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# BOUNDS ON THE ARITHMETIC DEGREE

A THESIS PRESENTED IN PARTIAL FULFILMENT  
OF THE REQUIREMENTS FOR  
THE DEGREE OF MASTER OF SCIENCE IN  
MATHEMATICS AT  
MASSEY UNIVERSITY

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1996

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## Abstract

In this thesis we study the arithmetic degree theory of polynomial ideals. The main objectives are: (i) to show whether we can generalize a lower bound on the arithmetic degree of monomial ideals to the arithmetic degree of arbitrary homogeneous ideals; and (ii) to explain whether some known bounds for the geometric degree can be restated in terms of bounds on the arithmetic degree. We give a negative answer to all questions raised by constructing counterexamples. In some cases we provide a general method for constructing such counterexamples. Concerning properties of the arithmetic degree, we give a new Bezout-type theorem. Finally we take a brief look at open problems concerning the arithmetic degree under hypersurface sections.

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## Acknowledgements

- This thesis is dedicated to the memory of my supervisor Professor Wolfgang Vogel. His expertise, guidance and enthusiasm were an inspiration. He will be greatly missed.
- My special thanks to Dr. Le Tuan Hoa for his careful critical reviews regarding this work. The final manuscript was substantially improved by these reviews.
- I would like to thank Dr. Kee Teo for taking over the position as my supervisor, with all that entails, at such a late stage. I am very grateful.
- Thanks to Mark Johnston for the help he has given me in learning to use text editors and the appropriate compilers. His patience is appreciated.
- I am also grateful for all of the encouragement and support given to me by fellow residents of Rm 2.40, in particular to Nick, Mary, and Anton.
- Thanks to all of the staff in the Department of Mathematics at Massey University for creating a friendly and supportive environment in which to work.
- I could not begin to put into words the thanks owed to my husband Brian. Thank you for making my world a much better place over the past 7 years.
- Finally, I would like to thank my parents Tom and Kathy for all the support they have given me, both financially and emotionally, in my educational career and otherwise.

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## Table of Contents

Abstract	iii
Acknowledgements	iv
Table of Contents	v
<b>0 Introduction</b>	<b>1</b>
0.1 Degrees of an ideal . . . . .	1
0.2 This work: an outline . . . . .	3
<b>1 Preliminaries and Notation</b>	<b>5</b>
1.1 Ideals . . . . .	5
1.2 Primary Decomposition . . . . .	6
1.3 Monomial Ideals, Their Primary Decomposition, and the Splitting Lemma . . . . .	6
1.4 Dimension . . . . .	9
1.5 Length-Multiplicity . . . . .	10
1.6 Degree, Geometric Degree, and Arithmetic Degree . . . . .	12
<b>2 On a lower bound for the arithmetic degree</b>	<b>14</b>
<b>3 Remarks on the division problem and Bezout-type theorems</b>	<b>17</b>
3.1 Introduction . . . . .	17
3.2 Remark on Hilbert Nullstellensatz . . . . .	18

3.3	Remark on a Bezout-type theorem, Part I . . . . .	22
3.4	Remark on a Bezout-type theorem, Part II . . . . .	23
4	A Bezout-type theorem for the arithmetic degree	26
4.1	Introduction . . . . .	26
4.2	New Results . . . . .	27
5	Arithmetic degree under hyperplane section	32
6	Conclusions and suggestions for future work	36
Appendix A	Some relationships between geometry and algebra	38
Appendix B	Computer output	40
Index		47
Bibliography		49

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# Introduction

## 0.1 Degrees of an ideal

A degree of an ideal  $I$  in a polynomial ring  $S$  is an integer which best represents  $I$  in  $S$ . In some sense it is the measure of the size of  $I$  over  $S$ . Which integer best represents  $I$  depends on what information we are trying to extract from  $I$ . In general, a degree of  $I$  is formed by attaching a cycle  $div(I) = \sum_{i=1}^s l_i [p_i]$  to  $I$ , where  $l_i$  are positive integers and  $[p_i]$  are prime ideals, both derived from  $I$ . We then associate a degree to  $div(I)$ .

The oldest and most studied notion of the degree of an ideal  $I$  is the classical degree, denoted by  $deg(I)$ . In this degree,  $l_i$  is the length-multiplicity,  $mult_I(p_i)$ , of the primes  $p_i$  of  $dim(S/I)$  in the primary decomposition of  $I$ , weighted by the degree of  $p_i$ . That is,

$$deg(I) = \sum_{i=1}^s mult_I(p_i) \cdot deg(p_i).$$

As an invariant the classical degree has had many uses, one of which has been to prove several Bezout-type theorems which provide information about the number of intersection components of an ideal's associated projective subvariety in  $n$ -space. (For details on relationships between geometric and algebraic notions, please see Appendix A.)

Computational complexity issues in algebraic geometry necessitate refinements in this notion of the degree. Two such refinements are the geometric degree and the arithmetic degree. The geometric degree of an ideal  $I$  is given by an expression

similar to that of the degree of  $I$ , however, it takes into consideration contributions made by all of the minimal prime ideals of  $I$ . The arithmetic degree of  $I$  is the most recent of the notions of the degree and it represents contributions made by all prime ideals.

These new notions of degree were initially developed to provide new bounds for invariants which could not be bound by the classical degree. For example, the classical degree was not helpful in providing an upper bound for the reduction number of an ideal as it proved not to be large enough. In [Vas96], the author has used the arithmetic degree to provide such an upper bound.

The geometric degree and arithmetic degree have also been used to improve bounds already given by the classical degree. Improved bounds have been found for several invariants. For example, in [BM93], the authors provide the following lower bound to  $d(I)$ , the maximum of the degrees of a minimal set of generators of  $I$ .

**PROPOSITION 0.1.1** *Let  $k$  be any field and let  $I \subset k[x_0, \dots, x_n]$ . Then*

$$\text{geom-deg}(I) \leq \sum_{i=0}^{n+1} d(I)^i.$$

It is shown in [BM93] that the arithmetic degree cannot be used to further improve this bound. However, the arithmetic degree is used to provide the following improved lower bound for the Castelnuovo-Mumford regularity of  $I$ ,  $m(I)$ .

**THEOREM 0.1.2** *Let  $k$  be any field and let  $I \subset k[x_0, \dots, x_n]$ . Then*

$$\text{arith-deg}(I) \leq \sum_{i=0}^{n+1} \binom{m(I) + i - 1}{i} \leq \sum_{i=0}^{n+1} m(I)^i.$$

[STV95] provides us with an improvement on the Hilbert Nullstellensatz using the geometric degree.

Once developed to provide new and improved bounds on invariants, it was noted that this new idea of the arithmetic degree was interesting in its own right. It takes into consideration contributions made by all associated primes of an ideal and may therefore provide us with more information about the ideal than the geometric degree or the classical degree.

In studying properties of the arithmetic degree we have returned to problems studied for the classical degree and looked at how the arithmetic degree can be used

in these problems. Such problems include: finding upper and lower bounds for the arithmetic degree; studying the arithmetic degree under hypersurface section; and studying properties of the arithmetic degree of the sum of two ideals, that is, studying Bezout-type theorems for the arithmetic degree. One or more of each of these problems has been studied in each of the following papers on arithmetic degrees: [STV95], [MV96], [MVY96].

## 0.2 This work: an outline

[BM93] raises the question of finding bounds on the geometric degree and the arithmetic degree of an ideal in terms of its generators. [STV95] proves the following lower and upper bounds on the arithmetic degree of a monomial ideal  $I$  in terms of generators of  $I$ .

**THEOREM 0.2.1** *Let  $I$  be a proper monomial ideal in  $S = k[x_1, \dots, x_n]$  with minimal set of monomial generators  $m_1, m_2, \dots, m_s$ , and let  $e := \dim(I) + s - n$ . Then*

$$\max\{\deg(m_i) : i = 1, \dots, s\} \leq \text{arith-deg}(I) \leq \prod_{i=1}^s \deg(m_i) - e.$$

It is shown that the upper bound in this theorem does not generalize to arbitrary homogeneous ideals. In Chapter 2 of this thesis we show that the lower bound also does not generalize to arbitrary homogeneous ideals. Hence, the question of bounding the arithmetic degree of an arbitrary homogeneous ideal in terms of its generators remains open.

In Chapter 3 we look at bounds proven for the geometric degree, as taken from Section 5 of [STV95], and show that similar bounds do not hold in general for the arithmetic degree. In Section 3.2 we look at an improvement on the Hilbert Nullstellensatz which uses the geometric degree. Through the detailed calculation of a counterexample, we show that the arithmetic degree cannot be used to further improve the given bound. In Section 3.3 we look at a Bezout-type theorem for the geometric degree and show through a counterexample that in this Bezout-type theorem, the geometric degree cannot be replaced by the arithmetic one. Finally, in Section 3.4 we look at an algebraic version of a corollary to the Bezout-type theorem given in Section 3.3 and show that once again the geometric degree cannot be replaced by the arithmetic one. Each of these cases illustrates a need for study of the relationship between the geometric and arithmetic degrees.

In Section 0.1 we stated that one problem used to determine properties on the arithmetic degree was the problem of finding Bezout-type theorems for the arithmetic degree. [MV96] provides a Bezout-type result for the arithmetic degree via iterated hypersurface sections. In Chapter 4 we use properties of the arithmetic degree as discussed in Section 3.4 to provide another Bezout-type theorem and two corollaries. There the ideals are assumed to be pure dimensional.

In Chapter 5 we consider Bezout-type theorems for the arithmetic degree of an arbitrary scheme cut by a hypersurface; that is, we study the arithmetic degree under hypersurface section. We take a brief look at an open problem on the arithmetic degree under hypersurface section and use Corollary 2.7 of [MVY96] to provide a new result concerning the arithmetic degree under hypersurface section.

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# Preliminaries and Notation

As stated in Chapter 0, this thesis is concerned with the arithmetic degree of an ideal and with the bounds on such a degree. Chapter 1 is an assembly of the basic definitions and results (without proofs) needed to define this degree. It is intended as a review and for quick reference. Further details can be found in the sources: [Nor53], [Eis95], [ZS58], [ZS60], and [BM93], where a general ring theory and its connection to algebraic geometry is given.

For the purpose of this thesis we are interested mainly in those definitions and results which relate to ideals in the polynomial ring in  $n$  variables. Therefore, throughout this chapter let  $S := k[x_1, \dots, x_n]$ , where  $k$  is a field, be a polynomial ring in  $n$  variables.

The arithmetic degree of an ideal is defined in terms of ideals, their primary decompositions, and the length multiplicities of associated primes. Hence, we begin by reviewing the definition of an ideal.

## 1.1 Ideals

An *ideal*  $I$  in a commutative ring  $R$  is an additive subgroup of  $R$  such that if  $r \in R$  and  $s \in I$  then  $rs \in I$ . An ideal  $I$  in the polynomial ring  $S$  is defined to be *homogeneous* if it is generated by homogeneous polynomials of  $S$ , that is, if there are *forms* (homogeneous polynomials)  $f_1, \dots, f_s \in S$  such that any element  $a \in I$  can be written in the form  $a = f_1g_1 + \dots + f_s g_s$ , where  $g_1, \dots, g_n \in S$ . In this case we write  $I = (f_1, \dots, f_s)$ . We study here only homogeneous ideals in  $S$ .

An ideal  $P$  in  $R$  is said to be *prime* if  $P$  is proper and if  $ab \in P$  implies  $a \in P$  or  $b \in P$  for  $a, b \in R$ . A proper ideal  $Q$  in  $R$  is a *primary ideal* if for  $a, b \in R$ ,  $ab \in Q$  and  $a \notin Q$  we have  $b^r \in Q$  for some  $r > 0$ . All prime ideals are primary. The *radical* of an ideal  $Q$  is defined to be  $\sqrt{Q} = \{x : x^r \in Q \text{ for some } r \in \mathbb{Z}^+\}$ . The radical of a primary ideal is prime. If  $Q$  is a primary ideal and  $P = \sqrt{Q}$ , then we say that  $Q$  is  *$P$ -primary*.

## 1.2 Primary Decomposition

As stated previously in order to define the arithmetic degree we must first define primary decomposition. The theory of primary decomposition states that in  $S := k[x_1, \dots, x_n]$  a homogeneous ideal  $I$  can be decomposed into an intersection of primary ideals:  $I = \bigcap_{i=1}^r Q_i$ . If none of the  $Q_i$  can be dropped, the decomposition is said to be *irredundant*. An irredundant decomposition in which the  $\sqrt{Q_i} = P_i$  are distinct is called a *normal decomposition*. Every primary decomposition can be refined into one which is normal. Therefore, from now on we assume that all primary decompositions are normal. Although an ideal may have many normal primary decompositions, the set of all prime ideals  $P_i = \sqrt{Q_i}$  in a primary decomposition of  $I$  depends only on  $I$ . This set is denoted by  $\text{Ass}(S/I)$  and its elements are called *associated prime ideals belonging to  $I$* . Note that all  $P_i, Q_i$  raised in this way are homogeneous.

Let  $I$  be an ideal in  $S$  and  $P$  a prime ideal belonging to  $I$ .  $P$  is said to be a *minimal prime ideal* of  $I$  if no other prime ideal belonging to  $I$  is strictly contained in  $P$ . The minimal prime ideals of  $I$  are said to be *isolated* and the prime ideals of  $I$  which are not minimal are said to be *embedded*.

## 1.3 Monomial Ideals, Their Primary Decomposition, and the Splitting Lemma

When finding examples and counterexamples to problems which require a primary decomposition of an ideal, it is often simplest to work with monomial ideals (when possible). The splitting lemma is a useful tool for finding the primary decomposition of a monomial ideal. With a few tricks, it may also be used to find the primary decomposition of some homogeneous ideals generated by monomials and a linear

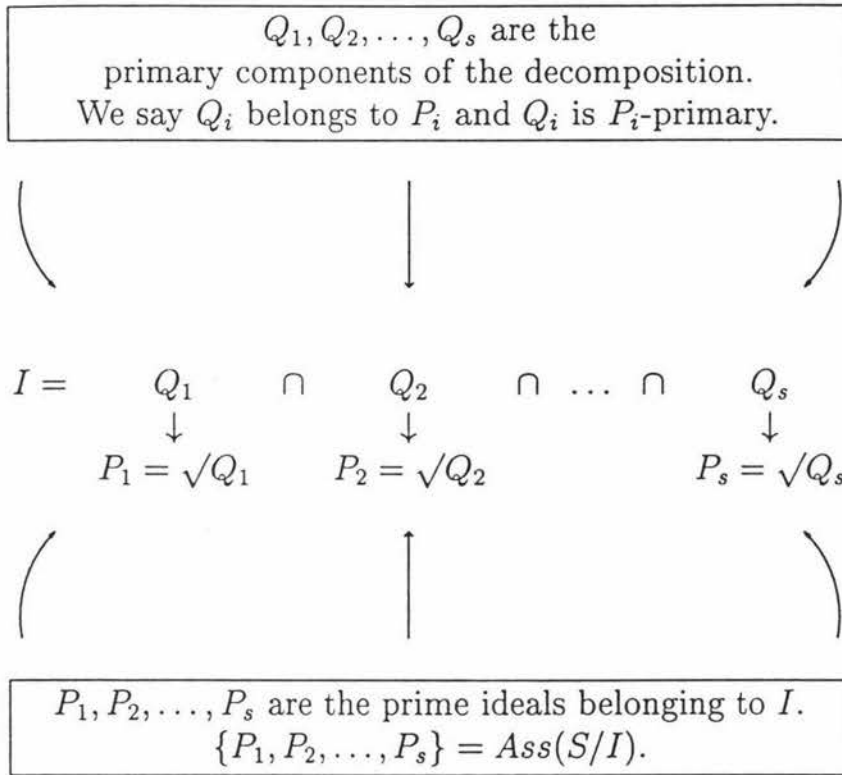


Figure 1.1: Primary decomposition of an ideal  $I$ .

form.

A *monomial* is a product of variables:  $a_j x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$ , where  $i_1, i_2, \dots, i_n$  are non-negative integers and  $a_j$  is any element of  $k$ . A *monomial ideal* is then an ideal generated by monomials. That is,  $I = (m_1, \dots, m_s)$ , where  $m_l$  are monomials for all  $l = 1, \dots, s$ .

The following propositions describe the structure of monomial prime and primary ideals.

**PROPOSITION 1.3.1** *Let  $P$  be a monomial ideal in  $k[x_1, \dots, x_n]$ . Then  $P$  is a prime ideal if and only if  $P = (x_{i_1}, \dots, x_{i_r})$ , where  $i_j \in \{1, \dots, n\}$  for  $j = 1, \dots, r$ .*

**PROPOSITION 1.3.2** *Let  $P = (x_{i_1}, \dots, x_{i_r})$ ,  $i_j \in \{1, \dots, n\}$  for  $j = 1, \dots, r$ , be a prime ideal. Then a monomial ideal  $Q$  of  $k[x_1, \dots, x_n]$  is  $P$ -primary if and only if  $Q = (x_{i_j}^{\rho_j}, \dots, x_{i_r}^{\rho_r}, m_1, m_2, \dots, m_s)$  where  $\rho_j \geq 1$  for  $j = 1, \dots, r$  and  $m_l$  are monomials in  $x_{i_1}, \dots, x_{i_r}$  for  $l = 1, \dots, s$ .*

Two monomials  $\lambda$  and  $\tau$  are said to be *relatively prime* if, when  $\lambda = x_{i_0}^{n_{i_0}} \dots x_{i_j}^{n_{i_j}}$  and  $\tau = x_{k_0}^{m_{k_0}} \dots x_{k_r}^{m_{k_r}}$ , where  $n_{i_0}, \dots, n_{i_j}, m_{k_0}, \dots, m_{k_r} > 0$ , then  $\{x_{i_0}, \dots, x_{i_j}\} \cap$

$\{x_{k_0}, \dots, x_{k_r}\} = \emptyset$ . The following lemma gives an algorithm for finding the primary decomposition of monomial ideals.

**LEMMA 1.3.3 (Splitting Lemma)** *Let  $\lambda, \tau, m_1, \dots, m_s$  be monomials in  $S := k[x_1, \dots, x_n]$ . If  $\lambda$  and  $\tau$  are relatively prime, then*

$$(\lambda \cdot \tau, m_1, \dots, m_s) = (\lambda, m_1, \dots, m_s) \cap (\tau, m_1, \dots, m_s).$$

We now look at an example illustrating how to find a primary decomposition of a monomial ideal using the splitting lemma.

**EXAMPLE 1.3.4** Let  $I = (x^2y, x^2z, xy^2, xyz^2) \subset k[x, y, z]$  where  $k$  is a field. Then, by several applications of Lemma 1.3.3, a primary decomposition of  $I$  is found as follows.

$$\begin{aligned} I &= (x^2y, x^2z, xy^2, xyz^2) \\ &= (x^2, x^2z, xy^2, xyz^2) \cap (y, x^2z, xy^2, xyz^2) && \text{(by Lemma 1.3.3)} \\ &= (x^2, xy^2, xyz^2) \cap (y, x^2z) \\ &= (x^2, x, xyz^2) \cap (x^2, y^2, xyz^2) \cap (y, x^2) \cap (y, z) && \text{(by 2 applications} \\ &&& \text{of Lemma 1.3.3)} \\ &= (x) \cap (x^2, y^2, xyz^2) \cap (y, x^2) \cap (y, z) \\ &= (x) \cap (x^2, y^2, x) \cap (x^2, y^2, yz^2) \cap (y, x^2) \cap (y, z) && \text{(by Lemma 1.3.3)} \\ &= (x) \cap (x, y^2) \cap (x^2, y^2, yz^2) \cap (y, x^2) \cap (y, z) \\ &= (x) \cap (x^2, y^2, yz^2) \cap (y, x^2) \cap (y, z) && \text{(as } (x) \subset (x, y^2)) \\ &= (x) \cap (x^2, y^2, y) \cap (x^2, y^2, z^2) \cap (y, x^2) \cap (y, z) && \text{(by Lemma 1.3.3)} \\ &= (x) \cap (y, z) \cap (x^2, y) \cap (x^2, y^2, z^2). \end{aligned}$$

By Proposition 1.3.2,  $(x)$ ,  $(y, z)$ ,  $(x^2, y)$  and  $(x^2, y^2, z^2)$  are all primary and the associated primes of  $I$  are  $(x)$ ,  $(y, z)$ ,  $(x, y)$  and  $(x, y, z)$ . No primary component may be dropped from this intersection and the primes are all distinct. Therefore we have a normal primary decomposition for  $I$ . Also, we have  $(x)$  and  $(y, z)$  are isolated, and  $(x, y)$  and  $(x, y, z)$  are embedded.  $\square$

**EXAMPLE 1.3.5** In this example we look at how the splitting lemma may be used to find a primary decomposition of a homogeneous ideal generated by monomials and a linear form.

Let  $S = k[x_1, \dots, x_5]$  and

$$I = (x_1^2x_3, x_1^2x_4, x_1^2x_5, x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1 - x_3) \subset S.$$

Let  $I^* = I/(x_1 - x_3) \hookrightarrow k[x_2, \dots, x_5]$ . Then

$$I^* = (x_3^3, x_3^2x_4, x_3^2x_5, x_2x_3^2, x_2x_3x_4, x_2x_3x_5)$$

which is monomial. Therefore, we can use the splitting lemma on  $I^*$  to obtain the following.

$$\begin{aligned} I^* &= (x_3^3, x_3^2x_4, x_3^2x_5, x_2x_3^2, x_2x_3x_4, x_2x_3x_5) \\ &= (x_3^2, x_2x_3x_4, x_2x_3x_5) \cap (x_3^3, x_4, x_3^2x_5, x_2x_3^2, x_2x_3x_5) \\ &= (x_2, x_3^2) \cap (x_3^2, x_3x_4, x_2x_3x_5) \cap (x_3^2, x_4, x_2x_3x_5) \cap (x_3^3, x_4, x_5, x_2x_3^2) \\ &= (x_2, x_3^2) \cap (x_2, x_3^2, x_3x_4) \cap (x_3^2, x_3x_4, x_3x_5) \cap (x_2, x_3^2, x_4) \cap (x_3^2, x_4, x_3x_5) \\ &\quad \cap (x_3^3, x_2, x_4, x_5) \cap (x_3^2, x_4, x_5) \\ &= (x_2, x_3^2) \cap (x_3) \cap (x_3^2, x_4, x_5) \cap (x_3, x_4) \cap (x_3^3, x_2, x_4, x_5) \\ &= (x_3) \cap (x_2, x_3^2) \cap (x_3^2, x_4, x_5) \cap (x_3^3, x_2, x_4, x_5). \end{aligned}$$

Therefore,

$$\begin{aligned} I &= (x_3, x_1 - x_3) \cap (x_2, x_3^2, x_1 - x_3) \cap (x_3^2, x_4, x_5, x_1 - x_3) \\ &\quad \cap (x_3^3, x_2, x_4, x_5, x_1 - x_3) \\ &= (x_1, x_3) \cap (x_2, x_3^2, x_1 - x_3) \cap (x_3^2, x_4, x_5, x_1 - x_3) \cap (x_3^3, x_2, x_4, x_5, x_1 - x_3). \quad \square \end{aligned}$$

## 1.4 Dimension

A proper prime ideal  $P$  is said to be of *coheight*  $d$  if there exists an ascending chain of  $d + 1$  prime ideals  $P = P_0 \subset P_1 \subset \dots \subset P_d$ , where all inclusions are strict; and if, at the same time, there is no such chain with  $d + 2$  members. Let  $I$  be an ideal in  $S$  with associated primes  $P_1, P_2, \dots, P_s$ . Then the *coheight of  $I$* , denoted by  $\text{coht}(I)$ , is defined to be the largest of the coheights of  $P_1, P_2, \dots, P_s$ . The *Krull dimension of  $S/I$*  is defined to be the coheight of  $I$  and is denoted by  $\text{dim}(S/I)$ .

An associated prime  $P$  of an ideal  $I \subset S$  is said to be *top-dimensional* if  $\text{dim}(S/P) = \text{dim}(S/I)$ . If  $\text{dim}(S/P) = \text{dim}(S/I)$  for all associated primes  $P$  of  $I$ , then  $I$  is said to be *pure dimensional* or *unmixed*.

### EXAMPLE 1.3.4 (revisited)

We have

$$\begin{aligned} I &= (x^2y, x^2z, xy^2, xyz^2) \\ &= (x) \cap (y, z) \cap (x^2, y) \cap (x^2, y^2, z^2). \end{aligned}$$

$(x) \subset (x, y) \subset (x, y, z)$  is a maximal strictly increasing chain of proper prime ideals containing  $(x)$ . Hence,

$$\dim(S/(x)) = 2.$$

Similarly,  $(y, z) \subset (x, y, z)$  and  $(x, y) \subset (x, y, z)$  are maximal strictly increasing chains of proper prime ideals containing  $(y, z)$  and  $(x, y)$  respectively. Therefore,

$$\dim(S/(y, z)) = \dim(S/(x, y)) = 1.$$

There are no proper prime ideals of  $S$  which strictly contain  $(x, y, z)$ , so

$$\dim(S/(x, y, z)) = 0.$$

Thus,

$$\dim(S/I) = \max\{0, 1, 2\} = 2.$$

We have  $\dim(S/(x)) = \dim(S/I)$ . Hence  $(x)$  is top-dimensional. The Krull dimensions of  $S/(y, z)$ ,  $S/(x, y)$  and  $S/(x, y, z)$  do not equal the Krull dimension of  $S/I$ . Thus  $(x)$  is the only top-dimensional prime of  $I$  and  $I$  is not pure dimensional.  $\square$

The *Krull dimension* of a ring  $R$ , denoted  $\dim R$ , is the supremum of lengths of chains of distinct prime ideals in  $R$ . Therefore the Krull dimension of  $S = k[x_1, \dots, x_n]$  is  $n$ .

## 1.5 Length-Multiplicity

Let  $Q \subset S$  be a  $P$ -primary ideal. Then the length multiplicity of  $Q$ , denoted  $\text{mult}(Q)$ , is the length  $l$  of a maximal strictly increasing chain of  $P$ -primary ideals

$$Q = J_l \subset J_{l-1} \subset \dots \subset J_1 = P.$$

Hence, the length multiplicity of a prime ideal is one.

Let  $I$  be a homogeneous ideal of  $S$ . Given any homogeneous prime  $P$  in  $S$ , let  $J_P$  be the intersection of the primary components of  $I$  with associated primes contained strictly in  $P$ . If there are no primes  $p$  belonging to  $I$  with  $p \subset P$ , then let  $J_P = S$ . The *length-multiplicity of  $P$  with respect to  $I$* , denoted  $\text{mult}_I(P)$ , is then the length of a maximal strictly increasing chain of ideals

$$I \subseteq J_l \subset J_{l-1} \subset \dots \subset J_2 \subset J_1 \subset J_P,$$

where each  $J_k$ ,  $k = 1, \dots, l$ , equals  $q_k \cap J_P$  for some  $P$ -primary ideal  $q_k$ . We have  $\text{mult}_I(P) > 0$  if and only if  $P$  is an associated prime of  $I$ , and, if  $P$  is an isolated prime of  $I$ , then  $\text{mult}_I(P) = \text{mult}(Q)$  where  $Q$  is the  $P$ -primary component of  $I$ .

The following is an algorithm for finding the length multiplicities of monomial ideals, as taken from [HV94].

**ALGORITHM 1.5.1** *Let  $Q$  be a  $P$ -primary ideal belonging to  $I$ .*

**STEP 1** *Take a maximal strictly increasing chain of primary ideals from  $Q$  to  $P$ .*

$$Q \subset \dots \subset Q_{i-1} \subset Q_i \subset \dots \subset P.$$

**STEP 2** *Intersect each primary ideal in the above chain with  $J_P$ .*

$$Q \cap J_P \subseteq \dots \subseteq Q_{i-1} \cap J_P \subseteq Q_i \cap J_P \subseteq \dots \subseteq P \cap J_P.$$

**STEP 3** *Eliminate duplicates in order to obtain a strictly increasing chain of ideals.*

$$Q \cap J_P := J_l \subset J_{l-1} \subset \dots \subset J_1 \subset J_P.$$

Then  $\text{mult}_I P = l$ .

**EXAMPLE 1.3.4 (revisited)**

We have

$$\begin{aligned} I &= (x^2y, x^2z, xy^2, xyz^2) \\ &= (x) \cap (y, z) \cap (x^2, y) \cap (x^2, y^2, z^2). \end{aligned}$$

We have  $(x)$  and  $(y, z)$  are isolated and prime. Hence  $\text{mult}_I(x) = \text{mult}(x) = 1$  and  $\text{mult}_I(y, z) = \text{mult}(y, z) = 1$ . We have  $(x, y)$  is the associated prime of  $(x^2, y)$  and  $(x, y)$  is embedded. Therefore  $J_{(x,y)} = (x)$ . A maximal strictly increasing chain of  $(x, y)$ -primary ideals from  $(x^2, y)$  to  $(x, y)$  is  $(x^2, y) \subset (x, y)$ . Intersecting each ideal in this chain with  $J_{(x,y)}$  we obtain  $(x^2, xy) \subset (x)$ . Thus,  $\text{mult}_I(x, y) = 1$ . We have  $(x, y, z)$  is the associated prime of  $(x^2, y^2, z^2)$  and  $(x, y, z)$  is embedded. Hence  $J_{(x,y,z)} = (x) \cap (y, z) \cap (x^2, y) = (xy, x^2z)$ . A maximal strictly increasing chain of  $(x, y, z)$ -primary ideals from  $(x^2, y^2, z^2)$  to  $(x, y, z)$  is

$$\begin{aligned} (x^2, y^2, z^2) &\subset (x^2, y^2, z^2, xyz) \subset (x^2, y^2, z^2, xy) \subset (x^2, y^2, z^2, xy, xz) \\ &\subset (x, y^2, z^2) \quad \subset (x, y^2, z^2, yz) \quad \subset (x, y, z^2) \\ &\subset (x, y, z). \end{aligned}$$

We have

$$\begin{aligned}
 (x^2, y^2, z^2) \cap J_{(x,y,z)} &= (x^2, y^2, z^2) \cap (xy, x^2z) \\
 &= (x^2y, x^2z, xy^2, xyz^2) = I = J_2 \\
 (x^2, y^2, z^2, xyz) \cap J_{(x,y,z)} &= (x^2, y^2, z^2, xyz) \cap (xy, x^2z) \\
 &= (x^2y, x^2z, xy^2, xyz) = J_1 \\
 (x^2, y^2, z^2, xy) \cap J_{(x,y,z)} &= (x^2, y^2, z^2, xy) \cap (xy, x^2z) \\
 &= (xy, x^2z) = J_{(x,y,z)}.
 \end{aligned}$$

As  $(x^2, y^2, z^2, xy) \cap J_{(x,y,z)} = J_{(x,y,z)}$ , we have the intersection of any ideal in our chain containing  $(x^2, y^2, z^2, xy)$  with  $J_{(x,y,z)}$  is  $J_{(x,y,z)}$ . Therefore, intersecting each ideal in our chain with  $J_{(x,y,z)}$  and eliminating duplicates gives

$$I = (x^2y, x^2z, xy^2, xyz^2) \subset (x^2y, x^2z, xy^2, xyz) \subset (xy, x^2z) = J_{(x,y,z)}$$

and  $\text{mult}_I(x, y, z) = 2$ .  $\square$

## 1.6 Degree, Geometric Degree, and Arithmetic Degree

We are now able to define the degrees of an ideal.

The earliest notion of the degree is the classical degree which was first defined in 1890. Its definition is as follows.

$$\text{deg}(I) := \sum \text{mult}(Q) \cdot \text{deg}P$$

where  $Q$  is  $P$ -primary and the sum is taken over all *top-dimensional prime ideals*  $P$  belonging to  $I$ .

The classical degree takes into consideration contributions made only by top-dimensional primes, and ignores those made by other isolated and embedded primes. The notion of the geometric degree is relatively new and takes into consideration contributions made by all isolated components of an ideal. It is defined as follows.

$$\text{geom-deg}(I) := \sum \text{mult}(Q) \cdot \text{deg}P$$

where  $Q$  is  $P$ -primary and the sum is taken over all *isolated prime ideals*  $P$  belonging to  $I$ .

The notion of the arithmetic degree is the most recent of the notions of the degree and it takes into consideration contributions made by all components of the ideal. It is defined by

$$\text{arith-deg}(I) := \sum \text{mult}_I(P) \cdot \text{deg}P$$

where the sum is taken over all *prime ideals*  $P$  belonging to  $I$ .

**EXAMPLE 1.3.4 (revisited)**

We have

$$\begin{aligned} I &= (x^2y, x^2z, xy^2, xyz^2) \\ &= (x) \cap (y, z) \cap (x^2, y) \cap (x^2, y^2, z^2). \end{aligned}$$

Therefore,

$$\begin{aligned} \text{deg}(I) &= \text{mult}(x) \cdot \text{deg}(x) = 1 \cdot 1 = 1, \\ \text{geom-deg}(I) &= \text{mult}(x) \cdot \text{deg}(x) + \text{mult}(y, z) \cdot \text{deg}(y, z) \\ &= 1 \cdot 1 + 1 \cdot 1 = 2, \text{ and} \\ \text{arith-deg}(I) &= \text{mult}_I(x) \cdot \text{deg}(x) + \text{mult}_I(y, z) \cdot \text{deg}(y, z) \\ &\quad + \text{mult}_I(x, y) \cdot \text{deg}(x, y) + \text{mult}_I(x, y, z) \cdot \text{deg}(x, y, z) \\ &= 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 = 5. \quad \square \end{aligned}$$

---

## On a lower bound for the arithmetic degree

In the paper *Bounds on degrees of projective schemes* by B. Sturmfels, N.V. Trung, and W. Vogel [STV95], a lower and an upper bound on the arithmetic degree of a monomial ideal in terms of its generators are found. [STV95, Section 3] is devoted to proving the following theorem.

**THEOREM 2.0.1** *Let  $I$  be a proper monomial ideal in  $k[x_1, \dots, x_n]$  with minimal set of monomial generators  $m_1, m_2, \dots, m_s$ , and let  $e := \dim(S/I) + s - n$ . Then*

$$\max\{\deg(m_i) : i = 1, \dots, s\} \leq \text{arith-deg}(I) \leq \prod_{i=1}^s \deg(m_i) - e.$$

It is proven, by way of example, that the upper bound in this theorem is generally false for an arbitrary homogeneous ideal. Whether or not the lower bound generalizes to arbitrary homogeneous ideals is said to be unknown.

The purpose of this chapter is to show that the lower bound of Theorem 2.0.1 is not true in general for arbitrary homogeneous ideals.

We recall the definition of the sum and product of homogeneous ideals. Let  $I = (f_1, \dots, f_r), J = (g_1, \dots, g_s)$  be homogeneous ideals in a ring  $R$ . The *sum* of  $I$  and  $J$ , denoted by  $I + J$ , is defined to be  $I + J = (f_1, \dots, f_r, g_1, \dots, g_s)$ . We sometimes write  $I + J = (I, J)$ . The *product* of  $I$  and  $J$ , denoted by  $IJ$ , is defined to be  $IJ = \{\sum_{i=1}^t a_i b_i : t \in \mathbb{Z}^+, a_1, \dots, a_t \in I, b_1, \dots, b_t \in J\}$ . Let  $I_1, \dots, I_r \subset R$ . If  $I_i = I$  for all  $i = 1, \dots, r$ , we write  $\prod_{i=1}^r I_i = I^r$ .

**THEOREM 2.0.2** *Let  $F, G$  be forms of equal degree in  $k[x_3, \dots, x_n]$ , where  $k$  is a field and  $n \geq 4$ . Let  $H$  be an irreducible form in  $k[x_1, \dots, x_n]$  which can be expressed as*

$$H = x_1 \cdot F - x_2 \cdot G.$$

Let  $P = (x_1, x_2)$  and  $Q_r = (P^r, H)$  where  $r \geq 2$ . Then:

- (i)  $Q_r$  is  $P$ -primary,
- (ii)  $Q_r$  has length-multiplicity  $r$ , and
- (iii)  $\text{arith-deg}(Q_r) = r$ .

**PROOF:** (i) We have  $Q_r \subseteq P$  and  $P^r \subset Q_r$ . Hence  $\sqrt{Q_r} = P$  and  $P$  is an associated prime of  $Q_r$ . Assume that  $Q_r$  is not  $P$ -primary. Then there exists a  $g \in P \setminus Q_r$  and an  $f \notin P$  such that  $g \cdot f \in Q_r$ . Consider  $g, f$  as elements in  $(k[x_3, \dots, x_n])[x_1, x_2]$  and  $Q_r$  as a homogeneous ideal in this ring. We may write  $g = g_{i_1} + \dots + g_{i_s}$ , where  $g_{i_j} \notin Q_r$  is a form of degree  $i_j$  in  $x_1, x_2$  (we write  $\text{deg}_{(x_1, x_2)} g_{i_j} = i_j$ ) and, as  $P^r \subseteq Q_r$ ,  $i_1 \leq \dots \leq i_t \leq r - 1$ . We may also write  $f = f_0 + \dots + f_t$  where  $f_0 \neq 0$ . Since  $g \cdot f \in Q_r$ , we have  $f_0 \cdot g_{i_1} \in Q_r$ . The forms  $h$  of  $Q_r$  with  $\text{deg}_{(x_1, x_2)} h < r$  must be the multiples of  $H$ ; hence,  $f_0 \cdot g_{i_1} \in (H)$ . Since  $H$  is irreducible and  $f_0 \notin P$ ,  $g_{i_1} \in (H) \subset Q_r$ . This is a contradiction.

(ii) Suppose  $x_1^{r-1} \in Q_r$ . Then there exists  $f_0, \dots, f_{r+1}, g_0 \in S$  such that  $x_1^{r-1} = f_0 x_1^r + f_1 x_1^{r-1} x_2 + \dots + f_{r+1} x_2^r + g_0 H$ . We have  $\text{deg}_{(x_1, x_2)} f_0 x_1^r \geq r$ ,  $\text{deg}_{(x_1, x_2)} f_1 x_1^{r-1} x_2 \geq r, \dots$ , and  $\text{deg}_{(x_1, x_2)} f_{r+1} x_2^r \geq r$ . Hence  $f_0 = f_1 = \dots = f_{r+1} = 0$  and we have  $x_1^{r-1} = g_0 H = g_0(x_1 F - x_2 G) = g_0 x_1 F - g_0 x_2 G$ . As  $G \in k[x_3, \dots, x_n]$ , we have  $g_0 x_2 G = 0$  which implies  $g_0 = 0$ . Hence  $x_1^{r-1} = 0$  which is a contradiction. Therefore,  $x_1^{r-1} \notin Q_r$ . Similarly we find  $x_1^{r-2} \notin (Q_r, x_1^{r-1}), \dots, x_1^2 \notin (Q_r, x_1^3)$  and  $x_1 \notin (Q_r, x_1^2)$ . Thus

$$Q_r \subset (Q_r, x_1^{r-1}) \subset \dots \subset (Q_r, x_1^2) \subset (x_1, x_2) = P.$$

By [ZS58, Corollary 2, p.237], we have that this is a saturated chain of  $P$ -primary ideals and the length of  $Q_r$  is  $r$ .

(iii)  $Q_r$  is  $P$ -primary by (i). Also,  $\text{deg}(P) = 1$  as  $P$  is a monomial prime ideal. Therefore

$$\begin{aligned} \text{arith-deg} Q_r &= \text{deg} Q_r \\ &= \text{mult}(Q_r) \cdot \text{deg} P \\ &= r \cdot 1 && \text{by part (ii)} \\ &= r. && \square \end{aligned}$$

Theorem 2.0.2 provides counterexamples which show that the lower bound of Theorem 2.0.1 is generally false for an arbitrary homogeneous ideal. We state one.

**EXAMPLE 2.0.3** Take  $r = 2$  in Theorem 2.0.2. We have

$$Q_2 = (P^2, H) = (x_1^2, x_1x_2, x_2^2, H)$$

and

$$\text{arith-deg}Q_2 = 2$$

by Theorem 2.0.2(iii). In order to obtain the required contradiction we need only to take  $H$  such that  $\text{deg}H \geq 3$ . For instance,  $n = 4$  and  $H = x_1x_3^2 - x_2x_4^2$ .  $\square$

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## Remarks on the division problem and Bezout-type theorems

### 3.1 Introduction

In [STV95, Section 5], the authors proved two theorems and a corollary involving bounds on geometric degrees. It was claimed there that these bounds proven for geometric degrees do not hold in general for the arithmetic degree. However there was no detailed proof. In this chapter we give a detailed computation of [STV95, Example 3.2] which provides a necessary counterexample to the bounds in terms of arithmetic degrees. We also give better stated counterexamples for a similar problem arising from [STV95, Corollary 5.4].

The following example is taken from [STV95] and will be referred to throughout the rest of this chapter.

**EXAMPLE 3.1.1** Let  $I$  be the ideal in  $k[x_1, x_2, x_3, x_4]$  generated by the forms

$$f := x_1x_4 - x_2x_3, \quad g := x_1^{b-a}x_3^a - x_2^b, \quad h := x_3^b - x_2^ax_4^{b-a},$$

for any integers  $b > a > 0$ . Then

$$\text{geom-deg}(I) = \text{deg}(I) = a + b$$

and

$$\text{arith-deg}(I) = a + b + \binom{b - a + 1}{3}.$$

For  $b \gg a$ , the arithmetic degree of  $I$  exceeds  $\deg(f) \cdot \deg(g) \cdot \deg(h) = 2b^2$ .  $\square$

### 3.2 Remark on Hilbert Nullstellensatz

If  $f, f_1, \dots, f_m$  are polynomials in  $S$  such that  $f \in (f_1, \dots, f_m)$  then  $f = g_1 f_1 + \dots + g_m f_m$  for some  $g_1, \dots, g_m \in S$ . In order to construct  $g_1, \dots, g_m$  it is important to have a bound on  $\deg(g_i f_i)$ . Using the geometric degree, [STV95, Section5] provides the following version of the so called effective Hilbert Nullstellensatz:

**THEOREM 3.2.1** *Let  $f, f_1, \dots, f_m$  be polynomials in  $S := k[x_1, \dots, x_n]$  such that  $f \in (f_1, \dots, f_m)$  and the homogenized ideal  $I := ({}^h f_1, \dots, {}^h f_m)$  in  $R := k[x_0, \dots, x_n]$  has no embedded components containing  $x_0$ . Put  $d_i = \deg(f_i)$  and suppose  $d_2 \geq \dots \geq d_m \geq d_1$ . Then there exists  $g_1, \dots, g_m \in S$  such that  $f = g_1 f_1 + \dots + g_m f_m$  with*

$$\deg(g_i f_i) \leq d_1 d_2 \dots d_t - \text{geom-deg}(\bar{I}) + \deg(f)$$

if  $\dim(R/I) > 0$ , where  $t$  is the maximal height of an isolated prime of  $I$  and  $\bar{I}$  is the intersection of all primary components of  $I$  whose associated primes do not contain  $x_0$ .

Here the *homogenization* of  $f$  is defined to be  ${}^h f := x_0^{\deg(f)} \cdot f(x_1/x_0, \dots, x_n/x_0)$ , and the *height* of a proper prime ideal  $P$ , denoted by  $htP$ , is defined to be the supremum of lengths of chains of distinct prime ideals strictly contained in  $P$ .

As  $\text{geom-deg}(\bar{I}) \leq \text{arith-deg}(\bar{I})$ , it is natural to ask whether we can replace  $\text{geom-deg}(\bar{I})$  by  $\text{arith-deg}(\bar{I})$  in the above theorem. The purpose of this section is to provide a negative answer to this question.

Consider

$$\begin{aligned} f_1 &= x_1 x_4 - x_2 x_3, \\ f_2 &= x_1^{b-a} x_3^a - x_2^b, \quad \text{and} \\ f_3 &= x_3^b - x_2^a x_4^{b-a} \in k[x_1, \dots, x_4] = S. \end{aligned}$$

We have  $d_1 = 2$ ,  $d_2 = b$  and  $d_3 = b$ , and  $d_2 \geq d_3 \geq d_1$  as required by Theorem 3.2.1.

Let  $I = ({}^h f_1, {}^h f_2, {}^h f_3) = (f_1, f_2, f_3) \cdot R \subset R = k[x_0, \dots, x_4]$ . Therefore  $I$  has no embedded components containing  $x_0$  as required.

We claim  $I = P \cap Q$ , where  $P = (x_1 x_4 - x_2 x_3, x_1^{b-a} x_3^a - x_2^b, x_1^{b-a-1} x_3^{a+1} - x_2^{b-1} x_4, \dots, x_1 x_3^{b-1} - x_2^{a+1} x_4^{b-a-1}, x_3^b - x_2^a x_4^{b-a})$  is prime and  $\sqrt{Q} = (x_1, x_2, x_3, x_4)$ .

In order to prove our claim we need the following definition and lemmas.

Let  $A, B$  be ideals in a ring  $K$  and let  $C \subseteq B$ . Then the *ideal quotient*  $A : C$  is defined by  $A : C = \{k \in K : kC \subseteq A\}$ .

**LEMMA 3.2.2** *Let  $J$  be an ideal in  $S = k[x_1, \dots, x_n]$  with primary decomposition  $J = P \cap q_1 \cap \dots \cap q_l$ , where each  $q_i$  is  $p_i$ -primary. Let  $m = (x_1, \dots, x_n)$ . If  $P = \cup_{r \geq 1} J : m^r$ , then  $p_1 = \dots = p_l = m$ .*

**PROOF:** Let  $P = \cup_{r \geq 1} J : m^r = \cup_{r \geq 1} (P \cap q_1 \cap \dots \cap q_l) : m^r$ .

Consider  $q_i : m^r = \{x \in S : x \cdot m^r \subseteq q_i\}$  for  $i = 1, \dots, l$  and for some  $r \geq 1$ . If  $p_i = m$ , then  $q_i : m^r = S$  by the definition of  $p_i$ -primary. Suppose  $p_i \neq m$ . Let  $x \in q_i : m$ . Then  $x \cdot m \subseteq q_i \subset m$ . Hence,  $x \in q_i$  and  $q_i : m = q_i$ .

Therefore, we have

$$\cup_{r \geq 1} q_i : m^r = \begin{cases} S & m = p_i \\ q_i & m \neq p_i \end{cases}.$$

Without loss of generalization, assume  $m = p_i$  for  $i = 1, \dots, s - 1$ , and  $m \neq p_i$  for  $i = s, \dots, t$ . Then  $P = \cup_{r \geq 1} (P \cap q_1 \cap \dots \cap q_l) : m^r = P \cap q_s \cap \dots \cap q_t$  where  $p_j \neq m$  for  $j = s, \dots, t$ .

Thus,  $P \subseteq p_j$  for  $j = s, \dots, t$ . Also,  $p_j \in \text{Ass}(S/J)$ . Therefore,  $p_j = \text{Ann}(x) = J : x$  for some  $x \in S/J$ . Suppose  $x \notin P$ . Let  $y \in J : x$ . Then  $y \cdot x \in J \subseteq P$  and  $y \in P$ . We have  $p_j = J : x \subseteq P$  and  $P \subseteq p_j$ ; hence  $p_j = P$ . Suppose  $x \in P$ . Let  $y \in m$ . As  $P = \cup_{r \geq 1} J : m^r$ , there exists an  $r = r_0$  such that  $y^{r_0} \cdot x \in J$ . Hence  $y^{r_0} \in J : x = p_j$ . Since  $p_j$  is prime we have  $y \in p_j$ . Hence  $p_j = m$ .  $\square$

**LEMMA 3.2.3** *Consider the following ideals in  $k[x_1, x_2, x_3, x_4]$  :*

$$\begin{aligned} J &= (x_1x_4 - x_2x_3, x_1^{b-a}x_3^a - x_2^b, x_3^b - x_2^ax_4^{b-a}), \\ P &= (x_1x_4 - x_2x_3, x_1^{b-a}x_3^a - x_2^b, x_1^{b-a-1}x_3^{a+1} - x_2^{b-1}x_4, \dots, x_1x_3^{b-1} - x_2^{a+1}x_4^{b-a-1}, \\ &\quad x_3^b - x_2^ax_4^{b-a}), \\ m &= (x_1, x_2, x_3, x_4). \end{aligned}$$

*Then  $P \subseteq \cup_{r \geq 1} J : m^r$ .*

The following result will be needed in the proof of Lemma 3.2.3.

**LEMMA 3.2.4** *Let  $J$  be defined as in 3.2.3. Then  $x_1^s x_4^s - x_2^s x_3^s \in J$  for all  $s \geq 1$ .*

**PROOF:** We have

$$x_1^s x_4^s - x_2^s x_3^s = (x_1 x_4 - x_2 x_3)[(x_1 x_4)^{s-1} + (x_1 x_4)^{s-2}(x_2 x_3) + (x_1 x_4)^{s-3}(x_2 x_3)^2 + \dots + (x_2 x_3)^{s-1}] \in J. \quad \square$$

**PROOF OF LEMMA 3.2.3:** In order to prove  $P \subseteq \cup_{r \geq 1} J : m^r$ , we show that for some  $r \gg 0$  we have  $p \cdot a \in J$  for all  $p \in P$  and for all  $a \in m^r$ . For any  $r > 0$ , we have  $(x_1 x_4 - x_2 x_3) \cdot a \in J$  for all  $a \in m^r$ . Hence, we must show

$$x_1^{r-j} x_2^k x_3^l x_4^{j-k-l} (x_1^{b-a-i} x_3^{a+i} - x_2^{b-i} x_4^i) \subseteq J$$

for some  $r \gg 0$ , where

$$\begin{aligned} i &= 0, 1, \dots, b-a, \\ j &= 0, 1, \dots, n, \\ k &= 0, 1, \dots, j, \\ l &= 0, 1, \dots, j-k. \end{aligned}$$

We have

$$\begin{aligned} &x_1^{r-j} x_2^k x_3^l x_4^{j-k-l} (x_1^{b-a-i} x_3^{a+i} - x_2^{b-i} x_4^i) \\ &= x_1^{r-j+b-a-i} x_2^k x_3^{a+i+l} x_4^{j-k-l} - x_1^{r-j} x_2^{k+b-i} x_3^l x_4^{i+j-k-l} =: A. \end{aligned}$$

Consider the following cases.

Case 1:

$$\begin{aligned} A &= x_1^{r-j-i} x_2^k x_3^{l+i} x_4^{j-k-l} (x_1^{b-a} x_3^a - x_2^b) + x_1^{r-j-i} x_2^{b+k} x_3^{l+i} x_4^{j-k-l} \\ &\quad - x_1^{r-j} x_2^{k+b-i} x_3^l x_4^{i+j-k-l} \\ &= x_1^{r-j-i} x_2^k x_3^{l+i} x_4^{j-k-l} (x_1^{b-a} x_3^a - x_2^b) - x_1^{r-j-i} x_2^{k+b-i} x_3^l x_4^{j-k-l} (x_1^i x_4^i - x_2^i x_3^i). \end{aligned}$$

Therefore, by Lemma 3.2.4 and the definition of  $J$ ,  $A \in J$  provided  $r - i - j \geq 0$ .

Case 2:

$$\begin{aligned} A &= x_1^{r-j} x_2^{k-i} x_3^l x_4^{i+j-k-l} (x_1^{b-a} x_3^a - x_2^b) - x_1^{r-j+b-a} x_2^{k-i} x_3^{a+l} x_4^{i+j-k-l} \\ &\quad + x_1^{r-j+b-a-i} x_2^k x_3^{a+i+l} x_4^{j-k-l} \\ &= x_1^{r-j} x_2^{k-i} x_3^l x_4^{i+j-k-l} (x_1^{b-a} x_3^a - x_2^b) - x_1^{r-j+b-a-i} x_2^{k-i} x_3^{a+l} x_4^{j-k-l} (x_1^i x_4^i - x_2^i x_3^i). \end{aligned}$$

Therefore, by Lemma 3.2.4 and the definition of  $J$ ,  $A \in J$  provided  $k - i \geq 0$ .

Case 3:

$$\begin{aligned}
 A &= x_1^{r-j+b-a-i} x_2^k x_3^{i+l-(b-a)} x_4^{j-k-l} (x_3^b - x_2^a x_4^{b-a}) \\
 &\quad + x_1^{r-j-b+a-i} x_2^{a+k} x_3^{i+l-(b-a)} x_4^{(b-a)+j-k-l} - x_1^{r-j} x_2^{k+b-i} x_3^l x_4^{i+j-k-l} \\
 &= x_1^{r-j+b-a-i} x_2^k x_3^{i+l-(b-a)} x_4^{j-k-l} (x_3^b - x_2^a x_4^{b-a}) \\
 &\quad + x_1^{r-j} x_2^{a+k} x_3^{l+i-(b-a)} x_4^{j-k-l+i} (x_1^{b-a-i} x_4^{b-a-i} - x_2^{b-a-i} x_3^{b-a-i}).
 \end{aligned}$$

Therefore, by Lemma 3.2.4 and the definition of  $J$ ,  $A \in J$  provided  $i+l-(b-a) \geq 0$ .

Case 4:

$$\begin{aligned}
 A &= x_1^{r-j} x_2^{k+b-a-i} x_3^l x_4^{i+j-k-l-(b-a)} (x_3^b - x_2^a x_4^{b-a}) - x_1^{r-j} x_2^{k+b-a-i} x_3^{b+l} x_4^{i+j-k-l-(b-a)} \\
 &\quad + x_1^{r-j+b-a-i} x_2^k x_3^{a+i+l} x_4^{j-k-l} \\
 &= x_1^{r-j} x_2^{k+b-a-i} x_3^l x_4^{i+j-k-l-(b-a)} (x_3^b - x_2^a x_4^{b-a}) \\
 &\quad + x_1^{r-j} x_2^k x_3^{a+i+l} x_4^{i+j-k-l-(b-a)} (x_1^{b-a-i} x_4^{b-a-i} - x_2^{b-a-i} x_3^{b-a-i}).
 \end{aligned}$$

Therefore, by Lemma 3.2.4 and the definition of  $J$ ,  $A \in J$  provided  $i+j-k-l-(b-a) \geq 0$ .

Let  $r = 4(b-a)$ .

Then, by Case 1,  $A \in J$  for all  $j$  such that  $3(b-a) \geq j$ . By Case 2,  $A \in J$  for all  $k$  such that  $k \geq b-a$ , and, by Case 3, we have  $A \in J$  provided  $l \geq b-a$ .

We have left to show that  $A \in J$  for  $3(b-a) < j \leq 4(b-a)$ ,  $0 \leq k < b-a$ , and  $0 \leq l < b-a$ .

Suppose  $3(b-a) < j \leq 4(b-a)$ ,  $0 \leq k < b-a$ , and  $0 \leq l < b-a$ . By Case 4,  $A \in J$  provided  $i+j-k-l-(b-a) \geq 0$ . Therefore, using  $0 \leq i \leq b-a$  along with the assumed conditions, we have  $A \in J$  if  $0+3(b-a)-(b-a)-(b-a)-(b-a) \geq 0$ . This is true.

Hence, for  $r = 4(b-a)$ ,  $A \in J$  for all possible values of  $i, j, k$ , and  $l$ .

Therefore  $P \subseteq \cup_{r \geq 1} J : m^r$ .  $\square$

By Lemma 3.2.2 and Lemma 3.2.3,  $J = (f_1, f_2, f_3) = P \cap Q \subset S = k[x_1, \dots, x_4]$ , where  $P = (x_1 x_4 - x_2 x_3, x_1^{b-a} x_3^a - x_2^b, x_1^{b-a-1} x_3^{a+1} - x_2^{b-1} x_4, \dots, x_1 x_3^{b-1} - x_2^{a+1} x_4^{b-a-1}, x_3^b - x_2^a x_4^{b-a})$  is prime and  $\sqrt{Q} = (x_1, x_2, x_3, x_4)$ . Therefore  $I = (f_1, f_2, f_3) \cdot R = P \cap Q \subset R = k[x_0, \dots, x_4]$  as claimed.

We have  $\dim(R/I) = \dim(S/J) + 1 > 0$  (actually one can show that  $\dim(S/J) = 2$ ), and  $I$  satisfies all conditions of Theorem 3.2.1.

We have  $\bar{I} = I$ . Hence, by Example 3.1.1,

$$\text{arith-deg}(\bar{I}) = \text{arith-deg}(I) = a + b + \binom{b - a + 1}{3}$$

and for  $b \gg a$ ,  $\text{arith-deg}(\bar{I}) > 2b^2$ .

$P$  is the only isolated prime of  $I$  and  $htP = 2$ .

Let  $f = f_1 \in (f_1, f_2, f_3)$ . Then the left side of Theorem 3.2.1 is greater than or equal to  $\text{deg}f$  while  $d_1d_2 - \text{arith-deg}(\bar{I}) + \text{deg}f < 2b - 2b^2 + \text{deg}f < \text{deg}f$  for  $b \gg a$ . This means in general the inequality  $\text{deg}(f_i g_i) \leq d_1d_2 - \text{arith-deg}(\bar{I}) + \text{deg}f$  is not true and the upper bound of Theorem 3.2.1 cannot be improved by replacing the geometric degree with the arithmetic one.

### 3.3 Remark on a Bezout-type theorem, Part I

The second bound on the geometric degree taken from [STV95] which we want to consider is a Bezout-type theorem for the geometric degree. First we recall the following notions.

A sequence  $a_1, \dots, a_t$  of elements of a ring  $A$  is called a *regular sequence* if the following conditions hold: (i)  $(a_1, \dots, a_t) \neq A$ ; and (ii)  $(a_1, \dots, a_{i-1}) : a_i = (a_1, \dots, a_{i-1})$  for  $i = 1, \dots, t$ . The maximal length of all regular sequences of  $A$  is called the *depth* of  $A$  and is denoted by  $\text{depth}A$ . If  $P$  is a prime ideal of  $A$ , the *localization of  $A$  at  $P$* , denoted by  $A_P$ , is the ring consisting of all elements  $a/x$ , where  $a \in A$  and  $x \notin P$ . We say  $A$  is *Cohen-Macaulay* if for all prime ideals  $P$  in  $A$  we have  $\text{depth}A_P = \text{dim}A_P$ .

**THEOREM 3.3.1** *Let  $I_1, \dots, I_r$  be homogeneous ideals in  $S$  such that  $S_P/I_i S_P$  is Cohen-Macaulay for all isolated primes  $P$  of  $I_1 + \dots + I_r$  and  $i = 1, \dots, r$ . Then*

$$\text{geom-deg}(I_1 + \dots + I_r) \leq \prod_{i=1}^r \text{geom-deg}(I_i).$$

The purpose of this section is to show that in this Bezout-type theorem one cannot replace the geometric degree by the arithmetic one.

Consider the ideals  $I_1 = (x_1x_4 - x_2x_3)$ ,  $I_2 = (x_1^{b-a}x_3^a - x_2^b)$ , and  $I_3 = (x_3^b - x_2^a x_4^{b-a})$  in  $S = k[x_1, x_2, x_3, x_4]$ .

Let  $P$  be an isolated prime of  $(I_1 + I_2 + I_3) = (x_1x_4 - x_2x_3, x_1^{b-a}x_3^a - x_2^b, x_3^b - x_2^a x_4^{b-a})$ .  $S$  is a Cohen-Macaulay ring by [Eis95, Proposition 18.9]. Hence  $(S/(x_1x_4 -$

$x_2x_3) \cdot S)_P = (S_P/I_1 \cdot S_P)$ ,  $(S/(x_1^{b-a}x_3^a - x_2^b) \cdot S)_P = (S_P/I_2 \cdot S_P)$  and  $(S/(x_3^b - x_2^ax_4^{b-a}) \cdot S)_P = (S_P/I_3 \cdot S_P)$  are Cohen-Macaulay by [SV80, Lemma 1.1(ii)] and [Eis95, Proposition 18.8]. Therefore  $I_1$ ,  $I_2$  and  $I_3$  satisfy the assumptions of Theorem 3.3.1.

$I_1$ ,  $I_2$  and  $I_3$  are all principal ideals and principal ideals are unmixed. Therefore, by [Vog84, 1.35],  $\text{arith-deg}(I_1) = \text{geom-deg}(I_1) = \text{deg}(I_1) = \text{deg}(x_1x_4 - x_2x_3) = 2$ ,  $\text{arith-deg}(I_2) = \text{geom-deg}(I_2) = \text{deg}(I_2) = \text{deg}(x_1^{b-a}x_3^a - x_2^b) = b$ , and  $\text{arith-deg}(I_3) = \text{geom-deg}(I_3) = \text{deg}(I_3) = \text{deg}(x_3^b - x_2^ax_4^{b-a}) = b$ .

We have  $b > a$ . Therefore,  $\text{geom-deg}(I_1 + I_2 + I_3) = a + b \leq 2b \leq 2b^2 = \text{deg}(I_1) \cdot \text{deg}(I_2) \cdot \text{deg}(I_3)$  as required by Theorem 3.3.1.

However, for  $b \gg a$ ,  $\text{arith-deg}(I_1 + I_2 + I_3) > 2b^2 = \text{deg}(I_1) \cdot \text{deg}(I_2) \cdot \text{deg}(I_3)$ . Therefore, even under the assumptions of Theorem 3.3.1 the inequality  $\text{arith-deg}(I_1 + \dots + I_r) \leq \prod_{i=1}^r \text{deg}(I_i)$  does not hold in general.

### 3.4 Remark on a Bezout-type theorem, Part II

Corollary 5.4 of [STV95] provides a geometric interpretation of Theorem 3.3.1. We present an algebraic version of this corollary. A table detailing the relationships between relevant geometric and algebraic concepts is given in Appendix A.

First we recall the following notion. Let  $I$  be an ideal in the polynomial ring  $S = k[x_1, \dots, x_n]$ .  $I$  is said to be *locally Cohen-Macaulay* if  $(S/I)_P$  is Cohen-Macaulay for all homogeneous prime ideals  $P \neq (x_1, \dots, x_n)$  in  $S$ .

**COROLLARY 3.4.1** *Let  $I, J \subset S := k[x_1, \dots, x_n]$  be locally Cohen-Macaulay ideals with  $\dim(S/I) + \dim(S/J) > n + 1$ . Then*

$$\text{geom-deg}(I + J) \leq \text{geom-deg}(I) \cdot \text{geom-deg}(J).$$

The purpose of this section is to show that under the same assumptions of Corollary 3.4.1 the  $\text{arith-deg}(I + J)$  is not necessarily less than or equal to  $\text{arith-deg}(I) \cdot \text{arith-deg}(J)$ . We first show how Example 3.1.1 can be used to provide a counterexample to Corollary 3.4.1 when restated in terms of arithmetic degrees, but only if we consider the so-called cone construction; that is, only if we add a new variable which does not appear in any element of minimal sets of generators of the ideals  $I$  and  $J$ . Then we provide another counterexample which does not require the use of cone constructions.

We need the following definition and fact.

**DEFINITION 3.4.2** Let  $I$  be an ideal generated by the elements  $f_1, \dots, f_t$  in the polynomial ring  $S$ . If  $(f_1, \dots, f_{i-1}) : f_i = (f_1, \dots, f_{i-1})$  for all  $i = 1, \dots, t$ , then  $I$  is a complete intersection.

**FACT 3.4.3** Let  $S = k[x_1, \dots, x_n]$ .  $I = (f_1, \dots, f_t) \subseteq S$  is a complete intersection if and only if  $\dim(S/I) = n - t$ .

Consider the use of Example 3.1.1 in providing the necessary contradiction to Corollary 3.4.1 restated in terms of arithmetic degrees.

Let  $S = k[x_1, x_2, x_3, x_4]$ . Let  $I = (f, g) = (x_1x_4 - x_2x_3, x_1^{b-a}x_3^a - x_2^b)$  and  $J = (h) = (x_3^b - x_2^ax_4^{b-a})$ . Then  $I$  and  $J$  are both complete intersections and are therefore locally Cohen-Macaulay. However, the dimension condition is not satisfied as, by Fact 3.4.3, we have  $\dim(S/I) = 4 - 2 = 2$  and  $\dim(S/J) = 4 - 1 = 3$ . This gives  $\dim(S/I) + \dim(S/J) = 2 + 3 = 5 = n + 1$ . Hence we must work in a polynomial ring of higher dimension.

Let  $I = (f, g), J = (h) \subset k[x_1, x_2, x_3, x_4, x_5]$  where  $f, g$ , and  $h$  are as before. Both ideals are still complete intersections and hence locally Cohen-Macaulay. The dimensions are now  $\dim(S/I) = 5 - 2 = 3$  and  $\dim(S/J) = 5 - 1 = 4$  giving  $\dim(S/I) + \dim(S/J) = 7 > 6 = n + 1$ .  $I$  and  $J$  have only top-dimensional primes. Therefore,  $\text{arith-deg}(I) \cdot \text{arith-deg}(J) = \text{deg}(I) \cdot \text{deg}(J) = 2b^2$  and for  $b \gg a$ ,  $\text{arith-deg}(I + J)$  exceeds  $2b^2$ .

Thus this provides a counterexample to Corollary 3.4.1 stated in terms of arithmetic degrees. However, we would like an example which does not require the use of cone constructions.

**REFINED PROBLEM 3.4.4** Let  $I, J \subset S = k[x_1, \dots, x_n]$  such that  $I$  is not generated by  $I \cap k[y_1, \dots, y_m]$ , where  $m < n$  and  $y_i$  is a linear combination of  $x_1, \dots, x_n$  for  $i = 1, \dots, m$ . If  $I$  and  $J$  are locally Cohen-Macaulay and if  $\dim(S/I) + \dim(S/J) > n + 1$ , then is  $\text{arith-deg}(I + J) \leq \text{arith-deg}(I) \cdot \text{arith-deg}(J)$ ?

In general this inequality is false. Take  $I, J \subset S = k[x_1, \dots, x_n]$  such that the following conditions are satisfied:  $I$  and  $J$  are locally Cohen-Macaulay, all variables  $x_1, \dots, x_n$  appear at least once in the forms defining a minimal basis of  $I$ ,  $\dim(S/I) \geq 3$ ,  $J = (l)$  where  $l$  is a linear form defining a non-zero divisor on  $S/I$ , and  $I + J$  has an embedded component. Then,  $\dim(S/I) + \dim(S/J) \geq 3 + (n - 1) > n + 1$ . Hence our assumptions are satisfied.  $I$  and  $J$  are pure

dimensional, so  $\text{arith-deg}(I) \cdot \text{arith-deg}(J) = \text{deg}(I) \cdot \text{deg}(J)$ . By applying [Vog84, 1.36(iii), p.47],  $\text{deg}(I) \cdot \text{deg}(J) = \text{deg}(I+J)$ . Since  $I+J$  has an embedded component,  $\text{deg}(I+J) < \text{arith-deg}(I+J)$ . Therefore, under these conditions,  $\text{arith-deg}(I+J) > \text{arith-deg}(I) \cdot \text{arith-deg}(J)$ , and we have the required counterexample to our refined problem.

[HSV91, Example 10, p228], provides a surface  $X$  whose defining ideal satisfies these conditions for  $I$ .

Consider the surface  $X$  of  $P_k^5$  given parametrically by  $(uv^3, uv^2w, v^4, v^3w, vw^3, w^4)$ . Let  $I$  be the defining prime ideal of  $X$  in  $S := k[x_1, x_2, x_3, x_4, x_5, x_6]$ . Then a run of the computer program Macaulay [BS90] provides the following:

$I = (x_2x_3 - x_1x_4, x_2x_5 - x_1x_6, x_4x_5 - x_3x_6, x_4^3 - x_3^2x_5, x_2x_4^2 - x_1x_3x_5, x_2^2x_4 - x_1^2x_5, x_3x_5^2 - x_4^2x_6, x_1x_5^2 - x_2x_4x_6, x_5^3 - x_4x_6^2)$ ,  $\dim(S/I) = 3$ ,  $\text{deg}(I) = 5 = \text{codim}(S/I) + 2$ , and  $\text{depth}(I) = 1$ , where the *codimension* of  $S/I$  is defined to be the minimum of the heights of the associated primes of  $I$ , and the *depth* of  $I$  is defined to be the length of any maximal regular sequence in  $I$ . (See Appendix B for computer output.) Therefore, by [HSV91, Theorem B(ii)],  $I$  is locally Cohen-Macaulay. We may also note that the variables  $x_1, \dots, x_6$  are all used in defining  $I$ .

Let  $J = (x_4)$ . We have  $x_4 \notin I$ . Therefore,  $x_4$  is a linear form defining a non-zero divisor on  $S/I$ .

We have  $\dim(S/J) = 5$ . Therefore,  $\dim(S/I) + \dim(S/J) = 3 + 5 = 8 > 7 = n + 1$ .

Hence,  $I$  and  $J$  satisfy the necessary conditions.

$I$  and  $J$  are unmixed primes. Therefore,  $\text{arith-deg}(I) \cdot \text{arith-deg}(J) = \text{deg}(I) \cdot \text{deg}(J) = 5 \cdot 1 = 5$ .

$$\begin{aligned} I + J &= (x_2x_3 - x_1x_4, x_2x_5 - x_1x_6, x_4x_5 - x_3x_6, x_4^3 - x_3^2x_5, \\ &\quad x_2x_4^2 - x_1x_3x_5, x_2^2x_4 - x_1^2x_5, x_3x_5^2 - x_4^2x_6, x_1x_5^2 - x_2x_4x_6, \\ &\quad x_5^3 - x_4x_6^2, x_4) \\ &= (x_4, x_2x_3, x_2x_5 - x_1x_6, x_3x_6, x_3^2x_5, x_1x_3x_5, x_1^2x_5, x_3x_5^2, x_1x_5^2, x_5^3). \end{aligned}$$

We have  $x_3x_5x_i \in I+J$  for  $i = 1, \dots, 6$ . Thus,  $x_3x_5 \in [I+J] : (x_1, x_2, x_3, x_4, x_5, x_6)$ . However,  $x_3x_5 \notin I+J$ . Hence,  $I+J \subset [I+J] : (x_1, x_2, x_3, x_4, x_5, x_6)$ , and by [Nor53, Theorem 6, Ch1],  $(x_1, x_2, x_3, x_4, x_5, x_6) \in \text{Ass}(S/[I+J])$ .

Therefore  $I+J$  has an embedded component and  $\text{arith-deg}(I+J) > \text{deg}(I+J) = \text{deg}(I) \cdot \text{deg}(J) = \text{arith-deg}(I) \cdot \text{arith-deg}(J)$ .

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## A Bezout-type theorem for the arithmetic degree

### 4.1 Introduction

In Chapter 3, Section 3 we posed the following refined problem.

**REFINED PROBLEM 3.4.4** *Let  $I, J \subset S = k[x_1, \dots, x_n]$  such that  $I$  is not generated by  $I \cap k[y_1, \dots, y_m]$ , where  $m < n$  and  $y_i$  is a linear combination of  $x_1, \dots, x_n$  for  $i = 1, \dots, m$ . If  $I$  and  $J$  are locally Cohen-Macaulay and if  $\dim(S/I) + \dim(S/J) > n + 1$ , then is  $\text{arith-deg}(I + J) \leq \text{arith-deg}(I) \cdot \text{arith-deg}(J)$ ?*

It was shown that this is not true in general. In this chapter we will look for conditions under which we have the reverse inequality in Refined Problem 3.4.4. First we introduce the following definitions.

**DEFINITION 4.1.1** *Let  $A$  be a local ring (that is,  $A$  has a unique maximal ideal) of dimension  $d$  with maximal ideal  $m$ . A system  $\{f_1, \dots, f_d\}$  of  $d$  elements of  $m$  such that  $m^t \subset (f_1, \dots, f_d)$  for  $t \gg 0$ , is called a system of parameters of  $A$ .*

**DEFINITION 4.1.2** *Let  $R$  be a ring and  $I$  an ideal in  $R$ . A sequence  $\{f_1, \dots, f_t\}$  of elements of  $R$  is called an  $R/I$ -sequence if*

1.  $(f_1, \dots, f_t) \cdot (R/I) \neq R/I$ , and
2. For  $i = 1, \dots, t$ ,  $f_i$  is a nonzero divisor on  $(R/I)/(f_1, \dots, f_{i-1}) \cdot (R/I)$ .

## 4.2 New Results

Let  $k$  be a field and  $S := k[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables over  $k$ .

**THEOREM 4.2.1** *Let  $I, J$  be pure dimensional homogeneous ideals of  $S$  such that  $\dim(S/(I + J)) = \dim(S/I) + \dim(S/J) - n$ . Then we have*

- (i)  $\text{arith-deg}(I) \cdot \text{arith-deg}(J) \leq \text{arith-deg}(I + J)$ .
- (ii) *If  $I+J$  is not pure dimensional then*  
 $\text{arith-deg}(I) \cdot \text{arith-deg}(J) < \text{arith-deg}(I + J)$ .
- (iii)  $\text{arith-deg}(I) \cdot \text{arith-deg}(J) = \text{arith-deg}(I + J)$   
*if and only if  $I+J$  is pure dimensional and  $(S/I)_P$  and  $(S/J)_P$  are Cohen-Macaulay for all isolated components  $P$  of  $I + J$ .*

**COROLLARY 4.2.2** *Let  $I$  be a pure dimensional homogeneous ideal of  $S$  and let  $J \subset S$  be an ideal generated by a part of a system of parameters of  $S/I$ . Then we have*

- (i)  $\text{arith-deg}(I) \cdot \text{arith-deg}(J) \leq \text{arith-deg}(I + J)$ .
- (ii) *If  $I+J$  is not pure dimensional then*  
 $\text{arith-deg}(I) \cdot \text{arith-deg}(J) < \text{arith-deg}(I + J)$ .
- (iii)  $\text{arith-deg}(I) \cdot \text{arith-deg}(J) = \text{arith-deg}(I + J)$   
*if and only if  $I+J$  is pure dimensional and  $(S/I)_P$  is Cohen-Macaulay for all isolated components of  $I + J$ .*

**COROLLARY 4.2.3** *Let  $I$  be a pure dimensional homogeneous ideal of  $S$  and let  $J \subset S$  be an ideal generated by a maximal  $S/I$ -sequence of length  $t$ .*

- (i) *If  $t < \dim(S/I)$  then  $\text{arith-deg}(I) \cdot \text{arith-deg}(J) < \text{arith-deg}(I + J)$ .*
- (ii) *If  $t = \dim(S/I)$  then  $\text{arith-deg}(I) \cdot \text{arith-deg}(J) = \text{arith-deg}(I + J)$ .*

The proof of Theorem 4.2.1 will make use of the following lemma.

**LEMMA 4.2.4** *Let  $I, J$  be pure dimensional ideals in  $S$ . If  $\dim(S/(I + J)) = \dim(S/I) + \dim(S/J) - n$ , then*

$$\text{deg}(I) \cdot \text{deg}(J) \leq \text{deg}(I + J).$$

**PROOF** This follows from [AHV85, Corollary 2(i)].  $\square$

**PROOF OF THEOREM 4.2.1**

$$\begin{aligned}
 \text{(i) } \text{arith-deg}(I + J) &= \sum_{p_i \in \text{Ass}(S/(I+J))} \text{mult}_{I+J}(p_i) \cdot \text{deg}(p_i) \\
 &= \sum_{\dim(S/p_i) = \dim(S/(I+J))} \text{mult}_{I+J}(p_i) \cdot \text{deg}(p_i) \\
 &\quad + \sum_{\dim(S/p_i) \neq \dim(S/(I+J))} \text{mult}_{I+J}(p_i) \cdot \text{deg}(p_i) \\
 &= \text{deg}(I + J) + \sum_{\dim(S/p_i) \neq \dim(S/(I+J))} \text{mult}_{I+J}(p_i) \cdot \text{deg}(p_i) \\
 &\geq \text{deg}I \cdot \text{deg}J + \sum_{\dim(S/p_i) \neq \dim(S/(I+J))} \text{mult}_{I+J}(p_i) \cdot \text{deg}(p_i) \\
 &\quad \text{(by Lemma 4.2.4)} \\
 &\geq \text{deg}I \cdot \text{deg}J \qquad \qquad \qquad (*) \\
 &= \text{arith-deg}(I) \cdot \text{arith-deg}(J).
 \end{aligned}$$

(ii) If  $I + J$  is not pure dimensional then there is at least one  $p_i$  with  $\dim(S/p_i) \neq \dim(S/(I + J))$ . Hence we have the strict inequality in (\*).

(iii) By (ii) the equality holds only if  $I + J$  is pure-dimensional. In this case  $\text{arith-deg}(I + J) = \text{deg}(I + J)$ . Therefore the last statement follows from [AHV85, Corollary 2(iii)].  $\square$

**EXAMPLE 4.2.5** Let

$$\begin{aligned}
 S &= k[x_1, \dots, x_5], \\
 I &= (x_3x_4 - x_2x_5, x_4^3 - x_3x_5^2, x_3^3 - x_2^2x_4, x_2x_4^2 - x_3^2x_5), \\
 J &= (x_2, x_5).
 \end{aligned}$$

Then

$$\begin{aligned}
 I + J &= (x_3x_4 - x_2x_5, x_4^3 - x_3x_5^2, x_3^3 - x_2^2x_4, x_2x_4^2 - x_3^2x_5, x_2, x_5) \\
 &= (x_3x_4, x_4^3, x_3^3, x_2, x_5)
 \end{aligned}$$

which is  $(x_2, \dots, x_5)$ -primary. Therefore,  $I + J$  is pure dimensional. We have  $\dim(S/I) = 3$ ,  $\dim(S/J) = 3$ , and  $\dim(S/(I + J)) = 1$ . Thus  $\dim(S/I) + \dim(S/J) - n = 3 + 3 - 5 = 1 = \dim(S/(I + J))$  as required.

$$\begin{aligned}
 (x_3x_4, x_4^3, x_3^3, x_2, x_5) &\subset (x_3x_4, x_4^2, x_3^3, x_2, x_5) \subset (x_3x_4, x_4^2, x_3^2, x_2, x_5) \\
 &\subset (x_3^2, x_4, x_2, x_5) \qquad \subset (x_2, x_3, x_4, x_5)
 \end{aligned}$$

is a maximal strictly increasing chain of  $(x_2, x_3, x_4, x_5)$ -primary ideals from  $I + J$  to  $(x_2, x_3, x_4, x_5)$ . Therefore,

$$\begin{aligned} \text{mult}_{I+J}(x_2, x_3, x_4, x_5) &= 5, \\ \text{arith-deg}(I + J) &= \text{mult}_{I+J}(x_2, x_3, x_4, x_5) \cdot \text{deg}(x_2, x_3, x_4, x_5) \\ &= 5 \cdot 1 = 5, \\ \text{arith-deg}(I) &= \text{deg}(I) = 4, \text{ and} \\ \text{arith-deg}(J) &= \text{deg}(J) \\ &= \text{mult}_J(x_2, x_5) \cdot \text{deg}(x_2, x_5) = 1. \end{aligned}$$

Therefore,  $\text{arith-deg}(I) \cdot \text{arith-deg}(J) = 4 < 5 = \text{arith-deg}(I + J)$  and  $I + J$  is pure dimensional. The reason for the inequality in this case is that  $(S/I)_{(x_1, \dots, x_5)} \simeq (k[s^4, s^3t, st^3, t^4])_{(x_1, \dots, x_5)}$  which is not a Cohen-Macaulay ring.  $\square$

The proof of Corollary 4.2.2 will make use of the following lemma, facts and propositions.

**LEMMA 4.2.6** *Let  $I$  and  $J$  be two homogeneous ideals of  $S$ . Then*

$$\dim(S/(I + J)) \geq \dim(S/I) + \dim(S/J) - n.$$

**PROOF** Let  $P \in \text{Ass}(S/I)$  with  $\dim(S/P) = \dim(S/I)$  and let  $Q \in \text{Ass}(S/J)$  with  $\dim(S/Q) = \dim(S/J)$ . Then  $I + J \subset P + Q$  and  $\dim(S/(I + J)) \geq \dim(S/(P + Q))$ . Therefore by [ZS60, Theorem 27, ChVII],  $\dim(S/(I + J)) \geq \dim(S/(P + Q)) \geq \dim(S/P) + \dim(S/Q) - n = \dim(S/I) + \dim(S/J) - n$ .  $\square$

**FACT 4.2.7** *Let  $J \subset S$  be the ideal generated by the forms  $F_1, \dots, F_t \in S$ . Then*

$$\dim(S/J) \geq n - t.$$

**FACT 4.2.8** *Let  $I$  be a homogeneous ideal of  $S$  and let  $J \subset S$  be an ideal generated by a part of a system of parameters of  $S/I$  of length  $t$ . Then*

$$\dim(S/(I + J)) = \dim(S/I) - t.$$

**PROPOSITION 4.2.9** *Let  $I$  be a homogeneous ideal of  $S$  and let  $J \subset S$  be an ideal generated by a part of a system of parameters of  $S/I$  of length  $t$ . Then*

$$\dim(S/J) = n - t.$$

**PROOF** We have  $\dim(S/(I + J)) \geq \dim(S/I) + \dim(S/J) - n$  by Lemma 4.2.6 and  $\dim(S/(I + J)) = \dim(S/I) - t$  by Fact 4.2.8. Therefore,  $\dim(S/I) - t \geq \dim(S/I) + \dim(S/J) - n$  and  $n - t \geq \dim(S/J)$ . We also have  $\dim(S/J) \geq n - t$  by Fact 4.2.7. Hence,  $\dim(S/J) = n - t$ .  $\square$

**PROPOSITION 4.2.10** *Let  $I$  be a homogeneous ideal of  $S$  and let  $J \subset S$  be an ideal generated by a part of a system of parameters of  $S/I$  of length  $t$ . Then*

$$\dim(S/(I + J)) = \dim(S/I) + \dim(S/J) - n.$$

**PROOF** We have  $\dim(S/J) = n - t$  by Proposition 4.2.9 and  $\dim(S/(I + J)) = \dim(S/I) - t$  by Fact 4.2.8. Therefore,  $\dim(S/I) + \dim(S/J) - n = \dim(S/I) + (n - t) - n = \dim(S/I) - t = \dim(S/(I + J))$ .  $\square$

**PROOF OF COROLLARY 4.2.2** Let  $J = (F_1, \dots, F_t)$ . Then  $\dim(S/J) = n - t$  by Proposition 4.2.9.  $J$  is therefore a complete intersection by Fact 3.4.3. Hence  $J$  is pure dimensional and  $S/J$  is a Cohen-Macaulay ring. Also,  $\dim(S/(I + J)) = \dim(S/I) + \dim(S/J) - n$  by Proposition 4.2.10 so all assumptions of the theorem are satisfied. Hence,

- (i)  $\text{arith-deg}(I) \cdot \text{arith-deg}(J) \leq \text{arith-deg}(I + J)$ .
- (ii) If  $I + J$  is not pure dimensional then  
 $\text{arith-deg}(I) \cdot \text{arith-deg}(J) < \text{arith-deg}(I + J)$ .
- (iii)  $\text{arith-deg}(I) \cdot \text{arith-deg}(J) = \text{arith-deg}(I + J)$   
 if and only if  $I + J$  is pure dimensional and  $(S/I)_p$   
 is Cohen-Macaulay for all isolated components of  $I + J$ .  $\square$

The proof of Corollary 4.2.3 will make use of the following additional lemmas.

**LEMMA 4.2.11** *An  $S/I$ -sequence is a part of a system of parameters of  $S/I$ .*

**PROOF** Let  $f_1, \dots, f_t \in S$  be an  $S/I$ -sequence. Then, by the Definition 4.1.2,  $f_i$  is a non-zero divisor on  $(S/I)/(f_1, \dots, f_{i-1}) \cdot (S/I)$  for  $i = 1, \dots, t$ . Hence, by [ZS60, Corollary 2, ChVIII, p290],  $\dim(S/I \cdot (f_1)) = \dim(S/I) - 1$ ,  $\dim(S/I \cdot (f_1, f_2)) = \dim(S/I \cdot f_1) - 1$ , and by induction  $\dim(S/I(f_1, \dots, f_t)) = \dim(S/I) - t$ . Therefore, by [Nor53, Theorem 2, p64],  $f_1, \dots, f_t$  is a part of a system of parameters of  $S/I$ .  $\square$

**LEMMA 4.2.12** *Let  $I$  be a homogeneous ideal of  $S$  and let  $J \subseteq S$  be an ideal generated by a maximal  $S/I$ -sequence. Then*

*$I + J$  has an  $(x_1, \dots, x_n)$ -primary component.*

**PROOF** Let  $f_1, \dots, f_t$  be a maximal  $S/I$ -sequence in  $(x_1, \dots, x_n) \cdot S/I$ . Suppose  $(f_1, \dots, f_t)$  has no  $(x_1, \dots, x_n)$ -primary component. Let  $P_1, \dots, P_s$  be the prime ideals belonging to  $(f_1, \dots, f_t)$ . Then by [Nor53, Proposition 6, Ch1], there exists  $f_{t+1} \in (x_1, \dots, x_n)$  such that  $f_{t+1} \notin P_i$  for all  $i = 1, \dots, s$ . Therefore  $(f_1, \dots, f_t) : f_{t+1} = (f_1, \dots, f_t)$  by [Nor53, Theorem 6, Ch1] and  $f_{t+1}$  is a non-zero divisor on  $(S/I)/((f_1, \dots, f_t) \cdot (S/I))$ . Hence  $f_1, \dots, f_t, f_{t+1}$  is an  $S/I$ -sequence. This contradicts the fact that  $f_1, \dots, f_t$  is a maximal  $S/I$ -sequence. Therefore  $(f_1, \dots, f_t)$  has an  $(x_1, \dots, x_n)$ -primary component. Let  $F_j \in S$  be a fixed representative of  $f_j$  for all  $j = 1, \dots, t$ . Let  $J = (F_1, \dots, F_t)$ . Then  $(I, F_1, \dots, F_t)$  has an  $(x_1, \dots, x_n)$ -primary component.  $\square$

**PROOF OF COROLLARY 4.2.3** Let  $J = (F_1, \dots, F_t)$ . Then  $\dim(S/J) = n - t$  by Proposition 4.2.9. Hence,  $J$  is a complete intersection and pure dimensional. Also,  $\dim(S/(I + J)) = \dim(S/I) + \dim(S/J) - n$  by Proposition 4.2.10 so all assumptions of the Theorem 4.2.1 are satisfied.

(i) We have  $\dim(S/(I + J)) = \dim(S/I) - t$  by Fact 4.2.8 and  $\dim(S/I) > t$  by assumption. Hence,  $\dim(S/(I + J)) > 0$ . Also,  $I + J$  has an  $(x_1, \dots, x_n)$ -primary component by Lemma 4.2.12. Therefore,  $(I + J)$  is not pure dimensional and by part (ii) of Theorem 4.2.1

$$\text{arith-deg}(I) \cdot \text{arith-deg}(J) < \text{arith-deg}(I + J).$$

(ii) If  $t = \dim(S/I)$  then  $S/I$  is a Cohen-Macaulay ring,  $I + J$  is an  $(x_1, \dots, x_n)$ -primary ideal (hence pure dimensional), and by Corollary 4.2.2 we have

$$\text{arith-deg}(I) \cdot \text{arith-deg}(J) = \text{arith-deg}(I + J). \quad \square$$

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## Arithmetic degree under hyperplane section

In this chapter we consider the following problem.

**PROBLEM 5.0.1** *Let  $S := k[x_0, \dots, x_n]$ ,  $k$  a field. Let  $I$  be a homogeneous ideal of  $S$  and let  $f$  be a homogeneous element of  $S$ . Is it possible to bound  $\text{arith-deg}(I + (f))$  by  $\text{deg} f \cdot \text{arith-deg}(I)$ ?*

Consider the following examples.

**EXAMPLE 5.0.2**

$$\begin{aligned} \text{Let } S &= k[x_0, x_1, x_2, x_3], \\ I &= (x_0^2 x_1, x_0 x_2, x_0^2 x_3) = (x_0) \cap (x_0^2, x_2) \cap (x_1, x_2, x_3), \text{ and} \\ f &= x_3. \end{aligned}$$

Then

$$\begin{aligned} (I, f) &= (x_0^2 x_1, x_0 x_2, x_3) = (x_0, x_3) \cap (x_0^2, x_2, x_3) \cap (x_1, x_2, x_3), \\ \text{arith-deg}(I, f) &= \text{mult}_{(I, f)}(x_0, x_3) + \text{mult}_{(I, f)}(x_0, x_2, x_3) \\ &\quad + \text{mult}_{(I, f)}(x_1, x_2, x_3) \\ &= 1 + 1 + 1 = 3, \\ \text{arith-deg}(I) &= \text{mult}_{(I)}(x_0) + \text{mult}_{(I)}(x_0, x_2) + \text{mult}_{(I)}(x_1, x_2, x_3) \\ &= 1 + 1 + 1 = 3, \\ \text{deg} f &= \text{deg}(x_3) = 1, \text{ and} \end{aligned}$$

$$\text{arith-deg}(I + (f)) = \text{deg}f \cdot \text{arith-deg}(I). \square$$

**EXAMPLE 5.0.3**

$$\begin{aligned} \text{Let } S &= k[x_0, x_1, x_2, x_3], \\ I &= (x_0^2x_1, x_0x_2, x_0^2x_3) = (x_0) \cap (x_0^2, x_2) \cap (x_1, x_2, x_3), \text{ and} \\ f &= x_0 - x_2. \end{aligned}$$

Then

$$\begin{aligned} (I, f) &= (x_0^2x_1, x_0x_2, x_0^2x_3, x_0 - x_2) \\ &= (x_2^2, x_0 - x_2). \end{aligned}$$

Therefore,

$$\text{arith-deg}(I, f) = 2 < 3 = \text{deg}f \cdot \text{arith-deg}(I). \square$$

**EXAMPLE 5.0.4**

$$\begin{aligned} \text{Let } S &= k[x_0, x_1, x_2, x_3, x_4], \\ I &= (x_0^2x_2, x_0^2x_3, x_0^2x_4, x_0x_1x_2, x_0x_1x_3, x_0x_1x_4) \\ &= (x_0) \cap (x_0^2, x_1) \cap (x_2, x_3, x_4), \text{ and} \\ f &= x_0 - x_2. \end{aligned}$$

$$\text{Then } (I, f) = (x_0^2x_2, x_0^2x_3, x_0^2x_4, x_0x_1x_2, x_0x_1x_3, x_0x_1x_4, x_0 - x_2).$$

$$\text{Let } (I, f)^* = (I, f)/(x_0 - x_2) \hookrightarrow k[x_1, \dots, x_4]$$

Then

$$\begin{aligned} (I, f)^* &= (x_2^3, x_2^2x_3, x_2^2x_4, x_1x_2^2, x_1x_2x_3, x_1x_2x_4) \\ &= (x_2) \cap (x_1, x_2^2) \cap (x_2^2, x_3, x_4) \cap (x_2^3, x_1, x_3, x_4). \end{aligned}$$

Therefore,

$$\begin{aligned} (I, f) &= (x_0, x_2) \cap (x_1, x_2^2, x_0 - x_2) \cap (x_2^2, x_3, x_4, x_0 - x_2) \\ &\quad \cap (x_2^3, x_1, x_3, x_4, x_0 - x_2). \end{aligned}$$

In order to compute the arithmetic degree of  $(I, f)$ , we must first determine

$$\begin{aligned} &\text{mult}_{(I, f)}(x_0, x_2), \text{mult}_{(I, f)}(x_0, x_1, x_2), \text{mult}_{(I, f)}(x_0, x_2, x_3, x_4), \\ &\text{and } \text{mult}_{(I, f)}(x_0, x_1, x_2, x_3, x_4). \end{aligned}$$

It is easily seen that

$$\text{mult}_{(I,f)}(x_0, x_2) = \text{mult}_{(I,f)}(x_0, x_1, x_2) = \text{mult}_{(I,f)}(x_0, x_2, x_3, x_4) = 1.$$

Computing  $\text{mult}_{(I,f)}(x_0, x_1, x_2, x_3, x_4)$  is more difficult. Using Algorithm 1.5.1 we have

$$\begin{aligned} Q &= (x_2^3, x_1, x_3, x_4, x_0 - x_2), \\ P &= (x_0, x_1, x_2, x_3, x_4), \text{ and} \\ J_P &= (x_0, x_2) \cap (x_1, x_2^2, x_0 - x_2) \cap (x_2^2, x_3, x_4, x_0 - x_2) \\ &= (x_2^2, x_1x_2x_3, x_1x_2x_4, x_0 - x_2). \end{aligned}$$

A maximal strictly increasing chain of ideals from  $Q$  to  $P$  is

$$(1) \quad (x_2^3, x_1, x_3, x_4, x_0 - x_2) \subset (x_2^2, x_1, x_3, x_4, x_0 - x_2) \subset (x_0, x_1, x_2, x_3, x_4).$$

Intersecting each primary ideal in (1) with  $J_P$  we obtain

$$\begin{aligned} Q \cap J_P &= J_1 = (x_1x_2^2, x_2^2x_3, x_2^2x_4, x_1x_2x_3, x_1x_2x_4, x_2^3, x_0 - x_2) \\ &\subset (x_2^2, x_1x_2x_3, x_1x_2x_4, x_0x_2 - x_2^2) \\ &= (x_2^2, x_1x_2x_3, x_1x_2x_4, x_0 - x_2) \\ &= J_P. \end{aligned}$$

There are no duplicates in this chain, hence  $\text{mult}_{(I,f)}(x_0, x_1, x_2, x_3, x_4) = 1$ .

Using this information we have

$$\begin{aligned} \text{arith-deg}(I, f) &= \text{mult}_{(I,f)}(x_0, x_2) + \text{mult}_{(I,f)}(x_0, x_1, x_2) \\ &\quad + \text{mult}_{(I,f)}(x_0, x_2, x_3, x_4) + \text{mult}_{(I,f)}(x_0, x_1, x_2, x_3, x_4) \\ &= 1 + 1 + 1 + 1 = 4, \\ \text{arith-deg}(I) &= \text{mult}_{(I)}(x_0) + \text{mult}_{(I)}(x_0, x_1) + \text{mult}_{(I)}(x_2, x_3, x_4) \\ &= 1 + 1 + 1 = 3, \text{ and} \\ \text{arith-deg}(I, f) &= 4 > 3 = \text{deg}f \cdot \text{arith-deg}(I). \quad \square \end{aligned}$$

These three examples show that

$$\begin{aligned} \text{arith-deg}(I, f) &> \text{deg}f \cdot \text{arith-deg}(I), \\ \text{arith-deg}(I, f) &= \text{deg}f \cdot \text{arith-deg}(I), \text{ and} \\ \text{arith-deg}(I, f) &< \text{deg}f \cdot \text{arith-deg}(I) \end{aligned}$$

are all possible.

**PROBLEM 5.0.5** Find necessary and sufficient conditions for each of the above three inequalities.

Below we give a sufficient condition for the inequality

$$\text{arith-deg}(I + (f)) \geq \text{deg}(f) \cdot \text{arith-deg}(I).$$

First we recall the following refined definition of the arithmetic degree. Let  $r$  be an integer with  $r \geq -1$ . The  $r$ -th arithmetic degree of an ideal  $I$ , denoted by  $\text{arith-deg}_r(I)$ , is defined to be

$$\text{arith-deg}_r(I) := \sum \text{mult}_I(P) \cdot \text{deg}P$$

where the sum is taken over all prime ideals  $P$  belonging to  $I$  with  $\dim(S/P) = r+1$ .

**PROPOSITION 5.0.6** Let  $I$  be a homogeneous ideal of  $S := k[x_0, \dots, x_n]$  having no  $(x_0, \dots, x_n)$ -primary component. Let  $f$  be a non-zero divisor on  $S/I$ . Then

$$\text{arith-deg}(I + (f)) \geq \text{deg}(f) \cdot \text{arith-deg}(I)$$

**PROOF** By [MVY96, Corollary 2.7]

$$\text{arith-deg}_{r-1}(I + (f)) - \text{deg}(f) \cdot \text{arith-deg}_r(I) \geq 0$$

for all  $r \geq 0$ . Therefore

$$\sum_{r=0}^{\infty} \text{arith-deg}_{r-1}(I + (f)) \geq \text{deg}(f) \cdot \sum_{r=0}^{\infty} \text{arith-deg}_r(I).$$

Thus

$$\text{arith-deg}(I + (f)) \geq \text{deg}(f) \cdot \text{arith-deg}(I) - \text{deg}(f) \cdot \text{arith-deg}_{-1}(I).$$

We have  $\text{arith-deg}_{-1}(I) = 0$  as  $I$  has no  $(x_0, \dots, x_n)$ -primary component. Hence,

$$\text{arith-deg}(I + (f)) \geq \text{deg}(f) \cdot \text{arith-deg}(I). \quad \square$$

---

## Conclusions and suggestions for future work

This thesis has been concerned with the study of bounds on the arithmetic degree of an ideal. A question previously posed by [STV95] on the possible generalization of a lower bound on the arithmetic degree of a monomial ideal to the lower bound on the arithmetic degree of a homogeneous ideal has been answered in the negative. Detailed calculations of counterexamples to bounds on the geometric degree restated in terms of bounds on the arithmetic degree have been given. Finally, a new Bezout-type theorem for the arithmetic degree of ideals under certain conditions has been found. While it is hoped that each of these contributions will help further the understanding of bounds on the arithmetic degree of an ideal, considerably more research is needed in this area.

In Chapter 2 we show through a class of counterexamples that the lower bound of Theorem 2.0.1 for the arithmetic degree of monomial ideals does not generalize to a lower bound for the arithmetic degree of arbitrary homogeneous ideals. However, very often examples show that the lower bound of Theorem 2.0.1 does hold for some homogeneous ideals. This raises the question: What are necessary and sufficient conditions for

$$\max\{\deg(g_i) : i = 1, \dots, s\} \leq \text{arith-deg}(I),$$

where  $I$  is a homogeneous ideal generated by  $g_1, \dots, g_s$ ?

In Section 0.1 we noted that the upper bound on the geometric degree in Proposition 0.1.1 does not generalize to an upper bound on the arithmetic degree. Theorem 2.3 and Proposition 4.1 of [STV95] show that the arithmetic degree goes up under Gröbner basis computations while the geometric degree goes down. In Chapter 3 of this thesis we provided counterexamples to the possible use of the arithmetic degree in three known results concerning bounds on the geometric degree. Each of these examples illustrate the need for future study on the relationship between the arithmetic degree and the geometric degree of an ideal. As the geometric degree and the arithmetic degree are two basic measures of the complexity of an ideal, the understanding of relationships between the two is important to the field of algebraic geometry.

Theorem 4.1 of [MV96] and results in Chapter 4 of this thesis provide a beginning to the study of Bezout-type theorems for the arithmetic degree of ideals. Our theorem, however, is quite restricted in its assumptions. A next step might be to look at what happens when the ideals  $I$  and  $J$  are not pure dimensional, but have no embedded components, and then to look at what happens when  $I$  and  $J$  have embedded components. Another avenue of investigation might be to look at the relationship between the  $\text{arith-deg}_r(I + J)$  and  $\text{arith-deg}_s(I) \cdot \text{arith-deg}_t(J)$  where the index  $r$  is related to  $s$  and  $t$  in some way.

Finally, the work done in Chapter 5, gives us some information on the arithmetic degree under hypersurface sections, but leaves us with the open problem of finding necessary and sufficient conditions for each of the following:

$$\begin{aligned} \text{arith-deg}(I, f) &> \text{deg}f \cdot \text{arith-deg}(I), \\ \text{arith-deg}(I, f) &= \text{deg}f \cdot \text{arith-deg}(I), \\ \text{arith-deg}(I, f) &< \text{deg}f \cdot \text{arith-deg}(I). \end{aligned}$$

In conclusion, this thesis has brought together a discussion of bounds on the arithmetic degree of ideals as begun by [BM93], [STV95], [MVY96], and [MV96]. The work was fruitful in terms of broadening the understanding of some problems raised by these papers, but as we have seen, much work has yet to be done. It is hoped that future research will help solve these important and difficult problems, and that the work in this thesis may serve as a signpost for such future developments.

## APPENDIX A

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## Some relationships between geometry and algebra

Geometry	Algebra
$P_k^n$ , $n$ -dimensional projective space.  $X \subset P_k^n$ , scheme defined over $k$ .	$k[x_0, \dots, x_n]$ , polynomial ring in $n + 1$ unknowns.  $I(X) \subset k[x_0, \dots, x_n]$ , homogeneous ideal generated by the forms defining $X$ .

Table A.1: Some relationships between geometry and algebra.

Geometry	Algebra
$X$ is reduced.	$I(X)$ is the intersection of prime ideals.
$X$ is irreducible.	$I(X)$ is a primary ideal.
$X$ is a subvariety (or variety.)	$I(X)$ is a prime ideal.
$X$ is a pure dimensional scheme.	$I(X)$ has only top-dimensional primes.
$X$ is a scheme, $C$ is an irreducible component of $X$ ; that is, $C$ is reduced and irreducible with $C \subseteq X$ .	$I(C)$ is a minimal prime ideal belonging to $I(X)$ .
$Y \cap Z$ , $Y$ and $Z$ schemes.	$I(Y) + I(Z)$ , $I(Y)$ and $I(Z)$ defining ideals of $Y$ and $Z$ respectively.
$C$ is an irreducible component of $Y \cap Z$ .	$I(C)$ is a minimal prime belonging to $I(Y) + I(Z)$ .
The projective variety $Y$ is a hypersurface.	$Y$ is the zero set of some irreducible homogeneous polynomial $f$ of positive degree in $k[x_0, \dots, x_n]$ .
The degree of $X$ .	The degree of $I(X)$ .
The geometric degree of $X$ .	The geometric degree of $I(X)$ .
The arithmetic degree of $X$ .	The arithmetic degree of $I(X)$ .
The dimension of $X$ .	The Krull dimension of $S/I(X) - 1$ .
$X \subset P_k^n$ is Cohen-Macaulay.	$(k[x_0, \dots, x_n]/I(X))_P$ is Cohen-Macaulay for all primes $P$ .
$X \subset P_k^n$ is locally Cohen-Macaulay.	$I(X) \subset k[x_0, \dots, x_n]$ is locally Cohen-Macaulay; that is $(k[x_0, \dots, x_n]/I(X))_P$ is Cohen Macaulay for all homogeneous prime ideals $P \neq (x_0, \dots, x_n)$ of $k[x_0, \dots, x_n]$ .

Table A.2: More relationships between geometry and algebra.

A P P E N D I X B

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## Computer output

In section 3.4 we considered the surface  $X$  of  $P_k^5$  given parametrically by  $(uv^3, uv^2w, v^4, v^3w, vw^3, w^4)$  and its defining prime ideal  $I$  in

$S := k[x_1, x_2, x_3, x_4, x_5, x_6]$ . We claimed that  $I = (x_2x_3 - x_1x_4, x_2x_5 - x_1x_6, x_4x_5 - x_3x_6, x_4^3 - x_3^2x_5, x_2x_4^2 - x_1x_3x_5, x_2^2x_4 - x_1^2x_5, x_1x_5^2 - x_4^2x_6, x_1x_5^2 - x_2x_4x_6, x_5^3 - x_4x_6^2)$ ,  $\dim(S/I) = 3$ ,  $\deg(I) = 5 = \text{codim}(I) + 2$ , and  $\text{depth}(S/I) = 1$ . This information was obtained by a run of the computer program Macaulay [BS90], the output of which follows.

```
% ring R
! characteristic (if not 31991) ?
! number of variables ? 9
! 9 variables, please ? uvwx[1]-x[6]
! variable weights (if not all 1) ? 1:3 4:6
! monomial order (if not rev. lex.) ? 3 6
; largest degree of a monomial : 512 471

% ideal J
! number of generators ? 6
! (1,1) ? x[1]-uv3
! (1,2) ? x[2]-uv2w
! (1,3) ? x[3]-v4
! (1,4) ? x[4]-v3w
```

```

! (1,5) ? x[5]-vw3
! (1,6) ? x[6]-w4

% std J J
; 4[126k]5.6.7.8.9.10.11.12.13.14.15.16.17.
; computation complete after degree 17

% elim J I

% type I
; x[5]3-x[4]x[6]2 x[4]x[5]-x[3]x[6] x[4]3-x[3]2x[5] x[3]x[5]2-x[4]2x[6]

; x[2]x[5]-x[1]x[6] x[2]x[4]2-x[1]x[3]x[5] x[2]x[3]-x[1]x[4]

; x[2]2x[4]-x[1]2x[5] x[1]x[5]2-x[2]x[4]x[6]

% deg I
; warning: no standard basis. Using initial terms of matrix
; codimension : 3
; degree : 5

% res I p
; 7.8.9.10.11..12..13..14..15..16..17..18...19...20...21....22....23....24.....
; 25.....26.....27.....28.....29.....30.....
; computation complete after degree 30

% pres p
;
;

; x[4]3-x[3]x[5]2 x[3]x[4]-x[2]x[5] x[3]3-x[2]2x[4] x[2]x[4]2-x[3]2x[5]

; x[1]x[4]-x[0]x[5] x[1]x[3]2-x[0]x[2]x[4] x[1]x[2]-x[0]x[3]

```

```

; x[1]2x[3]-x[0]2x[4] x[0]x[4]2-x[1]x[3]x[5]
;
;
; 0      0      0      0      0      -x[0]  0      0      0
; x[0]   x[1]   0      0      0      0      0      0      0
; 0      0      0      0      0      0      0      0      0
; 0      0      0      -x[0]  0      0      0      0      0
; -x[2]  -x[3]  x[0]x[4]  0      x[2]x[4]  x[3]x[5]  0      0      x[1]x[3]
; 0      0      0      0      -x[5]  0      x[0]   x[1]   0
; x[4]   x[5]   0      x[3]x[5]  -x[4]2  0      x[1]x[3]  x[0]x[4]  0
; 0      0      -x[5]  0      0      0      -x[2]  -x[3]  -x[4]
; 0      0      -x[1]  x[2]   -x[3]  x[4]   0      0      -x[0]

; 0      0      0      0      0      -x[2]  0      -x[1]  -x[3]
; 0      0      0      0      x[2]x[4]  x[3]x[5]  x[3]2  0      x[4]2
; x[0]   x[1]   0      0      -x[5]  0      -x[4]  0      0
; 0      0      -x[0]  -x[1]  -x[3]  x[4]   -x[2]  0      x[5]
; 0      0      x[3]2  0      0      0      0      x[4]2  0
; -x[2]  -x[3]  -x[4]  0      0      0      0      0      0
; x[3]2  x[2]x[4]  0      x[4]2  0      0      0      0      0
; 0      0      0      0      0      0      0      0      0
; 0      0      0      x[3]  0      0      0      x[5]  0
;
;
;

```

```

; 0      x[3]2  0      x[3]x[5] x[4]2  0      0      0      0      0
; x[4]2  0      0      0      x[3]x[5] x[2]x[4] x[3]2  0      x[3]x[5] 0
; 0      0      0      0      0      0      0      0      0      0
; 0      0      0      -x[4]  -x[5]  0      -x[3]  0      -x[5]  0
; 0      0      0      0      0      x[3]  0      0      -x[4]  0
; 0      0      0      x[2]  x[3]  0      0      0      0      0
; 0      0      0      0      0      0      0      0      0      x[3]
; 0      0      0      0      0      0      0      -x[4]  0      -x[2]
; 0      0      0      0      0      0      0      x[3]  0      0
; 0      -x[4]  0      0      0      0      0      0      0      -x[1]
; 0      0      0      0      0      -x[5]  -x[4]  0      0      x[0]
; 0      x[2]  0      0      0      0      x[3]  -x[1]  x[5]  0
; -x[5]  0      0      0      -x[4]  x[3]  x[2]  x[0]  -x[4]  0
; 0      0      -x[4]  0      0      -x[1]  0      0      0      0
; 0      0      -x[3]  -x[0]  -x[1]  0      0      0      -x[1]  0
; 0      -x[0]  x[5]  0      0      0      -x[1]  0      0      0
; x[3]  0      0      0      x[2]  0      0      0      x[2]  0
; -x[1]  0      x[2]  0      -x[0]  0      0      0      0      0

```

```

; x[2]x[4] -x[1]x[3] 0 0 -x[1]x[4]
; 0 x[0]x[3] 0 0 x[0]x[4]
; 0 0 -x[4] -x[2] x[3]
; x[3] -x[0] 0 -x[1] 0
; x[2] 0 0 x[0] -x[1]
; 0 0 -x[1] 0 0
; 0 x[4] 0 x[5] 0
; 0 0 0 x[4] -x[5]
; 0 -x[2] x[5] -x[3] 0
; -x[5] 0 0 0 0
; 0 0 0 0 0
; 0 x[0] 0 x[1] 0
; 0 0 0 0 0
; -x[0] 0 0 0 0
; 0 0 0 0 0
; 0 0 0 0 0
; 0 0 x[0] 0 0
; 0 0 0 0 0
;

```

;

```

; 0      x[0]  0      0      x[2]  0
; 0      0      -x[1]  0      0      x[5]
; 0      0      0      0      x[1]  x[0]
; 0      0      0      -x[1]  0      -x[3]
; 0      -x[1]  0      0      0      x[2]
; x[0]   0      0      0      -x[4]  0
; 0      0      x[0]  0      x[5]  0
; -x[3]  x[5]   -x[2]  x[4]  0      0
; 0      x[1]  0      x[0]  -x[3]  -x[2]
; x[5]   0      x[4]  0      0      0
; -x[1]  0      0      0      0      -x[4]
; 0      0      -x[3]  -x[5]  0      0
; 0      -x[3]  0      -x[2]  0      0
; -x[3]  0      0      x[4]  0      0
; -x[2]  -x[4]  0      0      0      0
;

```

;

```

; x[4]
; -x[2]
; -x[5]
; x[3]
; x[0]
; -x[1]
;
;

```

% betti p

```

; total : 1  9  18  15  6  1
;
; 0 :    1  -  -  -  -  -
; 1 :    -  -  -  -  -  -
; 2 :    -  -  -  -  -  -
; 3 :    -  -  -  -  -  -
; 4 :    -  -  -  -  -  -
; 5 :    -  -  -  -  -  -
; 6 :    -  -  -  -  -  -
; 7 :    -  3  -  -  -  -
; 8 :    -  -  -  -  -  -
; 9 :    -  -  -  -  -  -
; 10 :   -  -  2  -  -  -
; 11 :   -  6  -  -  -  -
; 12 :   -  -  -  -  -  -
; 13 :   -  -  -  -  -  -
; 14 :   -  - 16  -  -  -
; 15 :   -  -  -  -  -  -
; 16 :   -  -  -  -  -  -
; 17 :   -  -  - 15  -  -
; 18 :   -  -  -  -  -  -
; 19 :   -  -  -  -  -  -
; 20 :   -  -  -  -  6  -
; 21 :   -  -  -  -  -  -
; 22 :   -  -  -  -  -  -
; 23 :   -  -  -  -  -  1

```

By [Eis95, Theorem 19.9] we have  $\text{depth}(I)$  is equal to the number of variables in  $S$  less the length of the free resolution of  $I$ . Therefore, using the above we have  $\text{depth}(I) = 6 - 5 = 1$ .

---

# Index

- codimension, 25, 41
- Cohen-Macaulay, 22, 39
- coheight, 9
- complete intersection, 24
- cone construction, 23-24
- degree, 1, 12
  - arithmetic, 1-3, 13
  - classical, 1-2, 12-13
  - geometric, 1-2, 12-13
- depth, 22, 25, 46
- dimension, 9-10, 23-24, 39
  - Krull dimension, 9-10, 39
  - pure dimensional, 9-10
  - top-dimensional, 9-10, 39
- form, 5, 38
- height, 18, 25
- ideal, 5
  - homogeneous, 5
  - homogenization of, 18
  - ideal quotient, 19
  - monomial, 7
    - $P$ -primary, 7-8
    - prime, 7-8
- prime, 7-8
- primary, 6
  - $P$ -primary, 6
- prime, 6
  - associated, 6
  - embedded, 6, 8
  - isolated, 6, 8
  - minimal, 6
  - product of, 14
  - sum of, 14
- length-multiplicity, 10-12
  - with respect to  $I$ , 10-12
- local ring, 26
- localization, 22
- locally Cohen-Macaulay, 23, 39
- monomial, 7
  - ideal, 7
    - $P$ -primary, 7-8
    - prime, 7-8
  - relatively prime, 7-8
- primary decomposition, 6-9
  - irredundant, 6

normal, 6, 8

regular sequence, 22

$R/I$ -sequence, 26-27, 30-31

splitting lemma, 6-9

system of parameters, 26-27, 29-30

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