forecast evaluation which, based on the probability integral transform (PIT) (Rosenblatt, 1952), examines whether the transformed "generalized residuals"[18] are independent and identically distributed uniform over (0,1) (i.e., *i.i.d.* U(0, 1)). An *i.i.d.* U(0, 1) would imply that density forecasts have no deficiencies. In a pioneering study, Diebold *et al* (1998) observe that if the generalized residuals are *i.i.d.* U(0, 1), the histogram of the generalized residuals should be close to uniform. Accordingly, serial correlations in the correlograms of the power of the difference between the generalized residuals

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[18] The series of probability integral transform is actually the cumulative distribution function of random variables. Hong and Li (2005) and Egorov *et al* (2006) term the series as "generalized residuals".

and their mean should vanish. Although this less informal, graphical method is easy to implement, it does not provide guidance as to why a forecasting model is rejected.

To overcome the problem, Berkowitz (2001) proposes a likelihood ratio (LR) test as a formal testing procedure for density forecast evaluation. The LR test has been used by several researchers, including Sarno and Valente (2005), De Raaij and Raunig (2005) and Christoffersen and Mazzotta (2005). However, the LR test first transforms the generalized residuals to the standard normally distributed variables, and then imposes an autoregressive (AR) process on the transformed variables to test *i.i.d.* uniformity of the generalized residuals. As such, it may not have power against non-normal distributions of the generalized residuals that exhibit, for example, asymmetric dependence (Chen and Fan, 2004).

Because the non-normality of financial returns has been widely reported in empirical studies, there has appeared the need for developing appropriate techniques to evaluate non-Gaussian density forecasting models. Recently, several density forecast evaluation methods taking account of higher-order moments have emerged, including the parametric method via the copula approach (Chen and Fan, 2004)[19] and the nonparametric method (Hong *et al*, 2004, Hong and Li, 2005, and Egorov *et al*, 2006). In addition, a number of studies have extended Diebold *et al*'s (1998) idea by using the hybrid transformed PIT method to evaluate density forecasts. For example, Corradi and Swanson (2005, 2006a and 2006b) employ the conditional Komogorov test with the stationary bootstrap method. Mitchell and Hall (2005), Bao and Lee (2006) and Bao *et al* (2006) use the KLIC measure with the stationary bootstrap method.

There is another stream of the density-forecast-evaluation literature which does not use the PIT framework. Sarno and Valente (2004) and Li and Tkack (2006) propose to evaluate an

[19] An empirical study of Chen and Fan's (2004) method for the univariate density forecast evaluation is found in Li and Xu (2008).

individual density forecast based on its (integrated squared) distance from a nonparametric estimate of the density function. Similarly, based on the KLIC distance measure, Zheng (2000) develops a consistent test for a parametric density function compared with a empirical kernel density estimators, and Fan *et al* (2006) improve Zheng's (2000) work by a kernel-based bootstrap test. Finally, likelihood ratio test is also a useful tool outside the PIT framework, and a weighted version of Vuong's (1989) likelihood ratio test can be found in Amisano and Giacomini (2007).

While univariate density forecast evaluation has been extensively studied, similar studies for multivariate density forecast evaluation are relatively fewer. The few studies include Diebold *et al* (1999) and Clements and Smith (2002). The method proposed in Diebold *et al* (1999) is graphical, and the procedure suggested by Clements and Smith (2002) involve Monte Carlo comparisons based on both the PIT and the Komogorov-Smirnov test. However, two more important points to note are that these studies only focus on Gaussian multivariate density forecast evaluation, and the issue of data reuse in the context of multivariate density forecast evaluation has never been addressed.

4.1.3 Design of the test procedure

In view of multivariate density forecast evaluation being still in a nascent condition, I opt to undertake research on this issue. As exhibited in Figure 4.1, my objective is to choose a multivariate predictive density that provides the most accurate out-of-sample approximation of the true multivariate density, given multiple multivariate predictive densities. As mentioned earlier, in empirical applications of the multivariate predictive density, it is more common to consider several parametric models to fit data and compare the results obtained from different models. To address the model selection issue in this case, I extend Vuong's (1989) likelihood ratio test and advance from a test for two competing models to a test for more than two competing models along the lines of Hansen's (2005) SPA test.[20] Hence, unlike Vuong (1989) and Rivers and Vuong (2002), the null hypothesis I entertain in my test procedure is that the benchmark is not inferior to any of other alternative models in terms of the KLIC, while in Vuong (1989) and Rivers and Vuong (2002), the null hypothesis is that the two models perform equally well.

[20] Chen and Fan (2005, 2006b) propose a similar test procedure along the line of White's (2000) reality check test for semiparametric copula model selection, and name the modified likelihood ratio test as *pseudo* likelihood ratio test. My test procedure is different from theirs in two aspects. First, it is applied to the full parametric copula-based multivariate density forecast evaluation. Second, it uses Hansen's (2005) SPA test to deal with the data snooping problem.

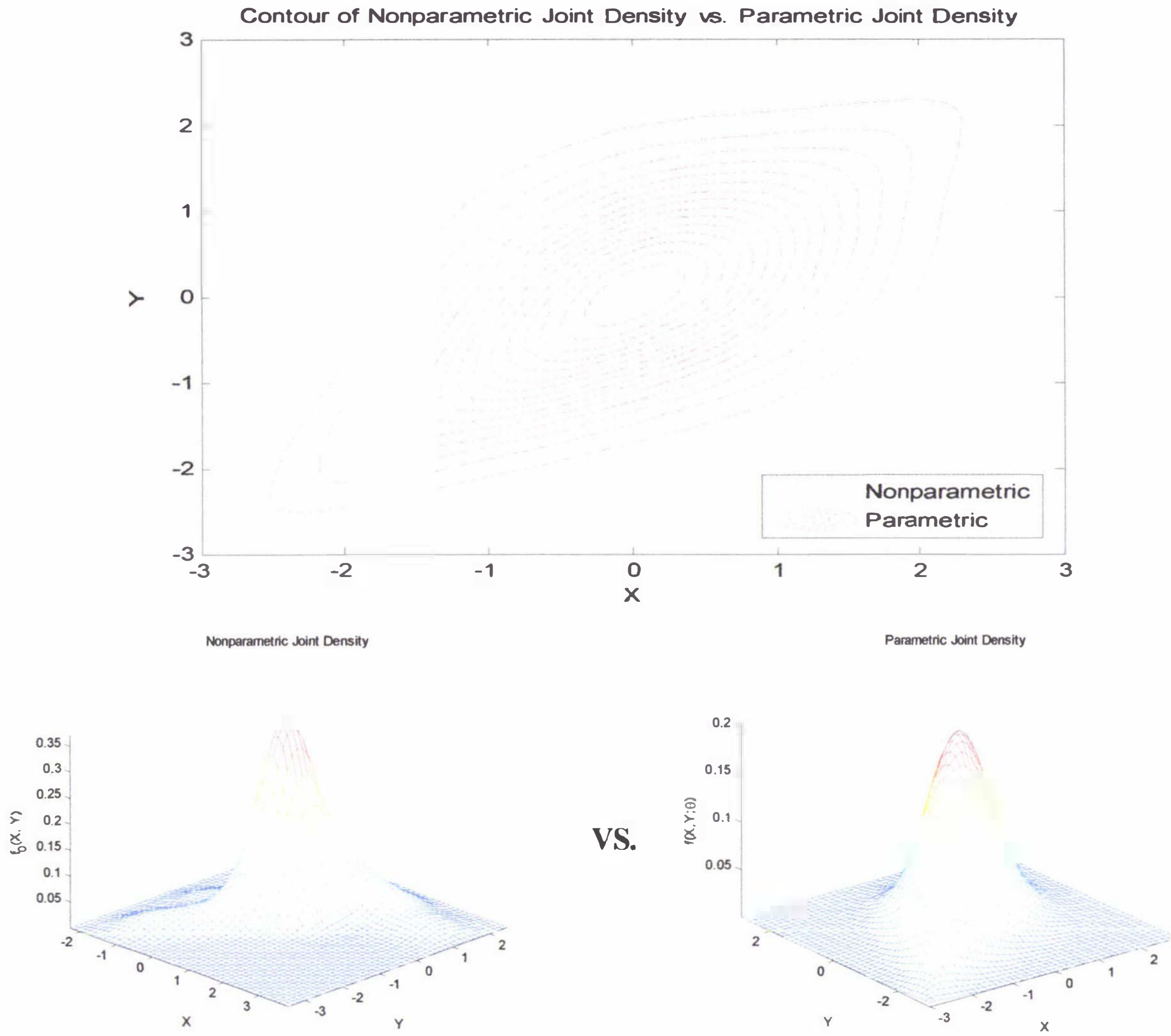

Figure 4.1

Note: The first row exhibits how the parametric joint density departs from the true (nonparametric) joint density. The second row shows the surfaces of the nonparametric and parametric joint densities.

My testing procedure is in a similar spirit of Zheng (2000), but extends that work to the multivariate case along the following four steps shown in Figure 4.2:

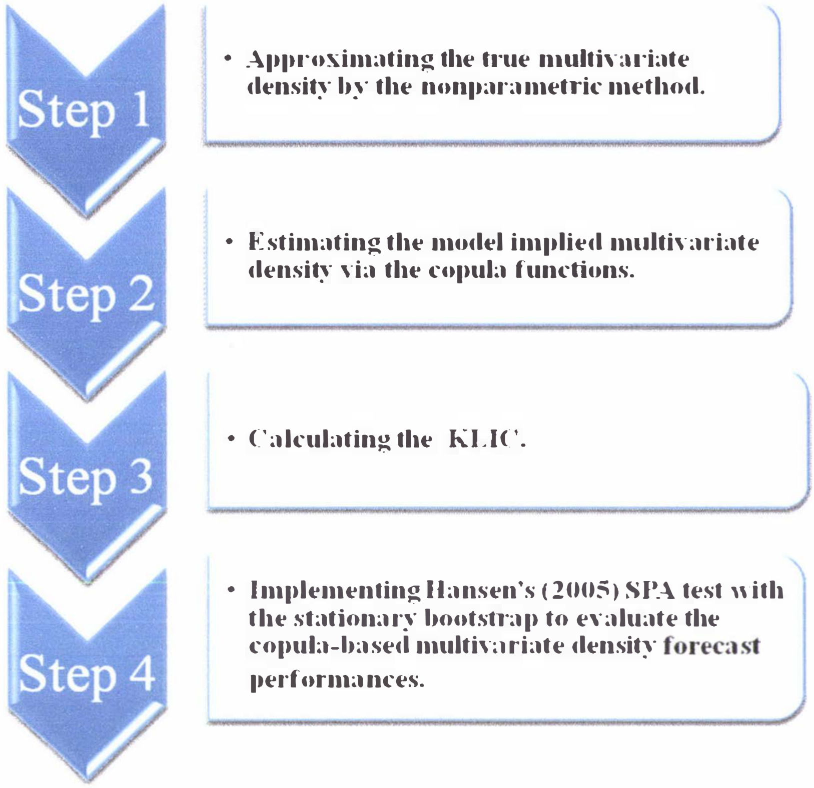

Figure 4.2

Note: This figure shows the test procedure of the copula-based multivariate density forecast evaluation.

First, I estimate the true, but unknown, multivariate density empirically by the nonparametric kernel estimator. Then I allow the *parametric* multivariate density construction to go beyond the normality framework by using the Archimedean copula functions. Specifically, I consider five Archimedean copulas (the Clayton, the Frank, the Gumbel, the BB1, and the BB7 copulas). These copulas have often been employed to capture the higher-order moments of portfolio returns in empirical studies. See Ané and Kharoubi (2003), Patton (2006a, 2006b), and Rosenberg and Schuermann (2006) among others. For the sake of multiple model comparisons, I also include two elliptical copulas (the Gaussian, and the *t* copulas) in the set of candidate copula models. As in Chapter 3, the candidate distribution functions considered for marginals are the standard normal, the Student-*t*, and the Hansen's (1994) skewed-*t* distribution. To avoid the potential "preselection omission" problem mentioned in Chapter 3, however, I subject all possible copula-marginal complexes to tests. Therefore, the advantages of my method are that the multiple comparisons of copula-based multivariate density (CBMD) models not only allow for the marginal and copula models to be misspecified under both the null and the alternative hypothesis, but also enable me to see how different marginals chosen may affect multivariate density forecasts.

In the third step, I calculate the KLIC 'distance' between the nonparametric multivariate density and the copula-model-implied multivariate density.[21] Finally, to address the data snooping bias, I perform Hansen's (2005) superior predictive ability (SPA) test in multiple model comparisons to evaluate out-of-sample forecasting as well as in-sample model fitting. The details of the data snooping problem and the SPA test were already given in Chapter 3.

In a word, this research plans to provide, to my best knowledge, the first-ever full ranking statistical test for evaluating non-Gaussian multivariate density forecasts via the KLIC. This proposed test should be practically straightforward to use and implement by financial risk managers.

The rest of this chapter is organized as follows. Section 4.2 explains the KLIC for multivariate density forecasting. The copula-based multivariate density is described in Section 4.3. Section 4.4 is about the SPA test. Section 4.5 applies my proposed testing procedure to the exchange rate data and discusses the results. Section 4.6 discusses the economic value of multivariate density forecast evaluation via a numerical example of multivariate option pricing. Concluding remarks are made in Section 4.7.

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[21] Strictly speaking, the KLIC is not a metric (distance function), i.e. $\mathrm{KLIC}(f_1{:}f_2) \neq \mathrm{KLIC}(f_2{:}f_1)$, as noted in White (1994, p9). However, Bao et al (2007) demonstrate that the KLIC could be used as if it is a metric as long as the benchmark is fixed and all other alternative models are compared against the benchmark.

4.2 KLIC for Multivariate Density Forecast Model

This section explains the KLIC which will be used as a loss function in comparing the forecasting performances of competing multivariate density models. Throughout this chapter, I denote the *cdf* of a random variable by an uppercase letter and the corresponding *pdf* by a lowercase letter.

Consider a time series $\{y_t\}_{t=1}^{n}$ which is governed by a true but unknown density function $f_0(y_t)$. Since $f_0(y_t)$ is unknown, it may be characterized by a one-step-ahead density forecast model with a finite-dimensional vector of parameters $\boldsymbol{\varphi} \in \boldsymbol{\Theta}$, where $\boldsymbol{\Theta}$ is a compact parameter subset of the real line $\mathbf{R}$:

$$f_t(y_t;\boldsymbol{\varphi}) = \partial\, \mathbf{P}(y_t \leq y;\boldsymbol{\varphi})/\partial y \qquad (4.1)$$

where $\mathbf{P}$ denotes probability. According to Vuong (1989), the univariate KLIC distance measure between the true density $f_0(y_t)$ and the model-implied density $f_t(y_t;\boldsymbol{\varphi})$ is defined as:

$$KLIC_t^U(f_0 : f_t;\boldsymbol{\varphi}) = \int f_0(y_t)\log\left[\frac{f_0(y_t)}{f_t(y_t;\boldsymbol{\varphi})}\right]\mathrm{d}y_t = \mathbf{E}\;[\log f_0(y_t) - \log f_t(y_t;\boldsymbol{\varphi})] \qquad (4.2)$$

where $\mathbf{E}$ [·] denotes the expectation operator. Equation (4.2) is referred to as the univariate loss function. The smaller the loss, the better the model-implied density forecast.

I now extend the univariate KLIC measure to the m-dimensional multivariate case. For a true multivariate density $f_0(y_{1,t}, y_{2,t}, \ldots, y_{m,t})$ and a given model-implied multivariate density $f_t(y_{1,t}, y_{2,t}, \ldots, y_{m,t};\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a finite-dimensional vector of parameters assumed to be identified on $\boldsymbol{\Theta}$, the multivariate KLIC becomes:

$$KLIC_t^M(f_0 : f_t;\boldsymbol{\theta}) = \int_{\mathbf{R}^m} f_0(y_{1,t}, y_{2,t}, \ldots, y_{m,t}) \log\left[\frac{f_0(y_{1,t}, y_{2,t}, \ldots, y_{m,t})}{f_t(y_{1,t}, y_{2,t}, \ldots, y_{m,t};\boldsymbol{\theta})}\right] \mathrm{d}y_{1,t}\mathrm{d}y_{2,t}\ldots\mathrm{d}y_{m,t}$$

$$= \mathbf{E}\ [\log f_0(y_{1,t}, y_{2,t}, \ldots, y_{m,t}) - \log f_t(y_{1,t}, y_{2,t}, \ldots, y_{m,t};\boldsymbol{\theta})] \qquad (4.3)$$

Equation (4.3) may be referred to as the multivariate loss function. The test of interest is to see whether the multivariate loss function equals zero. Under the regularity conditions, Equation (4.3) can be consistently estimated by

$$\overline{KLIC}^M(f_0 : f_t;\boldsymbol{\theta}) = \frac{1}{n}\sum_{t=1}^{n}[\log f_0(y_{1,t}, y_{2,t}, \ldots, y_{m,t}) - \log f_t(y_{1,t}, y_{2,t}, \ldots, y_{m,t};\boldsymbol{\theta})] \qquad (4.4)$$

See Mitchell and Hall (2005), Bao and Lee (2006), and Bao et al (2007).

The model-implied multivariate density can be estimated via a specified copula function. The true multivariate density, however, can hardly be represented by a simple mathematical function. Thus, in the spirit of Zheng (2000)[22] and for the purpose of *multiple* comparisons of competing models, I use the nonparametric multivariate kernel estimator suggested by Scott (1992) to approximate the true multivariate density:[23]

$$\hat{f}_0(y_{1,t}, y_{2,t}, \ldots, y_{m,t}) = \frac{1}{n \cdot \varpi_1 \cdot \varpi_2 \cdot \ldots \cdot \varpi_m}\sum_{t=1}^{n}\left\{\prod_{i=1}^{m} K\left(\frac{y_i - y_{i,t}}{\varpi_i}\right)\right\} \qquad (4.5)$$

---

[22] A similar proposition of evaluating an individual model-implied density forecast based on its distance from a nonparametric estimate of the true density function can also be found in Li and Tkacz (2006) and Corradi and Swanson (2006b). In addition, a semiparametric estimate of the true density function for the univariate KLIC calculation can be found in Bao and Lee (2006), and Bao et al (2007).

[23] According to Vuong (1989) and Rivers and Vuong (2003), one may propose that, when comparing the performance of the $KLIC^M$ between *two* competing models, there is no need to estimate $f_0(y_{1,t}, y_{2,t}, \ldots, y_{m,t})$ nonparametrically. That is,

$KLIC^M(f_0: f_{1,t}; \theta_1) - KLIC^M(f_0: f_{2,t}; \theta_2)$

$= \mathbf{E}[\log f_0(y_{1,t}, y_{2,t}, \ldots, y_{m,t}) - \log f_{1,t}(y_{1,t}, y_{2,t}, \ldots, y_{m,t}; \theta_1)]$

$- \mathbf{E}\ [\log f_0(y_{1,t}, y_{2,t}, \ldots, y_{m,t}) - \log f_{2,t}(y_{1,t}, y_{2,t}, \ldots, y_{m,t}; \theta_2)]$

$= \mathbf{E}[\log f_0(y_{1,t}, y_{2,t}, \ldots, y_{m,t})] - \mathbf{E}\ [\log f_{1,t}(y_{1,t}, y_{2,t}, \ldots, y_{m,t}; \theta_1)]$

$- \mathbf{E}\ [\log f_0(y_{1,t}, y_{2,t}, \ldots, y_{m,t}) + \mathbf{E}\ [\log f_{2,t}(y_{1,t}, y_{2,t}, \ldots, y_{m,t}; \theta_2)]$

$= \mathbf{E}\ [\log f_{2,t}(y_{1,t}, y_{2,t}, \ldots, y_{m,t}; \theta_2)] - \mathbf{E}\ [\log f_{1,t}(y_{1,t}, y_{2,t}, \ldots, y_{m,t}; \theta_1)]$

$= \mathbf{E}\ [\log f_{2,t}(y_{1,t}, y_{2,t}, \ldots, y_{m,t}; \theta_2) - \log f_{1,t}(y_{1,t}, y_{2,t}, \ldots, y_{m,t}; \theta_1)]$

However, as summarized in Rivers and Vuong (2003), this proposition is only applicable to the *pairwise* comparison, and not to the *multiple* comparison, of competing models.

where $K(\cdot)$ denotes the Gaussian kernel, ϖ_i is the kernel bandwidth for each kernel density, and y_i is the domain which is a set of points around $y_{i,t}$. According to Scott (1992), the kernel bandwidth is calculated as $\hat{\varpi}_i = [4/(m+2)]^{1/(m+4)} \cdot \hat{\sigma}_i \cdot n^{-1/(m+4)}$ (i = 1, 2,..., m) where $\hat{\sigma}_i$ is the standard deviation of $y_{i,t}$.

4.3 Constructing the Copula-Based Multivariate Density

Now turn to the model-implied multivariate density. It is constructed using the copula function. Recall from Chapter 2 that the copula-based multivariate density is:

$$f_t(y_{1,t}, y_{2,t},..., y_{m,t};\boldsymbol{\theta}) = f_1(y_{1,t};\theta_{M1}) \times f_2(y_{2,t};\theta_{M2}) \times ... \times f_m(y_{m,t};\theta_{Mm})$$

$$\times c[F_1(y_{1,t};\theta_{M1}), F_2(y_{2,t};\theta_{M2}),..., F_m(y_{m,t};\theta_{Mm});\theta_C] \tag{4.6}$$

where $f_i(y_{i,t};\theta_{Mi})$ is the density function corresponding to the marginal distribution $F_i(y_{i,t};\theta_{Mi})$ (i = 1, 2,..., m),

$$c[F_1(y_{1,t};\theta_{M1}), F_2(y_{2,t};\theta_{M2}),..., F_m(y_{m,t};\theta_{Mm});\theta_C] = \frac{\partial^m C[F_1(y_{1,t};\theta_{M1}), F_2(y_{2,t};\theta_{M2}),..., F_m(y_{m,t};\theta_{Mm});\theta_C]}{\partial F_1(y_{1,t};\theta_{M1})\partial F_2(y_{2,t};\theta_{M2})...\partial F_m(y_{m,t};\theta_{Mm})} \tag{4.7}$$

is the density of a copula, and $\boldsymbol{\theta}$ is a vector containing the parameters θ_{Mi} of marginal models and the parameters θ_C of copula model, i.e. $\boldsymbol{\theta} = \{\theta_{Mi}, \theta_C\}, \boldsymbol{\theta} \in \boldsymbol{\Theta}$.

Let $u = F_1(y_{1,t};\theta_{M1})$ and $\upsilon = F_2(y_{2,t};\theta_{M2})$ be two marginals in a bivariate copula. In this study, I consider five bivariate Archimedean copulas (the Clayton, the Frank, the Gumbel, the BB1, and the BB7 copula) and two bivariate elliptical copulas (the Gaussian, and the *t* copulas) for constructing the model-implied multivariate density. For the general definitions of the Archimedean and the elliptical copulas, one is referred to Chapter 2 (or to Nelsen (1999) and

Embrechts et al (2003)). The functional forms of the seven bivariate copulas and their corresponding densities are displayed in Table 2.1 of Chapter 2. That table also summarizes the tail dependence structures of these copulas. An attractiveness of Archimedean copulas is that they are able to capture coskewness, cokurtosis and nonlinear dependence simultaneously, whereas elliptical copulas cannot capture these features of a non-Gaussian multivariate distribution. It is worth noting that the Frank copula is not a regular elliptical copula although it is symmetric without tail dependence.

4.4 Hansen's (2005) SPA Test

The details of the SPA test have been introduced in Chapter 3. To avoid unnecessary repetition, I only mention the differences here. The loss function is now evaluated by the multivariate KLIC. For the benchmark model it is $L_0 = KLIC_{0,t}^{M}$, and for the candidate model k (k = 1,..., g) it is $L_k = KLIC_{k,t}^{M}$. Then $d_{k,t}$, the forecasting performance of the candidate model k relative to the benchmark model at time t, is defined as $d_{k,t} \equiv L_0 - L_k = KLIC_{0,t}^{M} - KLIC_{k,t}^{M}$. Now stacking $d_{k,t}$ into a vector of relative performance, $\mathbf{d}_t = (d_{1,t}, d_{2,t}, \ldots, d_{g,t})'$, we can formulate the null hypothesis of interest as

$$H_0: \max_{k=1,\ldots,g} \mathbf{E}(\mathbf{d}_t) \leq 0 \qquad (4.8)$$

Therefore, when we compare multiple models against a benchmark jointly, the null hypothesis that the benchmark CBMD model is superior to other CBMD models is rejected by small bootstrapped p-values.[24] Note that the remaining part of the SPA test is the same as that in Section 3.3 of Chapter 3.

4.5 Empirical Results

4.5.1 Data

I obtained, from Datastream, data on the daily spot exchange rates of four currencies: the Australia dollar (AD), the British pound (BP), the Japanese yen (JY) and the Swiss franc (SF), all against the US dollar. The reason for choosing these currencies is that they are highly liquid in foreign exchange markets and have been frequently studied in empirical researches. The sample period spans from 01 March, 1986 to 31 October, 2005, a total of 5172 observations. However, I split the whole sample into two equal sub-samples in empirical investigation. The first half, called the training set, is reserved for model estimation and in-sample fitting evaluation. The second half, called the test set, is used for out-of-sample density forecast evaluation based on the estimated parameters from the training set. Exchange rate returns are calculated as $y_t = 100\log(X_t / X_{t-1})$, where X_t denotes the exchange rate at time t.

The results of preliminary statistic analysis of the four exchange rate return series for both the in-sample and out-of-sample periods are provided in Panel A of Table 4.1. The four series have common statistical profiles in higher order moments: All of them are asymmetrically distributed with heavy tails in each sub-period. Negative skewness is found for the AD rate in the

[24] Chen and Fan (2005) point out that some commonly used parametric classes of copulas, such as the Clayton, the Frank, the Gumbel and the Gaussian copula, are generalized nested only when the closest member to the true copula in each class is the independence copula (product copula). Otherwise, they are generalized non-nested. For the elliptical copulas, such as the Gaussian and the t copula, they are generalized nested if the closest member to the true copula in the t copula family is the one with an infinite degree of freedom (i.e. a Gaussian copula). Meanwhile, Chen and Fan (2005, 2006b) theoretically demonstrate that White's (2000) reality check test is suited to both generalized (footnote 20 continued) non-nested and generalized nested copula-based models when doing the multiple model comparison based on the KLIC. Since Hansen's (2005) SPA test is an extension of White's (2000) reality check test, the implementation of the SPA test in my test procedure is therefore valid.

in-sample period, and for the BP rate in both sub-periods. The LM statistics of Engle (1982) with up to 10 lags reveal the presence of ARCH effects in each series. In addition, linear and rank correlations between these return series in the two sub-periods are given in Panel B of Table 4.1. The significant rank correlation estimates, including Spearman's ρ and Kendall's τ, provide evidence that the joint distribution for every pair of exchange rate returns is non-Gaussian.

Table 4.1: Summary Statistics of Daily Exchange Rate Returns

	In-Sample (03 January 1986 – 01 December 1995)				Out-of-Sample (04 December 1995 – 31 October 2005)			
	AD	BP	JY	SF	AD	BP	JY	SF
	Panel A. Preliminary Analysis							
Observations	2585	2585	2585	2585	2585	2585	2585	2585
Maximum	4.8502	3.4090	5.4217	5.3882	5.1682	2.0553	7.6773	3.0210
Minimum	-3.9505	-4.2758	-5.8305	-4.8500	-3.4808	-2.1953	-3.9549	-2.4522
Mean	0.0031	0.0024	0.0268	0.0218	0.0004	0.0055	-0.0054	-0.0038
Std.	0.6072	0.7057	0.7306	0.8317	0.6857	0.4995	0.6913	0.6700
Skewness	-0.6768	-0.2414	0.3433	0.0519	0.1187	-0.0619	0.8395	0.1750
Kurtosis	8.3993	5.7572	9.8349	6.0066	6.2006	4.0360	11.7813	4.1664
Engle (10)	**222.0105**	**98.8335**	**78.4050**	**113.1627**	**40.5934**	**43.6706**	**88.1183**	**25.0135**
	Panel B. Linear and Rank Correlations							
Pearson's ρ_P								
AD	—	**0.1791**	**0.0647**	**0.0884**	—	**0.3385**	**0.3031**	**0.3346**
BP		—	**0.5123**	**0.7179**		—	**0.2693**	**0.6004**
JY			—	**0.6120**			—	**0.3722**
SF				—				—
Spearman's ρ_S								
AD	—	**0.1749**	**0.0890**	**0.1024**	—	**0.3252**	**0.2732**	**0.3312**
BP		—	**0.5233**	**0.7176**		—	**0.2885**	**0.5886**
JY			—	**0.6098**			—	**0.3830**
SF				—				—

Kendall's τ_K								
AD	—	**0.1198**	**0.0622**	**0.0713**	—	**0.2261**	**0.1896**	**0.2324**
BP		—	**0.3744**	**0.5425**		—	**0.1987**	**0.4236**
JY			—	**0.4453**			—	**0.2668**
SF				—				—

Note: The exchange rate returns are calculated as $100\times\log(X_t/X_{t-1})$. AD, BP, JY and SF denote, respectively, the exchange rate of the Australian dollar, the British pound, the Japanese yen and the Swiss franc, against the US dollar. Engle (10) represents the LM test of Engle (1982) using 10 lags for the presence of ARCH effects. The critical value of the LM test is 18.307 at the 5% level. The linear correlation coefficient ρ_P is calculated as $\rho_P = \dfrac{\sigma_{xy}}{\sigma_x\sigma_y}$. The rank correlation coefficients ρ_S and τ_K are calculated as $\rho_S = \dfrac{12}{n(n^2-1)}\sum_{i=1}^{n}\left(rank(x_i)-\dfrac{n+1}{2}\right)\left(rank(y_i)-\dfrac{n+1}{2}\right)$ and $\tau_K = \binom{n}{2}^{-1}\sum_{1\le i<j\le n} sign[(x_i - x_j)(y_i - y_j)]$ respectively. Figures in bold type indicate significance at the 5% level.

4.5.2 In-sample ML estimation

I use the two-stage full parametric ML method with the fixed scheme (which has been mentioned in section 3.4.1 in Chapter 3) to estimate the model parameters in $\boldsymbol{\theta}$. I first model the single exchange rate returns by an AR(1)-GARCH (1, 1) process. This popular method has been employed in many empirical studies on foreign exchange markets. See Hsieh (1988, 1989a, 1989b, 1993) for the univariate case, and Chen and Fan (2006b) and Patton (2006a, 2006b) for the multivariate case.

Regarding the first stage of model estimation, an AR(1)–GARCH(1, 1) process in Equation (4.9) below is specified to obtain standardized returns z_t:

$$\begin{aligned} y_t &= \mu_t + e_t \\ \mu_t &= \omega + \phi \cdot y_{t-1} \\ e_t &= \sqrt{h_t} \cdot z_t \\ h_t &= a_0 + b_0 e_{t-1}^2 + c_0 h_{t-1} \end{aligned} \qquad (4.9)$$

For the conditional distribution of z_t, I consider three distributional restrictions as possible candidates: the standard normal distribution, the Student-t distribution, and Hansen's (1994) skewed-t distribution. The details of the *pdf* and *cdf* of Hansen's (1994) skewed-t distribution were already provided in Chapter 3, and so are not repeated here. The *cdf* of each restriction will be then used as marginal distributions for copula estimation in the second stage.

Table 4.2 presents the in-sample ML estimation results of the marginal parameters θ_{M_i} in the AR(1)–GARCH(1, 1) model with different distributions of standardized returns. Similar to Chen and Fan (2006b) and Patton (2006a, 2006b), the parameters (ω and ϕ) in the mean equation are not statistically significant in most cases, especially for the BP and the SF rate. This indicates that the exchange rate changes cannot be directly predicted using the mean equation only. However, the parameters in the variance equation are significant at the 5% level, exhibiting the presence of conditional heteroscedasticity, in all cases. The small values of the unconditional kurtosis parameter η for both the Student-t and the skewed-t distribution are also significant, suggesting that the four exchange rate returns follow heavy-tailed distributions. The estimation results of the unconditional skewness parameter λ for the skewed-t distribution, on the other hand, are mixed. As far as the in-sample period is concerned, strong evidence of negative and positive skewness is found for the AD and the JY rate respectively. In contrast, such evidence is not present for the BP and the SF rate, indicating that they are symmetrically distributed. Meanwhile, to investigate the adequateness of the marginal models, I perform the goodness-of-fit test proposed by Patton (2006a, 2006b) and Jondeau and Rockinger (2006) for the three distributions considered. The details of the test have been described in Chapter 3. As shown in Table 4.3, the test results are mixed for the LM test. The best marginal model fitting is found in the SF rate, followed by the JY and the AD rate. For the SF rate, all the marginal models pass the test, confirming that there are three possible distributions for it. For the BP rate in most cases, the second and the fourth centered moments seem to be non *i.i.d.*[25] In despite of the mixed results mentioned above, the K-S test results of all three different marginal distributions for each series consistently accept the null that all marginals are uniformly distributed. Since my test procedure is designed for the presence of model misspecification, I then retain all the three marginal restrictions at the second stage of the two-stage MLE of all candidate copulas, and assess all CBMD model performances in the final step.

[25] For modeling the BP return series, I have tried to add the lags up to 10 in the mean equation, i.e. AR(10), for different marginal distributions. However, there is no improvement at all in those model performances.

Table 4.2: In-Sample ML Estimation of the AR (1)-GARCH (1,1) Model with Different Distributions of Standardized Errors

Parameter	AD			BP		
	Normal	Student-t	Skewed-t	Normal	Student-t	Skewed-t
ω	0.0157	**0.0371**	-0.0144	0.0138	0.0208	0.0077
	(0.0110)	(0.0090)	(0.0213)	(0.0124)	(0.0113)	(0.0204)
ϕ	0.0044	-0.0198	-0.0217	-0.0149	-0.0317	-0.0323
	(0.0212)	(0.0184)	(0.0187)	(0.0215)	(0.0184)	(0.0183)
a_0	**0.0266**	**0.0153**	**0.0135**	**0.0064**	**0.0051**	**0.0049**
	(0.0033)	(0.0043)	(0.0033)	(0.0013)	(0.0023)	(0.0017)
b_0	**0.1036**	**0.1054**	**0.0999**	**0.0378**	**0.0432**	**0.0427**
	(0.0092)	(0.0198)	(0.0215)	(0.0036)	(0.0085)	(0.0055)
c_0	**0.8259**	**0.8771**	**0.8821**	**0.9494**	**0.9505**	**0.9512**
	(0.0151)	(0.0195)	(0.0061)	(0.0051)	(0.0094)	(0.0008)
η		**3.3698**	**3.4475**		**4.1125**	**4.1314**
		(0.2945)	(0.6074)		(0.4227)	(0.4939)
λ			**-0.0609**			-0.0177
			(0.0214)			(0.0228)
Loglik	-2232	-2059	-2054	-2664	-2554	-2553
	JY			SF		
	Normal	Student-t	Skewed-t	Normal	Student-t	Skewed-t
ω	0.0212	0.0023	0.0354	0.0207	0.0139	0.0278
	(0.0137)	(0.0108)	(0.0183)	(0.0153)	(0.0143)	(0.0195)
ϕ	-0.0187	**-0.0477**	**-0.0466**	-0.0165	-0.0183	-0.0192
	(0.0203)	(0.0172)	(0.0174)	(0.0200)	(0.0187)	(0.0184)
a_0	**0.0463**	**0.0287**	**0.0266**	**0.0188**	**0.0218**	**0.0218**
	(0.0056)	(0.0102)	(0.0022)	(0.0032)	(0.0091)	(0.0038)
b_0	**0.0749**	**0.0757**	**0.0685**	**0.0354**	**0.0315**	**0.0315**
	(0.0058)	(0.0205)	(0.0072)	(0.0047)	(0.0087)	(0.0061)
c_0	**0.8405**	**0.9065**	**0.9043**	**0.9374**	**0.9392**	**0.9391**
	(0.0138)	(0.0211)	(0.0043)	(0.0075)	(0.0184)	(0.0016)
η		**2.8540**	**3.0691**		**4.5894**	**4.5998**
		(0.2293)	(0.4061)		(0.5060)	(0.5573)
λ			**0.0487**			0.0245
			(0.0230)			(0.0238)
Loglik	-2787	-2559	-2557	-3138	-3048	-3045

Note: This table contains the ML estimation results for marginals with different distributional restrictions. The specified models are given by Equation (4.9). Loglike is the log likelihood function. Standard errors are given in parentheses. Figures in bold indicate significance at the 5% level. AD, BP, JY and SF denote, respectively, the exchange rate of the Australian dollar, the British pound, the Japanese yen and the Swiss franc, against the US dollar.

Table 4.3: Goodness-of-Fit Tests for Different Distributional Restrictions

	AD	BP	JY	SF
Panel A: Normal distribution				
1st Moment LM Test	0.06	0.11	0.06	0.46
2nd Moment LM Test	0.27	0.002*	0.56	0.15
3rd Moment LM Test	0.61	0.37	0.08	0.78
4th Moment LM Test	0.07	0.01*	0.39	0.24
K-S(20)	0.62	0.65	0.62	0.64
Panel B: Student-t distribution				
1st Moment LM Test	0.05	0.10	0.03*	0.49
2nd Moment LM Test	0.44	0.004*	0.53	0.12
3rd Moment LM Test	0.78	0.34	0.05	0.83
4th Moment LM Test	0.04*	0.05	0.12	0.20
K-S(20)	0.66	0.68	0.67	0.66
Panel C: Skewed-t distribution				
1st Moment LM Test	0.04*	0.08	0.05	0.34
2nd Moment LM Test	0.76	0.002*	0.56	0.12
3rd Moment LM Test	0.40	0.24	0.07	0.78
4th Moment LM Test	0.25	0.01*	0.67	0.13
K-S(20)	0.63	0.66	0.63	0.65

Note: This table reports *p*-values of the goodness-of-fit test statistics for the normal, the Student-*t*, and the skewed-*t* distributional restrictions of the general univariate model. The test contains two parts. The first part of the test which is labeled as *k*th Moment LM Test (k = 1,..., 4) in each panel evaluates whether the *k*th centered moments of the transformed residuals u_t are serially correlated. I regress $(u_t - \bar{u}_t)^k$ on 20 own lags. Under the null of no autocorrelation of the residuals, the statistics (n-20) R^2 is distributed as a χ^2_{20} where n and R^2 are the sample size and the coefficient of determination of the regression. The second part of the test which is labeled as K-S(20) in each panel employs Kolmogorov-Smirnov test within each bin to evaluate whether u_t are Uniform(0, 1), where the total number of the bin is 20. Under the null, the statistics is also distributed as a χ^2_{20}. * indicates significance at the 5% level. AD, BP, JY and SF denote, respectively, the exchange rate of the Australian dollar, the British pound, the Japanese yen and the Swiss franc, against the US dollar.

The second-stage estimation concerns the copula parameters θ_C in the seven copula models under investigation. I let each of them connect, respectively, the normal, the Student-t, and the skewed-t marginals. Table 4.4 reports the second-stage estimation results of the copula parameters. All estimated parameters are significant at the 5% level. However, statistical significance of model parameters does not necessarily promise that the model will have good (or relatively better) density forecasting performance, especially in the presence of possible data snooping. One needs to employ Hansen's SPA test to evaluate the in-sample fitting and out-of-sample forecasting abilities of these twenty-seven multivariate density models, before drawing a conclusion.

Table 4.4: In-Sample ML Estimation of Copulas

Copula	Marginal	Exchange rate pair					
		AD-vs-BP	AD-vs-JY	AD-vs-SF	BP-vs-JY	BP-vs-SF	JY-vs-SF
		Archimedean Copula					
Clayton (α)	Normal	**0.1015**	**0.0575**	**0.0559**	**0.6420**	**1.3573**	**1.0485**
		(0.0170)	(0.0162)	(0.0159)	(0.0368)	(0.0492)	(0.0446)
	Student-t	**0.4497**	**0.5020**	**0.2966**	**1.6049**	**2.4097**	**2.0406**
		(0.0433)	(0.0518)	(0.0441)	(0.0578)	(0.0669)	(0.0631)
	Skewed-t	**0.1866**	**0.0994**	**0.0986**	**0.8398**	**1.5464**	**1.1538**
		(0.0251)	(0.0233)	(0.0231)	(0.0363)	(0.0481)	(0.0412)
Frank (α)	Normal	**1.4063**	**0.8019**	**0.7825**	**4.9260**	**7.7192**	**6.0211**
		(0.1462)	(0.1588)	(0.1474)	(0.1509)	(0.1763)	(0.1584)
	Student-t	**2.2361**	**2.2923**	**1.3114**	**6.3897**	**8.8011**	**7.3539**
		(0.1788)	(0.2148)	(0.1968)	(0.1659)	(0.1907)	(0.1732)
	Skewed-t	**1.0988**	**0.5641**	**0.6252**	**3.9966**	**6.7436**	**5.0131**
		(0.1218)	(0.1224)	(0.1218)	(0.1361)	(0.1643)	(0.1448)
Gumbel (α)	Normal	**1.1225**	**1.0453**	**1.0638**	**1.4929**	**2.1086**	**1.6410**
		(0.0168)	(0.0130)	(0.0141)	(0.0246)	(0.0351)	(0.0274)
	Student-t	**1.3562**	**1.3877**	**1.2295**	**2.1024**	**2.6721**	**2.2667**
		(0.0279)	(0.0315)	(0.0260)	(0.0380)	(0.0460)	(0.0402)
	Skewed-t	**1.1208**	**1.0703**	**1.0741**	**1.5562**	**2.1032**	**1.7343**
		(0.0150)	(0.0133)	(0.0134)	(0.0243)	(0.0342)	(0.0276)
BB1 (α, β)	Normal	**0.0527**	**0.0405**	**0.0293**	**0.2369**	**0.2603**	**0.4540**
		(0.0168)	(0.0165)	(0.0203)	(0.0333)	(0.0347)	(0.0083)
		1.0891	**1.0312**	**1.0506**	**1.3451**	**1.8756**	**1.3820**
		(0.0182)	(0.0127)	(0.0157)	(0.0282)	(0.0421)	(0.0230)
	Student-t	**0.1330**	**0.1745**	**0.0915**	**0.3922**	**0.5017**	**0.6299**
		(0.0462)	(0.0496)	(0.0460)	(0.0569)	(0.0581)	(0.0651)
		1.2913	**1.3113**	**1.1951**	**1.8089**	**2.2087**	**1.8073**

		(0.0342)	(0.0357)	(0.0302)	(0.0510)	(0.0618)	(0.0526)
	Skewed-t	**0.0897**	**0.0390**	**0.0330**	**0.3235**	**0.4322**	**0.4781**
		(0.0285)	(0.0254)	(0.0250)	(0.0406)	(0.0450)	(0.0452)
		1.0836	**1.0563**	**1.0620**	**1.3729**	**1.7783**	**1.4502**
		(0.0176)	(0.0154)	(0.0156)	(0.0296)	(0.0422)	(0.0329)
BB7 (α, β)	Normal	**1.0915**	**1.0279**	**1.0541**	**1.3090**	**1.9479**	**1.3216**
		(0.0223)	(0.0138)	(0.0444)	(0.0327)	(0.0536)	(0.0355)
		0.0834	**0.0517**	**0.0452**	**0.4619**	**0.8068**	**0.8177**
		(0.0165)	(0.0160)	(0.0168)	(0.0376)	(0.0536)	(0.0486)
	Student-t	**1.4070**	**1.4542**	**1.2808**	**2.0695**	**2.6403**	**2.0527**
		(0.0426)	(0.0446)	(0.0379)	(0.0630)	(0.0764)	(0.0669)
		0.3028	**0.3659**	**0.1967**	**1.1033**	**1.7217**	**1.5638**
		(0.0455)	(0.0607)	(0.0447)	(0.0715)	(0.0902)	(0.0806)
	Skewed-t	**1.1042**	**1.0743**	**1.0797**	**1.4733**	**2.0011**	**1.5782**
		(0.0222)	(0.0196)	(0.0197)	(0.0370)	(0.0523)	(0.0415)
		0.1341	**0.0644**	**0.0612**	**0.6027**	**1.0949**	**0.8824**
		(0.0262)	(0.0236)	(0.0233)	(0.0410)	(0.0566)	(0.0469)
				Elliptical Copula			
Gaussian (ρ)	Normal	**0.1792**	**0.0728**	**0.0934**	**0.5368**	**0.7305**	**0.6211**
		(0.0187)	(0.0195)	(0.0194)	(0.0123)	(0.0074)	(0.0103)
	Student-t	**0.4052**	**0.4161**	**0.2630**	**0.7437**	**0.8279**	**0.7846**
		(0.0224)	(0.0250)	(0.0314)	(0.0078)	(0.0050)	(0.0064)
	Skewed-t	**0.1783**	**0.0861**	**0.0993**	**0.5490**	**0.7343**	**0.6342**
		(0.0185)	(0.0193)	(0.0192)	(0.0120)	(0.0073)	(0.0099)
t (ρ, δ)	Normal	**0.1747**	**0.1034**	**0.1036**	**0.5518**	**0.7657**	**0.6432**
		(0.0230)	(0.0243)	(0.0238)	(0.0129)	(0.0081)	(0.0105)
		5.4962	**7.8888**	**4.9663**	**34.7561**	**5.6550**	**32.8901**
		(0.0974)	(0.1752)	(0.0422)	(0.6002)	(0.5881)	(0.5756)
	Student-t	**0.2825**	**0.2374**	**0.1609**	**0.7382**	**0.8364**	**0.7881**
		(0.0319)	(0.0504)	(0.0331)	(0.0114)	(0.0071)	(0.0084)
		2.4804	**2.0153**	**2.3596**	**3.9633**	**2.8791**	**4.6812**
		(0.2349)	(0.3552)	(0.1946)	(0.6325)	(0.3080)	(0.7124)

Skewed-*t*	**0.1789**	**0.0924**	**0.1018**	**0.5547**	**0.7482**	**0.6434**
	(0.0202)	(0.0217)	(0.0211)	(0.0144)	(0.0094)	(0.0123)
	9.2728	**5.4864**	**7.4827**	**5.1580**	**3.5400**	**4.3341**
	(2.0507)	(0.7469)	(1.3300)	(0.6677)	(0.3370)	(0.4764)

Note: This table shows the in-sample two-stage ML estimation results of the copula functions for each pair of exchange rate returns. The copula models are provided in Table 2.1 of Chapter 2. Standard errors are given in parentheses. Figures in bold indicate significance at the 5% level. AD, BP, JY and SF denote, respectively, the exchange rate of the Australian dollar, the British pound, the Japanese yen and the Swiss franc, against the US dollar.

4.5.3 In-sample CBMD model performance

Implementing the SPA test, I calculate the KLIC distance measure between the "true" multivariate density and the copula-based multivariate density for each pair of exchange rate returns. This yields 21 estimates of the KLIC distance measure for each pair as there are 21 candidate copulas.

Based on Equation (4.5), I approximate the true multivariate density by the nonparametric multivariate kernel estimator. As a result, Figures 4.3-4.6 display the three-dimensional plots and contours of the in-sample and out-of-sample nonparametric multivariate kernel densities for the 6 pairs of returns. These nonparametric multivariate kernel densities are taken to be the "true" densities where the functional forms of their marginal distributions are unknown.

When making multiple model comparisons via the SPA test, I take in turn each of the 21 models as a benchmark model and compare it with the remaining 20 models, in terms of the KLIC distance measure. The null hypothesis of the SPA test is that the best of the remaining 20 models is no better then the benchmark model. In estimating the bootstrap p-values of the SPA test statistic, the number of resampling is chosen to be 1000 and the random block size for stationary bootstrapping is estimated following Politis and White's (2004) suggestion. The decision rule for determining the "best" in-sample model fitting is based on the largest p-value.

Table 4.5 sets out the p-values of the in-sample SPA test statistic based on the KLIC loss function. For ease of exposition, I will use the name of the copula and the name of the marginal distributions when referring to a copula-marginal complex. For example, a "Clayton-normal copula" refers to the Clayton copula with the normal distribution being marginals. Now, according to the bootstrapped p-values, for the AD-vs-BP pair of exchange rate returns, the best in-sample multivariate density model fitting is found with the BB7-normal copula, as it has the largest p-value of 0.234. Thus, I cannot reject the null hypothesis that the other 20 copulas are no better than this one. Then, using the estimated parameters for the BB7-normal copula reported in Table 4.4, I obtain its lower and upper tail dependence coefficients as $\tau^L = 2^{-1/\beta} = 2^{-1/0.0834} =$ 0.0003 and $\tau^U = 2 - 2^{-1/\alpha} = 2^{-1/1.0915} = 0.1129$. This result implies that the bivariate distribution of the AD and BP rates is asymmetric, with upper tail dependence being much greater than lower tail dependence. By the same way of inference, the AD-vs-JY pair has its bivariate density best characterized by the Clayton-skewed-t copula with lower tail dependence $\tau^L = 2^{-1/\alpha} = 2^{-1/0.0994}$ = 0.0009 but no upper tail dependence. The remaining four pairs, AD-vs-SF, BP-vs-JY, BP-vs-SF and JY-vs-SF, all pick the Gumbel copula as their best in-sample bivariate density fitting model, albeit three with the normal and one with the Student-t distribution as marginals. There is therefore no lower but upper tail dependence for these four pairs in the in-sample period, and the upper tail dependence coefficients ($2 - 2^{1/\alpha}$) may be calculated as 0.0814 (the AD-vs-SF pair), 0.6095 (the BP-vs-JY pair), 0.6108 (the BP-vs-SF pair) and 0.4744 (the JY-vs-SF pair) respectively.

In addition, no Gaussian copula with any one of the three marginal distributions emerges as the best in-sample fitting models for the six bivariate densities. Note also that with the skewed-t distribution as marginals, the copula-based multivariate density models show the worst in-sample model fitting as the corresponding p-values are 0.000 in most cases.

In summary, the bootstrapped p-values demonstrate that the in-sample bivariate densities modeled by Archimedean copulas for the six pairs of exchange rate returns outperform those modeled by the Gaussian and t copulas.

4.5.4 Out-of-sample CBMD forecast evaluation

In this section, I discuss the results of the out-of-sample copula-based multivariate density forecast evaluation. In-sample analysis in the preceding section shows that the non-normality dependence structure of a multivariate density can be captured by Archimedean copulas. My interest here is to see whether Archimedean copulas will do the same good job for forecasting the possible non-Gaussian multivariate density of exchange rate returns in the out-of-sample period.

The evaluation results are reported in Table 4.6. For the AD-vs-JY pair, the Clayton copula-based density model consistently outperforms others for forecasting, although its marginals are different from those in the in-sample period. Interestingly, the remaining five pairs now favor the symmetric copula-based multivariate density models. For the AD-vs-SF pair, the *t* copula becomes the best bivariate density forecasting model, while for the AD-vs-BP, the BP-vs-JY, the BP-vs-SF and the JY-vs-SF pair, the Frank copula, constructed by either the normal or the Student-*t* or the skewed-*t* marginals, is the champion.

There exist discrepancies in the best copula-based multivariate density model performance between in-sample fitting and out-of-sample forecasting. This indicates that the data generating process might be different between the two sub-sample periods.

However, one thing in common is that Archimedean copulas dominate the Gaussian copula in modeling multivariate densities. Also consistent with the in-sample evaluation results, no single particular form of the marginal distributions combined with any copulas would always lead to the best density forecasting performance. So my approach to testing all possible kinds of copula-marginal complexes has proved to be appropriate.

Finally, some copula-based density models perform the worst for out-of-sample density forecasting: their bootstrapped p-values are 0.000. These zero p-values may be taken to imply that the corresponding models are seriously misspecified.

Table 4.5: Results of the In-Sample SPA Test Based on the KLIC Loss Function

Copula	Marginal	Exchange Rate Pair					
		AD-vs-BP	AD-vs-JY	AD-vs-SF	BP-vs-JY	BP-vs-SF	JY-vs-SF
		Archimedean Copula					
Clayton (α)	Normal	0.025	0.178	0.030	0.198	0.001	0.034
	Student-t	0.075	0.253	0.409	0.123	0.000	0.000
	Skewed-t	0.000	**0.978**	0.000	0.000	0.000	0.000
Frank (α)	Normal	0.137	0.385	0.178	0.000	0.001	0.043
	Student-t	0.084	0.121	0.225	0.697	0.000	0.000
	Skewed-t	0.000	0.968	0.000	0.000	0.000	0.018
Gumbel (α)	Normal	0.171	0.429	**0.445**	0.182	**0.203**	**0.300**
	Student-t	0.008	0.086	0.292	**0.798**	0.000	0.001
	Skewed-t	0.000	0.000	0.000	0.000	0.000	0.006
BB1 (α, β)	Normal	0.011	0.375	0.186	0.000	0.001	0.001
	Student-t	0.116	0.143	0.259	0.611	0.000	0.000
	Skewed-t	0.000	0.000	0.000	0.000	0.000	0.000
BB7 (α, β)	Normal	**0.234**	0.426	0.270	0.040	0.000	0.017
	Student-t	0.000	0.167	0.353	0.542	0.000	0.000
	Skewed-t	0.000	0.000	0.000	0.000	0.000	0.000
		Elliptical Copula					
Gaussian (ρ)	Normal	0.176	0.181	0.251	0.000	0.001	0.000
	Student-t	0.109	0.217	0.238	0.000	0.000	0.000
	Skewed-t	0.000	0.964	0.000	0.000	0.000	0.000
t (ρ, δ)	Normal	0.013	0.067	0.002	0.040	0.005	0.000
	Student-t	0.000	0.195	0.315	0.542	0.000	0.000
	Skewed-t	0.000	0.000	0.000	0.000	0.000	0.000

Note: This table exhibits the p-values of the in-sample SPA test based on the KLIC loss function with stationary bootstrapping. The number of resamples is 1000. The choice of the random block length for the stationary bootstrapping is made following Politis and White's (2004). The p-values in bold indicate the most preferred model. AD, BP, JY and SF denote, respectively, the exchange rate of the Australian dollar, the British pound, the Japanese yen and the Swiss franc, against the US dollar.

Table 4.6: Results of the Out-of-Sample SPA Test Based on the KLIC Loss Function

Copula	Marginal	Exchange rate pair					
		AD-vs-BP	AD-vs-JY	AD-vs-SF	BP-vs-JY	BP-vs-SF	JY-vs-SF
		Archimedean copula					
Clayton (α)	Normal	0.250	**1.000**	0.080	0.158	0.003	0.007
	Student-t	0.000	0.174	0.000	0.010	0.147	0.071
	Skewed-t	0.000	0.000	0.116	0.000	0.000	0.000
Frank (α)	Normal	**1.000**	0.002	0.331	0.033	0.003	0.165
	Student-t	0.000	0.003	0.000	**0.209**	0.032	**0.182**
	Skewed-t	0.000	0.040	0.000	0.000	**0.781**	0.000
Gumbel (α)	Normal	0.000	0.000	0.000	0.008	0.002	0.002
	Student-t	0.002	0.000	0.000	0.000	0.070	0.057
	Skewed-t	0.000	0.000	0.000	0.000	0.000	0.000
BB1 (α, β)	Normal	0.000	0.000	0.000	0.016	0.005	0.000
	Student-t	0.000	0.000	0.000	0.000	0.000	0.000
	Skewed-t	0.000	0.000	0.000	0.000	0.000	0.000
BB7 (α, β)	Normal	0.000	0.000	0.000	0.043	0.001	0.075
	Student-t	0.000	0.000	0.000	0.000	0.209	0.000
	Skewed-t	0.000	0.000	0.000	0.000	0.000	0.000
		Elliptical copula					
Gaussian (ρ)	Normal	0.000	0.000	0.000	0.017	0.000	0.034
	Student-t	0.000	0.000	0.000	0.079	0.093	0.114
	Skewed-t	0.000	0.012	0.000	0.032	0.000	0.000
t (ρ, δ)	Normal	0.117	0.000	**0.438**	0.007	0.000	0.155
	Student-t	0.000	0.000	0.000	0.000	0.012	0.000
	Skewed-t	0.000	0.000	0.000	0.000	0.000	0.000

Note: This table exhibits the p-values of the out-of-sample SPA test based on the KLIC loss function with stationary bootstrapping. The number of resamples is 1000. The choice of the random block length for the stationary bootstrapping is made following Politis and White's (2004). The p-values in bold indicate the most preferred model. AD, BP, JY and SF denote, respectively, the exchange rate of the Australian dollar, the British pound, the Japanese yen and the Swiss franc, against the US dollar.

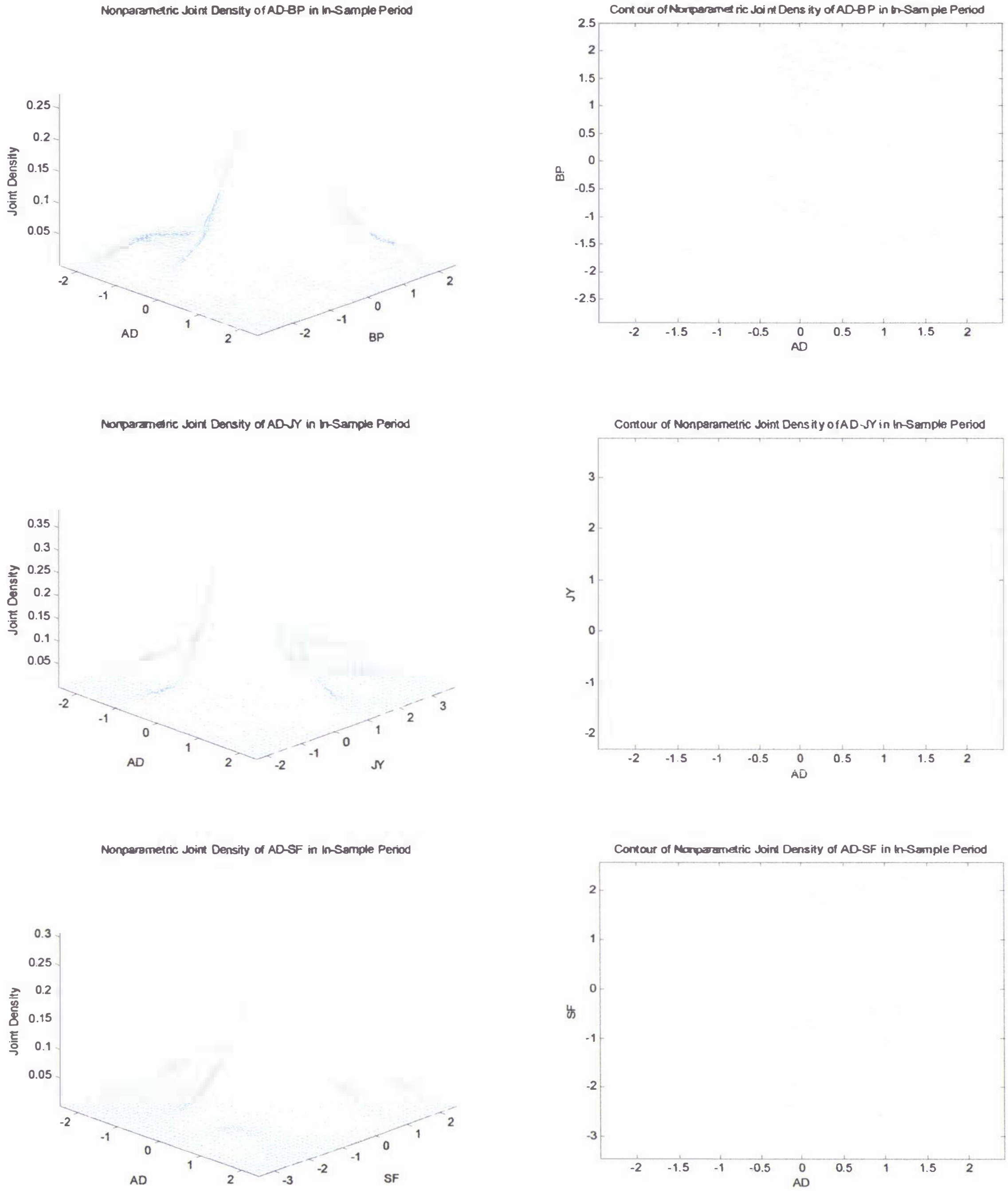


Figure 4.3

Note: This figure shows three-dimensional plots and corresponding contours of bivariate nonparametric kernel densities of exchange rate return pairs AD-BP, AD-JY, and AD-SF in in-sample period respectively.

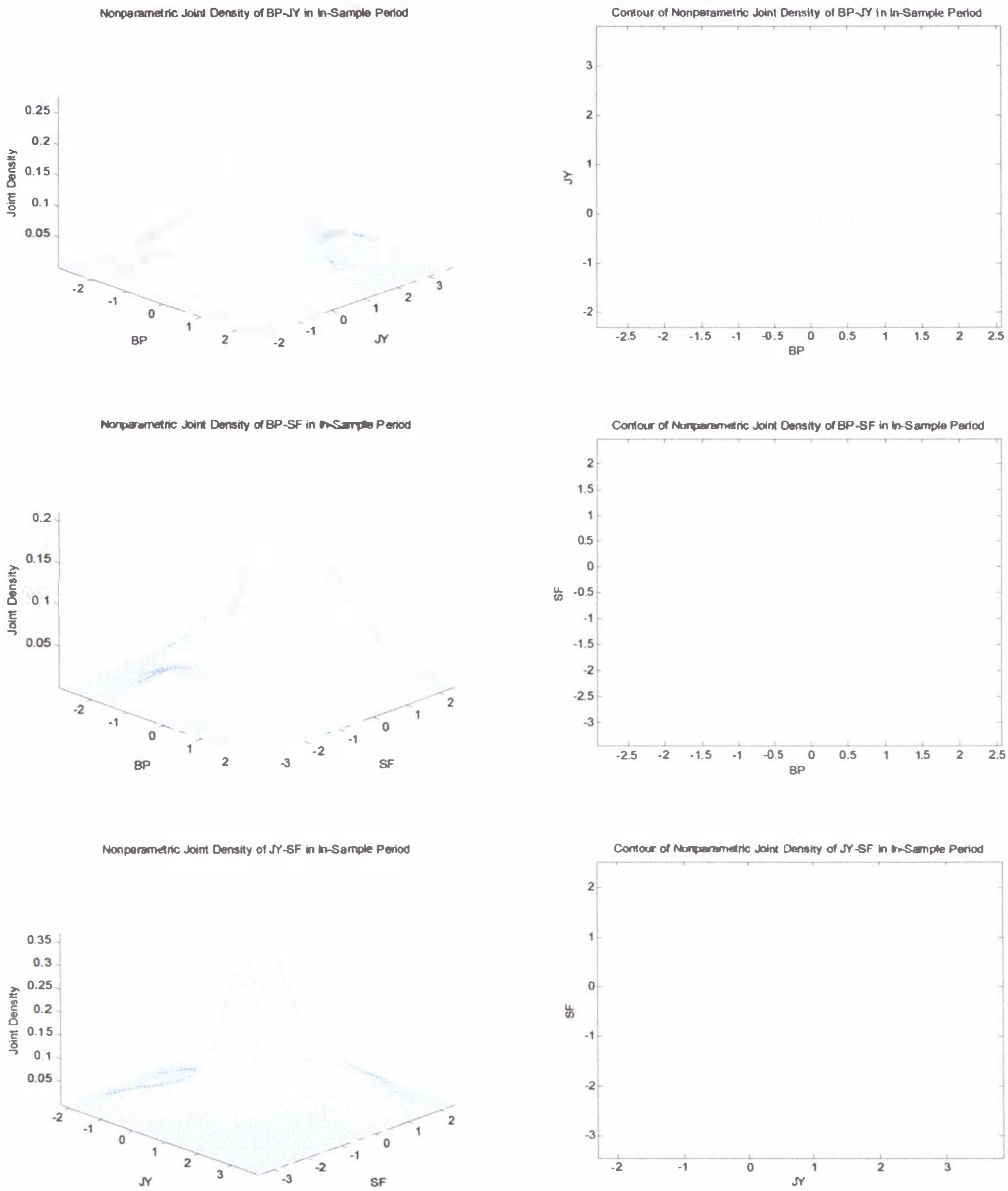


Figure 4.4

Note: This figure shows three-dimensional plots and corresponding contours of bivariate nonparametric kernel densities of exchange rate return pairs BP-JY, BP-SF, and JY-SF in in-sample period respectively.

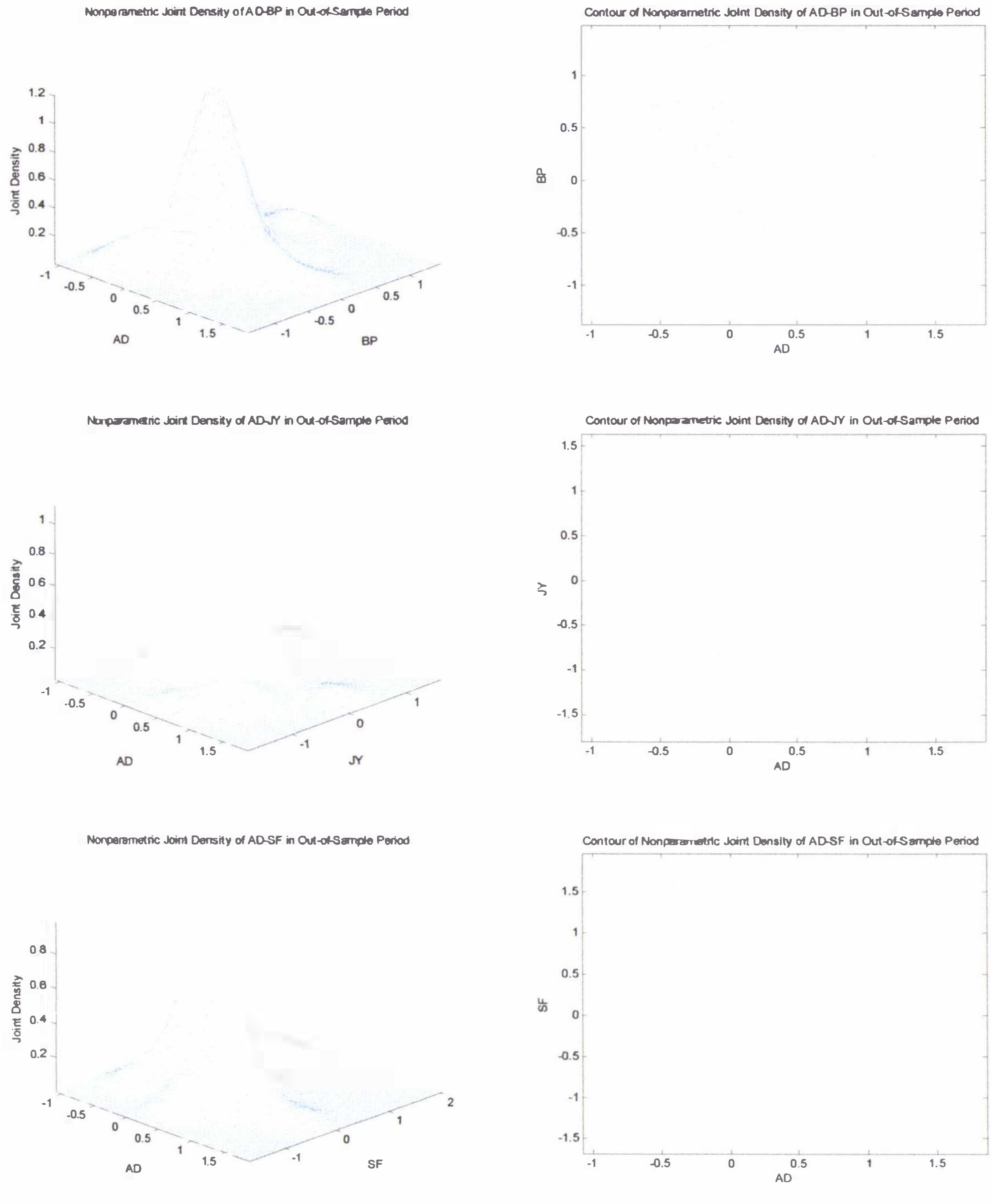


Figure 4.5

Note: This figure shows three-dimensional plots and corresponding contours of bivariate nonparametric kernel densities of exchange rate return pairs AD-BP, AD-JY, and AD-SF in out-of-sample period respectively.

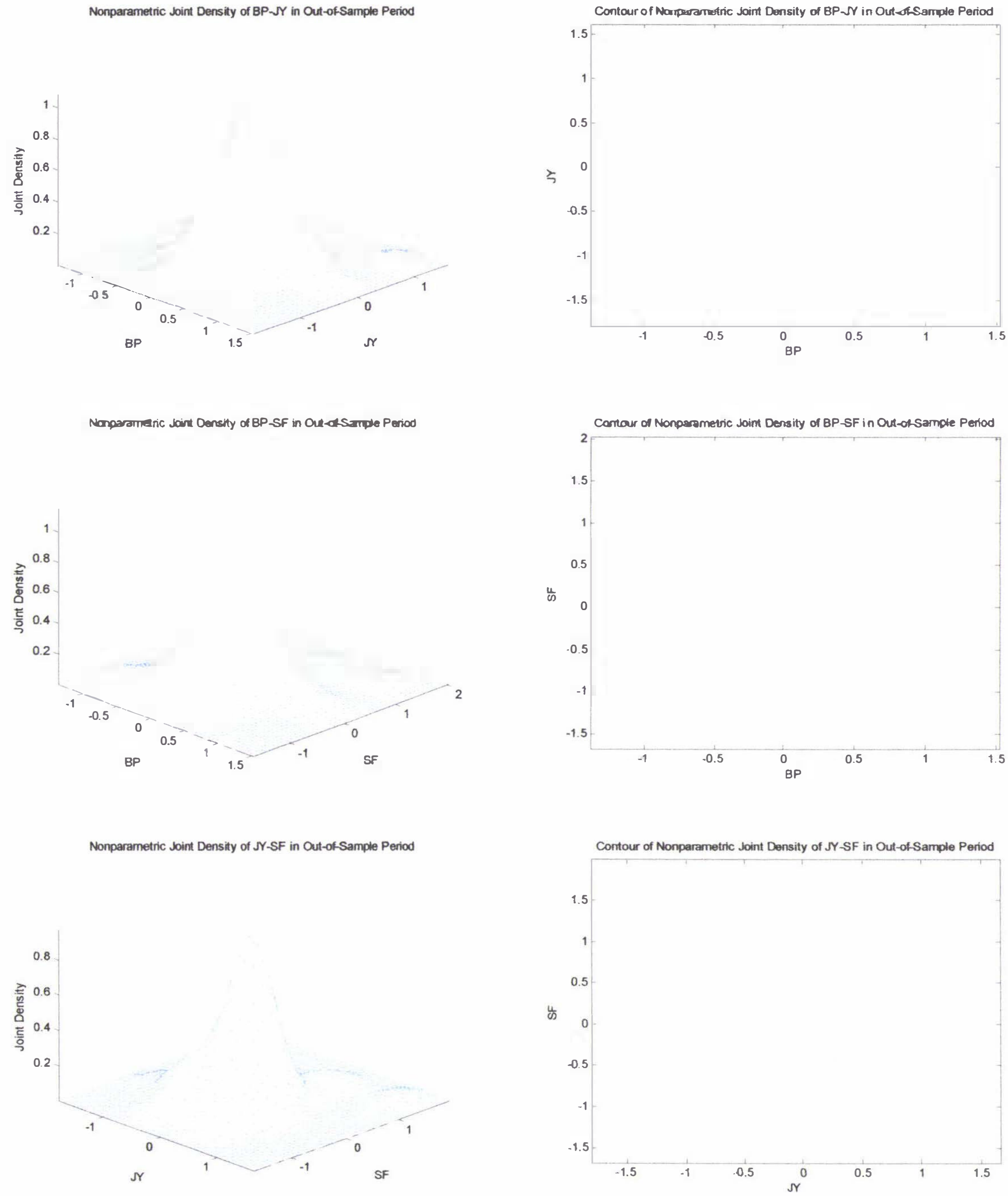


Figure 4.6

Note: This figure shows three-dimensional plots and corresponding contours of bivariate nonparametric kernel densities of exchange rate return pairs BP-JY, BP-SF, and JY-SF in out-of-sample period respectively.

4.6 The Economic Value of Archimedean CBMD Forecast Evaluation

In this section, I further investigate the economic value of Archimedean CBMD forecast evaluation by providing a numerical example of multivariate option pricing. I consider a bivariate cash-or-nothing digital put option on the exchange rates of the Australian dollar (AD) and the British pound (BP) against the US dollar.

Initially, according to Rosenberg (2003), the price of a bivariate option, p, is calculated by

$$p = e^{-(r_{AD}-r_{BP})(T-t)} \int_0^{K_{AD}} \int_0^{K_{BP}} D(X_{AD,t}, X_{BP,t}) \cdot f^*(X_{AD,t}, X_{BP,t} \mid \mathbf{0}) dX_{AD,t} dX_{BP,t} \qquad (4.10)$$

where $D(\cdot)$ is the payoff function, r_{AD} and r_{BP} are the risk-free interest rates of the two countries respectively, T-t is the time until expiration, $f^*(\cdot)$ is the bivariate pricing kernel (risk-neutral density), $X_{AD,t}$ and $X_{BP,t}$ are respectively the spot dollar prices of the two currencies at time t, K_{AD} and K_{BP} are the strike prices for the univariate options, and $\mathbf{0}$ is the parameter set for the function $f^*(\cdot)$. For a bivariate cash-or-nothing digital put option, Equation (4.10) becomes

$$p = D \cdot e^{-(r_{AD}-r_{BP})(T-t)} \int_0^{K_{AD}} \int_0^{K_{BP}} f^*(X_{AD,t}, X_{BP,t} \mid \mathbf{0}) dX_{AD,t} dX_{BP,t} \qquad (4.11)$$

where D is the predetermined cash payoff. For illustration, I assume $D = \$1$, and r_{AD} and r_{BP} are identical. Then the price of the bivariate digital put option is simplified as

$$p = \int_0^{K_{AD}} \int_0^{K_{BP}} f^*(X_{AD,t}, X_{BP,t} \mid \mathbf{0}) dX_{AD,t} dX_{BP,t} \qquad (4.12)$$

Cherubini and Luciano (2002) demonstrate that, in a complete market and for a bivariate digital option pricing, the bivariate pricing kernel is a bivariate probability measure. Therefore, the formula of the bivariate digital put option can be described by a pure bivariate cumulative distribution function and can be further expressed by a copula function:

$$p = \int_0^{K_{AD}} \int_0^{K_{BP}} f(X_{AD,t}, X_{BP,t} \mid \boldsymbol{\theta}) dX_{AD,t} dX_{BP,t} = \Pr(X_{AD,t} \leq K_{AD}, X_{BP,t} \leq K_{BP})$$

$$= C(F(X_{AD,t} \leq K_{AD}), F(X_{BP,t} \leq K_{BP}); \boldsymbol{\theta}) \qquad (4.13)$$

where $f(\cdot)$ is the bivariate objective density, and $C(\cdot)$ is the copula function. From Equation (4.13) we can see that the price of the bivariate put option p and the bivariate objective density function $f(\cdot)$ are closely associated. Hence, the multivariate density forecast evaluation can provide an adequate basis for pricing the multivariate option.

According to the evaluation results reported in Table 4.6, the best out-of-sample CBMD forecast for the AD-BP pair is the Frank-copula-based density with the normal marginals while the worst one is the full Gaussian-copula-based density. This indicates that if an option's pricing is based on the full Gaussian-copula-based density, the option will be mispriced. Therefore, for comparison, I calculate a Frank-copula-based digital put option price as well as a full Gaussian-copula-based one. Since the marginals for both copulas are normal, I then estimate the marginals by $u = F(X_{AD,t} \leq K_{AD}) = \Phi(-d_{AD})$ and $\upsilon = F(X_{BP,t} \leq K_{BP}) = \Phi(-d_{BP})$ where d_{AD} and d_{BP} are calculated using the equation of the univariate foreign currency put option based on the Black-Scholes formula (see more details in Garman and Kohlhagen (1983)):

$$d_{AD} \equiv \frac{\ln(X_{AD,t} / K_{AD}) + [r_{US} - r_{AD} - (\sigma_{AD}^2 / 2)] \cdot (T-t)}{\sigma_{AD} \sqrt{T-t}} \qquad (4.14)$$

$$d_{BP} \equiv \frac{\ln(X_{BP,t} / K_{BP}) + [r_{US} - r_{AD} - (\sigma_{BP}^2 / 2)] \cdot (T-t)}{\sigma_{BP} \sqrt{T-t}}$$

where r_{US} is the risk-free interest rate in the United States, and σ_{AD} and σ_{BP} are the standard deviations of the two exchange rate returns. For numerical analysis, suppose that

$X_{AD,t} = 0.75$, $X_{BP,t} = 1.83$,

$K_{AD} = 0.78$, $K_{BP} = 1.88$,

$r_{AD} = r_{US}$, $r_{BP} = r_{US}$,

$\sigma_{AD} = 0.69$, $\sigma_{BP} = 0.50$,

T-t = 3 months = 0.25 year.

Then

$d_{AD} = [\ln(0.75/0.78) - (0.69^2/2)\cdot 0.25]/(0.69\cdot\sqrt{0.25}) = -0.2862$

$d_{BP} = [\ln(1.83/1.88) - (0.50^2/2)\cdot 0.25]/(0.50\cdot\sqrt{0.25}) = -0.2328$

Therefore $u = F(-d_{AD}) = \Phi(-0.2862) = 0.6162$, $\upsilon = F(-d_{BP}) = \Phi(-0.2328) = 0.5921$.

If the returns of the US/AD and US/BP rates are linearly correlated within the traditional mean-variance framework, then the bivariate digital put option can be priced by a full Gaussian copula with a linear correlation coefficient ρ. Suppose $\rho = 0.5$. Then

$$
\begin{aligned}
p^{Gaussian} &= C(\Phi(-d_{BP}), \Phi(-d_{JY}); \rho) = C^{Gaussian}(0.6162, 0.5921; 0.5) \\
&= \Phi_{\rho}(0.6162, 0.5921; 0.5) \qquad (4.15) \\
&= 0.4416
\end{aligned}
$$

However, if we extend bivariate modeling beyond the traditional multivariate normal distribution to a Frank copula, where bivariate returns are nonlinearly associated, then the price of the bivariate digital put option becomes

$p^{Frank} = C^{Frank}(\Phi(-d_{BP}), \Phi(-d_{JY}); \theta) = C^{Frank}(0.6162, 0.5921; \theta)$.

Suppose the parameter θ for the Frank copula is 2.5. Thus,

$$p^{Frank} = C^{Frank}(0.6162, 0.5921; 2.5)$$

$$= -\frac{1}{\theta}\ln\left[1+\frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right] = -\frac{1}{2.5}\ln\left[1+\frac{(e^{-2.5\times 0.6162}-1)(e^{-2.5\times 0.5921}-1)}{e^{-2.5}-1}\right] \quad (4.16)$$

$$= 0.4310$$

Now compared with the Frank copula, the Gaussian copula clearly overprices the put option.

Next, suppose a portfolio (valued in the US dollar) contains two foreign risky assets, and an investor invest in 1,000,000 units of the portfolio. To hedge against simultaneous appreciations in the AD and the BP, the investor decides to buy a 3-month bivariate digital put option with a strike price equal to \$45 per unit of the portfolio. The initial investment based on the Frank copula is 1,000,000 × 0.4310 = \$431,000, and that based on the full Gaussian copula is 1,000,000 × 0.4416 = \$441,600. At the expiration date, if the price per unit of the portfolio is lower than its strike price, say \$30, then the investor can sell the 3-month bivariate digital put option for \$45 per unit to realize a gain of \$45 - \$30 = \$15 per unit. When the initial cost of the option is taken into account, the net gain based on the Frank copula is \$15,000,000 - \$431,000 = \$14,569,000, and that based on the full Gaussian copula is \$15,000,000 - \$441,600 = \$14,558,400. Therefore, as the option is mispriced by the full Gaussian copula, the investor will suffer a net loss which is \$14,569,000 - \$14,558,400 = \$10,600.

4.7 Conclusions

In this chapter, I report the research with an objective to show the importance and the way of studying, largely neglected, non-Gaussian multivariate density forecast evaluation. I propose a full ranking statistical testing procedure, which allows for the presence of misspecification in the multiple models, in terms of the KLIC to evaluate the Archimedean-copula-based multivariate density forecast. My study uses data from foreign exchange markets.

In comparing competing models, I consider five Archimedean copulas (the Clayton, the Frank, the Gumbel, the BB1, and the BB7 ones) and two elliptical copulas (the Gaussian, and the *t* ones) as candidates for tests. These copulas have been popularized in the literature. Meanwhile, I specify three models as the possible forms of marginal distributions: the standard normal, the Student-*t*, and Hansen's (1994) skewed-*t*. Further, in order to avoid the data snooping problem, I conduct Hansen's (2005) SPA test via the KLIC loss function to evaluate model performances.

The main empirical results of my research are summarized as follows. Firstly, Archimedean copulas outperform the Gaussian copula for modeling multivariate densities, in terms of both in-sample fitting and out-of-sample forecasting. This result provides evidence that the multivariate density of financial time series, at least of exchange rates studied in this chapter, may be non-Gaussian. However, the existing literature has paid attention only to the evaluation of Gaussian multivariate density forecast: a serious omission. So, non-normality should now become a focal issue for academic research on multivariate density forecast evaluation.

Secondly, any possible copula-marginal complexes of model could become the best performer in density forecast. This result conveys a message to risk managers that care must be taken in selecting the copula-based multivariate density forecasting models and comprehensive tests for both copula and marginal functions at the same time should be conducted.

Finally, discrepancies between the in-sample and out-of-sample copula-based multivariate density model performances are present in most cases. This result indicates structural change in the data generating process from the in-sample to the out-of-sample period. Although Inoue and Kilian (2004) argue that in-sample tests of predictability are more powerful than out-of-sample tests after correcting for data mining, the former should be supplemented the latter which may still provide additional and useful information for model selection.

Chapter 5 Estimating dynamic asymmetric tail dependences with time-varying investors' heterogeneous beliefs in Asian developed futures markets[26]

5.1 Introduction and Literature Review

This chapter's focus is on estimating dynamic asymmetric tail dependences in the Asian developed futures markets, using conditional two-parameter Archimedean copulas. Specifically, to model the marginal distribution for estimating the conditional two-parameter Archimedean copula, I consider a conditional skewed-*t* distribution in accordance with time-varying investors' heterogeneous beliefs (See Section 1.1 of Chapter 1 for discussion on the link between the skewed-*t* distribution and investors' heterogeneous beliefs). The main purpose of this chapter is to see whether the PVaR estimation can be improved by this model.

My work is motivated by the following considerations. First, the dependence structure of international financial markets is increasingly important in various fields of finance such as optimal assets allocation, multivariate asset pricing, and portfolio value at risk. In the real world, the market dependence structure is usually asymmetric due to investors' heterogeneity. A number of recent empirical studies have uncovered that correlations between international equity markets are higher during market downturns than during market upturns.[27] This suggests that, if all stock prices tend to fall together as tail events occur, the value of diversification might be

[26] A revised version of this chapter, titled "Estimation of Dynamic Asymmetric Tail Dependences: An Empirical Study on Asian Developed Futures Markets", has been accepted by *Applied Financial Economics* for publication

[27] For instance, Erb et al (1994), King et al (1994), De Santis and Gerard (1997), Longin and Solnik (1995, 2001), Ang and Bekaert (2002) and, Ang and Chen (2002).

overstated by those not taking the increase in downside dependence into account (Ang and Chen, 2002). As a consequence, international diversification is less beneficial than expected.

Second, when hedging dependent risk, portfolio managers should care not only about movements of individual markets, but also about comovements among them. Following financial liberalization and market integration in the last two decades, asymmetric dependence among international equity markets has become increasingly significant. This asymmetry suggests that downside dependent risk deserves particular attention of portfolio managers.

Third, many empirical researches have showed that investors' heterogeneity plays a key role in determining financial disturbances and contagions.[28]

Fourth, the Asian developed futures markets, such as the Hong Kong Futures Exchange, the Osaka Stock Exchange, and the Singapore International Monetary Exchange, have attracted an extensive research interest. However, previous studies mainly focus on individual index futures listed on these three futures markets.[29] Little attention has been paid to asymmetric dependence, especially tail dependence, between these markets. My work is therefore to fill this void.

Previous studies on dynamic dependence between markets are mainly based on the multivariate GARCH method, where multivariate returns are assumed to be normally distributed and linearly correlated (e.g. De Santis and Gerard (1997) and Kroner and Ng (1998)). However, this method is unable to capture both non-normality and asymmetric (tail) dependence when rare events and market contagion occur. Informally, tail dependence can be understood as the probability of an extremely large negative (positive) return on one asset/portfolio given that the

[28] See Hong and Stein (2003) and Levy (2007).
[29] For example, Chen et al (1999), Cheng et al (2000), Kim et al (2002), and So and Tse (2004) among others.

other asset/portfolio has yielded an extremely large negative (positive) return. Patton (2006a) shows that the multivariate GARCH model with the normal or Student's t distribution cannot capture (conditional) tail dependence. He concludes that "a hedge constructed using linear correlation may not offer the degree of protection it would under a multivariate normal or Student's t distribution". It is also worth noting that there exists another method, which is based on the multivariate extreme value theory, to estimate tail dependence (see Longin and Solnik (2001) and Bae et al (2003)). This method assumes an asymptotically dependent structure between random variables in the tails of the multivariate distribution. However, Poon et al (2004) points out that there is no evidence of asymptotic dependence in stock market returns for most countries. They further conclude that the multivariate extreme theory method based on the assumption of asymptotic dependence can lead to overestimating financial risk.

In order to overcome the drawbacks mentioned above, an alternative way to deal with asymmetric tail dependence is therefore called for. Recently, there has been a growing interest in applying copula theory in finance. A copula is a multivariate distribution function which can fully and flexibly capture tail dependence among two or more random variables. Major financial applications of the copula model can be found in Bouye et al (2000), Bradley and Taqqu (2003), Embrechts et al (2003), and Cherubini et al (2004), among others. Although the copula model is a new tool for evaluating multivariate dependence, empirical fronts have been expanded in many financial directions including asymmetric patterns of financial market comovements. See, for instance, Caillautt and Guégan (2005), Patton (2006a), Hu (2006), and Rodriguez (2007).

In this research, I use Hang Seng index futures (Hang Seng for short) traded on the Hong Kong Futures Exchange, Nikkei 225 index futures (Nikkei 225 for short) traded on the Osaka Stock Exchange, and Morgan Stanley Capital International index futures (MSCI SIN for short) traded on the Singapore International Monetary Exchange, as proxies of the Asian developed futures markets. The Hang Seng index is a value-weighted index consisting of 33 major Hong Kong stocks, which comprise approximately 65% of the total market value and trading volume. It is the benchmark of the Hong Kong stock market and is widely used by fund managers as their performance reference. The Hang Seng index futures provides investors with a set of effective instruments to manage portfolio risk and has developed gradually into one of the most active futures contracts in the world. The Nikkei 225 index is a price-weighted arithmetic average of the current stock prices of 225 companies and is the most widely quoted barometer of the Japanese stock market. It is also the basis of the most popular futures contracts in Japan in terms of daily average volume and open interest. The Singapore Morgan Stanley Capital International index represents about 50% of the underlying Singapore stock market and comprises of a basket of 36 Singapore stocks. The Singapore Morgan Stanley Capital International index futures are more popular among investors and traders. Fund managers often use the index as a benchmark and hedge their equity stocks against the index futures.

In the spirit of Patton (2006a), I allow asymmetric tail dependence to be time varying via the conditional Archimedean copula model. However, my methodologies are different from Patton (2006a) and other studies in three aspects. First, when estimating the marginal distribution, I parameterize the dynamics of the conditional third and fourth moments along with the threshold heteroscedasticity process by using Hansen's (1994) skewed-t model. Compared with a large body of existing researches, which assumed that standardized innovations are subject to either a log-normal or a standard normal distribution, my methods can fully reflect the characteristics of underlying returns.

Second, another novelty in my methodology is that I employ three conditional two-parameter Archimedean copulas known as BB1, BB4 and BB7,[30] to trace the dynamics of asymmetric tail dependence, rather than static one-parameter Archimedean copulas which have been popular in the existing literatures. As shown in the left column of Figure 4 in Chapter 2, the advantage of the conditional two-parameter Archimedean copula is that it can simultaneously capture the dynamics of upper and lower tail dependence. This enables me to redress the possible biasedness or inaccuracy of the static one-parameter Archimedean copula that assumes only one tail dependence (either upper or lower) between two random variables. To my knowledge, the conditional BB1 and the conditional BB4 copula have never been employed in previous empirical studies.

Third, unlike previous studies which commonly rely on the goodness-of-fit test to evaluate the performance of the estimated model, I implement a test procedure based on Hansen's (2005) superior predictive ability (SPA) test to assess model fitting. I compare the three sophisticated conditional two-parameter Archimedean copula models with the full Gaussian copula model (i.e. Gaussian copula with standard normal marginals), using the loss function of the Kullback-Leibler Information Criterion (KLIC) as a measure of differences between multivariate densities. This test procedure has been introduced in Chapter 4 and can help us to diagnose the problem of model overfitting.

This chapter is organized as follows. In Sections 5.2 I introduce data and conduct preliminary analysis. The employed copula models and the estimation method are outlined in Sections 5.3 and 5.4 respectively. Section 5.5 presents and discusses empirical results. Section 5.6 investigates possible improvement in the estimation of PVaR based on the conditional two-parameter Archimedean copula model. Concluding remarks are given in Section 5.7.

[30] The BB7 copula is also called Joe-Clayton copula in Patton (2006a).

5.2. Data

5.2.1 Preliminary Analysis

All daily price of the index futures are collected from Datastream, from 07 September 1998 to 28 February 2005. Although my sample period begins after the Asian financial crisis in 1997, it still contains several events which could lead to tail dependence of financial markets, such as the collapse of the Long-Term Capital Management (LTCM) in September 1998, the 9/11 terrorist attack in 2001, and the SARS triggered in March 2003. To illustrate, consider the 9/11 terrorist attack as an example. The raw data indicates that, on the next day of the attack, the daily prices of the Nikkei 225, the Hang Seng and the MSCI SIN index futures all dropped, and by 4.97%, 8.34% and 5.27% respectively. This can be seen as initial evidence that there exists lower tail dependence between these markets. In the following discussions of descriptive statistics, I will further present crude estimates of lower and upper tail dependence for a return pair.

Panel A of Table 5.1 presents a range of descriptive statistics for all index future return series. The return is defined as $Y_t = 100 \cdot \log(X_t / X_{t-1})$ where X_t (t = 1, 2,..., n, and n is the sample size) is the price of an index futures. The statistics of first and second moments of each return series indicate that empirical distributions are not standard normal. Daily mean returns of Hang Seng and MSCI SIN are positive around 0.04%, in contrast to Nikkei 225 with a negative mean –0.01%. The average of all unconditional standard deviations is about 1.58%. The values of skewness ranging between –0.0477 and 0.2352, and the values of kurtosis ranging between 4.8551 and 6.3724, further reveal that each return series is asymmetrically distributed with fat tails. Although the Ljung-Box statistics (Q_x) for up to 20 lags calculated for each raw return show the absence of linear autocorrelation, the results for squared returns (Q_{xx}) strongly suggest the presence of nonlinear autocorrelation. Meanwhile, the LM tests of Engle (1982) also significantly exhibit a heteroscedastic effect in the data. Therefore, these two test statistics indicate that the three return series are non-normally distributed with a heteroscedasticity process.

Panel B of Table 5.1 reports linear and rank correlations between two of the three futures markets. The first part of panel B is Pearson's ρ_P. Embrechts et al (2000, 2002) point out that Pearson's ρ_P is only a linear correlation measure for elliptically distributed random variables, and is a deficient measure for financial risk management due to the non-normality of financial data. Poon et al (2004) also find that Pearson's correlation measure is a poor measure when taking account of tail dependence. Embrechts et al (2002) thus propose to use rank correlation such as Spearman's ρ_S or Kendall's τ_K instead of Pearson's ρ_P as a measure of nonlinear relations. From the second and third parts of panel B, one can see that both statistics of Spearman's ρ_S and Kendall's τ_K are all highly significant indicating strong nonlinear relations between the return series, especially between the Hong Kong and Singaporean markets.

5.2.2 Informal Evidence of Asymmetric Tail Dependence

I now turn to co-skewness and co-kurtosis of bivariate returns by employing a multivariate normality test proposed by Mardia (1970). The test statistics reported in Table 5.2 for both co-skewness and co-kurtosis are highly significant at the 5% level. In particular, the strongest co-skewness and co-kurtosis are found for the Nikkei 225-vs-MSCI SIN pair. These results imply possible asymmetric comovements between the three markets.

I next provide crude estimates of asymmetric tail dependence for this return pair. For the ease of exposition, let us define a Nikkei 225 return extreme as being greater than 4% in absolute terms, while a MSCI SIN return is taken to be an extreme if it exceeds 5% in absolute terms. The two absolute threshold values (4% and 5%) approximately equal three times the corresponding standard deviations which are reported in panel A of Table 5.1. Now consider events in which bad news hits the MSCI SIN market and causes its returns to fall below -5%. There are 15 such events, characterized by 15 negative extreme returns, for the MSCI SIN market. Of these 15 shocks, 13 spread to the Nikkei 225 market and cause its returns to fall below -4%. Thus, a crude estimate of lower tail dependence between the two markets is 13/15 = 0.8667. Regarding the upper side, consider events where good news hits the MSCI SIN market and causes its returns to rise above 5%. 22 such positive extreme returns are found for the MSCI SIN market. Of these 22 shocks, 11 spread to the Nikkei 225 market and cause its returns to rise above 4%. Then, a crude estimate of upper tail dependence between the two markets is 11/22 = 0.5000.

Now turn to the case where good/bad news hits the Nikkei 225 market first, with contagion spread from the Nikkei 225 market to MSCI SIN market. By the same token, the lower and upper tail dependence parameters may be calculated as 0.7000 and 0.3333 respectively.

The above crude estimates provide informal evidence for the markets under my investigation that there is stronger dependence in the lower tail of the bivariate return distribution than in the upper tail. In the next section, I will introduce a formal measure of asymmetric tail dependence via the two-parameter Archimedean copula model.

Table 5.1: Preliminary Analysis of Index Futures Returns

	Index Futures Returns		
	Hang Seng	Nikkei 225	MSCI SIN
	Panel A. Summary Statistics		
Observations	1690	1690	1690
Maximum	8.7515	8.0043	11.1124
Minimum	-8.7116	-7.5986	-7.0921
Mean	0.0352	-0.0137	0.0365
Standard Deviation	1.7239	1.4910	1.5324
Skewness	0.0508	-0.0477	0.2352
Kurtosis	5.6085	4.8551	6.3724
Ljung-Box			
$Q_x(20)$	24.4408	19.6409	21.1178
$Q_{xx}(20)$	312.4551*	229.2490*	288.0914*
Engle (20)	134.8376*	118.6739*	125.3843*
	Panel B. Linear and Rank Correlations		
Pearson's ρ_P			
Hang Seng	—	0.4452 (0.0000)	0.5598 (0.0000)
Nikkei 225		—	0.3858 (0.0000)
MSCI SIN			—
Spearman's ρ_S			
Hang Seng	—	0.4162 (0.0000)	0.5003 (0.0000)
Nikkei 225		—	0.3668 (0.0000)
MSCI SIN			—
Kendall's τ_K			
Hang Seng	—	0.2886 (0.0000)	0.3572 (0.0000)
Nikkei 225		—	0.2566 (0.0000)
MSCI SIN			—

Note: Panel A reports the summary statistics of the index futures return series on the three leading Asian futures markets over the period 07 September 1998 to 28 February 2005. The daily percentages of the return series are measured as $100 \times \log(X_t / X_{t-1})$. The Ljung-Box statistics provides tests for the presence of autocorrelations of raw returns and squared returns as well as the LM test of Engle (1982) for the presence of ARCH effects. The critical value of the Ljung-Box test and the LM test of Engle (1982) using 20 lags is 31.410. * indicates significance at the 5% level. Panel B reports the linear and the rank correlations between two of the three index futures returns. The Pearson's ρ_P is calculated as $\rho_P = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$. The Spearman's ρ_S and the Kendall's τ_K are calculated as $\rho_S = \frac{12}{n(n^2-1)} \sum_{t=1}^{n} \left(rank(x_t) - \frac{n+1}{2} \right) \left(rank(y_t) - \frac{n+1}{2} \right)$ and $\tau_K = \binom{n}{2}^{-1} \sum_{1 \le i \le j \le n} sign[(x_i - x_j)(y_i - y_j)]$ respectively. The associated p-values are provided in parentheses.

Table 5.2: Multivariate Normality Test for the Bivariate Returns

	Index Futures Returns					
	Hang Seng		Nikkei 225		MSCI SIN	
	Panel A: Statistics of Bivariate Skewness Test					
	$b_{1,2}$	A	$b_{1,2}$	A	$b_{1,2}$	A
Hang Seng	—	—	0.1140	32.1079*	0.1682	47.3745*
Nikkei 225			—	—	0.2012	56.6642*
MSCI SIN					—	—
	Panel B: Statistics of bivariate kurtosis test					
	$b_{2,2}$	B	$b_{2,2}$	B	$b_{2,2}$	B
Hang Seng	—	—	13.6176	28.9160*	14.5407	33.6594*
Nikkei 225			—	—	14.7089	34.5238*
MSCI SIN					—	—

Note: This table exhibits Mardia's (1970) multivariate normality test which is to exam whether the multivariate skewness and kurtosis are significant at a certain level. Panel A reports the statistics of bivariate skewness test $A = n \cdot b_{1,2}/6 \sim \chi^2(4)$, with a critical value 9.49 at the 95% confidence interval where n is the sample size, and $b_{1,2}$ is a basic point for testing 2-demensional multivariate skewness. Panel B reports the statistics of bivariate kurtosis test $B = \frac{b_{2,2} - 8(n-1)/(n+1)}{\sqrt{64/n}} \sim N(0,1)$, with a critical value 1.96 at the 95% confidence interval where n is the sample size, and $b_{2,2}$ is a basic point for testing 2-demensional multivariate kurtosis. * indicates significance at the 5% level.

5.3 Models

5.3.1 Conditional Archimedean Copula

An m-dimensional copula function $C(F_1(y_{1,t}), F_2(y_{2,t}), \ldots, F_m(y_{m,t}))$ is a multivariate distribution function on $[0,1]^m$ with standard uniform marginal distributions $F_i(y_{i,t})$ (i = 1, 2,…, m). A class of copulas, known as Archimedean copulas, enable us to model the dependence structure beyond linear correlation, and provide a high degree of flexibility in analyzing market comovements. Note, however, that several one-parameter Archimedean copulas, such as the Clayton, Frank, and Gumbel copulas, are relevant only where there exists either lower or upper tail dependence. Since both lower and upper tail dependence between the markets under my investigation may be present and asymmetric, I employ three two-parameter Archimedean copulas referred to as the BB1, the BB4 and the BB7 copula. The formulas of the three copula functions are given in panel A of Table 2.1 in Chapter 2[31].

For estimating the time-varying asymmetric dependence structure, Patton (2006a) extends the standard definition of copula to the conditional case, and defines the conditional copula as a multivariate distribution of variables. Each variable has a uniform distribution conditional on a sigma algebra $\mathcal{F}$ generated by a set of all past information:

$$\mathcal{F}_{t-1} = \sigma\{y_{1,t-1}, y_{2,t-1}, \dots, y_{m,t-1}, y_{1,t-2}, y_{2,t-2}, \dots, y_{m,t-2}, \dots\} \text{ for } t = 1, 2, \dots, n.$$

Let F be an m-dimensional distribution function with continuous marginal distributions $F_1, F_2, \dots, F_m$. Then an extension of Sklar's (1959) theorem is:

$$F_t(y_{1,t}, y_{2,t}, \dots, y_{m,t} | \mathcal{F}_{t-1}) = C_t[F_{1,t}(y_{1,t} | \mathcal{F}_{t-1}), F_{2,t}(y_{2,t} | \mathcal{F}_{t-1}), \dots, F_{m,t}(y_{m,t} | \mathcal{F}_{t-1}) | \mathcal{F}_{t-1}] \quad (5.1)$$

If each marginal F_i and the copula C are differentiable, one can obtain the density of the conditional copula as:

$$\begin{aligned} & c_t[F_{1,t}(y_{1,t} | \mathcal{F}_{t-1}), F_{2,t}(y_{2,t} | \mathcal{F}_{t-1}), \dots, F_{m,t}(y_{m,t} | \mathcal{F}_{t-1}) | \mathcal{F}_{t-1}] \\ & = \partial^m C_t[F_{1,t}(y_{1,t} | \mathcal{F}_{t-1}), F_{2,t}(y_{2,t} | \mathcal{F}_{t-1}), \dots, F_{m,t}(y_{m,t} | \mathcal{F}_{t-1}) | \mathcal{F}_{t-1}] \\ & \quad / [\partial F_{1,t}(y_{1,t} | \mathcal{F}_{t-1}) \partial F_{2,t}(y_{2,t} | \mathcal{F}_{t-1}) \dots \partial F_{m,t}(y_{m,t} | \mathcal{F}_{t-1})] \qquad (5.2) \end{aligned}$$

Then the conditional joint density $f_t(y_{1,t}, y_{2,t}, \dots, y_{m,t} | \mathcal{F}_{t-1})$ is yielded by

$$\begin{aligned} f_t(y_{1,t}, y_{2,t}, \dots, y_{m,t} | \mathcal{F}_{t-1}) = {} & f_{1,t}(y_{1,t} | \mathcal{F}_{t-1}) \times f_{2,t}(y_{2,t} | \mathcal{F}_{t-1}) \times \dots \times f_{m,t}(y_{m,t} | \mathcal{F}_{t-1}) \\ & \times c_t[F_{1,t}(y_{1,t} | \mathcal{F}_{t-1}), F_{2,t}(y_{2,t} | \mathcal{F}_{t-1}), \dots, F_{m,t}(y_{m,t} | \mathcal{F}_{t-1}) | \mathcal{F}_{t-1}] \qquad (5.3) \end{aligned}$$

where $f_{i,t}(y_{i,t} | \mathcal{F}_{t-1})$ is the univariate conditional density.

[31] The functions of densities for the three two-parameter Archimedean copulas are very long. Matlab codes are available from the author upon request.

5.3.2 Conditional tail dependence

Now let $U_{i,t} = F_{i,t}(y_{i,t})$ (i = 1, 2) denote the marginals. Lower and upper tail dependence are then defined as:

$$\tau^L = \lim_{u \to 0} \Pr(U_{1,t} \leq u \mid U_{2,t} \leq u) = \lim_{u \to 0} \Pr(U_{2,t} \leq u \mid U_{1,t} \leq u) = \lim_{u \to 0} \frac{C(u,u)}{u} \tag{5.4}$$

$$\tau^U = \lim_{u \to 1} \Pr(U_{1,t} > u \mid U_{2,t} > u) = \lim_{u \to 1} \Pr(U_{2,t} > u \mid U_{1,t} > u) = \lim_{u \to 1} \frac{1 - 2u + C(u,u)}{1 - u}$$

The closed forms of lower and upper tail dependence for the three two-parameter Archimedean copulas are provided in panel A of Table 2.1 in Chapter 2. Note that these asymmetric tail dependences listed in the table are constant. To explore the possibility that they are time-varying and dynamic, I follow Patton (2006a) by specifying dynamic tail dependences as an ARMA (1,1) process:

$$\begin{aligned} \tau_t^L &= \Lambda\left(\omega^L + \delta^L \tau_{t-1}^L + \psi^L \cdot | u_{t-1} - \upsilon_{t-1} |\right) \\ \tau_t^U &= \Lambda\left(\omega^U + \delta^U \tau_{t-1}^U + \psi^U \cdot | u_{t-1} - \upsilon_{t-1} |\right) \end{aligned} \tag{5.5}$$

where u_t and υ_t are marginal distributions, $\delta^L \tau_{t-1}^L$ and $\delta^U \tau_{t-1}^U$ are autoregressive terms, $\psi^L \cdot | u_{t-1} - \upsilon_{t-1} |$ and $\psi^U \cdot | u_{t-1} - \upsilon_{t-1} |$ are forcing variables, and $\Lambda(k) = 1/[1 + \exp(-k)]$ is the logistic transformation to ensure $\tau_t^L, \tau_t^U \in (0, 1)$ at all time. Moreover, for the three two-parameter copulas, their parameters also become time-varying and are the functions of conditional tail dependences:

BB1: $\alpha_t = -1/[\beta_t \cdot \log_2(\tau_t^L)]$ $\quad\quad$ $\beta_t = 1/\log_2(2 - \tau_t^U)$

BB4: $\alpha_t = -\ln(2 - \tau_t^U)/\ln(\tau_t^L)$ $\quad\quad$ $\beta_t = -1/\log_2(\tau_t^U)$

BB7: $\alpha_t = 1/\log_2(2 - \tau_t^U)$ $\quad\quad$ $\beta_t = -1/\log_2(\tau_t^L)$

Once these parameters are determined, one can embed one of the three sets of parameters in the corresponding copula function and then estimate the copula function by a two-stage estimation procedure.

5.4 Estimation Method

5.4.1 Two-stage maximum likelihood estimator

Joe (1997, 2005) proposed a two-stage estimation procedure to estimate the unknown parameters of a copula. In the first step, for a sample size n with m observed random variables, $(Y_{1,t}, Y_{2,t}, \ldots, Y_{m,t})_{t=1}^{n}$, one can estimate the parameters of each marginal $\boldsymbol{\theta}_M$ parametrically.

$$\hat{\boldsymbol{\theta}}_M = \arg\max \sum_{t=1}^{n} \sum_{i=1}^{m} \ln f_{i,t}(y_{i,t} \mid \mathcal{F}_{t-1}) \qquad (5.6)$$

Next, based on the estimated parameters $\hat{\boldsymbol{\theta}}_M$ and a given density of the copula, the parameter estimates of each copula $\boldsymbol{\theta}_C$ can be obtained via the maximum likelihood method in the second step.

$$\hat{\boldsymbol{\theta}}_C = \arg\max \sum_{t=1}^{n} \ln c_t[F_{i,t}(y_{i,t}) \mid \mathcal{F}_{t-1}, \hat{\boldsymbol{\theta}}_M] \qquad (5.7)$$

Joe (2005) shows that the two-stage estimation method generally has good efficiency properties. In addition, to ensure the consistency and the asymptotic normality of the two-stage maximum likelihood estimation, the asymptotic covariance matrix is consistently estimated by the so-called "sandwich estimator" proposed by Newey and McFadden (1994) and White (1994). This method is also considered in Patton (2006b)

5.4.2 Marginal model

In this research, I estimate the marginals assumed to follow Hansen's (1994) skewed-*t* distribution with a threshold GARCH (1, 1) process:

$$\begin{aligned}
Y_t &= \mu + e_t \\
e_t &= \sqrt{h_t} \cdot z_t \\
e_t^+ &= \max(e_t, 0) \qquad\qquad (5.8)\\
e_t^- &= \max(-e_t, 0) \\
h_t &= a_0 + b_0 (e_{t-1}^+)^2 + c_0 (e_{t-1}^-)^2 + d_0 h_{t-1}
\end{aligned}$$

$$z_t \,|\, \mathcal{F}_{t-1} \sim skt\,(\eta, \lambda)$$

where $Y_t = 100 \cdot \log(X_t / X_{t-1})$, and *skt* denotes Hansen's (1994) skewed-*t* distribution.

The formulas of the *pdf* and the *cdf* of Hansen's (1994) skewed-t distribution were given in Chapter 3. For modeling the TGARCH (1, 1) process with the conditional skewed-*t* distribution, I follow Jondeau and Rockinger's (2003) proposition:

$$\begin{aligned}
\eta_t &= \Xi(a_1 + b_1 e_{t-1}^+ + c_1 e_{t-1}^-) \\
\lambda_t &= \Xi(a_2 + b_2 e_{t-1}^+ + c_2 e_{t-1}^-)
\end{aligned} \qquad (5.9)$$

Note that $\Xi(k) = L + ((U - L)/(1 + \exp(-k)))$ is a logistic function forcing $\eta_t \in (2, \infty)$ and $\lambda_t \in (-1, 1)$, where L and U are lower and upper bounds respectively. Accordingly, the standardized residuals are subject to the conditional skewed-*t* distribution, i.e.

$z_t \,|\, \mathcal{F}_{t-1} \sim skt(\eta_t, \lambda_t)$.

5.5 Empirical Results

5.5.1 Dynamic marginal distributions

Table 5.3 presents the results of the marginal models in which asymmetries on conditional heteroscedasticity are allowed for and higher moments are time-varying. Based on the robust standard errors, all parameters are highly significant. The autoregressive effect in the volatility specification is strong as d_0 is around 0.9265 suggesting extreme clustering effects. The parameter of negative returns c_0 is positive and greater than the parameter of positive returns b_0 indicating the presence of the leverage-effect for these three index futures returns. Meanwhile, the condition for covariance-stationarity is satisfied since $(b_0 + c_0)/2 + d_0 < 1$ for all the three series. To capture the dynamic nonnormality of the residuals, I let the conditional skewness λ_t and the kurtosis η_t depend on past information of negative and positive returns. The significance of the estimated results implies persistent dynamics of the higher-order moments over time. Figures 5.1-5.3 exhibit the conditional density and the conditional lower- and higher-order moments for each filtered return series.

Turning to panel A of Table 5.4, one can see that the average value of the time varying kurtosis parameter η_t is around 6.1896 with a range between 2.3621 and 16, exhibiting time-varying leptokurtosis. Likewise, the ranges of maximum and minimum values of the time varying skewness parameters λ_t are between 0.1540 and 0.4643 and between –0.2812 and –0.0199 respectively. The averages for λ_t are all negative except for MSCI SIN indicating that investors take care more about loss than about gain. For diagnosing the adequateness of the marginal model, I also employ the goodness-of-fit test suggested by Patton (2006a, 2006b) and Jondeau and Rockinger (2006). The details of the goodness-of-fit test have been introduced in

Chapter 3 and are not repeated again. Panel B of Table 5.4 presents the p-values of the test. As shown in the table, the specified marginal model for each series passes the test at the 5% level except for the forth moment of the Hang Seng series. These results indicate that these marginal distributions are generally adequate.

Table 5.3: Parameter Estimates of the TGARCH (1,1) Model and the Conditional Skewness and Kurtosis

Parameter	Index Futures Returns		
	Hang Seng	Nikkei 225	MSCI SIN
Conditional Variance			
a_0	**0.0080**	**0.0401**	**0.0215**
	(0.0001)	(0.0015)	(0.0007)
b_0	**0.0069**	**0.0386**	**0.0644**
	(0.0003)	(0.0011)	(0.0020)
c_0	**0.0660**	**0.1270**	**0.1082**
	(0.0006)	(0.0033)	(0.0027)
d_0	**0.9623**	**0.9084**	**0.9088**
	(0.0001)	(0.0025)	(0.0025)
Conditional Kurtosis			
a_1	**1.8329**	**1.0621**	**1.6901**
	(0.0287)	(0.0331)	(0.0418)
b_1	**0.2312**	**0.7091**	**0.2151**
	(0.0045)	(0.0073)	(0.0104)
c_1	**0.0122**	**0.4305**	**-0.1542**
	(0.0045)	(0.0084)	(0.0094)
Conditional Skewness			
a_2	**0.0398**	**0.0032**	**0.0729**
	(0.0002)	(0.0003)	(0.0011)
b_2	**-0.0182**	**0.0717**	**-0.0974**
	(0.0006)	(0.0001)	(0.0006)
c_2	**-0.0400**	**0.0234**	**-0.0526**
	(0.0005)	(0.0007)	(0.0010)
Loglike	3130.5328	2967.2045	2918.8259

Note: This table contains results of maximum likelihood estimator for marginal models. The specified models are

$Y_t = \mu + e_t, e_t = \sqrt{h_t} \cdot z_t, e_t^+ = \max(e_t, 0), e_t^- = \max(-e_t, 0), h_t = a_0 + b_0(e_{t-1}^+)^2 + c_0(e_{t-1}^-)^2 + d_0 h_{t-1}$

$z_t \mid \mathcal{F}_{t-1} \sim skt(\eta_t, \lambda_t)$

$\eta_t = \Xi(a_1 + b_1 e_{t-1}^+ + c_1 e_{t-1}^-), \lambda_t = \Xi(a_2 + b_2 e_{t-1}^+ + c_2 e_{t-1})$

The asymptotic standard errors are calculated according to Newey and McFadden (1994) and White (1994). Loglike is log likelihood. Figures in bold indicate significance at 5% level.

Table 5.4: Summary Statistics and Goodness-of-Fit Test for the Filtered Returns

	Filtered Index Futures Returns								
	Hang Seng			Nikkei 225			MSCI SIN		
	Panel A. Summary Statistics								
	λ_t	η_t	z_t	λ_t	η_t	z_t	λ_t	η_t	z_t
Maximum	0.1540	16.0000	3.5737	0.0000	16.0000	3.8723	0.4643	16.0000	4.7751
Minimum	-0.0199	2.5847	-6.1508	-0.2812	2.0328	-6.8158	-0.0364	2.4689	-5.4601
Mean	-0.0021	5.4580	-0.0119	-0.0276	6.7664	-0.0062	0.0044	6.3444	-0.0138
Std.			0.9895			0.9761			1.0038
Skewness			-0.2658			-0.3296			-0.0709
Kurtosis			5.1048			5.1222			5.1143
	Panel B. Goodness-of-Fit Test								
1st Moment		0.48			0.64			0.68	
2ndMoment		0.09			0.58			0.73	
3rd Moment		0.61			0.88			0.58	
4th Moment		0.02*			0.85			0.61	
K-S (20)		0.66			0.67			0.65	

Note: Panel A reports the summary statistics of the conditional skewness λ_t, the conditional kurtosis η_t, and the filtered returns z_t. Panel B reports the *p*-values of the goodness-of-fit test statistics for the conditional skewed-*t* distributional restriction of the general univariate model. The test contains two parts. The first part of the test, which is labeled as *k*th Moment, is the *k*th Moment LM Test (k = 1,..., 4) evaluating whether the *k*th centered moments of the transformed residuals u_t are serially correlated. I regress $(u_t - \bar{u}_t)^k$ on 20 own lags. Under the null of no autocorrelation of the residuals, the statistics $(n\text{-}20)\cdot R^2$ is distributed as a χ^2_{20} where n and R^2 are the sample size and the coefficient of determination of the regression. The second part of the test, which is labeled as K-S(20), employs Kolmogorov-Smirnov test within each bin to evaluate whether u_t are Uniform(0, 1), where the total number of the bin is 20. Under the null, the statistics is also distributed as a χ^2_{20}. * indicates significance at the 5% level.

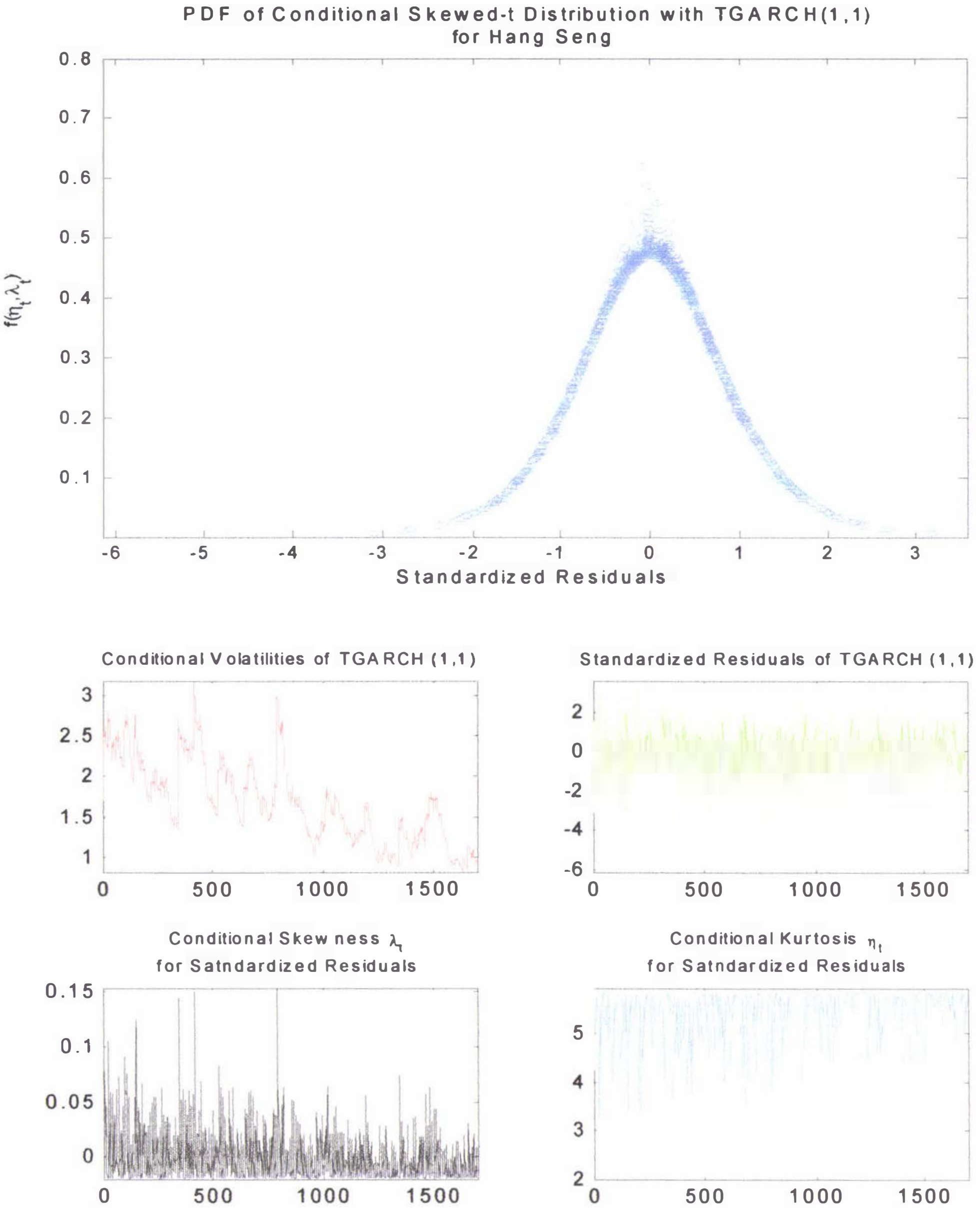


Figure 5.1

Note: Plots of the conditional density and the conditional lower- and higher-order moments for the filtered index futures returns of Hang Seng.

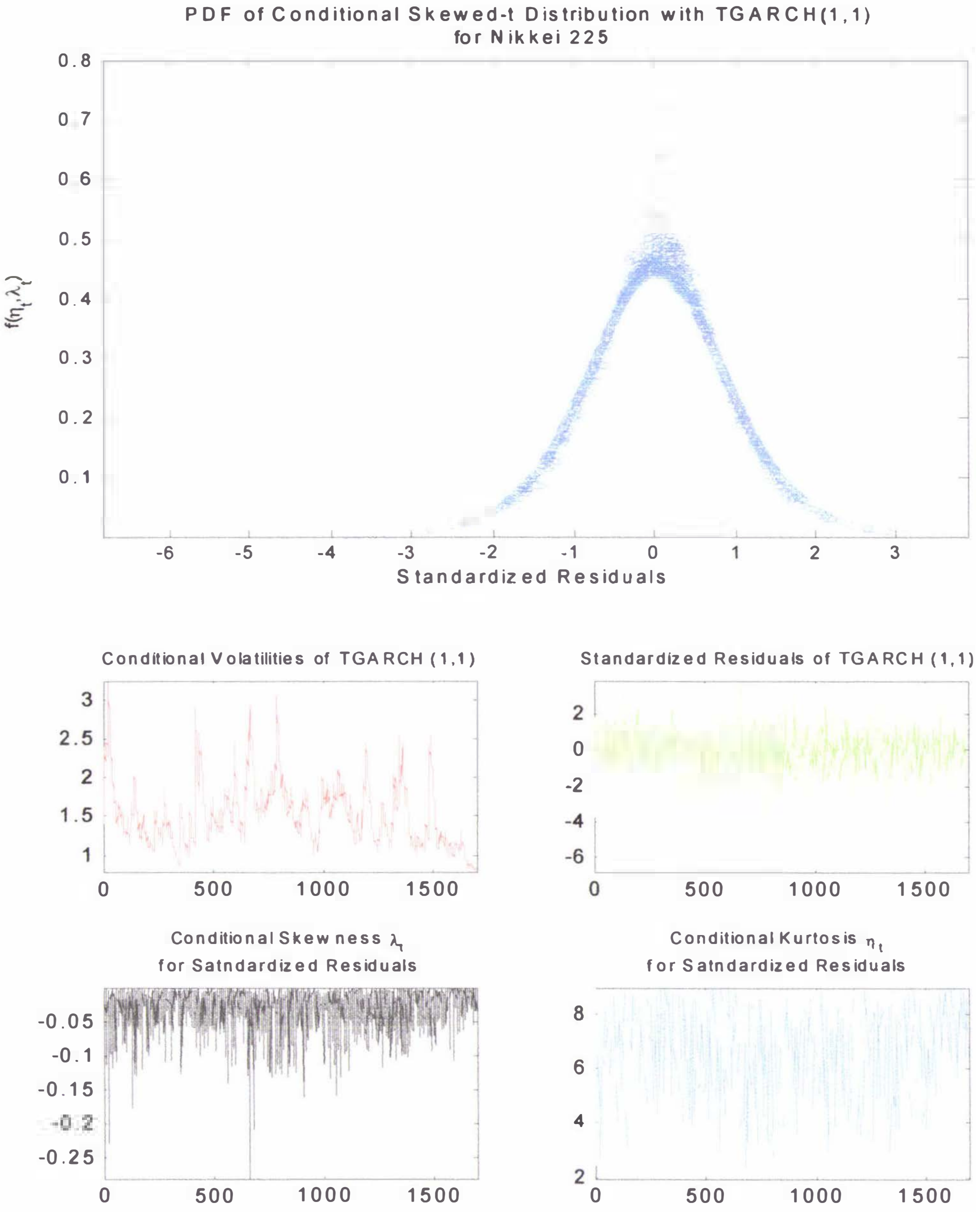


Figure 5.2

Note: Plots of the conditional density and the conditional lower- and higher-order moments for the filtered index futures returns of Nikkei 225.

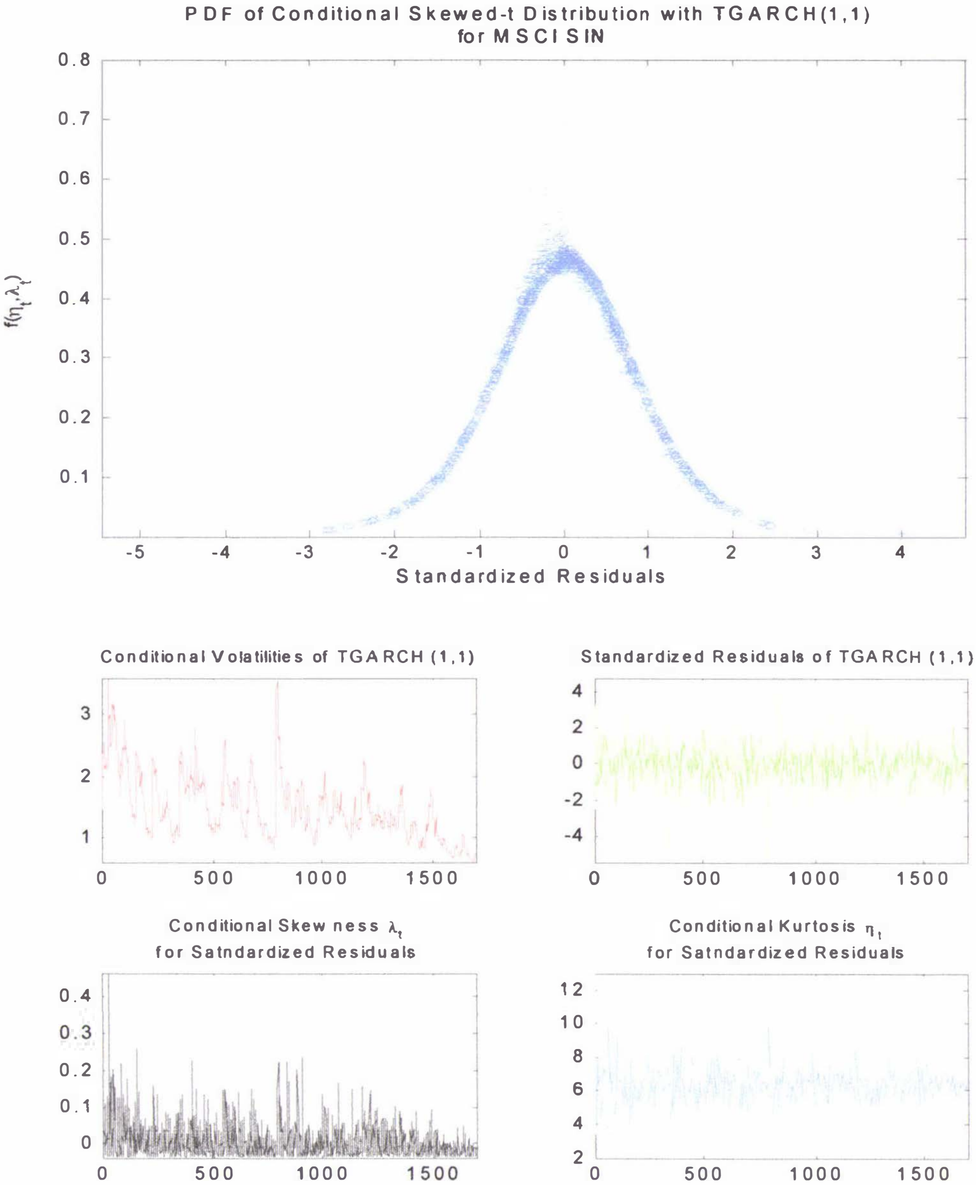


Figure 5.3

Note: Plots of the conditional density and the conditional lower- and higher-order moments for the filtered index futures returns of MSCI SIN.

5.5.2 Dynamic asymmetric tail dependences

Now let us investigate tail dependence with the static two-parameter Archimedean copulas. Panel A of Table 5.5 shows that unconditional tail dependence exhibits constant asymmetry. All the lower tail dependences are greater than the upper tail dependences except for the Hang Seng-Nikkei 225 pair in the BB7 model. For multiple model comparisons in the next section, I also report the estimation results of the full Gaussian copula $C_{Gaussian}(u,\upsilon,;\rho)=\Phi_{\rho}(\Phi^{-1}(u),\Phi^{-1}(\upsilon))$ in panel B of Table 5.5 where the marginals are subject to the standard normal distribution. Note that the standardized residuals z_t for the full Gaussian copula are filtered by the RiskMetrics model: $y_t=\mu+e_t$, $e_t=\sqrt{h_t}\cdot z_t$, $h_t=0.36e_t^2+0.94h_t$, $z_t\sim N(0,1)$.

In spite of the significant statistic values, however, information provided in Table 5.5 is insufficient to trace dynamic dependence. Table 5.6 sets out the results of the two-stage maximum likelihood estimation for the conditional two-parameter copulas. Although the intercept of conditional lower tail dependence ω^L in the BB1 model for the Hang Seng-MSCI SIN pair is insignificant, all other parameters in both the BB1 and the BB7 model are highly significant based on the asymptotic standard errors, suggesting that tail dependence is time-varying in all cases. Meanwhile, there are some common features between these two models in conditional upper tail dependence, but different features in conditional lower tail dependence. First, all of the intercepts ω^U are negative. Second, the dynamic upper tail dependence has a positive relationship with autoregressive terms ($\delta^U>0$) and a negative relationship with forcing variables ($\psi^U<0$). For the lower tails in the BB1 and the BB7 model, the parameters of the forcing variables ψ^L are all negative except for the Nikkei 225-MSCI SIN pair in the BB1 model.

The computation of the dynamic BB4 model is somewhat difficult. Each parameter varies sensitively with different starting values selected. Results in Table 5.6 thus are obtained with my best effort. Among the mixed results, the best fit is only found in the Nikkei 225-MSCI SIN pair while the significant parameters are only in the conditional lower tail dependences for the remaining pairs.

Table 5.7 summarizes the estimation results for the conditional tail dependence and time-varying parameters. As displayed in the table, the different average values of the time-varying lower and upper tail dependences provide us with the information of dynamic asymmetries. Five of the nine cases show that the average lower tail dependence parameters are greater than the average upper tail dependence parameters.

Table 5.5: Parameter Estimates of the Unconditional Copulas

Model	Parameter	Hang Seng vs. Nikkei 225	Hang Seng vs. MSCI SIN	Nikkei 225 vs. MSCI SIN
		Panel A: Two-Parameter Archimedean Copula		
BB1	α	**0.3391**	**0.4517**	**0.3496**
		(0.0039)	(0.0039)	(0.0038)
	β	**1.1868**	**1.2484**	**1.1399**
		(0.0041)	(0.0043)	(0.0038)
	τ^L	0.8204	0.7782	0.8085
	τ^U	0.2068	0.2576	0.1631
	Loglike	-187.3493	-295.7946	-155.2558
BB4	α	**0.6023**	**0.8200**	**0.5488**
		(0.0012)	(0.0017)	(0.0011)
	β	0.0316	0.0313	0.0315
		(0.0591)	(0.0626)	(0.0634)
	τ^L	0.3164	0.4294	0.2828
	τ^U	2.9767×10^{-10}	2.4123×10^{-10}	2.7765×10^{-10}
	Loglike	-160.9679	-252.5507	-138.3157
BB7	α	**1.2343**	**1.3168**	**1.1763**
		(0.0024)	(0.0026)	(0.0025)
	β	**0.4830**	**0.6666**	**0.4581**
		(0.0003)	(0.0004)	(0.00004)
	τ^L	0.2381	0.3535	0.2203
	τ^U	0.2466	0.3072	0.1973
	Loglike	-184.9232	-292.8262	-153.8232
		Panel B: Full Gaussian Copula		
Gaussian	ρ	**0.4359**	**0.5234**	**0.3775**
		(0.0009)	(0.0011)	(0.0008)
	Loglike	-200.1896	-300.0184	-147.3357

Note: This table shows the estimated parameters via the two-stage maximum likelihood estimates for both the unconditional two-parameter Archimedean and the Gaussian copulas. τ^L and τ^U are the unconditional lower and upper tail dependences respectively. Asymptotic standard errors are given in the parentheses and calculated by the propositions of Newey and McFadden (1994) and White (1994). Loglike represents log likelihood. Figures in bold indicate significant at 5% level.

Table 5.6: Parameter Estimates of the Conditional Two-Parameter Archimedean Copulas

Model	Parameter	Hang Seng vs. Nikkei 225	Hang Seng vs. MSCI SIN	Nikkei 225 vs. MSCI SIN
	ω^L	**0.1691**	-0.0086	**-2.2256**
		(0.0173)	(0.0175)	(0.1002)
	δ^L	**0.8676**	**0.8727**	**-0.2853**
		(0.0125)	(0.0399)	(0.0679)
	ψ^L	**-1.5320**	**-0.4759**	**0.9406**
BB1		(0.0024)	(0.0782)	(0.0123)
	ω^U	**-0.5871**	**-0.5755**	**-0.8652**
		(0.0299)	(0.0043)	(0.0516)
	δ^U	**0.4714**	**0.1426**	**0.1168**
		(0.0177)	(0.0036)	(0.0292)
	ψ^U	**-0.5642**	**-1.5895**	**-2.7935**
		(0.0244)	(0.0431)	(0.0091)
	Loglike	-190.8177	-299.9736	-159.1086
	ω^L	**-0.9998**	**-4.7465**	**-1.2881**
		(0.0164)	(0.0781)	(0.0509)
	δ^L	**0.9984**	**0.9918**	**0.9984**
		(0.0164)	(0.0163)	(0.0110)
	ψ^L	**-0.9741**	**-4.4554**	**0.2964**
BB4		(0.0160)	(0.0733)	(0.0308)
	ω^U	0.9392	1.5398	**0.6420**
		(0.8713)	(1.6326)	(0.0018)
	δ^U	0.3500	**-0.9537**	**-0.8627**
		(0.5635)	(0.1115)	(0.0142)
	ψ^U	**0.8512**	0.9933	**1.6269**
		(0.3691)	(1.2673)	(0.0089)
	Loglike	-9729.1119	-10973.8094	-9507.0763
	ω^L	**0.1487**	**-0.2260**	**-2.2458**
		(0.0131)	(0.00003)	(0.0094)
	δ^L	**0.8758**	**0.1394**	**-0.8142**
		(0.0108)	(0.0005)	(0.0071)
	ψ^L	**-1.2031**	**-1.2675**	**-0.0641**
BB7		(0.0087)	(0.0023)	(0.0094)
	ω^U	**-0.3658**	**-0.4062**	**-0.2008**
		(0.0100)	(0.0064)	(0.0283)

δ^U	**0.5224**	**0.1717**	**0.3919**
	(0.0149)	(0.0062)	(0.0249)
ψ^U	**-0.7457**	**-1.2619**	**-3.0209**
	(0.0268)	(0.0046)	(0.0447)
Loglike	-189.3858	-296.6299	-158.7936

Note: This table reports the estimated parameters of the conditional two-parameter Archimedean copulas via two-stage maximum likelihood estimates. The conditional tail dependences are specified as $\tau_t^L = \Lambda\left(\omega^L + \delta^L \tau_{t-1}^L + \psi^L \cdot |u_{t-1} - \upsilon_{t-1}|\right), \tau_t^U = \Lambda\left(\omega^U + \delta^U \tau_{t-1}^U + \psi^U \cdot |u_{t-1} - \upsilon_{t-1}|\right)$ where τ_t^L and τ_t^U are the conditional lower and upper tail dependences respectively, $\Lambda(k) = 1/[1 + \exp(-k)]$ is the logistic transformation to ensure $\tau_t^U, \tau_t^L \in (0, 1)$ at all time. Asymptotic standard errors are given in the parentheses and calculated by the suggestions of Newey and McFadden (1994) and White (1994). Loglike represents log likelihood. Figures in bold indicate significance at the 5% level.

Table 5.7: Summaries of the Conditional Tail Dependence and the Time-Varying Parameters for the Conditional Two-Parameter Archimedean Copulas

		Hang Seng vs. Nikkei 225				Hang Seng vs. MSCI SIN				Nikkei 225 vs. MSCI SIN			
		τ_t^L	τ_t^U	α_t	β_t	τ_t^L	τ_t^U	α_t	β_t	τ_t^L	τ_t^U	α_t	β_t
	Maximum	0.5793	0.5479	0.9442	1.8582	0.5793	0.5479	0.8663	1.8582	0.5793	0.5479	0.6832	1.8582
BB1	Minimum	0.0239	0.1412	0.1634	1.1181	0.1708	0.1029	0.3348	1.0825	0.0961	0.0229	0.2375	1.0169
	Mean	0.1989	0.2037	0.3689	1.1841	0.2958	0.2587	0.4560	1.2545	0.1774	0.1590	0.3549	1.1407
			$- < +$				$- > +$				$- > +$		
	Maximum	0.5793	0.9122	0.7426	7.5431	0.5793	0.9591	0.7426	16.5905	0.5793	0.9434	0.7426	11.8892
BB4	Minimum	0.0000	0.5000	0.0001	1.0000	0.0000	0.2061	0.0001	0.4389	0.0000	0.2123	0.0001	0.4472
	Mean	0.0006	0.8520	0.0011	4.4450	0.0003	0.6964	0.0010	2.4038	0.0006	0.6262	0.0019	1.8108
			$- < +$				$- < +$				$- < +$		
	Maximum	0.5638	0.5793	1.9737	1.2094	0.5479	0.5793	1.9737	1.1520	0.5479	0.5793	1.9737	1.1520
BB7	Minimum	0.0504	0.1478	1.1246	0.2319	0.1897	0.1552	1.1319	0.4169	0.0826	0.0210	1.0155	0.2780
	Mean	0.2535	0.2430	1.2316	0.5197	0.3602	0.3079	1.3223	0.6853	0.2235	0.1918	1.1811	0.4629
			$- > +$				$- > +$				$- > +$		

Notes: This table provides the summaries of conditional lower and upper tail dependences and of time varying parameters based on the results of Table 5.6. Symbol "–" and "+" indicate the average values of conditional lower and upper tail dependences respectively.

5.5.3 Model evaluation

I next use the test procedure (detailed in Chapter 4), based on the KLIC measure of differences between a true multivariate density $f^{True}(y_{1,t}, y_{2,t}, \ldots, y_{m,t})$ and a parametric multivariate density $f(y_{1,t}, y_{2,t}, \ldots, y_{m,t};\ \boldsymbol{\theta})$ ($\boldsymbol{\theta} = \{\hat{\boldsymbol{\theta}}_M,\ \hat{\boldsymbol{\theta}}_C\}$), to evaluate the model fitting. The test procedure is designed for multiple comparisons via Hansen's (2005) SPA test. The null hypothesis that the benchmark of the copula models is superior to others is rejected by a small p-value. For the test, the true and the parametric multivariate density are estimated by the nonparametric and copula methods respectively. Figure 5.4 shows the three-dimensional plots and contours of the nonparametric kernel densities for the three index futures return pairs. Note that for the reason to maintain the power of the SPA test, one needs to have the subsample size reasonably large enough (i.e. $n > 1000$).[32] However, in this research, I only collect 1690 observations. Meanwhile, Inoue and Kilian (2004) argue that the in-sample test of predictability is more powerful than the out-of-sample test after correcting for data snooping. I therefore only evaluate the in-sample model fitting.

I obtain the p-values of the SPA test by making multiple model comparisons. Along with the procedure of the test, each model is regarded as a benchmark model and then is compared with the remaining three models. For the stationary bootstrap, I set the number of bootstraps to be 1000. The estimated p-values are reported in Table 5.8 and the decision rule is basically dependent on the largest p-value. For instance, for the Hang Seng-MSCI SIN pair, the BB7 model is the most preferred with the largest p-value 0.121. Hence, I fail to reject the null hypothesis that the other remaining three models are no

[32] See Section 4 of Hansen (2005).

better than it. In contrast, for the same pair, the models structured by the BB4 and Gaussian copulas are clearly dominated by the other remaining models with the smallest p-value 0.000. In general, model overfitting is only found in the Hang Seng-Nikkei 225 pair. The BB7 model for the Hang Seng-MSCI SIN pair and the BB1 model for the Nikkei 225-MSCI SIN pair are all superior to the full Gaussian one. Figures 5.5 and 5.6 exhibit the conditional tail dependence parameters and the time-varying parameters for these two cases. Back to Table 5.7, the average values of the conditional lower and upper tail dependence parameters for the above two successful cases further reveal that the probability of downside market comovements is greater than the probability of upside market comovements between these markets.

5.6 Portfolio VaR and Diversification Benefit

Based on the above model evaluation, I now investigate whether the portfolio VaR estimation can be improved by using more sophisticated copula models.

5.6.1 Portfolio VaR

The definition of PVaR is the same as in Chapter 3. Here, I again follow Bradley and Taqqu's (2003) suggestion to estimate the PVaR.

$$\mathrm{VaR}_{\xi}(R_t) = \mu_{R_t} + \sigma_{R_t} q_{\xi} \qquad (5.10)$$

where μ_{R_t} and σ_{R_t} are the portfolio expected return and the portfolio variance respectively. In Equation (5.10), q_{ξ} is the ξ quantile of the standardized version of R_t, i.e.

$q_{\xi} = -F^{-1}_{\tilde{R}_t}(\xi)$ where $\tilde{R}_t = (R_t - \mu_{R_t})/\sigma_{R_t}$. Because there is no closed form for the standardized quantile q_{ξ}, I then approximate it by the Monte Carlo simulation method.

5.6.2 Monte Carlo simulation for the standardized quantile

The first step of the standardized quantile calculations involves the simulation of a vector of bivariate standardized random variables ($z_{1,t}$, $z_{2,t}$) whose dependence structure obeys a copula function. To start the simulation, one should first calculate the partial derivation of the bivariate copula function $C(u,\upsilon)$, i.e. $C_{2|1}(u \mid \upsilon) = \partial C(u,\upsilon)/\partial \upsilon$ where u and υ are the marginal distributions. For both the BB1 and the BB7 copulas, it might be hard to work out $C_{2|1}(u \mid \upsilon)$ by hand. However, one can easily handle the problem in Matlab. For instance, for the BB1 copula with parameters α (in terms of a) and β (in terms of b), one can obtain $C_{2|1}(u \mid \upsilon)$ via the following commands:

```
syms C u v a b;
C = (1+((u^(-a)-1)^b+(v^(-a)-1)^b)^(1/b))^(-1/a);%The BB1 copula.
dC_dv = diff(C,'v'); %Calculating C2|1(u|v).
dC_dv = (1+((u^(-a)-1)^b+(v^(-a)-1)^b)^(1/b))^(-1/a)*...
        ((u^(-a)-1)^b+(v^(-a)-1)^b)^(1/b)*(v^(-a)-1)^b*...
        v^(-a)/v/(v^(-a)-1)/((u^(-a)-1)^b+(v^(-a)-1)^b)/...
        (1+((u^(-a)-1)^b+(v^(-a)-1)^b)^(1/b))
```

Then the simulation procedure for the conditional BB1 copula is as follows:

1. Generate υ from $U(0, 1)$ with the sample size n.
2. Let $s = C_{2|1}(u \mid \upsilon)$, and generate s from $U(0, 1)$ with the sample size n.
3. Obtain the copula-based standardized marginal u by solving the inverse conditional copula function $u = C^{-1}_{2|1}(s \mid \upsilon, \hat{\boldsymbol{\theta}}_C)$ where $\hat{\boldsymbol{\theta}}_C = \{\alpha_t, \beta_t\}$ (t = 1, 2, …, n). Because there is no closed form of the inverse conditional copula function for the

BB1 copula, I then calculate the inversion numerically using the bisection method.

4. Obtain the standardized returns $z_{1,t}$ and $z_{2,t}$ via the inverse marginal distributions of the conditional skewed-t distribution $F_{skt}^{-1}(u \mid \hat{\boldsymbol{\theta}}_{M,u})$ and $F_{skt}^{-1}(\upsilon \mid \hat{\boldsymbol{\theta}}_{M,\upsilon})$ where $\hat{\boldsymbol{\theta}}_{M,u} = \{\eta_{u,t}, \lambda_{u,t}\}$ and $\hat{\boldsymbol{\theta}}_{M,\upsilon} = \{\eta_{\upsilon,t}, \lambda_{\upsilon,t}\}$ (t = 1, 2, ..., n). Then I calculate the standardized quantile by estimating the quantile of the sorted standardized portfolio returns $w \cdot z_{1,t} + (1-w) \cdot z_{2,t}$ where w is the portfolio weight.

5. Repeat the above steps r times and finally obtain the approximated standardized quantile by the average of r simulations. Then the PVaR is easily estimated via Equation (5.10).

Similarly, I can also do the simulation of q_{ξ} for the conditional BB7 copula. Figure 5.7 presents the scatter plots of the bivariate simulated random variables generated from the conditional BB1 copula and the conditional BB7 copula with time-varying skewed-t-distributed marginals respectively.

Following the above procedure, I first simulate the standardized quantile q_{ξ} by generating n = 1,690 realizations with r = 10,000 trials, and then calculate the PVaR via Equation (5.10). Note that the portfolios are all equally weighted. Panel A of Table 5.9 reports the results of the conditional BB1 copula-based PVaRs for the Nikkei 225-MSCI SIN pair and the conditional BB7 copula-based PVaRs for the Hang Seng-MSCI SIN pair as well as the simple full Gaussian copula-based PVaRs at both the 95% and 99% confidence levels. Consistent with the finding of Ané and Kharoubi (2003), the results show that at a lower confidence level the Gaussian copula-based PVaRs overestimate the risk, and at a higher confidence level the Gaussian copula-based PVaRs underestimate the risk.

In addition, I also calculate the diversification benefit, which is the percentage decrease in the PVaR, using this formula: |Gaussian copula-based PVaR – conditional copula-based PVaR | / (Gaussian copula-based PVaR). The diversification benefits reported in panel B of Table 5.9 clearly exhibit a high degree of conservatism associated with the full Gaussian copula-based PVaR at the 99% level. In other words, the conditional Archimedean-copula-based PVaR estimations are improved, which implies that a greater amount of diversification benefits can be gained at a higher confidence level.

Table 5.8: Results of the SPA Test

Model	Index Futures Returns Pair		
	Hang Seng vs. Nikkei 225	Hang Seng vs. MSCI SIN	Nikkei 225 vs. MSCI SIN
BB1	0.329	0.057	**0.049**
BB4	0.000	0.000	0.000
BB7	0.681	**0.121**	0.037
Gaussian	**0.765**	0.000	0.000

Note: This table exhibits the p-values of Hansen's (2005) SPA test. The test is based on the KLIC loss function with stationary bootstrap. The number of the resample is 1000. The random sample block length for the stationary bootstrap is determined by Politis and White's (2004) suggestion. Each model is regarded as a benchmark model comparing with other three models. The p-values in bold indicate that the best of the remaining models is no better than the benchmark model.

Table 5.9: Portfolio VaR and Diversification Benefit

Equally Weighted Portfolio of Index Futures Returns			
Hang Seng- MSCI SIN		Nikkei 225- MSCI SIN	
95%	99%	95%	99%
Panel A. Portfolio VaR			
-1.9915 (BB7)	-3.3340 (BB7)	-1.7073 (BB1)	-2.7842 (BB1)
-2.0308 (Gaussian)	-2.8803 (Gaussian)	-1.7088 (Gaussian)	-2.4193 (Gaussian)
Panel B. Diversification Benefit			
1.94%	15.75%	0.09%	15.08%

Note: Panel A reports the conditional copula-based portfolio VaRs estimated by the Monte Carlo simulation for the Hang Seng- MSCI SIN and the Nikkei 225- MSCI SIN pairs respectively. The portfolios are all equally weighted. The Monte Carlo simulation procedures are based on 5,000 realizations with 10,000 trials. Panel B reports the diversification benefits, which is the percentage decrease in the portfolio VaR, calculated by the value of |Gaussian copula-based portfolio VaR – conditional copula-based portfolio VaR| / (Gaussian copula-based portfolio VaR).

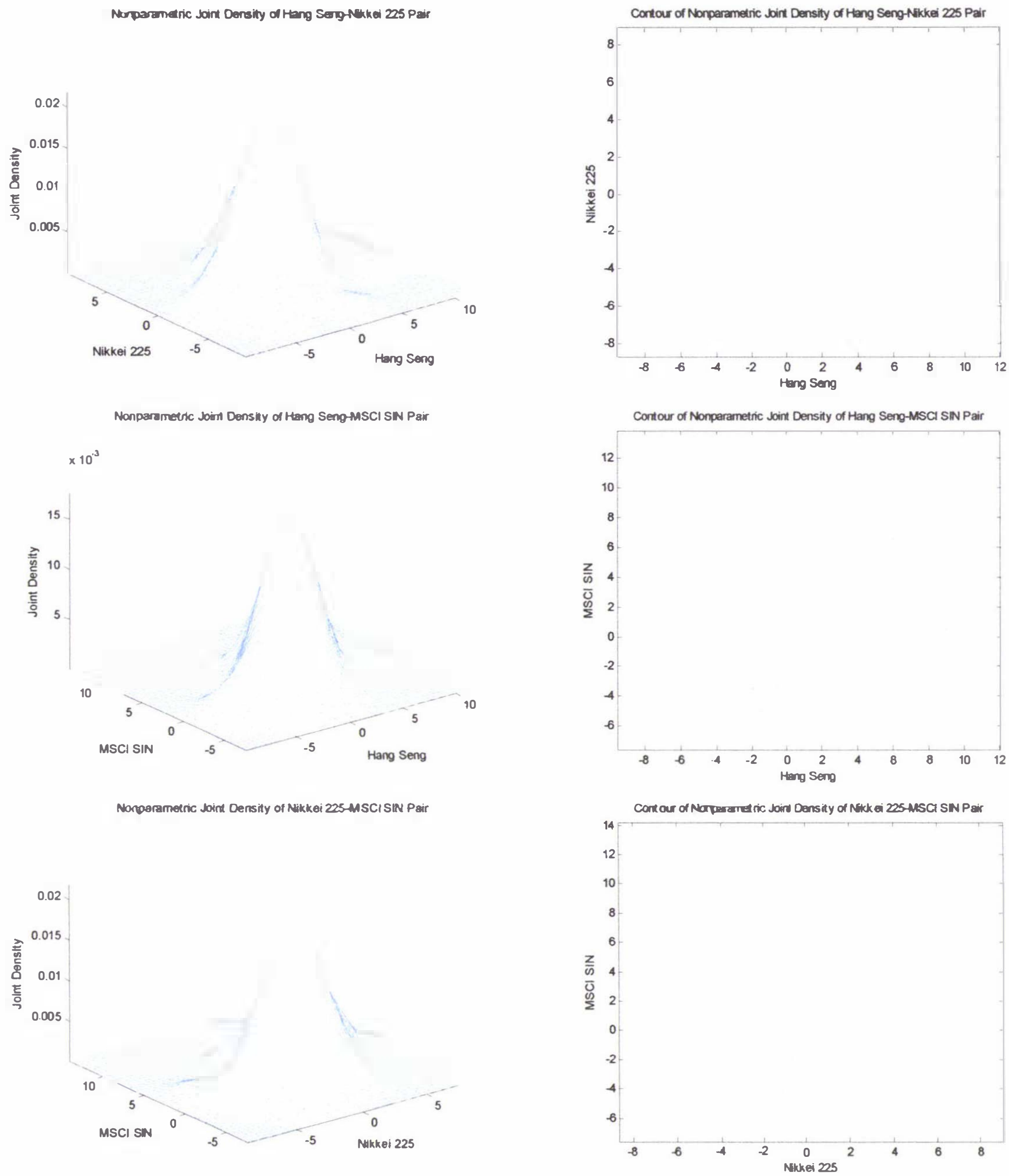


Figure 5.4

Note: Three-dimensional plots and contours of the nonparametric joint density for the three index futures returns.

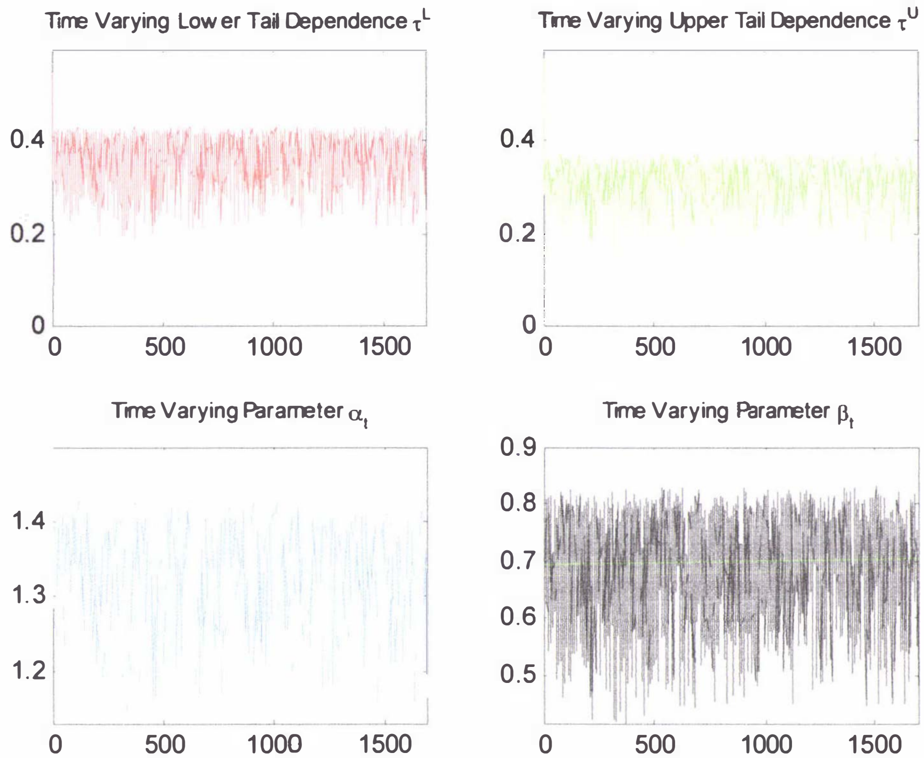


Figure 5.5

Note: Plots of time-varying tail dependences and time-varying parameters of the conditional BB7 copula for the Hang Seng-MSCI SIN pair.

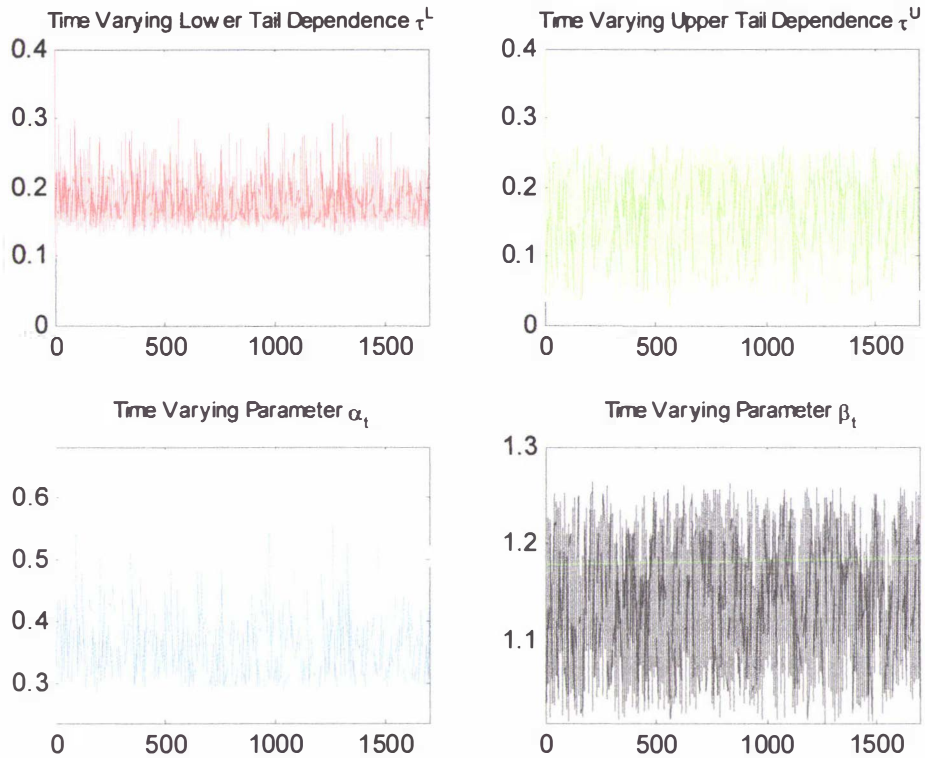
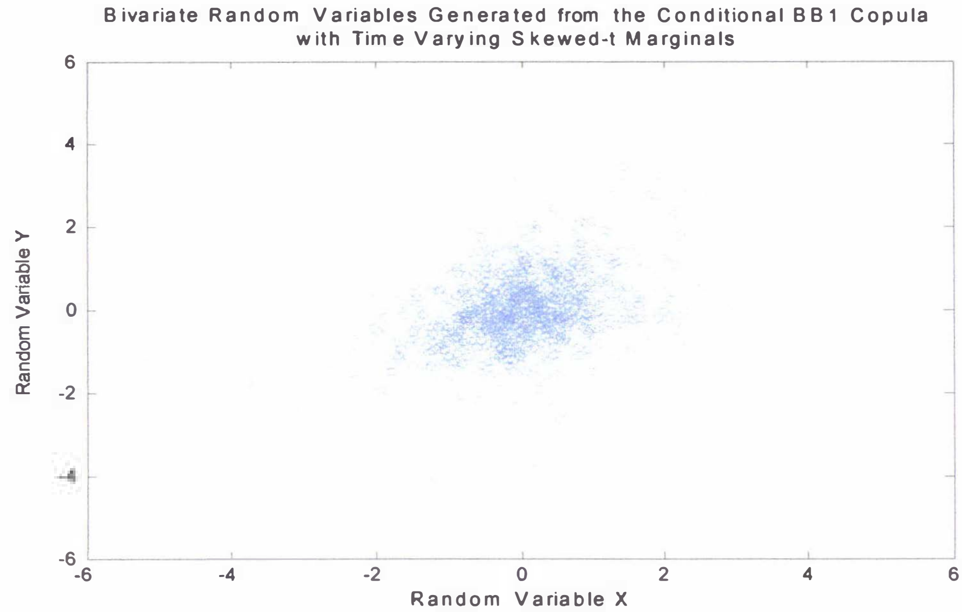
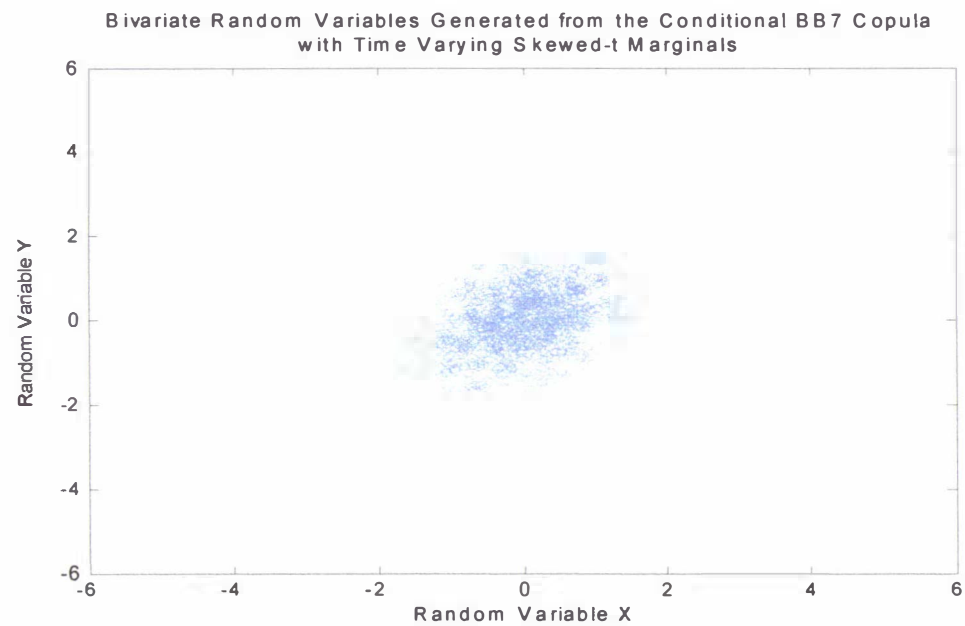


Figure 5.6

Note: Plots of time-varying tail dependences and time-varying parameters of the conditional BB1 copula for the Nikkei 225-MSCI SIN pair.



(a)



(b)

Figure 5.7

Note: Scatter plots of the bivariate simulated random variables, which are generated from the conditional two-parameter copula with time-varying skewed-*t*-distributed marginals. The parameters are based on the results of Table 5.3 and 5.6. The number of the realizations is 5,000. (a) the BB1 copula. (b) the BB7 copula.

5.7 Concluding Remarks

In this Chapter, I employ three two-parameter Archimedean copulas (BB1, BB4, and BB7) to investigate dynamic asymmetric tail dependences between two of three Asian developed futures markets, Hong Kong, Japan and Singapore, during the post Asian financial crisis period. I use the Hang Seng, Nikkei 225, and Morgan Stanley Capital International index futures (MSCI SIN) traded on these three futures markets respectively. For the Archimedean copula estimation, I first model the marginals using conditional Hansen's (1994) skewed-t distribution, and find that the higher moments of each filtered index futures return are time dependent. This indicates that investors' heterogeneous beliefs are time-varying. I next extend a class of two-parameter copulas with time-varying tail dependence to capture dynamic asymmetry. The estimated results provide strong evidence of asymmetric dependence across all the Asian developed futures markets. Moreover, based on model evaluation, I find that the BB7 copula for the Hang Seng-MSCI SIN pair and the BB1 copula for the Nikkei 225-MSCI SIN pair outperform the simple full Gaussian copula. These better-fitted models help to reveal that the probability of dependence in bear markets is higher than in bull markets, suggesting downside dependent risk in these markets. Finally, after model evaluation, I estimate the copula-based PVaRs and the diversification benefits at both the lower and higher confidence levels. The results indicate that the conditional-copula-based PVaR estimations are improved, and the models suggest that a greater amount of diversification benefits can be gained at a higher confidence level. Therefore, these sophisticated copula models are adequate and can be considered for financial risk management.

Chapter 6 Conclusions

6.1 Summaries

Traditional analysis of portfolio theory is limited to that of the first- and second-order moments of the multivariate distribution of asset returns. Within Markowitz's (1952) mean-variance framework, asset returns are assumed to be linearly correlated and normally distributed, so that investors' beliefs are homogeneous (which implies that they view upside and downside risks with equal distaste). While traditional portfolio theory provides a simple way to quantify the market risk, it often does not comport with empirical evidence. According to previous studies, portfolio returns often follow asymmetric distributions with heavy tails rather than normal ones, and investors typically do distinguish between upside and downside risks. The resultant coskewness and cokurtosis in the multivariate return distribution cannot be accounted for by traditional portfolio theory. In the context of modern financial risk management, Lo (1999) argues that optimal total risk management should be a synthesis of three P's, *price* (for considering how much an agent must pay for hedging various risks), *probabilities* (for assessing the likelihood of various risks), and *preferences* (for deciding how much risk to bear and how much to hedge). So, there is a need to find reasonably good approximations to various multivariate asset return distributions in the real world. In response to such a need, many attempts have recently been made to fit copulas, especially Archimedean copulas, to multivariate financial data. Archimedean copulas enable practitioners to construct flexible multivariate non-Gaussian distributions with nonlinear dependence and various asymmetries.

Although the Archimedean-copula-based portfolio theory has been acknowledged as an efficient tool over traditional one for financial risk management, several relevant questions remain untouched and unanswered. This dissertation deals with three of them, all involving the Archimedean copula-based models. The first is whether the Archimedean copula-based PVaR model outperforms the Gaussian copula-based PVaR model in out-of-sample forecasting, and is studied in Chapter 3. There, I fit the PVaR models to three international equity indexes (FTSE 100, Nikkei 225, and S&P 500) using three Archimedean copulas (the Clayton, the Gumble, and the BB1 copula) and one elliptical copula (the Gaussian copula). These four copulas are considered to be able to capture four possible types of market comovements between two of the three international equity markets. To see how sensitive the copula-based PVaR measure is to different marginals specified, I choose for the marginal each of three popular distributions (the Gaussian, the Student's *t*, and Hansen's (1994) skewed-*t* distribution) as the possible underlying data generating process for each copula. In order to address the data snooping problem, I perform Hansen's (2005) SPA test based on the quantile loss function with stationary bootstrap when evaluating the in-sample and the out-of-sample copula-based PVaR model performance. My empirical findings carry three messages pertaining to the application of the copula-based PVaR model. First, the Archimedean-copula-based PVaR model, especially the Clayton copula-based model, has better forecasting performance than the Gaussian-copula-based PVaR model in most cases in both the in-sample and out-of-sample periods. Second, the disparity between the in-sample and the out-of-sample evaluation results of the copula-based PVaR models clearly indicates that in-sample analysis should be combined with out-of-sample evaluation.

Third, choosing a portfolio model simply based on the minimum value of PVaR without data snooping check can be potentially dangerous to risk management.

The second question to which Chapter 4 is devoted is how to evaluate the non-Gaussian multivariate density forecast. In that chapter, I propose a test procedure, by extending Vuong's (1989) likelihood ratio test, to evaluate the Archimedean-copula-based multivariate density forecasts, and apply the procedure to foreign exchange markets. My test procedure has three merits. Fist, it is conducive to fully ranking competing sophisticated models with the non-Gaussian-distributed multivariate densities. Second, it allows for model misspecification in both marginal and copula functions under the null and the alternative hypothesis. Third, it can investigate how sensitive each candidate copula model's forecasting performance is to different marginals chosen as the possible data generating processes. To experiment on the test, I consider five Archimedean copulas (the Clayton, the Frank, the Gumbel, the BB1, and the BB7 copula) and two elliptical copulas (the Gaussian, and the *t* copula) as candidates for the test. As in Chapter 3, I choose for the margin each of three distributions (the standard normal, the Student's *t*, and Hansen's (1994) skewed-*t* distribution) as the possible underlying data generating process for each copula. Further, in order to avoid model misspecification (data snooping), I conduct Hansen's (2005) SPA test via the KLIC loss function when evaluating model forecasting performance. The test results provide strong evidences that the multivariate densities modeled by Archimedean copulas outperform those modeled by elliptical copulas in both the in-sample and out-of-sample periods. Unlike previous studies on multivariate density forecast evaluation where the multivariate densities are assumed to be normal, my proposed test allows the multivariate density to go beyond the normal dependence structure, and moreover is practically useful. Thus, the test is straightforward to apply in financial risk management.

Chapter 5 studies the third question: Will the PVaR estimation be improved if the Archimedean copula model takes into account conditional asymmetric tail dependence and time-varying investors' heterogeneous beliefs as represented by the conditional skewed-t distribution? There, I employ three two-parameter Archimedean copulas (the BB1, the BB4, and the BB7 copula) to investigate dynamic asymmetric tail dependence between two of three Asian developed futures markets represented by the Hang Seng index futures, the Nikkei 225 index futures, and the MSCI SIN index futures, during the post-crisis period. For modeling the marginal distribution, I consider Hansen's (1994) conditional skewed-*t* distribution and find that the higher moments of each filtered index futures return are time dependent. This indicates that investors' heterogeneous beliefs are time-varying. I next modify the three two-parameter copulas by allowing their tail dependence parameters to be time-varying, in order to capture possible dynamic asymmetrics. The estimation results provide strong evidence of conditional asymmetric dependence across all the Asian developed futures markets. Moreover, I apply the test for multivariate density forecast evaluation as proposed in Chapter 4 to the examination of model fitting. I find that the BB7 copula for the Hang Seng-MSCI SIN pair and the BB1 copula for the Nikkei 225-MSCI SIN pair outperform the simple symmetric Gaussian copula. Based on the model evaluation results, I estimate PVaRs and diversification benefits at both the lower and higher confidence levels. The results clearly show that the specified conditional copula models can improve the PVaR estimation, and a greater amount of diversification benefits can be reaped at a higher confidence level.

To sum up, my research work makes four contributions to the financial economics literature. First, I provide important empirical evidence that the Archimedean copula-based PVaR model generally has better forecasting performance than the Gaussian copula-based PVaR model. Therefore, financial risk managers should consider the use of the Archimedean copula-based PVaR model when attempting to forecast extreme downside dependent risk.

Second, I propose a practicable method for evaluating multivariate density forecast based on the Archimedean copula functions. This method is useful not only for examining the quality of multivariate density forecast *per se*, but also for checking the misspecification of the forecasting model.

Third, I show portfolio managers that the conditional two-parameter Archimedean copula-based PVaR model outperforms the simple full Gaussian copula-based one at a higher confidence level, in the cases where tail dependence is dynamic and investors' heterogeneity is time varying. So the former model should be preferred to the latter for the purpose of international diversification.

Fourth, regarding the full parametric estimation of the copula-based model, my work cautions risk managers not to overlook the problem of model risk. The choice of the distribution function for marginals does affect the performance of the copula model. In addition, the disparity between the empirical in-sample and out-of-sample results suggests that in-sample model fitting should be supplemented by out-of-sample evaluation. In a word, the selection of the copula-based model needs to be done with care.

6.2 Limitations and Further Research

It should be emphasized that my work reported in this dissertation is among the first steps in the application of Archimedean copulas to the problems of financial risk management, and more research needs to be done on the application and the estimation method of the copula-based model in the following four directions. *First*, in most previous researches (also including my studies presented in this dissertation), we saw that valuation solutions yielded by copula functions are easy to apply and flexible for the bivariate case - e.g., a portfolio only containing two risk factors. In reality, however, a risk manager might be interested in the dependence structure between multivariate risk factors in a portfolio for the purpose of risk diversification. Therefore, the multivariate copula-based model is desired. Notice that, to derive its density function from the multivariate copula, it requires a substantial shift from a simple closed form to a more sophisticated method (for instance, higher-order partial derivatives), especially to the two-parameter Archimedean copula functions. *Second*, throughout the whole dissertation, I have been only concerned with the equally weighted copula-based portfolio model. In practice, for the utility maximization and optimal asset allocation, investors might want to know whether a model which fits one portfolio the best will also fit other portfolios. Hence, it would be interesting to evaluate the performances of the copula-based models over a set of portfolio weights. *Third*, although I have taken the data snooping bias into account throughout the whole dissertation to mitigate model risk, model risk can be a serious problem for exotic positions due to factors such as unobservable variables (e.g. volatilities), interactions between risk factors, and numerical approximations etc. As we have seen, the adequateness of model fitting directly affects the accuracy of the risk measure. Therefore, for the parametric copula model estimation, an alternative way to mitigate model risk (parameter uncertainty) is the Bayesian approach. In this context, we can employ the Markov Chain Monte Carlo (MCMC) technique with the Gibbs sampling method by setting up conjugate priors for the parameters and draw one parameter from the prior distribution conditional on other given parameters. We then can obtain the parameter's posterior distribution by the product of the prior and the likelihood function of the parameter, and further confirm its accuracy. *Fourth*, for the copula model construction, it might be necessary to estimate risk measures across diverse positions with two (or more) different types of market risk. Although we all traditionally assume that a portfolio contains same type of market risks (that is, the distributions of risk factors in the portfolio are identical), it is hard to sustain such an assumption in the real world. Hence, one may build (copula-based) models to estimate the risk measures of two very diverse types of market risk. For instance, one marginal risk model might be based on a normal distribution and the other on, say, a skewed-t distribution. However, the cost of these considerations might be increased by the difficulty and the complexity of computation. I leave these challenging works for the future.

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