Two Elliptic Generator Kleinian Groups

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ABSTRACT

This thesis studies the discreteness of Kleinian groups and the geometry of their associated orbit spaces: hyperbolic 3-manifolds and 3-orbifolds.

Thurston's geometrization theorem states that the interior of every compact 3-manifold can be decomposed into pieces which have geometric structures. Most of these pieces have hyperbolic structures. Every hyperbolic 3-manifold can be described as $\mathbb{H}^3/G$, where $\mathbb{H}^3$ is the 3-dimensional hyperbolic space and G is a torsion free Kleinian group. The compact 3-manifolds other than hyperbolic 3-manifolds have been classified. Therefore the study of Kleinian groups holds the key to understanding 3-manifolds in general.

We investigate various conditions for determining the discreteness of a Kleinian group. Such a group is discrete if and only if all its two generator subgroups are discrete, thus we study the question of discreteness through two generator groups. Here we consider Kleinian groups generated by two elements of finite order. Once the order of these generators is fixed, these groups lie in a one complex-dimensional parameter space with a highly fractal boundary. Computational investigations of various types into the size and structure of this parameter space form an integral part of this thesis. In particular an analysis of Dehn surgery on 2-bridge knots and links gives us data indicating the boundary of the parameter space "internally" analogous to the boundary in the Riley slice (the parabolic case). We then investigate this boundary "externally" and numerically using the rational pleating rays and rational words, which were developed by Keen and Series to study the Riley slice boundary. We are then able to identify some boundary points via the interplay between algebraic and geometric convergence of sequences of Kleinian groups. We find an interesting connection between Conway's notation for 2-bridge knots and links and rational words.

We then apply these descriptions of parameter spaces to advance the identification problem for two generator arithmetic Kleinian groups, in fact we give a conjecturally complete list of certain families of such groups that were previously identified as being the most difficult cases.

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A few years ago, I gave a talk which turned out to be a disaster. I felt so low and doubted my ability to give any future talks. It was my co-supervisor Professor David Gauld who comforted me and assured me that with practice, I could succeed in giving talks. David encouraged me to give a talk at the post-graduate student conference that year and has always been happy to give me a chance to talk whenever possible; and he has always replied to my questions with great patience and wisdom. Aside from all of this, David is the person who has introduced me to the joys of tramping in the mountains of this beautiful country.

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CONTENTS

LIST OF FIGURES

LIST OF TABLES

1. INTRODUCTION

The classification theorem for surfaces (2-manifolds) states that every compact connected orientable surface is either a sphere S^2 or a connected sum of n tori, where $n \geq 1$ [52]. The classification of surfaces is a result of the *uniformization theorem*, which says any surface admits a metric from one of the three geometries: spherical S^n, Euclidean E^n and hyperbolic $\mathbb{H}^n$. The uniformization theorem was proved by the work of Klein in the 1880s, then Koebe and Poincaré in the early 1900s [53]. 3-manifolds are much more complicated. It was not until 1982 that Thurston proposed an analogue for 3-manifolds: *Thurston's geometrization conjecture*. It states that every compact orientable 3-manifold can be decomposed into submanifolds that can be locally modeled on one of eight geometries. See [55], page 368-371 for the details of the eight geometries. Among these geometries, hyperbolic geometry is the most prevalent; other geometries only appear in special cases [55, page 358]. The conjecture was proved by Perelman in 2003. But this is not the end of the story of 3-manifolds. The compact 3-manifolds, other than hyperbolic 3-manifolds, have been classified (see [54], [55]), but we still have not had a classification of hyperbolic 3-manifolds.

A hyperbolic 3-manifold is a 3-manifold such that given a point in the 3-manifold, there exists a small neighbourhood of this point that is isometric to an open subset of $\mathbb{H}^3$. Every hyperbolic 3-manifold can be described as $\mathbb{H}^3/G$, where $\mathbb{H}^3$ is the 3-dimensional hyperbolic space and G is a torsion free Kleinian group. Hence the study of Kleinian groups holds the key to understanding hyperbolic 3-manifolds. The n-dimensional hyperbolic space $\mathbb{H}^n$ can be modeled on the unit ball B^n with the Poincaré metric d, where

$$\cosh d(x, y) = 1 + \frac{2(|x - y|)^2}{(1 - |x|^2)(1 - |y|^2)}.$$

We use Δ to denote B^2, which is the open unit disk and which we identify with the hyperbolic plane $\mathbb{H}^2$. A Kleinian group is a subgroup of the group $\mathcal{M}$ of fractional linear transformations of the form:

$$g(z) = \frac{az + b}{cz + d} \qquad a, b, c, d \in \mathbb{C}, ad - bc = 1, \tag{1.1}$$

which we associate with the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad a, b, c, d \in \mathbb{C}, ad - bc = 1. \tag{1.2}$$

The group $\mathcal{M}$ is naturally associated with the group of isometries of $\mathbb{H}^3$. This action and its extension to $\mathbb{H}^3$ is extensively discussed in A.Beardon's Graduate book [3, page 35]. A subgroup G of $\mathcal{M}$ is *elementary* if and only if there exists a finite G-orbit in $\mathbb{R}^3$. That is for some point $x \in \mathbb{R}^3$, the set

$$G(x) = \{g(x) \in \mathbb{R}^3, g \in G\}$$

is finite. Otherwise, it is *non-elementary*. The group G is discrete if and only if

$$\inf \{\|g - identity\| : g \in G, g \neq identity\} > 0.$$

Here $g - identity$ is viewed as a 2×2 matrix via (1.2) and we have the Hilbert-Schmidt norm:

$$\left\| \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right\|^2 = x^2 + y^2 + z^2 + w^2.$$

A Kleinian group is discrete and non-elementary. Discrete elementary groups are known and classified [3, page 83-90]; but the discreteness of a Kleinian group is not easy to establish. The following theorem gives us a way of studying it.

Theorem 1.0.1. *[3, Theorem 5.4.2] A non-elementary subgroup G of $\mathcal{M}$ is discrete if and only if for each f and g in G, the two generator subgroup $\langle f, g\rangle$ is discrete.*

Thus the problem of the discreteness of G boils down to consideration of the two-generator subgroups of G. The study of such two generator subgroups and the discreteness conditions for them has a rich history. For example, see [3], [6], [11], [16], [18], [25], [27], [47] and the references therein. This thesis is an effort toward the understanding of these two generator groups, particularly those generated by elements of finite order.

A necessary or sufficient condition for a two generator group $\langle f, g\rangle$ to be discrete can be established by many ways. For example, you can look at the fixed points of the two generators (see, for example, Theorem 5.1.2, [3]); or start with two elementary cyclic groups $\langle f\rangle$ and $\langle g\rangle$ and check whether or not they satisfy the Klein's combination theorem [4, page 499]. Our approach is in line with that of Gehring and Martin, that is to study the representative matrices of f and g. They have shown that the three complex numbers

$$\gamma(f, g) = \mathrm{tr}[f, g] - 2, \quad \beta(f) = \mathrm{tr}^2(f) - 4, \quad \beta(g) = \mathrm{tr}^2(g) - 4$$

define $\langle f, g\rangle$ uniquely up to conjugacy whenever $\gamma(f, g) \neq 0$ [15, Lemma 2.2]; where $tr(f)$ and $tr(g)$ denote the traces of representative matrices of f and g respectively, and $[f, g]$ denotes the multiplicative *commutator* $fgf^{-1}g^{-1}$. These three numbers are called the *parameters* of the two generator group $\langle f, g\rangle$ and we write

$$\text{par}(\langle f, g\rangle) = (\gamma(f, g), \beta(f), \beta(g)).$$

Two subgroups G_0 and G_1 of G are conjugate if for some h in G, $G_0 = hG_1h^{-1}$ and they are isomorphic groups [3, page 65]; two elements f and g in G are conjugate if and only if $\text{tr}^2(f)$=$\text{tr}^2(g)$.

Therefore, the space of all two generator Kleinian groups may be viewed as a subset of the three complex dimensional space $\mathbb{C}^3$ via the map

$$\langle f, g\rangle \to \text{par}(\langle f, g\rangle) = (\gamma, \beta, \beta');$$

and the discreteness of $\langle f, g\rangle$ might be decided in terms of (γ, β, β'). For example, the well-known Jørgensen's inequality.

Theorem 1.0.2. *[3, Theorem 5.4.1] (Jørgensen's inequality) Suppose that the fractional linear transformations f and g generate a discrete non-elementary group with $\gamma(f, g) = \gamma$ and $\beta(f) = \beta$. Then*

$$|\gamma| + |\beta| \geq 1. \tag{1.3}$$

See [13], [15], [47], [40] for more such inequalities.

Lemma 1.0.3. *[16, Lemma 2.29] Suppose that $\langle f, h\rangle$ is a discrete subgroup of $\mathcal{M}$ with $\gamma \neq 0$ and $\gamma(f, h) \neq \beta(f)$. Then there exist elliptics h_1 and h_2 of order 2 such that $\langle f, h_1\rangle$ and $\langle f, h_2\rangle$ are discrete with*

$$\gamma(f, h_1) = \gamma(f, h) \quad \textit{and} \quad \gamma(f, h_2) = \beta(f) - \gamma(f, h).$$

According to Lemma 1.0.3,

$$(\gamma, \beta, \beta') \ \text{ is discrete} \Rightarrow (\gamma, \beta, -4) \ \text{ is discrete.}$$

In this way, the three complex dimensional space of two generator groups is projected to the two complex dimensional slice $\beta' = -4$. Except for a finite set of parameters, which correspond to elementary groups, this projection preserves the properties of $\langle f, g\rangle$ being discrete and non-elementary. See [40] for a list of these exceptional points. Note if $\gamma = 0$ or $\gamma = \beta$, then a two generator group is elementary [40]

Gehring and Martin further simplify the problem by fixing β and studying the subset of the complex plane consisting of commutator parameters for Kleinian groups [18, page 4-6]. Along these

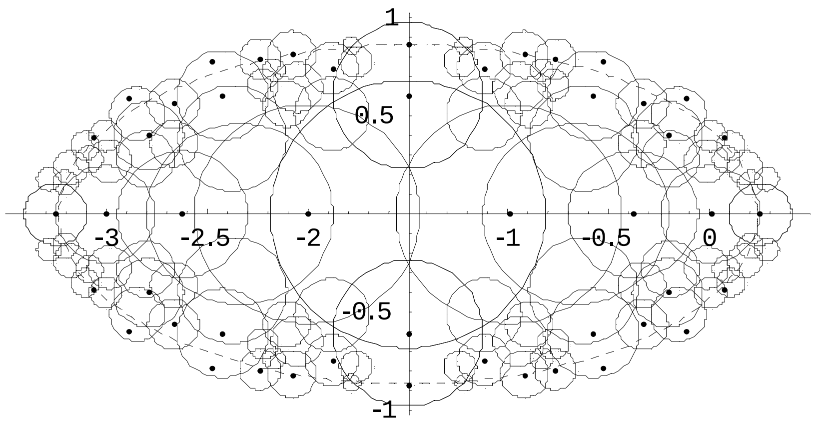


Fig. 1.1: disk covering on the $(2, 3)$ commutator plane

lines, they have made substantial progress when one of the generators is elliptic of order $m \geq 3$, that is when

$$\beta = -4\sin^2(\pi/m), m \geq 3.$$

For example, in Fig (1.1), where $m = 3$, each disk contains no points corresponding to discrete groups other than those indicated by points. Thus the union of these disks contains only finitely many points corresponding to discrete groups and covers only part of the region $\hat{\mathbb{C}} \setminus \mathscr{U}'$, where $\mathscr{U}'$ consists of those points corresponding to a group free on its two generators, see Section 4.3. Part of this thesis is to investigate how close to the boundary of $\mathscr{U}'$ this covering comes.

In our discussion, we use an (n, m) commutator plane to refer to the complex plane that contains the commutator parameters of the groups with two elliptic generators of orders n and m, respectively.

When $|\gamma|$ is large enough, every point corresponds to a two generator Kleinian group $\langle f, g \rangle$ such that (see (2.3)):

$$\langle f, g \rangle = (f, g : f^m = g^2 = Identity).$$

Note $\langle f, g \rangle$ is free on its two generators, that is, isomorphic to the free product of two cyclic groups. Assume $|\gamma| = \gamma_0$ is a circle such that every point outside the circle corresponds to a group free on its two generators. We investigate the region that is outside the area covered by these disks but within the circle $|\gamma| = \gamma_0$ in this thesis. See Figure 1.2. We want to find the points in this region that correspond to discrete two generator groups.

Our starting point is to consider $(m, 0)$ Dehn surgery on hyperbolic 2-bridge knots and links, where m is an integer and $m > 2$. If a 2-bridge knot (or link) is not a torus knot (or link), then its complement in S^3 admits a hyperbolic structure, so is a hyperbolic 3-manifold and the knot is hyperbolic. An $(m, 0)$ Dehn surgery on a hyperbolic 2-bridge knot or link K modifies

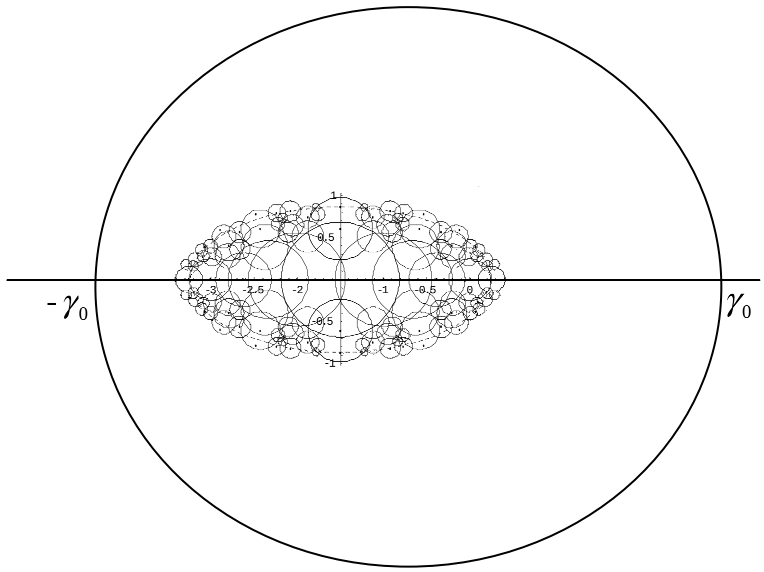


Fig. 1.2: Unknown region in the $(2, 3)$ commutator plane

the complement of K, the resulting space is an orbifold, denoted by $\mathcal{O}$. If such an orbifold $\mathcal{O}$ is hyperbolic, then the (orbifold) fundamental group $\pi_1(\mathcal{O})$ is isomorphic to a Kleinian group with two elliptic generators of orders m. We then use Lemma 2.26 of [16] to get a commutator parameter for a Kleinian group with two elliptic generators of orders 2 and m respectively. Such a Kleinian group is not free so the commutator parameter lies inside the circle $|\gamma| = \gamma_0$. We discuss all of these and the results from $(m, 0)$ Dehn surgery on 2-bridge knots and links in Chapter 3.

The results from $(m, 0)$ Dehn surgery suggest strong analogy with the Riley slice (see Figure 4.2). The Riley slice is the parameter space of groups generated by two parabolic elements. Ohshika and Miyachi have proved (Theorem 1.2, [29]) that the boundary of the Riley slice $\mathscr{R}$ in $\mathbb{C}$ is a Jordan curve. Every point in the closure of $\mathscr{R}$ represents a discrete and free group while there are numerous isolated points in the complement of the closure, $\mathbb{C} \setminus \bar{\mathscr{R}}$, corresponding to the groups that are discrete but not free. For example there are infinitely many points in $\mathbb{C} \setminus \bar{\mathscr{R}}$ corresponding to the fundamental groups of the complements of 2-bridge knots and links [11, page 258].

We view, for the purpose of discussion, a parabolic element as an elliptic of infinite order (see page 39), we conjecture (and discuss) that methods used in [35], such as rational words and rational pleating rays, also apply to the parameter space of a group generated by two elliptic elements. We have written a computer program in Mathematica accordingly to investigate this conjecture numerically. We discuss this investigation in Sections 4.2, 4.3 of Chapter 4. The results of the investigation show that there is indeed a similar boundary for a $(2, m)$ commutator plane. See Figure 4.10. On one side of the boundary, every point corresponds to a discrete group that is the free product of two generators of orders 2 and m, respectively; while on the other side, there are infinitely many isolated points corresponding to the fundamental groups of the orbifolds $\mathcal{O}$ obtained by $(m, 0)$ Dehn surgery on 2-bridge knots and links. Keen is in the process of generalising the rational pleating rays to Kleinian groups with two elliptic generators.

Our method can be extended to study any (n, m) commutator plane with $n > 1, m > 1$ and n, m not both equal to 2. A two generator Kleinian group with two elliptic generators of order 2 is a dihedral group $D_n, n = 2, 3, \ldots, \infty$, which is elementary [3, page 83-90]. There are computational issues with our method, which is based on the trace polynomial of a rational word (see, page 43). As the expression of a word in terms of their generators gets longer, the trace polynomial gets more complicated and it takes longer for the computer program to generate. Nevertheless, it gives us an idea of what an (n, m) commutator plane would look like.

In Chapter 3, we find that there are sequences of points converging to the boundary points of $\mathscr{U}'$ in a $(2, m)$ plane; and that each of these sequences of points corresponds to a sequence of 2-bridge knots and links. We discuss the limit link of such a sequence of knots and links in Section 4.4. We find the limit point, which lies on the boundary of a $(2, m)$ plane, for a sequence of points by doing $(m, 0)$ Dehn surgery on its limit link. This is based on the algebraic convergence theory for a sequence of orbifold groups.

Obtaining boundary points using limit links may be view as investigating the boundary "internally" from within the region bounded by the data obtained from $(m, 0)$ Dehn surgery data; while using rational words may be view as studying the boundary "externally". For instance we found 21 boundary points of $\mathscr{U}'$ (Table 4.1) through rational words which can also be obtained by doing $(m, 0)$ Dehn surgery on a limit link. This ensures that we are on the right track in studying the boundary of an (m, n) commutator plane.

The most interesting result of this thesis is the computational and mathematical investigation of the boundary of $\mathscr{U}'$ in an (m, n) commutator plane. We still don't have a boundary like the one in Riley slice because of the limitations of our method. But all the evidence has shown that the data from $(m, 0)$ Dehn surgery on knots and links serves as a good approximation for the boundary (see Figure 4.10). Plot the data in a $(2, 3)$ commutator plane from $(3, 0)$ Dehn surgery on knots and links and combine it with Figure 1.2, and we have Figure 1.3.

With this boundary, we have greatly reduced the unknown region in a $(2, m)$ commutator plane. We believe, just as in the Riley slice, that the boundary in a $(2, m)$ plane is also a Jordan curve. Indeed it seems that the methods of [29] will carry over in this more general setting without great modification.

We use the data in a $(2, m)$ commutator plane as an approximation for the boundary in the plane and use it to identify certain arithmetic Kleinian groups. We discuss this in Chapter 5.

In Chapter 2, we lay the foundation knowledge for the discussion of this thesis. We put our emphasis on the basic knowledge concerning Kleinian groups, while assuming a familiarity with general topology. See Appendix A for some basic terminology. We refer to [26] for a detailed discussion in topology.

Some interesting questions have arisen through our investigation. We discuss these questions and some possible future research in Chapter 6 and draw a closure for the whole thesis.

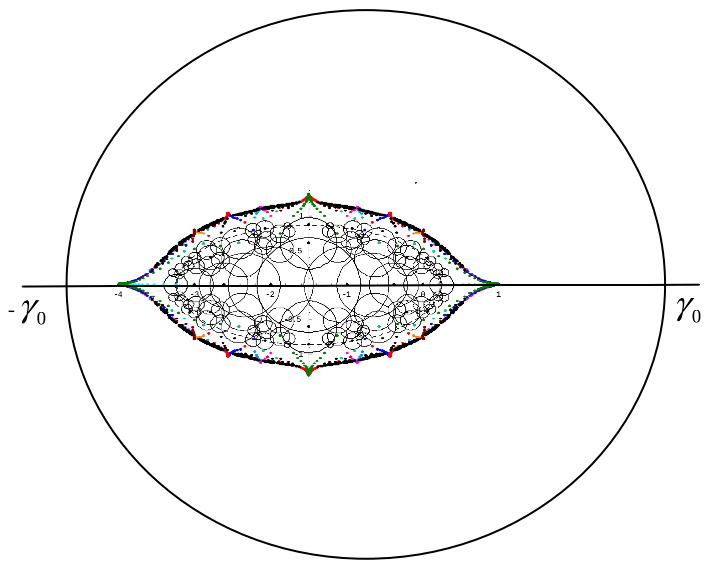


Fig. 1.3: The $(2, 3)$ commutator plane

2. PRELIMINARY TOPICS

2.1 *Kleinian groups*

Kleinian groups have been studied for more than 100 years. In his 1872 paper, *Erlangen Program*, Klein claimed that the essential properties of a given geometry can be studied through a group of transformations that preserve those properties. He observed that a Kleinian group (named after Klein by Poincaré in 1883) is isomorphic to a group of isometries of $\mathbb{H}^3$. Many important properties of Kleinian groups have been developed over the years; some of these we shall discuss in this section. All the definitions and theorems in this section can be found in [3], [5] or [23]. See the historical notes at the end of every chapter in [23] for a brief history of Kleinian groups.

Let $\mathcal{M}$ be the group of linear transformations of the form:

$$f(z) = \frac{az+b}{cz+d} \qquad a,b,c,d \in \mathbb{C}, ad-bc = 1, \tag{2.1}$$

which we associate with the matrix

$$A = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{C}) \qquad a,b,c,d \in \mathbb{C}, ad-bc = 1. \tag{2.2}$$

There are three types of elements in $\mathcal{M}$:

- f is *parabolic* if and only if f acts as a translation and has a unique fixed point in $\mathbb{C} = \hat{\mathbb{C}} \cup \infty$, that is $f(z)$ is conjugate to $m_1(z) = z+1$;
- f is *elliptic* if and only if f acts as a rotation and has two fixed points in $\hat{\mathbb{C}}$, that is $f(z)$ is conjugate to $m_k(z) = kz, |k| = 1, k \neq 1$;
- f is *loxodromic* if and only if f acts as a rotation and dilation and has two fixed points in $\hat{\mathbb{C}}$, that is $f(z)$ is conjugate to $m_k(z) = kz, |k| \neq 1$

These three types may also be defined in terms of traces of the associated matrices:

- f is *parabolic* if and only if $\operatorname{tr}^2(f) = 4$ and $f \neq identity$;
- f is *elliptic* if and only if $0 \leq \operatorname{tr}^2(f) < 4$;

- f is *loxodromic* if and only if $\mathrm{tr}^2(f) \in \mathbb{C} \setminus [0,4]$; if $\mathrm{tr}^2(f) \in (4, +\infty)$, f is *hyperbolic*.

An element f of a group $G \in \mathcal{M}$ is primitive if it is conjugate to

$$f = \begin{pmatrix} \pm e^{i\pi/m} & 0 \\ 0 & \pm e^{-i\pi/m} \end{pmatrix}$$

and $\beta(f) = -4\sin^2(\pi/m)$.

Let Λ_0 be the set of points fixed by *all the loxodromic* elements of G:

$$\Lambda_0 = \left\{ z \in \hat{\mathbb{C}} : \exists g \in G,\ g(z) = z, \quad g \ \ is \ \ loxodromic \right\}.$$

If Λ_0 contains more than 2 points, then G is non-elementary; otherwise, G is elementary. If G is non-elementary as well as discrete, then G is a *Kleinian group.* A Kleinian group is a group of conformal self-mappings of $\hat{\mathbb{C}}$ and is isomorphic to a group of isometries of $\mathbb{H}^3$. A subgroup of $\mathcal{M}$ is discrete if and only if its identity is isolated in the group. In other words, there is no converging sequence in the group.

The closure Λ of Λ_0 in $\hat{\mathbb{C}}$ is called the *limit set* $\Lambda(G)$ of G and the complement of Λ in $\hat{\mathbb{C}}$ is called the *ordinary set* $\Omega(G)$ of G. The limit set $\Lambda(G)$ of G is the smallest non-empty G-invariant closed subset of $\hat{\mathbb{C}}$ and G acts *discontinuously* on $\Omega(G)$. Note, the discontinuous action of a subgroup of $\mathcal{M}$ in $\hat{\mathbb{C}}$ implies discreteness, but the converse is not necessarily true. For example, the fundamental group of the complement of a 2-bridge knot is a discrete group whose limit set is the whole sphere. When G is *torsion free*, which implies that G contains no elliptic elements, it acts *properly* discontinuously. Every Kleinian group G acts discontinuously on $\mathbb{H}^3$ and $\mathbb{H}^3/G$ is a 3-manifold when G is torsion free; otherwise it is a 3-orbifold.

Definition 2.1.1. *Let X be a topological space and let G be a group of homeomorphism of X onto itself. We say that the action of G at a point $x \in X$ is*

- ***discontinuous*** *if there is a neighborhood U of x, so that $g(U) \cap U = \emptyset$ for all but finitely many $g \in G$;*

- ***properly discontinuous*** *if there is a neighborhood U of x, so that $g(U) \cap U = \emptyset$ for all $g \in G, g \neq$ identity.*

A Kleinian group G is of the second kind if $\Omega(G)$ is not empty, otherwise it is of the first kind. The volume of $\mathbb{H}^3/G$, is called the co-volume of G. It follows from Theorem 12.1.14 of [23] that if G is of the second kind, then its co-volume is infinite; and if G has finite co-volume then G is of the first kind. Note that there are Kleinian groups of the first kind with infinite co-volume [23, page 582-583].

We can study a Kleinian group G by drawing a picture of Ω/G, which somehow illustrates the action of G. Normally such a picture is given by a fundamental domain, which, in some sense, illustrates the topology of Ω/G.

Let X be a topological space with a group G of transformations which acts discontinuously. An open connected subset D of X is called a *fundamental domain* of G if and only if

- the members of $\{g(D) : g \in G\}$ are mutually disjoint; and
- $X = \cup \left\{g(\bar{D}) : g \in G\right\}$.

Not every fundamental domain D would have the property that $\bar{D}/G$ is topologically equivalent to Ω/G. See Example 9.2.1. on page 206-207 of [3] for such a domain. The property of a fundamental domain which determines the equivalence of $\bar{D}/G$ and Ω/G is its *local finiteness*.

Definition 2.1.2. *A collection of subsets $\mathcal{S}$ of X is* locally finite *if for each x of X, there is an open neighborhood U of x such that U meets only finitely many members of $\mathcal{S}$. A fundamental domain D is locally finite if the set $\left\{g(\bar{D}), g \in G\right\}$ is locally finite in X.*

Suppose $X = \mathbb{H}^n, S^n$ or E^n and G is a discontinuous group of isometries of X. We can construct a *convex fundamental polyhedron* whose interior is a locally finite fundamental domain for G as follows. Let u be a point of X whose stabilizer G_u is trivial. For each $g \neq \textit{identity}$ in G, define

$$H_g(u) = \{x \in X : d(x,u) < d(x, g(u))\}.$$

The *Dirichlet domain* $D(u)$ for G, with center u, is

$$D(u) = \cap \{H_g(u) : g \neq \textit{identity} \text{ in } G\}$$

and the closure of the Dirichlet domain is a convex fundamental polyhedron.

A fundamental domain D is *proper* if the volume of the boundary of D is 0. Every convex, locally finite fundamental domain for a discrete group G of isometries of X is proper. Suppose G is a Kleinian group, then the volume of $\mathbb{H}^3/G$ is the volume of any proper fundamental domain for G in $\mathbb{H}^3$.

Definition 2.1.3. *A* tessellation *of X is a collection $\mathcal{D}$ of n-dimensional convex polyhedra in X such that*

- *the interiors of the polyhedra in $\mathcal{D}$ are mutually disjoint;*
- *the union of the polyhedra in $\mathcal{D}$ is X; and*

- *the collection $\mathcal{D}$ is locally finite.*

We now use the *isometric circle* of an element in a Kleinian group to prove the existence of γ_0, which is discussed on page 4. Let

$$g(z) = \frac{az+b}{cz+d} \qquad a,b,c,d \in \mathbb{C}, ad-bc=1,$$

where $c \neq 0$. *The isometric circle* of an element g is defined as

$$\mathcal{C}_g = \{z \in \mathbb{C} : |cz+d| = 1\}.$$

The isometric circle of the inverse g^{-1} of g is

$$\mathcal{C}_{g^{-1}} = \{z \in \mathbb{C} : |cz-a| = 1\}.$$

The element g acts as an isometry on $\mathcal{C}_g$ and $g(\mathcal{C}_g) = \mathcal{C}_{g^{-1}}$. If g is parabolic, $\mathcal{C}_g$ and $\mathcal{C}_{g^{-1}}$ are tangent at the fixed point of g; if g is elliptic, $\mathcal{C}_g$ and $\mathcal{C}_{g^{-1}}$ intersect at the two fixed points g; if g is loxodromic, $\mathcal{C}_g$ and $\mathcal{C}_{g^{-1}}$ are disjoint and each of the isometric circles enclose a fixed point of g.

Let f be a primitive elliptic of order m and g an elliptic of order 2. We may assume that f has the following form by conjugation:

$$f = \begin{pmatrix} e^{i\pi/m} & 0 \\ 0 & e^{-i\pi/m} \end{pmatrix}.$$

So the lines $\overrightarrow{OA}$ and $\overrightarrow{OB}$ bound a fundamental domain for $\langle f \rangle$ in $\hat{\mathbb{C}}$. See Figure 2.1. Let the circle

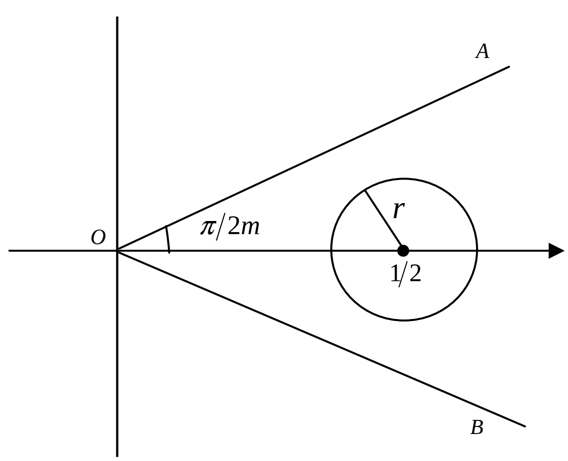


Fig. 2.1: A fundamental domain for $\langle f, g \rangle$

centered at 1/2 with radius r be the isometric circle $\mathcal{C}_g$ of g. Then g may be written as

$$g = \begin{pmatrix} 1/2r & -(r+1/4r) \\ 1/r & -1/2r \end{pmatrix}.$$

By a simple calculation (see Lemma 2.2.2), we have

$$\gamma(f,g) = -(e^{i\pi/m} - e^{-i\pi/m})^2(-(r+1/4r))1/r,$$

which is simplified to

$$\gamma(f,g) = -4\sin^2(\frac{\pi}{m})(1/4r^2+1). \tag{2.3}$$

According to (2.3), if the radius r of the isometric circle $\mathcal{C}_g$ of g is sufficiently small such that $\mathcal{C}_g$ always lies inside the fundamental domain of $\langle f \rangle$, then the group $\langle f, g \rangle$ is a free product of f and g according to Klein's combination theorem [4, page 499].

From (2.3), we can have a γ_0 value in a $(2,m)$ commutator plane such that every point outside the circle $|\gamma| \geq \gamma_0$ corresponds to a group that is a free product of f and g. If f is parabolic, it has been shown that $\gamma_0 \geq 4$ [42, page 1388]; but according to (2.3), γ_0 is 16. While these estimates are not very good, they show there is a circle in a $(2,m)$ commutator plane such that every point outside the circle corresponds to a group free on its two generators.

2.2 A computer program for generating polynomials

Our computer program is based on the Corollary 2.2.1.

Corollary 2.2.1. *[16, Corollary 7.19] Suppose that f and h are in $\mathcal{M}$ and that h is elliptic of order 2. If g is in $\langle f, h \rangle$, then*

$$\gamma(f,g) = p(\gamma(f,h), \beta(f)) \quad \text{and} \quad \beta(g) = q((\gamma(f,h), \beta(f)))$$

where p and q are polynomials in both variables with integer coefficients.

We start from the following lemma.

Lemma 2.2.2. *Let*

$$f = \begin{pmatrix} u & 0 \\ 0 & 1/u \end{pmatrix} \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $|u| \neq 0, ad - bc = 1$. Then $\gamma(f,g) = -(u-1/u)^2 bc$.

Proof The inverses of f and h are

$$f^{-1} = \begin{pmatrix} 1/u & 0 \\ 0 & u \end{pmatrix} \qquad g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

An elementary computation shows that

$$fgf^{-1}g^{-1} = \begin{pmatrix} ad - bcu^2 & -ab + abu^2 \\ (cd/u^2) - cd & -bc/u^2 + ad \end{pmatrix}.$$

Then

$$\begin{aligned} tr(fgf^{-1}g^{-1}) &= 2ad - bcu^2 - bc/u^2 && (2.4)\\ &= 2(1+bc) - bcu^2 - bc/u^2 && (2.5)\\ &= 2 + 2bc - bcu^2 - bc/u^2 && (2.6) \end{aligned}$$

The result follows from $\gamma(f,g) = tr(fgf^{-1}g^{-1}) - 2$. □

Suppose $\langle f, h\rangle$ is a discrete subgroup of $\mathcal{M}$. Recall that every two generator Kleinian group can be represented by three complex numbers (γ, β, β') and there is a projection from the three complex dimensional space of two generator groups to the two complex dimensional slice $\beta' = -4$. So we assume h is elliptic of order 2; we also assume f is non-parabolic. Without loss of generality, we can assume by conjugation that f and h have the form:

$$f = \begin{pmatrix} u & 0 \\ 0 & 1/u \end{pmatrix} \qquad h = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

where $u \neq 0$ and $a^2 - bc = 1$. Note $\beta(h) = -4$. Let

$$U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$$

be an element in $\langle f, h\rangle$ (we say U is a word in $\langle f, h\rangle$). Then according to Corollary 2.2.1, $\gamma(f,U) = p(\gamma(f,h), \beta(f))$. Let $x = \beta(f) = (u - 1/u)^2$. According to Lemma 2.2.2, $z = \gamma(f,h) = -xbc$ and $y = \gamma(f,U) = -xu_2u_3$. We want to symbolically solve the equations $\beta(f) = (u-1/u)^2, \gamma(f,h) = -xbc$ and $\gamma(f,U) = -xu_2u_3$ to find y as a polynomial in $\gamma(f,h)$ and $\beta(f)$. Gehring and Martin have written a program in Mathematica to solve the following system of equations for y in terms of x, z.

$$\begin{cases} x = (u - 1/u)^2 \\ y = -xu_2u_3 \\ -a^2 - bc = 1 \\ z = -xbc \end{cases} \tag{2.7}$$

The following is the program in Mathematica, where $U[[1,2]] * U[[2,1]] = u_2u_3$ and u_2, u_3 are the polynomails in terms of a, b, c, u.

$$f^{-1} = \text{Inverse}[f] \qquad h^{-1} = \text{Inverse}[h]$$

gamma[$U\_$] := Block[s, x, z, y,

$s = U[[1,2]] * U[[2,1]];$

$eqn = \{x == (u - 1/u)^2,$

$y == -s * x,$

$-a * a - b * c == 1,$

$z == -b * c * x\};$

$S1 = \text{Solve}[eqn, y, \{a, b, c, u\}];$

$S2 = S1[[1,1]];$

$S3 = \text{Factor}[S2[[2]]]]$

3. DEHN SURGERY ON 2 BRIDGE KNOTS AND LINKS

3.1 Two bridge knots and links

A knot in $S^3 = R^3 \cup \infty$ is a homeomorphic image of a circle; a link is a disjoint union of one or more knots. In practice, knots and links are studied through their diagrams. A knot (or link) diagram is a regular projection of the knot (or link) into a plane with no three points on the knot (or link) projected to the same point. A point in a knot diagram which corresponds to two points in a knot is called a *double* point and we say that the knot crosses itself at this point. See Figure 3.1 [41]. A knot (or link) can have many different diagrams, see Figure 3.2 [41]. There is at least one knot diagram which contains minimum number of double points among all such knot diagrams. This minimum number is called the crossing number of a knot. Undercrossings and Overcrossings are identified in the obvious manner from the knot or link projection. For a more detailed discussion on knots and links, see, for example, [44] and [17] . We are interested in 2-bridge knots and links.

Definition 3.1.1. *[44, page 114] An overpass is a subarc of the diagram which contains an overcrossing but no undercrossing points, and the number of maximal overpasses is called the* ***bridge number*** *of the projection. The bridge index $b(k)$ of k is the least bridge number of all diagrams of a knot k. See Figure 3.2*

A 2-bridge knot (or link) is a knot (or link) with bridge-index 2. Given a knot diagram, it is not a trivial task to identify whether or not it is a 2-bridge knot. Fortunately, we have found *KnotInfo*, a website that is created and maintained by Livingston with the assistance of Cha [9]. We have since identified all diagrams of 2-bridge knots and links up to 12 crossings using KnotInfo's knot invariant calculator. The diagrams we have used in this research are from [44] (up to 10 crossings) and [9] (11 and 12 crossings).

Fig. 3.1: Two knot diagrams: Trefoil (left) and Figure Eight (right)

Fig. 3.2: Two diagrams of the trefoil: one with bridge number 3, the other 2

Let p/q be a rational number and $0 < p/q < 1$. The continued fraction of p/q is

$$p/q = [c_1, c_2, c_3 \ldots, c_r],$$

which is

$$p/q = \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cdots \cfrac{1}{c_{r-1} + \cfrac{1}{c_r}}}}}$$

where c_i are all positive. Every 2-bridge knot or link may be put in the form shown as in Figure 3.3 [44, page 303]. Conway's notation is defined as $[c_r \ \ldots \ c_2 \ c_1]$ [24, page 331-332].

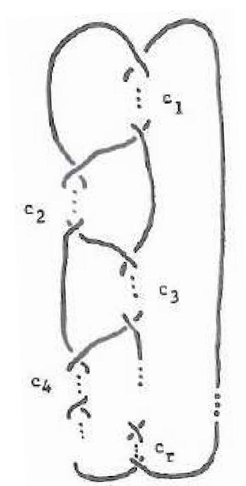


Fig. 3.3: A 2-bridge knot

A knot or link is *alternating* if it possesses a diagram in which the crossings alternate over-under when one travels along the diagram in a fixed direction. An alternating knot or link is *prime* if it admits a reduced diagram (e.g., the diagram cannot be manipulated to have less crossings) such that no circle in the projection plane intersects the diagram twice and contains crossings to either side ([1, page 4], [21, page 39]). All 2-bridge knots are alternating and prime. It is a consequence of Definition 5.2 and Corollary 5.3 of [39], that if a 2-bridge knot is not a torus knot, then its complement in S^3 admits a hyperbolic structure of finite volume. In future we will simply say it is a hyperbolic 3-manifold of finite volume. The fundamental group of the complement of a 2-bridge knot in S^3 is generated by two conjugate parabolic elements [45, page 507]. A *torus knot* is a simple closed curve on an unknotted torus in S^3; a torus link is a disjoint union of one or more torus knots.

A link is *split* if there exists a 2-sphere in the link complement which separates some components of the link from the rest [20, page 4]. An alternating link is splitable if and only if any and every alternating projection is disconnected, which means that every link diagram can be pulled apart so it is a disjoint union of smaller diagrams. A 2-bridge link is non-splitable, prime, and alternating [1, page 4]. It follows from Corollary 2 of [37], the complement of a 2-bridge link which is not a torus link is a hyperbolic 3-manifold of finite volume. The fundamental group of such a 3-manifold is a two generator group with two parabolic generators.

Theorem 3.1.2. *[38, Theorem 8] Let M be a* ***cusped*** *hyperbolic 3-manifold of finite volume. Then the fundamental group of M is generated by two parabolic elements if and only if M is the complement of a 2-bridge link in S^3 that is not a 2-braid. The two generators correspond to meridians.*

A cusped 3-manifold can be thought of, topologically, as the interior of a compact manifold with toral boundary [46, Definition 2.1]. A 2-braid knot or link is a $(p, 2)$-torus knot or link [7, page 619], which is a 2-bridge torus knot or link. A knot or link is hyperbolic when its complement in S^3 is hyperbolic, which means the complement is locally (in a small neighborhood of a point) isometric to H^3.

3.2 Dehn surgery

Given a 3-manifold, a Dehn surgery, named after German mathematician Max Dehn, on the manifold is a specific construction used to modify the manifold. Suppose the following data is given [44, page 257-259]:

(a) a 3-manifold M, perhaps with boundary;

(b) a link $L = L_1 \cup \ldots \cup L_n$ of simple closed curves in the interior M° of M;

(c) disjoint closed tubular neighborhoods N_i of the L_i in M°;

(d) a specified simple closed curve J_i in each ∂N_i, where $J_i = m_i\mu_i + n_i l_i$, (μ_i, l_i) is a meridian-longitude pair of N_i and m_i, n_i are the number of intersections of J_i with μ_i and l_i respectively.

We may then construct a 3-manifold

$$M' = (M - (N_1^\circ \cup \ldots \cup N_n^\circ)) \underset{h}{\cup} (N_1 \cup \ldots \cup N_n),$$

where h is a union of homeomorphisms $h_i : \partial N_i \to \partial N_i \subset M$, each of which takes a meridian curve μ_i of N_i onto the specified J_i with m_i, n_i are coprime integers.

Definition 3.2.1. *[44, page 258] The 3-manifold M' is said to be the result of a Dehn surgery on M along the link L with surgery instructions (c) and (d).*

The Dehn surgery coefficient is defined as m_i/n_i. In our discussion, an (m_i, n_i) Dehn surgery on a knot or link k means a Dehn surgery with surgery coefficient m_i/n_i on S^3 along k. An (m_i, n_i) Dehn surgery can also be used to construct 3-orbifolds. For example, the modified S^3 is an orbifold when $n_i = 0$ [10, p.66].

The fundamental group of the complement of a hyperbolic 2-bridge knot (or link) in S^3, simply a knot (or link) group, is a two generator group with two parabolic generators. When we do $(m, 0)$ Dehn surgery on such a knot (or link), we would have a fundamental group with two elliptic generators of orders m for the resulting orbifold. We shall call such a fundamental group an $(m, 0)$ knot (or link) group. A knot group or an $(m, 0)$ knot group is a Kleinian group of finite co-volume, it is not free on its two generators (see the discussion in Section 4.4 of the Chapter 4). Hence the commutator parameter of a $\mathbb{Z}_2$-extension (see page 21) of an $(m, 0)$ knot group is a complex number lying inside the circle $|\gamma| = \gamma_0$. Note that an $(m, 0)$ knot group has index 2 in its $\mathbb{Z}_2$-extension so any of its $\mathbb{Z}_2$-extensions is also discrete. We want to find as many points as we can in an $(2, m)$ commutator plane which correspond to a $\mathbb{Z}_2$-extension of an $(m, 0)$ knot group. This is the reason we are interested in 2-bridge knots and links. Recall a group free on two generators is isomorphic to the free product of two cyclic groups.

The program that we use to do (m, n) Dehn surgery is called SnapPea. SnapPea is a free software designed to study hyperbolic 3-manifolds and 3-orbifolds. The primary developer is Jeffrey Weeks, who created the first version as part of his doctoral thesis. It may be downloaded at *http://www.geometrygames.org/SnapPea/*. Computational issues and precision etc are discussed in the SnapPea help files.

SnapPea takes a hyperbolic knot or link as an input and finds a representation of its complement in S^3 as a hyperbolic 3-manifold. Then it computes a list of properties of the complement, such as the fundamental group, fundamental domain , co-volume, etc. It can also be used to modify the complement by doing (m, n) Dehn surgery. In our case, we do $(m, 0)$ Dehn surgery on hyperbolic 2-bridge knots or links to get the Kleinian groups that are generated by two elliptic elements of order m. A retriangulation in SnapPea gives a new set of generators and relations for the group by constructing a new ideal triangulation of the manifold or orbifold using a different base point. In our study, we need to do retriangulation many times to get different Nielsen classes (see page 22) of two generators of order m. Note that the Nielsen class determines the complex commutator parameter.

3.3 The arithmetic knot groups

An arithmetic knot group is an arithmetic Kleinian group; an arithmetic Kleinian group is of finite co-volume and may be defined in terms of algebra and number theory and they are extensively discussed in the graduate text book by Maclachlan and Reid [10]. This enables the use of techniques from these branches of mathematics to study Kleinian groups of finite co-volume. "...for these classes of arithmetic Kleinian groups and arithmetic Fuchisian groups (subgroups of $SL(2,\mathbb{C})$, many important features, topological, geometric, group-theoretic, can be determined from the arithmetic data going into the definition of the group. Thus it is important to be able to identify, among all Kleinian groups, those that are arithmetic..." [10, page 254]. Theorem 3.3.1 states a criterion for arithmeticity of Kleinian groups with two elliptic generators of order 2 and m, respectively, where $m = 3, 4, 6$.

Theorem 3.3.1. *[14, Theorem 5.14] Let $G = \langle f, g\rangle$ be a subgroup of $PSL(2,\mathbb{C})$ with f a primitive elliptic of order $m = 3, 4,$ or 6, g an elliptic of order 2 and $\gamma(f, g) \neq 0, \beta(f)$. Then G is a discrete subgroup of an arithmetic group if*

- *$\gamma(f, g)$ is a root of a monic polynomial $p(z)$ with integer coefficients;*
- *if $\gamma(f, g)$ and $\bar{\gamma}(f, g)$ are complex, then all other roots of $p(z)$ lie in the interval $(\beta(f), 0)$;*
- *if $\gamma(f, g))$ is real, then all other roots of $p(z)$ lie in the interval $(\beta(f), 0)$.*

If G is also of finite co-volume, then G is arithmetic (see page 3623 of [14]).

Lemma 3.3.2. *[16, Lemma 2.26] Suppose that $\langle f, g\rangle$ is a discrete subgroup of $\mathcal{M}$ with $\gamma(f, g) \neq 0$ and $\beta(f) = \beta(g)$ and that γ is a complex number for which*

$$\gamma(f, g) = \gamma(\gamma - \beta(f)). \tag{3.1}$$

Then there exists an elliptic h of order 2 such that $\langle f, h\rangle$ is discrete with

$$\gamma(f, h) = \gamma \quad \text{and} \quad hfh^{-1} = g^{\eta} \quad \text{where} \quad \eta = \pm 1. \tag{3.2}$$

$\langle f, h\rangle$ is called the $\mathbb{Z}_2$***-extension*** of $\langle f, g\rangle$ and $\langle f, g\rangle$ has index 2 in $\langle f, h\rangle$.

Drawing a 2-bridge knot (or link) k and do $(m, 0)$ Dehn surgery in SnapPea (in the case of 2-bridge links, we do $(m, 0)$ Dehn surgery on both components of the link), where $m \geq 3$. There are a few things that we are interested in.

- First of all, we need to find a pair of elliptic generators of orders m for the $(m, 0)$ knot or link group. SnapPea outputs such a generating pair straight away for an $(m, 0)$ link group but not for an $(m,0)$ knot group. We need Neilsen's equivalence in the knot case. Basically, Nielsen's equivalence states that if (a,b) is a pair of generators for a group (SnapPea uses symbols a,b for a generating pair, we use them accordingly in the subsequent discussion), so are $(ab,b),(aB,b),(Ab,b),(AB,b)$ and $(a,ab)(a,Ab),(a,aB),(a,AB)$ [40, page 35]. It is easy to check that the commutator parameter of the group $\langle a,b\rangle$ remains the same under Neilsen's equivalence. The generators a,b given by SnapPea are 2×2 complex matrices with determinant 1, and A,B are the **inverse** matrices of a,b, respectively.

SnapPea produces a generating pair for the majority of $(m,0)$ knot or link groups but not for all. For example, SnapPea didn't work at every attempt to get a generating pair for the $(3,0)$ knot group from $(3,0)$ Dehn surgery on the knot $12a_{247}$ in [9];

- record the group presentation for the group with these two generators. For example, in the case that (ab,b) is the pair of two elliptic generators, the group presentation is $(ab)^m = b^m = w(a,b) = identity$ where $w(a,b)$ is a word in a,b,A,B;

- find the trace of the commutator $[a,b] = aba^{-1}b^{-1}$ corresponding to this group presentation and subtract 2 to get a commutator parameter $\gamma(a,b)$ for the group;

- Use (3.1) and $\gamma(a,b)$ to find the commutator parameters for *both* $\mathbb{Z}_2$ -extensions of the $(m,0)$ knot group. Define these two parameters as γ_1 and γ_2, respectively.

We want to know whether or not a $\mathbb{Z}_2$-extension of $\langle a,b\rangle$ is arithmetic. According to Theorem 3.3.1, first of all we need to find a polynomial that has γ_1 (or γ_2) as a root.

- Assume that (ab,b) are an elliptic generating pair both of orders m and let $x=ab, y=b$. So the relations $(ab)^m = b^m = w(a,b) = identity$ become $x^m = y^m = w(x,y) = identity$;

- Let $c \in SL(2,\mathbb{C})$ with c of order 2, and set $y = cxc^{-1}$ (or $y^{-1} = cxc^{-1}$). Then $\langle x,c\rangle$ is a $\mathbb{Z}_2$-extension of $\langle x,y\rangle$. Rewrite $w(x,y)$ in terms of x,c;

- Use the word $w(x,c)$ in our computer program as discussed in Section 2.2 to generate a polynomial $P(\gamma_i) = \gamma(a,w(a,c))$. Since $w(a,c) = identity$, so $P(\gamma_i)=0$. $P(\gamma_i)$ depends on $y = cxc^{-1}$ or $y^{-1} = cxc^{-1}$.

Once we have a polynomial which has γ_1, or γ_2 as a root, we check the rest of its roots to see if they satisfy the last two conditions in Theorem 3.3.1. If they do, then the $\mathbb{Z}_2$-extension is arithmetic since the co-volume of the orbifold from $(m,0)$ Dehn surgery on a knot or link is finite.

We take the knot 5_2 as an example for the above very dry description. The following shows how to do a $(3,0)$ Dehn surgery on the knot 5_2 in SnapPea:

(1) double click the “snappea.exe” to open a SnapPea window;

(2) click “file → New Link Projection” in the SnapPea window to draw the knot 5_2;

(3) click “Complement” in the Link Projection window;

(4) type in $(0, 3, 0)$ for “Cusp, Meridian, Longitude” in the new window then click “Recompute” which is at the left bottom of the window. After this step you have modified the complement of the knot 5_2 from a hyperbolic 3-manifold to a hyperbolic 3-orbifold $\mathcal{O}$;

(5) then go back to the first window you have opened in the first step and click “view → Fundamental Group”;

(6) SnapPea then opens a new window with some relations for the fundamental group of $\mathcal{O}$; in this case it gives $aabAbaaBAAB = bbb = identity$ where b is elliptic of order 3 and a is loxodromic;

(7) click “Matrix Representation” to see if any of ab, aB, Ab, AB is elliptic of order 3; in this case, ab is;

(8) in the case that none of ab, aB, Ab, AB is elliptic of order 3, we go back to the first window click “view” then “Retriangulation” and click ”Randomize” in the window newly popped out; then repeat the process from (5).

We now have a group presentation for the fundamental group of $\mathcal{O}$: $(ab)^3 = b^3 = w(a, b)$, where $w(a, b) = aabAbaaBAAB$. The commutator parameter corresponding to this representation is $-1.21507985450 - 1.30714127877i$, so $\gamma_1 = -2.662358979 + 0.562279512i$ and $\gamma_2 = -0.33764102136 - 0.56227951209i$.

Let $ab = x$ and $b = y$ then $a = xy^{-1}, A = yx^{-1}, B = y^{-1}$. The relation of the group in terms of x, y is:

$$x^3 = y^3 = w(x, y) = identity$$

where

$$w(x, y) = xy^{-1}xyx^{-1}yxy^{-1}xy^{-1}x^{-1}yx^{-1}y^{-1}.$$

Define $y = cxc^{-1}$, rewrite $w(x, y)$ in terms of x, c:

$$w(x, c) = xcx^{-1}c^{-1}xcxc^{-1}x^{-1}cxc^{-1}xcx^{-1}c^{-1}xcx^{-1}c^{-1}x^{-1}cxc^{-1}x^{-1}cx^{-1}c^{-1} = Identity.$$

Using w as the word in our computer program, we have generated the following polynomial.

$$z(3 + z)(5 + 11z + 6z^2 + z^3)^2(1 + 6z + 25z^2 + 37z^3 + 25z^4 + 8z^5 + z^6) \tag{3.3}$$

which has $-2.662358980 + 0.5622795i$ as a root. As we can see the polynomial in Table 3.1 for knot 5_2 is just a factor of the polynomial in (3.3). This is because $-2.662358980 + 0.5622795i$ is also a root of $(5 + 11z + 6z^2 + z^3)$ and this is sufficient to prove the arithmeticity of the group

Tab. 3.1: The arithmetic $\mathbb{Z}_2$-extension of $(3,0)$ knot groups (up to 9 crossings)

K	Relation	$\mathbb{Z}_2$extension	Polynomial	$\gamma-$ value
5_2	$(ab)^3=b^3=aabAbaaBAAB$	$y=cxc^{-1}$	$5+11z+6z^2+z^3$	$-2.662358980+0.5622795i$
		$y^{-1}=cxc^{-1}$	$1+2z+3z^2+z^3$	$-0.33764102-0.5622795i$
5_2	$a^3=(ab)^3=abbaBAbaBabAB$	$y=cxc^{-1}$	$1+5z+4z^2+z^3$	$-1.87743883-0.74486177i$
		$y^{-1}=cxc^{-1}$	$5+8z+5z^2+z^3$	$-1.12256117+0.74486177i$
6_1	$b^3=(aB)^3=aabAABaaaBAAb$	$y=cxc^{-1}$	$1+z+4z^2+4z^3+z^4$	$0.00755236-0.51311580i$
		$y^{-1}=cxc^{-1}$	$7+23z+22z^2+8z^3+z^4$	$-3.00755236+0.51311580i$
6_2	$b^3=(aB)^3=aaBABaaBAba-$	$y=cxc^{-1}$	$1+8z+14z^2+13z^3+6z^4+z^5$	$-0.66222154+0.8997821i$
	$bAAbabAB$	$y^{-1}=cxc^{-1}$	$5+32z+49z^2+31z^3+9z^4+z^5$	$-2.33777846-0.89978214i$
6_2	$b^3=(ab)^3=aabAABaabAbaa-$	$y=cxc^{-1}$	$1+11z+27z^2+23z^3+8z^4+z^5$	$-2.8151578-0.7124188i$
	$BAAb$	$y^{-1}=cxc^{-1}$	$5+11z+18z^2+17z^3+7z^4+z^5$	$-0.1848422+0.7124188i$
6_3	$a^3=(ab)^3=aBabbaBAbABab-$	$y=cxc^{-1}$	$1+10z+13z^2+6z^3+z^4$	$-2.20710678-0.97831834i$
	$baBabABBAb$	$y^{-1}=cxc^{-1}$	$7+14z+13z^2+6z^3+z^4$	$-0.79289322+0.97831834i$
7_2	$a^3=(aB)^3=aBBBabbbABBA-$	$y=cxc^{-1}$	$4+19z+21z^2+8z^3+z^4$	$-3.21021099+0.41374653i$
	bbb	$y^{-1}=cxc^{-1}$	$1-z+3z^2+4z^3+z^4$	$0.21021099-0.41374653i$
7_2	$a^3=(ab)^3=aBAbaBAbaabAB-$	$y=cxc^{-1}$	$1+11z+13z^2+6z^3+z^4$	$-1.70657555-1.00144119i$
	$abABabAb$	$y^{-1}=cxc^{-1}$	$4+13z+13z^2+6z^3+z^4$	$-1.29342445+1.00144119i$
7_3	$a^3=(ab)^3=abbABBabbbaBB-$	$y=cxc^{-1}$	$1+16z+38z^2+29z^3+9z^4+z^5$	$-3.02268039+0.62320413i$
	$AbbaBB$	$y^{-1}=cxc^{-1}$	$2+4z+7z^2+11z^3+6z^4+z^5$	$0.02268039-0.62320413i$
7_3	$a^3=(ab)^3=aBBabbABBabbb-$	$y=cxc^{-1}$	$1+7z+11z^2+12z^3+6z^4+z^5$	$-0.3705294560+0.840164i$
	$aBBAbb$	$y^{-1}=cxc^{-1}$	$2+22z+43z^2+30z^3+9z^4+z^5$	$-2.62947054-0.840164i$
7_4	$a^3=(ab)^3=aBBAbbaBabbAB-$	$y=cxc^{-1}$	$11+16z+7z^2+z^3$	$-2.87743883-0.74486177i$
	$BabbABBAbb$	$y^{-1}=cxc^{-1}$	$1+z+2z^2+z^3$	$-0.12256117+0.74486177i$
7_7	$a^3=(ab)^3=aBAbABabbaBabA-$	$y=cxc^{-1}$	$13+31z+24z^2+8z^3+z^4$	$-2.24812606-1.03398206i$
	$BBAbABabbaBAbABBAbaBabb$	$y^{-1}=cxc^{-1}$	$1+5z+6z^2+4z^3+z^4$	$-0.75187394+1.03398206i$
8_1	$(ab)^3=b^3=aaaabAAABaaaB-$	$y=cxc^{-1}$	$5+32\ z+55z^2+36z^3+10z^4+z^5$	$-3.34814666+0.31569802i$
	$AAAb$	$y^{-1}=cxc^{-1}$	$1-z-z^2+6z^3+5z^4+z^5$	$0.34814666-0.31569802i$
8_1	$b^3=(ab)^3=abABabABaBAb-$	$y=cxc^{-1}$	$5+26z+38z^2+25z^3+8z^4+z^5$	$-1.649127617-1.0584827i$
	$aBAbaBAbaabAB$	$y^{-1}=cxc^{-1}$	$1+14z+25z^2+19z^3+7z^4+z^5$	$-1.350872383+1.0584827i$
9_{23}		$y=cxc^{-1}$	$1+10z+23z^2+19z^3+7z^4+z^5$	$-1.97873393-1.12211796i$
		$y^{-1}=cxc^{-1}$	$11+34z+40z^2+25z^3+8z^4+z^5$	$-1.02126607+1.12211796i$

$(-2.662358980+0.5622795i,-3,-4)$. Indeed, every polynomial in Table 3.1 is a factor of a word polynomial. By setting $y^{-1}=cxc^{-1}$, we have another word polynomial:

$$z(3+z)(1+2z+3z^2+z^3)^2(19+63\ z+97z^2+83z^3+40z^4+10z^5+z^6)$$

which maps $-0.33764102-0.5622795i$ to 0.

Using this method, the arithmeticity of the $\mathbb{Z}_2$-extensions of $(3,0)$ knot groups up to 9 crossings were checked and the results are in Table 3.1. The K in the first column of the Table 3.1 means knot. The relation for the $(3,0)$ group from the knot 9_{23} in Table 3.1 is:

$$a^3=(ab)^3=abABabbaBAbaBabABabbaBAbABabABBAbaBAbABabbaBAbaBabABabbaBAbaBab$$
$$ABBAbaBAbABabABBAbaB$$

Theorem 3.3.3. *Let $G=\langle f,g\rangle$ be a Kleinian group generated by elements of orders 2 and 3, respectively; then $par(G)=(\gamma,-3,-4)$. Table 3.1 contains a list of γ such that G is arithmetic and $\mathbb{H}^3/G$ is an orbifold which is 2-fold covering obtained from $(3,0)$ Dehn surgery on a two bridge knot. Table 3.2 contains a complete list of such arithmetic groups arising from knots of up to nine crossings.*

Tab. 3.2: The volumes of the arithmetic $\mathbb{Z}_2$-extension of $(3, 0)$ knot groups in Table 3.1

Knots	volume
5_2	0.1571178938
6_1	0.327122942
6_2	0.517468087
6_3	0.686161456
7_2	0.463638985
7_3	0.647167695
7_4	0.785589469
7_7	1.144930301
8_1	0.5671652085
9_{23}	1.893859885

The arithmeticity of the $\mathbb{Z}_2$-extensions of the $(3, 0)$ knot groups up to 9 crossings can also be derived from the list of polynomials in Chapter 6, where we will consider the arithmeticity of the $\mathbb{Z}_2$-extensions of the link groups up to 9 crossings.

3.4 The $(2, m)$ Dehn surgery data

Input a 2-bridge knot or link in SnapPea. There are four possibilities concerning the commutator parameter of the $(m, 0)$ knot group. We have no theoretical explanation for these possibilities.

- There are two commutator parameters for the group. For example, $-1.21507985450 - 1.30714127877i$ and $-2.66235897860 + 0.56227951210i$ for the $(3, 0)$ knot group from the knot 5_2. This case is the most prevalent;
- there are two conjugate parameters . For example, $-2.70710678118655 + 1.38355107i$ and $-2.70710678118655 - 1.38355107i$ for the knot 6_3;
- there is just one commutator parameter, such as $-2.3577351476 + 1.507636189i$ for the knot 6_1;
- Snappea fails to find the commutator parameter (it certainly exists), such as in the case of the knot $12a_{247}$

The $(m, 0)$ Dehn surgery was done on 2-bridge knots up to 12 crossings (links up to 9 crossings) for $m = 3, 4, 5, 6, 7$. Using Lemma 3.2, we have obtained all commutator parameters of $\mathbb{Z}_2$-extensions of the $(m, 0)$ knot groups. We call these commutator parameters the $(2, m)$ Dehn surgery data , which means the commutator parameters of the $\mathbb{Z}_2$-extensions of $(m, 0)$ knot groups. Figures 3.4 to 3.8 are the plots of the $(2, m)$ Dehn surgery data on the $(2, m)$ commutator plane for $m = 3, 4, 5, 6, 7$; Figure 3.9 is a combination of these figures.

There are two obvious phenomena in Figures 3.4 to 3.8:

(A): there are sequences of commutators converging;

(B): the pictures are fractal. That is, wherever you look at a picture, you can see sequences of commutators converging;

(C): the pictures are symmetrical with respect to the x-axis and $\beta(f)/2$, where f is elliptic of order m.

Remark.

(A) is a consequence of Thurston's Dehn surgery Theorem, and connects the algebraic and geometric limits of Kleinian groups as discussed in Section 4.4;

(B) is a conjectural observation;

(C) is proved in [16] where it is shown that if $\mathrm{par}(G) = (\gamma, \beta, -4)$ is a discrete group, then so are groups with parameters $(\beta-\gamma, \beta, -4)$, $(\bar{\gamma}, \bar{\beta}, -4)$, $(\bar{\beta}-\bar{\gamma}, \bar{\beta}, -4)$. When $\beta = -4\sin^2(\pi/p), \bar{\beta} = \beta$ and we see the four symmetric points: complex conjugate and reflection across the line $\mathrm{Re}(\gamma) = \beta/2$.

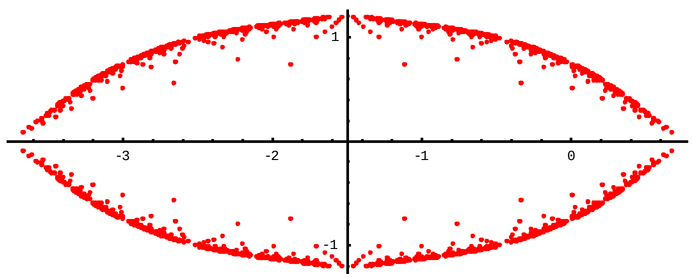


Fig. 3.4: The $(2,3)$ commutator plane
(each point corresponds to a discrete group resulting from $(3,0)$ Dehn surgery on a 2-bridge knot or link)

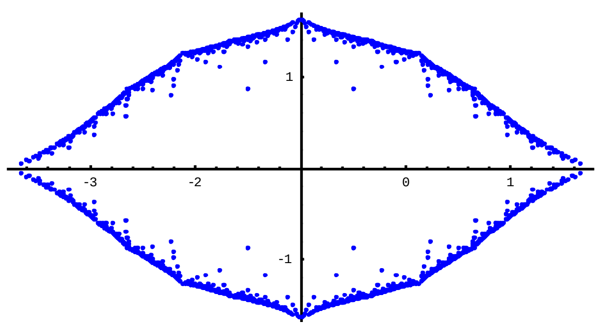


Fig. 3.5: The $(2,4)$ commutator plane
(each point corresponds to a discrete group resulting from $(4,0)$ Dehn surgery on a 2-bridge knot or link)

Ten of the sequences in the $(2,3)$ Dehn surgery data have been investigated. These ten sequences (in terms of their corresponding knots) are in Table 3.3. In subsequent discussions, A knot sequence will refer to a sequence of commutators in the $(2,3)$ Dehn surgery data.

As we see in Table 3.3, the knots and links in a knot sequence are connected by their Conway notation. That is one of the entries of Conway's notation increases from one knot to the next (for

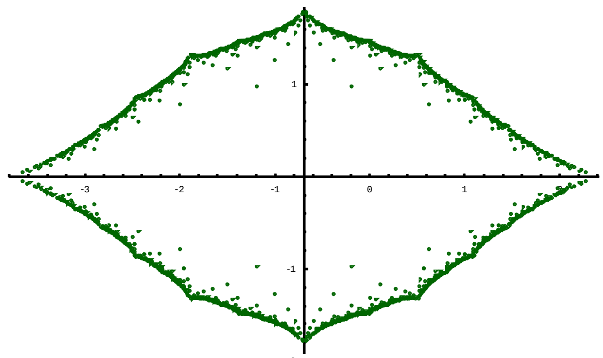


Fig. 3.6: The $(2, 5)$ commutator plane
(each point corresponds to a discrete group resulting from $(5, 0)$ Dehn surgery on a 2-bridge knot or link)

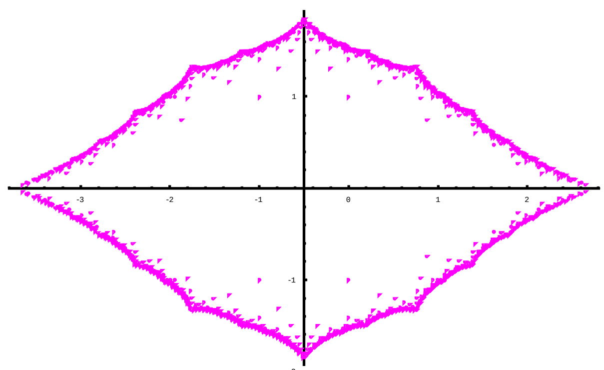


Fig. 3.7: The $(2, 6)$ commutator plane
(each point corresponds to a discrete group resulting from $(6, 0)$ Dehn surgery on a 2-bridge knot or link)

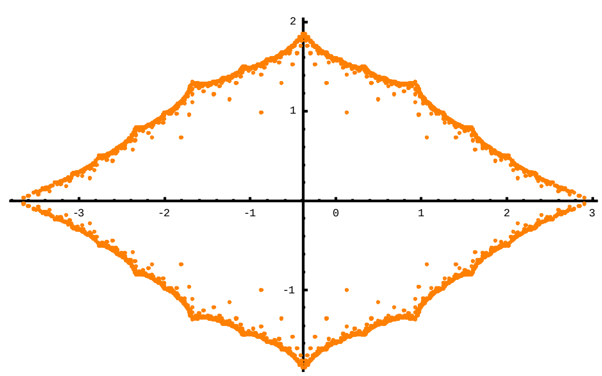


Fig. 3.8: The $(2, 7)$ commutator plane
(each point corresponds to a discrete group resulting from $(7, 0)$ Dehn surgery on a 2-bridge knot or link)

simplicity, we omit the square brackets in Conway's notation in the table). Since a knot diagram can be obtained from its Conway's notation as in Figure 3.3, this connection implies that every knot or link comes from adding one more half twist to the previous knot. This pattern is a consequence

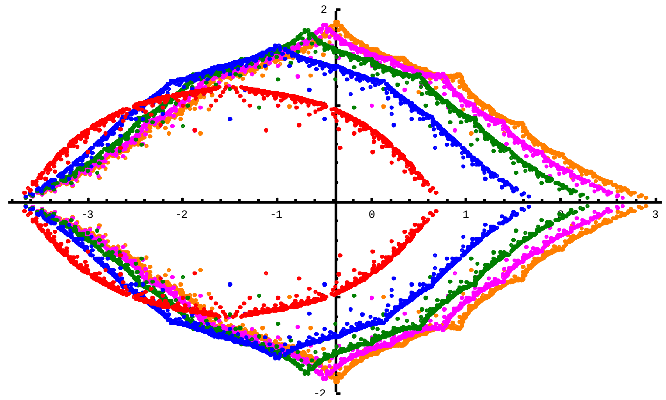


Fig. 3.9: The $(2, m)$ commutator planes, $m = 2, 3, 4, 5, 6, 7$

Tab. 3.3: Knot and link sequences

	First few terms	$4_1 \to 5_2 \to 6_1 \to 7_2 \to 8_1 \to 9_2 \to 10_1 \to 11_{247} \to 12_{803}$
Sequence 1	Conway's Notations	$22 \to 32 \to 42 \to 52 \to 62 \to 72 \to 82 \to 92 \to 102$
	First few terms	$6_2 \to 7_5 \to 8_6 \to 9_7 \to 10_20 \to 11_{246} \to 12_{802}$
Sequence 2	Conway's Notations	$312 \to 322 \to 332 \to 342 \to 352 \to 362 \to 372$
	First few terms	$5^2_1 \to 6_2 \to 7^2_1 \to 8_2 \to 9^2_1 \to 10_2$
Sequence 3	Conway's Notations	$212 \to 312 \to 412 \to 512 \to 612 \to 712$
	First few terms	$6_3 \to 7^2_2 \to 8_7 \to 9^2_2 \to 10_5$
Sequence 4	Conway's Notations	$2112 \to 3112 \to 4112 \to 5112 \to 6112$
	First few terms	$7_4 \to 8_4 \to 9_5 \to 10_4 \to 11_{343} \to 12_{1149}$
Sequence 5	Conway's Notations	$313 \to 413 \to 513 \to 613 \to 713 \to 813$
	First few terms	$7_3 \to 8_3 \to 9_4 \to 10_3 \to 11_{342} \to 12_{1166}$
Sequence 6	Conway's Notations	$43 \to 44 \to 54 \to 64 \to 74 \to 84$
	First few terms	$6^2_3 \to 7_5 \to 8^2_3 \to 9_6$
Sequence 7	Conway's Notations	$222 \to 322 \to 422 \to 522$
	First few terms	$6_3 \to 7_6 \to 8_8 \to 9_8 \to 10_{34} \to 11_{59} \to 12_{169}$
Sequence 8	Conway Notations	$2212 \to 2312 \to 2412 \to 2512 \to 2612 \to 2712$
	First few terms	$6^2_2 \to 7_3 \to 8^2_2 \to 9_3$
Sequence 9	Conway's Notations	$33 \to 43 \to 53 \to 63$
	First few terms	$5^2_1 \to 6^2_3 \to 7^2_3 \to 8^2_6 \to 9^2_{10}$
Sequence 10	Conway's Notations	$212 \to 222 \to 232 \to 242 \to 252$

of Thurston's hyperbolic Dehn surgery theorem. See Section 4.4 for a detailed discussion. Knots or links with crossing numbers greater than 13 and less than 126 have been added to the knot sequences from 1 to 7 and $(3, 0)$ Dehn Surgery has been done on them. SnapPea could not compute the identity word $w(a, b)$ correctly for the knots with more than 126 crossings when the data was collected.

In general, SnapPea works well for the knots or links with less than 40 crossing with some exceptions, such as the two bridge knot $12a_{247}$. But it is not stable for a knot with more than 40 crossings. Since the commutator parameters in a knot sequence do not change much after 40 crossings, only one or two knots were chosen for every 10 knots and links. For example, for knots and links with crossing numbers from 40 to 49 in the sequence 7, $(3, 0)$ Dehn Surgery was done on the link with 40 crossings and the knot with 41 crossings. There was no particular rule for choosing these knots: we just chose one that SnapPea works on. The sequences in the $(2, 3)$ commutator plane are shown in Figures 3.10 to 3.13. The right picture in Figure 3.13 is from $(2, 3, p)$ triangle groups, where p is an integer and $p \geq 7$. It is a spherical triangle group if $p = 3, 4, 5$ and an euclidean triangle group when $p = 6$.

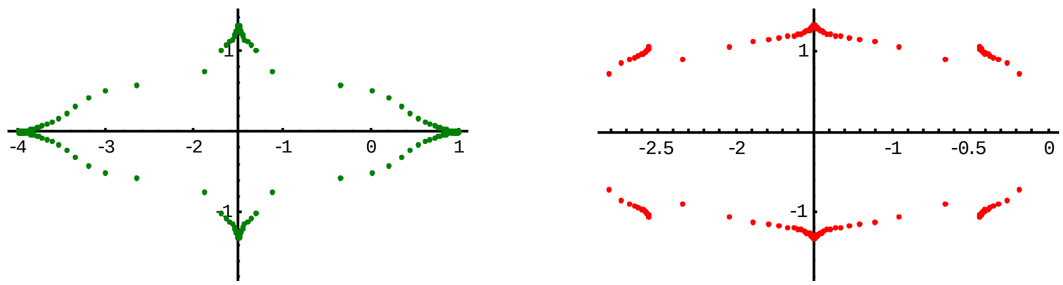


Fig. 3.10: Sequence 1 and 2

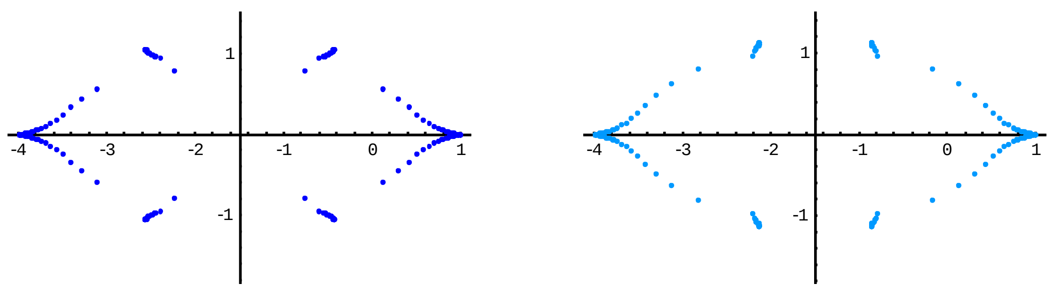


Fig. 3.11: Sequence 3 and 4

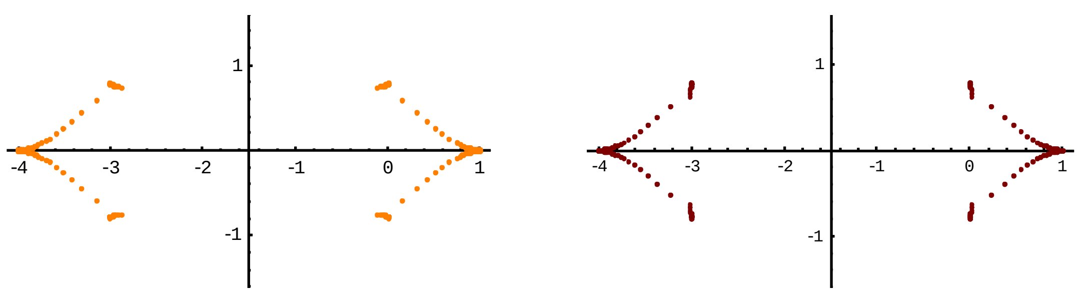


Fig. 3.12: Sequence 5 and 6

We now explain the second picture of Figure 3.13. Suppose f in $\mathcal{M}$ is non-parabolic. Then the axis of f is the hyperbolic line in $\mathbb{H}^3$ which has the fixed points of f as its endpoints.

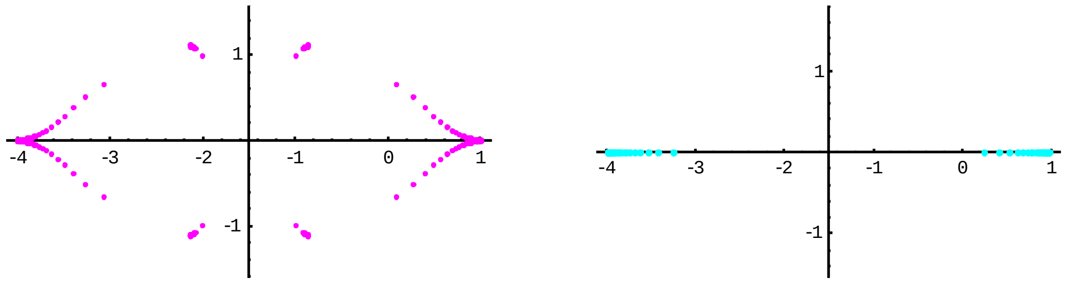


Fig. 3.13: Sequence 7 and 8

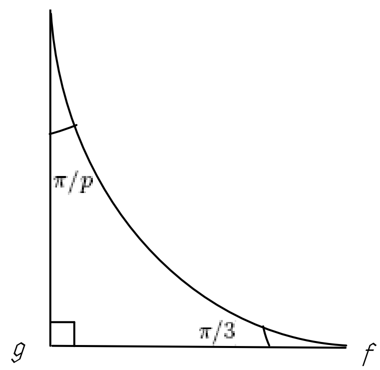


Fig. 3.14: A $(2, 3, p)$ hyperbolic triangle

Lemma 3.4.1. *[16, Lemma 4.2] Suppose that $f, g \in \mathcal{M}$ have disjoint pairs of fixed points and that α is the geodesic in H^3 which is orthogonal to the axes of f and g. Then*

$$\sinh^2(\delta + i\theta) = \frac{4\gamma(f,g)}{\beta(f)\beta(g)} \tag{3.4}$$

where $\delta = \delta(f,g)$ and $\theta = \theta(f,g)$ is the angle between the spheres or hyperplanes which contain $axis(f) \cup \alpha$ and $axis(g) \cup \alpha$ respectively.

Figure 3.14 is a $(2, 3, p)$ hyperbolic triangle. Since $axis(f)$ and $axis(g)$ are parallel in this case, so $\theta = 0$. Substituting $\beta(g) = -4$ and $\beta(f) = -3$, the equation (3.4) becomes:

$$\sinh^2(\delta) = \frac{\gamma(f,g)}{3} \tag{3.5}$$

Since $\sinh^2(\delta) = \cosh^2(\delta) - 1$ and $\cosh(\delta) = \cos(\pi/p)/\sin(\pi/3)$(Theorem 7.11.3 of [3]), we have

$$\gamma(f,g) = 4\cos^2(\pi/p) - 3. \tag{3.6}$$

Substituting $7 \le p \le 60$ in (3.6), we have the second picture of Figure 3.13.

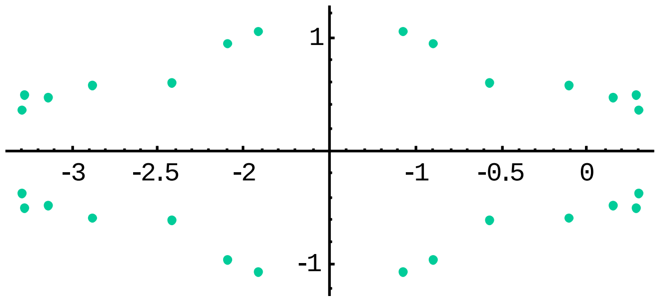


Fig. 3.15: The points in [2]

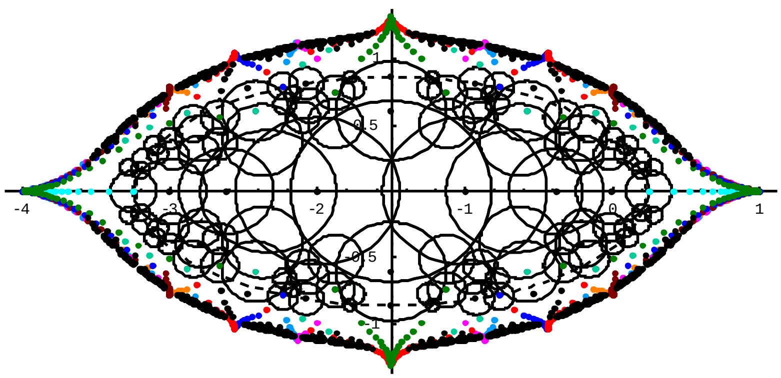


Fig. 3.16: The $(2, 3)$ commutator plane

Figure 3.15 contains the $(2, 3)$ commutator parameters for the Kleinian groups from [2], page 311. Using a disk covering procedure([16]), Gehring and Martin have proved that there are no other points except the isolated points in the disks in Figure 1.1 corresponding to discrete groups. Figure 3.16 is a combination of the results in the $(2, 3)$ commutator plane we have discussed so far. The black points in Figure 3.16 are from $(3, 0)$ Dehn surgery, which are not in the sequence from 1 to 7.

Figure 3.16 is quite an amazing picture. Figure 1.1 sits perfectly inside the region bounded by $(2, 3)$ Dehn surgery data and covers most of the region. We observe from Figure 3.16 that if we keep on using the disk covering procedure to study the region $|\gamma| \leq \gamma_0$, soon the disks would reach the region covered by infinitely many points of the $(2, 3)$ Dehn surgery data. So a new approach is needed.

We observe from Figure 3.9 that as $p \to \infty$, the $(2, m)$ Dehn surgery data approaches the Riley slice (see Figure 4.2), which is the parameter space of the groups generated by two parabolic elements. We have the following conjecture.

Conjecture 3.4.2. *The rational words and rational pleating rays, which were used by Keen and Series in [35] to investigate the boundary of the Riley slice, may be generalised to study the parameter spaces of the groups with two elliptic generators.*

We discuss this in the next Chapter.

4. SIMPLE CLOSED CURVES, RATIONAL WORDS AND 2-BRIDGE KNOTS AND LINKS

Keen and Series have proved in [35] that the endpoint of a rational pleating ray (page 44) lies in the boundary of the Riley slice (see Figure 4.2). The $(2, m)$ Dehn surgery data discussed in the last chapter leads us to the conjecture that the rational pleating rays also apply to the parameter spaces of the groups with two elliptic generators. We discuss in this chapter the rational pleating rays and the numerical investigation of the conjecture. In the final section, we discuss the boundary of a $(2, m)$ commutator plane through the limit link of a sequence of knots and links, as discussed in the last chapter. We reveal a beautiful conjectural connection between these limit links with the rational pleating rays.

4.1 Some basics

The theorems and definitions discussed in this section belong to three categories. The first one is about covering spaces of a surface and the relationship between the groups of transformations acting on these covering spaces; the second is a brief discussion of the Farey diagram. These two categories lay the foundation for the discussion in Section 4.2 and Section 4.3. The last category is about algebraic and geometric convergence of a sequence of Kleinian groups, which lays the foundation for Section 4.4.

4.1.1 Covering spaces of a surface

Let $p : \tilde{X} \to X$ be a covering map; let $p(\tilde{x}_0) = x_0, \tilde{x}_0 \in \tilde{X}, x_0 \in X$. Then $\pi_1(\tilde{X}, \tilde{x}_0)$ is isomorphic to a subgroup N of $\pi_1(X, x_0)$ [26, Theorem 54.6]. N is called the *defining subgroup* of the covering map p. If N is normal, p is called a *regular* covering map.

Theorem 4.1.1. *[26, Theorem 81.5] Let X be path connected and locally path connected; let G be a group of homeomorphism of X. The quotient map $\pi : X \to X/G$ is a covering map if and only if the action of G is properly discontinuous. In this case, the covering map π is regular.*

Theorem 4.1.2. *[5, page 43, Theorem B.6] Let $p : \tilde{X} \to X$ be a regular covering with defining subgroup N. Then there is a group G of homeomorphisms of $\tilde{X}$ onto itself, where ${}^{\circ}\Omega(G) = \tilde{X}$,* *and* $X = \tilde{X}/G$. *Further, if we choose a base point* $\tilde{x}_0$ *in* $\tilde{X}$, *and* $x_0 = p(\tilde{x}_0)$ *on* X, *then there is a canonical isomorphism* $\Phi : \pi_1(X, x_0)/N \to G$.

It is a direct consequence of Theorem 4.1.2 that if $\tilde{X}$ is simply connected, then $\pi_1(X, x_0)$ is isomorphic to G and G is called the group of the deck transformations. Recall that $\Omega(G)$ is the ordinary set of G (see Chapter 2) and $^\circ\Omega(G)$ is the subset of $\Omega(G)$ on which the action of G is properly discontinous.

Let $p : \tilde{X}_1 \to X$ and $q : \tilde{X}_2 \to X$ be regular coverings of the same base X, with defining subgroups N and M respectively, where $N \subset M$. Then there is a regular covering $r : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ with $q \circ r = p$ [5, page 48].

Theorem 4.1.3. *[5, page 48, Theorem E.2] Let* G *be the group of deck transformations on* $\tilde{X}_1$ *as a covering of* X *and* J *be the group of deck transformations on* $\tilde{X}_1$ *as a covering of* $\tilde{X}_2$, *Let* H *be the group of deck transformations on* $\tilde{X}_2$ *as a covering of* X. *There is a homomorphism* $\Psi : G \to H$, *with Kernel* J.

J is a subgroup of G [5, page 48] and is isomorphic to $\pi(\tilde{X}_2)$ if $\tilde{X}_1$ is simply connected (Theorem 4.1.2). A regular covering space $\tilde{X}$ has a group G acting properly discontinuously in it; if G acts discontinuously, but not properly, then we get a *branched* regular covering.

Suppose G is a group of homomorphisms acting discontinuously, but not properly, on the space $\tilde{X}$. Set $X = \tilde{X}/G$, and let $p : \tilde{X} \to X$ be the natural projection. Let $^\circ\tilde{X}$ be the subset of $\tilde{X}$ at which G acts properly discontinuously, $\{\tilde{x}_m\} = \tilde{X} \setminus {}^\circ\tilde{X}$ where $m \in \mathbb{N}$. Every point of $\{\tilde{x}_m\}$ is the fixed point of an elliptic element of G and $\{\tilde{x}_m\}$ is discrete in $\tilde{X}$ [5, page 25]. The points $\{x_m\}$, where $x_m = p(\tilde{x_m})$, are called the *special* or ramification points.

Theorem 4.1.4. *[5, page 49] The map* $p : \tilde{X} \to X$ *is a branched regular covering with branch points* $\{\tilde{x}_m\}$, *special or ramification points* $\{x_m = p(\tilde{x}_m)\}$, *covering space* $\tilde{X}$, *base* X *and projection* p, *if the following hold.*

- $p :\ {}^\circ\tilde{X} \to {}^\circ X$ *is a regular covering with deck group* G;
- *for every branch point* $\tilde{x}_m$, *the stabilizer of* $\tilde{x}_m$, $G_{\tilde{x}_m}$ *is finite; and there is a neighborhood of* $\tilde{x}_m$, *called the nice neighborhood of* $\tilde{x}_m$, *which is precisely invariant under Stab*$(\tilde{x}_m)$ *in* G.

4.1.2 Farey diagram

Let m/n and r/s be two rational numbers such that m, n (resp. r, s) are coprime. Define an operation $\oplus$ such that $m/n \oplus r/s = (m+r)/(n+s)$; we call $\oplus$ Farey addition. Suppose $m/n < r/s$. Then $m/n < (m+r)/(n+s) < r/s$. A Farey diagram is constructed as follows [32, page 1-4].

- Draw a horizontal axis with vertical lines at distances 1 unit apart. Join the neighboring points $\ldots, -1, 0, 1, \ldots$ by semi-circles. Notice that 0 is written as $0/1$, 1 as $1/1$, ∞ as $1/0$, etc.

- Take any two neighboring points in the last step and combine them by the operation $\oplus$. Join these new points up by semi-circles to their neighboring points.

- Again, combine each pair of neighbors using $\oplus$ and join the new points up by semi-circles with their neighbors.

The Farey diagram is obtained by continuing this process forever. See Figure 4.1[29, page 6]. If m/n and r/s are neighbors in the Farey diagram, then $ms - rn = \pm 1$.

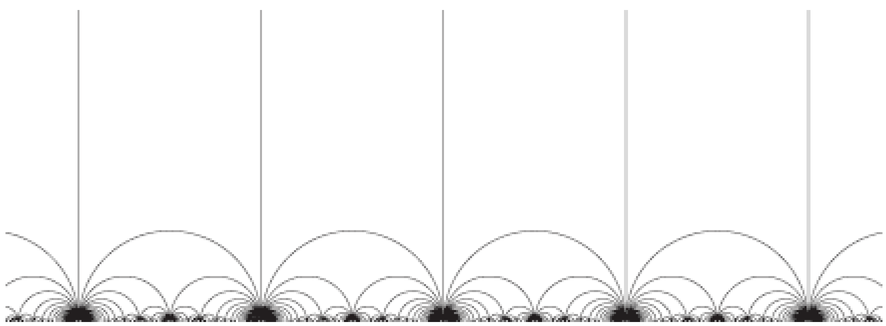


Fig. 4.1: Farey diagram

4.1.3 Algebraic and geometric convergence

Definition 4.1.5. *[56, Chapter 9, page 225] A sequence $\{G_i\}$ of discrete subgroups of a group $\mathcal{M}$ of linear transformations* ***converges geometrically*** *to a group G if*

- *each $g \in G$ is the limit of a sequence $\{g_i\}$, with $g_i \in G_i$, and*

- *the limit of every convergent sequence $\left\{g_{i_j}\right\}$, with $g_{i_j} \in G_{i_j}$, is in G.*

Definition 4.1.6. *[43, page 52] The sequence of discrete groups $\{G_i\}$* ***converges polyhedrally*** *to the group G if G is discrete and for some point $O \in H^3$, the sequence of fundamental polyhedra $\{D_i\}$ of $\{G_i\}$ centered at O converges uniformly to the fundamental polyhedron of G centered at O.*

Proposition 4.1.7. *[43, page 56] The sequence $\{G_i\}$ of Kleinian groups* ***converges geometrically*** *to a Kleinian group G if and only if $\{G_i\}$* ***converges polyhedrally*** *to G. The geometric and polyhedral limits of $\{G_i\}$ are the same.*

Definition 4.1.8. *[56, Chapter 9, page 225] Let H be an abstract group, and $\phi_i : H \to \mathcal{M}$ be a sequence of representations of H into $\mathcal{M}$. The sequence $\{\phi_i\}$* ***converges algebraically*** *if for every $h \in H, \{\phi_i(h)\}$ converges. The limit $\phi : H \to \mathcal{M}$ is called the algebraic limit of $\{\phi\}$.*

Proposition 4.1.9. *[56, Chapter 9, page 228] Let H be an abstract group. If $\{\phi_i(H)\}$* ***converges algebraically*** *to $\phi(H)$ and $\{\phi_i(H)\}$* ***converges geometrically*** *to H', then $\phi(H) \subset H'$.*

Let G be a Kleinian group, then G acts discontinuously on H^3 [3, Theorem 5.3.2]. The quotient H^3/G is a hyperbolic 3-manifold M if G is torsion free, otherwise it is a hyperbolic 3-orbifold $\mathcal{O}$. Let $X \in H^3$ be the set of fixed points of elliptic elements of G, then X/G in $\mathcal{O}$ is called the singular locus $\Lambda_{\mathcal{O}}$. For example, when we do $(m, 0)$ Dehn surgery on a knot complement, the knot itself becomes the singular locus.

4.2 Simple closed curves on a four times punctured sphere

Let $S = \Delta/F$ be a surface, where F is a finitely generated Fuchsian group and Δ is the open unit disc. A Fuchsian group is a discrete subgroup of $\mathcal{M}$ with an invariant disc. The surface S inherits the hyperbolic metric in Δ. Let $\pi_1(S)$ be the fundamental group with base point z_0 and let z be a lift of z_0 in Δ. Assume λ is a closed geodesic on S that is the projection of λ' in Δ, where λ' is a geodesic segment joining z and one of its images, say $f(z)$. We say that f is a word in F representing λ. An expression of f in terms of the generators of F can be read off the cutting sequence of λ' in Δ. See [22], page 451-455 for a detailed discussion on this. In this section, we first explain how to get an expression for a representive word of a simple closed curve on a four times punctured sphere. We then find an expression for a representive word of a simple closed curve on a sphere with four cones.

Up to conjugation, a pair of parabolic elements of $\mathrm{SL}(2, \mathbb{C})$ with different fixed points may be written as:

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad Y = \begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix} \tag{4.1}$$

with $\rho \in \mathbb{C}$. Let $G_\rho = \langle X, Y \rangle$; if G_ρ is a free Kleinian group of the second kind, which means the ordinary set $\Omega(G_\rho)$ of G_ρ is not empty, then the quotient $\Omega(G_\rho)/G_\rho$ is either a four times punctured sphere with the punctures identified in pairs or a pair of triply punctured spheres ([6]).

Definition 4.2.1. *The Riley slice is:*

$$\mathscr{R} = \{\rho \in \mathbb{C} : \Omega(G_\rho)/G_\rho \text{ is a four times punctured sphere}\}.$$

The *Riley slice* $\mathscr{R}$ is a connected open subset in $\mathbb{C}$ and topologically, it is an annulus [35, page 72]. See Figure 4.2 (the lines in Figure 4.2 will be discussed in Section 4.3). If $\Omega(G_\rho)/G_\rho$ a pair of triply punctured spheres, then ρ lies on the boundary of $\mathscr{R}$ [35, page 75].

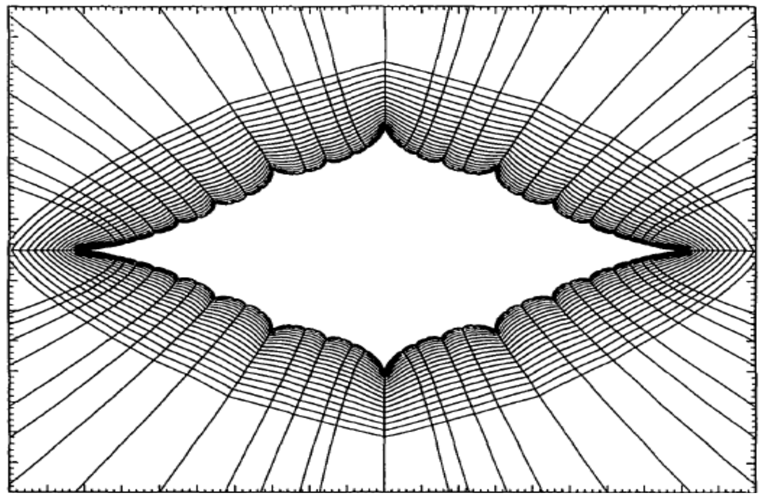


Fig. 4.2: The Riley slice
(rays are rational pleating rays, starting from infinity and terminating at a point of the boundary of $\mathscr{R}$)

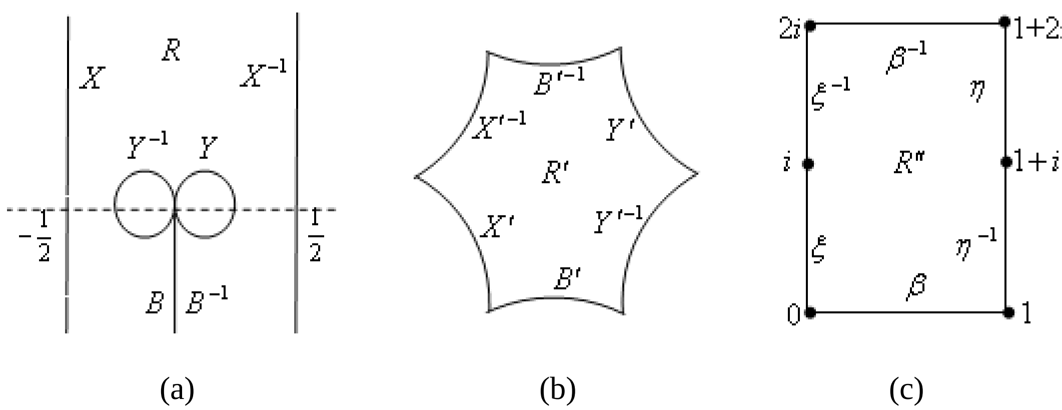


Fig. 4.3: Three fundamental domains

(a) in Figure 4.3 is a fundamental domain R for G_ρ with $\rho < -4$, whose sides are paired by the generators X and Y as indicated on page 77 of [35]. The label $W, W \in \{X^\pm, Y^\pm\}$ means the side W is mapped to W^{-1} by W. There is a labeling error in Fig.2 of [35], page 77, which is corrected in [57], page 307. Figure 4.3 is a corrected version of Fig.2.

The quotient Ω/G_ρ is a four times punctured sphere S°. Cutting open the fundamental domain R along the negative imaginary axis, we obtain a simply connected hexagonal region that lifts to a fundamental domain R' in the universal covering space Δ of S°. See (b) of Figure 4.3. Since the vertices of R' project to punctures on S°, they lie on $\partial\Delta$. Denote by G' the group generated by the side pairings X', Y', B'. The labeling convention here is the same as for (a) of Figure 4.3. This labeled fundamental domain R' and its images under G' tessellate the hyperbolic plane Δ.

Let λ' be an oriented geodesic segment from z to $h'(z)$, where $z, h'(z) \in \Delta$ and $h' \in G'$. Then λ' cuts through a well-defined sequence of adjacent $R', g'_1R', g'_2R', \dots g'_kR'$ where $g'_i \in G', i = 1, 2 \dots k, z \in R', h'(z) \in g'_kR'$. Let $\alpha_1 \dots \alpha_k$ be the ordered sequence of labels of the edges of g'_iR' cut by λ' and let α_i be the label of the common side of g'_iR' and $g'_{i+1}R'$ which is inside $g'_{i+1}R'$. The sequence $\alpha_1 \dots \alpha_k$ is called the G'-*cutting sequence* of λ' and is a word in the generators of G' representing h'. If λ' passes through a vertex of R', we replace λ' by another geodesic segment λ'' parallel to it and the resulting cutting sequence by λ'' is a word conjugate to h [57, page 310]. Since we are interested in the trace of a word and two conjugate words have the same trace, we say the cutting sequence given by λ'' represents h as well. Note any permutation of the sequence $\alpha_1 \dots \alpha_k$ is also a conjugate of h. Therefore a presentation of h doesn't depend on where a geodesic is in the plane and where on the geodesic we start to read the cutting sequence, as long as we fix the orientation of the geodesic. In the case that the orientation reverses, we have the representation for the inverse of h.

Let $\Gamma = \langle \xi, \eta, \beta \rangle$, where $\beta = z + 2i$ and ξ, η are the rotations through an angle of π about i and $1 + i$ respectively. See (c) of Figure 4.3. Let L be the lattice in the plane given by $L = \{z = m + ni : m, n \in \mathbb{Z}\}$. Then $(\mathbb{C} - L)/\Gamma$ is also a four times punctured sphere S°. A loop on S° is called *admissible* if it is not homotopically equivalent to a loop around the punctures. Every homotopy class of simple closed admissible loops on S° is the projection of a line of rational slope in $\mathbb{C} - L$ [35, Proposition 2.1]. A more detailed proof of the Proposition 2.1 can be found on page 307-308 of [57]). A fundamental domain R'' for Γ is shown in (c) of Figure 4.3 and this labeled fundamental domain and its images tessellate $\mathbb{C} - L$.

Let λ be any simple closed admissible loop on S°. Its lift to any of the three covering spaces Ω, Δ or $\mathbb{C} - L$ of S° is simple. Therefore it appears on each of the fundamental domains R, R', R'' as a collection of pairwise disjoint arcs with endpoints on the labeled sides. There is a homeomorphism of the interiors of R' and R'' that respects the identifications of the sides and takes punctures to punctures [35, page 78]. Hence any simple closed curve can be equally well represented on either R' or R''.

Let $\lambda_{p/q}$ be a simple closed curve on S° that corresponds to the projection of a rational line of slope p/q in $\mathbb{C} - L$. Denote by $W_{p/q}$ the element in the fundamental group of S° representing $\lambda_{p/q}$; called a rational word. Since any simple closed curve on S° can be equally well represented on either R' or R'', the representation of $W_{p/q}$ in the generators of G' can be read off the Γ-cutting sequence of the lift of $\lambda_{p/q}$ in $\mathbb{C} - L$ by replacing the labels $\beta^{\pm 1}$ with $B'^{\pm 1}$, $\xi^{\pm 1}$ with $X'^{\pm 1}$ and $\eta^{\pm}$ with $Y'^{\pm}$. The Γ-cutting sequence of a rational line is defined in the same manner as the G'-cutting sequence of λ'.

According to Theorem 4.1.3, the kernel of the homomorphism h from G' to G_ρ is $\pi_1(\Omega(G_\rho))$. Suppose l is the loop on S° that is the projection of the geodesic arc l' joining B' to B'^{-1}. Then the lifting of l in Ω is again a loop (this is easy to see from the fact that B' and B'^{-1} are obtained by cutting R along the edge $B = B^{-1}$). The word in G' corresponding to l' is B' or B'^{-1} depending the orientation of l'. It follows that $h(B') = identity$. Therefore to get a representation for a rational word $W_{p/q}$ in G_ρ, we omit $B'^{\pm 1}$ and replace $X'^{\pm 1}$ with $X^{\pm 1}$ and $Y'^{\pm 1}$ with $Y^{\pm 1}$ in the representation of $W_{p/q}$ in G' [57, page 309]. We may obtain a representation for a rational word $W_{p/q}$ in G_ρ directly from the Γ-cutting sequences by omitting $\beta^{\pm 1}$ and replacing ξ with X and η with Y.

We now make a connection between what we have discussed above and the two elliptic generator groups. Up to conjugation, a pair of elliptic elements of $SL(2, \mathbb{C})$ with distinct fixed points may always be written in the form:

$$a = \begin{pmatrix} e^{i\pi/n} & 1 \\ 0 & e^{-i\pi/n} \end{pmatrix} \qquad b = \begin{pmatrix} e^{i\pi/m} & 0 \\ \mu & e^{-i\pi/m} \end{pmatrix} \tag{4.2}$$

with $\mu \in \mathbb{C}, n, m \in \mathbb{N}$, and n, m are the orders of a and b, respectively. We write $G_\mu = \langle a, b \rangle$. Notice that as $n, m \to \infty$, traces of a, b converge to ± 2 and a, b become parabolic at the limit. Therefore, we follow the often used convention of considering a parabolic element as an elliptic element with infinite order. Hence we expect that some of the results and techniques in the studying of Kleinian groups with two parabolic generators may also apply to the groups with two elliptic generators. We discuss this in the rest of this section and Section 4.3.

We are interested in the set of μ such that G_μ is a Kleinian group formed from the free product of the generators a and b and is of the second kind. Notice that (4.3) is a group presentation for G_μ

$$G_\mu =< a, b : a^n = b^m = Identity > . \tag{4.3}$$

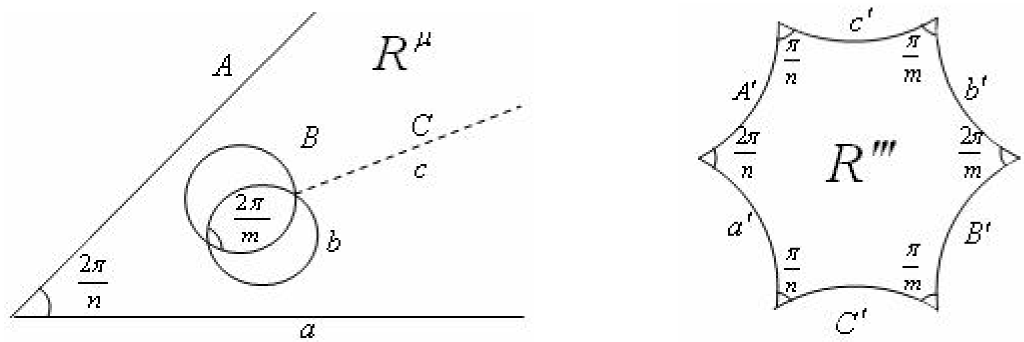
Suppose G_μ is a group such that the region R^μ (see Figure 4.4) bounded by the sides a and A and outside the isometric circles of b and B is a fundamental domain R^μ for G_μ; where A (resp. B) is the inverse of a (resp. b). Then G_μ is the free product of a, b and acts discontinuously on $\Omega(G_\mu) = \bigcup_g (R^\mu)$ where $g \in G_\mu$ [3, Theorem 5.3.15]. Since discontinuousness implies discreteness [3, page 95], it follows that G_μ is discrete. Note that G_μ is Kleinian group of the second kind and the quotient $\Omega(G_\mu)/G_\mu$ is a sphere with four cones and each cone point corresponds to the projection of an elliptic fixed point. Let

$$\mathscr{U} = \{\mu \in \mathbb{C} : \Omega(G_\mu)/G_\mu \text{ is a sphere with four cones}\} . \tag{4.4}$$

Every $\mu \in \mathscr{U}$ corresponds to a free Kleinian group of the second kind (see [30], [31]). We expect that the arguments on the Riley slice $\mathscr{R}$ on page 72 of [35] also apply to the set $\mathscr{U}$.

Theorem 4.2.2. *The set $\mathscr{U}$ is topologically equivalent to the Riley slice $\mathscr{R}$, that is $\mathscr{U}$ is an annulus in $\mathbb{C}$.*

This result will be presented else where (Martin, personal communication). We investigate this result numerically in the next section.



(1) (2)

Fig. 4.4: A fundamental domain of G_μ

Cutting the fundamental domain R^μ open along the line joining ∞ and one of the fixed points of b (see (1) of Figure 4.4), we obtain a simply connected hexagonal region that lifts (this means by a homeomorphism) to a fundamental domain R''' (see (2) of Figure 4.4) in the universal branched cover Δ of Ω/G_μ. Since the vertices of R''' project to the singular points on Ω/G_μ, they lie inside Δ. Note the fundamental domain R^μ of G_μ is not simply connected, there is a non-trivial loop around the isometric circles of b. This loop lifts to an arc joining sides c' and C' in R'''. Denote by F the group generated by the side pairings a', b', c'.

Let $\{x_m\} \subset \Delta$ and $\{y_m\} \subset \Omega(G_\mu)$ be the sets of fixed points of elliptic elements of F and G_μ, respectively. Let $^\circ\Delta = \Delta \setminus \{x_m\}$ and $^\circ\Omega(G_\mu) = \Omega(G_\mu) \setminus \{y_m\}$. Then F (or G_μ) acts properly discontinuously on $^\circ\Delta$ (or $^\circ\Omega(G_\mu)$). The quotient space $^\circ\Delta/F$ (or $^\circ\Omega(G_\mu)/G_\mu$) is a sphere with four singular points removed; denote by S^* such a sphere and S^* is topologically equivalent to a four times punctured sphere S°. Note that a fundamental domain for F in $^\circ\Delta$ is the fundamental domain R''' with its vertices removed. Let $\lambda_{p/q}$ be a simple closed curve on S^* corresponding to the projection of the rational line p/q in $\mathbb{C} \setminus L$. Denote the word representing $\lambda_{p/q}$ by $W_{p/q}$. Following the arguments on page 38, a representation of the word $W_{p/q}$ in F can be read off the Γ-cutting sequence of the rational line p/q by replacing ξ by a', η by b' and β by c'.

Let $p : {}^{\circ}\Delta \to {}^{\circ}\Delta/F$ and $q : {}^{\circ}\Omega(G_\mu) \to {}^{\circ}\Omega(G_\mu)/G_\mu$. They both are regular coverings (Theorem 4.1.1) with defining subgroups N and M, respectively. Since $N \subset M$ [5, page 49-50], there is a homomorphism h^* from F to G_μ and the kernel of the homomorphism is $\pi(\Omega/G_\mu)/N$ (Theorem 4.1.3 and Theorem 4.1.2).

Suppose l is a loop on S^* that is the projection of a geodesic arc l' joining C' to c'. Then the lift of l in $\Omega(G_\mu)$ is again a loop. The word in F corresponding to l is C' or c' depending on the orientation of l. It follows that $h^*(C) = identity$. Therefore to get a representation for a rational word $W_{p/q}$ in G, we omit c', C' and replace a' with a and b' with b in the representation of $W_{p/q}$ in F. Observe that we can get a representation for a rational word $W_{p/q}$ in G_μ directly from the Γ cutting sequence by omitting $\beta^\pm$ and replacing ξ with a and η with b. Therefore Lemma 4.2.3 [57, page 310-311], concerning rational words in G_ρ, also holds for G_μ.

Lemma 4.2.3. *$W_{p/q} \cong W_{r/s}$ if and only if $r/s = p/q + 2n$ or $r/s = -(p/q + 2n)$.*

According to Lemma 4.2.3, every rational word is conjugate to one of the words $W_{p/q}$, where $0 \le p/q \le 1$. The cutting sequence of a rational line is obviously periodic and a rational word $W_{p/q}$ has length $2q$ [35, Proposition 2.2]. The length of a word is the number of the generators in the word.

We now give an example to show how to get a representation for a word $W_{p/q}$. We first get a tessellation of $\mathbb{C} \setminus L$ by the labeled fundamental domain R'' ((c) of Figure 4.3) of Γ. Then in the light of the discussion above, we omit the labels $\beta^\pm$ and replace ξ by a and η by b. A portion of the tessellated $\mathbb{C} \setminus L$ is shown in Figure 4.5. The red rectangle in Figure 4.5 is the re-labeled fundamental domain R''.

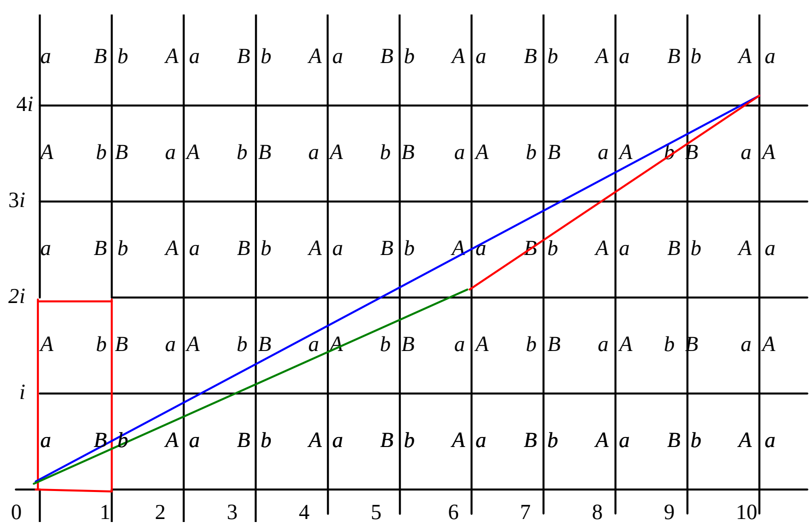


Fig. 4.5: An example of three rational words

The blue line $\lambda'_{2/5}$ in Figure 4.5 is a rational line with slope 2/5 and cuts through a sequence of rectangles. We write down the label of the side through which $\lambda'_{2/5}$ enters a hexagon. Starting from the red hexagon, we have a cutting sequence $abaBAbabAB$. This sequence is the representation of the word $W_{2/5}$ in the generators of G_μ. The word $W_{2/5}$ corresponds to the simple closed curve $\lambda_{2/5}$, which is the projection of $\lambda'_{2/5}$ on S^*. The green and red lines in Figure 4.5 are the rational lines $\lambda'_{1/3}, \lambda'_{1/2}$, respectively. The cutting sequences of these two lines are the words $W_{1/3} = abaBAB$ and $W_{1/2} = abAB$. It is easy to see from Figure 4.5 that $W_{2/5} = abaBAbabAB$ can be derived from concatenating of the words $W_{1/3}$ and $W_{1/2}$ and changing the 6th symbol B into its inverse, b. This fact is stated in the new proposition below.

Proposition 4.2.4. *Given $0 \leq m/n < r/s < 1$ with $rn - ms = 1$ and set $p/q = (m+r)/(n+s)$. The word $W_{p/q}$ can be formed from the words $W_{m/n}$ and $W_{s/t}$ as follows. Concatenate the words and change the $q+1^{st}$ symbol to its inverse.*

Proof. Draw three lines $y_1 = rx/s$ (red), $y_2 = px/q$ (blue), $y_3 = mx/n$ (green), see Figure 4.6.

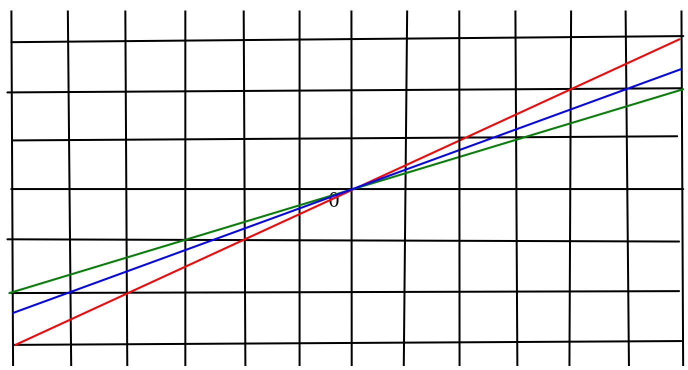


Fig. 4.6: Three rational lines

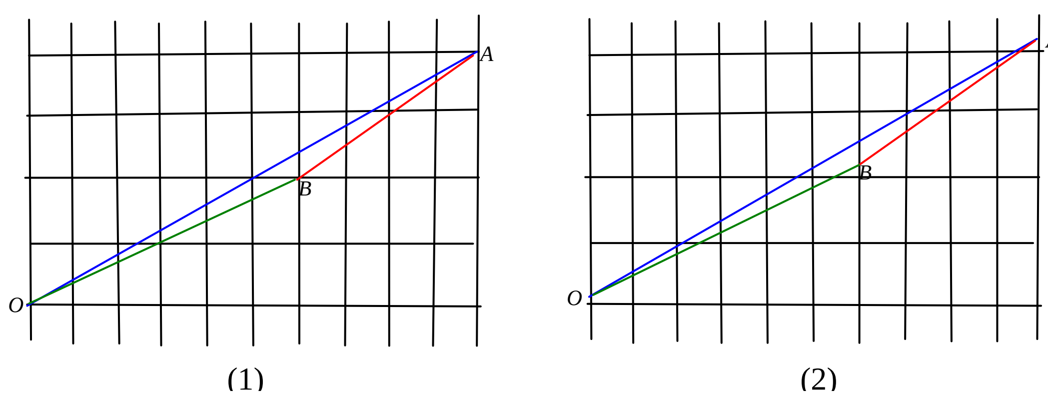


Fig. 4.7: Concatenating words

Since r/s and p/q are neighbors in the Farey Diagram, there is no lattice with integer coordinates in the regions bounded by y_1 and y_2 with $x \in (-(q+s),\ 0) \cup (0,\ (q+s))$. This is also true for the regions bounded by y_2 and y_3 with $x \in (-(q+n),\ 0) \cup (0,\ (q+n))$. Shift y_1 so that it passes the point $A = (2q, 2p)$, see (1) of Figure 4.7. A simple calculation shows that the shifted line y_1' intersects the line y_3 at $B = (2n, 2m)$ and for $x \in (q-s, 2q)$, there is no integer lattice in the region bounded by y_2 and y_1'. Let $O = (0,0)$. Since $q + n = 2n + s > 2n > q - s = n$, we see there is no integer lattice in the triangle ΔOAB. Note that (q, p) is the only integer lattice in the line segment OA. Shift ΔOAB along its y-coordinates upwards a small distance ϵ such that (q, p) is the only integer lattice inside the shifted $\Delta' OAB$. See (2) of Figure 4.7. The cutting sequences of two rational lines are the same as long as there is no integer lattice in the region bounded by them. Therefore the cutting sequence of OA may be obtained by concatenating the cutting sequences of OB and BA and changing the label at $x = q$ to its inverse. Since the lengths of cutting sequences of the lines OA, OB, BA are $2q, 2s, 2n$, respectively, we see that these cutting sequences determine the representations of the rational words $W_{p/q}, W_{r/s}, W_{m/n}$, respectively. □

4.3 Parabolic elements and pinching curves on surfaces

A geometrically finite Kleinian group is a group that has a fundamental polyhedron in $\mathbb{H}^3$ with finitely many sides. For a geometrically finite group G, it has been proved [28, Theorem 5.1] that if $\lambda_1, \ldots, \lambda_k$ are a set of mutually disjoint and simple closed curves on $\Omega(G)/G$, represented by primitive non-elliptic and non-conjugate elements $g_1, \ldots, g_k$ of G, then there is a group G', and an isomorphism $\phi : G \to G'$ taking parabolic elements of G to parabolic elements of G', for which the images $\phi(g_1), \ldots, \phi(g_k)$ are parabolic. Note that the loops in a homotopy class represented by a parabolic (or elliptic) element have arbitrarily small length, e.g, the greatest lower bound of the length is 0 [31, page 596]. So geometrically ϕ implies that the curves λ_i have been "shrunk" or "pinched" to punctures ([33]). A parabolic element obtained by pinching a curve on a surface is called an *accidental* parabolic.

Let μ be a point in $\mathscr{U}$ such that G_μ has a fundamental domain such as in Figure 4.4 (or more generally recall the discussion after (4.3)), then G_μ is a geometrically finite Kleinian group according to Theorem C.2 on page 149 of [5]. Suppose λ is a simple closed curve on S^* corresponding to $W_{p/q}$ and is pinched to a point. Algebraically, this means $\mathrm{Tr}(W_{p/q})$, the trace of the word $W_{p/q}$, has become ± 2 and $W_{p/q}$ is parabolic. Note that $\mathrm{Tr}(W_{p/q})$ is a polynomial $P(\mu)$ in terms of μ . There are a couple of questions:

- $\mathrm{Tr}(W_{p/q}) = 2$ or $\mathrm{Tr}(W_{p/q}) = -2$?
- Which root of $\mathrm{Tr}(W_{p/q}) = \pm 2$ corresponds to the accidental parabolic element?

To answer these questions, we need to have a look at $\mathscr{R}$ (Figure 4.2). See Sections 6, 7 of [34] and Section 5 [35] for an explanation of the "horizontal" lines; we discuss here the "vertical" lines in Figure 4.2. A "vertical" line is a set of points such that the trace of the word $W_{p/q}$ takes real value and $\mathrm{Tr}(W_{p/q}) \leq -2$. Such a line is called a *rational pleating ray*, and is denoted $\mathscr{P}_{p/q}$. When $0 < p/q < 1$, every $\mathscr{P}_{p/q}$ consists of exactly two connected components: a pair of conjugate "vertical" rays which asymptotically have arguments $-e^{i\pi p/q}$ and $-e^{-i\pi p/q}$, with unique complex conjugate endpoints on $\partial\mathscr{R}$, respectively [57, Theorem 2.4]. The conjugate endpoints are the roots of $\mathrm{Tr}(W_{p/q}) = -2$ [35, Proposition 4.2].

It has been proved that the conjugate endpoints of a rational pleating ray are the only points on $\partial\mathscr{R}$ such that $W_{p/q}$ is parabolic. Firstly, suppose $W_{p/q}$ in G_ρ represents a simple closed curve that is pinched to a point; then there is a unique point ρ_1 on the boundary $\mathscr{R}$ such that $W_{p/q}$ in G_ρ becomes parabolic [33, page 2]. Secondly, $\bar{\rho}_1$ is the unique point on $\partial\mathscr{R}$ such that $W_{p/q}$ in $G_{\bar{\rho}}$ becomes parabolic [57, Corollary 2.2].

Let $\rho \in \bar{\mathscr{R}}$. Then there are at most three distinct conjugacy classes of parabolics in G_ρ [6, page 261]. Suppose G_ρ contains three distinct conjugacy classes of parabolic elements, then ρ necessarily lies on $\partial\mathscr{R}$ [35, page 75]. Therefore the rest of the roots of $\mathrm{Tr}(W_{p/q}) = -2$ and all the roots of $\mathrm{Tr}(W_{p/q}) = -2$ are in the complement of the closure of $\mathscr{R}$ in $\mathbb{C}$.

We are interested in the boundary of $\mathscr{U}$ in $\mathbb{C}$. Recall that a parabolic element may be viewed as an elliptic element with infinite order. In addition to the $(2, m)$ Dehn surgery data discussed in Chapter 3, we have the following conjecture.

Conjecture 4.3.1. *Let $\langle a, b\rangle$ be a two elliptic generator Kleinian group and a, b have the form as in (4.2) and suppose $W_{p/q}$ is a rational word in $\langle a, b\rangle$. Then there is only one pair of conjugate roots of $Tr(W_{p/q}) = -2$ on the boundary of $\mathscr{U}$.*

A computer program in Mathematica (see Appendix B) has been written to investigate this conjecture numerically.

Proposition 4.3.2. *Lemma 4.2.3 and Proposition 4.2.4 give rise to a computer program, written by Zhang, and included as Appendix B.*

Following the arguments in the parabolic case, we expect that all the roots of $\mathrm{Tr}(W_{p/q}) = \pm 2$ lie either on $\partial\mathscr{U}$ or in $\mathbb{C} \setminus \mathscr{U}$; and only one pair of conjugate roots of $\mathrm{Tr}(W_{p/q}) = -2$ lie on $\partial\mathscr{U}$. But unfortunately we do not have an algorithm to choose the conjugate roots which are indeed in the boundary. So we use the Mathematica program to generate as many rational words as we can in a reasonable time and plot all the roots of $\mathrm{Tr}(W_{p/q}) = \pm 2$ in the hope that we shall have a nice picture of $\partial\mathscr{U}$.

Let a, b as in (4.2) on page 39. Then $\gamma(a, b)$ is a quadratic polynomial $\mu^2 - \kappa\mu$ where κ is a real number. Our numerical experiments have shown that $\mathscr{P}_{1-p/q} = \kappa - \mathscr{P}_{p/q}$. When a, b are elliptic elements of the same order m, then $\kappa = -\beta(a)$. If a, b are both parabolic, then $\mathscr{P}_{1-p/q} = -\mathscr{P}_{p/q}$.

This has been proved on page 314 of [57]. In general, $\mathscr{P}_{1-p/q}$ and $\mathscr{P}_{p/q}$ is symmetric with respect to $\kappa/2$. Therefore we only need to either use $W_{0/1}$ and $W_{1/2}$ or $W_{1/2}$ and $W_{1/1}$ as the initial points in the computer program.

Let $0 \leq p/q \leq 1$, $W_{1/1} = aB$ and $W_{1/2} = abAB$. We have generated 1025 rational words of G_μ where G_μ as in 4.3 on page 39 and a, b are of orders 3. It takes about 6 hours for the program to generate these words. The yellow dots in Figure 4.8 are the roots of $\mathrm{Tr}(W_{p/q}) = -2$ of these words and their symmetrical images. The blue dots are the roots of $\mathrm{Tr}(W_{p/q}) = 2$ for 129 rational words. It is clear from Figure 4.8 that $\mathrm{Tr}(W_{p/q}) = -2$ at $\partial\mathscr{U}$.

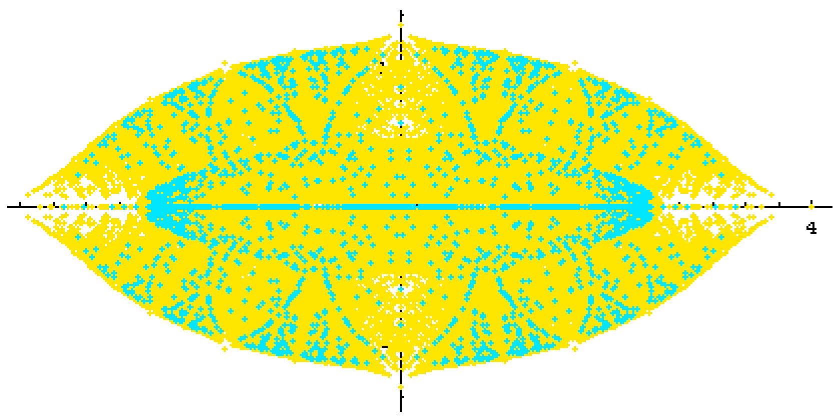


Fig. 4.8: The $(3, 3)$ parameter space

When $m = n$, we have $\gamma(a, b) = -\mu(-\mu - \beta(b))$, where a, b are as defined as in (4.2) on page 39. Suppose $< a, c >$ is a $\mathbb{Z}_2$-extension of G_μ and c is elliptic of order 2, then $\gamma(a, c) = -\mu$.

Lemma 4.3.3. *Let $\mathscr{U}'$ be the reflection of $\mathscr{U}$ with respect to the y-axis. Then $\mathscr{U}'$ is the set of $\gamma(a, c)$ such that $\langle a, c \rangle$ is a Kleinian group that is the free product of two elliptic generators of order n and 2, respectively.*

Proof. We prove by contradiction. Let $G = \langle a, c \rangle$ be a Kleinian group that is the free product of the two elliptic generators a, c of orders n and 2, respectively, and $\gamma(a, c) \notin \mathscr{U}'$. Let $b = cac^{-1}$ and $H = \langle a, b \rangle$. Normalise a, b so that they are defined as in (4.2). Let $\mu \in \mathbb{C} \setminus \mathscr{U}$ correspond to the group H. Note μ is the reflection of $\gamma(a, c)$ with respect to y-axis and G is a $\mathbb{Z}_2$-extension of H. Since $\mu \notin \mathscr{U}$, this implies H is not the free product of a and b. This is a contradiction as H is a subgroup of G. □

It follows from Exercise 12 on page 588 of [23], that $G_{\mu'}$ is of the second kind if G_μ is of the second kind. .

$\mathrm{Tr}(W_{p/q})$ is a polynomial of μ and $\mathscr{P}_{p/q}$ is a set of μ such that $\mathrm{Tr}(W_{p/q})$ is real and less than -2. When a, b are elliptics of the same orders, we replace μ in $\mathrm{Tr}(W_{p/q})$ with $-\gamma(a, c)$ so $\mathrm{Tr}(W_{p/q})$

is a polynomial of $\gamma(a,c)$. This polynomial takes real negative values and is less than -2 when $\gamma = -\mu$ and $\mu \in \mathscr{P}_{p/q}$. Let $\mathscr{P}'_{p/q} = -\mathscr{P}_{p/q}$, then $\mathscr{P}'_{p/q}$ is a rational pleating ray in $\mathscr{U}'$.

Let $\mathrm{Tr}(W_{p/q})$ be a polynomial of $\gamma(a,c)$. We solve $\mathrm{Tr}(W_{p/q}) = -2$. Then there should be a pair of conjugate roots on $\partial\mathscr{U}'$, but we do not know which pair. Since the $(2,m)$ Dehn surgery data is a very good approximation to the boundary in a $(2,m)$ commutator plane (see Figure 4.10), we use it to find the pair. We give an example to show how this is done.

Let $\gamma = \gamma(a,c)$, where a, b are elliptic of order 3, c is of order 2 and $b = cac^{-1}$, $p/q = 5/7$. Firstly we generate $\mathrm{Tr}(W_{5/7})$ in terms of μ, we then replace μ by $-\gamma(a,c)$ and we have the following polynomial.

$$\mathrm{Tr}(W_{5/7}) = -1 - 13\gamma - 34\gamma^2 - 52\gamma^3 - 53\gamma^4 - 31\gamma^5 - 9\gamma^6 - \gamma^7.$$

The roots of $\mathrm{Tr}(W_{5/7}) = -2$ are $0.0647842, -0.189454 \pm 0.877922i, -1.46443 \pm 0.390392i$ and $-2.87851 \pm 0.212617i$. Plot all these roots (red points in Figure 4.9) and combine the plot with the $(2,3)$ Dehn surgery data, and we have Figure 4.9.

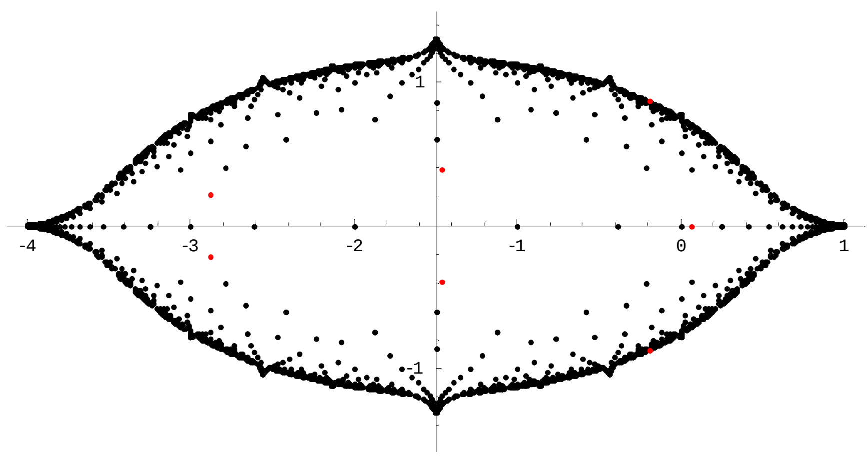


Fig. 4.9: The roots of $\mathrm{Tr}(W_{5/7}) = -2$ and the $(2,3)$ Dehn surgery data

It is clear from Figure 4.9, $-0.189454 \pm 0.877922i$ are the conjugate roots in the boundary of $\mathscr{U}'$. Using this technique, we have found 21 points that lie on the boundary of $\mathscr{U}'$. These boundary points and their corresponding rational words are in Table 4.1.

Figure 4.10 is a combination of the pleating rays $\mathscr{P}'_{p/q}$, which have the complex numbers in Table 4.1 as their endpoints on $\partial\mathscr{U}'$ (red points in Figure 4.10), the $(2,3)$ Dehn surgery Data and the symmetrical images of the roots of $\mathrm{Tr}(W_{p/q}) = -2$ in Figure 4.8 with respect to $y - axis$. We show here how we get the pleating ray $(P)'_{2/5}$, the rest of the pleating rays in Figure 4.10 are obtained in the same manner.

Tab. 4.1: $W_{p/q}$ and the corresponding boundary points

$W_{p/q}$	Corresponding boundary points	$W_{p/q}$	Corresponding boundary points
$W_{1/2}$	$-1.5 \pm 1.3228756555322954i$		
$W_{1/3}$	$-2.5651977173836396 \pm 1.0434274358930324i$	$W_{2/3}$	$-0.43480228261636056 \pm 1.0434274358930324i$
$W_{1/4}$	$-3.0081126741970383 \pm 0.7866998385255016i$	$W_{3/4}$	$0.008112674197038539 \pm -0.7866998385255016i$
$W_{1/5}$	$-3.235107158073311 \pm 0.5955249695866256i$	$W_{4/5}$	$0.23510715807331098 \pm 0.5955249695866208i$
$W_{2/5}$	$-2.137183626116933 \pm 1.126044952699754i$	$W_{3/5}$	$-0.8628163738830659 \pm 1.1260449526997542i$
$W_{1/6}$	$-3.3743129691129123 \pm 0.4508042902874505i$	$W_{5/6}$	$0.3743129691129023 \pm 0.4508042902874115i$
$W_{1/7}$	$-3.473803914477452 \pm 0.3394104928915261i$	$W_{6/7}$	$0.4738039144774501 \pm 0.33941049289154285i$
$W_{2/7}$	$-2.810546383518051 \pm 0.8779220210343224i$	$W_{5/7}$	$-0.18945361648194758 \pm 0.877922021034319i$
$W_{3/7}$	$-1.9393406909368742 \pm 1.1514507031871966i$	$W_{4/7}$	$-1.0606593090631287 \pm 1.151450703187187i$
$W_{1/8}$	$-3.5529789549033057 \pm 0.254434312179971i$	$W_{7/8}$	$0.5529789549033908 \pm 0.2544343121801925i$
$W_{3/8}$	$-2.299177912959644 \pm 1.0726002520591051i$	$W_{5/8}$	$-0.7008220870403765 \pm 1.072600252059128i$

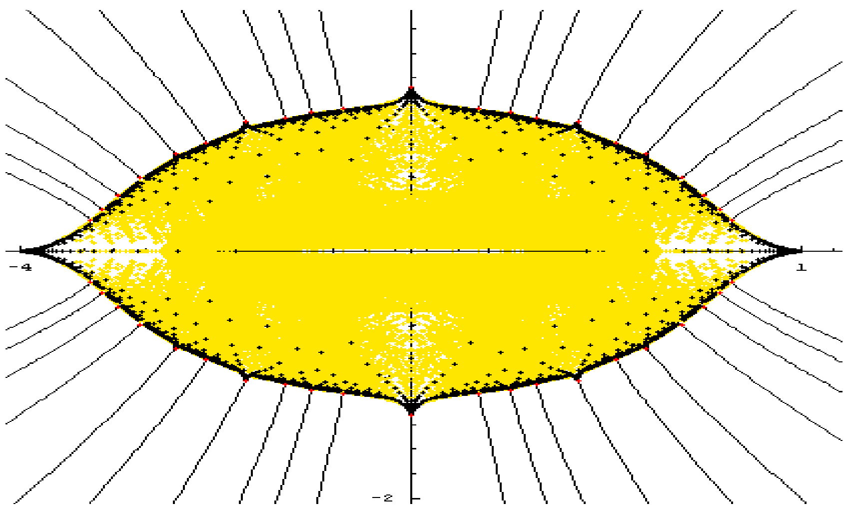


Fig. 4.10: The (2, 3) commutator plane

Let $\langle a, c\rangle$ be as defined as above and $\gamma(a, c) = -\mu$. Firstly We generate $\mathrm{Tr}(W_{2/5})$,

$$\mathrm{Tr}(W_{2/5}) = 2 - 16\mu + 32\mu^2 - 24\mu^3 + 8\mu^4 - \mu^5.$$

Next replace μ by $-\gamma$ and separate $\mathrm{Tr}(W_{2/5})$ into imaginary part (Im) and real part (Re):

$$\mathrm{Tr}(W_{2/5}) = 2 + 16x + 32x^2 + 24x^3 + 8x^4 + x^5 - 32y^2 - 72xy^2 - 48x^2y^2 - 10x^3y^2 + 8y^4 + 5xy^4 +$$

$$+i(16y + 64xy + 72x^2y + 32x^3y + 5x^4y - 24y^3 - 32xy^3 - 10x^2y^3 + y^5).$$

Plot $Im(\mathrm{Tr}(W_{2/5})) = 0$ and $Re(\mathrm{Tr}(W_{2/5})) \leq -2$. The combination of these two plots and the (2, 3) Dehn surgery data are shown in Figure 4.11. The trace polynomial $\mathrm{Tr}(W_{2/5})$ takes real values in the curves in Figure 4.11 and the real part of $\mathrm{Tr}(W_{2/5})$ smaller than -2 in the light blue regions. The two red points are conjugate roots $\mathrm{Tr}(W_{2/5}) = -2$ that we believe lie in $\partial\mathscr{U}'$. From what we have discussed above, the two connected components of $\mathscr{P}'_{2/5}$ are the two rays that have the red points as the endpoints in $\partial\mathscr{U}'$.

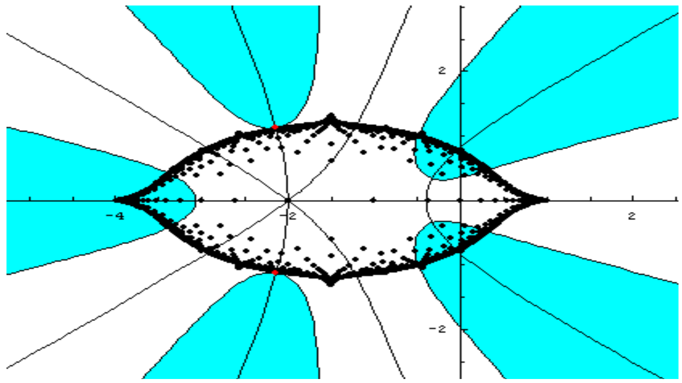


Fig. 4.11: The rational pleating ray $\mathscr{R}_{2/5}$

Unlike the $(2, m)$ commutator parameter planes, we have yet to find out more information about the (n, m) commutator plane, where n, m are the orders of two elliptic elements a, b, respectively, and none of n, m is 2. But since $\gamma(a, b) = P_\mu = \mu^2 - \kappa\mu$, we should be able to use the images of the roots of $\mathrm{Tr}(W_{p/q}) = -2$ under P_μ to study the (n, m) commutator plane.

We have generated 129 rational words for a, b of orders $3, 4$ respectively. In this case, $\kappa = 2.44948974278$. We plot the roots of $\mathrm{Tr}(W_{p/q}) = -2$ (blue dots) and $\mathrm{Tr}(W_{p/q}) = 2$ (orange dots), see Figure 4.12. The images of these dots under P_μ, which are in $(3, 4)$ commutator plane, are in Figure 4.13. Figure 4.14 is a combination of the images of roots of $\mathrm{Tr}(W_{p/q}) = -2$ of the 129 rational words and the $(3, 4)$ Dehn surgery data on two bridge links up to 11 crossings (red points). These images give us an idea of how the boundary of $(3, 4)$ would be. Here once again, we see the Dehn surgery on knots and links gives a very good estimation for $\partial\mathscr{U}'$.

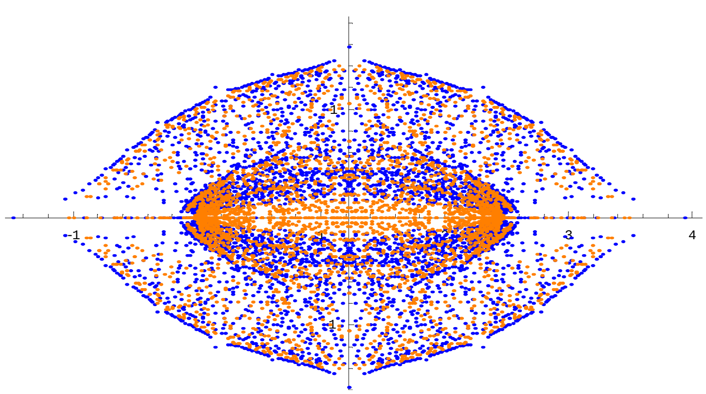


Fig. 4.12: The $(3, 4)$ parameter μ space

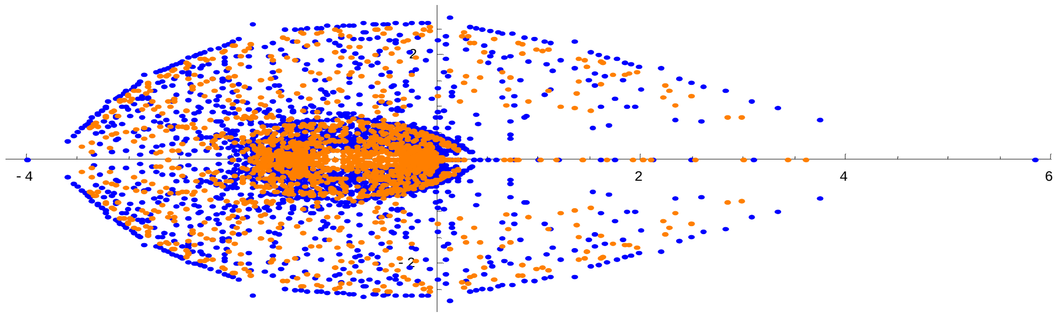


Fig. 4.13: The $(3, 4)$ commutator plane I

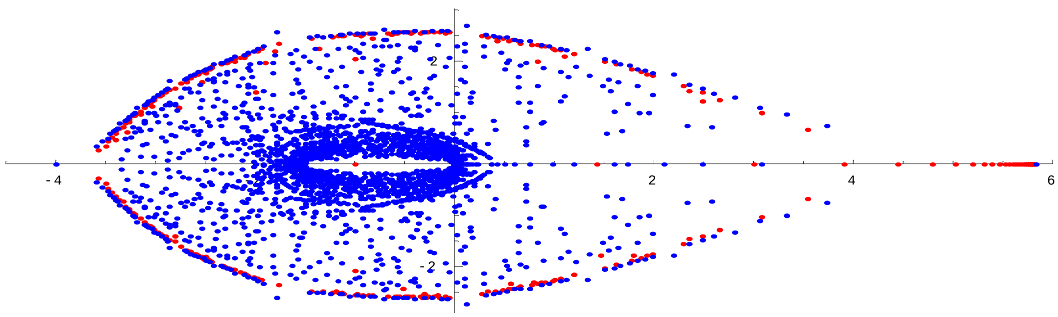


Fig. 4.14: The $(3, 4)$ commutator plane II

We have also done the same to the two generator groups with a, b which are of order $3, 5$, respectively. See Figures 4.15, 4.16, 4.17.

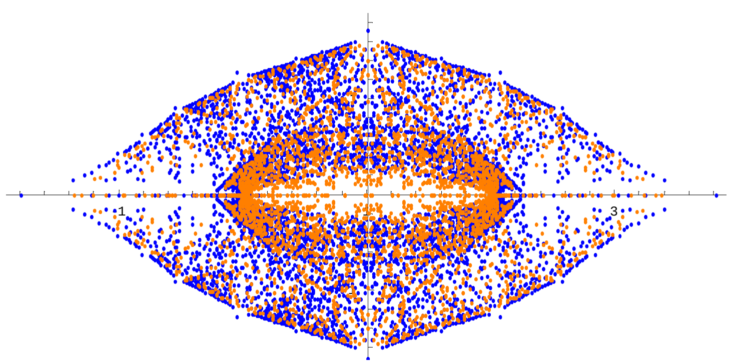


Fig. 4.15: The $(3, 5)$ parameter μ space

For the $(2, 3)$ commutator plane, we have only generated 1025 rational words and in $(3, 4)$ and $(3, 5)$ cases, we have only 129 rational words. This is because as we generate more and more words, the length of a word is getting longer and longer. So the trace polynomials $\mathrm{Tr}(W_{p/q})$ are getting very complicated. Very often, it takes a very long time for a computer to generate one trace polynomial. For example, in $(3, 4)$ case, it takes more than 1 hour to generate $\mathrm{Tr}(W_{27/50})$, which,

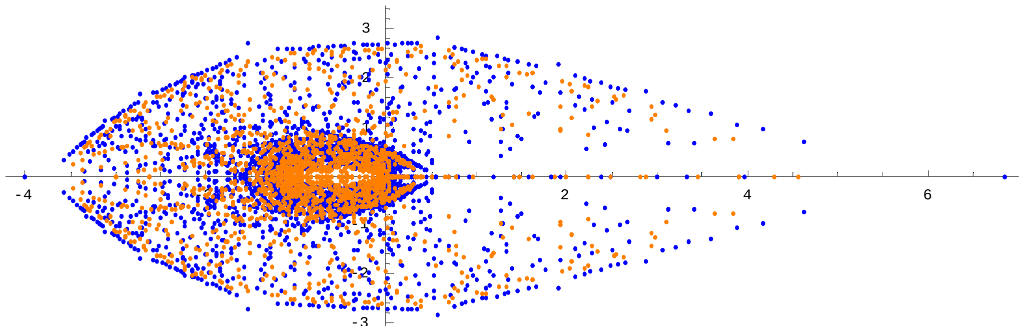


Fig. 4.16: The $(3, 5)$ commutator plane I

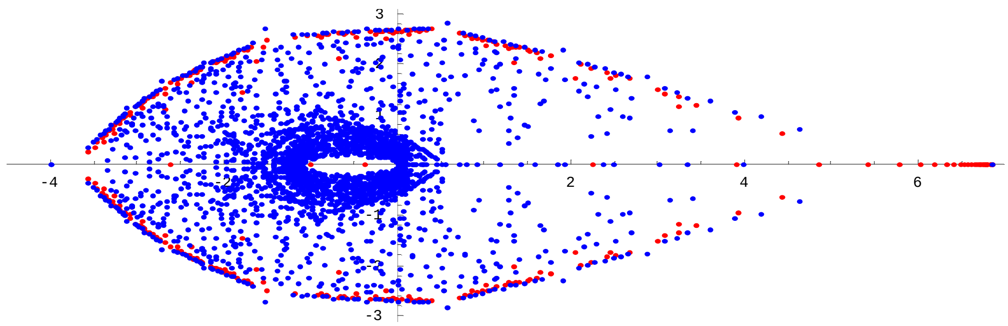


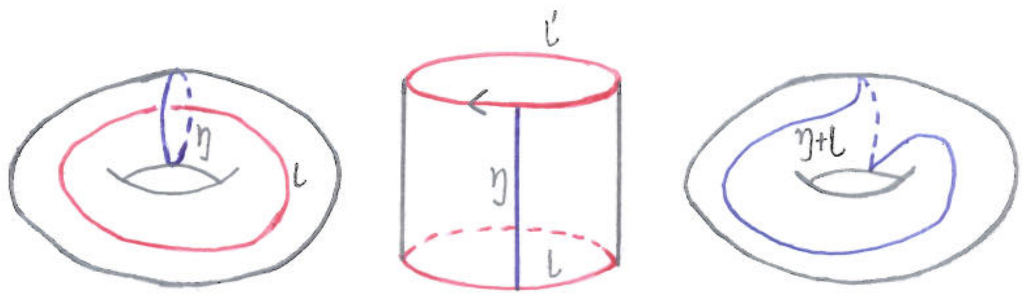
Fig. 4.17: The $(3, 5)$ commutator plane II

unlike in the $(2, 3)$ case, does not have integer coefficients. But even in the $(2, 3)$ case, we cannot generate as many rational words as we want in a reasonable time. Therefore, plotting the roots of $\mathrm{Tr}(W_{p/q}) = \pm 2$ is not ideal. We shall need to improve our algorithm in the future.

4.4 Knots and Links and the Boundary of the $(2, m)$ Commutator Plane

The $(2, m)$ Dehn surgery data gives a very good approximation of the boundary of a $(2, m)$ commutator plane. Using rational words $W_{p/q}$ and the $(2, 3)$ Dehn surgery data, we have indeed found some boundary points (Table 4.1). In this section, we discuss another way of obtaining boundary points for a $(2, m)$ commutator plane. We reveal an amazing conjectural connection between 2-bridge knots and links and the rational pleating rays.

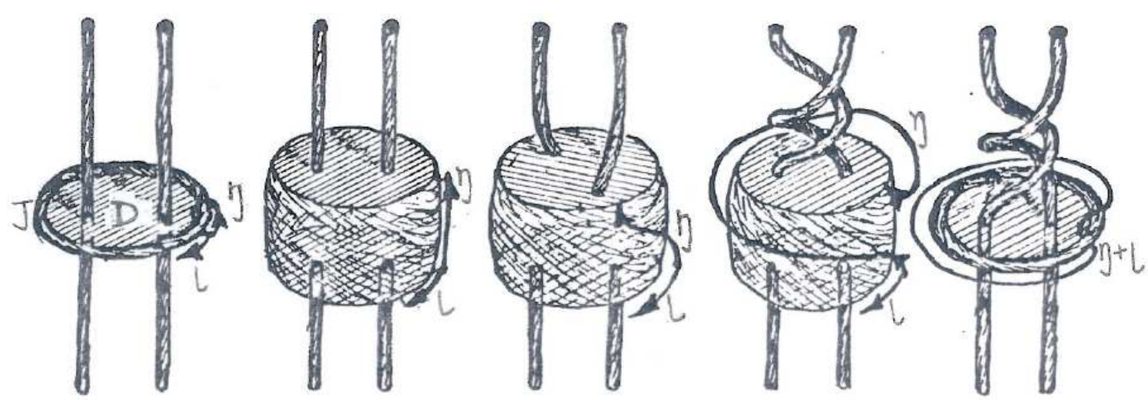
A *Dehn twist* is a certain type of self-homeomorphism of a surface. Let $\mathcal{T}$ be a torus and (η, l) be a meridian-longitude pair of $\mathcal{T}$, shown in (1) of Figure 4.18. Geometrically, a Dehn twist along the longitude l may be described as to cut the torus $\mathcal{T}$ open along l so it becomes a cylinder as shown in (2) of Figure 4.18. Next fix l in the cylinder, twist l' clockwise for 360° and glue l' back to l. The image of the meridian η under this cutting and gluing process is shown in (3) of Figure 4.18. It is a curve running along both the meridian η and longitude l once. Therefore a Dehn twist θ along l has $\theta(\eta) = \eta + l$.



(1) (2) (3)

Fig. 4.18: A Dehn twist on a torus

Let J be a trivial knot (a circle) in S^3. Then the complement of J in S^3 a 3-manifold whose boundary is a torus $\mathcal{T}$. Let (η, l) be a meridian-longitude pair of $\mathcal{T}$. The Dehn twist $\theta(\eta) = \eta + l$ on the torus $\mathcal{T}$ extends to a Dehn twist in $S^3 \setminus J$ supported in a cylinder neighborhood of a disk $D \subset S^3 \setminus J$ bounded by J (see Figure 4.19). Geometrically such a Dehn twist may be seen to cut $\mathcal{T}$ open along the disk D to get a cylinder covered on each end by a disk (see (1) and (2) of Figure 4.19, where the two parallel strings are in $S^3 \setminus J$). Then fix the bottom disk and twist the top disk clockwise 360° and glue the disks (page 62 and 63a of [54]). See (3)-(5) of Figure 4.19) The two parallel strings are mapped into two strings with a full twist.



(1) (2) (3) (4) (5)

Fig. 4.19: A Dehn twist in S^3

Now let K be a knot in S^3 cutting the disk D transversely (see Figure 4.20), where $\partial D = J$. Let (η, l) be a meridian-longitude pair of a neighborhood of J. Then $\theta^n(\eta) = \eta + nl$ is an n times Dehn twist and an n times Dehn twist is indeed a $(1, n)$ Dehn surgery on J. The following definition is from [58] , page 95.

Definition 4.4.1. *The knot $K_n = \theta^n(K)$ is said to be obtained by twisting K n times on D (or on J); n is a positive or negative integer, depending on the choice of orientation and on the sense of the twists. K_n is obtained from K by $(1, n)$ Dehn surgery on the component J of K, that is $K_n = K(J; 1/n)$.*

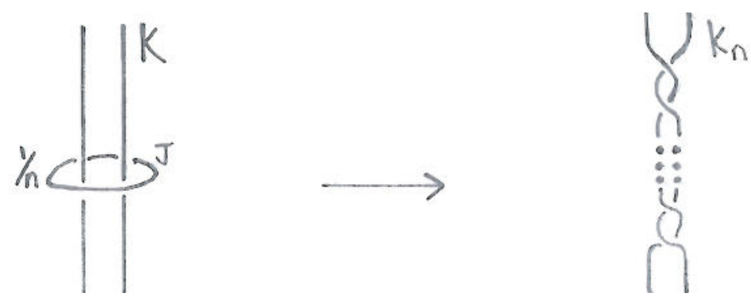


Fig. 4.20: An n times Dehn twist in S^3

Now, suppose M is a hyperbolic 3-manifold with toral boundary components. Suppose also we have chosen a meridian-longitude pair for each boundary torus. Let $M(u_1, u_2, \dots, u_n)$ denote the manifold obtained from M by a u_i Dehn surgery in the i-th boundary where $u_i = m_i/n_i$ and each pair m_i and n_i are coprime integers. We allow $u_i = 1/0 = \infty$, which means we do the"empty" Dehn filling. So $M = M(\infty, \infty, \dots, \infty)$. Let H be the space of hyperbolic 3-manifolds with finite co-volume, we have the following **Thurston's hyperbolic Dehn surgery theorem**([50]).

Theorem 4.4.2. *$M(u_1, u_2, \dots, u_n) \in H$ is hyperbolic as long as a finite set of slopes is avoided for the i-th cusp for each i. In addition, $M(u_1, u_2, \dots, u_n)$ converges to $M \in H$ as all $m_i^2 + n_i^2 \to \infty$ for all m_i/n_i corresponding to non-empty Dehn fillings u_i.*

Performing $(m_i, 0)$-Dehn surgery on a knot or link K gives rise to an orbifold with the underlying space S^3. The hyperbolic Dehn surgery theorem is also valid in this setting. The volume of the (m_i, n_i)-surgered manifold or orbifold is less than the volume of M and the volumes of surgered manifolds or orbifolds converges to that of M [10, page 66].

Let K be an arbitrary 2-bridge knot or link having a diagram as in Figure 3.3 on page 18. Let J be the boundary of a disk D that is cut by two strings of K transversely. We make the convention that the fourth string, which runs straightly down in Figure 3.3, does not pass through the disk D.

Let $1/n$ be the Dehn surgery coefficient on J. According to what we have discussed above, the $(1, n)$ Dehn surgery is indeed an n times Dehn twist and each Dehn twist adds two crossings for the two strings that cut through the disk D. Hence for each n, we have a knot (or link) K_n such that K_n has two more crossing than K_{n-1} at the position where D cuts through K. It is easy to see from Figure 4.21, that K_n is again a two bridge knot (or link).

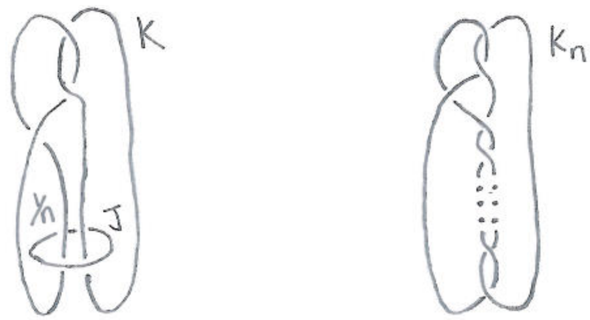


Fig. 4.21: K_n remains 2-bridge

Let M_n be the complement of K_n and M be the complement of $K \cup J$ in S^3. Suppose M is hyperbolic. Then according to Theorem 2.6 of [55] (or Theorem 4.4.2) M_n is again hyperbolic, except for a finite number of n. Let the sequence $\{M_n\}$ contain only hyperbolic manifolds from now on. As n tends to infinity, $M_n \to M$ (Theorem 4.4.2). This implies $\{K_n\}$ converges to $K \cup J$: we say $K \cup J$ is the limit link of the sequence $\{K_n\}$.

Since M_n and M are hyperbolic manifolds so $M_n = H^3/G_n$ and $M = H^3/G$ for some torsion free Kleinian groups G_n and G. Note that M_n (or M) is indeed the quotient of the fundamental domain D_n (or D) of G_n (or G). So geometrically, $M_n \to M$ implies that the fundamental domains $\{D_n\}$ of the sequence $\{G_n\}$ converge to the fundamental domain D of G. Therefore according to Proposition 4.1.7, $\{G_n\}$ converges to G geometrically; while algebraically, $\{G_n\}$ converges to a subgroup of G.

Let J be the boundary of disk D that is cut by two strings of K transversely and $L = K \cup J$; K_n is obtained from K by $(1, n)$ Dehn surgery on J. We have the following new lemma.

Lemma 4.4.3. *The sequence $\{\mathcal{O}_n\}$ of hyperbolic 3-orbifolds obtained from $\{M_n\}$ by $(m, 0)$ Dehn surgery on K_n converges to the hyperbolic 3-orbifold $\mathcal{O}$ obtained by $(m, 0)$ Dehn surgery on K of the link L.*

Proof. We show this by obtaining the same orbifold $\mathcal{O}_n$ by two different means.

- Firstly, do an n full twist on K to get K_n and then obtain $\mathcal{O}_n$ by $(m, 0)$ Dehn surgery on each component of K_n;

- Secondly, do $(m, 0)$ Dehn surgery on each component of K and then get $\mathcal{O}_n$ by n full twist on K.

In the second case, as $n \to \infty, K_n \to L$, we see $\{\mathcal{O}_n\}$ converges to $\mathcal{O}$. □

Let $\mathcal{O} = H^3/G$ and $\mathcal{O}_n = H^3/G_n$. Then according to Proposition 4.1.7, the sequence of the groups $\{G_n\}$ converges geometrically to G and algebraically to a subgroup of G. G_n (or G) is isomorphic to $\pi_1(\mathcal{O}_n)$ (or $\pi_1(\mathcal{O})$) [36, page 33].

Since K_n is a two bridge knot or link, $\pi_1(M_n)$ is a two parabolic generator group and these two generators correspond to the meridians of the boundary component(s) of M_n. In the case K_n is a two bridge knot, the two generators correspond to a pair of conjugate meridians.

Let D be the open disk bounded by a meridian η. An $(m, 0)$ Dehn surgery maps D into a cone with a singular point x of order m. This is easy to see geometrically by cutting D and twisting it m times and gluing it back. The boundary η of D is mapped to the boundary of a neighborhood of x, so a parabolic generator of $\pi_1(M_n)$ becomes an elliptic of order m. Since all the meridians of a boundary component are homotoptic to each other, the other parabolic generator of $\pi_1(M_n)$ becomes an elliptic of order m as well. In this way an $(m, 0)$ Dehn surgery on M_n changes the structure of the neighborhood of K_n, from locally homeomorphic to a 3-ball to locally homeomorphic to a quotient of a 3-ball over a cyclic group of order m. K_n becomes the singular locus $\Lambda_{\mathcal{O}_n}$ of the resulting orbifold $\mathcal{O}_n$.

The fundamental group of the 3-manifold M_n has presentation

$$\pi_1(M_n) = \langle a, b; W(a, b) = identity\rangle$$

where $W(a, b)$ is word in $\pi_1(M_n)$ in terms of a, b. Then *the orbifold fundamental group*, $\pi_1(\mathcal{O}_n)$, has the following presentation.

$$\pi_1(\mathcal{O}_n) = \langle a, b; W(a, b) = identity, a^m = b^m = identity\rangle .$$

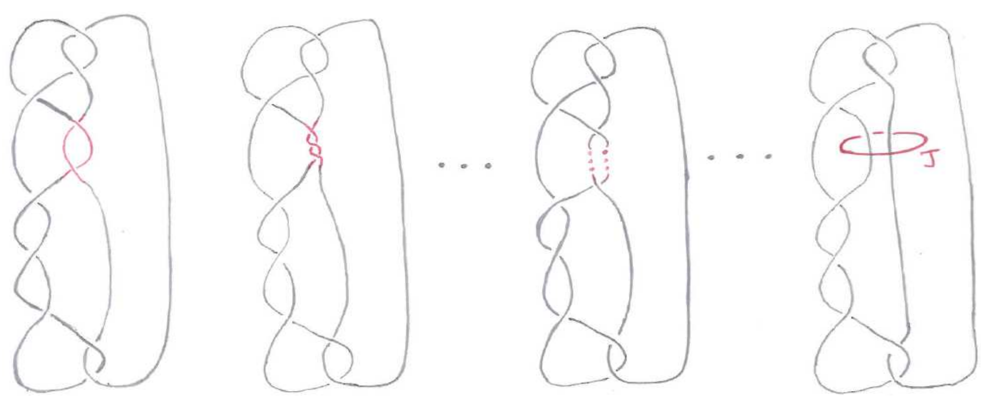
Normalize a, b as in (4.2). We now show that the points μ_n, corresponding to hyperbolic orbifold groups $\pi_1(\mathcal{O}_n)$, are isolated points in $\mathbb{C} \setminus \mathscr{U}$. That is, there is no μ corresponding to a discrete group in a neighborhood of μ_n.

Lemma 4.4.4. *Let $\mu_n \in \mathbb{C} \setminus \mathscr{U}$ correspond to a hyperbolic orbifold fundamental group. Then there is no other point corresponding to a discrete group in a neighborhood of μ_n so μ_n is isolated in $\mathbb{C} \setminus \mathscr{U}$.*

Proof. Suppose μ_n is not isolated. Then there is a sequence $\{\mu_i\}$ of complex numbers in $\mathbb{C}\setminus\mathscr{U}$, which corresponds to a sequence $\{G_i = \langle a_i, b_i\rangle\}$ of discrete groups where a_i, b_i are elliptic of order m, converging to μ_n as $i \to \infty$. So there is a sequence of words $\{W(a_i, b_i)\}$ such that $W(a_i, b_i) \neq Identity$ and $\{W_i\}$ converges to $W(a, b) = Identity$ as i tends to infinity. Let $G'_i = \langle a_i, W(a_i, b_i)\rangle$. $W(a_i, b_i)$ tends to the identity implies both $\beta(W_i)$ and $\gamma[a_i, W_i]$ tend to 0. According to Jørgensen's Inequality [3, Theorem 5.4.1], G'_i is not discrete. This contradicts the discreteness of G_i. □

Suppose the sequence $\{\pi_1(\mathcal{O}_n)\}$ converges algebraically to a group G_{μ_0}. It is a consequence of Lemma 4.4.4 that μ_0 lies in the boundary of $\mathscr{U}$. Note the two elliptic generators of $\mathcal{O}_n$ have the same orders. Let μ_n' be a commutator parameter of a $\mathbb{Z}_2$ -extension of $\mathcal{O}_n$. Then $\{\mu_n'\}$ converges to a point in $\partial\mathscr{U}'$.

We have mentioned in Section 3.4 of Chapter 3 that, in most cases, an orbifold fundamental group, resulting from $(m, 0)$ Dehn surgery on a hyperbolic 2-bridge knot or link, has two distinct commutator parameters γ. These two distinct commutator parameters give two distinct commutator parameters for the $\mathbb{Z}_2$-extensions of the orbifold group. By distinctness we mean the two commutator parameters are not a conjugate nor symmetric with respect to $\beta(a)$, where a is elliptic of order m. So if a sequence of orbifold fundamental groups converges algebraically, then there are two distinct sequences of commutator parameters γ converging to two different boundary points in a $(2, m)$ commutator plane. We now give an example to show an amazing connection between the rational words and knots and links.



K_1 K_2 K_n $L = K \cup J$

Fig. 4.22: Limit Points

Start with a two bridge knot or link, say, K_1 in Figure 4.22, add one more crossing to K_1 at the position marked red to get K_2 and do the same to K_2. Keep going in this manner and we get a sequence $\{K_n\}$ of knots and links. Do $(m, 0)$ Dehn surgery on each component of $\{K_n\}$, and we have a sequence of hyperbolic orbifolds $\{\mathcal{O}_i\}$ that converges (as $i \to \infty$) to the orbifold $\mathcal{O}$ that is obtained from $(m, 0)$ Dehn surgery on K of $L = K \cup J$ in Figure 4.22, provided $\mathcal{O}$ is hyperbolic. Then fundamental groups of $\{\mathcal{O}_i\}$ converge algebraically to a subgroup G of $\pi_1(\mathcal{O})$.

We do $(3, 0)$ Dehn surgery on K, where $L = K \cup J$ to get the commutator parameter γ for this limit subgroup. We need to be careful here since not every two generator subgroup of $\pi_1(\mathcal{O})$ is the limit subgroup. Since there are two distinct converging sequences of commutators in a $(2, m)$

commutator plane, there should be two distinct commutators for the limit group. We use $(2, m)$ Dehn surgery data as an approximation of $\partial\mathscr{U}'$ to choose γ as follows. Let $\gamma = \gamma_1(\gamma_1 - \beta(a))$, where a is elliptic of order m. If γ_1 appears in $\partial\mathscr{U}'$ and at the end of the two distinct commutator parameters for the $\mathbb{Z}_2$ -extension of $\{\mathcal{O}_i\}$, then γ is what we want, otherwise, keep doing Dehn surgery.

The continued fraction corresponding to a knot or link in Figure 4.22 is $[2, 1, n, 3, 2]$, where n is a positive integer and $n > 1$. At the limit of $\{K_n\}$, $L = K \cup J$, $n = \infty$. So we say the continuous fraction "breaks" into two parts: $[2, 1] = 1/3$ and $[2, 3] = 3/7$. Amazingly, we have found that the two distinct commutator parameters for the $\mathbb{Z}_2$-extensions of the limit subgroup are the roots of the rational words $W_{1/3}$ and $W_{3/7}$, respectively!

Start with an arbitrary 2-bridge knot or link, whose continued fraction is $[c_1, c_2, \ldots, c_k, \ldots, c_i]$; if we keep adding crossings at c_k, we then have a sequence of knots and links $\{K_n\}$, which is convergent to a link as c_k tends to infinity. At the limit link, the continued fraction "breaks" into two parts: $[c_1, \ldots, c_{k-1}]$ and $[c_{k+1}, \ldots, c_i]$. From our numerical experiments we have the following conjecture.

Conjecture 4.4.5. *The two distinct commutator parameters of the $\mathbb{Z}_2$-extensions of the limit subgroup of a fundamental group $\pi_1(\mathcal{O})$ are the roots of the two rational words corresponding to $[c_1, \ldots, c_{k-1}]$ and $[c_i, c_{i-1}, \ldots, c_{k+2}, c_{k+1}]$, respectively.*

The above observation can be seen in the sequences in Chapter 3. In the case that the first or last entry of a continued fraction becomes infinity, then one of commutator parameters of the $\mathbb{Z}_2$-extensions of the limit subgroup is the root of the rational word $W_{0/1}$. All parameters corresponding to the rational words in Table 4.1 can be found by $(3, 0)$ Dehn surgery on some limit link L using SnapPea. For example, the link L in Figure 4.22 gives the parameters corresponding to words $W_{1/3}$ and $W_{3/7}$, respectively. Since we can have a limit link for an arbitrary rational number, our experiment suggests that the points on the boundary of $\mathscr{U}'$ corresponding to Dehn surgery on hyperbolic links are dense. We hope to return to this problem later.

By now we have described the boundary of a $(2, m)$ commutator plane in three different ways.

(A): *We have done $(m, 0)$ Dehn surgery on knots and links up to 12 crossings and had approximation $\partial\mathscr{U}'$;*

(B): *we have found $\partial\mathscr{U}'$ by using the endpoints of the rational pleating rays in two elliptic generator groups;*

(C): *we have found some boundary points using the limit link of a sequence of two bridge links and knots.*

Remark.

(A) is an approximation to $\partial\mathscr{U}'$ from inside;

(C) gives points on $\partial\mathscr{U}'$;

(B) gives points on $\partial\mathscr{U}'$ from outside.

In the next chapter, we discuss an application of $\partial\mathscr{U}'$ in a $(2, m)$ commutator plane.

5. AN APPLICATION OF THE $(2, m)$ DEHN SURGERY DATA

Our applications lie in the field of arithmetic Klienian groups. In [11], Maclachlan and Martin proved that there are only finitely many arithmetic Kleinian groups with two elliptic generators of finite order. They initiated a programme to find them all. The results of this chapter contribute to that programme. We discuss in this chapter arithmetic Kleinian groups with two elliptic generators of order 2 and n, respectively, where $n = 3, 4, 6$. See Theorem 3.3.1 for a criterion for arithmeticity of these Kleinian groups. We use Dehn surgery data on knots and links as an approximation for the boundary for the $(2, m)$ commutator plane. The polynomials in our discussion are from [48]. For convenience, we use the following abbreviations.

- A $(2, m)$ group means a Kleinian group with two elliptic generators of orders 2 and m, where $m = 3, 4, 6$, respectively;
- $\gamma(2, m)$ is the commutator parameter of a $(2, m)$ group;
- $(2, m)$ arithmetic point refers to a $(2, m)$ arithmetic group.

5.1 $(2, 3)$ *Arithmetic Kleinian Groups*

Flammang and Rhin have proved in [48] that there are only 15909 polynomials with integer coefficients such that:

- every polynomial has only two complex conjugate roots;
- all the complex roots of the polynomials lie inside the ellipse $|z + 1| + |z - 2| = 5$;
- the rest of the roots are real and lie in the interval [-1, 2].

The degrees of these polynomials range from 2 to 10. See the website *www.mmas.univ-metz.fr/∽rhin* for a complete list of these polynomials.

Replacing z in Flammang and Rhin's polynomials by $z + 2$, we have all the polynomials with integer coefficients such that:

- every polynomial has only two complex conjugate roots;

- all the complex roots of the polynomials lie inside the ellipse $|z + 3| + |z| = 5$;

- the rest of the roots are real and lie in the interval [-3, 0].

The complex roots of the resulting polynomials (Figure 5.1) satisfy the conditions 1 and 2 in Theorem 3.3.1 when f is of order 3 ($\beta(f) = -3$). So every complex root corresponds to a discrete subgroup of an arithmetic group, generated by two generators of order $2, 3$, respectively. If such a subgroup G is also of finite co-volume, then G itself is arithmetic (see page 3623 of [14]).

The black points in Figure 5.1 are the $(2, 3)$ Dehn surgery data we have discussed in Section 3.4. We use the data as an approximation to the boundary of $\mathscr{U}'$ in the $(2, 3)$ commutator plane. Recall that $\mathscr{U}'$ is a set γ of complex numbers such that every γ is a commutator parameter of a $(2, m)$ group $G_{\mu'}$ that is of the second kind (see Lemma 4.3.3) and free on its two generators, which implies the co-volume of $G_{\mu'}$ is infinite; recall also that the $(2, m)$ Dehn surgery data is a very good approximation of the boundary of $\mathscr{U}'$ (see Figure 4.10). Since a $(2, m)$ arithmetic Kleinian group is of finite co-volume, a commutator parameter γ of an $(2, m)$ arithmetic Kleinian group must lie in the complement of $\mathscr{U}'$ in $\mathbb{C}$. We use this to delete the roots which are obviously outside the region bounded by the $(2, m)$ Dehn surgery data and obtain a list of all the possible commutator parameters for $(2, m)$ arithmetic Kleinian groups, where $m = 3, 4$ and 6.

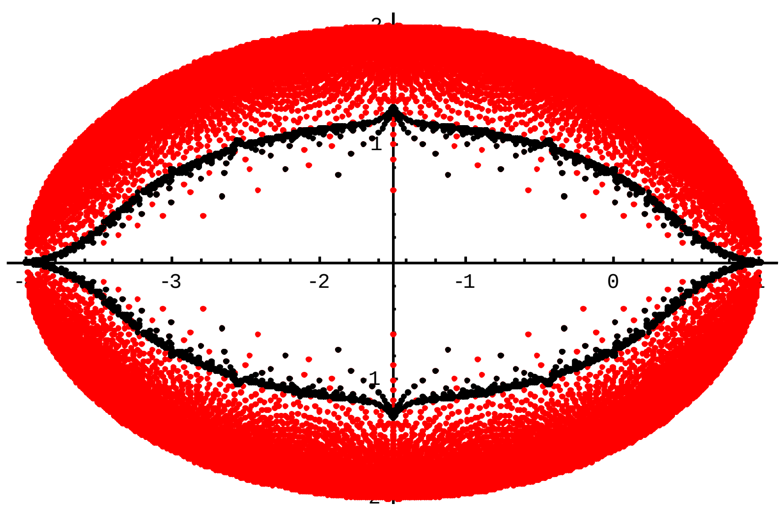


Fig. 5.1: Complex roots of the shifted Flammang and Rhin's polynomials (to the left by 2 units) and $(2, 3)$ Dehn surgery data on knots and links

There are 31818 complex roots of Flammang and Rihn's polynomials. We first delete all the roots which are apparently outside the boundary (the distance between these points and $(2, 3)$ Dehn surgery data is more than 0.04 unit). See Figure 5.2. Each of those roots should correspond to a group free on its two generators. We then analyse the remaining roots (a total of 798 red points,

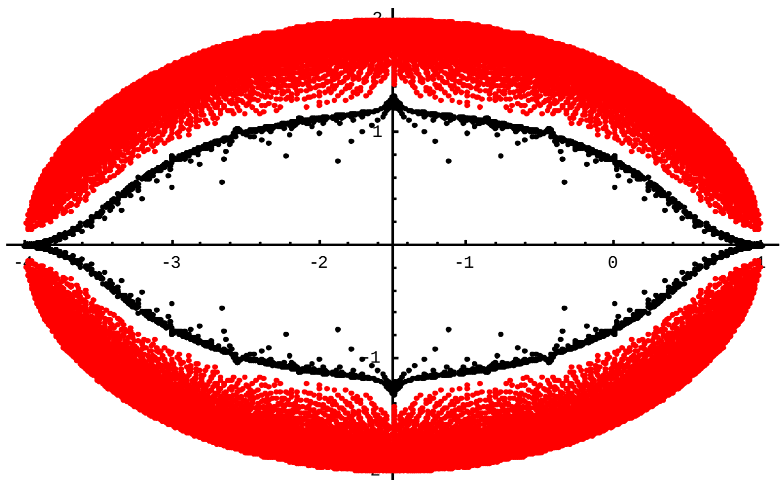


Fig. 5.2: The complex roots outside (2, 3) Dehn surgery data

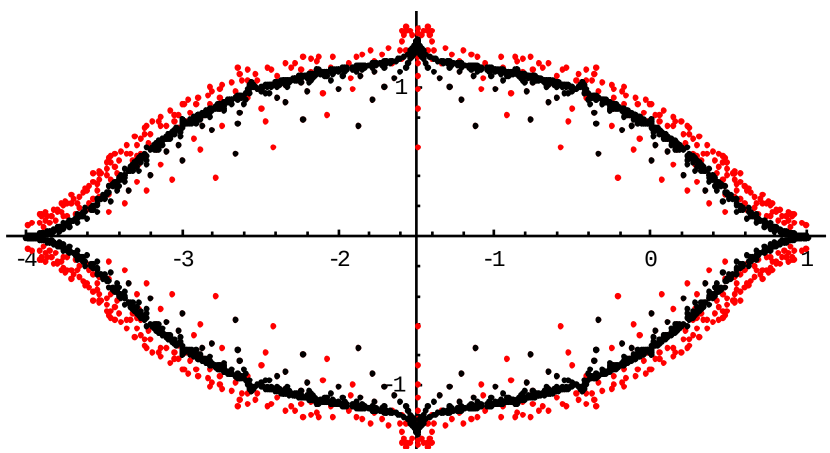


Fig. 5.3: The remaining complex roots after deleting the ones in Figure 5.2

see Figure 5.3) for more (2, 3) arithmetic points. We reserve the red color for the roots of these polynomials in the remaining discussion.

In Chapter 3, we found all the arithmetic groups from Dehn surgery on knots up to 9 crossings using the relations in the groups (see Table 3.1). The arithmeticity of these groups can also be obtained by identifying the points in Dehn surgery data with the complex roots of the polynomials. Since a point in Dehn surgery data corresponds to a group with finite co-volume, if it is also a root of one of Flammangs and Rhin's polynomials, then it is an arithmetic group. Using this technique, we have identified all the arithmetic groups from Dehn surgery on links up to 9 crossings. See Table 5.1. We also include some known arithmetic groups in Table 5.2 (from TABLE 1 on page 3624 and

Tab. 5.1: The $(2, 3)$ Arithmetic link groups up to 9 crossings

Links	Polynomial	γ − value	volume
	$3 + 5z + 4z^2 + z^3$	$-0.7672143840 + 0.7925519925i$	
5^2_1	$3 + 8z + 5z^2 + z^3$	$-2.2327856159 - 0.7925519925i$	0.263861110105
	$3 + 6z + 8z^2 + 5z^3 + z^4$	$-0.3486121811 + 0.7587449568i$	
6^2_2	$3 + 15z + 17z^2 + 7z^3 + z^4$	$-2.6513878189 - 0.7587449568i$	0.43256459895
	$2 + 2z + z^2$	$-1 + i$	
6^2_3	$1 + (2 + z)^2$	$-2 - i$	0.6106437295
	$5 + 17z + 23z^2 + 18z^3 + 7z^4 + z^5$	$-0.6018557750 + 0.9386737351i$	
7^2_1	$1 + 14z + 31z^2 + 24z^3 + 8z^4 + z^5$	$-2.3981442250 - 0.9386737351i$	0.711338779
	$1 + 2z + 3z^2 + 7z^3 + 5z^4 + z^5$	$0.1100450683 + 0.5718996068i$	
7^2_1	$5 + 38z + 60z^2 + 37z^3 + 10z^4 + z^5$	$-3.110045068334 - 0.5718996068i$	0.711338779
	$3 + 23z + 37z^2 + 25z^3 + 8z^4 + z^5$	$-1.8623969046 - 1.0758948385i$	
7^2_3	$3 + 17z + 26z^2 + 19z^3 + 7z^4 + z^5$	$-1.1376030954 + 1.0758948385i$	0.8667872855
	$2 + 2z + 3z^2 + z^3$	$-0.2393101466 + 0.8578736266i$	
8^2_4	$-4 - z + (2 + z)^3$	$-2.7606898534 - 0.8578736266i$	1.2529483161
	$2 + 6z + 4z^2 + z^3$	$-1.7718445063 - 1.1151425080i$	
8^2_6	$7 + 9z + 5z^2 + z^3$	$-1.2281554937 + 1.1151425080i$	1.0590977595
	$4 + 5z + 4z^2 + z^3$	$-0.6521896152 + 1.0288522541i$	
8^2_7	$2 + 8z + 5z^2 + z^3$	$-2.3478103848 - 1.0288522541i$	1.488219637
	$3 + 2z + 6z^2 + 11z^3 + 6z^4 + z^5$	$0.1722945573 + 0.5855931163i$	
9^2_4	$3 + 20z + 39z^2 + 29z^3 + 9z^4 + z^5$	$-3.1722945573 - 0.5855931163i$	1.181360578
	$3 + 8z + 11z^2 + 12z^3 + 6z^4 + z^5$	$-0.1397176074 + 0.8258575711i$	
9^2_9	$3 + 23z + 43z^2 + 30z^3 + 9z^4 + z^5$	$-2.8602823926 - 0.8258575711i$	1.754145367

Tab. 5.2: The $(2, 3)$ Arithmetic groups in [14]

Polynomial	γ − value	volume
$z^2 + 3z + 3$	$-1.5 + 0.8660i$	0.0845
$z^4 + 6z^3 + 12z^2 + 9z + 1$	$-1.5 + 0.6066i$	0.0390
$z^4 + 5z^3 + 7z^2 + 3z + 1$	$-0.2118 + 0.4013i$	
$z^4 + 7z^3 + 16z^2 + 12z + 1$	$-2.7881 - 0.4013i$	0.0408
$z^4 + 5z^3 + 8z^2 + 6z + 1$	$-0.9236 + 0.8147i$	
$z^4 + 7z^3 + 17z^2 + 15z + 1$	$-2.076379 - 0.8147i$	0.1274
$z^4 + 5z^3 + 6z^2 + 1$	$0.06115 + 0.3882i$	
$z^4 + 7z^3 + 15z^2 + 9z + 1$	$-3.06115 - 0.3882i$	0.1028

TABLE 6 on page 3631 of [14]) and Table 5.3 (from Table 1 on page 311 of [2]), respectively. Some points in TABLE 1 of [14] also appear in Table 5.1 and Table 5.3, so they are not in Table 5.2.

After deleting all the points in Tables 4.1, 5.1, 5.2 and 5.3, there are 642 points of the 798 points left. We have divided these 642 points further into 4 categories: see Figure 5.4 to Figure 5.7. The blue points in the figures are some boundary points.

After further investigation of the red points in Figure 5.4 and Figure 5.5, we have found a few more arithmetic groups, see Table 5.4. The first 6 points are from Figure 5.4, and the last 4 points are from Figure 5.5.

Except $0.3390928378 \pm 0.4466300999i$, which correspond to the knot 11_{363}; the arithmeticity of the rest of the points is not straightforward. The point $-0.4184706395 + 0.9391560347i$ appears among the sequence of points which correspond to Sequence 9 in Table 4.3 and right after the point corresponding to the knot 9_3. So add one twist to the knot 9_3, we have the link 9_3-addonetwist. We then do $(3, 3)$ Dehn surgery on this link. The point $-0.4184706395 + 0.9391560347i$ (as well as

Tab. 5.3: The $(2,3)$ Arithmetic groups in [2]

Polynomial	γ − value	volume
z^3+5z^2+7z+1	$-2.4196+0.6363i$	
z^3+4z^2+4z+2	$-0.5804+0.6063i$	0.1323
z^3+3z^2+z+1	$-0.1154+0.5897i$	
$z^3+6z^2+10z+2$	$-2.8846+0.5897i$	0.3308
$z^4+8z^3+21z^2+18z+1$	$-3.1385+0.4851i$	
$z^4+4z^3+3z^2+1$	$0.1385+0.4851i$	0.492289
$z^5+7z^4+18z^3+22z^2+13z+1$	$-0.8970+0.9590i$	
$z^5+8z^4+24z^3+32z^2+16z+2$	$-2.1030+0.9590i$	0.546596
$z^5+6z^4+10z^3+2z^2-2z+2$	$0.2984+0.3768i$	
$z^5+9z^4+28z^3+34z^2+13z+1$	$-3.2984+0.3768i$	0.616779
$z^5+5z^4+6z^3+z+1$	$0.2799+0.4869i$	
$z^5+10z^4+36z^3+54z^2+28z+2$	$-3.2799+0.4869i$	1.223051
$z^6+10z^5+38z^4+66z^3+51z^2+14z+1$	$-1.9247+1.0617i$	
$z^6+z^5+10z^4+36z^3+54z^2+28z+2$	$-1.0753+1.0617i$	0.847927

Tab. 5.4: Some arithmetic knot or link groups with more than 9 crossings

knots or Links	Polynomial	γ − value	volume
	$1-2z+3z^3+z^4$	$0.4674587293+0.2775899693i$	
9_3 − addonetwist	$7+29z+27z^2+9z^3+z^4$	$-3.4674587293+0.2775899693$	1.0197793525
	$1+7z+12z^2+6z^3+z^4$	$-2.5815293605-0.9391560347i$	
9_3 − addonetwist	$7+11z+12z^2+6z^3+z^4$	$-0.4184706395+0.9391560347i$	1.0197793525
	$3+z+(2+z)^3$	$-1.6588360981+1.1615413999i$	
9^2_{10} − addonetwist	$1+4z+3z^2+z^3$	$-1.3411639019+1.1615413999i$	1.3193055505
	$1+2z^2+z^3$	$0.1027847152+0.6654569512i$	
10^2 − conway424	$8+15z+7z^2+z^3$	$-3.1027847152-0.6654569516i$	1.7126470675
	$1-z+z^2+3z^3+z^4$	$0.3390928378+0.4466300999i$	
11_{363}	$13+34z+28z^2+9z^3+z^4$	$-3.3390928378-0.4466300999i$	1.474327445

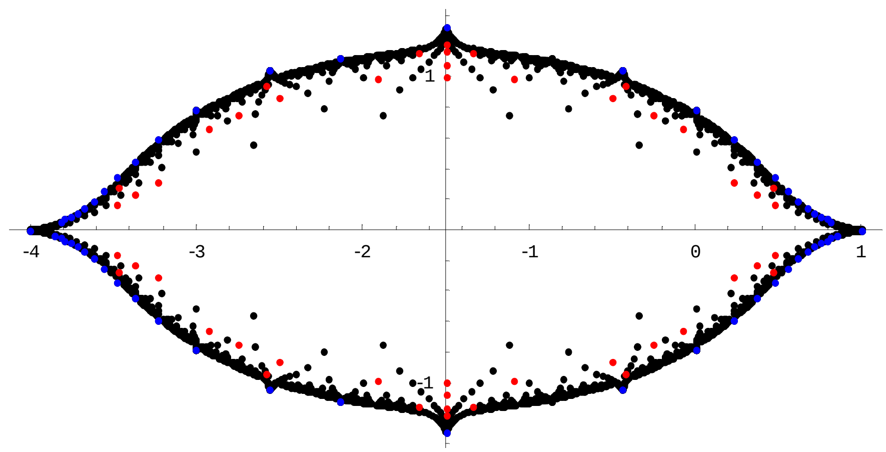


Fig. 5.4: The points inside the $(2, 3)$ Dehn surgery data

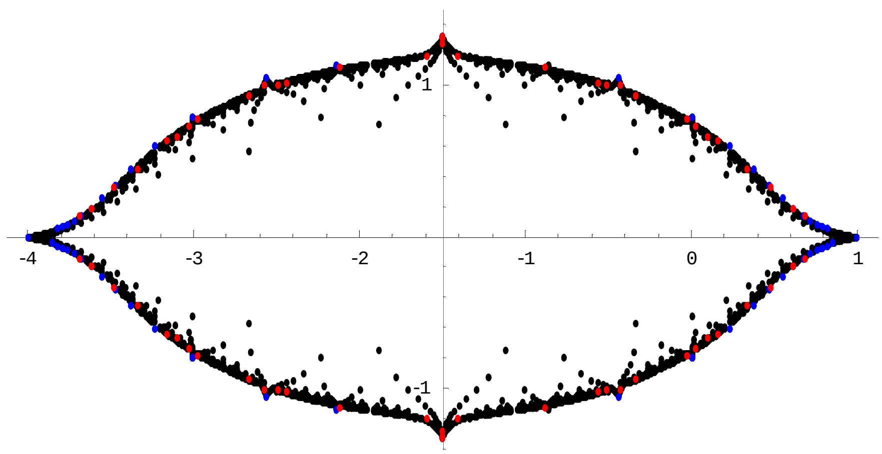


Fig. 5.5: The points less than 0.01 distance from Dehn surgery data

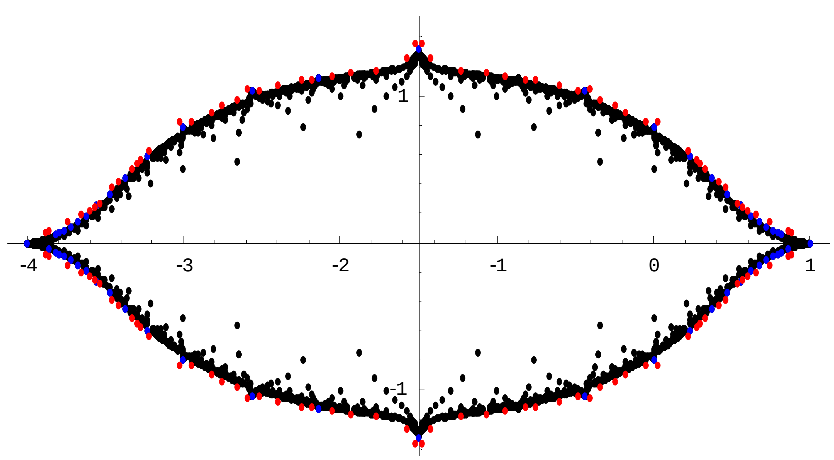


Fig. 5.6: The points greater than 0.01 but less than 0.04 distance from Dehn surgery data

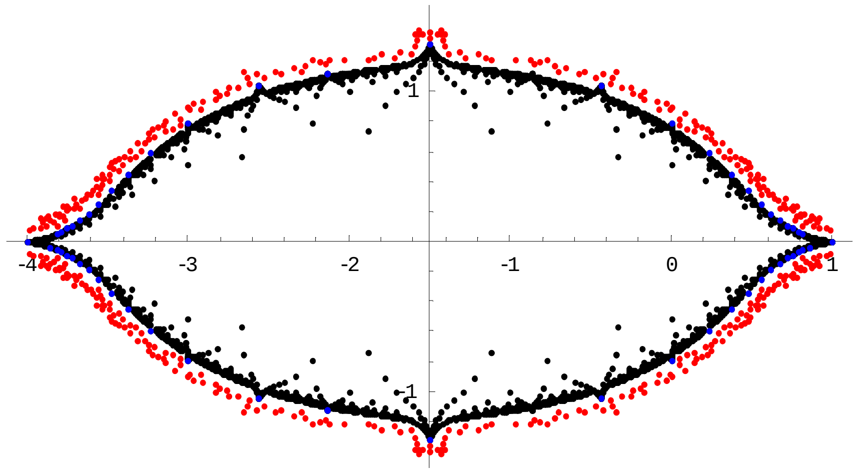


Fig. 5.7: The points greater than 0.04 but less than 0.1 distance from Dehn surgery data

the point $-2.5815293605 - 0.9391560347i$) indeed also corresponds to the link so it is arithmetic.

The point $-1.3411639019 + 1.1615413999i$ appears among the sequence of points corresponding to the Sequence 10 and right after the point corresponds to the link 9^2_{10}. Again I add one twist to 9^2_{10} and do $(3, 3)$ Dehn surgery on the resulting link 9^2_{10}-addonetwist to get the result.

It is not obvious that the point $0.1027847152 + 0.6654569512i$ is in some sequence of points. But it appears after two points that correspond to the link 8^2_3 and the knot 9_9, which have Conway's notation [4 2 2] and [4 2 3] respectively. With this information we drew the link 10^2-conway424 and found that $0.1027847152 + 0.6654569512i$ (as well as $-3.1027847152 - 0.6654569516i$) corresponds to the link and hence is arithmetic. We have the following new theroem.

Theorem 5.1.1. *Let $G = \langle f, g \rangle$ be a Kleinian group generated by elements of orders 2 and 3, respectively. Then $par(G) = (\gamma, -3, -4)$ and Tables 5.1, 5.4 contains a list of γ such that G is arithmetic and $\mathbb{H}^3/G$ is an orbifold which is 2-fold covering obtained from $(3, 0)$ Dehn surgery on a two bridge knot or link of up to 12 crossings.*

The rest of the points in Figure 5.4 , Figure 5.5 and some of points in Figure 5.6 (less than 0.025 unit distance from the Dehn surgery data) are in Tables 5.5, 5.6 and 5.7, respectively. We claim that except for the arithmetic points we have discussed above, all the other arithmetic points are to be found in these three tables. Further, except for the point $0.610682 + 0.187805i$, the rest of the points in Table 5.7 are outside the boundary though they are very close to it. Therefore our claim can be further refined as the majority of the arithmetic points shall be found in Tables 5.5 and 5.6.

Conjecture 5.1.2. *Suppose $G = \langle f, g \rangle$ is an arithmetic Kleinian group generated by elements of orders 2 and 3 respectively. Then $par(G) = (\gamma, -3, -4)$ and γ appears in one of the Tables 3.1, 5.1,* *5.2, 5.3, 5.4, 5.5, 5.6, 5.7.*

Remark *If our methods have reproduced the boundary of $\mathscr{U}'$ with a reasonable degree of accuracy, then this conjecture is also valid.*

Cooper([19]) currently is working on a program (tentatively) called $F - elliptic$, which is based heavily around the SnapPea code for constructing fundamental domains for given orbifolds. Given the orders of two elliptic generators and a complex γ parameter, it calculates the generators for a group generated by these two elliptic elements, from which it then attempts to construct a fundamental domain. While still in development, it gives strong indications as to whether a domain is infinite; and gives the volume for finite domains. Currently work is being done on the program to explicitly determine infinite domains and to compute a representation of the relevant group.

He has tested all the points in Tables 5.5, 5.6, 5.7 in his program. The results are recorded in the third columns of these tables. The question mark ? in the tables means that at the time of writing this, his program has no conclusion yet about these points. But remarkably, all the points with question mark ? are in one of the following two classes.

- Outside our boundary (although very close to it, we need to zoom in on the pictures to see this), therefore presumably they are groups free on two generators;

- on the extension of the rational pleating ray $\mathscr{P}'_{1/2} \in \mathscr{U}'$. The extension means the pleating ray crosses the boundary of $\mathscr{U}'$ to $\mathbb{C} \setminus \mathscr{U}'$. On this extension, the trace of the rational word $W_{1/2}$ is real and greater than -2. Every rational pleating ray $\mathscr{P}'_{p/q}$ has such an extension. Professor Keen in CUNY predicts that there are infinitely many web groups on the extension of a rational pleating ray. Therefore these points on $\mathscr{P}'_{1/2}$ may be the $\mathbb{Z}_2$-extensions of some web groups, which have infinite co-volume. We discuss web groups more in the next chapter.

Lemma 2.29 and its proof on page 183 [16] show that if γ corresponds to a $\mathbb{Z}_2$-extension of a 2-generator group of order n, then so does $\beta - \gamma$. If such a point γ is arithmetic, so are $\bar{\gamma}$, $\beta - \gamma$ and $\beta - \bar{\gamma}$. Hence Tables 5.5, 5.6 and 5.7 contain only points with positive imaginary parts and real parts greater than or equal to -1.5. In the following discussion, a conjugate of γ means a point that is $\bar{\gamma}, \beta - \gamma$ or $\beta - \bar{\gamma}$.

An interesting observation is that except for $1 - 4z + 9z^3 - 6z^4 - z^5 + 5z^6 - 4z^7 + z^8$, whose complex roots are $-1.5 \pm 1.30625i$ in Table 5.6, the degrees of the polynomials in [48], which have the points in Tables 5.5, 5.6 and 5.7 as their roots, are at most 6. See Figure 5.8 to Figure 5.11. Note also that all the arithmetic points we have found are also the roots of polynomials of degree at most 6.

Tab. 5.5: Possible (2, 3) arithmetic points 1

	$\gamma -$ value	volume
1	$-0.5 + 0.8660254037844386i$	0.675975
2	$0.36778434696203943 + 0.23154129443909352i$	0.192792
3	$-0.25459834415920657 + 0.7495282360734272i$	0.526161
4	$-1.5 + 0.9984886597841094i$	?
5	$-1.5 + 1.078987285547469i$	?
6	$-1.5 + 1.1696298511708287i$	?
7	$-1.5 + 1.2173950909909557i$	?
8	$0.4758232140570704 + 0.1646025518579301i$	0.219917
9	$0.2367560145079861 + 0.31257031944565544i$	0.153716
10	$-0.07087410303499064 + 0.6643508327148338i$	0.594148
11	$-1.0869846518685065 + 0.9878714402908879i$	0.263641

Tab. 5.6: Possible (2, 3) arithmetic points 2

	$\gamma -$ value	volume
1	$0.02763926928282956 + 0.7366101577206986i$	2.637428
2	$-0.33909070209486325 + 0.9288109156519962i$	1.779692
3	$-1.4063088180790835 + 1.1961583360709451i$	1.475220
4	$-0.5615369602180349 + 1.0175787389809263i$	1.916677
5	$-0.4284705219623066 + 1.0066416705006005i$	2.446200
6	$-0.8776303086557513 + 1.1140026635827618i$	3.422112
7	$-1.5 + 1.3062523498419483i$	?
8	$0.1616914183624485 + 0.6367094816799277i$	?
9	$-0.022557003004360254 + 0.7789602807998743i$	2.885081
10	$-0.5115227740791208 + 1.0018671311047347i$	1.541135
11	$-1.5 + 1.2764737527939292$	?

Tab. 5.7: Possible (2, 3) arithmetic points 3

	$\gamma -$ value	volume
1	$0.27340913844204096 + 0.5638210928291186i$	?
2	$-1.0635592045220754 + 1.1643804990998183i$	?
3	$-0.8223887278669912 + 1.1125277921303889i$	?
4	$0.6815193547604368 + 0.14728794302860906i$	?
5	$0.5718632313907062 + 0.25235161679552787i$	?
6	$0.6106821844759649 + 0.18780514128590894i$	1.966746
7	$0.531012602661534 + 0.2734943961017966i$	?
8	$-0.17611155789574795 + 0.891627543858532i$	?
9	$-1.231365882100651 + 1.1811791365746456i$	?
10	$-0.9434383278152931 + 1.1411511877204064i$	?
11	$0.5972652447639999 + 0.22322596645703782i$	?
12	$0.4762253743518343 + 0.335893920570061i$	?
13	$0.2951619639928418 + 0.539127458455876i$	?
14	$0.3257990493689751 + 0.5113802002036603i$	?

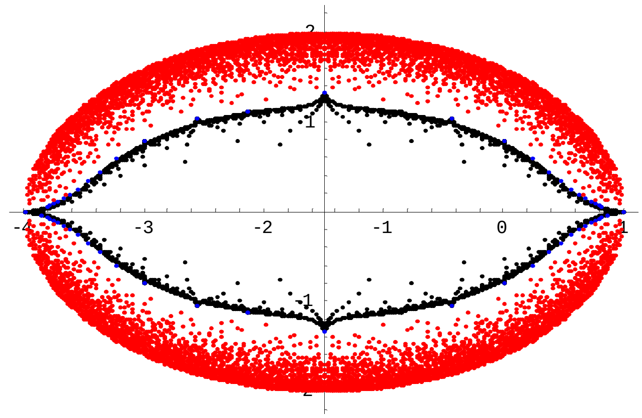


Fig. 5.8: The complex roots of polynomials of degree 7 with Dehn surgery data

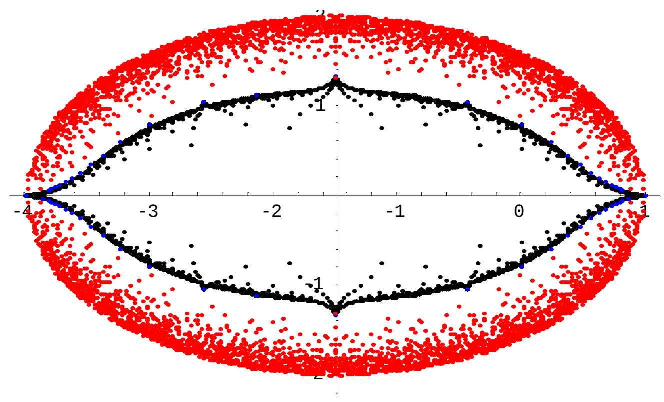


Fig. 5.9: The complex roots of polynomials of degree 8 with Dehn surgery data

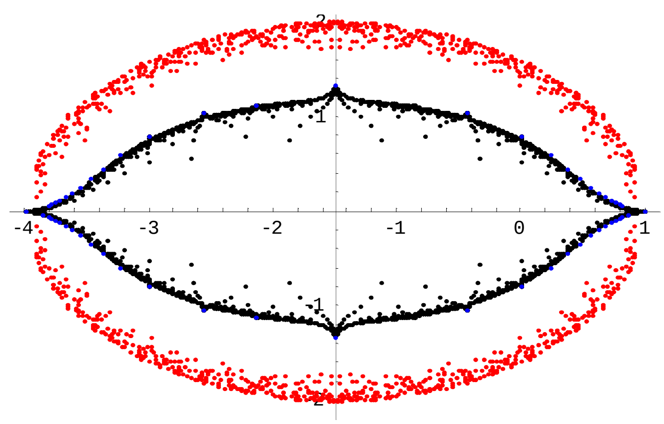


Fig. 5.10: The complex roots of polynomials of degree 9 with Dehn surgery data

5.2 $(2, 4)$ *Arithmetic Kleinian groups*

Replace z in Flammang and Rhin's polynomials by $z + 1$. Then we have all the polynomials with integer coefficients such that:

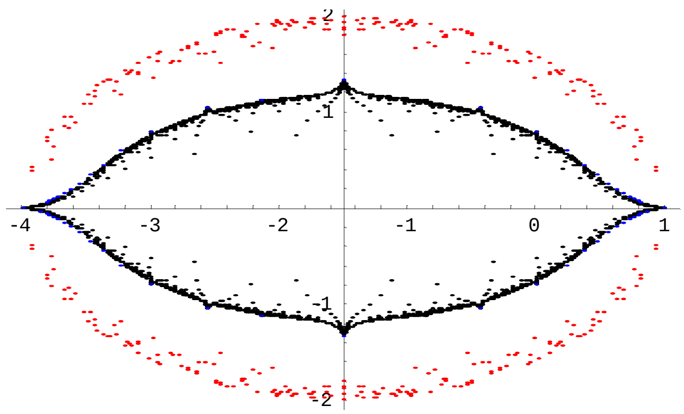


Fig. 5.11: The complex roots of polynomials of degree 10 with Dehn surgery data

- every polynomial has only two complex conjugate roots;
- all the complex roots of the polynomials lie inside the ellipse $|z+2|+|z-1|=5$;
- the rest of roots are real and lie in the interval [-2, 1].

Figure 5.12 contains the complex roots of such polynomials and $(2,4)$ Dehn surgery data with real parts greater or equal to -1 (black points). It shows that $(2,4)$ Dehn surgery data lies entirely inside the ellipse $|z+2|+|z-1|=5$. We are interested in those polynomials whose real roots lie in the interval $[-2,0]$. Recall that if a point in $(2,4)$ Dehn surgery data is also a complex root of such polynomial, then this point corresponds to an arithmetic group, so is its symmetric point with respect to $x=-1$. In this way, we shall find all the possible values for arithmetic $(2,4)$ groups. Figure 5.13 contains $(2,4)$ Dehn surgery data (black points) and the complex roots of those polynomials whose real roots lie in the interval $[-2,0]$.

We again delete all the points that are obviously outside the $(2,4)$ Dehn surgery data. The rest of points and their symmetric points with respect to -1 are shown in Figure 5.14. By comparing with $(2,4)$ Dehn surgery data, we have found some $(2,4)$ arithmetic groups, see Tables 5.8 and 5.12. Table 5.10 is from TABLEs 2 and 7 of [16]. Note that if a point in [16] corresponds to an arithmetic knot or link group, we record it in Table 5.8 or 5.12. We claim all the other possible $(2,4)$ arithmetic points are in Table 5.11 and their conjugates with respect to $y=0$ and $x=-1$. See Figure 5.15. The third column is the results from Cooper's program. We have the following theorem and conjecture which is new to this thesis.

Tab. 5.8: $(2,4)$ Arithmetic knot groups

Knots	Polynomial	$\gamma -$ value	volume
4_1	$1+z+z^2$	$-0.5+0.866025403784i$	0.253735
	$3+3z+z^2$	$-1.5-0.866025403784i$	
5_2	$1+z^2+z^3$	$0.232785615929046-0.7925519925163i$	0.593687
	$-3-8z-5z^2-z^3$	$-2.232785615929046+0.7925519925163i$	
5_2	$3+4z+3z^2+z^3$	$-0.658836097599065+1.161541397584019i$	0.593687
	$-1-4z-3z^2-z^3$	$-1.341163902400935-1.161541397584019i$	
6_1	$1-z+z^3$	$0.662358978521655-0.562279511872486i$	0.8248689425
	$-5-11z-6z^2-z^3$	$-2.662358978521655+0.562279511872486i$	
7_4	$1-z+z^2$	$0.5+0.866025403784i$	1.5224124095
	$7+5z+z^2$	$-2.5-0.866025403784i$	
7_7	$1+2z+z^2+z^3$	$-0.215079854501591+1.307141278681935i$	2.12109156625
	$-7-10z-5z^2-z^3$	$-1.784920145498409-1.307141278681935i$	
8_2	$3-z-3z^2+z^3+z^4$	$1.061146301206304+0.388297763934367i$	1.54264512295
	$1+9z+15z^2+7z^3+z^4$	$-3.061146301206304-0.388297763934367i$	
8_2	$1+3z+2z^2+2z^3+z^4$	$0.036402620323842+1.212379235839235i$	1.54264512295
	$3+13z+14z^2+6z^3+z^4$	$-2.036402620323842-1.212379235839235i$	
8_3	$1+z-z^2+z^3+z^4$	$0.651387818843013+0.758744956637659i$	1.622117246
	$3+15z+17z^2+7z^3+z^4$	$-2.651387818843013-0.758744956637659i$	
12_{762}	$3+z-4z^2+z^4$	$1.367784347021253-0.231541294370707i$	2.891603182
	$1+15z+20z^2+8z^3+z^4$	$-3.367784347021253+0.231541294370707i$	
12_{762}	$1+5z+5z^2+3z^3+z^4$	$-0.579434939376813-1.457427273182777i$	2.891603182
	$3+11z+11z^2+5z^3+z^4$	$-1.420565060623187+1.457427273182777i$	

Tab. 5.9: $(2,4)$ Arithmetic link groups

Knots	Polynomial	$\gamma -$ value	volume
5^2_1	$2+2z+2z^2+z^3$	$-0.228155493737054+1.115142507999893i$	0.7943233195
	$-2-6z-4z^2-z^3$	$-1.771844506262946-1.115142507999893i$	
6^2_2	$1+2z+z^2+2z^3+z^4$	$0.207106781177589+0.978318343467463i$	1.0292421841
	$1+10z+13z^2+6z^3+z^4$	$-2.207106781177589-0.978318343467463i$	
6^2_3	$2+z+z^2$	$-0.5+1.3228756555i$	1.3333723915
	$4+3z+z^2$	$-1.5-1.3228756555i$	
8^2_4	$2+z^2+z^3$	$0.347810384657162+1.028852254208391i$	2.232329456
	$-2-8z-5z^2-z^3$	$-2.347810384657162-1.028852254208391i$	
8^2_6	$4+5z+3z^2+z^3$	$-0.773301174242719+1.467711508709982i$	1.9267279507
	$-2-5z-3z^2-z^3$	$-1.226698825757281-1.467711508709982i$	
8^2_7	$3+2z+2z^2+z^3$	$-0.094732143056706+1.283742172094812i$	2.59921683
	$-1-6z-4z^2-z^3$	$-1.905267856943294-1.283742172094812i$	
9^2_4	$2-2z+z^3$	$0.884646177159086+0.589742805026357i$	1.985144812
	$-2-10z-6z^2-z^3$	$-2.884646177159086-0.589742805026357i$	

Tab. 5.10: $(2,4)$ Arithmetic groups in [16]

Polynomial	$\gamma -$ value	volume
$2+2z+z^2$	$-1+i$	0.2289
$1+6z+7z^2+4z^3+z^4$	$-1+1.272019649514069i$	0.0717
$5+4z+z^2$	$-2+i$	0.1526
$1+z^2$	i	
$1+4z+4z^2+3z^3+z^4$	$-0.4063088180790835+1.1961583360709451i$	0.4475
$1+8z+10z^2+5z^3+z^4$	$-1.5936911819209165-1.1961583360709451i$	
$1+z+2z^2+z^3$	$-0.12256116687665031+0.7448617666197513i$	0.1374
$-1-5z-4z^2-z^3$	$-1.8774388331233496-0.7448617666197513i$	
$1-z+z^2+z^3$	$0.4196433776070805+0.6062907292071993i$	0.0661
$-1-7z-5z^2-z^3$	$-2.4196433776070805-0.6062907292071993i$	
$1-2z^2+z^3+z^4$	$0.788104887228904+0.401357867370071144i$	0.3475
$1+12z+16z^2+7z^3+z^4$	$-2.7881048872289043-0.401357867370071144i$	

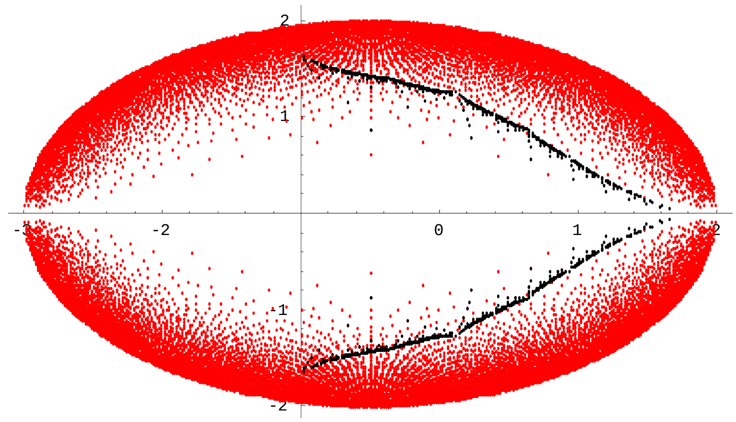


Fig. 5.12: Complex roots of the shifted Flammang and Rhin's polynomials (to the left by 1 unit) and (2, 4) Dehn surgery data on knots and links

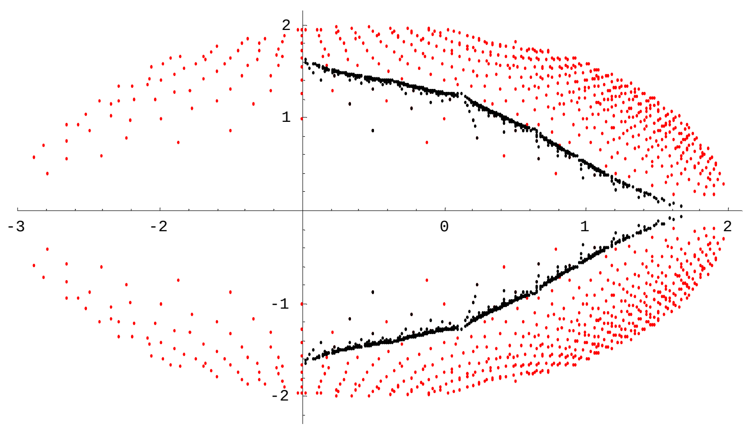


Fig. 5.13: Complex roots of the shifted Flammang and Rhin's polynomials (real roots in $[-2, 0]$) and (2, 4) Dehn surgery data on knots and links

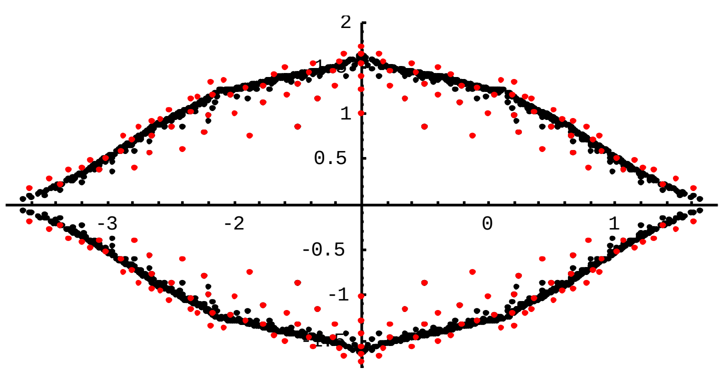


Fig. 5.14: Complex roots in Figure 5.13 and their symmetries with respect to -1, less than 0.1 distance from (2, 4) Dehn surgery data

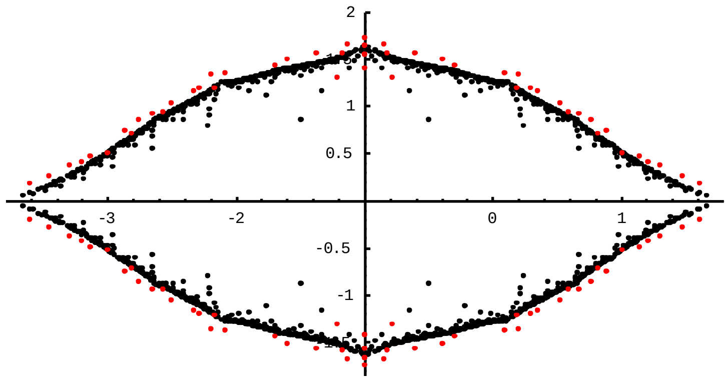


Fig. 5.15: The possible $(2, 4)$ commutators parameters

Tab. 5.11: Possible $(2, 4)$ Arithmetic points

	$\gamma -$ value	volume
1	$-1 + 1.41421356237730951i$	?
2	$-1 + 1.55377397403003774i$	?
3	$1.0075523593781792 + 0.5131157955970148i$	3.692968
4	$0.17660498209966358 + 1.2028208192854806i$	3.301824
5	$0.8151578023918029 + 0.7124188033466969i$	?
6	$0.5815293604578504 + 0.9391560346936633i$	?
7	$-0.3036766091486792 + 1.4359498641099564i$	?
8	$-0.8283528529755734 + 1.5766860923274049i$	?
9	$-1 + 1.6528916502810695i$	?
10	$-1 + 1.7320508075688772i$	?
11	$-0.7849201454990266 + 1.3071412786820455i$	0.824778

Theorem 5.2.1. *Let $G = \langle f, g\rangle$ be a Kleinian group generated by elements of orders 2 and 4, respectively. Then par$(G) = (\gamma, -2, -4)$ and Table 5.8 and Table 5.9 contains a list of γ such that G is arithmetic and $\mathbb{H}^3/G$ is an orbifold which is 2-fold covering obtained from $(4, 0)$ Dehn surgery on a two bridge knot or link of up to 12 crossings.*

Conjecture 5.2.2. *Suppose $G = \langle f, g\rangle$ is an arithmetic Kleinian group generated by elements of orders 2 and 4 respectively. Then par$(G) = (\gamma, -2, -4)$ and γ appears in Tables 5.8, 5.9, 5.10, 5.11.*

Remark *If our methods have reproduced the boundary of $\mathscr{U}$ with a reasonable degree of accuracy, then this conjecture is also valid.*

5.3 $(2, 6)$ Arithmetic Kleinian groups

The $(2, 6)$ Dehn surgery data is symmetric with respect to -0.5. In the case of the $(2, 4)$ Dehn surgery, the data with real parts greater than or equal to -1 lies entirely inside the ellipse $|z + 2| + |z - 1| = 5$. But a tiny fraction of $(2, 6)$ data with real parts greater than or equal to -0.5 lies outside the ellipse $|z + 1| + |z - 2| = 5$. See Figure 5.16. Since most of the data is inside the ellipse, we may use the technique in the last two sections to identify some $(2,6)$ arithmetic groups. Note that we don't need to shift the complex roots of Flammang and Rhin's polynomials in this case.

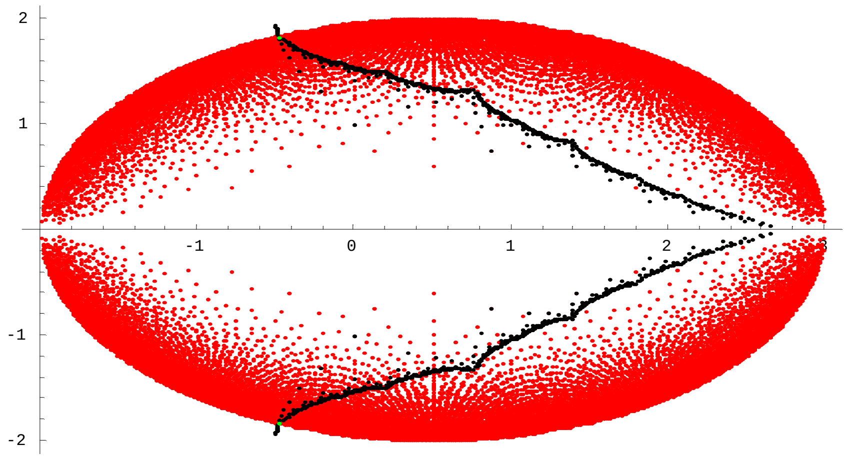


Fig. 5.16: The complex roots of Flammang and Rhin's polynomials and $(2,6)$ Dehn surgery data

The complex roots of those polynomials, whose real roots lie in the interval $[-1,0]$, correspond to discrete $(2,6)$ subgroups of arithmetic groups. If such a complex root is also in $(2,6)$ Dehn surgery data, then it is an arithmetic point; so is its symmetric point with respect to $x=-1$. In this way, we shall find the majority of possible values for arithmetic $(2,6)$ groups. Figure 5.17 contains $(2,6)$ Dehn surgery data (black points) and the complex roots of those polynomials with real roots in the interval $[-1,0]$.

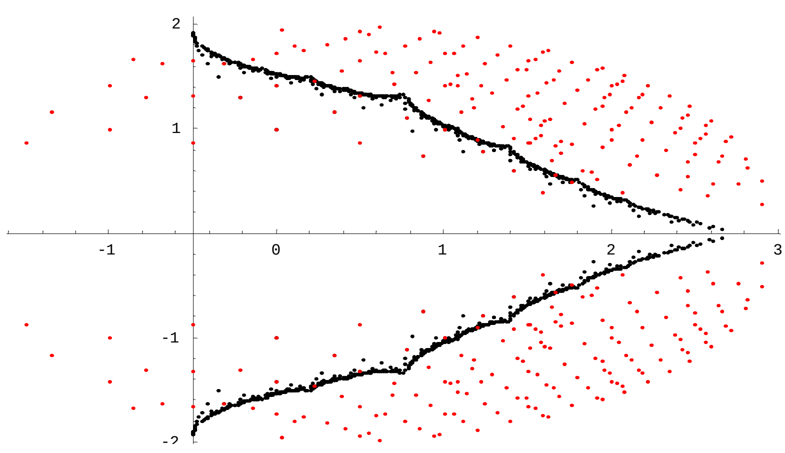


Fig. 5.17: The complex roots of Flammang and Rhin's polynomials with real parts in $[-1,0]$ and $(2,6)$ Dehn surgery data I

We first delete the points in Figure 5.17, whose distances from the $(2,6)$ Dehn surgery data are greater than 1 and outside the data. The rest of the points and their symmetric points with respect to -0.5 are shown in Figure 5.18.

By comparing with $(2, 6)$ Dehn surgery data, we found $(2, 6)$ arithmetic groups.

Theorem 5.3.1. *Let $G = \langle f, g \rangle$ be a Kleinian group generated by elements of orders 2 and 6, respectively. Then $par(G) = (\gamma, -1, -4)$ and Tables 5.12, 5.13 contains a list of γ such that G is arithmetic and $\mathbb{H}^3/G$ is an orbifold which is 2-fold covering obtained from $(6, 0)$ Dehn surgery on a two bridge knot or link of up to 12 crossings.*

Table 5.14 is from TABLEs 4 and 9 of [16]. The points in [16] that correspond to an arithmetic knot or link groups are in Table 5.12 or 5.13.

Figure 5.19 is a result of deleting the points in Tables 5.12, 5.9, 5.14 and their conjugates from the roots in Figure 5.18. As we shall see from Figure 5.19, except for the points $-0.5+1.6583123951777i$, $0.22669882575820183 + 1.4677115087102242i$ (green dots) and their conjugates, all the other points are outside the Dehn surgery data.

Conjecture 5.3.2. *Suppose $G = \langle f, g \rangle$ is an arithmetic Kleinian group generated by elements of orders 2 and 6 respectively. Then $par(G) = (\gamma, -1, -4)$. If the imaginary part of γ is in the interval $[-1.8, 1.8]$ then γ either appears in one of Tables 5.12, 5.13, 5.14, or is one of the points $-0.5 + 1.6583123951777i$, $0.22669882575820183 + 1.4677115087102242i$.*

Remark *If our methods have reproduced the boundary of $\mathscr{U}'$ with a reasonable degree of accuracy, then this conjecture is also valid.*

The interval $[-1.8, 1.8]$ comes from the point $-0.484424188457453+1.826090783122572i$, which is from the knot 11_{a247} (the green point in Figure 5.16). Cooper's computer program has confirmed that the volume of $0.22669882575820183 + 1.4677115087102242i$ is 5.990900. While $-0.5 + 1.6583123951777i$ lies on the extension of the rational pleating ray $\mathscr{P}_{1/2}$, so it may be a web group with infinite co-volume.

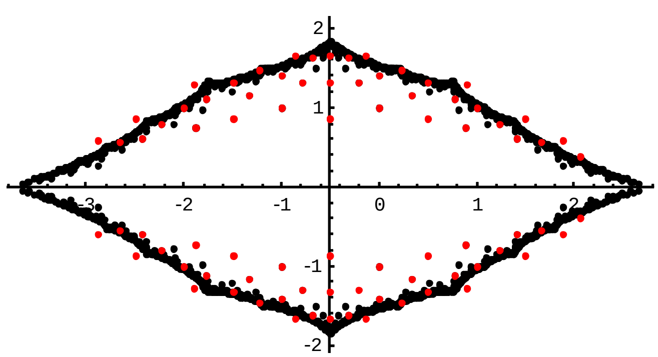


Fig. 5.18: The complex roots of Flammang and Rhin's polynomials with real parts in $[-1, 0]$ and $(2, 6)$ Dehn surgery data II

Tab. 5.12: (2, 6) Arithmetic knot groups

Knots	Polynomial	$\gamma-$ value	volume
	$1+z^2$	i	
$4-1$	$2+2z+z^2$	$-1-i$	0.61064372945
	$1-z^2+z^3$	$0.87743883318303-0.74486176647792i$	
$5-2$	$-1-5z-4z^2-z^3$	$-1.87743883318303+0.74486176647792i$	1.0212663095
	$1+2z+z^2+z^3$	$-0.21507985450633+1.30714127867195$	
$5-2$	$-1-3z-2z^2-z^3$	$-0.78492014549368-1.30714127867195i$	1.0212663095
	$2-2z^2+z^3$	$1.419643377618046-0.606290729215630i$	
$7-3$	$-1-7z-5z^2-z^3$	$-2.419643377618046+0.606290729215630i$	1.8534210792
	$1+z-z^2+z^3$	$0.771844506351892+1.115142508039886i$	
$7-3$	$-2-6z-4z^2-z^3$	$-1.771844506351892-1.115142508039886i$	1.8534210792
	$1+z-2z^2+z^3$	$1.232785615944199+0.792551992504666i$	
$7-4$	$-3-8z-5z^2-z^3$	$-2.232785615944199-0.792551992504666i$	2.11088888085
	$1+2z-3z^2+z^3$	$1.6623589786223731+0.5622795120623013i$	
11_{a333}	$-5-11z-6z^2-z^3$	$-2.662358978622373-0.5622795120623013i$	4.3993010265

Tab. 5.13: (2, 6) Arithmetic link groups

Knots	Polynomial	$\gamma-$ value	volume
	$1+z+z^3$	$0.341163901915408+1.161541399987772i$	
5_1^2	$-1-4z-3z^2-z^3$	$-1.341163901915408-1.161541399987772i$	1.3193055505
	$2+z^2$	$1.4142135623730951i$	
6_3^2	$3+2z+z^2$	$-1-1.4142135623730951i$	2.0076820065
	$2-2z+z^2$	$1+i$	
8_4^2	$5+4z+z^2$	$-2-i$	3.0532186475
	$1+3z+z^2+z^3$	$-0.319448459741747+1.633170240908653i$	
8_6^2	$-2-4z-2z^2-z^3$	$-0.680551540258253-1.633170240908653i$	2.6468597495
	$2-z+z^2$	$0.5+1.322875655532295i$	
8_7^2	$4+3z+z^2$	$-1.5-1.322875655532295i$	3.5556597115

Tab. 5.14: (2, 6) Arithmetic groups in [16]

Polynomial	$\gamma-$ value	volume
$1+z+z^2$	$-0.5+0.8660254037844388$	0.0845
$2+z+z^2$	$-0.5+1.3228756555322954i$	0.1691
$1-z+z^2$	$0.5+0.866025403784438i$	
$3+3z+z^2$	$-1.5-0.8660254037844386i$	0.1526

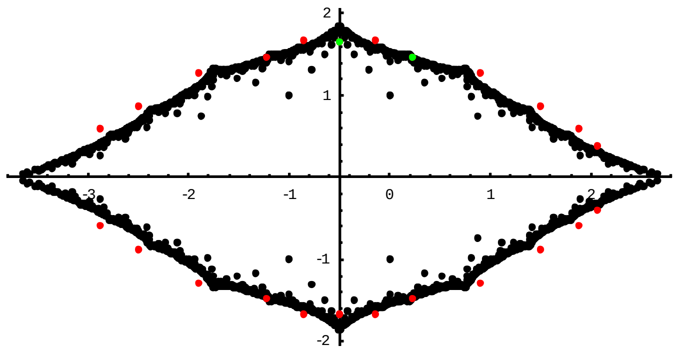


Fig. 5.19: The complex roots of Flammang and Rhin's polynomials with real parts in $[-1, 0]$ and $(2, 6)$ Dehn surgery data III

6. CONCLUSION AND FURTURE RESEARCH

The most interesting result of this thesis should be the Dehn surgery data on 2-bridge knots and links. It has led to the generalization of the rational pleating rays to parameter spaces of Kleinian groups with two elliptic generators and the discovery of the boundaries in these parameter spaces. It has also led to a fairly convincing identification of all the arithmetic Kleinian groups generated by elements of order $2\&3, 2\&4$ and $2\&6$. Unfortunately, the boundary images are not perfect. We discuss in this chapter what could be done to improve them in the future, as well as some research possibilities.

6.1 Future Research

Let $\rho \in \mathscr{P}_{p/q}$. Keen and Series have proved [35, page 79-82] that G_ρ contains exactly two non-conjugate *F-peripheral* groups. An *F-peripheral* group is a Fuchsian group such that one of the disks bounded by its invariant circle contains no limit points of G_ρ. The disk that is disjoint from $\Lambda(G_\rho)$ is called the *peripheral disk* of F and is denoted by $\Delta(F)$.

Let F_1 and F_2 be two F-peripheral subgroups of G. The invariant circles c_1 and c_2 of F_1 and F_2, respectively, intersect in the fixed points of $W_{p/q}$. The arc σ_1 in c_1 between the fixed points of $W_{p/q}$ that is contained in $\Delta(F_2)$ contains no limit points of $\Lambda(G_\rho)$. The same is true for the arc σ_2 in c_2 between the fixed points of $W_{p/q}$. The intersection of c_i with $\Lambda(G_\rho)$ is the limit set of F_i. So F_i is a Fuchsian group of second kind, which means the limit set of F_i is not the whole circle. $\Lambda(G_\rho)$ is the union of the limit sets of F_1, F_2 and the images of these limit sets.

At the endpoint ρ_0 of a pleating ray $\mathscr{P}_{p/q}$, $W_{p/q}$ becomes parabolic. So c_1 is tangent to c_2. Let G_{ρ_0} be the corresponding group . By Theorem 1 of [33], the limit set of G_ρ is circle packing; that is, every component of its regular set is a round disc. As $\mathscr{P}_{p/q}$ crosses the boundary into the inside of the "eye" in Riley's slice, $\left|Tr(W_{p/q})\right| < 2$. Therefore if G_ρ is discrete then $W_{p/q}$ is an elliptic of order n. Keen predicts that in this case, G_ρ is again a direct product of F_1, F_2 and c_1 is still tangent to c_2. Hence the limit set of G_ρ is circle packing. She is in the process of verifying this claim.

Recall that our assumption is that the results on Kleinian groups with two parabolic generators work in general on Kleinian groups with two elliptic generators. Suppose $\mathscr{P}_{p/q} \in \mathscr{U}$ and G_μ is a

Kleinian group on the extension of $\mathscr{P}_{p/q}$, which is in the complement of $\mathscr{U} \in \mathbb{C}$. Then G_μ is a direct product of F_1', F_2', where

$$F_1' = \langle a, W_{p/q} : a^m = (W_{p/q})^n = identity \rangle$$

and

$$F_2' = \langle b, W_{p/q} : b^m = (W_{p/q})^n = identity \rangle .$$

Hence $G_\mu = \langle a, b : a^m = b^m = (W_{p/q})^n = identity \rangle$ and G_μ is a *generalised triangle group*. According to Keen's claim, the limit set of G_μ is circle packing hence it is a *web group*. This implies that there are more web groups in the compliment of $\mathscr{U}$ than Martin previously thought (personal communication).

A *web group* is a finitely generated Kleinian group whose components are bounded by Jordan curves [49, page 687]. In contrast to the orbifold groups we have discussed in this thesis, a web group is a kleinian group of the second kind; that is, its ordinary set is not empty. Therefore it has infinite co-volume (Theorem 12.1.15, [23]).

The commutator parameter of a $\mathbb{Z}_2$-extension of a web group lies on the rational pleating rays $\mathscr{P}'_{p/q}$ in a $(2, m)$ commutator plane. The co-volume of a $\mathbb{Z}_2$-extension of a web group is infinite; but not every group on the extension of $\mathscr{P}'_{p/q}$ has infinite co-volume. In [2], Williams finds 21 generalised triangle groups, which have a 2-bridge knot or link with an unknotting tunnel as their singular sets. He proved that these are arithmetic Kleinian groups. An unknotting tunnel is an arc in the 3-sphere that begins and ends on a given knot or link but does not intersect with the knot [8, page 617]. Every point in Figure 3.15 is on a rational pleating ray. For example, the point $-2.4196 + i0.6063$ is on $\mathscr{P}_{1/3}$. See Figure 6.1. Therefore it is worthwhile to study the relationship between the generalised triangle groups and unknotting tunnels of 2-bridge knots and links.

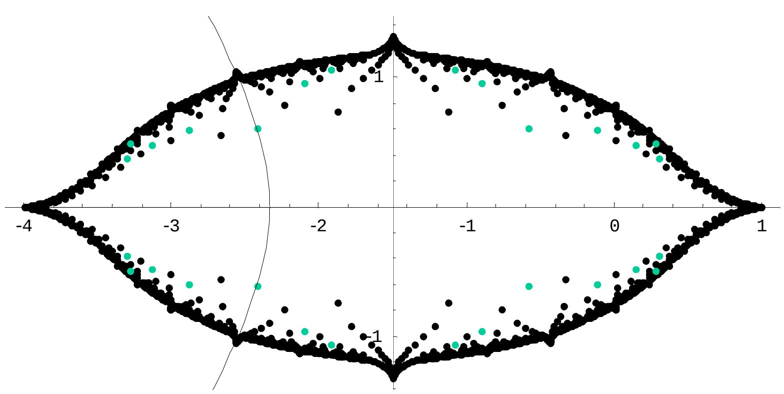


Fig. 6.1: The point $-2.4196 + 0.6063i$ and $\mathscr{P}_{1/3}$

Ohshika and Miyachi have proved (Theorem 1.2, [29]) that the boundary of the Riley slice $\mathscr{R}$

in $\mathbb{C}$ is a Jordan curve. We believe this is also true for $\mathscr{U}$. The techniques used in [29] could be used to explore a proof for this.

$(3,4)$ and $(3,5)$ Dehn surgery have been done for two bridge links up to 11 crossing. It is intended to draw diagrams for two bridge links with more than 12 crossings and do $(3,4)$ and $(3,5)$ to improve the pictures for $(3,4)$ and $(3,5)$ commutator planes.

Another way to improve $(3,4)$ and $(3,5)$ commutator planes is to generate as many rational words as we could in a reasonable time. But since the trace polynomials are getting more and more complicated as the length of rational words is getting longer, it is not likely we could get satisfactory pictures using trace polynomials. This is also true for any (n,m) commutator plane. Therefore it would be desirable to have a program to generate the boundary points without solving trace polynomials. This technique has been used in [12] to plot the boundary of the Maskit slice [12, page 288]. This will be explored further.

A $(3,4)$ (or $(3,5)$) commutator plane is quite different from a $(2,n)$ commutator plane. An interesting question is: what does a rational pleating ray look like in $(3,4)$ commutator plane?

In Chapter 4, we have described a conjectural connection between a 2-bridge notation of knots and links and rational pleating rays. There should be a theoretical explanation waiting to be discovered.

Dehn surgery data in this thesis has been obtained by drawing a 2-bridge knot in SnapPea and repeatedly clicking its retriangulation link by hand. To get suitable commutator for a knot group, the link had to be clicked many times. It was very time consuming. It worthwhile to explore the possibilities of a computer program for such a job.

APPENDIX

A. SOME BASICS IN TOPOLOGY

All the definitions and theorems discussed in this section are from [26] or [23].

Definition A.0.1. *A metric space is a set X together with a metric d on X.*

A function $f : X \to Y$ between metric spaces *preserves distance* if and only if

$$d_Y(f(x), f(y)) = d_X(x, y) \text{for all } x, y \text{ in} X.$$

Definition A.0.2. *An isometry from a metric space X to a metric space Y is a distance preserving bijection $f : X \to Y$.*

A bijection is a one-to-one and onto function. The inverse of an isometry is an isometry and the composite of two isometries is an isometry. Two metric space X and Y are said to be *isometric* if and only if there is an isometry $f : X \to Y$.

Definition A.0.3. *A topology on a set X is a collection $\Im$ of subsets of X having the following properties:*

(1) $\emptyset$ and X are in $\Im$.

(2) The union of the elements of any subcollection of $\Im$ is in $\Im$.

(3) The intersection of the elements of any finite subcollection of $\Im$ is in $\Im$

A set X for which a topology $\Im$ has been specified is called a ***topological space*** ([26], page 76).

Definition A.0.4. *Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is said to be a **quotient map** provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.*

Definition A.0.5. *If X is a space and A is a set and if $p : X \to A$ is a surjective map, then there exists exactly one topology $\Im$ on A relative to which p is a quotient map; it is called the **quotient topology** induced by p.*

Definition A.0.6. *Let X be a topological space and define an equivalence relation $\mathcal{R}$ on X. Let X^* be a partition of X, according to $\mathcal{R}$, into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a quotient space of X.*

Definition A.0.7. *A topological space X is called a **Hausdorff space** if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1, and U_2 of x_1 and x_2, respectively, that are disjoint.*

Definition A.0.8. *An n-manifold (without boundary) is a Hausdorff space M that is locally homeomorphic to E^n, that is for each point u of M, there is an open neighborhood U of u in M such that U is homeomorphic to an open subset of E^n.*

A compact n-manifold M is an n-manifold such that every sequence of points of M has a convergent subsequence. The orientability of a n-manifold measures whether one can consistently choose a "clockwise" orientation for all loops in the manifold ([51])

Definition A.0.9. *Let X and Y be topological spaces; let $p : X \to Y$ be a bijection. If both the function p and the inverse function*

$$p^{-1} : X \to Y$$

*are continuous, then f is called a **homeomorphism**.*

Definition A.0.10. *Let $p : E \to B$ be a continuous surjective map. The open set U of B is said to be **evenly covered** by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ will be called a partition of p^{-1} into slices.*

Definition A.0.11. *Let $p : E \to B$ be continuous and surjective. If every point b of B has a neighborhood U that is evenly covered by p, then p is called a **covering map**, and E is said to be a **covering space** of B.*

Let X be a topological space and $f : [0, 1] \to X$ be a continuous map such that $f(0) = x_0$ and $f(1) = x_1$. We say f is a path in X from x_0 to x_1, where x_0 is the initial point and x_1 is the final point of the path f.

Definition A.0.12. *Two paths f and f', mapping the interval $I = [0, 1]$ into X, are said to be path homotopic if they have the same initial point x_0 and the same final point x, and if there is a continuous map $F : I \times I \to X$ such that*

$$F(s, 0) = f(s) \quad \textit{and} \quad F(s, 1) = f'(s),$$

$$F(0, t) = x_0 \quad \textit{and} \quad F(1, t) = x_1,$$

for each $s \in I$ and each $t \in I$. We call F a path homotopy between f and f'. See Figure A.1. If f is path homotopic to f we write $f \cong_p f'$.

The relation $\cong_p$ is an equivalence relation, e.g., $f \cong_p f$; $f' \cong_p f$; if $f \cong_p f'$, $f' \cong_p f''$, then

$$f \cong_p f''.$$

If f is a path, we denote its path-homotopy equivalence class by $[f]$.

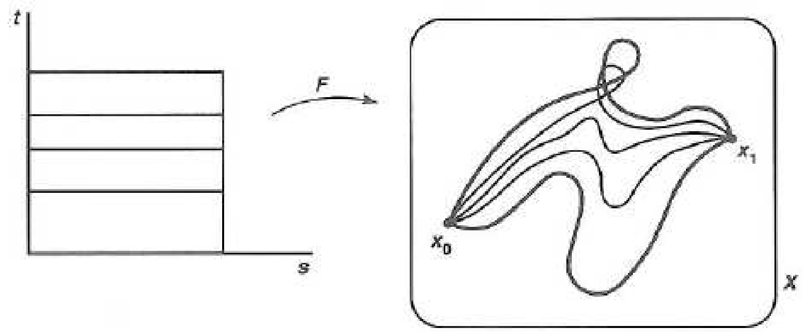


Fig. A.1: path homotopy

Definition A.0.13. *If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2, we define the product $f * g$ of f and g to be the path h given by the equations $h(s) = f(2s)$ for $s \in [0, 1/2]$ and $h(s) = g(2s - 1)$ for $s \in [1/2, 1]$.*

The function h is a path in X from x_0 to x_2 whose first half is the path f and whose second half is the path g. The product operation $*$ can be extended to an operation on path-homotopy classes, which is

$$[f] * [g] = [f * g].$$

B. GENERATING TRACE POLYNOMIALS OF RATIONAL WORDS

$L1 = \{1/2,\ 1/1\}$; (*$L1$ *is a list of rational numbers**)

$L2 = \{\{X, Y, x, y\},\ \{X, y\}\}$;

(*$L2$ *is a list of rational words. Instead of writing them as dot products I write a rational word as a list of Xs and Ys and do the dot products later**)

$\text{Timing}[\text{For}[j = 1, j < 8, j++,$

(*j *determines the number of rational numbers(or words) being generated**)

$d = \text{Length}[L1];$

$\text{For}[i = 1, i < (2d - 1), i = i + 2,$

$\text{M1}=\frac{(\text{Numerator}[L1[[i]]]+\text{Numerator}[L1[[i+1]]])}{(\text{Denominator}[L1[[i]]]+\text{Denominator}[L1[[i+1]]])}$

(**Farey addition to get a rational number from two known ones**)

$L1 = \text{Insert}[L1, M1, i + 1];$

$T2 = \text{Join}[L2[[i]], L2[[i + 1]]];$

$d1 = \text{Denominator}[M1] + 1;$

$R = \text{Switch}[T2[[d1]], x, X, X, x, y, Y, Y, y];$

$T2 = \text{ReplacePart}[T2, R, d1];$

$L2 = \text{Insert}[L2, T2, i + 1]]]]$

(**We now have a certain number of rational words in terms of* X, Y, x, y *and we define* X, Y, x, y *as follows**)

$$X = \begin{pmatrix} e^{i\pi/m} & 1 \\ 0 & e^{-i\pi/m} \end{pmatrix} \qquad Y = \begin{pmatrix} e^{i\pi/m} & 0 \\ \mu & e^{-i\pi/m} \end{pmatrix}$$

$x = \text{Inverse}[X] \qquad y = \text{Inverse}[Y]$

(**We then find the traces of these rational words. A rational word can be a dot product of many matrices of* X, Y, x, y. *When the number of matrices is large enough, for example 15 if* X, Y *are elliptic of order* 3, *it takes very long time for computer to calculate the dot product. So basically the following code is , first to break the matrices in a rational words into smaller groups, for example 15 matrices in a group and calculate each group and then calculate the results from each group* *)

```
Q = {};
partblock = 5;
blocksize = 15;
im = {{1,0},{0,1}};
myDot[v_] := Module[{vb},
            If[Length[v] > blocksize,
            vb = Partition[ Join[v, Table[im, Abs[Mod[Length[v], -partblock]]]],
            Ceiling[Length[v]/partblock]];
            Simplify[Apply[Dot, Map[myDot, vb]], TimeConstraint → Infinity],
            Simplify[Apply[Dot, v], TimeConstraint → Infinity] ] ]
```

(**I credit this myDot function to Don Taylor, whom I know through Mathgroup. You can send any question you have to "mathgroup@smc.vnet.net" and with some luck, you would get some very good advice**)

```
Timing[For[j = 1, j <Length[L2] + 1, j + +,
    Clear[f1];
    f1 = Simplify[Tr[myDot[L2[[j]]]], TimeConstraint → Infinity];
    Q = Append[Q, f1]]]
```

(**After the above code, we have a number of trace polynomials, we then solve these polynomials and plot the results*)*

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INDEX