On Two Problems of Arithmetic Degree Theory

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To the memory

of

Wolfgang Vogel

whose guidance was invaluable.

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ABSTRACT

The reader of this thesis should already have a basic understanding of ideal theory. For this reason it is recommended that a good introduction to this subject would be gained from reading D. G. Northcott's book "Ideal Theory", paying special attention to chapters one and three. This thesis consists of three chapters, with chapter one providing the definitions and theorems which will be used throughout. Then I will be considering two problems on the arithmetic degree of an ideal, one posed by Sturmfels, Trung and Vogel and the other by Renschuch. These problems will be described in the introductions to chapters two and three.

CHAPTER 1

PRELIMINARY RESULTS

Let I be a homogeneous ideal of the polynomial ring $S = F[x_0, \ldots, x_n]$ where F is any field.

Let P be a prime ideal belonging to I.

If P is isolated, we know from the corollary of theorem 3 of Northcott's book [6, p.19], that the primary component corresponding to P is the same for all normal decompositions of I.

However, if P is embedded, then this is not true, as the following example [6, p.30] shows.

Consider the ideal (x^2, xy) in the ring $F[x, y], F$ any field.

It is shown in Northcott's book [6, p. 30] that

$(x) \cap (y + ax, x^2)$ (where a is any element of F),

$(x) \cap (y + bx, x^2)$ (where $b \in F,\ b \neq a$),

and $(x) \cap (x^2, xy, y^2)$

are all normal decompositions of (x^2, xy) with $(y + ax, x^2)$, $(y + bx, x^2)$, (x^2, xy, y^2) all (x, y) – primary.

So the primary component corresponding to an embedded prime ideal need not be unique.

Therefore, if we have two normal primary decompositions of I, one having a primary component Q_1 corresponding to an embedded prime P, and the other having a primary component Q_2 corresponding to P, $Q_1 \neq Q_2$, then in general, the classical length multiplicity of Q_1 does not equal the classical length multiplicity of Q_2.

However, in arithmetic degree theory, we do have a way of defining the length multiplicity of an embedded component of an ideal which is well-defined.

The definitions that are needed to do this are given later in this chapter.

We will also need some basic facts about the Hilbert function from the classical degree theory.

Let $V(n+1,t)$ denote the F–vector space consisting of all forms of degree t in $x_0,\ldots,x_n$.

Then $\dim_F V(n+1,t) = \binom{t+n}{n}, \quad t \geq 0,\ n \geq 0.$

Let $V(I,t)$ be the F–vector space consisting of all forms in $V(n+1,t)$ which are in I.

Definition 1. The function $H(I,-) : Z^+ \to Z^+$ [10, p.43] defined by $H(I,t) = \dim_F V(n+1,t) - \dim_F V(I,t)$ is called the Hilbert function of I.

For large enough t, the Hilbert function is a polynomial $P(I,t)$ in t with coefficients in Z. The degree $d\,(0 \leq d \leq n)$ of this polynomial is called the dimension of I and is denoted by $\dim(I)$.

The polynomial $P(I,t)$ can be written in the following form:
$P(I,t) = h_0(I)\binom{t}{d} + h_1\binom{t}{d-1} + \ldots + h_d$ [10, p.45] where $h_0(I)$ is a positive integer.

The leading coefficient of $P(I,t)$, namely $h_0(I)$, is called the degree of I.

There is of course a great deal of theory on the Hilbert polynomial, but for our purposes the following definition and theorem will suffice.

Let $I = (f_1,\ldots,f_t)$.

Definition 2. I is said to be a complete intersection if $(f_1,\ldots,f_{i-1}) : f_i = (f_1,\ldots,f_{i-1})$ for all $i = 1,\ldots,t$.

Theorem 1 [10, p.46]. Let the generators $f_1,\ldots f_t$ of I be forms of degrees $s_1,\ldots,s_t$ respectively. If I is a complete intersection then $h_0(I) = s_1\ldots s_t$.

We will now state the other definitions, theorems and propositions that will be used in chapters two and three.

Definition 3 [5, p.1]. Given any homogeneous ideal I and prime ideal P in S, we define J to be the intersection of the primary components of I with associated primes strictly contained in P. We let $J = S$ if there are no primes p belonging to I with $p \subsetneq P$.

Let Q be a P–primary ideal belonging to I.

Definition 4 [3]. We define the length–multiplicity of Q, denoted by $\mathrm{mult_I(P)}$, as the length of a maximal strictly increasing chain of ideals, $I \subseteq J_\ell \subset J_{\ell-1} \subset \ldots \subset J_2 \subset J_1 \subset J$ where each J_k equals $q \cap J$ for some P–primary ideal q.

As we will be making repeated use of an algorithmic approach to calculate $\mathrm{mult_I(P)}$ it is convenient to state it here, followed by a theorem.

Step 1. Take a maximal strictly increasing chain of primary ideals from Q to P.

$$Q \subset \ldots \subset Q_{i-1} \subset Q_i \subset \ldots P. \tag{1}$$

Step 2. Intersect each primary ideal in (1) with J.

$$Q \cap J \subseteq \ldots \subseteq Q_{i-1} \cap J \subseteq Q_i \cap J \subseteq \ldots \subseteq P \cap J = J. \tag{2}$$

Step 3. Eliminate duplicates in (2) in order to get a strictly increasing chain of ideals in the sense of definition 4.

$$Q \cap J =: J_\ell \subset J_{\ell-1} \subset \ldots \subset J_1 \subset J. \tag{3}$$

Note: If P is an isolated prime ideal of I, then $\mathrm{mult_I(P)}$ gives the classical length multiplicity of Q.

Theorem 2 [5, p.2]. Using the above notation we have $\ell = \text{mult}_{\text{I}}\,(\text{P})$.

Definition 5 [2, p.1]. A polynomial of the form $a_{(i)}x_1^{i1}\,x_2^{i2}\ldots x_n^{in}$, where $i_1, i_2, \ldots i_n$ are any non-negative integers and $a_{(i)}$ is any element of F, is a monomial.

Definition 6 [2, p.1]. If A is an ideal of S then A is a monomial ideal of S if and only if A is generated by monomials. That is, $A = (m_1, \ldots, m_s)$, where m_ℓ are monomials for $\ell = 1, \ldots, s$.

Proposition 1 [2, p.2]. Let P_1 be a monomial ideal of $S = F\,[x_0, \ldots, x_n]$; P_1 is a prime ideal if and only if $P_1 = (x_{i_0}, \ldots, x_{i_r})$, $i_j \in \{0, \ldots, n\}$ for $j = 0, \ldots, r$.

Proposition 2 [2, p.2]. Let P_1, Q_1 be monomial ideals of $S = F\,[x_o, \ldots, x_n]$ where P_1 is prime and, say $P_1 = (x_{i_0}, \ldots, x_{i_r})$, $i_j \in \{0, \ldots, n\}$ for $j = 0, \ldots, r$. Q_1 is P–primary if and only if $Q_1 = (x_{i_0}^{t_0}, \ldots, x_{i_r}^{t_r}, m_0, \ldots, m_s)$ where $t_j \geq 1$ for $j = 0, \ldots, r$, and m_ℓ are monomials in $x_{i_0}, \ldots, x_{i_r}$ for $\ell = 0, \ldots, s$.

Definition 7. Consider a primary decomposition of $I = Q_1 \cap \ldots \cap Q_k$ where Q_i is P_i–primary. The arithmetic degree of I, denoted by arith-deg (I), is given by arith-deg $(I) := \sum_{i=1}^{k} \text{mult}_I\,(P_i)$ degree (P_i).

Let $I = (f_1, \ldots, f_t)$.

Definition 8. $M\,(I) := \max\limits_{i=1 \text{ to } t} \left\{\text{degree } (f_i)\right\}$.

Theorem 3 (criterion of $\mathrm{mult_I}(\mathrm{P}) = 1$) [1, p.2].

Let R be a Noetherian ring.

Let A and B be ideals in R such that $B \subsetneqq A$.

Let P be a prime ideal such that all primes belonging to A and B are contained in P.

Necessary and sufficient conditions, that there exists no ideal, say C, with $B \subsetneqq C \subsetneqq A$, and all primes that belong to C are also contained in P, are the following:

(i) there exists an element x in A such that $A = B + R \cdot x$.

(ii) $PA \subseteqq B$.

Definition 9 [2, p.3]. Two monomials λ and τ are said to be relatively prime if, when

$$\lambda = x_{i_0}^{n_{i_0}} \dots x_{i_j}^{n_{i_j}} \text{ and } \tau = x_{k_0}^{m_{k_0}} \dots x_{k_r}^{m_{k_r}},$$

$$\text{then } \{x_{i_0}, \dots, x_{i_j}\} \cap \{x_{k_0}, \dots, x_{k_r}\} = \phi.$$

Theorem 4 [2, p.3]. Let $S = F[x_0, \dots, x_n]$ be a ring of polynomials in $n+1$ indeterminates. Let $\lambda, \tau, m_0, \dots, m_r$ be monomials in F. If λ and τ are relatively prime, then:

$$(\lambda \cdot \tau, m_0, \dots, m_r) = (\lambda, m_0, \dots, m_r) \cap (\tau, m_0, \dots, m_r).$$

CHAPTER 2

ON A PROBLEM OF B. RENSCHUCH

Consider the polynomial ring $S = F[x_0, x_1, x_2]$ where F is any field.

Let $m \geq 1$ be an integer.

We set

$$Q = (x_0^3, x_1^2, x_0 x_1)$$

$$A = (x_0^3,\ x_0^2 x_1,\ x_0 x_1 x_2^2,\ x_1^4,\ x_1^2 x_2^2)$$

$$B = (x_0,\ x_1,\ x_2)$$

$$Q_m = (A, B^m)$$

Also we will define $I_m := Q \cap Q_m$.

This example was discussed by Dr. B. Renschuch of Germany [7, p.92]. It is an example stated in the classical paper of G. Hermann [4]. However we want to study the ideal I_m again, in order to prove the following theorem.

Theorem 5. Arithmetic-degree of I_m does not depend on the integer m for $m \geq 6$. More precisely, arith-deg $(I_m) = 14$ for $m \geq 6$.

Remark: In a letter written to Professor W. Vogel on 12 July 1995, Dr. B. Renschuch said that because of a time constraint he was unable to show why the length multiplicity of Q_m does not depend on m if $m \geq 6$. The aim of this chapter is to give two proofs for Theorem 5. The following two theorems will provide the first proof. The second proof will solve the problem stated by Dr. Renschuch by proving that the length multiplicity of $Q_m = 10$ for all $m \geq 6$.

Theorem 6. $I_m = A$ if and only if $m \geq 6$.

Proof.

(i) Suppose $m = 1$. Then $B^1 = (x_0, x_1, x_2)$.

So $Q_1 = (x_0, x_1, x_2) = B$

Clearly $A \subsetneq Q$.

Therefore $A \subsetneq Q \cap Q_1$.

(ii) $m = 2$

Then $B^2 = (x_0^2, x_0 x_1, x_0 x_2, x_1^2, x_1 x_2, x_2^2)$.

So $Q_2 = (x_0^2, x_0 x_1, x_1^2, x_0 x_2, x_1 x_2, x_2^2) = B^2$.

$Q \cap Q_2 = Q$ since Q is contained in Q_2.

Therefore $A \subsetneq Q \cap Q_2$.

(iii) $m = 3$

Then $B^3 = (x_0^3, x_0^2 x_1, x_0^2 x_2, x_0 x_1^2, x_0 x_1 x_2, x_0 x_2^2, x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3)$.

So $Q_3 = (x_0^3, x_0^2 x_1, x_0 x_1 x_2, x_1^3, x_0^2 x_2, x_0 x_1^2, x_0 x_2^2, x_1^2 x_2, x_1 x_2^2, x_2^3) = B^3$.

$A \subsetneq Q \cap Q_3$ i.e. $Q \cap Q_3$ contains the element $x_0 x_1^2$ which is not in A.

(iv) $m = 4$

Then $B^4 = (x_0^4, x_0^3 x_1, x_0^3 x_2, x_0^2 x_1^2, x_0^2 x_1 x_2, x_0^2 x_2^2, x_0 x_1^3, x_0 x_1^2 x_2, x_0 x_1 x_2^2,$

$x_0 x_2^3, x_1^4, x_1^3 x_2, x_1^2 x_2^2, x_1 x_2^3, x_2^4)$.

So $Q_4 = (x_0^3, x_0^2 x_1, x_0 x_1 x_2^2, x_1^4, x_1^2 x_2^2, x_0^2 x_2^2, x_0 x_1^3, x_0 x_1^2 x_2, x_0 x_2^3, x_1^3 x_2, x_1 x_2^3, x_2^4)$.

$A \subsetneq Q \cap Q_4$ i.e. $x_0 x_1^3$ is contained in $Q \cap Q_4$ but is not contained in A.

(v) $m = 5$

Then $B^5 = (x_0^5, x_0^4 x_1, x_0^4 x_2, x_0^3 x_1^2, x_0^3 x_1 x_2, x_0^3 x_2^2, x_0^2 x_1^3, x_0^2 x_1^2 x_2, x_0^2 x_1 x_2^2, x_0^2$

$x_2^3, x_0 x_1^4, x_0 x_1^3 x_2, x_0 x_1^2 x_2^2, x_0 x_1 x_2^3, x_0 x_2^4, x_1^5, x_1^4 x_2, x_1^3 x_2^2, x_1^2 x_2^3, x_1 x_2^4, x_2^5)$.

So $Q_5 = (x_0^3, x_0^2 x_1, x_0 x_1 x_2^2, x_1^4, x_1^2 x_2^2, x_0^2 x_2^3, x_0 x_1^3 x_2, x_0 x_2^4, x_1^5, x_1 x_2^4, x_2^5)$.

i.e. $x_0 x_1^3 x_2 \in Q \cap Q_5, x_0 x_1^3 x_2 \notin A$.

$A \subsetneq Q \cap Q_5$.

(vi) $m = 6$

Then $B^6 = (x_0^6, x_0^5 x_1, x_0^5 x_2, x_0^4 x_1^2, x_0^4 x_1 x_2, x_0^4 x_2^2, x_0^3 x_1^3, x_0^3 x_1^2 x_2, x_0^3 x_1 x_2^2, x_0^3 x_2^3, x_0^2 x_1^4,$

$x_0^2\,x_1^3\,x_2, x_0^2\,x_1^2x_2^2, x_0^2\,x_1\,x_2^3, x_0^2\,x_2^4, x_0x_1^5, x_0\,x_1^4\,x_2, x_0\,x_1^3x_2^2, x_0\,x_1^2\,x_2^3, x_0\,x_1x_2^4, x_0\,x_2^5, x_1^6,$

$x_1^5\,x_2, x_1^4\,x_2^2, x_1^3\,x_2^3, x_1^2\,x_2^4, x_1\,x_2^5, x_2^6)$.

So $Q_6 = (x_0^3, x_0^2\,x_2^4, x_0\,x_2^5, x_1\,x_2^5, x_2^6, x_0^2\,x_1, x_0\,x_1\,x_2^2, x_1^4, x_1^2\,x_2^2)$.

We now apply theorem 4 of chapter 1 to Q_6 which gives

$Q_6 = (x_0^2, x_0\,x_2^5, x_1\,x_2^5, x_2^6, x_0\,x_1\,x_2^2, x_1^4, x_1^2\,x_2^2) \cap (x_0^3, x_0^2\,x_2^2, x_0\,x_2^5, x_2^6, x_1)$

$= (x_0, x_1\,x_2^5, x_2^6, x_1^4, x_1^2\,x_2^2) \cap (x_0^2, x_1\,x_2^2, x_0\,x_2^5, x_2^6, x_1^4) \cap (x_0^2, x_1, x_0\,x_2^5, x_2^6)$

$\cap (x_0^3, x_1, x_2^2)$

$= (x_0, x_1, x_2^6) \cap (x_0, x_1^4, x_2^5, x_1^2\,x_2^2) \cap (x_0^2, x_1, x_2^6, x_0\,x_2^5) \cap (x_0^2, x_1^4, x_2^2)$

$\cap (x_0, x_1, x_2^6) \cap (x_0^2, x_1, x_2^5) \cap (x_0^3, x_1, x_2^2)$

$= (x_0, x_1, x_2^6) \cap (x_0, x_1^2, x_2^5) \cap (x_0, x_1^4, x_2^2) \cap (x_0, x_1, x_2^6)$

$\cap (x_0^2, x_1, x_2^5) \cap (x_0^2, x_1^4, x_2^2) \cap (x_0, x_1, x_2^6) \cap (x_0^2, x_1, x_2^5)$

$\cap (x_0^3, x_1, x_2^2)$.

Now $(x_0, x_1^4, x_2^2) \cap (x_0^2, x_1^4, x_2^2) = (x_0^2, x_1^4, x_2^2)$.

Using this and eliminating duplicates we have

$Q_6 = (x_0, x_1, x_2^6) \cap (x_0, x_1^2, x_2^5) \cap (x_0^2, x_1, x_2^5) \cap (x_0^2, x_1^4, x_2^2) \cap (x_0^3, x_1, x_2^2)$.

Applying theorem 4 to Q gives

$Q = (x_0, x_1^2) \cap (x_0^3, x_1)$.

$(x_0, x_1, x_2^6) \cap (x_0, x_1^2, x_2^5) \cap (x_0, x_1^2) = (x_0, x_1^2)$.

$(x_0^2, x_1, x_2^5) \cap (x_0^3, x_1, x_2^2) \cap (x_0^3, x_1) = (x_0^3, x_1)$.

So $Q \cap Q_6 = (x_0^2, x_1^4, x_2^2) \cap (x_0, x_1^2) \cap (x_0^3, x_1)$.

Applying theorem 4 to A gives

$A = (x_0^2, x_0\,x_1\,x_2^2, x_1^4, x_1^2\,x_2^2) \cap (x_0^3, x_1) = (x_0^2, x_0\,x_1, x_1^4, x_1^2\,x_2^2) \cap (x_0^2, x_1^4, x_2^2) \cap$
$(x_0^3, x_1) = (x_0, x_1^4, x_1^2\,x_2^2) \cap (x_0^2, x_1) \cap (x_0^2, x_1^4, x_2^2) \cap (x_0^3, x_1) = (x_0\,x_1^2) \cap$
$(x_0, x_1^4, x_2^2) \cap (x_0^2, x_1) \cap (x_0^2, x_1^4, x_2^2) \cap (x_0^3, x_1) = (x_0^2, x_1^4, x_2^2) \cap (x_0^3, x_1) \cap$
$(x_0, x_1^2) = Q \cap Q_6$.

(vii) $m \geq 7$

From the definition of Q_m, $Q_1 \supsetneq Q_2 \supsetneq Q_3 \supsetneq Q_4 \cdots \supsetneq Q_m$ for any positive integer m.

Therefore $Q \cap Q_6 \supseteq Q \cap Q_m$ for $m \geq 6$.

Also from the definition of Q_m, $A \subset Q_m$ for all $m \geq 1$ and certainly $A \subset Q$ so $Q \cap Q_m \supseteq A$ for $m \geq 1$.

Since $Q \cap Q_6 = A$ from (vi)

we get $A = Q \cap Q_6 \supseteq Q \cap Q_m \supseteq A$ for all $m \geq 6$.

Hence $Q \cap Q_m = A$ for $m \geq 6$.

Theorem 7. Arithmetic-degree of $I_6 = 14$.

Proof: From Theorem 6 we know that $A = Q \cap Q_6$. Propositions 1 and 2 of chapter 1 show that $Q \cap Q_6$ is a primary decomposition of A. i.e. Q is $(x_0 x_1)$–primary, Q_6 is (x_0, x_1, x_2)–primary.

Thus arith-degree (A) = $\text{mult}_A\,(x_0, x_1) \cdot$ degree (x_0, x_1) + $\text{mult}_A\,(x_0, x_1, x_2)\cdot$ degree (x_0, x_1, x_2). We have degree $(x_0, x_1) = 1$ and degree $(x_0, x_1, x_2) = 1$ by theorem 1. Thus we have arith-degree $(A) = \text{mult}_A\,(x_0, x_1) + \text{mult}_A\,(x_0, x_1, x_2)$. We apply the algorithm that was given in the first chapter to calculate $\text{mult}_A\,(x_0, x_1)$.

A maximal strictly increasing chain from Q to (x_0, x_1) is $(x_0^3, x_1^2, x_0 x_1) \subsetneq (x_0^3, x_1) \subsetneq (x_0^2, x_1) \subsetneq (x_0, x_1)$. (It is easy to see that this chain is maximal by applying theorem 3 chapter 1). Next we find J which in this case equals the whole ring $S = F\,[x_0, x_1, x_2]$.

Intersecting each ideal in the above chain with J will leave the chain unchanged, so from theorem 2 we have $\text{mult}_A\,(x_0, x_1) = 4$.

We now calculate $\text{mult}_A\,(x_0, x_1, x_2)$.

A maximal strictly increasing chain from Q_6 to $P = (x_0, x_1, x_2)$ is:

$Q_6 = (x_0^3, x_0^2\,x_1, x_0\,x_1\,x_2^2, x_1^4, x_1^2\,x_2^2, x_0^2\,x_2^4, x_0\,x_2^5, x_2^6, x_1\,x_2^5)$

$\subsetneq (x_0^3, x_0^2\,x_1, x_0\,x_1\,x_2^2, x_1^4, x_1^2\,x_2^2, x_0^2\,x_2^4, x_0\,x_2^5, x_2^6, x_1\,x_2^4)$

$\subsetneq (x_0^3, x_0^2\,x_1, x_0\,x_1\,x_2^2, x_1^4, x_1^2\,x_2^2, x_0^2\,x_2^4, x_0\,x_2^5, x_2^6, x_1\,x_2^3)$

$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_0^2 x_2^4, x_0 x_2^5, x_2^6, x_1 x_2^2)$

$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_2^6, x_1 x_2^2, x_0 x_2^4) \subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_2^6, x_1 x_2^2, x_0 x_2^4, x_0^2 x_2^3)$

$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_2^6, x_1 x_2^2, x_0 x_2^4, x_0^2 x_2^2) \subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_0 x_2^4, x_0^2 x_2^2, x_2^5)$

$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_0 x_2^4, x_0^2 x_2^2, x_2^4) \subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_0^2 x_2^2, x_2^4, x_0 x_2^3)$

$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2) \subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1^3 x_2)$

$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1^2 x_2) \subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1 x_2)$

$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1 x_2, x_0 x_1^3)$

$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1 x_2, x_0 x_1^2) \subsetneq (x_0^3, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1)$

$\subsetneq (x_0^3, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1, x_1^3 x_2) \subsetneq (x_0^3, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1, x_1^2 x_2)$

$\subsetneq (x_0^3, x_1^4, x_2^4, x_0 x_2^2, x_0 x_1, x_1 x_2) \subsetneq (x_0^3, x_1^4, x_2^4, x_0 x_2^2, x_0 x_1, x_1 x_2, x_0^2 x_2)$

$\subsetneq (x_0^3, x_1^4, x_2^4, x_0 x_1, x_1 x_2, x_0 x_2) \subsetneq (x_0^2, x_1^4, x_2^4, x_0 x_1, x_1 x_2, x_0 x_2)$

$\subsetneq (x_0, x_1^4, x_2^4, x_1 x_2) \subsetneq (x_0, x_1^3, x_2^4, x_1 x_2) \subsetneq (x_0, x_1^2, x_2^4, x_1 x_2)$

$\subsetneq (x_0, x_1, x_2^4) \subsetneq (x_0, x_1, x_2^3) \subsetneq (x_0, x_1, x_2^2) \subsetneq (x_0, x_1, x_2) = P.$

Then we find that $J = (x_0^3, x_1^2, x_0 x_1) = Q$.

Next we intersect each ideal in the chain with Q and eliminate duplicities.

We know from the proof of theorem 6 that we can express $Q_6 \cap Q$ as $(x_0^2, x_1^4, x_2^2) \cap (x_0, x_1^2) \cap (x_0^3, x_1)$. If we apply theorem 4 to the ideals in $(x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2) \cap (x_0^3, x_1^2, x_0 x_1)$ we get $(x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2) \cap (x_0^3, x_1^2, x_0 x_1)$

$= (x_0^2, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2) \cap (x_0^3, x_1, x_2^4, x_0 x_2^2) \cap (x_0, x_1^2) \cap (x_0^3, x_1)$

$= (x_0^2, x_1, x_2^4, x_0 x_2^2) \cap (x_0^2, x_1^4, x_2^2) \cap (x_0, x_1^2) \cap (x_0^3, x_1)$

$= (x_0^2, x_1^4, x_2^2) \cap (x_0, x_1^2) \cap (x_0^3, x_1) = Q_6 \cap Q.$

So all the ideals in the chain from $Q_6 \cap Q$ to $(x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2) \cap Q$ can be written as $(x_0^2, x_1^4, x_2^2) \cap (x_0, x_1^2) \cap (x_0^3, x_1)$.

The element $x_0 x_1^3 x_2 \in (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1^3 x_2) \cap Q$ but is not contained in the ideal $(x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2) \cap Q$.

The element $x_0 x_1^2 x_3 \in (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1^2 x_2) \cap Q$ but is not contained in the ideal $(x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1^3 x_2) \cap Q$.

The element $x_0 x_1 x_2 \in (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1 x_2) \cap Q$ but is not contained in $(x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1^2 x_2) \cap Q$.

The element $x_0 x_1^3 \in (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1 x_2, x_0 x_1^3) \cap Q$ but is not contained in $(x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1 x_2) \cap Q$.

The element $x_0 x_1^2 \in (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1 x_2, x_0 x_1^2) \cap Q$ but is not contained in $(x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1 x_2, x_0 x_1^3) \cap Q$.

The element $x_0 x_1 \in (x_0^3, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1) \cap Q$ but is not contained in $(x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1 x_2, x_0 x_1^2) \cap Q$.

The element $x_1^3 x_2 \in (x_0^3, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1, x_1^3 x_2) \cap Q$ but is not contained in $(x_0^3, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1) \cap Q$.

The element $x_1^2 x_2 \in (x_0^3, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1, x_1^2 x_2) \cap Q$ but is not contained in $(x_0^3, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1, x_1^3 x_2) \cap Q$.

If we apply theorem 4 to the ideal $(x_0^3, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1, x_1^2 x_2) \cap Q$ we get $(x_0, x_1^4, x_1 x_2^2, x_2^4, x_1^2 x_2) \cap Q = (x_0, x_1^4, x_2^2, x_1^2 x_2) \cap Q = (x_0, x_1^4, x_2) \cap Q = (x_0, x_1^4, x_2^4, x_1 x_2) \cap Q$.

Therefore all the ideals in the chain from $(x_0^3, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1, x_1^2 x_2) \cap Q$ to $(x_0, x_1^4, x_2^4, x_1 x_2) \cap Q$ are equal.

The element $x_1^3 \in (x_0, x_1^3, x_2^4, x_1 x_2) \cap Q$ but is not contained in $(x_0, x_1^4, x_2^4, x_1 x_2) \cap Q$.

Finally we have $(x_0, x_1^2, x_2^4, x_1 x_2) \cap Q = Q$.

Thus we are left with the chain $Q_6 \cap Q$

$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1^3 x_2) \cap Q$

$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1^2 x_2) \cap Q$

$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1 x_2) \cap Q$

$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1 x_2, x_0 x_1^3) \cap Q$

$\subsetneq (x_0^3, x_0^2\,x_1, x_1^4, x_1\,x_2^2, x_2^4, x_0\,x_2^2, x_0\,x_1\,x_2, x_0\,x_1^2) \cap Q$

$\subsetneq (x_0^3, x_1^4, x_1\,x_2^2, x_2^4, x_0\,x_2^2, x_0\,x_1) \cap Q$

$\subsetneq (x_0^3, x_1^4, x_1\,x_2^2, x_2^4, x_0\,x_2^2, x_0\,x_1, x_1^3\,x_2) \cap Q$

$\subsetneq (x_0^3, x_1^4, x_1\,x_2^2, x_2^4, x_0\,x_2^2, x_0\,x_1, x_1^2\,x_2) \cap Q$

$\subsetneq (x_0, x_1^3, x_2^4, x_1\,x_2) \cap Q$

$\subsetneq Q.$

So from theorem 2 of chapter 1 we have $\text{mult}_A\,(x_0, x_1, x_2) = 10$.

Now $\text{mult}_A\,(x_0, x_1) + \text{mult}_A\,(x_0, x_1, x_2) = 4 + 10 = 14$. So arith-deg $(A) = 14$.

Now Q_m is (x_0, x_1, x_2) – primary for any $m \geq 6$ so $Q \cap Q_m$ is a primary decomposition of A for $m \geq 6$. Since arith-deg $(A) = 14$ and length multiplicity of $Q = 4$ it follows that length multiplicity of $Q_m = 10$ for any $m \geq 6$.

We will now describe a second proof which solves the problem stated at the beginning of this chapter. Our second proof needs some parts of the first proof. Also, in this second proof, we will assume that m is always ≥ 6.

Claim.

$$Q_m = (A, B^m) = (A, x_2^m, x_0^2\,x_2^{m-2}, x_0\,x_2^{m-1}, x_1\,x_2^{m-1}).$$

Proof: We will use induction on m.

We know from the proof of theorem 6 that the claim is true for $m = 6$. Suppose that the claim is true for $m \geq 6$.

Then $Q_{m+1} = (A, B^{m+1}) = (A, x_0\,x_2^m, x_1\,x_2^m, x_2^{m+1}, x_0^3\,x_2^{m-2}, x_0^2\,x_1\,x_2^{m-2},$
$x_0^2\,x_2^{m-1}, x_0^2\,x_2^{m-1}, x_0\,x_1\,x_2^{m-1}, x_0\,x_2^m, x_0\,x_1\,x_2^{m-1}, x_1^2\,x_2^{m-1}, x_1\,x_2^m)$
$= (A, x_2^{m+1}, x_0^2\,x_2^{m-1}, x_0\,x_2^m, x_1\,x_2^m)$.

Therefore $Q_m = (A, x_2^m, x_0^2\,x_2^{m-2}, x_0\,x_2^{m-1}, x_1\,x_2^{m-1})$ for $m \geq 6$.

We now construct a strictly increasing maximal chain from Q_m to $P = (x_0, x_1, x_2)$.

$$Q_m \subsetneq (A, x_2^m, x_0^2 x_2^{m-2}, x_0 x_2^{m-1}, x_1 x_2^{m-2}) \subsetneq \cdots \subsetneq$$

$$(x_0^3, x_0^2 x_1, x_1^4, x_2^m, x_0^2 x_2^{m-2}, x_0 x_2^{m-1}, x_1 x_2^2) \subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_2^m, x_1 x_2^2, x_0 x_2^{m-2})$$

$$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_2^m, x_1 x_2^2, x_0 x_2^{m-2}, x_0^2 x_2^{m-3})$$

$$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_2^m, x_1 x_2^2, x_0 x_2^{m-3})$$

$$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_2^m, x_1 x_2^2, x_0 x_2^{m-3}, x_0^2 x_2^{m-4})$$

$$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_2^m, x_1 x_2^2, x_0 x_2^{m-4})$$

$$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_2^m, x_1 x_2^2, x_0 x_2^{m-4}, x_0^2 x_2^{m-5})$$

$$\subsetneq \cdots$$

$$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_2^m, x_1 x_2^2, x_0 x_2^4)$$

$$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_2^m, x_1 x_2^2, x_0 x_2^4, x_0^2 x_2^3)$$

$$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_2^m, x_1 x_2^2, x_0 x_2^4, x_0^2 x_2^2)$$

$$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_0 x_2^4, x_0^2 x_2^2, x_2^{m-1})$$

$$\subsetneq \cdots$$

$$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_0^2 x_2^2, x_2^4)$$

$$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_0^2 x_2^2, x_2^4, x_0 x_2^3)$$

$$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2)$$

$$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1^3 x_2)$$

$$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1^2 x_2)$$

$$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1 x_2)$$

$$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1 x_2, x_0 x_1^3)$$

$$\subsetneq (x_0^3, x_0^2 x_1, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1 x_2, x_0 x_1^2)$$

$$\subsetneq (x_0^3, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1)$$

$$\subsetneq (x_0^3, x_1^4, x_1 x_2^2, x_2^4, x_0 x_2^2, x_0 x_1, x_1^3 x_2)$$

$$
\begin{aligned}
&\subsetneq (x_0^3, x_1^4, x_1\, x_2^2, x_2^4, x_0\, x_2^2, x_0\, x_1, x_1^2\, x_2)\\
&\subsetneq (x_0^3, x_1^4, x_2^4, x_0\, x_2^2, x_0\, x_1, x_1\, x_2)\\
&\subsetneq (x_0^3, x_1^4, x_2^4, x_0\, x_2^2, x_0\, x_1, x_1\, x_2, x_0^2\, x_2)\\
&\subsetneq (x_0^3, x_1^4, x_2^4, x_0\, x_1, x_1\, x_2, x_0\, x_2)\\
&\subsetneq (x_0^2, x_1^4, x_2^4, x_0\, x_1, x_1\, x_2, x_0\, x_2)\\
&\subsetneq (x_0, x_1^4, x_2^4, x_1 x_2)\\
&\subsetneq (x_0, x_1^3, x_2^4, x_1 x_2)\\
&\subsetneq (x_0, x_1^2, x_2^4, x_1 x_2)\\
&\subsetneq (x_0, x_1, x_2^4)\\
&\subsetneq (x_0, x_1, x_2^3)\\
&\subsetneq (x_0, x_1, x_2^2)\\
&\subsetneq (x_0, x_1, x_2)\\
&= P.
\end{aligned}
$$

The fact that this chain is maximal follows from Theorem 3 chapter 1, as the reader can readily verify. We now intersect each ideal in our above chain with $J = (x_0^3, x_1^2, x_0\, x_1) = Q$ to form a new chain from $Q_m \cap Q$ to $P \cap Q$.

From (vi) and (vii) of the proof of Theorem 6 we know that $Q_m \cap Q = (x_0^2, x_1^4, x_2^2) \cap (x_0, x_1^2) \cap (x_0^3, x_1)$.

From the proof of Theorem 7 we have $(x_0^3, x_0^2\, x_1, x_1^4, x_1\, x_2^2, x_2^4, x_0\, x_2^2) \cap Q = (x_0^2, x_1^4, x_2^2) \cap (x_0, x_1^2) \cap (x_0^3, x_1)$. So all ideals between $Q_m \cap Q$ and $(x_0^3, x_0^2\, x_1, x_1^4, x_1\, x_2^2, x_2^4, x_0\, x_2^2) \cap Q$ are equal.

From $(x_0^3, x_0^2\, x_1, x_1^4, x_1\, x_2^2, x_2^4, x_0\, x_2^2, x_0\, x_1^3\, x_2)$ onwards, the ideals in our chain are the same as the ideals in the chain given in the proof of Theorem 7.

Therefore $\text{mult}_A\,(x_0, x_1, x_2) = 10$. Also from Theorem 7, $\text{mult}_A\,(x_0, x_1) = 4$, so arith-deg $(I_6) = 14$.

CHAPTER 3

ON A PROBLEM OF STURMFELS, TRUNG AND VOGEL

Let I be a monomial ideal in the polynomial ring $S = F[x_0, \ldots, x_n]$ with minimal set of monomial generators $(m_1, \ldots, m_s)$.

Sturmfels, Trung and Vogel [9, Theorem 3.1] proved that

$$\text{arith-deg}\,(I) \geq \max\{\text{degree}\,(\mathrm{m_i}) : \mathrm{i} = 1, \ldots, \mathrm{s}\} \tag{1}$$

It was an open problem in [9] to extend this result for ideals which are not monomial. T. Smith [8] has constructed examples showing that this problem is not true in general. However, the aim of this chapter is to describe families of non-monomial ideals for which (1) is true. (See our problem at the end of this chapter on page 36.)

Theorem 7.

Let n be any positive integer.

Let the ideal I_n be given by $I_n = (x_0^n, x_1, x_2, x_3^2) \cap (x_0 x_3 - x_1 x_2)$ in the polynomial ring $F[x_0, x_1, x_2, x_3]$. (F is any field).

Then $I_n = (x_0^n x_3 - x_0^{n-1} x_1 x_2, x_0 x_1 x_3 - x_1^2 x_2, x_0 x_2 x_3 - x_1 x_2^2, x_0 x_3^2 - x_1 x_2 x_3)$.

Proof: Any element in I_n can be written in the form $w\,(x_0 x_3 - x_1 x_2)$ where w is an element of the ring $F[x_0, x_1, x_2, x_3]$.

(i) Suppose that w is a single term.

Then, since $w x_0 x_3$, $w x_1 x_2$ belong to (x_0^n, x_1, x_2, x_3^2), $w\,(x_0 x_3 - x_1 x_2)$ must be generated by one of

$x_0^n x_3 - x_0^{n-1} x_1 x_2, x_0 x_1 x_3 - x_1^2 x_2, x_0 x_2 x_3 - x_1 x_2^2, x_0 x_3^2 - x_1 x_2 x_3$.

(ii) Suppose w contains t terms i.e. $w = w_1 + w_2 + \ldots + w_t$.

Also we assume that there are no $w_k\,(1 \leq k \leq t)$ such that $w_k x_0 x_3, w_k x_1 x_2$ both belong to (x_0^n, x_1, x_2, x_3^2). If w_k contains x_1 or x_2 or x_3 then $w_k\,(x_0\,x_3 - x_1\,x_2)$ would be generated by $x_0\,x_1\,x_3 - x_1^2\,x_2$ or $x_0\,x_2\,x_3 - x_1\,x_2^2$ or $x_0\,x_3^2 - x_1\,x_2\,x_3$.

If each w_k contains αx_0^i (where $\alpha \in F$) and if $i \geq n-1$, then w_k is generated by $x_0^n\,x_3 - x_0^{n-1}x_1\,x_2$.

If each w_k contains αx_0^i (where $\alpha \in F$) and if $i \leq n-2$, then $w\,(x_0\,x_3 - x_1\,x_2) \notin (x_0^n, x_1, x_2, x_3^2)$ and thus $w \notin I_n$. Therefore the only generators of I_n are those given in Theorem 7.

Note: If $n = 1$ then $I_n = (x_0\,x_3 - x_1\,x_2)$.

Theorem 8. arith-deg $(I_n) = n + 1$.

Proof:

$$\text{arith-deg}\,(I_n) = \text{mult}_{I_n}\,(x_0\,x_3 - x_1\,x_2) \cdot \deg\,(x_0\,x_3 - x_1\,x_2) +$$
$$\text{mult}_{I_n}\,(x_0, x_1, x_2, x_3) \cdot \deg\,(x_0, x_1, x_2, x_3)$$
$$= \deg\,(x_0\,x_3 - x_1\,x_2) + \text{mult}_{I_n}\,(x_0, x_1, x_2, x_3).$$

We know that $\deg\,(x_0\,x_3 - x_1\,x_2) = 2$ from Theorem 1 of chapter 1. Therefore we must show that $\text{mult}_{I_n}\,(x_0, x_1, x_2, x_3) = n - 1$. A strictly increasing maximal chain from (x_0^n, x_1, x_2, x_3^2) to (x_0, x_1, x_2, x_3) is given by;

$(x_0^n, x_1, x_2, x_3^2) \subsetneq (x_0^n, x_1, x_2\,x_0^{n-1}x_3, x_3^2) \subsetneq (x_0^n, x_1, x_2, x_0^{n-2}x_3, x_3^2)$
$\subsetneq (x_0^n, x_1, x_2, x_0^{n-3}x_3, x_3^2) \subsetneq \cdots \subsetneq$
$(x_0^n, x_1, x_2, x_0x_3, x_3^2) \subsetneq (x_0^{n-1}, x_1, x_2, x_0\,x_3, x_3^2) \subsetneq (x_0^{n-2}, x_1, x_2, x_0\,x_3, x_3^2)$
$\subsetneq (x_0^{n-3}, x_1, x_2, x_0\,x_3, x_3^2) \subsetneq \cdots \subsetneq (x_0^2, x_1, x_2, x_0x_3, x_3^2) \subsetneq (x_0, x_1, x_2, x_3^2)$
$\subsetneq (x_0, x_1, x_2, x_3)$.

For example, if we apply Theorem 3 of chapter 1 to the ideals $(x_0^n, x_1, x_2, x_0x_3, x_3^2)$, $(x_0^{n-1}, x_1, x_2, x_0\,x_3, x_3^2)$, we have

(i) $(x_0^{n-1}, x_1, x_2, x_0\,x_3, x_3^2) = (x_0^n, x_1, x_2, x_0x_3, x_3^2) + F\,[x_0, x_1, x_2, x_3] \cdot x_0^{n-1}$.

(ii) $(x_0, x_1, x_2, x_3)\,(x_0^{n-1}, x_1, x_2, x_0\,x_3, x_3^2) \subseteq (x_0^n, x_1, x_2, x_0x_3, x_3^2)$.

We also note that the length of this chain is $2n$. i.e. from (x_0^n, x_1, x_2, x_3^2) to $(x_0^n, x_1, x_2, x_0 x_3, x_3^2)$ there are n ideals and from $(x_0^{n-1}, x_1, x_2, x_0\,x_3, x_3^2)$ to (x_0, x_1, x_2, x_3) there are n ideals.

Next we calculate J which is the ideal $(x_0\,x_3 - x_1\,x_2)$. Now we intersect each ideal in our chain with J.

The ideal $(x_0^n, x_1, x_2, x_0\,x_3, x_3^2) \cap (x_0\,x_3 - x_1\,x_2) = (x_0\,x_3 - x_1\,x_2)$ so we can confine our attention to the first $n-1$ ideals in our chain.

The element $x_0^{n-1} x_3 - x_0^{n-2} x_1\,x_2$ is contained in $(x_0^n, x_1, x_2, x_0^{n-1} x_3, x_3^2) \cap J$ but is not in $(x_0^n, x_1, x_2, x_3^2) \cap J$.

If we choose any two ideals $(x_0^n, x_1, x_2, x_0^{n-i} x_3, x_3^2) \cap J$, $(x_0^n, x_1, x_2, x_0^{n-(i+1)} x_3, x_3^2) \cap J$, $(1 \leq i \leq n-3)$ then $x_0^{n-(i+1)} x_3 - x_0^{n-(i+2)} x_1\,x_2$ belongs to $(x_0^n, x_1, x_2, x_0^{n-(i+1)} x_3, x_3^2) \cap J$ and is not in $(x_0^n, x_1, x_2, x_0^{n-i} x_3, x_3^2) \cap J$.

Therefore no two of these ideals are equal, so from Theorem 2 of chapter 1 [5, p.2] we have $\mathrm{mult}_{I_n}(x_0, x_1, x_2, x_3) = n-1$. Therefore arith-deg $(I_n) = n+1$. This completes the proof of Theorem 8.

Remark: The maximum $m\,(I_n)$ of degrees of the polynomials generating I_n is given by

$$m\,(I_n) = \begin{cases} 2 & : n = 1 \\ n+1 & : n \geq 2\,. \end{cases}$$

Therefore arith-deg $(I_n) = m\,(I_n)$ for $n \geq 1$.

For the remainder of this chapter we will consider a similar but more complicated example.

Theorem 9

Let S be the polynomial ring $F\,[x_0, x_1, x_2, x_3]$ over the field F and let n and r be any positive integers. Let the following ideal I_{nr} of S be given by

$$I_{nr} = (x_0\,x_3 - x_1\,x_2) \cap (x_0^n, x_1^r, x_2^2, x_3^2). \tag{2}$$

(i) arith-deg $(I_{nr}) = 2rn$

(ii) the maximum $m(I_{nr})$ of degrees of the polynomials generating I is given by

$$m(I_{nr}) = \begin{cases} 2 & : n = r = 1 \\ 4 & : \text{if } n = 1, r = 2 \text{ or } r = 1, n = 2 \\ r+1 & : n = 1, r > 2 \\ n+1 & : r = 1, n > 2 \\ n+r & : n, r \geq 2 \end{cases}$$

Remark: arith-deg of $I_{nr} \geqq m(I_{nr})$ for all positive integers n and r.

Proof:

First we prove (i)

Let $Q_{nr} = (x_0^n, x_1^r, x_2^2, x_3^2)$. Then arith-deg $(I_{nr}) = \deg(J) + \mathrm{mult}_{I_{nr}}(P)$ (where P is the prime ideal of Q_{nr} i.e. $P = (x_0, x_1, x_2, x_3)$, and where $J = (x_0x_3 - x_1x_2)$).

Since $\deg J = 2$ we need to prove the following key lemma.

Lemma 1.

$$\mathrm{mult}_{Inr}(P) = 2rn - 2 \tag{3}$$

Before we prove (3) we will give a maximal strictly increasing chain of primary ideals from Q_{nr} to P.

$Q_{nr} \subsetneqq (Q_{nr}, x_0^{n-1}x_1^{r-1}x_2\,x_3) \subsetneqq (Q_{nr}, x_0^{n-1}x_1^{r-1}x_2)$

$\subsetneqq (Q_{nr}, x_0^{n-1}x_1^{r-1}x_2, x_o^{n-1}\,x_1^{r-1}\,x_3) \subsetneqq (Q_{nr}, x_0^{n-1}x_1^{r-1})$

$\subsetneqq (Q_{nr}, x_0^{n-1}x_1^{r-1}, x_0^{n-1}\,x_1^{r-2}x_2\,x_3) \subsetneqq (Q_{nr}, x_0^{n-1}x_1^{r-1}, x_0^{n-1}\,x_1^{r-2}\,x_2)$

$\subsetneqq (Q_{nr}, x_0^{n-1}x_1^{r-1}, x_0^{n-1}\,x_1^{r-2}\,x_2, x_0^{n-1}\,x_1^{r-2}x_3) \subsetneqq (Q_{nr}, x_0^{n-1}x_1^{r-2}) \subsetneqq \cdots$

$\cdots \subsetneqq (Q_{nr}, x_0^{n-1}x_1) \subsetneqq (Q_{nr}, x_0^{n-1}x_1, x_0^{n-2}x_1^{r-1}x_2\,x_3)$

$\subsetneqq (Q_{nr}, x_0^{n-1}x_1, x_0^{n-2}\,x_1^{r-1}\,x_2) \subsetneqq (Q_{nr}, x_0^{n-1}x_1, x_0^{n-2}x_1^{r-1}x_2, x_0^{n-2}x_1^{r-1}x_3)$

$\subsetneqq (Q_{nr}, x_0^{n-1}x_1, x_0^{n-2}\,x_1^{r-1}) \subsetneqq (Q_{nr}, x_0^{n-1}x_1, x_0^{n-2}x_1^{r-1}, x_0^{n-2}\,x_1^{r-2}\,x_2\,x_3)$

$$\subsetneq (Q_{nr}, x_0^{n-1}x_1, x_0^{n-2}x_1^{r-1}, x_0^{n-2}x_1^{r-2}x_2) \subsetneq (Q_{nr}, x_0^{n-1}x_1, x_0^{n-2}x_1^{r-1}, x_0^{n-2}x_1^{r-2}x_2, x_0^{n-2}x_1^{r-2}x_3)$$
$$\subsetneq (Q_{nr}, x_0^{n-1}x_1, x_0^{n-2}x_1^{r-2}) \subsetneq \cdots \subsetneq (Q_{nr}, x_0^{n-1}x_1, x_0^{n-2}x_1^{2})$$
$$\subsetneq (Q_{nr}, x_0^{n-1}x_1, x_0^{n-2}x_1^{2}, x_0^{n-2}x_1x_2x_3) \subsetneq (Q_{nr}, x_0^{n-1}x_1, x_0^{n-2}x_1^{2}, x_0^{n-2}x_1x_2)$$
$$\subsetneq (Q_{nr}, x_0^{n-1}x_1, x_0^{n-2}x_1^{2}, x_0^{n-2}x_1x_2, x_0^{n-2}x_1x_3) \subsetneq (Q_{nr}, x_0^{n-2}x_1) \subsetneq \cdots$$
$$\subsetneq (Q_{nr}, x_0^{n-3}x_1) \subsetneq \cdots \subsetneq (Q_{nr}, x_0^{n-4}x_1) \subsetneq \cdots \subsetneq (Q_{nr}, x_0^{2}x_1)$$
$$\subsetneq (Q_{nr}, x_0^{2}x_1, x_0x_1^{r-1}x_2x_3) \subsetneq (Q_{nr}, x_0^{2}x_1, x_0x_1^{r-1}x_2)$$
$$\subsetneq (Q_{nr}, x_0^{2}x_1, x_0x_1^{r-1}x_2, x_0x_1^{r-1}x_3) \subsetneq (Q_{nr}, x_0^{2}x_1, x_0x_1^{r-1})$$
$$\subsetneq \cdots \subsetneq (Q_{nr}, x_0^{2}x_1, x_0x_1^{2}) \subsetneq (Q_{nr}, x_0^{2}x_1, x_0x_1^{2}, x_0x_1x_2x_3)$$
$$\subsetneq (Q_{nr}, x_0^{2}x_1, x_0x_1^{2}, x_0x_1x_2) \subsetneq (Q_{nr}, x_0^{2}x_1, x_0x_1^{2}, x_0x_1x_2, x_0x_1x_3)$$
$$\subsetneq (Q_{nr}, x_0x_1) \subsetneq (Q_{nr}, x_0x_1, x_0^{n-1}x_2x_3) \subsetneq (Q_{nr}, x_0x_1, x_0^{n-1}x_2)$$
$$\subsetneq (Q_{nr}, x_0x_1, x_0^{n-1}x_2, x_0^{n-1}x_3) \subsetneq (x_0^{n-1}, x_1^{r}, x_2^{2}, x_3^{2}, x_0x_1)$$
$$\subsetneq (x_0^{n-1}, x_1^{r}, x_2^{2}, x_3^{2}, x_0x_1, x_0^{n-2}x_2x_3) \subsetneq (x_0^{n-1}, x_1^{r}, x_2^{2}, x_3^{2}, x_0x_1, x_0^{n-2}x_2)$$
$$\subsetneq (x_0^{n-1}, x_1^{r}, x_2^{2}, x_3^{2}, x_0x_1, x_0^{n-2}x_2, x_0^{n-2}x_3)$$
$$\subsetneq (x_0^{n-2}, x_1^{r}, x_2^{2}, x_3^{2}, x_0x_1) \subsetneq \cdots \subsetneq (x_0^{2}, x_1^{r}, x_2^{2}, x_3^{2}, x_0x_1)$$
$$\subsetneq (x_0^{2}, x_1^{r}, x_2^{2}, x_3^{2}, x_0x_1, x_0x_2x_3) \subsetneq (x_0^{2}, x_1^{r}, x_2^{2}, x_3^{2}, x_0x_1, x_0x_2)$$
$$\subsetneq (x_0^{2}, x_1^{r}, x_2^{2}, x_3^{2}, x_0x_1, x_0x_2, x_0x_3) \subsetneq (x_0, x_1^{r}, x_2^{2}, x_3^{2})$$
$$\subsetneq (x_0, x_1^{r}, x_2^{2}, x_3^{2}, x_1^{r-1}x_2x_3) \subsetneq (x_0, x_1^{r}, x_2^{2}, x_3^{2}, x_1^{r-1}x_2)$$
$$\subsetneq (x_0, x_1^{r}, x_2^{2}, x_3^{2}, x_1^{r-1}x_2, x_1^{r-1}x_3) \subsetneq (x_0, x_1^{r-1}, x_2^{2}, x_3^{2})$$
$$\subsetneq (x_0, x_1^{r-1}, x_2^{2}, x_3^{2}, x_1^{r-2}x_2x_3) \subsetneq (x_0, x_1^{r-1}, x_2^{2}, x_3^{2}, x_1^{r-2}x_2)$$
$$\subsetneq (x_0, x_1^{r-1}, x_2^{2}, x_3^{2}, x_1^{r-2}x_2, x_1^{r-2}x_3) \subsetneq (x_0, x_1^{r-2}, x_2^{2}, x_3^{2})$$
$$\subsetneq \cdots \subsetneq (x_0, x_1, x_2^{2}, x_3^{2}) \subsetneq (x_0, x_1, x_2^{2}, x_3^{2}, x_2x_3)$$
$$\subsetneq (x_0, x_1, x_2, x_3^{2}) \subsetneq (x_0, x_1, x_2, x_3) = P.$$

For example, if we apply Theorem 3 from chapter 1 to the ideals $(Q_{nr}, x_0^{n-1}x_1, x_0^{n-2}x_1^{r-1}, x_0^{n-2}x_1^{r-2}x_2)$, $(Q_{nr}, x_0^{n-1}x_1, x_0^{n-2}x_1^{r-1}, x_0^{n-2}x_1^{r-2}x_2, x_0^{n-2}x_1^{r-2}x_3)$ we have

(i) $(Q_{nr}, x_0^{n-1}x_1, x_0^{n-2}x_1^{r-1}, x_0^{n-2}x_1^{r-2}x_2, x_0^{n-2}x_1^{r-2}x_3) =$

$(Q_{nr}, x_0^{n-1}x_1, x_0^{n-2}x_1^{r-1}, x_0^{n-2}x_1^{r-2}x_2) + F[x_0, x_1, x_2, x_3] \cdot x_0^{n-2}x_1^{r-2}x_3.$

(ii) $(x_0, x_1, x_2, x_3)\,(Q_{nr}, x_0^{n-1}\,x_1, x_0^{n-2}\,x_1^{r-1}, x_0^{n-2}\,x_1^{r-2}\,x_2, x_0^{n-2}x_1^{r-2}\,x_3) \subseteqq$
$(Q_{nr}, x_0^{n-1}\,x_1, x_0^{n-2}\,x_1^{r-1}, x_0^{n-2}\,x_1^{r-2}\,x_2)$.

Hence we cannot extend our chain between these two ideals.

Another example to consider is the ideals $(Q_{nr}, x_0\,x_1, x_0^{n-1}\,x_2, x_0^{n-1}x_3)$, $(x_0^{n-1}, x_1^r, x_2^2, x_3^2, x_0\,x_1)$.

(i) $(x_0^{n-1}, x_1^r, x_2^2, x_3^2, x_0\,x_1) = (Q_{nr}, x_0\,x_1, x_0^{n-1}\,x_2, x_0^{n-1}x_3) + F\,[x_0, x_1, x_2, x_3] \cdot x_0^{n-1}$.

(ii) $(x_0, x_1, x_2, x_3)\,(x_0^{n-1}, x_1^r, x_2^2, x_3^2, x_0\,x_1) \subseteqq (Q_{nr}, x_0\,x_1, x_0^{n-1}x_2, x_0^{n-1}\,x_3)$.

Other examples can be checked by the same method. Hence it can be shown that this chain is maximal.

Claim A**:** The above chain is of length $4rn$.

Proof:

From $(Q_{nr}, x_0^{n-1}\,x_1^{r-1}\,x_2\,x_3)$ to $(Q_{nr}, x_0^{n-1}\,x_1)$ we have $4\,(r-1)$ ideals. So from Q_{nr} to $(Q_{nr}, x_0^{n-1}\,x_1^2, x_0^{n-1}\,x_1\,x_2, x_0^{n-1}\,x_1\,x_3)$ we have $4\,(r-1)$ ideals.

From $(Q_{nr}, x_0^{n-1}\,x_1)$ to $(Q_{nr}, x_0^{n-1}\,x_1, x_0^{n-2}\,x_1^2, x_0^{n-2}\,x_1\,x_2, x_0^{n-2}\,x_1\,x_3)$ we have $4\,(r-1)$ ideals.

From $(Q_{nr}, x_0^{n-2}\,x_1)$ to $(Q_{nr}, x_0^{n-2}\,x_1, x_0^{n-3}\,x_1^2, x_0^{n-3}\,x_1\,x_2, x_0^{n-3}\,x_1\,x_3)$ we have $4\,(r-1)$ ideals .... etc.

From $(Q_{nr}, x_0^2\,x_1)$ to $(Q_{nr}, x_0^2\,x_1, x_0\,x_1^2, x_0\,x_1\,x_2, x_0\,x_1\,x_3)$ we have $4\,(r-1)$ ideals.

So from $(Q_{nr}, x_0^{n-1}\,x_1)$ to $(Q_{nr}, x_0^2\,x_1, x_0\,x_1^2, x_0\,x_1\,x_2, x_0\,x_1\,x_3)$ we have $(n-2)\;[4\,(r-1)\;]$ ideals.

From $(Q_{nr}, x_0\,x_1)$ to $(Q_{nr}, x_0\,x_1, x_0^{n-1}\,x_2, x_0^{n-1}x_3)$ we have 4 ideals.

From $x_0^{n-1}, x_1^r, x_2^2, x_3^2, x_0\,x_1)$ to $(x_0^2, x_1^r, x_2^2, x_3^2, x_0\,x_1, x_0\,x_2, x_0\,x_3)$ we have $4\,(n-2)$ ideals.

From $(x_0, x_1^r, x_2^2, x_3^2)$ to $(x_0, x_1^2, x_2^2, x_3^2, x_1\, x_2, x_1\, x_3)$ we have $4\,(r-1)$ ideals.

From (x_0, x_1, x_2^2, x_3^2) to (x_0, x_1, x_2, x_3) we have 4 ideals.

Adding these terms we get

$$4\,(r-1) + (n-2)\,(4\,(r-1)\,) + 4 + 4\,(n-2) + 4\,(r-1) + 4 =$$

$$4r - 4 + 4rn - 4n - 8r + 8 + 4 + 4n - 8 + 4r - 4 + 4 = 4rn$$

Alternative proof of claim A.

Claim B**:** Q_{nr} is a complete intersection.

Proof:

We must first show the following to be true.

(i) $(0) \,:\, (x_0^n) \;=\; (0)$.

(ii) $(x_0^n) \,:\, (x_1^r) \;=\; (x_0^n)$.

(iii) $(x_0^n, x_1^r) \,:\, (x_2^2) \;=\; (x_0^n, x_1^r)$.

(iv) $(x_0^n, x_1^r, x_2^2) \,:\, (x_3^2) \;=\; (x_0^n, x_1^r, x_2^2)$.

(i) Let y be any element in S. Then $yx_0^n \in (0) \Leftrightarrow y = 0$.

(ii) $yx_1^r \in (x_0^n) \Leftrightarrow y \in (x_0^n)$.

(iii) $yx_2^2 \in (x_0^n, x_1^r) \Leftrightarrow y \in (x_0^n, x_1^r)$.

(iv) $yx_3^2 \in (x_0^n, x_1^r, x_2^2) \Leftrightarrow y \in (x_0^n, x_1^r, x_2^2)$.

We can now prove length $Q_{nr} = 4rn$.

Proof:

degree $Q_{nr} = 4rn$ (from above claim and from Theorem 3)

$=$ length $Q_{nr} \cdot$ degree P (If Q_{nr} is P-primary then degree

$Q_{nr} =$ length $Q_{nr} \cdot$ degree P)

$=$ length Q_{nr}

Remark: If Q_{nr} is a monomial P–primary ideal of F, then degree Q_{nr} = length Q_{nr}. We intersect each ideal in the claim given on pages 20 – 21. We can now prove Lemma 1 by applying the following 32 claims.

Claim 1.

$x_0^{n-1}x_1^{r-1}x_2x_3 - x_0^{n-2}x_1^r x_2^2 \in (Q_{nr}, x_0^{n-1}x_1^{r-1}x_2x_3) \cap J$ but is not contained in the ideal $Q_{nr} \cap J$. Therefore $Q_{nr} \cap (x_0x_3 - x_1x_2) \subsetneqq (Q_{nr}, x_0^{n-1}x_1^{r-1}x_2x_3) \cap J$.

Claim 2.

$(Q_{nr}, x_0^{n-1}x_1^{r-j}) \cap J \subsetneqq (Q_{nr}, x_0^{n-1}x_1^{r-j}, x_0^{n-1}x_1^{r-(j+1)}x_2x_3) \cap J$ for $1 \leq j \leq r-2$.

Proof:

The element $x_0^{n-1}x_1^{r-(j+1)}x_2x_3 - x_0^{n-2}x_1^{r-j}x_2^2 \in (Q_{nr}, x_0^{n-1}x_1^{r-j}, x_0^{n-1}x_1^{r-(j+1)}x_2x_3) \cap J$ but is not contained in $(Q_{nr}, x_0^{n-1}x_1^{r-j}) \cap J$.

Claim 3.

$(Q_{nr}, x_0^{n-i}x_1) \cap J \subsetneqq (Q_{nr}, x_0^{n-i}x_1, x_0^{n-(i+1)}x_1^{r-1}x_2x_3) \cap J$ for $1 \leq i \leq n-2$

Proof:

The element $x_0^{n-(i+1)}x_1^{r-1}x_2x_3 - x_0^{n-(i+2)}x_1x_2^2$ $\in (Q_{nr}, x_0^{n-i}x_1, x_0^{n-(i+1)}x_1^{r-1}x_2x_3) \cap J$ but is not contained in $(Q_{nr}, x_0^{n-i}x_1) \cap J$.

Claim 4.

$(Q_{nr}, x_0^{n-i}x_1, x_0^{n-(i+1)}x_1^{r-j}) \cap J \subsetneqq (Q_{nr}, x_0^{n-i}x_1, x_0^{n-(i+1)}x_1^{r-j}, x_0^{n-(i+1)}x_1^{r-(j+1)}x_2x_3) \cap J$ for $1 \leq i \leq n-2,\ 1 \leq j \leq r-2$.

Proof:

The element $x_0^{n-(i+1)}x_1^{r-(j+1)}x_2x_3 - x_0^{n-(i+2)}x_1^{r-j}x_2^2$ $\in (Q_{nr}, x_0^{n-i}x_1, x_0^{n-(i+1)}x_1^{r-j}, x_0^{n-(i+1)}x_1^{r-(j+1)}x_2x_3) \cap J$ but is not contained in

$(Q_{nr}, x_0^{n-i} x_1, x_0^{n-(i+1)} x_1^{r-j}) \cap J.$

Claim 5.

$(Q_{nr}, x_0^{n-1} x_1^{r-1} x_2 x_3) \cap J \subsetneqq (Q_{nr}, x_0^{n-1} x_1^{r-1} x_2) \cap J.$

Proof:

The element $x_0^n x_1^{r-2} x_3 - x_0^{n-1} x_1^{r-1} x_2 \in (Q_{nr}, x_0^{n-1} x_1^{r-1} x_2) \cap J$ but is not contained in $(Q_{nr}, x_0^{n-1} x_1^{r-1} x_2 x_3) \cap J$.

Claim 6.

$(Q_{nr}, x_0^{n-1} x_1^{r-j}, x_0^{n-1} x_1^{r-(j+1)} x_2 x_3) \cap J \subsetneqq (Q_{nr}, x_0^{n-1} x_1^{r-j}, x_0^{n-1} x_1^{r-(j+1)} x_2)$
$\cap J$ for $1 \leq j \leq r-2$.

Proof:

The element $x_0^n x_1^{r-(j+2)} x_3 - x_0^{n-1} x_1^{r-(j+1)} x_2 \in (Q_{nr}, x_0^{n-1} x_1^{r-j} x_0^{n-1} x_1^{r-(j+1)} x_2) \cap J$ but is not contained in $(Q_{nr}, x_0^{n-1} x_1^{r-j}, x_0^{n-1} x_1^{r-(j+1)} x_2 x_3) \cap J$.

Claim 7.

$(Q_{nr}, x_0^{n-i} x_1, x_0^{n-(i+1)} x_1^{r-1} x_2 x_3) \cap J \subsetneqq (Q_{nr}, x_0^{n-i} x_1, x_0^{n-(i+1)} x_1^{r-1} x_2)$
$\cap J$ for $1 \leq i \leq n-2$.

Proof:

The element $x_0^{n-i} x_1^{r-2} x_3 - x_0^{n-(i+1)} x_1^{r-1} x_2 \in (Q_{nr}, x_0^{n-i} x_1, x_0^{n-(i+1)} x_1^{r-1} x_2) \cap J$ but is not contained in $(Q_{nr}, x_0^{n-i} x_1, x_0^{n-(i+1)} x_1^{r-1} x_2 x_3) \cap J$.

Claim 8.

$(Q_{nr}, x_0^{n-i} x_1, x_0^{n-(i+1)} x_1^{r-j}, x_0^{n-(i+1)} x_1^{r-(j+1)} x_2 x_3) \cap J$
$\subsetneqq (Q_{nr}, x_0^{n-i} x_1, x_0^{n-(i+1)} x_1^{r-j}, x_0^{n-(i+1)} x_1^{r-(j+1)} x_2) \cap J$
for $1 \leq i \leq n-2,\ 1 \leq j \leq r-3$.

Proof:

The element $x_0^{n-i}\,x_1^{r-(j+2)}\,x_3 - x_0^{n-(i+1)}\,x_1^{r-(j+1)}\,x_2 \in$ $(Q_{nr}, x_0^{n-i}\,x_1, x_0^{n-(i+1)}\,x_1^{r-j}, x_0^{n-(i+1)}\,x_1^{r-(j+1)}x_2) \cap J$ but is not contained in $(Q_{nr}, x_0^{n-i}\,x_1, x_0^{n-(i+1)}\,x_1^{r-j}, x_0^{n-(i+1)}\,x_1^{r-(j+1)}\,x_2\,x_3) \cap J$.

Claim 9.

$(Q_{nr}, x_0^{n-i}\,x_1, x_0^{n-(i+1)}\,x_1^2, x_0^{n-(i+1)}\,x_1\,x_2\,x_3) \cap J$

$= (Q_{nr}, x_0^{n-i}\,x_1, x_0^{n-(i+1)}\,x_1^2, x_0^{n-(i+1)}\,x_1\,x_2) \cap J$ for $1 \le i \le n-2$.

Proof:

Let $A = (Q_{nr}, x_0^{n-i}\,x_1, x_0^{n-(i+1)}\,x_1^2, x_0^{n-(i+1)}\,x_1\,x_2\,x_3) \cap J$

$B = (Q_{nr}, x_0^{n-i}\,x_1, x_0^{n-(i+1)}\,x_1^2, x_0^{n-(i+1)}\,x_1\,x_2) \cap J.$

Suppose that $A \subsetneq B$.

Then there is an element $b \in B$ such that $b \notin A$. Since $b \in B$, b must $\in$ $(Q_{nr}, x_0^{n-i}x_1, x_0^{n-(i+1)}\,x_1^2, x_0^{n-(i+1)}\,x_1\,x_2)$. So b can be written in the form

$$v_1\,x_0^n + v_2\,x_1^r + v_3\,x_2^2 + v_4\,x_3^2 + v_5\,x_0^{n-i}\,x_1 + v_6\,x_0^{n-(i+1)}\,x_1^2 + v_7\,x_0^{n-(i+1)}\,x_1\,x_2$$

where $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ are arbitrary but fixed elements of $F\,[x_0, x_1, x_2, x_3]$.

Let $C := (Q_{nr}, x_0^{n-i}\,x_1, x_0^{n-(i+1)}\,x_1^2, x_0^{n-(i+1)}x_1\,x_2\,x_3)$

The terms $v_1\,x_0^n, v_2\,x_1^r, v_3\,x_2^2, v_4\,x_3^2, v_5\,x_0^{n-i}\,x_1, v_6\,x_0^{n-(i+1)}\,x_1^2$ of b are all elements of C. If $v_7\,x_0^{n-(i+1)}\,x_1\,x_2 \in C$, then the element $b \in C$ by definition.

Then we have $b \in J$ (since $b \in (Q_{nr}, x_0^{n-i}x_1, x_0^{n-(i+1)}\,x_1^2, x_0^{n-(i+1)}\,x_1x_2)$ $\cap J = B$).

So $b \in C \cap J = A$ which would be a contradiction. If $v_7 = 0$, $v_7\,x_0^{n-(i+1)}\,x_1\,x_2 \in$ C.

If $v_7 = x_0\,(w_0)$ $(w_0 \in F\,[x_0, x_1, x_2, x_3]\,)$ then $v_7\,x_0^{n-(i+1)}\,x_1\,x_2 = w_0 x_0^{n-i}\,x_1\,x_2$ which is generated by $x_0^{n-i}\,x_1$, a generator of C.

If $v_7 = x_1\,(w_1)$ then $v_7\,x_0^{n-(i+1)}\,x_1\,x_2 = w_1 x_0^{n-(i+1)}\,x_1^2\,x_2$ which is generated by $x_0^{n-(i+1)}\,x_1^2$, a generator of C.

If $v_7 = x_2\,(w_2)$ then $v_7\,x_0^{n-(i+1)}\,x_1\,x_2$ is generated by x_2^2 and if $v_7 = x_3\,(w_3)$ then $v_7\,x_0^{n-(i+1)}\,x_1\,x_2$ is generated by $x_0^{n-(i+1)}\,x_1\,x_2\,x_3$ and both $x_2^2, x_0^{n-(i+1)}\,x_1\,x_2\,x_3$ are generators of C.

Therefore $v_7 \in F\backslash\{0\}$. Since $b \in J$, b can also be written as $q\,(x_0\,x_3 - x_1\,x_2)$. One of the terms of q must be $x_0^{n-(i+1)}\,v_7$ i.e. $q = \ldots + \ldots + \ldots x_0^{n-(i+1)}\,v_7$.

But this means that $x_0^{n-(i+1)}\,v_7\,(x_0\,x_3)$ is a term of $b\,x_0^{n-(i+1)}\,v_7\,x_0\,x_3 = v_7\,x_0^{n-i}\,x_3$.

$v_7\,x_0^{n-i}\,x_3$ is not an element of $(Q_{nr}, x_0^{n-i}\,x_1, x_0^{n-(i+1)}\,x_1^2, x_0^{n-(i+1)}\,x_1\,x_2)$. Thus it is impossible to construct an element $b \in B$ such that $b \notin A$.

Claim 10.

$(Q_{nr}, x_0^{n-1}\,x_1^{r-1}\,x_2) \cap J \subsetneqq (Q_{nr}, x_0^{n-1}\,x_1^{r-1}\,x_2, x_0^{n-1}\,x_1^{r-1}\,x_3) \cap J.$

Proof:

The element $x_0^{n-1}\,x_1^{r-1}\,x_3 - x_0^{n-2}\,x_1^{r}\,x_2 \in (Q_{nr}, x_0^{n-1}\,x_1^{r-1}x_2, x_0^{n-1}\,x_1^{r-1}x_3) \cap J$ but is not contained in $(Q_{nr}, x_0^{n-1}\,x_1^{r-1}\,x_2) \cap J$.

Claim 11.

$(Q_{nr}, x_0^{n-1}\,x_1^{r-j}, x_0^{n-1}x_1^{r-(j+1)}x_2) \cap J = (Q_{nr}, x_0^{n-1}\,x_1^{r-j}, x_0^{n-1}\,x_1^{r-(j+1)}\,x_2, x_0^{n-1}\,x_1^{r-(j+1)}\,x_3) \cap J \qquad \text{for} \qquad 1 \le j \le r-2.$

Proof:

Let $\quad A = (Q_{nr}, x_0^{n-1}\,x_1^{r-j}, x_0^{n-1}\,x_1^{r-(j+1)}\,x_2) \cap J$

$\qquad B = (Q_{nr}, x_0^{n-1}\,x_1^{r-j}, x_0^{n-1}\,x_1^{r-(j+1)}\,x_2, x_0^{n-1}\,x_1^{r-(j+1)}\,x_3) \cap J.$

Suppose $A \subsetneqq B$.

Then there is an element $b \in B$ such that $b \notin A$. Since $b \in B$, b must $\in (Q_{nr}, x_0^{n-1}\,x_1^{r-j}, x_0^{n-1}x_1^{r-(j+1)}\,x_2, x_0^{n-1}\,x_1^{r-(j+1)}\,x_3)$.

So b can be written in the form

$$v_1\,x_0^n + v_2\,x_1^r + v_3\,x_2^2 + v_4\,x_3^2 + v_5\,x_0^{n-1}\,x_1^{r-j} + v_6\,x_0^{n-1}\,x_1^{r-(j+1)}\,x_2 + v_7\,x_0^{n-1}\,x_1^{r-(j+1)}\,x_3$$

where v_1 to v_7 are arbitrary but fixed elements of $F\,[x_0, x_1, x_2, x_3]$.

Let $i = (Q_{nr}, x_0^{n-1}\,x_1^{r-j}, x_0^{n-1}\,x_1^{r-(j+1)}\,x_2)$.

The terms $v_1\,x_0^n, v_2\,x_1^r, v_3\,x_2^2, v_4\,x_3^2, v_5\,x_0^{n-1}\,x_1^{r-j}, v_6\,x_0^{n-1}\,x_1^{r-(j+1)}\,x_2$ of b are all elements of C.

If $v_7\,x_0^{n-1}\,x_1^{r-(j+1)}\,x_3 \in C$, then the element $b \in C$ by definition.

Then we have $b \in J$ (since $b \in B$). So $b \in C \cap J = A$ which would be a contradiction.

If $v_7 = 0,\ v_7\,x_0^{n-1}\,x_1^{r-(j+1)}\,x_3 \in C$.

If every term of v_7 contains an $x_k\,(k = 0, \ldots, 3)$ then $v_7\,x_0^{n-1}\,x_1^{r-(j+1)}\,x_3$ is an element of C. Thus v_7 must have a term α such that $\alpha \in F\backslash\{0\}$. Since $b \in J$, b can also be written as $q\,(x_0\,x_3 - x_1\,x_2)$. One of the terms of q must be $\alpha\,x_0^{n-2}\,x_1^{r-(j+1)}$ i.e. $q = \ldots + \ldots + \ldots + \alpha x_0^{n-2}\,x_1^{r-(j+1)}$.

But this means that $\alpha x_0^{n-2}\,x_1^{r-(j+1)}\,(-x, x_2)$ is a term of b.

$-\alpha\,x_0^{n-2}\,x_1^{r-(j+1)}\,(x_1\,x_2) = -\alpha x_0^{n-2} x_1^{r-j}\,x_2$.

$-\alpha x_0^{n-2} x_1^{r-j} x_2$ is not an element of $(Q_{nr}, x_0^{n-1} x_1^{r-j}, x_0^{n-1} x_1^{r-(j+1)} x_2, x_0^{n-1} x_1^{r-(j+1)} x_3)$.

Thus $\quad b \notin (Q_{nr}, x_0^{n-1}\,x_1^{r-j}, x_0^{n-1} x_1^{r-(j+1)}\,x_2, x_0^{n-1} x_1^{r-(j+1)} x_3)$.

$\Rightarrow b \notin B$ which is a contradiction.

Claim 12.

$\left(Q_{nr}, x_0^{n-i}\,x_1, x_0^{n-(i+1)} x_1^{r-1)} x_2\right) \cap J \subsetneqq (Q_{nr}, x_0^{n-i}\,x_1, x_0^{n-(i+1)}\,x_1^{r-1}\,x_2,$ $x_0^{n-(i+1)}\,x_1^{r-1}\,x_3\big) \cap J$ for $1 \leq i \leq n-2$.

Proof:

The element $x_0^{n-(i+1)}\,x_1^{r-1}\,x_3 - x_0^{n-(i+2)}\,x_1^r\,x_2$ $\in \quad (Q_{nr}, x_0^{n-i} x_1, x_0^{n-(i+1)}\,x_1^{r-1}\,x_2, x_0^{n-(i+1)} x_1^{r-1}\,x_3) \cap J$ but is not contained in $(Q_{nr}, x_0^{n-i} x_1, x_0^{n-(i+1)}\,x_1^{r-1} x_2) \cap J$.

Claim 13.

$$(Q_{nr}, x_0^{n-i}\, x_1, x_0^{n-(i+1)} x_1^{r-j}, x_0^{n-(i+1)} x_1^{r-(j+1)} x_2) \cap J =$$

$$(Q_{nr}, x_0^{n-i}\, x_1, x_0^{n-(i+1)}\, x_1^{r-j}, x_0^{n-(i+1)}\, x_1^{r-(j+1)}\, x_2, x_0^{n-(i+1)} x_1^{r-(j+1)} x_3)$$

$$\cap J \qquad 1 \le i\ \le n-2,\ 1 \le j \le r-2.$$

Proof: Almost identical to proof of Claim 11.

Claim 14.

$$(Q_{nr}, x_0^{n-1}\, x_1^{r-1} x_2, x_0^{n-1}\, x_1^{r-1} x_3) \cap J = (Q_{nr}, x_0^{n-1} x_1^{r-1}) \cap J.$$

Proof:

$$\text{Let } A = (Q_{nr}, x_0^{n-1}\, x_1^{r-1} x_2, x_0^{n-1}\, x_1^{r-1} x_3) \cap J.$$

$$B = (Q_{nr}, x_0^{n-1}\, x_1^{r-1}) \cap J.$$

Suppose $A \subsetneq B$. Then we have an element $b \in B$ such that $b \notin A$. Since $b \in B$, $b \in (Q_{nr}, x_0^{n-1}\, x_1^{r-1})$. So b can be written in the form

$$v_1\, x_0^n + v_2\, x_1^r + v_3\, x_2^2 + v_4\, x_3^2 + v_5 x_0^{n-1}\, x_1^{r-1} \text{ where } v_1, \ldots, v_5$$

are arbitrary but fixed elements of $F\,[x_0, x_1, x_2, x_3]$.

In a similar manner to the proof of Claim 11, one of the terms of v_5 belongs to $F \backslash \{0\}$. Call this element α.

But this means that b cannot be written in the form $q\,(x_0\, x_3 - x_1\, x_2)$, $q \in F\,[x_0, x_1, x_2, x_3]$ (as $\alpha x_0^{n-1}\, x_1^{r-1}$ does not cancel with other terms of b nor does it contain $x_0\, x_3$ or $x_1\, x_2$.) So $b \notin J$ which is a contradiction.

Claim 15.

$$(Q_{nr}, x_0^{n-1}\, x_1^{r-j}, x_0^{n-1} x_1^{r-(j+1)}\, x_2, x_0^{n-1} x_1^{r-(j+1)}\, x_3) \cap J$$

$$= (Q_{nr}, x_0^{n-1} x_1^{r-(j+1)}\,) \cap J \qquad 1 \le j \le r-2.$$

Claim 16.

$$(Q_{nr}, x_0^{n-i}\,x_1, x_0^{n-(i+1)}x_1^{r-1}\,x_2, x_0^{n-(i+1)}x_1^{r-1}\,x_3) \cap J$$
$$= (Q_{nr}, x_0^{n-i}x_1, x_0^{n-(i+1)}\,x_1^{r-1}) \cap J \qquad 1 \le i \le n-2.$$

Claim 17.

$$(Q_{nr}, x_0^{n-i}\,x_1, x_0^{n-(i+1)}x_1^{r-j}, x_0^{n-(i+1)}x_1^{r-(j+1)}\,x_2, x_0^{n-(i+1)}x_1^{r-(j+1)}\,x_3)$$
$$\cap J = (Q_{nr}, x_0^{n-i}x_1, x_0^{n-(i+1)}x_1^{r-(j+1)}) \cap J \;\; 1 \le i \le n-2, 1 \le j \le r-2.$$

The proofs of Claims 15, 16 and 17 are almost identical to the proof of Claim 14.

Claim 18.

$$(Q_{nr}, x_0\,x_1) \cap J \subsetneqq (Q_{nr}, x_0\,x_1, x_0^{n-1}x_2\,x_3) \cap J.$$

Proof:

$$x_0^{n-1}x_2\,x_3 - x_0^{n-2}x_1\,x_2^2 \in (Q_{nr}, x_0\,x_1, x_0^{n-1}x_2\,x_3) \cap J$$

but is not contained in $\qquad (Q_{nr}, x_0\,x_1) \cap J.$

Claim 19.

$$(Q_{nr}, x_0\,x_1, x_0^{n-1}x_2\,x_3) \cap J = (Q_{nr}, x_0\,x_1, x_0^{n-1}x_2) \cap J.$$

Proof: Very similar to proof of Claim 14.

Claim 20.

$$(Q_{nr}, x_0\,x_1, x_0^{n-1}x_2) \cap J \subsetneqq (Q_{nr}, x_0\,x_1, x_0^{n-1}x_2, x_0^{n-1}\,x_3) \cap J \;\text{ if } n \ge 3.$$

Proof: If $n \geq 3$ then $x_0^{n-1} x_3 - x_0^{n-2} x_1 x_2 \in (Q_{nr}, x_0 x_1, x_0^{n-1} x_2, x_0^{n-1} x_3) \cap J$ but is not contained in $(Q_{nr}, x_0 x_1, x_0^{n-1} x_2) \cap J$.

Claim 21.

$$(Q_{nr}, x_0 x_1, x_0^{n-1} x_2, x_0^{n-1} x_3) \cap J = (x_0^{n-1}, x_1^r, x_2^2, x_3^2, x_0 x_1) \cap J.$$

Proof: Very similar to proof of Claim 14.

Claim 22.

$(x_0^{n-i}, x_1^r, x_2^2, x_3^2, x_0 x_1) \cap J \subsetneq (x_0^{n-i}, x_1^r, x_2^2, x_3^2 x_0 x_1, x_0^{n-(i+1)} x_2 x_3)$ $\cap J$ for $1 \leq i \leq n-2$.

Proof: The element $x_0^{n-(i+1)} x_2 x_3 - x_0^{n-(i+2)} x_1 x_2^2 \in$ $(x_0^{n-i}, x_1^r, x_2^2, x_3^2, x_0 x_1, x_0^{n-(i+1)} x_2 x_3) \cap J$ but is not contained in $(x_0^{n-i}, x_1^r, x_2^2, x_3^2, x_0 x_1) \cap J$.

Claim 23.

$(x_0^{n-i}, x_1^r, x_2^2, x_3^2, x_0 x_1, x_0^{n-(i+1)} x_2 x_3) \cap J =$
$(x_0^{n-i}, x_1^r, x_2^2, x_3^2, x_0 x_1, x_0^{n-(i+1)} x_2) \cap J$ for $i \leq i \leq n-2$.

Proof: Very similar to proof of Claim 14.

Claim 24.

$(x_0^{n-i}, x_1^r, x_2^2, x_3^2, x_0 x_1, x_0^{n-(i+1)} x_2) \cap J \subsetneq$
$(x_0^{n-i}, x_1^r, x_2^2, x_3^2, x_0 x_1, x_0^{n-(i+1)} x_2, x_0^{n-(i+1)} x_3) \cap J$
for $1 \leq i \leq n-3$.

Proof:

$x_0^{n-(i+1)} x_3 - x_0^{n-(i+2)} x_1 x_2 \in (x_0^{n-i}, x_1^r, x_2^2, x_3^2, x_0 x_1, x_0^{n-(i+1)} x_2, x_0^{n-(i+1)} x_3)$ $\cap J$ but is not contained in $(x_0^{n-i}, x_1^r, x_2^2, x_3^2, x_0 x_1, x_0^{n-(i+1)} x_2) \cap J$.

Claim 25.

$$(x_0^2, x_1^r, x_2^2, x_3^2, x_0\,x_1, x_0\,x_2) \cap J = (x_0^2, x_1^r, x_2^2, x_3^2, x_0\,x_1, x_0\,x_2, x_0\,x_3) \cap J.$$

Proof: Very similar to proof of Claim 11.

Claim 26.

$$(x_0^{n-i}, x_1^r, x_2^2, x_3^2, x_0\,x_1, x_0^{n-(i+1)}\,x_2, x_0^{n-(i+1)}x_3) \cap J$$

$$= (x_0^{n-(i+1)}, x_1^r, x_2^2, x_3^2, x_0\,x_1) \cap J \text{ for } 1 \leq i \leq n-3.$$

Proof: Very similar to proof of Claim 14.

Claim 27.

$$(x_0^2, x_1^r, x_2^2, x_3^2, x_0\,x_1, x_0\,x_2, x_0\,x_3) \cap J = (x_0, x_1^r, x_2^2, x_3^2) \cap J.$$

Proof: Very similar to proof of Claim 14.

Claim 28.

$(x_0, x_1^{r-j}, x_2^2, x_3^2) \cap J \subsetneq (x_0, x_1^{r-j}, x_2^2, x_3^2, x_1^{r-(j+1)}x_2\,x_3) \cap J$

for $0 \leq j \leq r-2$.

Proof: The element $x_0\,x_1^{r-(j+2)}\,x_3^2 - x_1^{r-(j+1)}\,x_2x_3 \in$

$(x_0, x_1^{r-j}, x_2^2, x_3^2, x_1^{r-(j+1)}x_2x_3) \cap J$ but is not contained in $(x_0, x_1^{r-j}, x_2^2, x_3^2) \cap J$.

Claim 29.

$(x_0, x_1^{r-j}, x_2^2, x_3^2, x_1^{r-(j+1)}x_2\,x_3) \cap J \subsetneq (x_0, x_1^{r-j}, x_2^2, x_3^2, x_1^{r-(j+1)}x_2)$

$\cap J$ for $0 \leq j \leq r-3$.

Proof:

$$x_0x_1^{r-(j+2)}x_3 - x_1^{r-(j+1)}x_2 \in (x_0, x_1^{r-j}, x_2^2, x_3^2, x_1^{r-(j+1)}x_2)$$

$\cap\, J$ but is not contained in $(x_0, x_1^{r-j}, x_2^2, x_3^2, x_1^{r-(j+1)}\,x_2\,x_3) \cap J$.

Claim 30.

$(x_0, x_1^{r-j}, x_2^2, x_3^2, x_1^{r-(j+1)}x_2) \cap J = (x_0, x_1^{r-j}, x_2^2, x_3^2, x_1^{r-(j+1)}x_2,$
$x_1^{r-(j+1)}x_3) \cap J$ for $0 \leq j \leq r-3$.

Proof: Similar to proof of Claim 14.

Claim 31.

$$(x_0, x_1^{r-j}, x_2^2, x_3^2, x_1^{r-(j+1)}x_2, x_1^{r-(j+1)}x_3) \cap J =$$
$$(x_0, x_1^{r-(j+1)}, x_2^2, x_3^2) \cap J \text{ for } 0 \leq j \leq r-3 .$$

Proof: Similar to proof of Claim 14.

Claim 32.

$(x_0, x_1^2, x_2^2, x_3^2, x_1x_2x_3) \cap J \subsetneq (x_0, x_1^2, x_2^2, x_3^2, x_1x_2) \cap J.$

Proof: $x_0 x_3 - x_1 x_2 \in (x_0, x_1^2, x_2^2, x_3^2, x_1x_2) \cap J$ but is not contained in $(x_0, x_1^2, x_3^2, x_1 x_2 x_3) \cap J.$

Also, as $x_0 x_3 - x_1 x_2 \in (x_0, x_1^2, x_2^2, x_3^2, x_1 x_2)$, $(x_0, x_1^2, x_2^2, x_3^2, x_1 x_2) \cap J = J.$ So all ideals after $(x_0, x_1^2, x_2^2, x_3^2, x_1 x_2) \cap J$ in our chain equal J.

We are now in a position to prove lemma 1. To find $\text{mult}_{I_{nr}}\ (x_0, x_1, x_2, x_3)$ we calculate the number of duplicate ideals in our chain, denote this number by e say, and then $4rn - e$ will be $\text{mult}_{I_{nr}}\ (x_0, x_1, x_2, x_3)$ [5, p.2].

So the claims we will use are claims 9, 11, 13, 14, 15, 16, 17, 19, 21, 23, 25, 26, 27, 30, 31. (Since the other claims only relate to ideals that are unequal). (Note Claim 21 and Claim 27 are the same for $n = 2$).

Claim 9 gives $n - 2$ ideals that are equal.

Claim 11 gives $r - 2$ ideals that are equal.

Claim 13 gives $(n - 2)\ (r - 2)$ ideals that are equal.

Claim 14 gives a pair of ideals that are equal.

Claim 15 gives $r-2$ ideals that are equal.

Claim 16 gives $n-2$ ideals that are equal.

Claim 17 gives $(n-2)\,(r-2)$ ideals that are equal.

Claim 19 gives a pair of ideals that are equal.

Claim 21 gives a pair of ideals that are equal.

Claim 23 gives $n-2$ ideals that are equal.

Claim 25 gives a pair of ideals that are equal.

Claim 26 gives $n-3$ ideals that are equal.

Claim 27 gives a pair of ideals that are equal.

Claim 30 gives $r-2$ ideals that are equal.

Claim 31 gives $r-2$ ideals that are equal.

Also since the last 6 ideals in our chain equal J we have another 6 ideals which are equal. So length-multiplicity $(x_0^n, x_1^r, x_2^2, x_3^2)$ is

$$4rn - [3\,(n-2) + 4\,(r-2) + 2\,(r-2)\,(n-2) + 5 + (n-3) + 6]$$

$$\Rightarrow 4rn - (2rn+2) = 2rn - 2.$$

Hence we have proved lemma 1 and thus (i) of theorem 9. We will now prove theorem 9 (ii). We must first find the generators of I_{nr}.

Claim C.

$$I_{nr} = (x_0^{n+1}x_3 - x_0^n\,x_1\,x_2, x_0\,x_1^r x_3 - x_1^{r+1}x_2, x_0\,x_2^2\,x_3 - x_1\,x_2^3,$$

$$x_0\,x_3^3 - x_1\,x_2x_3^2, x_0^n\,x_1^{r-1}x_3 - x_0^{n-1}x_1^r x_2, x_0^n\,x_2x_3 - x_0^{n-1}x_1x_2^2,$$

$$x_0x_1^{r-1}x_3^2 - x_1^r x_2x_3, x_0x_2x_3^2 - x_1x_2^2x_3, x_0^2x_3^2 - x_1^2x_2^2).$$

Proof: Any element in $Q_{nr} \cap J$ can be written in the form $w\,(x_0x_3 - x_1x_2)$ where w is an element of the ring $F\,[x_0, x_1, x_2, x_3]$.

(i) Suppose that w is just a single term. Then, since $wx_0\,x_3, wx_1\,x_2$ must belong to $Q_{nr}, w\,(x_0x_3 - x_1x_2)$ must be generated by one of

$$x_0^{n+1}x_3 - x_0^n\,x_1\,x_2, x_0\,x_1^r x_3 - x_1^{r+1}x_2, x_0\,x_2^2\,x_3 - x_1\,x_2^3,$$
$$x_0\,x_3^3 - x_1\,x_2x_3^2, x_0^n\,x_1^{r-1}x_3 - x_0^{n-1}x_1^r x_2, x_0^n\,x_2x_3 - x_0^{n-1}x_1x_2^2,$$
$$x_0x_1^{r-1}x_3^2 - x_1^r x_2x_3, x_0x_2x_3^2 - x_1x_2^2x_3.$$

(ii) Suppose that w contains two terms i.e. $w = w_1 + w_2$.

If w_1 and w_2 differ only in their coefficients then w can be expressed as a single term which is case (i). So we assume that w cannot be written as a single term. Now if $w_1x_0x_3, w_1x_1x_2, w_2x_0x_3, w_2x_1x_2$ all belong to Q_{nr} then w is generated by two of the generators already given.

If $w_1x_0x_3, w_1x_1x_2 \in Q_{nr}, w_2x_0x_3, w_2x_1x_2 \notin Q_{nr}$ or $w_1x_0x_3, w_1x_1x_2 \notin Q_{nr}, w_2x_0x_3, w_2x_1x_2 \in Q_{nr}$ then $w\,(x_0x_3 - x_1x_2) \notin Q_{nr} \cap J$. If both $w_1x_0x_3, w_1x_1x_2 \in Q_{nr}$ and only one of $w_2x_0x_3, w_2x_1x_2$ belong to Q_{nr}, then $w\,(x_0x_3 - x_1x_2) \notin Q_{nr} \cap J$.

Similarly if both $w_2x_0x_3, w_2x_1x_2 \in Q_{nr}$ and only one of $w_1x_0x_3, w_1x_1x_2 \in Q_{nr}$ then $w\,(x_0x_3 - x_1x_2) \notin Q_{nr} \cap J$.

If $w_1x_0x_3, w_2x_0x_3 \in Q$ but $w_1x_1x_2, w_2x_1x_2 \notin Q_{nr}$, then $w\,(x_0x_3 - x_1x_2)$ can only belong to Q_{nr} if $-w_1x_1x_2 = w_2x_1x_2$ i.e. $w_2 = -w_1$ which is contrary to our assumption.

Also if $w_1x_1\,x_2, w_2x_1x_2 \in Q_{nr}$ but $w_1x_0x_3, w_2x_0x_3 \notin Q_{nr}$ we have the same contradiction. Thus we must have $w_1x_0x_3, w_2x_1x_2 \in Q_{nr}, w_1x_1x_2, w_2x_0x_3 \notin Q_{nr}$ or $w_1x_1x_2, w_2x_0x_3 \in Q_{nr}, w_1x_0x_3, w_2x_1x_2 \notin Q_{nr}$.

Since it just depends on how we label w_1, w_2, we will only consider $w_1x_0x_3, w_2x_1x_2 \in Q_{nr}, w_1x_1x_2, w_2x_0x_3 \notin Q_{nr}$.

Now $w_1x_0x_3 - w_2x_1x_2 - w_1x_1x_2 + w_2x_0x_3 \in Q_{nr} \Leftrightarrow$

$w_1x_1x_2 = w_2x_0x_3 \Rightarrow w_1 = y_1x_0x_3,\ w_2 = y_2x_1x_2$

where $y_1, y_2 \in F[x_0, x_1, x_2, x_3]$.

So $y_1x_0x_1x_2x_3 = y_2x_0x_1x_2x_2 \Rightarrow y_1 = y_2$.

$$\begin{aligned} \text{Hence} \qquad w(x_0x_3 - x_1x_2) &= (w_1 + w_2)(x_0x_3 - x_1x_2) \\ &= (y_1x_0x_3 + y_1x_1x_2)(x_0x_3 - x_1x_2) \\ &= y_1(x_0x_3 + x_1x_2)(x_0x_3 - x_1x_2) \\ &= y_1(x_0^2x_3^2 - x_1^2x_2^2). \end{aligned}$$

So $x_0^2x_3^2 - x_1^2x_2^2$ is another generator of $Q_{nr} \cap J$.

(iii) Suppose that w has t terms $(t \geq 3)$. Since we are trying to find new generators, we assume that there are no w_k $(1 \leq k \leq t)$ such that $w_kx_0x_3, w_kx_1x_2$ both belong to Q_{nr} (case (i)). We also assume that there are no two w_k, w_ℓ $(1 \leq k \leq t,\ 1 \leq \ell \leq t,\ k \neq \ell)$ such that $(w_k + w_\ell)(x_0x_3 - x_1x_2) = \beta(x_0^2x_3^2 - x_1^2x_2^2)$ $(\beta \in F[x_0, x_1, x_2, x_3])$ (case (ii)).

So for each term w_k, at the most only one of $w_kx_0x_3,\ w_kx_1x_2$ can belong to Q_{nr}. Thus when we multiply $w(x_0x_3 - x_1x_2)$ out, we have at least t terms that do not belong to Q_{nr}.

Since $w(x_0x_3 - x_1x_2) \in Q_{nr}$, these terms must all cancel. But no $w_kx_0x_3$ can cancel with $w_\ell x_1x_2$ (case (ii)), so there must be at least three terms involved in each cancellation.

This means that we must have a w_k and w_ℓ $(k \neq \ell)$ that differ only in coefficients contrary to our assumption.

The generators of the ideal of Claim C are independent only if $n, r \geq 2$.

If $n = r = 1$ then $I_{11} = (x_0 x_3 - x_1 x_2)$.

If $n = 1, r \geq 2$ then

$I_{nr} = (x_0^2 x_3 - x_0 x_1 x_2, x_0 x_3^3 - x_1 x_2 x_3^2, x_0 x_1^{r-1} x_3 - x_1^r x_2, x_0 x_2 x_3 - x_1 x_2^2, x_0^2 x_3^2 - x_1^2 x_2^2)$,

and if $n \geq 2, r = 1$ then

$I_{nr} = (x_0 x_1 x_3 - x_1^2 x_2, x_0 x_2^2 x_3 - x_1 x_2^3, x_0^n x_3 - x_0^{n-1} x_1 x_2, x_0 x_3^2 - x_1 x_2 x_3, x_0^2 x_3^2 - x_1^2 x_2^2)$.

Thus from the definition of $m(I_{nr})$

$$m(I_{nr}) = \begin{cases} 2: & n = r = 1 \\ 4: & n = 1, r = 2 \text{ or r} = 1, \text{n} = 2 \\ r+1: & n = 1, r > 2 \\ n+1: & r = 1, n > 2 \\ n+r: & n, r \geq 2. \end{cases}$$

Analysing the examples of [8] and our theorems, we would like to finish with the following (open) problem.

Problem: Let I be a homogeneous ideal of $F[x_0, \ldots, x_n]$ which is not monomial. Under what assumptions do we have arith-deg $(I) \geq m(I)$, where $m(I)$ is the maximum of degrees of forms generating I.

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