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**q-series**  
**in**  
**Number Theory**  
**and**  
**Combinatorics**

A thesis presented in partial fulfillment

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# Abstract

Srinivasa Ramanujan (1887-1920) was one of the world's greatest mathematical geniuses. He work extensively in a branch of mathematics called “ $q$ -series”. Around 1913, he found an important formula which now is known as Ramanujan's  ${}_1\psi_1$  summation formula.

The aim of this thesis is to investigate Ramanujan's  ${}_1\psi_1$  summation formula and explore its applications to number theory and combinatorics. First, we consider several classical important results on elliptic functions and then give new proofs of these results using Ramanujan's  ${}_1\psi_1$  summation formula. For example, we will present a number of classical and new solutions for the problem of representing an integer as sums of squares (one of the most celebrated in number theory and combinatorics) in this thesis. This will be done by using  $q$ -series and Ramanujan's  ${}_1\psi_1$  summation formula. This in turn will give an insight into how Ramanujan may have proven many of his results, since his own proofs are often unknown, thereby increasing and deepening our understanding of Ramanujan's work.

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# Chapter 1

## Introduction

Srinivasa Ramanujan was an Indian mathematician who had to deal with a lack of education and resources, and so was largely self-taught. He was born on 22 December 1887 but had a short life. He died on 26 April 1920, and left behind three notebooks [92], a “lost notebook” [93], other manuscripts [93], and published papers [61]. For an extensive biography of Ramanujan, see [68].

The three notebooks and the “lost notebook” are filled with over three thousand results without proofs. Since 1974, B. C. Berndt has been finding proofs for these claims. It took him over twenty years to complete the task of editing Ramanujan’s notebooks [92]. His work has been published in five books [17]–[21]. Berndt said [68, p. 280], “*I still don’t understand it all. I may be able to prove it, but I don’t know where it comes from and where it fits into the rest of mathematics,*” after years of working through Ramanujan’s notebooks. He also said [68, p. 280], “*The enigma of Ramanujan’s creative process is still covered by a curtain that has barely been drawn.*” Therefore it is worthwhile to study Ramanujan’s work and to gain a better understanding of his works.

This thesis contains nine chapters. This chapter gives a brief overview. Chapter 2 contains preliminary results and will be used as a basis for Chapters 3–8. In Chapter 3, we study seven types of transformations such as the Gauss transformation, Landen transformations, and modular transformations. In Chapter 4, we study eighteen problems in the area of sums of squares and triangular numbers. Sixteen series theorems will be given

in Chapter 5. In Chapter 6, we study and discuss some applications of Ramanujan's  ${}_1\psi_1$  summation formula to number theory and combinatorics. Chapter 7 deals with Eisenstein series. Eighteen conjectures will be presented in Chapter 8. Chapter 9 summarises the work of this thesis and gives suggestions for future work.

Around 1913, Ramanujan found one of his famous formulae which is now called Ramanujan's  ${}_1\psi_1$  summation formula. The formula is

$$\begin{aligned}
 & \prod_{k=1}^{\infty} \frac{(1 + zq^{2k-1})(1 + q^{2k-1}/z)(1 - q^{2k})(1 - \alpha\beta q^{2k})}{(1 + \alpha zq^{2k-1})(1 + \beta q^{2k-1}/z)(1 - \alpha q^{2k})(1 - \beta q^{2k})} \\
 = & 1 + \left\{ \frac{1 - \alpha}{1 - \beta q^2} qz + \frac{1 - \beta}{1 - \alpha q^2} \frac{q}{z} \right\} \\
 & + \left\{ \frac{(1 - \alpha)(q^2 - \alpha)}{(1 - \beta q^2)(1 - \beta q^4)} (qz)^2 + \frac{(1 - \beta)(q^2 - \beta)}{(1 - \alpha q^2)(1 - \alpha q^4)} \left(\frac{q}{z}\right)^2 \right\} \\
 & + \left\{ \frac{(1 - \alpha)(q^2 - \alpha)(q^4 - \alpha)}{(1 - \beta q^2)(1 - \beta q^4)(1 - \beta q^6)} (qz)^3 \right. \\
 & \left. + \frac{(1 - \beta)(q^2 - \beta)(q^4 - \beta)}{(1 - \alpha q^2)(1 - \alpha q^4)(1 - \alpha q^6)} \left(\frac{q}{z}\right)^3 \right\} + \dots, \tag{1.0.1}
 \end{aligned}$$

where  $|\beta q| < |z| < 1/|\alpha q|$  and  $|q| < 1$ .

This formula is a fundamental result and is related directly or indirectly to all the main results presented in this thesis. We will give a proof of this formula in Chapter 2, together with more references and details.

We will define four functions,  $f_0, f_1, f_2$ , and  $f_3$ , which arise from Ramanujan's  ${}_1\psi_1$  summation formula and will construct twelve other functions from these, giving a total of sixteen functions. We will show that function  $f_0$  is not an elliptic function but that its derivative  $f'_0$  is. On the other hand we will show that functions  $f_1, f_2, f_3$  are elliptic functions. Then we will study a number of properties of these sixteen functions such as (a) Fourier series ex-

pansions of functions  $f_0, f_1, f_2$ , and  $f_3$ ; (b) infinite product expansions for  $f_1, f_2$ , and  $f_3$ ; (c) the derivatives of the four functions; (d) Fourier expansions of their squares; and (e) addition formulae of functions  $f_0, f_1, f_2$ , and  $f_3$ . We will obtain sixteen Lambert series by expansions of the four functions  $f_0, f_1, f_2, f_3$  at four points  $0, \pi, \pi\tau$ , and  $\pi + \pi\tau$ . The fundamental multiplicative identity and the Weierstrass  $\wp$  function will be introduced. We will define functions  $z, x, K$ , and  $E$ , then employ these to obtain the reciprocals and quotients of the functions  $f_1, f_2$ , and  $f_3$ . We will also demonstrate a connection between the twelve functions and Jacobian elliptic functions.

Most of the methods and ideas in Chapter 2 are based on Venkatachaliengar's monograph [100].

The use of transformations is another important tool in this thesis. In Chapter 3 we will study and apply various transformations to functions  $f_0, f_1, f_2, f_3$ , as well as  $z, x, 1 - x, dz/dx$ , and  $E$ . These results will be used in Chapters 5–8.

The problems of representing an integer as sums of squares and sums of triangular numbers is one of the most celebrated in number theory and combinatorics. In Chapter 4 we will study and give proofs of eighteen problems in this area. Some of these eighteen problems are known classical formulae. We will present a new identity which involves infinite products and Lambert series expansions, namely

$$\prod_{k=1}^{\infty} \frac{(1 - q^{2k})^4 (1 - q^{4k})^2}{(1 - q^{2k-1})^4 (1 - q^{4k-2})^2} = \sum_{j=1}^{\infty} \frac{j^2 q^{j-1}}{1 + q^{2j}}. \quad (1.0.2)$$

We will also give an arithmetic interpretation of this identity: the number of solutions in non-negative integers of

$$\frac{x_1(x_1+1)}{2} + \frac{x_2(x_2+1)}{2} + \frac{x_3(x_3+1)}{2} + \frac{x_4(x_4+1)}{2} + x_5(x_5+1) + x_6(x_6+1) = n,$$

where  $n = 0, 1, 2, 3, \dots$ , is

$$= \sum_{\substack{d|n+1 \\ d \text{ odd}}} (-1)^{\frac{d-1}{2}} \left( \frac{n+1}{d} \right)^2, \quad (1.0.3)$$

where we use  $d|n$  to denote  $d$  is a divisor of  $n$ . Another form of writing (1.0.3) will be given in Chapter 4. The significant point is that our method is based on Ramanujan's  ${}_1\psi_1$  summation formula and the fundamental multiplicative identity to obtain these eighteen formulae.

Ramanujan [92, Chapter 17, Entries 13–17] gave fourteen families of identities. In each case he gave only the first few examples, giving us the motivation to find the general solutions in each family of identities. The aim of Chapter 5 is to develop a powerful tool (four versatile functions  $f_0, f_1, f_2$ , and  $f_3$ ) to collect all of Ramanujan's examples together. We will first express the sixteen Lambert series as various polynomials in terms of  $z, x$ , and  $dz/dx$ . This will give a total of sixteen infinite families of identities which contain all of Ramanujan's examples. We will also prove that Ramanujan's Eisenstein series, namely  $P, Q$ , and  $R$ , can be expressed in terms of  $z, x$ , and  $dz/dx$ . The results in Chapter 5 are the key of this thesis and will be used in Chapters 6 and 8.

In Chapter 6 we will study and give proofs of a total of forty-four identities by using selected transformations given in Chapter 3 and results from Chapter 5. From these forty-four

identities, fourteen identities are new. For example

$$\prod_{k=1}^{\infty} \frac{(1+q^{2k-1})^4 (1-q^{2k})^4 (1-q^{8k})^2}{(1-q^{2k-1})^4 (1+q^{2k})^4 (1-q^{8k-4})^2} = \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1)^2 q^{2j-2}}{1-q^{4j-2}} + 2 \sum_{j=1}^{\infty} \frac{j^2 q^{j-1}}{1+q^{2j}}. \quad (1.0.4)$$

As for the identity (1.0.2) we showed earlier, we can give an arithmetic interpretation of identity (1.0.4): the number of solutions in integers of

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2y_1(y_1 + 1) + 2y_2(y_2 + 1) = n,$$

where  $n = 0, 1, 2, 3, \dots$ , is

$$= k(n) \sum_{\substack{d|n+1 \\ d \text{ odd}}} (-1)^{\frac{d-1}{2}} \left( \frac{n+1}{d} \right)^2,$$

where

$$k(n) = \begin{cases} 1 & : n \equiv 0 \pmod{4}, \\ 2 & : n \equiv 1 \pmod{2}, \\ 3 & : n \equiv 2 \pmod{4}. \end{cases}$$

The aim of Chapter 7 is to rewrite the sixteen Lambert series in the form of Eisenstein series. From these sixteen Eisenstein series we will obtain an alternative path to see how the sixteen Lambert series form one system.

In Chapter 8 we will present eighteen conjectures that lead to sums of  $2t$  squares and triangular numbers where  $t \geq 3$ . We will prove that all eighteen conjectures are true for the first five cases.

Finally, Chapter 9 summarises the work of this thesis and gives suggestions for future work.

Four useful derivations are given in the Appendices; they are Fourier series, infinite products, squares functions, and Jacobian elliptic functions. A list of symbols is given in the Index of Symbols.

# Chapter 2

## Preliminary results

### 2.1 Introduction

In this chapter, we summarise selected results of Venkatachaliengar [100] and S. Cooper [38], [39]. These results will be used throughout the thesis. Firstly, we introduce some notation.

### 2.2 Notation

Throughout this thesis, let  $\tau$  be a fixed complex number satisfying  $\text{Im } \tau > 0$  and let  $q = e^{i\pi\tau}$  so that  $|q| < 1$ . Define

$$(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j),$$

where  $n$  is a positive integer, and

$$(a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j).$$

We may extend the definition of  $(a; q)_n$  by defining

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}, \tag{2.2.1}$$

for all real numbers  $n$ .

We also use the following notation:

$$(a_1, a_2, \dots, a_n; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_n; q)_\infty.$$



## 2.3 Ramanujan's ${}_1\psi_1$ summation formula

Around 1913, Ramanujan discovered an important formula, now known as Ramanujan's  ${}_1\psi_1$  summation formula, which is recorded in his Second Notebook in [92, Chapter 16, Entry 17]. The formula is

**Theorem 2.3.1 (Ramanujan's  ${}_1\psi_1$  summation formula)**

$$\sum_{n=-\infty}^{\infty} \frac{(a; q)_n}{(b; q)_n} x^n = \frac{(ax; q)_{\infty} \left(\frac{q}{ax}; q\right)_{\infty} (q; q)_{\infty} \left(\frac{b}{a}; q\right)_{\infty}}{(x; q)_{\infty} \left(\frac{b}{ax}; q\right)_{\infty} (b; q)_{\infty} \left(\frac{q}{a}; q\right)_{\infty}}, \quad (2.3.1)$$

where  $|q| < 1$  and  $|b/a| < |x| < 1$ .

By using the ratio test, the series on the left of (2.3.1) converges for  $|x| < 1$  when  $n = 0, 1, 2, 3, \dots$ , and for  $|b/a| < |x|$  when  $n = -1, -2, -3, \dots$ . Therefore the series on the left of (2.3.1) converges for  $|b/a| < |x| < 1$ . As a function of  $x$  in (2.3.1), the right hand side has poles at  $x = bq^n/a$ , where  $n = 0, 1, 2, \dots$ , and  $x = q^{-n}$ , where  $n = 0, 1, 2, \dots$ . Under the condition  $|b/a| < |x| < 1$ , the two sets of poles are separated into  $\dots < bq^2/a < bq/a < b/a < 1 < q^{-1} < q^{-2} < \dots$ . Theorem 2.3.1 gives the Laurent expansion, in powers of  $x$ , in the annulus  $|b/a| < |x| < 1$ .

**Proof** Let  $a$ ,  $b$ , and  $q$  be fixed, and define

$$f(x) = \frac{(ax; q)_{\infty} \left(\frac{q}{ax}; q\right)_{\infty}}{(x; q)_{\infty} \left(\frac{b}{ax}; q\right)_{\infty}}. \quad (2.3.2)$$

By Laurent's theorem, we can expand (2.3.2) as a Laurent series

$$f(x) = \sum_{n=-\infty}^{\infty} C_n x^n, \quad (2.3.3)$$

valid in the annulus  $\left|\frac{b}{a}\right| < |x| < 1$  where  $f(x)$  is analytic. Note that the coefficients  $c_n$  depend on  $a$ ,  $b$ , and  $q$ , as well as on  $n$ . Next, consider the Laurent expansion of  $f(qx)$ , which exists for  $\left|\frac{b}{aq}\right| < |x| < \frac{1}{|q|}$ . Additionally assume that  $\left|\frac{b}{aq}\right| < |x| < 1$ . Now both  $f(x)$  and  $f(qx)$  are valid for  $\left|\frac{b}{aq}\right| < |x| < 1$  and  $|q| < 1$ . Then consider the ratio

$$\begin{aligned} \frac{f(x)}{f(qx)} &= \frac{(ax; q)_\infty \left(\frac{q}{ax}; q\right)_\infty (qx; q)_\infty \left(\frac{b}{aqx}; q\right)_\infty}{(x; q)_\infty \left(\frac{b}{ax}; q\right)_\infty (aqx; q)_\infty \left(\frac{1}{ax}; q\right)_\infty} \\ &= \frac{(1 - ax) \left(1 - \frac{b}{aqx}\right)}{\left(1 - \frac{1}{ax}\right) (1 - x)} \\ &= \frac{b - aqx}{q(1 - x)}. \end{aligned}$$

Hence we obtain the functional equation

$$q(1 - x)f(x) = (b - aqx)f(qx). \quad (2.3.4)$$

Substituting (2.3.3) into (2.3.4) gives

$$q(1 - x) \sum_{n=-\infty}^{\infty} C_n x^n = (b - aqx) \sum_{n=-\infty}^{\infty} C_n q^n x^n. \quad (2.3.5)$$

Equating the coefficients of  $x^n$  gives

$$qC_n - qC_{n-1} = bC_n q^n - aC_{n-1} q^n.$$

Simplifying and making  $C_n$  the subject, we obtain the recurrence relation

$$C_n = \frac{(1 - aq^{n-1})}{(1 - bq^{n-1})} C_{n-1}, \quad (2.3.6)$$

and iterating (2.3.6) we obtain

$$\begin{aligned} C_n &= \frac{(1 - aq^{n-1})(1 - aq^{n-2}) \cdots (1 - a)}{(1 - bq^{n-1})(1 - bq^{n-2}) \cdots (1 - b)} C_0 \\ &= \frac{(a; q)_n}{(b; q)_n} C_0, \quad n = 1, 2, 3, \dots \end{aligned} \quad (2.3.7)$$

Next replace  $n$  with  $1 - n$  in (2.3.6), and rearrange into

$$C_{-n} = \frac{(1 - bq^{-n})}{(1 - aq^{-n})} C_{-n+1}.$$

Iteration yields

$$\begin{aligned} C_{-n} &= \frac{(1 - bq^{-n})(1 - bq^{-n+1}) \cdots (1 - bq^{-1})}{(1 - aq^{-n})(1 - aq^{-n+1}) \cdots (1 - aq^{-1})} C_0 \\ &= \frac{(bq^{-n}; q)_\infty}{(b; q)_\infty} \frac{(a; q)_\infty}{(aq^{-n}; q)_\infty} C_0 \\ &= \frac{(a; q)_{-n}}{(b; q)_{-n}} C_0, \quad n = 1, 2, 3, \dots, \end{aligned} \quad (2.3.8)$$

using (2.2.1).

By combining the results of (2.3.7) and (2.3.8) we obtain

$$C_n = \frac{(a; q)_n}{(b; q)_n} C_0, \quad n \in \mathbb{Z}. \quad (2.3.9)$$

Putting (2.3.9) into (2.3.3) gives

$$f(x) = C_0 \sum_{n=-\infty}^{\infty} \frac{(a; q)_n}{(b; q)_n} x^n. \quad (2.3.10)$$

It remains for us to find  $C_0$ .

For this, we need to use Abel's continuity theorem [11, p. 504] which states that:

$$\text{Suppose } \lim_{n \rightarrow \infty} a_n = a. \text{ Then } \lim_{x \rightarrow 1^-} (1 - x) \sum_{n=-\infty}^{\infty} a_n x^n = a. \quad (2.3.11)$$

Equation (2.3.9) gives at once

$$\lim_{n \rightarrow \infty} C_n = C_0 \frac{(a; q)_\infty}{(b; q)_\infty}. \quad (2.3.12)$$

From equation (2.3.2),

$$\begin{aligned} \lim_{x \rightarrow 1^-} (1-x) f(x) &= \lim_{x \rightarrow 1^-} (1-x) \frac{(ax; q)_\infty \left(\frac{q}{ax}; q\right)_\infty}{(x; q)_\infty \left(\frac{b}{ax}; q\right)_\infty} \\ &= \frac{(a; q)_\infty \left(\frac{q}{a}; q\right)_\infty}{(q; q)_\infty \left(\frac{b}{a}; q\right)_\infty}. \end{aligned} \quad (2.3.13)$$

Therefore Abel's continuity theorem, (2.3.12), and (2.3.13) imply:

$$C_0 = \frac{(b; q)_\infty \left(\frac{q}{a}; q\right)_\infty}{(q; q)_\infty \left(\frac{b}{a}; q\right)_\infty}. \quad (2.3.14)$$

By substituting the result (2.3.14) into (2.3.10) we find:

$$f(x) = \frac{(b; q)_\infty \left(\frac{q}{a}; q\right)_\infty}{(q; q)_\infty \left(\frac{b}{a}; q\right)_\infty} \sum_{n=-\infty}^{\infty} \frac{(a; q)_n}{(b; q)_n} x^n. \quad (2.3.15)$$

Putting (2.3.15) into (2.3.2) and rearranging arrives at (2.3.1). Lastly, we employ analytic continuation to extend the result from  $\left|\frac{b}{aq}\right| < |x| < 1$  to  $\left|\frac{b}{a}\right| < |x| < 1$ . This completes the proof. ■

This proof is the same as the one given by Venkatachaliengar [100, pp. 24–27, pp. 29–30] and can be found in [8, pp. 503–504]. G. H. Hardy [60, p. 222] described it as “a remarkable formula with many parameters”. The first published proofs of this formula appeared in 1949 and 1950, by W. Hahn [59] and M. Jackson [66], respectively. Since then other proofs have been found, for example R. Askey [11], S. H. Chan [34], and A. J. Yee [104]. Ramanujan's  ${}_1\psi_1$  summation formula is a fundamental result and is related directly

or indirectly to all the main results presented in this thesis.

Note that if we replace  $q$  and  $x$  with  $q^2$  and  $z$ , respectively, in (2.3.1), then replace  $1/a, b/q^2$ , and  $-az/q$  with  $\alpha, \beta$ , and  $z$ , respectively, we arrive at (1.0.1).

**Corollary 2.3.2 (Jacobi triple product identity)**

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n^2-n}{2}} x^n = (x, qx^{-1}, q; q)_{\infty}, \quad (2.3.16)$$

where  $|q| < 1$  and  $x \neq 0$ .

**Proof** Set  $x \rightarrow x/a$ ,  $a \rightarrow \infty$  and let  $b = 0$  in (2.3.1). ■

**Corollary 2.3.3 ( $q$ -binomial theorem)**

$$\frac{(ax; q)_{\infty}}{(x; q)_{\infty}} = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n, \quad (2.3.17)$$

where  $|q| < 1$  and  $|x| < 1$ .

**Proof** Set  $b = q$  in (2.3.1). ■

**Corollary 2.3.4 (Geometric series)**

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad (2.3.18)$$

where  $|x| < 1$ .

**Proof** Set  $a = q$  in (2.3.17). ■

## 2.4 The Jordan-Kronecker function

The Jordan-Kronecker function is a special case of the series on the left hand side of (2.3.1) and is defined as follows

**Definition 2.4.1** [100, p. 37] *Let*

$$F(a, b) = F(a, b; q) = F(a, b|\tau) = \sum_{n=-\infty}^{\infty} \frac{a^n}{1 - bq^{2n}}, \quad (2.4.1)$$

where  $|q^2| < |a| < 1$  and  $b \neq q^{2k}$ ,  $k \in \mathbb{Z}$ .

In Ramanujan's  ${}_1\psi_1$  summation formula (2.3.1) replace  $q$ ,  $x$ ,  $a$ , and  $b$  with  $q^2$ ,  $a$ ,  $b$ , and  $bq^2$ , respectively, then divide by  $1 - b$  to obtain

$$F(a, b; q) = \frac{(ab, a^{-1}b^{-1}q^2, q^2, q^2; q^2)_{\infty}}{(a, a^{-1}q^2, b, b^{-1}q^2; q^2)_{\infty}}. \quad (2.4.2)$$

The product converges for all values of  $a$  and  $b$  except for  $a, b = q^{2k}$ , where  $k \in \mathbb{Z}$ .

The product extends the definition of the function to more general values of  $a$  by analytic continuation.

The following properties follow immediately from (2.4.2)

$$F(a, b; q) = F(b, a; q), \quad (2.4.3)$$

$$F(a, b; q) = -F\left(\frac{1}{a}, \frac{1}{b}; q\right), \quad (2.4.4)$$

$$F(a, b; q) = bF(aq^2, b; q) = aF(a, bq^2; q). \quad (2.4.5)$$

We can rewrite (2.4.1) as

$$F(a, b; q) = \sum_{n=-\infty}^{\infty} \frac{a^n}{1 - bq^{2n}}$$

$$\begin{aligned}
&= \frac{1}{1-b} + \sum_{n=1}^{\infty} \frac{a^n}{1-bq^{2n}} + \sum_{n=1}^{\infty} \frac{a^{-n}}{1-bq^{-2n}} \\
&= \frac{1}{1-b} + \sum_{n=1}^{\infty} \frac{a^n (1-bq^{2n} + bq^{2n})}{1-bq^{2n}} + \sum_{n=1}^{\infty} \frac{a^{-n}}{1-bq^{-2n}} \\
&= \frac{1}{1-b} + \frac{a}{1-a} + \sum_{n=1}^{\infty} \frac{a^n bq^{2n}}{1-bq^{2n}} - \sum_{n=1}^{\infty} \frac{a^{-n} b^{-1} q^{2n}}{1-b^{-1} q^{2n}} \\
&= \frac{1+b}{2(1-b)} + \frac{1+a}{2(1-a)} + \sum_{n=1}^{\infty} \frac{a^n bq^{2n}}{1-bq^{2n}} - \sum_{n=1}^{\infty} \frac{a^{-n} b^{-1} q^{2n}}{1-b^{-1} q^{2n}}.
\end{aligned} \tag{2.4.6}$$

Using (2.4.3) and (2.4.6), replacing  $a$  and  $b$  by  $e^{i\theta}$  and  $e^{iv}$  respectively,

$$\begin{aligned}
F(e^{i\theta}, e^{iv}; q) &= F(e^{iv}, e^{i\theta}; q) \\
&= \frac{1+e^{i\theta}}{2(1-e^{i\theta})} + \frac{1+e^{iv}}{2(1-e^{iv})} + \sum_{n=1}^{\infty} \left( \frac{e^{inv} e^{i\theta} q^{2n}}{1-e^{i\theta} q^{2n}} - \frac{e^{-inv} e^{-i\theta} q^{2n}}{1-e^{-i\theta} q^{2n}} \right).
\end{aligned} \tag{2.4.7}$$

The Bernoulli numbers  $B_n$  and Euler numbers  $E_n$  are defined by

$$\frac{u}{e^u - 1} = \sum_{n=0}^{\infty} \frac{B_n u^n}{n!}, \quad \text{for } |u| < 2\pi, \tag{2.4.8}$$

and

$$\frac{2e^u}{e^{2u} + 1} = \sum_{n=0}^{\infty} \frac{E_n u^n}{n!}, \quad \text{for } |u| < \frac{\pi}{2}. \tag{2.4.9}$$

Add  $u/2$  to both sides of (2.4.8) and then multiply by  $-1/u$  we have

$$\frac{1+e^u}{2(1-e^u)} = -\frac{1}{2} - \sum_{n=0}^{\infty} \frac{B_n u^{n-1}}{n!}. \tag{2.4.10}$$

Using (2.4.10), we expand (2.4.7) in powers of  $v$  to give

$$\begin{aligned}
&F(e^{i\theta}, e^{iv}; q) \\
&= \frac{i}{2} \cot \frac{\theta}{2} - \frac{1}{2} - \sum_{j=0}^{\infty} \frac{B_j (iv)^{j-1}}{j!} + \sum_{n=1}^{\infty} \frac{e^{i\theta} q^{2n}}{1-e^{i\theta} q^{2n}} \sum_{j=0}^{\infty} \frac{(inv)^j}{j!}
\end{aligned}$$

$$\begin{aligned}
& - \sum_{n=1}^{\infty} \frac{e^{-i\theta} q^{2n}}{1 - e^{-i\theta} q^{2n}} \sum_{j=0}^{\infty} \frac{(-inv)^j}{j!} \\
& = \frac{i}{2} \cot \frac{\theta}{2} - \frac{1}{2} - \frac{1}{iv} + \sum_{j=0}^{\infty} \left[ -\frac{B_{j+1}}{j+1} + \sum_{n=1}^{\infty} \left( \frac{n^j e^{i\theta} q^{2n}}{1 - e^{i\theta} q^{2n}} - \frac{(-1)^j n^j e^{-i\theta} q^{2n}}{1 - e^{-i\theta} q^{2n}} \right) \right] \frac{i^j v^j}{j!} \\
& = -\frac{1}{iv} + \frac{i}{2} \cot \frac{\theta}{2} + \sum_{n=1}^{\infty} \left( \frac{e^{i\theta} q^{2n}}{1 - e^{i\theta} q^{2n}} - \frac{e^{-i\theta} q^{2n}}{1 - e^{-i\theta} q^{2n}} \right) \\
& \quad + \left[ -\frac{1}{12} + \sum_{n=1}^{\infty} \left( \frac{ne^{i\theta} q^{2n}}{1 - e^{i\theta} q^{2n}} + \frac{ne^{-i\theta} q^{2n}}{1 - e^{-i\theta} q^{2n}} \right) \right] iv \\
& \quad + \sum_{j=2}^{\infty} \left[ -\frac{B_{j+1}}{j+1} + \sum_{n=1}^{\infty} \left( \frac{n^j e^{i\theta} q^{2n}}{1 - e^{i\theta} q^{2n}} - \frac{(-1)^j n^j e^{-i\theta} q^{2n}}{1 - e^{-i\theta} q^{2n}} \right) \right] \frac{i^j v^j}{j!}. \tag{2.4.11}
\end{aligned}$$

Additionally

$$\begin{aligned}
\sum_{n=1}^{\infty} \left( \frac{e^{i\theta} q^{2n}}{1 - e^{i\theta} q^{2n}} - \frac{e^{-i\theta} q^{2n}}{1 - e^{-i\theta} q^{2n}} \right) & = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (e^{im\theta} q^{2mn} - e^{-im\theta} q^{2mn}) \\
& = \sum_{m=1}^{\infty} \frac{q^{2m}}{1 - q^{2m}} (e^{im\theta} - e^{-im\theta}) \\
& = 2i \sum_{m=1}^{\infty} \frac{q^{2m}}{1 - q^{2m}} \sin m\theta \tag{2.4.12}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=1}^{\infty} \left( \frac{ne^{i\theta} q^{2n}}{1 - e^{i\theta} q^{2n}} + \frac{ne^{-i\theta} q^{2n}}{1 - e^{-i\theta} q^{2n}} \right) & = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (ne^{im\theta} q^{2mn} + ne^{-im\theta} q^{2mn}) \\
& = \sum_{m=1}^{\infty} \frac{q^{2m}}{(1 - q^{2m})^2} (e^{im\theta} + e^{-im\theta}) \\
& = 2 \sum_{m=1}^{\infty} \frac{q^{2m}}{(1 - q^{2m})^2} \cos m\theta. \tag{2.4.13}
\end{aligned}$$

By substituting (2.4.12) and (2.4.13) into (2.4.11) we find that

$$\begin{aligned}
F(e^{i\theta}, e^{iv}; q) & = -\frac{1}{iv} + 2i \left[ \frac{1}{4} \cot \frac{\theta}{2} + \sum_{m=1}^{\infty} \frac{q^{2m}}{1 - q^{2m}} \sin m\theta \right] \\
& \quad + \left[ -\frac{1}{12} + 2 \sum_{m=1}^{\infty} \frac{q^{2m}}{(1 - q^{2m})^2} \cos m\theta \right] iv
\end{aligned}$$



$$+ \sum_{j=2}^{\infty} \left[ -\frac{B_{j+1}}{j+1} + \sum_{m=1}^{\infty} \left( \frac{m^j e^{i\theta} q^{2m}}{1 - e^{i\theta} q^{2m}} - \frac{(-1)^j m^j e^{-i\theta} q^{2m}}{1 - e^{-i\theta} q^{2m}} \right) \right] \frac{i^j v^j}{j!}. \quad (2.4.14)$$

Next we consider four special cases of the function  $F(a, b; q)$ .

## 2.5 Four functions

In this section we present four functions and show how they may be derived from Ramanujan's  ${}_1\psi_1$  summation formula. We then obtain the Fourier series expression for  $f_0, f_1, f_2, f_3$  using the Jordan-Kronecker function (2.4.1). Similarly the infinite products formulae for  $f_1, f_2, f_3$  can be obtained by using (2.4.2). Lastly we determine the zeros of  $f_1, f_2, f_3$  and the poles of  $f_0, f_1, f_2, f_3$ .

Four special cases of the function  $F(a, b; q)$  are considered by examining  $a = e^{i\theta}$  and  $b = 1, e^{i\pi}, e^{i\pi\tau}, e^{i\pi+i\pi\tau}$ , respectively. Their properties follow from those of  $F(a, b; q)$  and Ramanujan's  ${}_1\psi_1$  summation formula.

Let

$$f_0(\theta) = f_0(\theta; q) = f_0(\theta|\tau) = \frac{1}{i} [v^0] F(e^{i\theta}, e^{iv}; q), \quad (2.5.1)$$

$$f_1(\theta) = f_1(\theta; q) = f_1(\theta|\tau) = \frac{1}{i} F(e^{i\theta}, e^{i\pi}; q), \quad (2.5.2)$$

$$f_2(\theta) = f_2(\theta; q) = f_2(\theta|\tau) = \frac{e^{i\theta/2}}{i} F(e^{i\theta}, e^{i\pi\tau}; q), \quad (2.5.3)$$

$$f_3(\theta) = f_3(\theta; q) = f_3(\theta|\tau) = \frac{e^{i\theta/2}}{i} F(e^{i\theta}, e^{i\pi+i\pi\tau}; q). \quad (2.5.4)$$

Note that we will use the notation of  $f_0(\theta), f_1(\theta), f_2(\theta), f_3(\theta)$  in this chapter and the other notations such as  $f_0(\theta; q)$  and  $f_0(\theta|\tau)$  will be used in the later chapters.

Here the notation  $[v^0] F(e^{i\theta}, e^{iv}; q)$  means expand the function of  $F(e^{i\theta}, e^{iv}; q)$  in powers of  $v$  and extract the constant term. The factors  $1/i$  and  $e^{i\theta/2}/i$  are included so that  $f_0, f_1, f_2$ , and  $f_3$  become real valued when  $\theta$  is real.

### 2.5.1 Fourier series expansion for $f_0, f_1, f_2$ , and $f_3$

In this subsection, we give the Fourier series expansion for  $f_0, f_1, f_2$ , and  $f_3$ . First we consider the constant term of (2.4.14) then the Fourier series for  $f_0$  becomes

$$f_0(\theta) = \frac{1}{2} \cot \frac{\theta}{2} + 2 \sum_{m=1}^{\infty} \frac{q^{2m}}{1 - q^{2m}} \sin m\theta. \quad (2.5.5)$$

$f_0$  is the Weierstrass zeta function [101, p. 445] with period  $2\pi$  and quasi-period  $2\pi\tau$ ; that is,  $f_0(\theta + 2\pi\tau) = f_0(\theta) - i$ .

Ramanujan used (2.5.5) to prove a number of identities for elliptic functions. For example, he proved that [94, p. 139]

$$f_0^2(\theta) = \frac{1}{4} \cot^2 \frac{\theta}{2} + 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{(1 - q^{2n})^2} \cos n\theta + 2 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} (1 - \cos n\theta). \quad (2.5.6)$$

More details of (2.5.6) and another proof are shown in Section 2.8.

The Fourier series expansion for  $f_1$  similarly follows from (2.4.6)

$$\begin{aligned} f_1(\theta) &= \frac{1}{i} \left[ \frac{1 + e^{i\theta}}{2(1 - e^{i\theta})} - \sum_{m=1}^{\infty} \frac{q^{2m}}{1 + q^{2m}} (e^{im\theta} - e^{-im\theta}) \right] \\ &= \frac{1}{2} \cot \frac{\theta}{2} - 2 \sum_{m=1}^{\infty} \frac{q^{2m}}{1 + q^{2m}} \sin m\theta. \end{aligned} \quad (2.5.7)$$

Moreover, by using (2.4.6) and (2.5.3),

$$\begin{aligned} f_2(\theta) &= \frac{e^{i\theta/2}}{i} \left[ \frac{1 + q}{2(1 - q)} + \frac{1 + e^{i\theta}}{2(1 - e^{i\theta})} + \sum_{m=1}^{\infty} \frac{e^{im\theta} q^{2m+1}}{1 - q^{2m+1}} - \sum_{m=1}^{\infty} \frac{e^{-im\theta} q^{2m-1}}{1 - q^{2m-1}} \right] \\ &= \frac{e^{i\theta/2}(1 + q)}{2i(1 - q)} + \frac{e^{i\theta/2}(1 + e^{i\theta})}{2i(1 - e^{i\theta})} - \frac{e^{i\theta/2}q}{i(1 - q)} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{i} \sum_{m=1}^{\infty} \frac{q^{2m-1}}{1 - q^{2m-1}} (e^{i(m-1/2)\theta} - e^{-i(m-1/2)\theta}) \\
& = \frac{1}{2} \csc \frac{\theta}{2} + 2 \sum_{m=1}^{\infty} \frac{q^{2m-1}}{1 - q^{2m-1}} \sin \left( m - \frac{1}{2} \right) \theta.
\end{aligned} \tag{2.5.8}$$

Similarly,

$$f_3(\theta) = \frac{1}{2} \csc \frac{\theta}{2} - 2 \sum_{m=1}^{\infty} \frac{q^{2m-1}}{1 + q^{2m-1}} \sin \left( m - \frac{1}{2} \right) \theta. \tag{2.5.9}$$

The series (2.5.5)–(2.5.9) converge for  $-\operatorname{Im}(2\pi\tau) < \operatorname{Im} \theta < \operatorname{Im}(2\pi\tau)$ .

We can obtain another set of twelve functions by replacing  $\theta$  by  $\theta + \pi, \theta + \pi\tau, \theta + \pi + \pi\tau$ , in (2.5.5), (2.5.7)–(2.5.9), respectively. Their Fourier series expansions can be found in Appendix A.

### 2.5.2 Infinite products for $f_1, f_2, f_3$

The infinite product formulae for  $f_1, f_2, f_3$  follow from (2.4.2). They are

$$f_1(\theta) = \frac{1}{i} \frac{(-e^{i\theta}, -q^2 e^{-i\theta}, q^2, q^2; q^2)_{\infty}}{(e^{i\theta}, q^2 e^{-i\theta}, -1, -q^2; q^2)_{\infty}} \tag{2.5.10}$$

$$= \frac{1}{2} \frac{(q^2; q^2)_{\infty}^2}{(-q^2; q^2)_{\infty}^2} \cot \frac{\theta}{2} \prod_{n=1}^{\infty} \frac{(1 + 2q^{2n} \cos \theta + q^{4n})}{(1 - 2q^{2n} \cos \theta + q^{4n})}, \tag{2.5.11}$$

$$f_2(\theta) = \frac{e^{\frac{i\theta}{2}}}{i} \frac{(qe^{i\theta}, qe^{-i\theta}, q^2, q^2; q^2)_{\infty}}{(e^{i\theta}, q^2 e^{-i\theta}, q, q; q^2)_{\infty}} \tag{2.5.12}$$

$$= \frac{1}{2} \frac{(q^2; q^2)_{\infty}^2}{(q; q^2)_{\infty}^2} \csc \frac{\theta}{2} \prod_{n=1}^{\infty} \frac{(1 - 2q^{2n-1} \cos \theta + q^{4n-2})}{(1 - 2q^{2n} \cos \theta + q^{4n})}, \tag{2.5.13}$$

$$f_3(\theta) = \frac{e^{\frac{i\theta}{2}}}{i} \frac{(-qe^{i\theta}, -qe^{-i\theta}, q^2, q^2; q^2)_{\infty}}{(e^{i\theta}, q^2 e^{-i\theta}, -q, -q; q^2)_{\infty}} \tag{2.5.14}$$

$$= \frac{1}{2} \frac{(q^2; q^2)_{\infty}^2}{(-q; q^2)_{\infty}^2} \csc \frac{\theta}{2} \prod_{n=1}^{\infty} \frac{(1 + 2q^{2n-1} \cos \theta + q^{4n-2})}{(1 - 2q^{2n} \cos \theta + q^{4n})}. \tag{2.5.15}$$

### 2.5.3 Elliptic functions

**Definition 2.5.1** [101, p. 429]

Let  $\omega_1$  and  $\omega_2$  be two complex numbers for which the ratio  $\omega_1/\omega_2$  is not a real number. A function satisfying the relations

$$f(z + 2\omega_1) = f(z), \quad f(z + 2\omega_2) = f(z),$$

for all complex values of  $z$  at which  $f(z)$  exists, is called a doubly periodic function of  $z$  with periods  $2\omega_1, 2\omega_2$ . A doubly periodic function that is meromorphic in the finite part of the complex plane is an elliptic function. A doubly periodic function  $f$  is completely defined by its restriction to the so called fundamental parallelogram, that is, a parallelogram with corners  $0, 2\omega_1, 2\omega_2, 2\omega_1 + 2\omega_2$ , or translation.

**Lemma 2.5.2**

$$f_0(-\theta) = -f_0(\theta), \quad (2.5.16)$$

$$f_0(\theta + 2\pi) = f_0(\theta), \quad (2.5.17)$$

$$f_0(\theta + 2\pi\tau) = \frac{1}{i} + f_0(\theta); \quad (2.5.18)$$

$$f_1(-\theta) = -f_1(\theta), \quad (2.5.19)$$

$$f_2(-\theta) = -f_2(\theta), \quad (2.5.20)$$

$$f_3(-\theta) = -f_3(\theta); \quad (2.5.21)$$

$$f_1(\theta + 2\pi m + 2\pi\tau n) = (-1)^n f_1(\theta), \quad (2.5.22)$$

$$f_2(\theta + 2\pi m + 2\pi\tau n) = (-1)^m f_2(\theta), \quad (2.5.23)$$

$$f_3(\theta + 2\pi m + 2\pi\tau n) = (-1)^{m+n} f_3(\theta), \quad (2.5.24)$$

where  $m$  and  $n$  are integers.

**Proof** The proofs of (2.5.16) and (2.5.17) follow immediately from (2.5.5).

Making use of (2.5.5), the left hand side of (2.5.18) becomes

$$\begin{aligned}
 & f_0(\theta + 2\pi\tau) \\
 &= \frac{1}{2} \cot \frac{\theta + 2\pi\tau}{2} + 2 \sum_{m=1}^{\infty} \frac{q^{2m}}{1 - q^{2m}} \sin m(\theta + 2\pi\tau) \\
 &= \frac{1}{i} \left[ \frac{1 + e^{i(\theta + 2\pi\tau)}}{2(1 - e^{i(\theta + 2\pi\tau)})} + \sum_{m=1}^{\infty} \left( \frac{e^{i(\theta + 2\pi\tau)m} q^{2m}}{1 - q^{2m}} - \frac{e^{-i(\theta + 2\pi\tau)m} q^{2m}}{1 - q^{2m}} \right) \right] \\
 &= \frac{1}{i} \left[ \frac{1 + e^{i\theta} q^2}{2(1 - e^{i\theta} q^2)} - \sum_{m=1}^{\infty} \frac{e^{i\theta m} (1 - q^{2m} - 1) q^{2m}}{1 - q^{2m}} - \sum_{m=1}^{\infty} \frac{e^{-i\theta m} (1 - q^{2m} + q^{2m})}{1 - q^{2m}} \right] \\
 &= \frac{1}{i} \left[ \frac{1 + e^{i\theta} q^2}{2(1 - e^{i\theta} q^2)} - \frac{e^{i\theta} q^2}{1 - e^{i\theta} q^2} - \frac{e^{-i\theta}}{1 - e^{-i\theta}} + \sum_{m=1}^{\infty} \frac{q^{2m}}{1 - q^{2m}} (e^{i\theta m} - e^{-i\theta m}) \right] \\
 &= \frac{1}{i} \left[ \frac{1}{2} + \frac{1}{1 - e^{i\theta}} + 2i \sum_{m=1}^{\infty} \frac{q^{2m}}{1 - q^{2m}} \sin m\theta \right] \\
 &= \frac{1}{i} \left[ 1 + \frac{1 + e^{i\theta}}{2(1 - e^{i\theta})} + 2i \sum_{m=1}^{\infty} \frac{q^{2m}}{1 - q^{2m}} \sin m\theta \right] \\
 &= \frac{1}{i} + \frac{1}{2} \cot \frac{\theta}{2} + 2 \sum_{m=1}^{\infty} \frac{q^{2m}}{1 - q^{2m}} \sin m\theta \\
 &= \frac{1}{i} + f_0(\theta). \quad (2.5.25)
 \end{aligned}$$

This completes the proof of (2.5.18).

Use of the infinite products for  $f_1, f_2, f_3$ , enable us to complete the proofs of (2.5.19)–(2.5.21).

Replacing  $\theta$  by  $\theta + 2\pi m + 2\pi\tau n$  in (2.5.10), (2.5.12), (2.5.14), and simplifying gives the

periodicity properties of (2.5.22)–(2.5.24), where  $m$  and  $n$  are integers. This completes the proof of Lemma 2.5.2. ■

The function  $f_0$  is not an elliptic function but its derivative  $f'_0$  is. From (2.5.22)–(2.5.24),  $f_1$  is doubly periodic with periods  $2\pi$  and  $4\pi\tau$ ,  $f_2$  is doubly periodic with periods  $4\pi$  and  $2\pi\tau$ , while  $f_3$  is doubly periodic with periods  $4\pi$  and  $2\pi\tau$ . That is, functions  $f_1, f_2, f_3$  are elliptic functions.

In Section 2.7 it is shown that there is a connection between  $f'_0$  and the Weierstrass  $\wp$  function. The functions  $f_1, f_2$ , and  $f_3$  turn out to be the Jacobian elliptic functions namely  $\text{cs}$ ,  $\text{ns}$ , and  $\text{ds}$ , respectively, after rescaling. More details are given in Section 2.13.

## 2.5.4 Zeros, poles, and residues

From the Fourier series expansion of  $f_0$  (2.5.5), it is clear that the function  $f_0$  has zeros at  $(2m+1)\pi$ , where  $m$  is any integer. There are two zeros in any period parallelogram by general elliptic function theory [101], but it is unlikely that there is a simple formula for the location of the zeros of  $f_0$ , apart from the ones at  $(2m+1)\pi$ .

From the infinite product expansion (2.5.10) we see that  $f_1$  has zeros when  $1 + q^{2n}e^{i\theta} = 0$ , that is, when  $\theta = (2m+1)\pi + 2n\pi\tau$ , where  $m$  and  $n$  are any integers.

Similarly, if we use (2.5.12) and (2.5.14), the zeros of  $f_2$  and  $f_3$  are  $2m\pi + (2n+1)\pi\tau$  and  $(2m+1)\pi + (2n+1)\pi\tau$ , respectively.

From (2.5.5) the Fourier expansion of  $f_0$  can be represented as

$$f_0(\theta) = \frac{1}{2} \cot \frac{\theta}{2} + 2 \sum_{m=1}^{\infty} \frac{q^{2m}}{1 - q^{2m}} \sin m\theta$$

$$\begin{aligned}
&= \frac{1}{2} \cot \frac{\theta}{2} + \frac{1}{i} \sum_{m=1}^{\infty} \frac{q^{2m}}{1 - q^{2m}} (e^{im\theta} - e^{-im\theta}) \\
&= \frac{1}{2} \cot \frac{\theta}{2} + \frac{1}{i} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (q^{2mn} e^{im\theta} - q^{2mn} e^{-im\theta}) \\
&= \frac{1}{2} \cot \frac{\theta}{2} + \frac{1}{i} \sum_{n=1}^{\infty} \left( \frac{q^{2n} e^{i\theta}}{1 - q^{2n} e^{i\theta}} - \frac{q^{2n} e^{-i\theta}}{1 - q^{2n} e^{-i\theta}} \right). \tag{2.5.26}
\end{aligned}$$

This shows that  $f_0$  has simple poles at  $\theta = 2m\pi + 2n\pi\tau$ , where  $m, n \in \mathbb{Z}$ .

From the infinite product expansions (2.5.11), (2.5.13), and (2.5.15), the poles of  $f_1$ ,  $f_2$ , and  $f_3$  are also the same as those of  $f_0$ . Therefore all four functions have simple poles at  $\theta = 2m\pi + 2n\pi\tau$ , where  $m, n \in \mathbb{Z}$ , and no other singularities.

Section 3 of Chapter 7 gives another way to find the poles of  $f_0, f_1, f_2, f_3$ .

A summary of the results of the zeros and poles of  $f_0, f_1, f_2, f_3$  as follows

	Zeros	Poles
$f_0(\theta)$	no explicit formula	$2m\pi + 2n\pi\tau$
$f_1(\theta)$	$(2m+1)\pi + 2n\pi\tau$	$2m\pi + 2n\pi\tau$
$f_2(\theta)$	$2m\pi + (2n+1)\pi\tau$	$2m\pi + 2n\pi\tau$
$f_3(\theta)$	$(2m+1)\pi + (2n+1)\pi\tau$	$2m\pi + 2n\pi\tau$

(2.5.27)

For any integer values of  $m$  and  $n$ , the residue of each function at each pole is 1.

## 2.6 The Fundamental multiplicative identity

In this section, we introduce the Fundamental multiplicative identity. It is a very useful tool used frequently throughout this thesis.

### Theorem 2.6.1 (The Fundamental Multiplicative Identity)

Let  $F$  be the Jordan-Kronecker function defined by (2.4.1).

Then

$$F(a, t)F(b, t) = t \frac{\partial}{\partial t} F(ab, t) + F(ab, t) [\rho_1(a) + \rho_1(b)], \quad (2.6.1)$$

where

$$\rho_1(a) = \frac{1}{2} + \sum_{n \neq 0} \frac{a^n}{1 - q^{2n}}. \quad (2.6.2)$$

**Proof** Using the definition (2.4.1), the left hand side of (2.6.1) becomes

$$\begin{aligned} F(a, t)F(b, t) &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{a^n b^m}{(1 - tq^{2n})(1 - tq^{2m})} \\ &= \sum_{n=-\infty}^{\infty} \frac{a^n b^n}{(1 - tq^{2n})^2} + \sum_{n \neq m} \frac{a^n b^m}{(1 - tq^{2n})(1 - tq^{2m})}. \end{aligned} \quad (2.6.3)$$

Next by considering the first sum of (2.6.3),

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{a^n b^n}{(1 - tq^{2n})^2} &= \sum_{n=-\infty}^{\infty} \frac{\partial}{\partial t} \frac{(ab/q^2)^n}{(1 - tq^{2n})} \\ &= \frac{\partial}{\partial t} F\left(\frac{ab}{q^2}, t\right). \end{aligned} \quad (2.6.4)$$

From (2.4.5), equation (2.6.4) can be rewritten as

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{a^n b^n}{(1 - tq^{2n})^2} &= \frac{\partial}{\partial t} [tF(ab, t)] \\ &= t \frac{\partial}{\partial t} F(ab, t) + F(ab, t). \end{aligned} \quad (2.6.5)$$

By letting  $m = n + k, k \neq 0$  in the second sum on the right hand side of (2.6.3) and using partial fractions, we obtain

$$\begin{aligned} &\sum_{n \neq m} \frac{a^n b^m}{(1 - tq^{2n})(1 - tq^{2m})} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k \neq 0} \frac{a^n b^{n+k}}{(1 - tq^{2n})(1 - tq^{2n+2k})} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k \neq 0} \left[ \frac{a^n b^{n+k}}{(1 - tq^{2n})(1 - q^{2k})} - \frac{q^{2k} a^n b^{n+k}}{(1 - q^{2k})(1 - tq^{2n+2k})} \right]. \end{aligned} \quad (2.6.6)$$



Now by setting  $j = n + k$  in the second double sum of (2.6.6) we obtain

$$\begin{aligned}
 & \sum_{n \neq m} \frac{a^n b^m}{(1 - tq^{2n})(1 - tq^{2m})} \\
 &= \left( \sum_{n=-\infty}^{\infty} \frac{a^n b^n}{1 - tq^{2n}} \right) \left( \sum_{k \neq 0} \frac{b^k}{1 - q^{2k}} \right) - \left( \sum_{j=-\infty}^{\infty} \frac{a^j b^j}{1 - tq^{2j}} \right) \left( \sum_{k \neq 0} \frac{q^{2k} a^{-k}}{1 - q^{2k}} \right) \\
 &= F(ab, t) \left[ \sum_{k \neq 0} \frac{b^k}{1 - q^{2k}} - \sum_{k \neq 0} \frac{q^{-2k} a^k}{1 - q^{-2k}} \right] \\
 &= F(ab, t) \left[ \sum_{k \neq 0} \frac{b^k}{1 - q^{2k}} + \sum_{k \neq 0} \frac{a^k}{1 - q^{2k}} \right] \\
 &= F(ab, t) [\rho_1(a) + \rho_1(b) - 1].
 \end{aligned} \tag{2.6.7}$$

Then we substitute (2.6.5) and (2.6.7) into (2.6.3) to prove (2.6.1). ■

The proof of (2.6.1) is the same as that given in Venkatachaliengar [100, p. 40] or Cooper [39, p. 66].

From (2.6.2) we have

$$\begin{aligned}
 \rho_1(a) &= \frac{1}{2} + \sum_{n \neq 0} \frac{a^n}{1 - q^{2n}} \\
 &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{a^n}{1 - q^{2n}} + \sum_{n=1}^{\infty} \frac{a^{-n}}{1 - q^{-2n}}
 \end{aligned} \tag{2.6.8}$$

$$\begin{aligned}
 &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{a^n (1 - q^{2n} + q^{2n})}{1 - q^{2n}} - \sum_{n=1}^{\infty} \frac{q^{2n} a^{-n}}{1 - q^{2n}} \\
 &= \frac{1+a}{2(1-a)} + \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} (a^n - a^{-n})
 \end{aligned} \tag{2.6.9}$$

$$\begin{aligned}
 &= \frac{1+a}{2(1-a)} + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} q^{2nk} a^n - \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} q^{2nk} a^{-n} \\
 &= \frac{1+a}{2(1-a)} + \sum_{k=1}^{\infty} \frac{aq^{2k}}{1 - aq^{2k}} - \sum_{k=1}^{\infty} \frac{a^{-1}q^{2k}}{1 - a^{-1}q^{2k}}.
 \end{aligned} \tag{2.6.10}$$

Since the series (2.6.10) converges for all values of  $z$  except  $z = q^{2m}$ ,  $m \in \mathbb{Z}$  (where there are poles of order 1), (2.6.1) is valid for all values of  $a$ ,  $b$ , and  $t$ .

The properties of

$$\rho_1(a) = -\rho_1(a^{-1}) \quad (2.6.11)$$

and

$$\rho_1(a) = \rho_1(aq^2) - 1 \quad (2.6.12)$$

can be deduced using (2.6.10).

If we let  $a = e^{i\theta}$  in (2.6.9) we obtain

$$\begin{aligned} \rho_1(e^{i\theta}) &= \frac{1 + e^{i\theta}}{2(1 - e^{i\theta})} + \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} (e^{i\theta n} - e^{-i\theta n}) \\ &= \frac{i}{2} \cot \frac{\theta}{2} + 2i \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sin n\theta \\ &= if_0(\theta). \end{aligned} \quad (2.6.13)$$

We end this section with the following two corollaries, which will be used to obtain formulae (4.2.9) and (4.3.9), respectively.

### Corollary 2.6.2

$$\frac{F(a, t) F(b, t)}{F(ab, t)} = \rho_1(a) + \rho_1(b) + \rho_1(t) - \rho_1(abt). \quad (2.6.14)$$

**Proof** If we divide both sides of (2.6.1) by  $F(ab, t)$  then we obtain

$$\frac{F(a, t) F(b, t)}{F(ab, t)} = t \frac{\partial}{\partial t} \log F(ab, t) + \rho_1(a) + \rho_1(b). \quad (2.6.15)$$

Next we use (2.4.2) to compute the logarithmic derivative of (2.6.15) and obtain

$$\begin{aligned}
 t \frac{\partial}{\partial t} \log F(ab, t) &= t \frac{\partial}{\partial t} \log \left[ \frac{(abt, a^{-1}b^{-1}t^{-1}q^2, q^2, q^2; q^2)_{\infty}}{(ab, a^{-1}b^{-1}q^2, t, t^{-1}q^2; q^2)_{\infty}} \right] \\
 &= \sum_{n=1}^{\infty} \left[ -\frac{abtq^{2n-2}}{1 - abtq^{2n-2}} + \frac{a^{-1}b^{-1}t^{-1}q^{2n}}{1 - a^{-1}b^{-1}t^{-1}q^{2n}} \right. \\
 &\quad \left. + \frac{tq^{2n-2}}{1 - tq^{2n-2}} - \frac{t^{-1}q^{2n}}{1 - t^{-1}q^{2n}} \right]. \tag{2.6.16}
 \end{aligned}$$

By using (2.6.10), we can rewrite (2.6.16) to become

$$t \frac{\partial}{\partial t} \log F(ab, t) = \rho_1(t) - \rho_1(abt). \tag{2.6.17}$$

By substituting (2.6.17) into (2.6.15), we complete the proof of (2.6.14). ■

The proof of (2.6.14) is the same as that given in Cooper and M. D. Hirschhorn in [44] or Cooper [42].

### Corollary 2.6.3

$$\begin{aligned}
 &\sum_{j=1}^{\infty} \frac{(u^{-j} - u^j)(v^{-j} - v^j)(w^{-j} - w^j)}{(q^{-j} - q^j)} \\
 &= \frac{q}{uvw} \frac{\left(u^2, v^2, w^2, \frac{q^2}{u^2}, \frac{q^2}{v^2}, \frac{q^2}{w^2}, q^2, q^2; q^2\right)_{\infty}}{uvw \left(uvwq, \frac{vwq}{u}, \frac{uwq}{v}, \frac{uvq}{w}, \frac{wq}{uv}, \frac{vq}{uw}, \frac{uq}{vw}, \frac{q}{uvw}; q^2\right)_{\infty}}, \tag{2.6.18}
 \end{aligned}$$

where  $|q| < |uvw|, |u|, |v|, |w| < 1$ .

**Proof** Using (2.6.11), equation (2.6.14) can be written as

$$\frac{F(a, t) F(b, t)}{F(ab, t)} = \rho_1(a) + \rho_1(b) - \rho_1(t^{-1}) - \rho_1(abt). \tag{2.6.19}$$

If we let  $a = \frac{uq}{vw}, b = \frac{vq}{uw}, t = \frac{wq}{uv}$  in (2.6.19) then we obtain

$$\frac{F\left(\frac{uq}{vw}, \frac{wq}{uv}\right) F\left(\frac{vq}{uw}, \frac{wq}{uv}\right)}{F\left(\frac{q^2}{w^2}, \frac{wq}{uv}\right)} = \rho_1\left(\frac{uq}{vw}\right) + \rho_1\left(\frac{vq}{uw}\right) - \rho_1\left(\frac{uv}{wq}\right) - \rho_1\left(\frac{q^3}{uvw}\right). \tag{2.6.20}$$

By using (2.6.12), it can be shown that

$$\rho_1 \left( \frac{uv}{wq} \right) = \rho_1 \left( \frac{uvq}{w} \right) - 1$$

and

$$\rho_1 \left( \frac{q^3}{uvw} \right) = \rho_1 \left( \frac{q}{uvw} \right) + 1.$$

Therefore equation (2.6.20) can be written as

$$\frac{F \left( \frac{uq}{vw}, \frac{wq}{uv} \right) F \left( \frac{vq}{uw}, \frac{wq}{uv} \right)}{F \left( \frac{q^2}{w^2}, \frac{wq}{uv} \right)} = \rho_1 \left( \frac{uq}{vw} \right) + \rho_1 \left( \frac{vq}{uw} \right) - \rho_1 \left( \frac{uvq}{w} \right) - \rho_1 \left( \frac{q}{uvw} \right). \quad (2.6.21)$$

Using (2.4.2) and (2.6.8) in (2.6.21) and simplifying, we find

$$\begin{aligned} & \frac{\left( \frac{q^2}{v^2}, v^2, \frac{q^2}{u^2}, u^2, \frac{q^2}{w^2}, w^2, q^2, q^2; q^2 \right)_{\infty}}{\left( \frac{uq}{vw}, \frac{vwq}{u}, \frac{vq}{uw}, \frac{uwq}{v}, \frac{wq}{uv}, \frac{uvq}{w}, \frac{q^3}{uvw}, \frac{uvw}{q}; q^2 \right)_{\infty}} \\ &= \sum_{n=1}^{\infty} \frac{q^n}{1 - q^{2n}} \left[ \left( \frac{u}{vw} \right)^n - \left( \frac{vw}{u} \right)^n + \left( \frac{v}{uw} \right)^n - \left( \frac{uw}{v} \right)^n \right. \\ & \quad \left. - \left( \frac{uv}{w} \right)^n + \left( \frac{w}{uv} \right)^n - \left( \frac{1}{uvw} \right)^n - (uvw)^n \right]. \end{aligned} \quad (2.6.22)$$

By factorising the terms in the series we find

$$\begin{aligned} & \frac{\left( 1 - \frac{q}{uvw} \right)}{\left( 1 - \frac{uvw}{q} \right)} \frac{\left( u^2, v^2, w^2, \frac{q^2}{u^2}, \frac{q^2}{v^2}, \frac{q^2}{w^2}, q^2, q^2; q^2 \right)_{\infty}}{\left( uvwq, \frac{vwq}{u}, \frac{uwq}{v}, \frac{uvq}{w}, \frac{wq}{uv}, \frac{vq}{uw}, \frac{uq}{vw}, \frac{q}{uvw}; q^2 \right)_{\infty}} \\ &= \sum_{n=1}^{\infty} \frac{(u^n - u^{-n})(v^n - v^{-n})(w^n - w^{-n})}{q^{-n} - q^n}. \end{aligned} \quad (2.6.23)$$

This proves (2.6.18).

Using the ratio test, the series of (2.6.18) on the left converges for  $|q| < |uvw|$ ,  $|u|, |v|, |w| < 1$ . ■

This proof of (2.6.18) was given in [42].

If we set  $u^2 = a$ ,  $v^2 = b$ ,  $w^2 = ab$  in (2.6.18) and simplify then we find that

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{q^n}{1 - q^{2n}} \left( \frac{1}{a^n b^n} - \frac{1}{a^n} - \frac{1}{b^n} + a^n + b^n - a^n b^n \right) \\ &= \frac{q}{ab} \frac{\left( a, \frac{q^2}{a}, b, \frac{q^2}{b}, ab, \frac{q^2}{ab}, q^2, q^2; q^2 \right)_{\infty}}{\left( aq, \frac{q}{a}, bq, \frac{q}{b}, abq, \frac{q}{ab}, q, q; q^2 \right)_{\infty}}. \end{aligned} \quad (2.6.24)$$

This was given by J. W. L. Glaisher [56] and an application of this formula can be found in Berndt [19, p. 303].

## 2.7 The Weierstrass $\wp$ function

We now derive the connection between the function  $\rho_1$  and the Weierstrass  $\wp$  function. The

Weierstrass  $\wp$  function with periods  $2\pi$  and  $2\pi\tau$  is defined by

$$\wp(\theta) = \frac{1}{\theta^2} + \sum_{(m,n) \neq (0,0)} \left[ \frac{1}{(\theta - 2\pi m - 2\pi\tau n)^2} - \frac{1}{(2\pi m + 2\pi\tau n)^2} \right].$$

It can be shown [39, p. 68] that

$$\wp(\theta) = \frac{-1}{12} + 2 \sum_{n=1}^{\infty} \frac{q^{2n}}{(1 - q^{2n})^2} - \sum_{n=-\infty}^{\infty} \frac{e^{i\theta} q^{2n}}{(1 - e^{i\theta} q^{2n})^2} \quad (2.7.1)$$

$$= \frac{-1}{12} + 2 \sum_{n=1}^{\infty} \frac{q^{2n}}{(1 - q^{2n})^2} + i \frac{d}{d\theta} \rho_1(e^{i\theta}). \quad (2.7.2)$$

Let

$$P = P(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{q^{2n}}{(1 - q^{2n})^2} \quad (2.7.3)$$

$$\begin{aligned} &= 1 - 24 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m q^{2mn} \\ &= 1 - 24 \sum_{m=1}^{\infty} \frac{m q^{2m}}{1 - q^{2m}}, \end{aligned} \quad (2.7.4)$$

then by (2.6.13)

$$\wp(\theta) = \frac{1}{4} \csc^2 \frac{\theta}{2} - 2 \sum_{m=1}^{\infty} \frac{mq^{2m}}{1-q^{2m}} \cos m\theta - \frac{P}{12} \quad (2.7.5)$$

$$= -\frac{d}{d\theta} f_0(\theta) - \frac{P}{12}. \quad (2.7.6)$$

The Weierstrass invariants  $e_1$ ,  $e_2$ , and  $e_3$  are defined by

$$e_1 = e_1(q) = \wp(\pi), \quad (2.7.7)$$

$$e_2 = e_2(q) = \wp(\pi\tau), \quad (2.7.8)$$

$$e_3 = e_3(q) = \wp(\pi + \pi\tau). \quad (2.7.9)$$

If we put  $\theta = \pi$  into (2.7.5) we find

$$e_1(q) = \frac{1}{4} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n nq^{2n}}{1-q^{2n}} - \frac{1}{12} + 2 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} \quad (2.7.10)$$

$$\begin{aligned} &= \frac{1}{6} + 4 \sum_{n=1}^{\infty} \frac{(2n-1)q^{4n-2}}{1-q^{4n-2}} \\ &= \frac{1}{6} + 4 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} - 4 \sum_{n=1}^{\infty} \frac{2nq^{4n}}{1-q^{4n}} \\ &= \frac{1}{6} + 4 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} - 4 \sum_{n=1}^{\infty} \frac{nq^{4n}}{1-q^{4n}} - 4 \sum_{n=1}^{\infty} \frac{nq^{4n}}{1-q^{4n}} \\ &= \frac{1}{6} + 4 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{4n}} - 4 \sum_{n=1}^{\infty} \frac{nq^{4n}}{1-q^{4n}} \\ &= \frac{1}{6} + 4 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1+q^{2n}}. \end{aligned} \quad (2.7.11)$$

Similarly, by putting  $\theta = \pi\tau$  and  $\pi + \pi\tau$  into (2.7.5) respectively, then simplifying, we obtain

$$e_2(q) = -\frac{1}{12} - 2 \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} \quad (2.7.12)$$

and

$$e_3(q) = -\frac{1}{12} + 2 \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1+q^{2n-1}}. \quad (2.7.13)$$

From equations (2.7.11)–(2.7.13) we obtain the sum

$$\begin{aligned} & e_1(q) + e_2(q) + e_3(q) \\ &= 4 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1+q^{2n}} + 2 \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1+q^{2n-1}} - 2 \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} \\ &= 2 \sum_{n=1}^{\infty} \frac{nq^n}{1+q^n} - 2 \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} \\ &= 2 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}} + 2 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} - 4 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} - 2 \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} \\ &= 2 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - 2 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \\ &= 0. \end{aligned} \quad (2.7.14)$$

Next we consider the relationship of each of  $f_1$ ,  $f_2$ , and  $f_3$  to the Weierstrass  $\wp$  function.

First let  $b \rightarrow 1/a$  in the fundamental multiplicative identity (2.6.1) to arrive at

$$\lim_{b \rightarrow 1/a} F(a, t) F(b, t) = \lim_{b \rightarrow 1/a} t \frac{\partial}{\partial t} F(ab, t) + \lim_{b \rightarrow 1/a} F(ab, t) (\rho_1(a) + \rho_1(b)). \quad (2.7.15)$$

The left hand side of (2.7.15) is simply  $F(a, t) F(1/a, t)$ . The first limit on the right hand side of (2.7.15) is

$$\begin{aligned} \lim_{b \rightarrow 1/a} t \frac{\partial}{\partial t} \sum_{n=-\infty}^{\infty} \frac{t^n}{1-abq^{2n}} &= \lim_{b \rightarrow 1/a} \sum_{n \neq 0} \frac{nt^n}{1-abq^{2n}} \\ &= \sum_{n \neq 0} \frac{nt^n}{1-q^{2n}} \\ &= t \frac{d}{dt} \rho_1(t). \end{aligned}$$

By using (2.6.11) and the infinite product formula (2.4.2) for the function  $F$ , we see that the remaining limit on the right hand side of (2.7.15) becomes

$$\begin{aligned}
 & \lim_{b \rightarrow 1/a} F(ab, t) (\rho_1(a) + \rho_1(b)) \\
 &= \lim_{b \rightarrow 1/a} (1 - ab) F(ab, t) \lim_{b \rightarrow 1/a} \frac{\rho_1(a) + \rho_1(b)}{1 - ab} \\
 &= \lim_{b \rightarrow 1/a} (1 - ab) \frac{(abt, q^2/abt, q^2, q^2; q^2)_\infty}{(t, q^2/t, ab, q^2/ab; q^2)_\infty} \lim_{b \rightarrow 1/a} \frac{\rho_1(a) - \rho_1(1/b)}{a - 1/b} \left(-\frac{1}{b}\right) \\
 &= (1) \rho'_1(a) (-a).
 \end{aligned}$$

Hence we have [100, p. 112]

$$F(a, t) F(1/a, t) = t \frac{d}{dt} \rho_1(t) - a \frac{d}{da} \rho_1(a). \quad (2.7.16)$$

If we set  $a = e^{i\alpha}$ ,  $t = e^{i\theta}$  and use (2.7.6) on (2.7.16), we obtain [39, p. 122]

$$F(e^{i\alpha}, e^{i\theta}) F(e^{-i\alpha}, e^{i\theta}) = \wp(\alpha) - \wp(\theta). \quad (2.7.17)$$

## 2.8 Fourier series for squares of four functions

In this section we obtain the Fourier series for squares of  $f_0, f_1, f_2, f_3$  using the Fundamental multiplicative identity.

First by letting  $a = b = e^{iv}$ ,  $t = e^{i\theta}$  in (2.6.1) we obtain

$$F^2(e^{iv}, e^{i\theta}) = \frac{1}{i} \frac{\partial}{\partial \theta} F(e^{2iv}, e^{i\theta}) + 2F(e^{2iv}, e^{i\theta}) \rho_1(e^{iv}). \quad (2.8.1)$$

By (2.4.8), we observe that

$$\cot u = \frac{1}{u} + \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j} B_{2j}}{(2j)!} u^{2j-1}. \quad (2.8.2)$$



Now if we use (2.8.2) to expand (2.6.13) in powers of  $v$  then

$$\begin{aligned}\rho_1(e^{iv}) &= \frac{i}{2} \cot \frac{v}{2} + 2i \sum_{m=1}^{\infty} \frac{q^{2m}}{1 - q^{2m}} \sin mv \\ &= \frac{i}{v} + 2i \sum_{j=1}^{\infty} \left[ \frac{B_{2j}}{4j} - \sum_{m=1}^{\infty} \frac{m^{2j-1} q^{2m}}{1 - q^{2m}} \right] \frac{(-1)^j v^{2j-1}}{(2j-1)!}.\end{aligned}\quad (2.8.3)$$

Using (2.8.3) and (2.4.14), equation (2.8.1) becomes

$$\begin{aligned}& \left\{ -\frac{1}{iv} + 2i \left[ \frac{1}{4} \cot \frac{\theta}{2} + \sum_{m=1}^{\infty} \frac{q^{2m}}{1 - q^{2m}} \sin m\theta \right] \right. \\ & + \left. \left[ -\frac{1}{12} + 2 \sum_{m=1}^{\infty} \frac{q^{2m}}{(1 - q^{2m})^2} \cos m\theta \right] iv \right. \\ & + \left. \sum_{j=2}^{\infty} \left[ -\frac{B_{j+1}}{j+1} + \sum_{m=1}^{\infty} \left( \frac{m^j e^{i\theta} q^{2m}}{1 - e^{i\theta} q^{2m}} - \frac{(-1)^j m^j e^{-i\theta} q^{2m}}{1 - e^{-i\theta} q^{2m}} \right) \right] \frac{i^j v^j}{j!} \right\}^2 \\ &= -\frac{1}{4} - \frac{1}{4} \cot^2 \frac{\theta}{2} + 2 \sum_{m=1}^{\infty} \frac{mq^{2m}}{1 - q^{2m}} \cos m\theta \\ & + \frac{1}{i} \frac{\partial}{\partial \theta} \left\{ \left[ -\frac{1}{12} + 2 \sum_{m=1}^{\infty} \frac{q^{2m}}{(1 - q^{2m})^2} \cos m\theta \right] 2iv \right. \\ & + \left. \sum_{j=2}^{\infty} \left[ -\frac{B_{j+1}}{j+1} + \sum_{m=1}^{\infty} \left( \frac{m^j e^{i\theta} q^{2m}}{1 - e^{i\theta} q^{2m}} - \frac{(-1)^j m^j e^{-i\theta} q^{2m}}{1 - e^{-i\theta} q^{2m}} \right) \right] \frac{2^j i^j v^j}{j!} \right\} \\ & + 2 \left\{ \frac{i}{v} + 2i \sum_{j=1}^{\infty} \left[ \frac{B_{2j}}{4j} - \sum_{m=1}^{\infty} \frac{m^{2j-1} q^{2m}}{1 - q^{2m}} \right] \frac{(-1)^j v^{2j-1}}{(2j-1)!} \right\} \\ & \times \left\{ -\frac{1}{2iv} + 2i \left[ \frac{1}{4} \cot \frac{\theta}{2} + \sum_{m=1}^{\infty} \frac{q^{2m}}{1 - q^{2m}} \sin m\theta \right] \right. \\ & + \left. \left[ -\frac{1}{12} + 2 \sum_{m=1}^{\infty} \frac{q^{2m}}{(1 - q^{2m})^2} \cos m\theta \right] 2iv \right. \\ & + \left. \sum_{j=2}^{\infty} \left[ -\frac{B_{j+1}}{j+1} + \sum_{m=1}^{\infty} \left( \frac{m^j e^{i\theta} q^{2m}}{1 - e^{i\theta} q^{2m}} - \frac{(-1)^j m^j e^{-i\theta} q^{2m}}{1 - e^{-i\theta} q^{2m}} \right) \right] \frac{2^j i^j v^j}{j!} \right\}.\end{aligned}\quad (2.8.4)$$

By comparing the coefficients of  $v^0$  in (2.8.4) we find

$$\begin{aligned}
 & \left[ \frac{1}{6} - 4 \sum_{m=1}^{\infty} \frac{q^{2m}}{(1-q^{2m})^2} \cos m\theta \right] - 4 \left( \frac{1}{4} \cot \frac{\theta}{2} + \sum_{m=1}^{\infty} \frac{q^{2m}}{1-q^{2m}} \sin m\theta \right)^2 \\
 = & -\frac{1}{4} - \frac{1}{4} \cot^2 \frac{\theta}{2} + 2 \sum_{m=1}^{\infty} \frac{mq^{2m}}{1-q^{2m}} \cos m\theta \\
 & + \frac{1}{12} - 2 \sum_{m=1}^{\infty} \frac{mq^{2m}}{1-q^{2m}} + \frac{1}{3} - 8 \sum_{m=1}^{\infty} \frac{q^{2m}}{(1-q^{2m})^2} \cos m\theta.
 \end{aligned} \tag{2.8.5}$$

By simplifying and rearranging (2.8.5) we arrive at

$$\begin{aligned}
 f_0^2(\theta) &= \left( \frac{1}{2} \cot \frac{\theta}{2} + 2 \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \sin n\theta \right)^2 \\
 &= \frac{1}{4} \cot^2 \frac{\theta}{2} + 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2} \cos n\theta + 2 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} (1 - \cos n\theta),
 \end{aligned} \tag{2.8.6}$$

which is the same as (2.5.6). The proof we have given here is due to Venkatachaliengar [100]. Ramanujan published an equivalent identity (2.8.6) in [61, p. 139] and gave a proof by using series manipulations. B. Van Der Pol [99], J. Ewell [52], and L. C. Shen [98] employed the heat equation to obtain identity (2.8.6). Other proofs of identity (2.8.6) can be found in [38] and [100]. Hardy and E. M. Wright [62, p. 312] and Hardy [60, p. 134] applied identity (2.8.6) to obtain the sums of four squares theorem.

Next if we let  $\alpha = \pi, \pi\tau, \pi + \pi\tau$ , respectively, in (2.7.17) we obtain [38, p. 72]

$$f_1^2(\theta) = \wp(\theta) - \wp(\pi), \tag{2.8.7}$$

$$f_2^2(\theta) = \wp(\theta) - \wp(\pi\tau), \tag{2.8.8}$$

$$f_3^2(\theta) = \wp(\theta) - \wp(\pi + \pi\tau). \tag{2.8.9}$$

By (2.7.7)–(2.7.9) and (2.8.7)–(2.8.9), we obtain [38, p. 72]

$$f_2^2(\theta) - f_1^2(\theta) = e_1 - e_2, \quad (2.8.10)$$

$$f_2^2(\theta) - f_3^2(\theta) = e_3 - e_2, \quad (2.8.11)$$

$$f_3^2(\theta) - f_1^2(\theta) = e_1 - e_3. \quad (2.8.12)$$

By using the definitions of  $e_1, e_2, e_3$  and (2.7.17) the following hold [39, p. 123] or [100, p. 66]

$$e_1 - e_2 = \frac{1}{4} \frac{(-q; q^2)_\infty^4 (q^2; q^2)_\infty^4}{(q; q^2)_\infty^4 (-q^2; q^2)_\infty^4}, \quad (2.8.13)$$

$$e_3 - e_2 = 4q \frac{(-q^2; q^2)_\infty^4 (q^2; q^2)_\infty^4}{(-q; q^2)_\infty^4 (q; q^2)_\infty^4}, \quad (2.8.14)$$

$$e_1 - e_3 = \frac{1}{4} \frac{(q; q^2)_\infty^4 (q^2; q^2)_\infty^4}{(-q; q^2)_\infty^4 (-q; q^2)_\infty^4}. \quad (2.8.15)$$

Note that since  $\text{Im } \tau > 0$  this implies that  $e_1 \neq e_2$ ,  $e_1 \neq e_3$ , and  $e_2 \neq e_3$ . Substituting (2.7.5) and (2.7.11) into (2.8.7), we see

$$f_1^2(\theta) = -\frac{1}{4} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n n q^{2n}}{1 - q^{2n}} + \frac{1}{4} \csc^2 \frac{\theta}{2} - 2 \sum_{n=1}^{\infty} \frac{n q^{2n}}{1 - q^{2n}} \cos n\theta. \quad (2.8.16)$$

Similarly, by substituting (2.7.12) and (2.7.13) into (2.8.8) and (2.8.9), respectively, and then using (2.7.5) gives us

$$f_2^2(\theta) = 2 \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^{2n}} + \frac{1}{4} \csc^2 \frac{\theta}{2} - 2 \sum_{n=1}^{\infty} \frac{n q^{2n}}{1 - q^{2n}} \cos n\theta, \quad (2.8.17)$$

$$f_3^2(\theta) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n n q^n}{1 - q^{2n}} + \frac{1}{4} \csc^2 \frac{\theta}{2} - 2 \sum_{n=1}^{\infty} \frac{n q^{2n}}{1 - q^{2n}} \cos n\theta. \quad (2.8.18)$$

## 2.9 Derivatives

The aim of this section is to give the first and second derivatives of  $f_0, f_1, f_2$ , and  $f_3$ .

### 2.9.1 The derivatives of $f_0, f_1, f_2$ , and $f_3$

From equations (2.7.6) and (2.8.7) we have

$$\begin{aligned} f'_0(\theta) &= -\wp(\theta) - \frac{P}{12} = -f_1^2(\theta) - \wp(\pi) - \frac{P}{12} \\ &= -f_1^2(\theta) - \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n n q^{2n}}{1 - q^{2n}} \end{aligned} \quad (2.9.1)$$

by (2.7.10). Next we will examine the derivatives of  $f_1, f_2$ , and  $f_3$ .

By changing the variable  $t$  to  $e^{i\theta}$ , the fundamental multiplicative identity (2.6.1) becomes

$$F(a, e^{i\theta}) F(b, e^{i\theta}) = \frac{1}{i} \frac{\partial}{\partial \theta} F(ab, e^{i\theta}) + F(ab, e^{i\theta})(\rho_1(a) + \rho_1(b)). \quad (2.9.2)$$

We let  $a = e^{i\pi\tau}$  and  $b = e^{i\pi + i\pi\tau}$ . From (2.6.8),  $\rho_1(e^{i\pi\tau}) = \rho_1(e^{i\pi + i\pi\tau}) = \frac{1}{2}$ . Therefore (2.9.2) becomes

$$F(q, e^{i\theta}) F(-q, e^{i\theta}) = \frac{1}{i} \frac{\partial}{\partial \theta} F(-q^2, e^{i\theta}) + F(-q^2, e^{i\theta}). \quad (2.9.3)$$

By applying the property of (2.4.5) on the right hand side of (2.9.3) we obtain

$$F(q, e^{i\theta}) F(-q, e^{i\theta}) = \frac{1}{i} \frac{\partial}{\partial \theta} [e^{-i\theta} F(-1, e^{i\theta})] + e^{-i\theta} F(-1, e^{i\theta}). \quad (2.9.4)$$

Employing (2.5.2)–(2.5.4), equation (2.9.4) becomes

$$\begin{aligned} ie^{-i\theta/2} f_2(\theta) ie^{-i\theta/2} f_3(\theta) &= \frac{1}{i} \frac{\partial}{\partial \theta} [ie^{-i\theta} f_1(\theta)] + ie^{-i\theta} f_1(\theta) \\ -e^{-i\theta} f_2(\theta) f_3(\theta) &= -ie^{-i\theta} f_1(\theta) + e^{-i\theta} f'_1(\theta) + ie^{-i\theta} f_1(\theta). \end{aligned}$$

Therefore we deduce that [39, p. 73]

$$f_1'(\theta) = -f_2(\theta) f_3(\theta). \quad (2.9.5)$$

Similarly by letting  $a = e^{i\pi}$ ,  $b = e^{i\pi+i\pi\tau}$  and  $a = e^{i\pi}$ ,  $b = e^{i\pi\tau}$ , respectively, in (2.9.2) this gives us

$$f_2'(\theta) = -f_1(\theta) f_3(\theta), \quad (2.9.6)$$

$$f_3'(\theta) = -f_1(\theta) f_2(\theta). \quad (2.9.7)$$

Note that if we set  $q = 0$  in equation (2.9.5) and (2.9.6) then we obtain the well known trigonometric results, respectively,

$$\left(\frac{1}{2} \cot \frac{\theta}{2}\right)' = -\frac{1}{4} \csc^2 \frac{\theta}{2}$$

and

$$\left(\frac{1}{2} \csc \frac{\theta}{2}\right)' = -\frac{1}{4} \cot \frac{\theta}{2} \csc \frac{\theta}{2}.$$

## 2.9.2 The second derivatives of $f_0$ , $f_1$ , $f_2$ , and $f_3$

### Lemma 2.9.1

$$f_0''(\theta) = 2f_1(\theta) f_2(\theta) f_3(\theta), \quad (2.9.8)$$

$$f_1''(\theta) = 2f_1^3(\theta) + 3e_1 f_1(\theta), \quad (2.9.9)$$

$$f_2''(\theta) = 2f_2^3(\theta) + 3e_2 f_2(\theta), \quad (2.9.10)$$

$$f_3''(\theta) = 2f_3^3(\theta) + 3e_3 f_3(\theta). \quad (2.9.11)$$

**Proof** Equation (2.9.8) is obtained by differentiating equation (2.9.1) and using (2.9.5).

By differentiating (2.9.5) and using the results of (2.9.6) and (2.9.7) we find

$$f_1''(\theta) = f_1(\theta) (f_2^2(\theta) + f_3^2(\theta)). \quad (2.9.12)$$

By using (2.8.10) and (2.8.12) into (2.9.12) then simplifying, the proof of (2.9.9) is completed. The proofs of (2.9.10) and (2.9.11) are similar. ■

**Lemma 2.9.2**

$$f_0(\theta) = \frac{1}{\theta} + \sum_{j=1}^{\infty} \left[ \frac{B_{2j}}{2j} - 2 \sum_{m=1}^{\infty} \frac{m^{2j-1} q^{2m}}{1 - q^{2m}} \right] \frac{(-1)^j \theta^{2j-1}}{(2j-1)!}, \quad (2.9.13)$$

$$\wp(\theta) = \frac{1}{\theta^2} + \sum_{j=1}^{\infty} \left( \frac{B_{2j+2}}{2j+2} - 2 \sum_{m=1}^{\infty} \frac{m^{2j+1} q^{2m}}{1 - q^{2m}} \right) \frac{(-1)^j \theta^{2j}}{(2j)!}. \quad (2.9.14)$$

**Proof** By employing the results of (2.6.13) and (2.8.3) we obtain (2.9.13). This proves the first part of the lemma.

Next by substituting (2.9.13) into (2.7.6) and simplifying gives (2.9.14). ■

**Theorem 2.9.3**

$$f_0'''(\theta) = -2 (f_1^2(\theta) f_2^2(\theta) + f_1^2(\theta) f_3^2(\theta) + f_2^2(\theta) f_3^2(\theta)) \quad (2.9.15)$$

$$= -6 \left( f_0'(\theta) + \frac{P}{12} \right)^2 + \frac{Q}{24}, \quad (2.9.16)$$

where

$$Q = Q(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^{2n}}{1 - q^{2n}}. \quad (2.9.17)$$

**Proof** Differentiating (2.9.8) with respect to  $\theta$  and using (2.9.5)–(2.9.7) to simplify gives (2.9.15).

Equation (2.9.15) may be rewritten as

$$f_0'''(\theta) = f_1^4(\theta) + f_2^4(\theta) + f_3^4(\theta) - (f_1^2(\theta) + f_2^2(\theta) + f_3^2(\theta))^2.$$

By substituting (2.8.7)–(2.8.9) into this we obtain

$$\begin{aligned} f_0'''(\theta) &= (\wp(\theta) - e_1)^2 + (\wp(\theta) - e_2)^2 + (\wp(\theta) - e_3)^2 \\ &\quad - (3\wp(\theta) - e_1 - e_2 - e_3)^2 \\ &= -6\wp^2(\theta) + 4\wp(\theta)(e_1 + e_2 + e_3) \\ &\quad - 2(e_1e_2 + e_1e_3 + e_2e_3) \\ &= -6\wp^2(\theta) + 4\wp(\theta)(e_1 + e_2 + e_3) \\ &\quad + e_1^2 + e_2^2 + e_3^2 - (e_1 + e_2 + e_3)^2. \end{aligned} \quad (2.9.18)$$

By using (2.7.14), (2.9.18) becomes

$$f_0'''(\theta) = -6\wp^2(\theta) + e_1^2 + e_2^2 + e_3^2. \quad (2.9.19)$$

Now if we use (2.9.13) and (2.9.14) to expand (2.9.19) in powers of  $\theta$  then equate the constant terms to give

$$e_1^2 + e_2^2 + e_3^2 = \frac{Q}{24}. \quad (2.9.20)$$

By substituting (2.9.20) into (2.9.19) and using (2.7.6) gives (2.9.16). ■

Equation (2.9.16) may be rewritten as [100, p. 2]

$$\wp''(\theta) = 6\wp^2(\theta) - \frac{Q}{24}. \quad (2.9.21)$$

Ramanujan [61, p. 139] also gave an equivalent result to (2.9.21).

#### Corollary 2.9.4

$$(\wp'(\theta))^2 = 4\wp^3(\theta) - \frac{Q}{12}\wp(\theta) - \frac{R}{216}, \quad (2.9.22)$$

where

$$R = R(q) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^{2n}}{1 - q^{2n}}. \quad (2.9.23)$$

**Proof** If we multiply both sides of (2.9.21) by  $\wp'(\theta)$  and integrate we obtain

$$(\wp'(\theta))^2 = 4\wp^3(\theta) - \frac{Q}{12}\wp(\theta) + c, \quad (2.9.24)$$

for some constant  $c$ .

If we employ (2.9.14) to expand (2.9.24) in powers of  $\theta$  and equate the constant term, this gives

$$c = -\frac{R}{216}. \quad (2.9.25)$$

By substituting (2.9.25) into (2.9.24) we complete the proof. ■

Equation (2.9.22) is the differential equation satisfied by the Weierstrass  $\wp$  function.

## 2.10 Higher derivatives

In this section we obtain sixteen Lambert series by expansions of the four functions  $f_0, f_1, f_2$ , and  $f_3$  at the four points  $0, \pi, \pi\tau$ , and  $\pi + \pi\tau$ .



Since  $f_0, f_1, f_2$ , and  $f_3$  are analytic at  $\pi, \pi\tau$ , and  $\pi + \pi\tau$ , the coefficients in the expansions about these points can be obtained by differentiation.

From equations (2.5.5), (2.5.7)–(2.5.9), we see that functions  $f_0, f_1, f_2, f_3$  have a simple pole at  $\theta = 0$  with residue 1. Therefore we can obtain the expansions about  $\theta = 0$  by differentiating the functions

$$\dot{f}_k(\theta) := f_k(\theta) - \frac{1}{\theta}, \quad \text{where } k = 0, 1, 2, 3.$$

By expanding (2.5.5), (2.5.7)–(2.5.9) in powers of  $\theta$  and then equating the coefficient of  $\frac{\theta^{2j-1}}{(2j-1)!}$ , respectively, we have

$$\dot{f}_0^{(2j-1)}(0) = 2(-1)^j \left[ \frac{B_{2j}}{4j} - \sum_{m=1}^{\infty} \frac{m^{2j-1} q^{2m}}{1 - q^{2m}} \right], \quad (2.10.1)$$

$$\dot{f}_1^{(2j-1)}(0) = 2(-1)^j \left[ \frac{B_{2j}}{4j} + \sum_{m=1}^{\infty} \frac{m^{2j-1} q^{2m}}{1 + q^{2m}} \right], \quad (2.10.2)$$

$$\dot{f}_2^{(2j-1)}(0) = 2(-1)^{j-1} \left[ \frac{(1 - 2^{1-2j}) B_{2j}}{4j} + \sum_{m=1}^{\infty} \frac{(m - \frac{1}{2})^{2j-1} q^{2m-1}}{1 - q^{2m-1}} \right], \quad (2.10.3)$$

$$\dot{f}_3^{(2j-1)}(0) = 2(-1)^{j-1} \left[ \frac{(1 - 2^{1-2j}) B_{2j}}{4j} - \sum_{m=1}^{\infty} \frac{(m - \frac{1}{2})^{2j-1} q^{2m-1}}{1 + q^{2m-1}} \right], \quad (2.10.4)$$

where  $j = 1, 2, 3, \dots$

Note that  $\dot{f}_k^{(2j)}(0) = 0$ , because  $\dot{f}_k$  is an odd function.

We obtain another set of twelve series by evaluating derivatives of  $f_0, f_1, f_2$ , and  $f_3$  at the points  $\theta = \pi, \pi\tau$ , and  $\pi + \pi\tau$ .

$$f_0^{(2j-1)}(\pi) = 2(-1)^j \left[ \frac{(2^{2j} - 1) B_{2j}}{4j} - \sum_{m=1}^{\infty} \frac{(-1)^m m^{2j-1} q^{2m}}{1 - q^{2m}} \right], \quad (2.10.5)$$

$$f_1^{(2j-1)}(\pi) = 2(-1)^j \left[ \frac{(2^{2j} - 1) B_{2j}}{4j} + \sum_{m=1}^{\infty} \frac{(-1)^m m^{2j-1} q^{2m}}{1 + q^{2m}} \right], \quad (2.10.6)$$

$$f_2^{(2j)}(\pi) = 2(-1)^j \left[ \frac{E_{2j}}{2^{2j+2}} - \sum_{m=1}^{\infty} \frac{(-1)^m \left(m - \frac{1}{2}\right)^{2j} q^{2m-1}}{1 - q^{2m-1}} \right], \quad (2.10.7)$$

$$f_3^{(2j)}(\pi) = 2(-1)^j \left[ \frac{E_{2j}}{2^{2j+2}} + \sum_{m=1}^{\infty} \frac{(-1)^m \left(m - \frac{1}{2}\right)^{2j} q^{2m-1}}{1 + q^{2m-1}} \right]; \quad (2.10.8)$$

$$f_0^{(2j-1)}(\pi\tau) = 2(-1)^{j-1} \sum_{m=1}^{\infty} \frac{m^{2j-1} q^m}{1 - q^{2m}}, \quad (2.10.9)$$

$$f_1^{(2j)}(\pi\tau) = 2i(-1)^{j-1} \sum_{m=1}^{\infty} \frac{m^{2j} q^m}{1 + q^{2m}}, \quad (2.10.10)$$

$$f_2^{(2j-1)}(\pi\tau) = 2(-1)^{j-1} \sum_{m=1}^{\infty} \frac{\left(m - \frac{1}{2}\right)^{2j-1} q^{m-\frac{1}{2}}}{1 - q^{2m-1}}, \quad (2.10.11)$$

$$f_3^{(2j)}(\pi\tau) = 2i(-1)^{j-1} \sum_{m=1}^{\infty} \frac{\left(m - \frac{1}{2}\right)^{2j} q^{m-\frac{1}{2}}}{1 + q^{2m-1}}; \quad (2.10.12)$$

$$f_0^{(2j-1)}(\pi + \pi\tau) = 2(-1)^{j-1} \sum_{m=1}^{\infty} \frac{(-1)^m m^{2j-1} q^m}{1 - q^{2m}}, \quad (2.10.13)$$

$$f_1^{(2j)}(\pi + \pi\tau) = 2i(-1)^{j-1} \sum_{m=1}^{\infty} \frac{(-1)^m m^{2j} q^m}{1 + q^{2m}}, \quad (2.10.14)$$

$$f_2^{(2j)}(\pi + \pi\tau) = 2(-1)^{j-1} \sum_{m=1}^{\infty} \frac{(-1)^m \left(m - \frac{1}{2}\right)^{2j} q^{m-\frac{1}{2}}}{1 - q^{2m-1}}, \quad (2.10.15)$$

$$f_3^{(2j-1)}(\pi + \pi\tau) = 2i(-1)^j \sum_{m=1}^{\infty} \frac{(-1)^m \left(m - \frac{1}{2}\right)^{2j-1} q^{m-\frac{1}{2}}}{1 + q^{2m-1}}. \quad (2.10.16)$$

Equations (2.10.1)–(2.10.16) hold for  $j \geq 1$ .

Also [39, pp. 127–128],

$$f_0^{(2j)}(\pi) = f_1^{(2j)}(\pi) = f_2^{(2j-1)}(\pi) = f_3^{(2j-1)}(\pi) = 0,$$

$$f_0^{(2j)}(\pi\tau) = f_1^{(2j-1)}(\pi\tau) = f_2^{(2j)}(\pi\tau) = f_3^{(2j-1)}(\pi\tau) = 0,$$

$$f_0^{(2j)}(\pi + \pi\tau) = f_1^{(2j-1)}(\pi + \pi\tau) = f_2^{(2j-1)}(\pi + \pi\tau) = f_3^{(2j)}(\pi + \pi\tau) = 0.$$

Equations (2.10.1)–(2.10.16) are called Fourier series expansions of Eisenstein series, which will be studied in more detail in Chapter 7. Equations (2.10.5)–(2.10.16) can be found in Cooper [39, p. 127], where  $(f_1^2)^{(2m)}(\pi)$ ,  $(f_1^2)^{(2m)}(\pi\tau)$ ,  $(f_1^2)^{(2m)}(\pi + \pi\tau)$  were used instead of  $f_0^{(2m)}(\pi)$ ,  $f_0^{(2m)}(\pi\tau)$ ,  $f_0^{(2m)}(\pi + \pi\tau)$ , respectively.

The following lemma completes this section.

**Lemma 2.10.1** For  $j = 1, 2, 3, \dots$ ,

$$\dot{f}_1^{(2j-1)}(0) + \dot{f}_2^{(2j-1)}(0) + \dot{f}_3^{(2j-1)}(0) = -(1 - 4^{1-j}) \dot{f}_0^{(2j-1)}(0), \quad (2.10.17)$$

$$f_0^{(2j-1)}(\pi) + f_0^{(2j-1)}(\pi\tau) + f_0^{(2j-1)}(\pi + \pi\tau) = (4^j - 1) \dot{f}_0^{(2j-1)}(0). \quad (2.10.18)$$

**Proof** We observe that

$$\begin{aligned} & \frac{(m - \frac{1}{2})^{2j-1} q^{2m-1}}{1 - q^{2m-1}} - \sum_{m=1}^{\infty} \frac{(m - \frac{1}{2})^{2j-1} q^{2m-1}}{1 + q^{2m-1}} - \sum_{m=1}^{\infty} \frac{m^{2j-1} q^{2m}}{1 + q^{2m}} \\ &= 4^{1-j} \sum_{m=1}^{\infty} \frac{(2m-1)^{2j-1} q^{4m-2}}{1 - q^{4m-2}} - \sum_{m=1}^{\infty} \frac{m^{2j-1} q^{2m}}{1 + q^{2m}} \\ &= 4^{1-j} \sum_{m=1}^{\infty} \frac{m^{2j-1} q^{2m}}{1 - q^{2m}} - 4^{1-j} \sum_{m=1}^{\infty} \frac{(2m)^{2j-1} q^{4m}}{1 - q^{4m}} - \sum_{m=1}^{\infty} \frac{m^{2j-1} q^{2m}}{1 + q^{2m}} \\ &= 4^{1-j} \sum_{m=1}^{\infty} \frac{m^{2j-1} q^{2m}}{1 - q^{2m}} - \sum_{m=1}^{\infty} \frac{m^{2j-1} q^{4m}}{1 - q^{4m}} - \sum_{m=1}^{\infty} \frac{m^{2j-1} q^{2m}}{1 - q^{4m}} \\ &= 4^{1-j} \sum_{m=1}^{\infty} \frac{m^{2j-1} q^{2m}}{1 - q^{2m}} - \sum_{m=1}^{\infty} \frac{m^{2j-1} q^{2m}}{1 - q^{2m}} \\ &= (4^{1-j} - 1) \sum_{m=1}^{\infty} \frac{m^{2j-1} q^{2m}}{1 - q^{2m}}. \end{aligned} \quad (2.10.19)$$

By (2.10.2)–(2.10.4) and using (2.10.19), the left hand side of (2.10.17) becomes

$$\begin{aligned}
 & \dot{f}_1^{(2j-1)}(0) + \dot{f}_2^{(2j-1)}(0) + \dot{f}_3^{(2j-1)}(0) \\
 = & 2(-1)^{j-1} \left[ \frac{B_{2j}}{4j} (1 - 4^{1-j}) + (4^{1-j} - 1) \sum_{m=1}^{\infty} \frac{m^{2j-1} q^{2m}}{1 - q^{2m}} \right] \\
 = & - (1 - 4^{1-j}) \dot{f}_0^{(2j-1)}(0).
 \end{aligned}$$

This completes the proof of (2.10.17).

The proof of (2.10.18) can be obtained in a similar way. ■

## 2.11 The functions $z$ , $x$ , $K$ , and $E$

In this section, we define functions of  $z$ ,  $x$ ,  $K$ , and  $E$ . We show that  $E$  and  $K$  may be expressed in terms of  $x$ ,  $z$ , and  $dz/dx$ . The results from this section will be used to throughout this thesis.

**Definition 2.11.1** *Let*

$$z = z(q) = \left( \sum_{n=-\infty}^{\infty} q^{n^2} \right)^2, \quad (2.11.1)$$

$$x = x(q) = \left( \frac{\sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}} \right)^4, \quad (2.11.2)$$

$$x' = x'(q) = \left( \frac{\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}} \right)^4. \quad (2.11.3)$$

Then by replacing  $q$  with  $q^2$  in the Jacobi triple product identity (2.3.16) and setting  $x = -q, -q^2, q$ , respectively, we find that

$$\sum_{n=-\infty}^{\infty} q^{n^2} = (-q, -q, q^2; q^2)_{\infty}, \quad (2.11.4)$$

$$\sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} = 2q^{\frac{1}{4}} (-q^2, -q^2, q^2; q^2)_{\infty}, \quad (2.11.5)$$

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = (q, q, q^2; q^2)_{\infty}. \quad (2.11.6)$$

Now using these results into (2.11.1)–(2.11.3) we obtain

$$z = (-q; q^2)_{\infty}^4 (q^2; q^2)_{\infty}^2, \quad (2.11.7)$$

$$x = 16q \frac{(-q^2; q^2)_{\infty}^8}{(-q; q^2)_{\infty}^8}, \quad (2.11.8)$$

$$x' = \frac{(q; q^2)_{\infty}^8}{(-q; q^2)_{\infty}^8}. \quad (2.11.9)$$

Using equations (2.8.13)–(2.8.15), equations (2.11.8) and (2.11.9) may be represented as

$$x = \frac{e_3 - e_2}{e_1 - e_2}, \quad (2.11.10)$$

$$x' = \frac{e_1 - e_3}{e_1 - e_2}. \quad (2.11.11)$$

It is clear from (2.11.10) and (2.11.11) that

$$x + x' = 1, \quad (2.11.12)$$

and hence we obtain Jacobi's formula [100, p. 62]

$$(q; q^2)_{\infty}^8 + 16q (-q^2; q^2)_{\infty}^8 = (-q; q^2)_{\infty}^8. \quad (2.11.13)$$

The functions  $K$  and  $E$  are discussed next. From [26, pp. 7–8] the complete elliptic integrals of the first and second kind are defined, respectively, as

$$\begin{aligned} K &= K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \\ &= \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}, \end{aligned} \quad (2.11.14)$$

and

$$\begin{aligned} E &= E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta} d\theta \\ &= \int_0^1 \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} dt, \end{aligned} \quad (2.11.15)$$

where  $x = k^2$ .

In the integral (2.11.14), we make the following change of variable  $t = \frac{2}{zk} f_2(\alpha + \pi\tau)$ .

Then  $dt = \frac{-2}{zk} f_1(\alpha + \pi\tau) f_3(\alpha + \pi\tau) d\alpha$ . If we set  $\theta = \alpha + \pi\tau$  in (2.5.12) we obtain

$$f_2(\alpha + \pi\tau) = \frac{q^{\frac{1}{2}} e^{\frac{i\alpha}{2}} (q^2 e^{i\alpha}, e^{-i\alpha}, q^2, q^2; q^2)_{\infty}}{i (q e^{i\alpha}, q e^{-i\alpha}, q, q; q^2)_{\infty}} \quad (2.11.16)$$

$$= 2q^{\frac{1}{2}} \frac{(q^2; q^2)_{\infty}^2}{(q; q^2)_{\infty}^2} \sin \frac{\theta}{2} \prod_{n=1}^{\infty} \frac{(1 - 2q^{2n} \cos \theta + q^{4n})}{(1 - 2q^{2n-1} \cos \theta + q^{4n-2})}. \quad (2.11.17)$$

If we use (2.11.7), (2.11.8), and (2.11.17), then  $f_2(\alpha + \pi\tau)$  increases from 0 to  $zk/2$  as  $\alpha$  increases from 0 to  $\pi$ . When  $t = 0$ ,  $\alpha = 0$  and when  $t = 1$ , then  $\alpha = \pi$ . Thus  $K$  can be rewritten as

$$K = \int_0^{\pi} \frac{\left(\frac{-2}{zk}\right) f_1(\alpha + \pi\tau) f_3(\alpha + \pi\tau) d\alpha}{\sqrt{\left[1 - k^2 \left(\frac{2}{zk} f_2(\alpha + \pi\tau)\right)^2\right] \left[1 - \left(\frac{2}{zk} f_2(\alpha + \pi\tau)\right)^2\right]}}. \quad (2.11.18)$$

Using the results of (2.8.8), (2.8.9), (2.11.7), and (2.11.8) we obtain

$$f_2^2(\theta) - f_3^2(\theta) = \frac{z^2 k^2}{4} \quad (2.11.19)$$

and

$$f_2^2(\theta) - f_1^2(\theta) = \frac{z^2}{4}. \quad (2.11.20)$$

By using (2.11.19) and (2.11.20), equation (2.11.18) becomes

$$K = \int_0^\pi \frac{\left(\frac{-2}{zk}\right) f_1(\alpha + \pi\tau) f_3(\alpha + \pi\tau) d\alpha}{\sqrt{\left[\frac{-4}{z^2} f_1^2(\alpha + \pi\tau)\right] \left[\frac{-4}{z^2 k^2} f_3^2(\alpha + \pi\tau)\right]}}. \quad (2.11.21)$$

Equations (2.5.10) and (2.5.14) imply

$$f_1(\alpha + \pi\tau) = \frac{(-qe^{i\alpha}, -qe^{-i\alpha}, q^2, q^2; q^2)_\infty}{i(qe^{i\alpha}, qe^{-i\alpha}, -1, -q^2; q^2)_\infty} \quad (2.11.22)$$

and

$$f_3(\alpha + \pi\tau) = \frac{q^{\frac{1}{2}} e^{\frac{i\alpha}{2}} (-q^2 e^{i\alpha}, -e^{-i\alpha}, q^2, q^2; q^2)_\infty}{i(qe^{i\alpha}, qe^{-i\alpha}, -q, -q; q^2)_\infty}. \quad (2.11.23)$$

Now if we substitute the infinite products (2.11.22) and (2.11.23) in (2.11.21) and select the positive sign under the square root, then equation (2.11.21) becomes

$$\begin{aligned} K &= \int_0^\pi \frac{z}{2} d\alpha \\ &= \frac{\pi}{2} z. \end{aligned} \quad (2.11.24)$$

Similarly, if we use the same change of variable in (2.11.15) we find that

$$\begin{aligned} E &= \int_0^\pi \sqrt{\frac{1 - k^2 \left(\frac{2}{zk} f_2(\theta + \pi\tau)\right)^2}{1 - \left(\frac{2}{zk} f_2(\theta + \pi\tau)\right)^2}} \left(\frac{-2}{zk}\right) f_1(\theta + \pi\tau) f_3(\theta + \pi\tau) d\theta \\ &= \int_0^\pi \sqrt{\frac{\frac{-4}{z^2} f_1^2(\theta + \pi\tau)}{\frac{-4}{z^2 k^2} f_3^2(\theta + \pi\tau)}} \left(\frac{-2}{zk}\right) f_1(\theta + \pi\tau) f_3(\theta + \pi\tau) d\theta \\ &= \frac{-2}{z} \int_0^\pi f_1^2(\theta + \pi\tau) d\theta \\ &= \frac{-2}{z} \int_0^\pi \left(\frac{1}{i} \sum_{n=-\infty}^\infty \frac{e^{i(\theta + \pi\tau)n}}{1 + q^{2n}}\right)^2 d\theta \\ &= \frac{1}{z} \int_{-\pi}^\pi \left(\frac{1}{2} + 2 \sum_{n=1}^\infty \frac{q^n}{1 + q^{2n}} \cos n\theta\right)^2 d\theta \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{z} \int_{-\pi}^{\pi} \left( \frac{1}{4} + 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{(1+q^{2n})^2} \cos^2 n\theta \right) d\theta \quad \text{by orthogonality} \\
&= \frac{1}{z} \left[ \frac{2\pi}{4} + 4\pi \sum_{n=1}^{\infty} \frac{q^{2n}}{(1+q^{2n})^2} \right] \\
&= \frac{2\pi}{z} \left[ \frac{1}{4} - 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^n n q^{2nm} \right] \\
&= \frac{4K}{z^2} \left[ \frac{1}{4} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n n q^{2n}}{1 - q^{2n}} \right]. \tag{2.11.25}
\end{aligned}$$

Therefore

$$E = \frac{2\pi}{z} \left[ \frac{1}{4} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n n q^{2n}}{1 - q^{2n}} \right]. \tag{2.11.26}$$

Equations (2.11.24) and (2.11.26) can be found in C. G. J. Jacobi [67, p. 159, eqn. (4) and p. 169, eqn. (6)]. From [26, p. 9, eqn. (1.3.13)], we have

$$\frac{dK}{dk} = \frac{E - k'^2 K}{k k'^2},$$

where  $k' = \sqrt{1 - k^2}$ . This can be rewritten in terms of  $x, z$ , and  $dz/dx$  as

$$\begin{aligned}
E &= 2x(1-x) \frac{dK}{dx} + (1-x) K \\
&= \pi x^{\frac{1}{2}} (1-x) \frac{d}{dx} \left( z x^{\frac{1}{2}} \right). \tag{2.11.27}
\end{aligned}$$

## 2.12 Reciprocals and quotients of the functions $f_1, f_2$ , and $f_3$

The aim of this section is to write  $f_k(\theta + \omega)$  in terms of  $f_1(\theta), f_2(\theta), f_3(\theta)$ , where  $\omega = \pi, \pi\tau, \pi + \pi\tau$  and  $k = 1, 2, 3$ .



**Lemma 2.12.1**

$$f_1(\theta + \pi) = -\frac{z^2\sqrt{1-x}}{4} \frac{1}{f_1(\theta)}, \quad (2.12.1)$$

$$f_2(\theta + \pi) = \frac{z}{2} \frac{f_3(\theta)}{f_1(\theta)}, \quad (2.12.2)$$

$$f_3(\theta + \pi) = \frac{z\sqrt{1-x}}{2} \frac{f_2(\theta)}{f_1(\theta)}, \quad (2.12.3)$$

$$f_1(\theta + \pi\tau) = -\frac{iz}{2} \frac{f_3(\theta)}{f_2(\theta)}, \quad (2.12.4)$$

$$f_2(\theta + \pi\tau) = \frac{z^2\sqrt{x}}{4} \frac{1}{f_2(\theta)}, \quad (2.12.5)$$

$$f_3(\theta + \pi\tau) = -\frac{iz\sqrt{x}}{2} \frac{f_1(\theta)}{f_2(\theta)}, \quad (2.12.6)$$

$$f_1(\theta + \pi + \pi\tau) = -\frac{iz\sqrt{1-x}}{2} \frac{f_2(\theta)}{f_3(\theta)}, \quad (2.12.7)$$

$$f_2(\theta + \pi + \pi\tau) = \frac{z\sqrt{x}}{2} \frac{f_1(\theta)}{f_3(\theta)}, \quad (2.12.8)$$

$$f_3(\theta + \pi + \pi\tau) = \frac{iz^2\sqrt{x(1-x)}}{4} \frac{1}{f_3(\theta)}. \quad (2.12.9)$$

**Proof** If we set  $\theta = \theta + \pi$  in (2.5.10) then the left hand side of (2.12.1) becomes

$$f_1(\theta + \pi) = \frac{(e^{i\theta}, q^2 e^{-i\theta}, q^2, q^2; q^2)_\infty}{i(-e^{i\theta}, -q^2 e^{-i\theta}, -1, -q^2; q^2)_\infty}. \quad (2.12.10)$$

Using (2.5.10), (2.11.7), and (2.11.9), the right hand side of (2.12.1) becomes

$$-\frac{z^2\sqrt{1-x}}{4} \frac{1}{f_1(\theta)} = \frac{(e^{i\theta}, q^2 e^{-i\theta}, q^2, q^2; q^2)_\infty}{i(-e^{i\theta}, -q^2 e^{-i\theta}, -1, -q^2; q^2)_\infty}. \quad (2.12.11)$$

Comparing the right hand sides of (2.12.10) and (2.12.11) completes the proof of (2.12.1).

Equations (2.12.2)–(2.12.9) can be proved similarly. ■

## 2.13 Jacobian elliptic functions

In this section, we demonstrate the connection between the twelve functions and the Jacobian elliptic functions, shown in tabular form at the end of this section.

Upon comparing the Fourier series (2.5.5), (2.5.7)–(2.5.9) and replacing  $\theta$  by  $\theta + \pi$ ,  $\theta + \pi\tau$ ,  $\theta + \pi + \pi\tau$  in (2.5.5), (2.5.7)–(2.5.9), respectively, with those in [101, pp. 511–512] gives

$$\left(\frac{2}{z}\right) f_1\left(\frac{2u}{z}\right) = \operatorname{cs}(u, k), \quad (2.13.1)$$

$$\left(\frac{2}{z}\right) f_2\left(\frac{2u}{z}\right) = \operatorname{ns}(u, k), \quad (2.13.2)$$

$$\left(\frac{2}{z}\right) f_3\left(\frac{2u}{z}\right) = \operatorname{ds}(u, k), \quad (2.13.3)$$

$$\left(\frac{-2}{z\sqrt{x'}}\right) f_1\left(\frac{2u}{z} + \pi\right) = \left(\frac{z}{2}\right) \frac{1}{f_1\left(\frac{2u}{z}\right)} = \operatorname{sc}(u, k), \quad (2.13.4)$$

$$\left(\frac{2}{z}\right) f_2\left(\frac{2u}{z} + \pi\right) = \frac{f_3\left(\frac{2u}{z}\right)}{f_1\left(\frac{2u}{z}\right)} = \operatorname{dc}(u, k), \quad (2.13.5)$$

$$\left(\frac{2}{z\sqrt{x'}}\right) f_3\left(\frac{2u}{z} + \pi\right) = \frac{f_2\left(\frac{2u}{z}\right)}{f_1\left(\frac{2u}{z}\right)} = \operatorname{nc}(u, k), \quad (2.13.6)$$

$$\left(\frac{2i}{z}\right) f_1\left(\frac{2u}{z} + \pi\tau\right) = \frac{f_3\left(\frac{2u}{z}\right)}{f_2\left(\frac{2u}{z}\right)} = \operatorname{dn}(u, k), \quad (2.13.7)$$

$$\left(\frac{2}{z\sqrt{x}}\right) f_2\left(\frac{2u}{z} + \pi\tau\right) = \left(\frac{z}{2}\right) \frac{1}{f_2\left(\frac{2u}{z}\right)} = \operatorname{sn}(u, k), \quad (2.13.8)$$

$$\left(\frac{2i}{z\sqrt{x}}\right) f_3\left(\frac{2u}{z} + \pi\tau\right) = \frac{f_1\left(\frac{2u}{z}\right)}{f_2\left(\frac{2u}{z}\right)} = \operatorname{cn}(u, k), \quad (2.13.9)$$

$$\left(\frac{2i}{z\sqrt{x'}}\right) f_1\left(\frac{2u}{z} + \pi + \pi\tau\right) = \frac{f_2\left(\frac{2u}{z}\right)}{f_3\left(\frac{2u}{z}\right)} = \operatorname{nd}(u, k), \quad (2.13.10)$$

$$\left(\frac{2}{z\sqrt{x}}\right) f_2\left(\frac{2u}{z} + \pi + \pi\tau\right) = \frac{f_1\left(\frac{2u}{z}\right)}{f_3\left(\frac{2u}{z}\right)} = \operatorname{cd}(u, k), \quad (2.13.11)$$

$$\left(\frac{2}{iz\sqrt{xx'}}\right) f_3\left(\frac{2u}{z} + \pi + \pi\tau\right) = \left(\frac{z}{2}\right) \frac{1}{f_3\left(\frac{2u}{z}\right)} = \operatorname{sd}(u, k). \quad (2.13.12)$$

Equations (2.13.1)–(2.13.12) are the twelve Jacobian elliptic functions. The Fourier series expansions of (2.13.1)–(2.13.12) originally appeared in Jacobi [67, Sections 39, 41, and 42]. The Fourier series expansions of equations (2.13.4)–(2.13.12) can also be found in S. C. Milne [88, pp. 11–12]. More Jacobian elliptic functions will be presented in Appendix D.

A table is presented to show the connection between the twelve functions and the Jacobian elliptic functions as follows:

Table 2.1. The connection between the twelve functions and the Jacobian elliptic functions.

$A$	$B$	$C$	$C \times F(A, B e^{i\theta})$		
-1	1	$\frac{1}{i}$	$f_1(\theta)$	$f_1(\theta) = \frac{K}{\pi} \operatorname{cs}\left(\frac{K\theta}{\pi}, k\right)$	$\operatorname{cs}(u, k) = \left(\frac{2}{\pi}\right) f_1\left(\frac{2u}{\pi}\right),$
$q$	1	$\frac{e^{i\theta/2}}{i}$	$f_2(\theta)$	$f_2(\theta) = \frac{K}{\pi} \operatorname{ns}\left(\frac{K\theta}{\pi}, k\right)$	$\operatorname{ns}(u, k) = \left(\frac{2}{\pi}\right) f_2\left(\frac{2u}{\pi}\right),$
$-q$	1	$\frac{e^{i\theta/2}}{i}$	$f_3(\theta)$	$f_3(\theta) = \frac{K}{\pi} \operatorname{ds}\left(\frac{K\theta}{\pi}, k\right)$	$\operatorname{ds}(u, k) = \left(\frac{2}{\pi}\right) f_3\left(\frac{2u}{\pi}\right),$
-1	-1	$\frac{1}{i}$	$f_1(\theta + \pi)$	$f_1(\theta + \pi) = \frac{-k'K^2}{\pi^2} \frac{1}{f_1(\theta)} = \frac{-k'K}{\pi} \operatorname{sc}\left(\frac{K\theta}{\pi}, k\right)$	$\operatorname{sc}(u, k) = \left(\frac{-2}{z\sqrt{x'}}\right) f_1\left(\frac{2u}{\pi} + \pi\right) = \left(\frac{z}{2}\right) \frac{1}{f_1\left(\frac{2u}{\pi}\right)},$
$q$	-1	$e^{i\theta/2}$	$f_2(\theta + \pi)$	$f_2(\theta + \pi) = \frac{K f_3(\theta)}{\pi f_1(\theta)} = \frac{K}{\pi} \operatorname{dc}\left(\frac{K\theta}{\pi}, k\right)$	$\operatorname{dc}(u, k) = \left(\frac{2}{\pi}\right) f_2\left(\frac{2u}{\pi} + \pi\right) = \frac{f_3\left(\frac{2u}{\pi}\right)}{f_1\left(\frac{2u}{\pi}\right)},$
$-q$	-1	$e^{i\theta/2}$	$f_3(\theta + \pi)$	$f_3(\theta + \pi) = \frac{k'K f_2(\theta)}{\pi f_1(\theta)} = \frac{k'K}{\pi} \operatorname{nc}\left(\frac{K\theta}{\pi}, k\right)$	$\operatorname{nc}(u, k) = \left(\frac{2}{z\sqrt{x'}}\right) f_3\left(\frac{2u}{\pi} + \pi\right) = \frac{f_2\left(\frac{2u}{\pi}\right)}{f_1\left(\frac{2u}{\pi}\right)},$
-1	$q$	$\frac{1}{i}$	$f_1(\theta + \pi\tau)$	$f_1(\theta + \pi\tau) = \frac{K f_3(\theta)}{i\pi f_2(\theta)} = \frac{K}{i\pi} \operatorname{dn}\left(\frac{K\theta}{\pi}, k\right)$	$\operatorname{dn}(u, k) = \left(\frac{2i}{\pi}\right) f_1\left(\frac{2u}{\pi} + \pi\tau\right) = \frac{f_3\left(\frac{2u}{\pi}\right)}{f_2\left(\frac{2u}{\pi}\right)},$
$q$	$q$	$\frac{q^{1/2} e^{i\theta/2}}{i}$	$f_2(\theta + \pi\tau)$	$f_2(\theta + \pi\tau) = \frac{kK^2}{\pi^2} \frac{1}{f_2(\theta)} = \frac{kK}{\pi} \operatorname{sn}\left(\frac{K\theta}{\pi}, k\right)$	$\operatorname{sn}(u, k) = \left(\frac{2}{z\sqrt{x}}\right) f_2\left(\frac{2u}{\pi} + \pi\tau\right) = \left(\frac{z}{2}\right) \frac{1}{f_2\left(\frac{2u}{\pi}\right)},$
$-q$	$q$	$\frac{q^{1/2} e^{i\theta/2}}{i}$	$f_3(\theta + \pi\tau)$	$f_3(\theta + \pi\tau) = \frac{-ikK f_1(\theta)}{\pi f_2(\theta)} = \frac{-ikK}{\pi} \operatorname{cn}\left(\frac{K\theta}{\pi}, k\right)$	$\operatorname{cn}(u, k) = \left(\frac{2i}{z\sqrt{x}}\right) f_3\left(\frac{2u}{\pi} + \pi\tau\right) = \frac{f_1\left(\frac{2u}{\pi}\right)}{f_2\left(\frac{2u}{\pi}\right)},$
-1	$-q$	$\frac{1}{i}$	$f_1(\theta + \pi + \pi\tau)$	$f_1(\theta + \pi + \pi\tau) = \frac{k'K f_2(\theta)}{i\pi f_3(\theta)} = \frac{k'K}{i\pi} \operatorname{nd}\left(\frac{K\theta}{\pi}, k\right)$	$\operatorname{nd}(u, k) = \left(\frac{2i}{z\sqrt{x'}}\right) f_1\left(\frac{2u}{\pi} + \pi + \pi\tau\right) = \frac{f_2\left(\frac{2u}{\pi}\right)}{f_3\left(\frac{2u}{\pi}\right)},$
$q$	$-q$	$q^{1/2} e^{i\theta/2}$	$f_2(\theta + \pi + \pi\tau)$	$f_2(\theta + \pi + \pi\tau) = \frac{kK f_1(\theta)}{\pi f_3(\theta)} = \frac{kK}{\pi} \operatorname{cd}\left(\frac{K\theta}{\pi}, k\right)$	$\operatorname{cd}(u, k) = \left(\frac{2}{z\sqrt{x}}\right) f_2\left(\frac{2u}{\pi} + \pi + \pi\tau\right) = \frac{f_1\left(\frac{2u}{\pi}\right)}{f_3\left(\frac{2u}{\pi}\right)},$
$-q$	$-q$	$q^{1/2} e^{i\theta/2}$	$f_3(\theta + \pi + \pi\tau)$	$f_3(\theta + \pi + \pi\tau) = \frac{ikK^2}{\pi^2} \frac{1}{f_3(\theta)} = \frac{ikK}{\pi} \operatorname{sd}\left(\frac{K\theta}{\pi}, k\right)$	$\operatorname{sd}(u, k) = \left(\frac{2}{iz\sqrt{xx'}}\right) f_3\left(\frac{2u}{\pi} + \pi + \pi\tau\right) = \left(\frac{z}{2}\right) \frac{1}{f_3\left(\frac{2u}{\pi}\right)},$

## 2.14 Special values

In this section we present four special values of  $f_0(\theta)$ ,  $f_1(\theta)$ ,  $f_2(\theta)$ , and  $f_3(\theta)$ . By substituting  $\theta = 0, \pi, \pi\tau, \pi + \pi\tau$  into (2.5.5), (2.5.10), (2.5.12), and (2.5.14), respectively, we obtain

	0	$\pi$	$\pi\tau$	$\pi + \pi\tau$
$f_0(\theta)$	$\infty$	0	$\frac{1}{2i}$	$\frac{1}{2i}$
$f_1(\theta)$	$\infty$	0	$\frac{z}{2i}$	$\frac{z\sqrt{1-x}}{2i}$
$f_2(\theta)$	$\infty$	$\frac{z}{2}$	0	$\frac{z\sqrt{x}}{2}$
$f_3(\theta)$	$\infty$	$\frac{z\sqrt{1-x}}{2}$	$\frac{z\sqrt{x}}{2i}$	0

(2.14.1)

The results of (2.14.1) will be used in Chapters 3–6.

## 2.15 Addition formulae

In this section we present addition formulae for the functions  $f_0$ ,  $f_1$ ,  $f_2$ , and  $f_3$ .

### Lemma 2.15.1

$$f_0(\alpha + \beta) = \frac{q \frac{d}{dq} [f_0(\alpha) - f_0(\beta)] + f'_0(\alpha) f_0(\beta) - f_0(\alpha) f'_0(\beta)}{f'_0(\alpha) - f'_0(\beta)}. \quad (2.15.1)$$

**Proof** Let  $a = e^{i\alpha}$ ,  $b = e^{i\beta}$ , and  $t = e^{iv}$  in the fundamental multiplicative identity (2.6.1)

so that

$$F(e^{i\alpha}, e^{iv})F(e^{i\beta}, e^{iv}) = \frac{1}{i} \frac{\partial}{\partial v} F(e^{i(\alpha+\beta)}, e^{iv}) + F(e^{i(\alpha+\beta)}, e^{iv})(\rho_1(e^{i\alpha}) + \rho_1(e^{i\beta})). \quad (2.15.2)$$

We apply  $\frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta}$  to obtain

$$\begin{aligned} & \frac{\partial}{\partial \alpha} F(e^{i\alpha}, e^v) F(e^{i\beta}, e^v) - F(e^{i\alpha}, e^v) \frac{\partial}{\partial \beta} F(e^{i\beta}, e^v) \\ &= \frac{1}{i} F(e^{i(\alpha+\beta)}, e^v) \left[ i \frac{\partial}{\partial \alpha} \rho_1(e^{i\alpha}) - i \frac{\partial}{\partial \beta} \rho_1(e^{i\beta}) \right]. \end{aligned} \quad (2.15.3)$$

Note that

$$i \frac{d}{d\alpha} \rho_1(e^{i\alpha}) = \wp(\alpha) + \frac{P}{12}.$$

Using (2.4.14) we find

$$\begin{aligned} & \frac{\partial}{\partial \alpha} F(e^{i\alpha}, e^{iv}) \\ &= 2i \left[ \frac{-1}{8} \csc^2 \frac{\alpha}{2} + \sum_{m=1}^{\infty} \frac{mq^{2m}}{1-q^{2m}} \cos m\alpha \right] - \left[ \sum_{m=1}^{\infty} \frac{2mq^{2m}}{(1-q^{2m})^2} \sin m\alpha \right] v \\ &+ \left\{ \sum_{j=2}^{\infty} \left[ \frac{B_{j+1}}{j+1} + \sum_{m=1}^{\infty} \left( \frac{me^{i\alpha} q^{2m}}{1-e^{i\alpha} q^{2m}} - \frac{(-1)^j me^{-i\alpha} q^{2m}}{1-e^{-i\alpha} q^{2m}} \right) \right] \frac{v^j}{j!} \right\}'. \end{aligned} \quad (2.15.4)$$

Note that

$$q \frac{d}{dq} f_0(\theta) = \sum_{m=1}^{\infty} \frac{2mq^{2m}}{(1-q^{2m})^2} \sin m\theta.$$

We equate  $[v^0]$  from equation (2.15.3) and use (2.4.14) and (2.15.4) to obtain

$$q \frac{d}{dq} [f_0(\alpha) - f_0(\beta)] + f'_0(\alpha) f_0(\beta) - f_0(\alpha) f'_0(\beta) = f_0(\alpha + \beta) [f'_0(\beta) - f'_0(\alpha)]. \quad (2.15.5)$$

Rearrangement of equation (2.15.5) completes the proof. ■

An equivalent formula to (2.15.1) can be found in Lawden [72, p. 161, eqn. (6.8.4)].

In [38, p. 74], the addition formulae of  $f_1$ ,  $f_2$ , and  $f_3$  are given by

$$\begin{aligned} f_1(\alpha + \beta) &= \frac{f_1(\alpha) f_2(\beta) f_3(\beta) - f_1(\beta) f_2(\alpha) f_3(\alpha)}{f_1^2(\beta) - f_1^2(\alpha)}, \\ f_2(\alpha + \beta) &= \frac{f_2(\alpha) f_3(\beta) f_1(\beta) - f_2(\beta) f_3(\alpha) f_1(\alpha)}{f_2^2(\beta) - f_2^2(\alpha)}, \\ f_3(\alpha + \beta) &= \frac{f_3(\alpha) f_1(\beta) f_2(\beta) - f_3(\beta) f_1(\alpha) f_2(\alpha)}{f_3^2(\beta) - f_3^2(\alpha)}. \end{aligned} \tag{2.15.6}$$

## 2.16 Summary

We have introduced and given a proof of Ramanujan's  ${}_1\psi_1$  summation formula. Most of the results presented in subsequent chapters will be directly or indirectly related to this formula. By using a special case of Ramanujan's  ${}_1\psi_1$  summation formula, we obtained an infinite product expansion of the Jordan-Kronecker function. We defined four functions,  $f_0$ ,  $f_1$ ,  $f_2$ , and  $f_3$  that arise from Ramanujan's  ${}_1\psi_1$  summation formula and constructed twelve other functions from these, giving a total of sixteen functions. We also obtained Fourier series expansions of functions  $f_0$ ,  $f_1$ ,  $f_2$ ,  $f_3$  and infinite product expansions for  $f_1$ ,  $f_2$ , and  $f_3$ .

We also presented sixteen Lambert series, and showed how they arise as derivatives of  $f_0$ ,  $f_1$ ,  $f_2$ , and  $f_3$ . Several of these Lambert series can be found in [2], [19], [66], [67], [72]–[75], [88], [92], [97]. Glaisher [57] and Zucker [105], [106] presented the complete set of sixteen Lambert series. The significant point is the sixteen Lambert series all originate from Ramanujan's  ${}_1\psi_1$  summation formula.

The fundamental multiplicative identity and the Weierstrass  $\wp$  function have been introduced. These were used to calculate the derivatives of the four functions, Fourier expansions of their squares, and their addition formulae.

The functions  $z$ ,  $x$ ,  $K$ , and  $E$  have been defined, and we obtained the reciprocals and quotients of the functions  $f_1$ ,  $f_2$ , and  $f_3$ . A connection between the twelve functions and Jacobian elliptic functions is also known.

The sixteen Lambert series, functions  $z$ ,  $x$ ,  $K$ , and  $E$  will be used frequently throughout this thesis.



# Chapter 3

## Transformations

### 3.1 Introduction

The aim of this chapter is to apply selected transformations to functions  $f_0, f_1, f_2$ , and  $f_3$ , as well as  $z, x, 1 - x$ , and  $E$ .

In Section 2 to 4 three transformations are applied including the Gauss transformation ( $\tau$  to  $\tau + 1$ ), the Landen transformation ( $\tau$  to  $2\tau$ ), and the modular transformation ( $\tau$  to  $-1/\tau$ ) to the functions  $f_0, f_1, f_2$ , and  $f_3$ , respectively. From each of these transformations, we establish connections between functions  $f_0, f_1, f_2$ , and  $f_3$ .

In Section 5 we apply seven transformations ( $q$  to  $-q$ ,  $q$  to  $q^{\frac{1}{2}}$ ,  $q$  to  $q^2$ ,  $q$  to  $-q^{\frac{1}{2}}$ ,  $q$  to  $-q^2$ ,  $q$  to  $iq^{\frac{1}{2}}$ , and  $q$  to  $-iq^{\frac{1}{2}}$ ) satisfied by  $z, x, 1 - x$ , and  $E$ .

### 3.2 The Gauss transformation $\tau \rightarrow \tau + 1$

If we change  $\tau$  to  $\tau + 1$ , then the Fourier series (2.5.5), (2.5.7)–(2.5.9) imply

$$f_0(\theta|\tau + 1) = f_0(\theta|\tau), \quad (3.2.1)$$

$$f_1(\theta|\tau + 1) = f_1(\theta|\tau), \quad (3.2.2)$$

$$f_2(\theta|\tau + 1) = f_3(\theta|\tau), \quad (3.2.3)$$

$$f_3(\theta|\tau + 1) = f_2(\theta|\tau). \quad (3.2.4)$$

### 3.3 The Landen transformation $\tau \rightarrow 2\tau$

#### Theorem 3.3.1

$$f_0(2\theta|2\tau) = \frac{1}{2} [f_0(\theta|\tau) + f_0(\theta + \pi|\tau)], \quad (3.3.1)$$

$$f_1(2\theta|2\tau) = \frac{1}{2} [f_1(\theta|\tau) + f_1(\theta + \pi|\tau)], \quad (3.3.2)$$

$$f_2(2\theta|2\tau) = \frac{f_2(\theta|\tau) f_3(\theta|\tau)}{2f_1(\theta|\tau)}, \quad (3.3.3)$$

$$f_3(2\theta|2\tau) = \frac{1}{2} [f_1(\theta|\tau) - f_1(\theta + \pi|\tau)]. \quad (3.3.4)$$

**Proof** If we replace  $\theta$  with  $\theta + \pi$  in (2.5.5) and (2.5.7) we obtain

$$f_0(\theta + \pi) = -\frac{1}{2} \tan \frac{\theta}{2} + 2 \sum_{m=1}^{\infty} \frac{(-1)^m q^{2m}}{1 - q^{2m}} \sin m\theta \quad (3.3.5)$$

and

$$f_1(\theta + \pi) = -\frac{1}{2} \tan \frac{\theta}{2} - 2 \sum_{m=1}^{\infty} \frac{(-1)^m q^{2m}}{1 + q^{2m}} \sin m\theta. \quad (3.3.6)$$

Equation (3.3.1) follows immediately from the Fourier series of (2.5.5) and (3.3.5). The proofs of (3.3.2) and (3.3.4) are similarly straightforward by employing the Fourier series of (2.5.7) and (3.3.6).

Using (2.5.10), (2.5.12), and (2.5.14), the right hand side of (3.3.3) may be manipulated as follows

$$\begin{aligned} \frac{f_2(\theta|\tau) f_3(\theta|\tau)}{2f_1(\theta|\tau)} &= \frac{e^{i\theta} (qe^{i\theta}, qe^{-i\theta}, -qe^{i\theta}, -qe^{-i\theta}, q^2, q^2 - 1, -q^2; q^2)_{\infty}}{2i (e^{i\theta}, q^2e^{-i\theta}, -e^{i\theta}, -q^2e^{-i\theta}, q, q - q, -q; q^2)_{\infty}} \\ &= \frac{e^{i\theta} (q^2e^{2i\theta}, q^2e^{-2i\theta}, q^4, q^4; q^4)_{\infty}}{i (e^{2i\theta}, q^4e^{-2i\theta}, q^2, q^2; q^4)_{\infty}} \\ &= f_2(2\theta|2\tau). \end{aligned}$$

By rearranging this we complete the proof of (3.3.3). ■

Adding (3.3.2) and (3.3.4), rearranging the result gives

$$f_1(\theta|\tau) = f_1(2\theta|2\tau) + f_3(2\theta|2\tau). \quad (3.3.7)$$

Similarly, by subtracting (3.3.4) from (3.3.2) and rearranging the result we find

$$f_1(\theta + \pi|\tau) = f_1(2\theta|2\tau) - f_3(2\theta|2\tau). \quad (3.3.8)$$

E. T. Whittaker and G. N. Watson [101, p. 507] wrote (3.3.3) as

$$k_1^{\frac{1}{2}} \operatorname{sn} \left( \frac{4\Lambda z}{\pi}, k_1 \right) = k \operatorname{sn} \left( \frac{2Kz}{\pi}, k \right) \operatorname{cd} \left( \frac{2Kz}{\pi}, k \right), \quad (3.3.9)$$

where

$$\begin{aligned} k_1^{\frac{1}{2}} &= \frac{k}{1 + k'}, \\ \Lambda &= \frac{1}{2} (1 + k') K. \end{aligned}$$

Using the results of (2.13.1)–(2.13.3), (3.3.3) can be represented as

$$\frac{2}{zx} f_2 \left( \frac{u}{z} + \pi\tau | 2\tau \right) = \frac{\operatorname{sn}(u, k) \operatorname{cn}(u, k)}{\operatorname{dn}(u, k)}, \quad (3.3.10)$$

which is a much simpler form. Equation (3.3.10) is equivalent to (3.3.9).

### 3.4 The modular transformation $\tau \rightarrow \frac{-1}{\tau}$

In this section the modular transformation is established for the functions of  $f_0, f_1, f_2$ , and  $f_3$ . Then we expand the right hand sides of the functions  $f_0, f_1, f_2$ , and  $f_3$  in powers of  $\theta$ , and equate the coefficients of  $\theta^{2j-1}/(2j-1)!$  or  $\theta^{2j}/(2j)!$  to obtain the modular transformation on the sixteen Lambert series.

**Lemma 3.4.1**

$$F(e^{i\alpha}, e^{i\theta}|\tau) = \frac{1}{\tau} \exp\left(\frac{-i\alpha\theta}{2\pi\tau}\right) F\left(e^{\frac{i\alpha}{\tau}}, e^{\frac{i\theta}{\tau}} \middle| \frac{-1}{\tau}\right). \quad (3.4.1)$$

**Proof** Recall  $q = e^{i\pi\tau}$ . Let  $p = e^{\frac{-i\pi}{\tau}}$ . From [8, p. 538], we have

$$\eta\left(\frac{-1}{\tau}\right) = \sqrt{-i\tau} \eta(\tau), \quad (3.4.2)$$

where

$$\eta(\tau) = q^{\frac{1}{12}} (q^2; q^2)_{\infty}. \quad (3.4.3)$$

From [101, pp. 470 and 475], we have

$$\theta_1(z, q|\tau) = -i(-i\tau)^{-\frac{1}{2}} e^{\frac{z^2}{i\pi\tau}} \theta_1\left(\frac{-z}{\tau} \middle| \frac{-1}{\tau}\right), \quad (3.4.4)$$

where

$$\theta_1(z, q) = -iq^{\frac{1}{4}} e^{iz} (q^2; q^2)_{\infty} (q^2 e^{2iz}; q^2)_{\infty} (e^{-2iz}; q^2)_{\infty}. \quad (3.4.5)$$

Rearranging equations (3.4.2) and (3.4.3) we obtain

$$(q^2; q^2)_{\infty} = \frac{p^{\frac{1}{12}}}{q^{\frac{1}{12}} \sqrt{-i\tau}} (p^2; p^2)_{\infty}. \quad (3.4.6)$$

Equations (3.4.4) and (3.4.5) implies

$$\begin{aligned} & q^{\frac{1}{4}} (e^{iz}, q^2 e^{-iz}, q^2; q^2)_{\infty} \\ &= (-i\tau)^{-\frac{1}{2}} \exp\left(\frac{z^2}{4\pi i\tau} + \frac{iz}{2\tau} + \frac{iz}{2}\right) \frac{(p^2; p^2)_{\infty}}{(q^2; q^2)_{\infty}} \left(e^{\frac{-iz}{\tau}}, p^2 e^{\frac{iz}{\tau}}, p^2; p^2\right)_{\infty}. \end{aligned} \quad (3.4.7)$$

Using the results of (3.4.6) and (3.4.7) on the left hand side of (3.4.1), we obtain

$$\begin{aligned} & F(e^{i\alpha}, e^{i\theta}|\tau) \\ &= \frac{q^{\frac{1}{4}} (e^{i(\alpha+\theta)}, q^2 e^{-i(\alpha+\theta)}, q^2; q^2)_{\infty} q^{\frac{1}{4}} (q^2; q^2)_{\infty}^3}{q^{\frac{1}{4}} (e^{i\alpha}, q^2 e^{-i\alpha}, q^2; q^2)_{\infty} q^{\frac{1}{4}} (e^{i\theta}, q^2 e^{-i\theta}, q^2; q^2)_{\infty}} \end{aligned}$$

$$\begin{aligned}
&= \frac{\exp\left(\frac{\alpha\theta}{2i\pi\tau}\right) p^{\frac{1}{4}} \left(e^{\frac{-i(\alpha+\theta)}{\tau}}, p^2 e^{\frac{i(\alpha+\theta)}{\tau}}, p^2; p^2\right)_{\infty} \left(\frac{1}{\sqrt{-i\tau}}\right)^3 p^{\frac{1}{4}} (p^2; p^2)_{\infty}^3}{p^{\frac{1}{4}} \left(e^{\frac{-i\alpha}{\tau}}, p^2 e^{\frac{i\alpha}{\tau}}, p^2; p^2\right)_{\infty} p^{\frac{1}{4}} \left(e^{\frac{-i\theta}{\tau}}, p^2 e^{\frac{i\theta}{\tau}}, p^2; p^2\right)_{\infty}} \\
&= \frac{-1}{\tau} \exp\left(\frac{\alpha\theta}{2i\pi\tau}\right) F\left(e^{\frac{-i\alpha}{\tau}}, e^{\frac{-i\theta}{\tau}} \middle| \frac{-1}{\tau}\right). \tag{3.4.8}
\end{aligned}$$

Substitute (2.4.4) into (3.4.8) to complete the proof. ■

### Theorem 3.4.2

$$f_0(\theta|\tau) = \frac{1}{\tau} f_0\left(\frac{\theta}{\tau} \middle| \frac{-1}{\tau}\right) + \frac{\theta}{2\pi i \tau}, \tag{3.4.9}$$

$$f_1(\theta|\tau) = \frac{1}{\tau} f_2\left(\frac{\theta}{\tau} \middle| \frac{-1}{\tau}\right), \tag{3.4.10}$$

$$f_2(\theta|\tau) = \frac{1}{\tau} f_1\left(\frac{\theta}{\tau} \middle| \frac{-1}{\tau}\right), \tag{3.4.11}$$

$$f_3(\theta|\tau) = \frac{1}{\tau} f_3\left(\frac{\theta}{\tau} \middle| \frac{-1}{\tau}\right). \tag{3.4.12}$$

**Proof** By employing (2.4.14), we can expand the right hand side of (3.4.1) in powers of  $\alpha$  to obtain

$$\begin{aligned}
F(e^{i\alpha}, e^{i\theta}|\tau) &= \frac{1}{\tau} \left(1 - \frac{i\alpha\theta}{2\pi\tau} + \frac{1}{2!} \left(\frac{i\alpha\theta}{2\pi\tau}\right)^2 - \dots\right) \\
&\times \left\{ -\frac{\tau}{i\alpha} + 2i \left( \frac{1}{4} \cot \frac{\theta}{2\tau} + \sum_{m=1}^{\infty} \frac{p^{2m}}{1-p^{2m}} \sin \frac{m\theta}{\tau} \right) \right. \\
&+ \left[ -\frac{1}{12} + 2 \sum_{m=1}^{\infty} \frac{mp^{2m}}{1-p^{2m}} \cos \frac{m\theta}{\tau} \right] \frac{i\alpha}{\tau} \\
&\left. + \sum_{j=2}^{\infty} \left[ -\frac{B_{j+1}}{j+1} + \sum_{m=1}^{\infty} \left( \frac{me^{i\theta/\tau} p^{2m}}{1-e^{i\theta/\tau} p^{2m}} - \frac{me^{-i\theta/\tau} p^{2m}}{1-e^{-i\theta/\tau} p^{2m}} \right) \right] \frac{i^j \alpha^j}{j! \tau^j} \right\}. \tag{3.4.13}
\end{aligned}$$

Equate the coefficient of  $\alpha^0$  on both sides of (3.4.13) to give

$$if_0(\theta|\tau) = \frac{i}{\tau} \left( \frac{1}{2} \cot \frac{\theta}{2\tau} + 2 \sum_{m=1}^{\infty} \frac{p^{2m}}{1-p^{2m}} \sin \frac{m\theta}{\tau} \right) + \frac{\theta}{2\pi\tau}.$$

Therefore, using the definition (2.5.1)

$$f_0(\theta|\tau) = \frac{1}{\tau} f_0 \left( \frac{\theta}{\tau} \middle| \frac{-1}{\tau} \right) + \frac{\theta}{2\pi i\tau}.$$

This completes the proof of (3.4.9).

Using (2.5.10) and (3.4.1), equation (2.5.2) may be presented as

$$\begin{aligned} f_1(\theta|\tau) &= \frac{1}{i} F(e^{i\pi}, e^{i\theta}|\tau) \\ &= \frac{1}{i\tau} \exp \left( \frac{-i\theta}{2\tau} \right) F \left( e^{\frac{\pi}{\tau}}, e^{\frac{\theta}{\tau}} \middle| \frac{-1}{\tau} \right). \end{aligned} \quad (3.4.14)$$

Using (2.4.5) and (2.5.3), this can be rewritten as

$$\begin{aligned} f_1(\theta|\tau) &= \frac{1}{i\tau} \exp \left( \frac{i\theta}{2\tau} \right) F \left( e^{\frac{-\pi}{\tau}}, e^{\frac{\theta}{\tau}} \middle| \frac{-1}{\tau} \right) \\ &= \frac{1}{\tau} f_2 \left( \frac{\theta}{\tau} \middle| \frac{-1}{\tau} \right). \end{aligned} \quad (3.4.15)$$

Hence proving (3.4.10).

If we replace  $\tau$  with  $\frac{-1}{\tau}$  and  $\theta$  with  $\frac{-\theta}{\tau}$  in (3.4.15), then rearrange we obtain

$$f_2(\theta|\tau) = \frac{-1}{\tau} f_1 \left( \frac{-\theta}{\tau} \middle| \frac{-1}{\tau} \right). \quad (3.4.16)$$

Then using (2.5.19) in (3.4.16) proves (3.4.11).

Similarly (3.4.12) can be obtained by using (2.5.4) and (2.5.14). ■

Venkatachaliengar [100, pp. 32–35] gave an elementary proof of (3.4.9). Cooper [38, pp. 7–8] used Liouville's theorem to prove (3.4.9).

**Corollary 3.4.3**

$$\dot{f}_0^{(2j+1)} \left( 0 \middle| \frac{-1}{\tau} \right) = \tau^{2j+2} \dot{f}_0^{(2j+1)} (0|\tau), \quad (3.4.17)$$

$$f_0^{(2j+1)} \left( \pi \middle| \frac{-1}{\tau} \right) = \tau^{2j+2} f_0^{(2j+1)} (\pi\tau|\tau), \quad (3.4.18)$$

$$f_0^{(2j+1)} \left( \pi\tau \middle| \frac{-1}{\tau} \right) = \tau^{2j+2} f_0^{(2j+1)} (\pi|\tau), \quad (3.4.19)$$

$$f_0^{(2j+1)} \left( \pi + \pi\tau \middle| \frac{-1}{\tau} \right) = \tau^{2j+2} f_0^{(2j+1)} (\pi + \pi\tau|\tau); \quad (3.4.20)$$

$$\dot{f}_1^{(2j+1)} \left( 0 \middle| \frac{-1}{\tau} \right) = \tau^{2j+2} \dot{f}_2^{(2j+1)} (0|\tau), \quad (3.4.21)$$

$$f_1^{(2j+1)} \left( \pi \middle| \frac{-1}{\tau} \right) = \tau^{2j+2} f_2^{(2j+1)} (\pi\tau|\tau), \quad (3.4.22)$$

$$f_1^{(2j)} \left( \pi\tau \middle| \frac{-1}{\tau} \right) = -\tau^{2j+1} f_2^{(2j)} (\pi|\tau), \quad (3.4.23)$$

$$f_1^{(2j)} \left( \pi + \pi\tau \middle| \frac{-1}{\tau} \right) = -\tau^{2j+1} f_2^{(2j)} (\pi + \pi\tau|\tau); \quad (3.4.24)$$

$$\dot{f}_2^{(2j+1)} \left( 0 \middle| \frac{-1}{\tau} \right) = \tau^{2j+2} \dot{f}_1^{(2j+1)} (0|\tau), \quad (3.4.25)$$

$$f_2^{(2j)} \left( \pi \middle| \frac{-1}{\tau} \right) = \tau^{2j+1} f_1^{(2j)} (\pi\tau|\tau), \quad (3.4.26)$$

$$f_2^{(2j+1)} \left( \pi\tau \middle| \frac{-1}{\tau} \right) = -\tau^{2j+2} f_1^{(2j+1)} (\pi|\tau), \quad (3.4.27)$$

$$f_2^{(2j)} \left( \pi + \pi\tau \middle| \frac{-1}{\tau} \right) = -\tau^{2j+1} f_1^{(2j)} (\pi + \pi\tau|\tau); \quad (3.4.28)$$

$$\dot{f}_3^{(2j+1)} \left( 0 \middle| \frac{-1}{\tau} \right) = \tau^{2j+2} \dot{f}_3^{(2j+1)} (0|\tau), \quad (3.4.29)$$

$$f_3^{(2j)} \left( \pi \middle| \frac{-1}{\tau} \right) = \tau^{2j+1} f_3^{(2j)} (\pi\tau|\tau), \quad (3.4.30)$$

$$f_3^{(2j)} \left( \pi\tau \middle| \frac{-1}{\tau} \right) = \tau^{2j+1} f_3^{(2j)} (\pi|\tau), \quad (3.4.31)$$

$$f_3^{(2j+1)} \left( \pi + \pi\tau \middle| \frac{-1}{\tau} \right) = \tau^{2j+2} f_3^{(2j+1)} (\pi + \pi\tau|\tau). \quad (3.4.32)$$

Equations (3.4.17)–(3.4.20), (3.4.23), (3.4.24), (3.4.26), and (3.4.28) hold for  $j \geq 1$ , and

equations (3.4.21), (3.4.22), (3.4.25), (3.4.27), (3.4.29)–(3.4.32) hold for  $j \geq 0$ .

For  $j = 0$  in (3.4.17)–(3.4.20), (3.4.23), (3.4.24), (3.4.26), and (3.4.28) we have

$$\dot{f}_0^{(1)}\left(0\left|\frac{-1}{\tau}\right.\right) = \tau^2 \dot{f}_0^{(1)}(0|\tau) - \frac{\tau}{2\pi i}, \quad (3.4.33)$$

$$\dot{f}_0^{(1)}\left(\pi\left|\frac{-1}{\tau}\right.\right) = \tau^2 \dot{f}_0^{(1)}(\pi\tau|\tau) - \frac{\tau}{2\pi i}, \quad (3.4.34)$$

$$\dot{f}_0^{(1)}\left(\pi\tau\left|\frac{-1}{\tau}\right.\right) = \tau^2 \dot{f}_0^{(1)}(\pi|\tau) - \frac{\tau}{2\pi i}, \quad (3.4.35)$$

$$\dot{f}_0^{(1)}\left(\pi + \pi\tau\left|\frac{-1}{\tau}\right.\right) = \tau^2 \dot{f}_0^{(1)}(\pi + \pi\tau|\tau) - \frac{\tau}{2\pi i}, \quad (3.4.36)$$

$$f_1^{(0)}\left(\pi\tau\left|\frac{-1}{\tau}\right.\right) = -\tau f_2^{(0)}(\pi|\tau) - \frac{1}{2i}, \quad (3.4.37)$$

$$f_1^{(0)}\left(\pi + \pi\tau\left|\frac{-1}{\tau}\right.\right) = -\tau f_2^{(0)}(\pi + \pi\tau|\tau) - \frac{1}{2i}, \quad (3.4.38)$$

$$f_2^{(0)}\left(\pi\left|\frac{-1}{\tau}\right.\right) = \tau f_1^{(0)}(\pi\tau|\tau) + \frac{\tau}{2i}, \quad (3.4.39)$$

$$f_2^{(0)}\left(\pi + \pi\tau\left|\frac{-1}{\tau}\right.\right) = -\tau f_1^{(0)}(\pi + \pi\tau|\tau) - \frac{\tau}{2i}. \quad (3.4.40)$$

**Proof** By using (2.10.1), equation (3.4.9) can be represented as

$$\frac{1}{\theta} + \sum_{j=1}^{\infty} \dot{f}_0^{(2j-1)}(0|\tau) \frac{\theta^{2j-1}}{(2j-1)!} = \frac{1}{\theta} + \sum_{j=1}^{\infty} \dot{f}_0^{(2j-1)}\left(0\left|\frac{-1}{\tau}\right.\right) \frac{\theta^{2j-1}}{(2j-1)!\tau^{2j}} + \frac{\theta}{2\pi i\tau}. \quad (3.4.41)$$

Equating the coefficient of  $\theta$  on both sides of (3.4.41) gives equation (3.4.33).

Now by equating the coefficient of  $\theta^{2j-1}/(2j-1)!$  on both sides of (3.4.41) we arrive at (3.4.17).

The other formulae can be proved in a similar way. ■

### 3.5 Seven transformations satisfied by $z$ , $x$ , $1 - x$ , and $E$

In this section, we apply seven transformations ( $q$  to  $-q$ ,  $q$  to  $q^{\frac{1}{2}}$ ,  $q$  to  $q^2$ ,  $q$  to  $-q^{\frac{1}{2}}$ ,  $q$  to  $-q^2$ ,  $q$  to  $iq^{\frac{1}{2}}$ , and  $q$  to  $-iq^{\frac{1}{2}}$ ) to  $z$ ,  $x$ ,  $1 - x$ , and  $E$ .



**Lemma 3.5.1**

$q$	$1 - x$	$x$	$z$	
$q \rightarrow -q$	$\frac{1}{x'}$	$-\frac{x}{x'}$	$z\sqrt{x'}$	
$q \rightarrow q^{\frac{1}{2}}$	$\frac{(1 - \sqrt{x})^2}{(1 + \sqrt{x})^2}$	$\frac{4\sqrt{x}}{(1 + \sqrt{x})^2}$	$z(1 + \sqrt{x})$	(3.5.1)
$q \rightarrow q^2$	$\frac{4\sqrt{x'}}{(1 + \sqrt{x'})^2}$	$\frac{(1 - \sqrt{x'})^2}{(1 + \sqrt{x'})^2}$	$\frac{1}{2}z(1 + \sqrt{x'})$	

**Proof** The proof of the transformation  $q \rightarrow -q$  on  $z$ ,  $x$ , and  $1 - x$  follows immediately by using (2.11.7)–(2.11.9).

Now observe that

$$\begin{aligned}
 & \left( \sum_{n=-\infty}^{\infty} q^{n^2} \right)^2 + \left( \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \right)^2 \\
 &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} q^{m^2+n^2} + \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-1)^{m+n} q^{m^2+n^2} \\
 &= 2 \sum_{m+n \text{ even}} q^{m^2+n^2}.
 \end{aligned}$$

Set  $m + n = 2i$  and  $m - n = 2j$ , then

$$\begin{aligned}
 \left( \sum_{n=-\infty}^{\infty} q^{n^2} \right)^2 + \left( \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \right)^2 &= 2 \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} q^{2i^2+2j^2} \\
 &= 2 \left( \sum_{n=-\infty}^{\infty} q^{2n^2} \right)^2.
 \end{aligned} \tag{3.5.2}$$

By using (2.11.1) and (2.11.3), equation (3.5.2) may be represented as

$$z(q^2) = \frac{1}{2}z(1 + \sqrt{x'}) \tag{3.5.3}$$

and setting  $\theta = \pi/2$  in (3.3.7) gives

$$f_1(\pi|2\tau) + f_3(\pi|2\tau) = f_1\left(\frac{\pi}{2}|\tau\right). \quad (3.5.4)$$

Square both sides of (3.5.4), use (2.11.9), (2.14.1), and (3.5.3); and rearrange to give

$$\begin{aligned} x'(q^2) &= \frac{1}{z^2(q^2)} \frac{(q^2; q^2)_\infty^4}{(-q^2; q^2)_\infty^4} \\ &= \frac{4\sqrt{x'}}{(1 + \sqrt{x'})^2}. \end{aligned} \quad (3.5.5)$$

By using (2.11.12),  $x(q^2)$  holds.

If we replace  $q$  with  $q^{\frac{1}{2}}$  in (3.5.5) then make  $x'(q^{\frac{1}{2}})$  the subject we obtain

$$x'(q^{\frac{1}{2}}) = \frac{(1 - \sqrt{x})^2}{(1 + \sqrt{x})^2}. \quad (3.5.6)$$

Similarly by (3.5.3) we have

$$z(q^{\frac{1}{2}}) = z(1 + \sqrt{x}).$$

By using (2.11.12),  $x(q^{\frac{1}{2}})$  holds. ■

B. C. Berndt [19, Chapter 17, pp. 125–126] also gave proofs of (3.5.1).

Next we investigate three transformations on function  $E$ .

First replacing  $q$  with  $-q$  in (2.11.27) gives

$$\begin{aligned} E\left(\sqrt{-\frac{x}{x'}}\right) &= x'(-q) x^{\frac{1}{2}}(-q) \pi \frac{d}{dx(-q)} \left[ z(-q) x^{\frac{1}{2}}(-q) \right] \\ &= x'(-q) x^{\frac{1}{2}}(-q) \pi \frac{dx}{dx(-q)} \frac{d}{dx} \left[ z(-q) x^{\frac{1}{2}}(-q) \right]. \end{aligned} \quad (3.5.7)$$

From (3.5.1) we have

$$\frac{dx(-q)}{dx} = \frac{-1}{(x-1)^2}. \quad (3.5.8)$$

Using the results of (3.5.1) and (3.5.8) equation (3.5.7) becomes

$$E \left( \sqrt{-\frac{x}{x'}} \right) = \pi \sqrt{x(x-1)} \frac{d}{dx} (z\sqrt{x}). \quad (3.5.9)$$

Similarly,

$$E \left( \sqrt{\frac{4\sqrt{x}}{(1+\sqrt{x})^2}} \right) = 2x^{\frac{3}{4}} (1-x^{\frac{1}{2}}) \pi \frac{d}{dx} (zx^{\frac{1}{4}}), \quad (3.5.10)$$

$$E \left( \sqrt{\frac{(1-\sqrt{x'})^2}{(1+\sqrt{x'})^2}} \right) = x' \pi \frac{d}{dx} [(1-\sqrt{x'})z]. \quad (3.5.11)$$

The following results can be easily obtained by applying the results of (2.11.27), (3.5.1), (3.5.7), (3.5.10), and (3.5.11).

$q$	$1-x$	$x$	$z$
$q \rightarrow -q^{\frac{1}{2}}$	$\frac{(1+\sqrt{x})^2}{(1-\sqrt{x})^2}$	$\frac{-4\sqrt{x}}{(1-\sqrt{x})^2}$	$z(1-\sqrt{x})$
$q \rightarrow -q^2$	$\frac{(1+\sqrt{x'})^2}{4\sqrt{x'}}$	$\frac{-(1-\sqrt{x'})^2}{4\sqrt{x'}}$	$z^4\sqrt{x'}$
$q \rightarrow iq^{\frac{1}{2}}$	$\frac{(\sqrt{x'}-i\sqrt{x})^2}{(\sqrt{x'}+i\sqrt{x})^2}$	$\frac{4i\sqrt{xx'}}{(\sqrt{x'}+i\sqrt{x})^2}$	$z(\sqrt{x'}+i\sqrt{x})$
$q \rightarrow -iq^{\frac{1}{2}}$	$\frac{(\sqrt{x'}+i\sqrt{x})^2}{(\sqrt{x'}-i\sqrt{x})^2}$	$\frac{-4i\sqrt{xx'}}{(\sqrt{x'}-i\sqrt{x})^2}$	$z(\sqrt{x'}-i\sqrt{x})$

(3.5.12)

$$E \left( \sqrt{\frac{-4\sqrt{x}}{(1-\sqrt{x})^2}} \right) = 2x^{\frac{3}{4}} (1+x^{\frac{1}{2}}) \pi \frac{d}{dx} (zx^{\frac{1}{4}}), \quad (3.5.13)$$

$$E \left( \sqrt{-\frac{(1-\sqrt{x'})^2}{4\sqrt{x'}}} \right) = \frac{1}{2} x'^{\frac{3}{4}} (1+x'^{\frac{1}{2}}) \pi \frac{d}{dx} [(1-x'^{\frac{1}{2}})z], \quad (3.5.14)$$

$$E \left( \sqrt{\frac{4i\sqrt{x}\sqrt{x'}}{(\sqrt{x'} + i\sqrt{x})^2}} \right) = 2\pi (xx')^{\frac{3}{4}} \left( x'^{\frac{1}{2}} - ix^{\frac{1}{2}} \right) \frac{d}{dx} \left[ z (xx')^{\frac{1}{4}} \right], \quad (3.5.15)$$

$$E \left( \sqrt{\frac{-4i\sqrt{x}\sqrt{x'}}{(\sqrt{x'} - i\sqrt{x})^2}} \right) = 2\pi (xx')^{\frac{3}{4}} \left( x'^{\frac{1}{2}} + ix^{\frac{1}{2}} \right) \frac{d}{dx} \left[ z (xx')^{\frac{1}{4}} \right]. \quad (3.5.16)$$

The results of this chapter will be used in Chapters 4–6 and 8.

# Chapter 4

## Sums of squares and triangular numbers

### 4.1 Introduction

Let  $k$  be a positive integer and  $r_k(n)$  denote the number of ways of expressing a non-negative integer  $n$  as a sum of  $k$  squares. That is,  $r_k(n)$  is the number of solutions in integers of

$$x_1^2 + x_2^2 + \cdots + x_k^2 = n.$$

We take into account the sign and order of  $x_1, x_2, \dots, x_k$ . For example,

$$5 = (\pm 1)^2 + (\pm 2)^2 = (\pm 2)^2 + (\pm 1)^2,$$

$$4 = 0^2 + (\pm 2)^2 = (\pm 2)^2 + 0^2,$$

and therefore  $r_2(5) = 8$ ,  $r_2(4) = 4$ , and also  $r_2(3) = 0$ .

Let  $t_k(n)$  denote the number of ways of expressing a non-negative integer  $n$  as a sum of  $k$  triangular numbers. Thus  $t_k(n)$  is the number of solutions in *non-negative* integers of

$$\frac{x_1(x_1 + 1)}{2} + \frac{x_2(x_2 + 1)}{2} + \cdots + \frac{x_k(x_k + 1)}{2} = n.$$

The first few triangular numbers are 0, 1, 3, 6, 10, 15,  $\dots$ . For example,

$$6 = 0 + 6 = 6 + 0 = 3 + 3,$$

$$16 = 1 + 15 = 15 + 1 = 6 + 10 = 10 + 6,$$

and therefore  $t_2(6) = 3$ ,  $t_2(16) = 4$ . Also  $t_2(5) = 0$ .

We define  $r_k(0) = t_k(0) = 1$ . Let

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2}, \quad (4.1.1)$$

and

$$\psi(q) = \sum_{j=0}^{\infty} q^{j(j+1)/2}. \quad (4.1.2)$$

Then the generating functions for  $r_k(n)$  and  $t_k(n)$  are

$$\sum_{n=0}^{\infty} r_k(n) q^n = \varphi^k(q) = \left( \sum_{j=-\infty}^{\infty} q^{j^2} \right)^k, \quad (4.1.3)$$

and

$$\sum_{n=0}^{\infty} t_k(n) q^n = \psi^k(q) = \left( \sum_{j=0}^{\infty} q^{j(j+1)/2} \right)^k, \quad (4.1.4)$$

respectively.

Suppose  $\lambda_1, \lambda_2, \dots, \lambda_k$  are positive integers with  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k$ . The function

$r_{(\lambda_1, \lambda_2, \dots, \lambda_k)}(n)$  will denote the number of solutions in integers of

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \dots + \lambda_k x_k^2 = n, \quad (4.1.5)$$

where  $n = 0, 1, 2, 3, \dots$ . The function  $t_{(\lambda_1, \lambda_2, \dots, \lambda_k)}(n)$  will denote the number of solutions

in non-negative integers of

$$\lambda_1 \frac{x_1(x_1 + 1)}{2} + \dots + \lambda_k \frac{x_k(x_k + 1)}{2} = n, \quad (4.1.6)$$

where  $n = 0, 1, 2, 3, \dots$ .

We also define  $r_{(\lambda_1, \lambda_2, \dots, \lambda_k)}(0) = t_{(\lambda_1, \lambda_2, \dots, \lambda_k)}(0) = 1$ .

Then the generating functions for  $r_{(\lambda_1, \lambda_2, \dots, \lambda_k)}(n)$  and  $t_{(\lambda_1, \lambda_2, \dots, \lambda_k)}(n)$  are

$$\begin{aligned} \sum_{n=0}^{\infty} r_{(\lambda_1, \lambda_2, \dots, \lambda_k)}(n) q^n &= \varphi(q^{\lambda_1}) \varphi(q^{\lambda_2}) \cdots \varphi(q^{\lambda_k}) \\ &= \left( \sum_{x_1=-\infty}^{\infty} q^{\lambda_1 x_1^2} \right) \left( \sum_{x_2=-\infty}^{\infty} q^{\lambda_2 x_2^2} \right) \cdots \left( \sum_{x_k=-\infty}^{\infty} q^{\lambda_k x_k^2} \right), \end{aligned} \quad (4.1.7)$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} t_{(\lambda_1, \lambda_2, \dots, \lambda_k)}(n) q^n &= \psi(q^{\lambda_1}) \psi(q^{\lambda_2}) \cdots \psi(q^{\lambda_k}) \\ &= \left( \sum_{x_1=0}^{\infty} q^{\lambda_1 \frac{x_1(x_1+1)}{2}} \right) \left( \sum_{x_2=0}^{\infty} q^{\lambda_2 \frac{x_2(x_2+1)}{2}} \right) \cdots \left( \sum_{x_k=0}^{\infty} q^{\lambda_k \frac{x_k(x_k+1)}{2}} \right), \end{aligned} \quad (4.1.8)$$

respectively.

If we set  $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 1$  on the left hand side of (4.1.5) and (4.1.6) we will use  $r_k(n)$  and  $t_k(n)$  instead of  $r_{(\lambda_1, \lambda_2, \dots, \lambda_k)}(n)$  and  $t_{(\lambda_1, \lambda_2, \dots, \lambda_k)}(n)$ .

The problems of representing an integer as sums of squares and triangular numbers have a long and interesting history. More details can be found in Dickson [48], Milne [88], and Section 4 of this chapter.

The aim of this chapter is to study the functions listed in the following table:

Sums of squares	Sums of triangular numbers
$r_2(n)$	$t_2(n)$
$r_4(n)$	$t_4(n)$
$r_6(n)$	$t_6(n)$
$r_8(n)$	$t_8(n)$
$r_{(1,2)}(n)$	$t_{(1,2)}(n)$
$r_{(1,3)}(n)$	$t_{(1,3)}(n)$
$r_{(1,7)}(n)$	$t_{(1,4)}(n)$
$r_{(1,1,3,3)}(n)$	$t_{(1,7)}(n)$
	$t_{(1,1,2,2)}(n)$
	$t_{(1,1,1,1,2,2)}(n)$

We present self-contained proofs based on Ramanujan’s  ${}_1\psi_1$  summation formula and the Fundamental multiplicative identity.

So far, there is no Lambert series for the representation of  $n$  by

$$\frac{x_1(x_1+1)}{2} + \frac{x_2(x_2+1)}{2} + \frac{x_3(x_3+1)}{2} + \frac{x_4(x_4+1)}{2} + x_5(x_5+1) + x_6(x_6+1)$$

and the explicit formula for  $t_{(1,1,1,1,2,2)}(n)$  in the literature. It appears that the result is new.

4.2 Sums of squares

From Jacobi’s triple product identity [67, p. 90] we have

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2} = \frac{(-q, q^2; q^2)_{\infty}}{(q, -q^2; q^2)_{\infty}}.$$

(4.2.1)

The following lemma is used to prove formulae (4.2.3), (4.2.4), (4.2.7), and (4.2.8).



**Lemma 4.2.1**

$$\frac{(-aq, -a^{-1}q, q^2, q^2; q^2)_\infty}{(aq, a^{-1}q, -q^2, -q^2; q^2)_\infty} = 1 + 2 \sum_{n=1}^{\infty} \frac{q^n}{1 + q^{2n}} (a^n + a^{-n}), \quad (4.2.2)$$

where  $|q| < |b| < |q^{-1}|$ .

**Proof** Let  $b = -1$  and replace  $a$  with  $aq$  in the Jordan Kronecker function (2.4.1) and (2.4.2) to obtain

$$\frac{(-aq, -a^{-1}q, q^2, q^2; q^2)_\infty}{(aq, a^{-1}q, -1, -q^2; q^2)_\infty} = \sum_{n=-\infty}^{\infty} \frac{a^n q^n}{1 + q^{2n}}.$$

By rearranging and simplifying, we further obtain (4.2.2). ■

**Theorem 4.2.2**

$$\left( \sum_{m=-\infty}^{\infty} q^{m^2} \right)^2 = 1 + 4 \sum_{j=1}^{\infty} \frac{q^j}{1 + q^{2j}}, \quad (4.2.3)$$

$$\left( \sum_{m=-\infty}^{\infty} q^{m^2} \right)^4 = 1 + 8 \sum_{j=1}^{\infty} \frac{j q^j}{1 + (-q)^j}, \quad (4.2.4)$$

$$\begin{aligned} \left( \sum_{m=-\infty}^{\infty} q^{m^2} \right)^6 &= 1 + 4 \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1)^2 q^{2j-1}}{1 - q^{2j-1}} \\ &\quad + 16 \sum_{j=1}^{\infty} \frac{j^2 q^j}{1 + q^{2j}}, \end{aligned} \quad (4.2.5)$$

$$\left( \sum_{m=-\infty}^{\infty} q^{m^2} \right)^8 = 1 + 16 \sum_{j=1}^{\infty} \frac{j^3 q^j}{1 - (-q)^j}, \quad (4.2.6)$$

$$\left( \sum_{m=-\infty}^{\infty} q^{m^2} \right) \left( \sum_{k=-\infty}^{\infty} q^{2k^2} \right) = 1 + 2 \sum_{j=1}^{\infty} \frac{q^{3j}}{1 + q^{4j}} + 2 \sum_{j=1}^{\infty} \frac{q^j}{1 + q^{4j}}, \quad (4.2.7)$$

$$\begin{aligned} \left( \sum_{m=-\infty}^{\infty} q^{m^2} \right) \left( \sum_{k=-\infty}^{\infty} q^{3k^2} \right) &= 1 + 2 \sum_{j=1}^{\infty} \left( \frac{q^{3j-2}}{1 - q^{3j-2}} - \frac{q^{3j-1}}{1 - q^{3j-1}} \right) \\ &\quad + 4 \sum_{j=1}^{\infty} \left( \frac{q^{12j-8}}{1 - q^{12j-8}} - \frac{q^{12j-4}}{1 - q^{12j-4}} \right), \end{aligned} \quad (4.2.8)$$

$$\left( \sum_{m=-\infty}^{\infty} q^{m^2} \right) \left( \sum_{k=-\infty}^{\infty} q^{7k^2} \right) = 1 - 2 \sum_{j=1}^{\infty} \frac{\chi_{(1,7)}(j) (-q)^j}{1 + (-q)^j}, \quad (4.2.9)$$

where

$$\chi_{(1,7)}(j) = \begin{cases} 1 & \text{if } j \equiv 1, 2 \text{ or } 4 \pmod{7}, \\ -1 & \text{if } j \equiv 3, 5 \text{ or } 6 \pmod{7}, \\ 0 & \text{if } j \equiv 0 \pmod{7}. \end{cases} \quad (4.2.10)$$

$$\left( \sum_{m=-\infty}^{\infty} q^{m^2} \right)^2 \left( \sum_{k=-\infty}^{\infty} q^{3k^2} \right)^2 = 1 + 4 \sum_{\substack{j=1 \\ 3 \nmid j}}^{\infty} \frac{j q^j}{1 - (-q)^j}. \quad (4.2.11)$$

**Proof** Proofs of (4.2.3)–(4.2.11) are given one at a time.

Proof of (4.2.3). Follows by setting  $a = -1$  in (4.2.2).

Proof of (4.2.4). Equation (4.2.2) can be represented as

$$\begin{aligned} \frac{(-aq, -a^{-1}q, q^2, q^2; q^2)_{\infty}}{(aq, a^{-1}q, -q^2, -q^2; q^2)_{\infty}} &= 1 + 2 \sum_{n=1}^{\infty} \frac{q^n a^n (1 + q^{2n} - q^{2n})}{1 + q^{2n}} + 2 \sum_{n=1}^{\infty} \frac{q^n a^{-n}}{1 + q^{2n}} \\ &= 1 + 2 \sum_{n=1}^{\infty} q^n a^n - 2 \sum_{n=1}^{\infty} \frac{q^{3n} a^n}{1 + q^{2n}} + 2 \sum_{n=1}^{\infty} \frac{q^n a^{-n}}{1 + q^{2n}} \\ &= \frac{1 + aq}{1 - aq} + 2 \sum_{n=1}^{\infty} \frac{q^n a^{-n} (1 - q^{2n} a^{2n})}{1 + q^{2n}}. \end{aligned} \quad (4.2.12)$$

If we multiply both sides by  $(1 - aq) / (1 + aq)$  in (4.2.12) and set  $a \rightarrow -1/q$  we arrive at

$$\frac{(q^2; q^2)_{\infty}^4}{(-q^2; q^2)_{\infty}^4} = 1 + 8 \sum_{n=1}^{\infty} \frac{(-1)^n n q^{2n}}{1 + q^{2n}}. \quad (4.2.13)$$

Replacing  $q^2$  with  $-q$  in (4.2.13) and then employing (4.2.1) gives (4.2.4).

Proof of (4.2.5). The ideas of the following proof can be traced back to V. Ramamani [90] although her methods are slightly different.

From (2.9.9), (2.9.10), and (2.14.1), it is shown

$$\begin{aligned} 8f_2''(\pi) - 8if_1''(\pi\tau) &= 16f_2^3(\pi) + 24e_2f_2(\pi) - 16if_1^3(\pi\tau) - 24ie_1f_1(\pi\tau) \\ &= 4z^3 - 12z(e_1 - e_2). \end{aligned} \quad (4.2.14)$$

By comparing (2.8.13) and (2.11.7) it can be shown that

$$e_1 - e_2 = \frac{z^2}{4}. \quad (4.2.15)$$

Therefore equation (4.2.14) can be rewritten as

$$8f_2''(\pi) - 8if_1''(\pi\tau) = z^3. \quad (4.2.16)$$

By (2.11.7) and (4.2.1)

$$z = \varphi^2(q) = \left( \sum_{m=-\infty}^{\infty} q^{m^2} \right)^2. \quad (4.2.17)$$

Substituting (4.2.17) into (4.2.16) gives

$$\left( \sum_{m=-\infty}^{\infty} q^{m^2} \right)^6 = 8f_2''(\pi) - 8if_1''(\pi\tau).$$

The Lambert series for  $f_2''(\pi)$  and  $f_1''(\pi\tau)$  can be obtained by setting  $j = 1$  into (2.10.7) and (2.10.10). This completes the proof of (4.2.5).

Proof of (4.2.6). We set  $j = 2$  into (2.10.5) and (2.10.9) to obtain

$$f_0'''(\pi) = -\frac{1}{8} - 2 \sum_{m=1}^{\infty} \frac{(-1)^m m^3 q^{2m}}{1 - q^{2m}}, \quad (4.2.18)$$

and

$$f_0'''(\pi\tau) = -2 \sum_{m=1}^{\infty} \frac{m^3 q^m}{1 - q^{2m}}. \quad (4.2.19)$$

By (2.9.15) and (2.14.1) equations (4.2.18) and (4.2.19) become

$$f_0'''(\pi) = -\frac{1}{8}z^4(1-x), \quad (4.2.20)$$

$$f_0'''(\pi\tau) = -\frac{1}{8}z^4x. \quad (4.2.21)$$

From (4.2.18) and (4.2.19) we observe that

$$\begin{aligned} -8[f_0'''(\pi) + f_0'''(\pi\tau)] &= 1 + 16 \sum_{m=1}^{\infty} \frac{(-1)^m m^3 q^{2m}}{1 - q^{2m}} + 16 \sum_{m=1}^{\infty} \frac{m^3 q^m}{1 - q^{2m}} \\ &= 1 + 16 \sum_{m=1}^{\infty} \frac{(2m)^3 q^{4m}}{1 - q^{4m}} - 16 \sum_{m=1}^{\infty} \frac{(2m-1)^3 q^{4m-2}}{1 - q^{4m-2}} \\ &\quad + 16 \sum_{m=1}^{\infty} \frac{(2m)^3 q^{2m}}{1 - q^{4m}} + 16 \sum_{m=1}^{\infty} \frac{(2m-1)^3 q^{2m-1}}{1 - q^{4m-2}} \\ &= 1 + 16 \sum_{m=1}^{\infty} \frac{(2m)^3 q^{2m}}{1 - q^{2m}} + 16 \sum_{m=1}^{\infty} \frac{(2m-1)^3 q^{2m-1}}{1 + q^{2m-1}} \\ &= 1 + 16 \sum_{m=1}^{\infty} \frac{m^3 q^m}{1 - (-q)^m}. \end{aligned} \quad (4.2.22)$$

Similarly from (4.2.20) and (4.2.21) we find that

$$-8[f_0'''(\pi) + f_0'''(\pi\tau)] = z^4. \quad (4.2.23)$$

Substituting (4.2.23) into (4.2.22) and using (4.2.17) we obtain (4.2.6).

Proof of (4.2.7). In (4.2.2) replace  $q$  with  $q^2$  and put  $a = q$  :

$$\frac{(-q^3, -q, q^4, q^4; q^4)_{\infty}}{(q^3, q, -q^4, -q^4; q^4)_{\infty}} = 1 + 2 \sum_{j=1}^{\infty} \frac{q^{3j}}{1 + q^{4j}} + 2 \sum_{j=1}^{\infty} \frac{q^j}{1 + q^{4j}}. \quad (4.2.24)$$

The left hand side of (4.2.24) simplifies into

$$\frac{(-q^3, -q, q^4, q^4; q^4)_{\infty}}{(q^3, q, -q^4, -q^4; q^4)_{\infty}} = \frac{(-q, q^2; q^2)_{\infty} (-q^2, q^4; q^4)_{\infty}}{(q; q^2)_{\infty} (-q^4; q^4)_{\infty}}. \quad (4.2.25)$$

Observe that

$$\begin{aligned}
 (-q, q^2)_\infty &= \frac{(-q; q)_\infty}{(-q^2; q^2)_\infty} \\
 &= \frac{(-q; q)_\infty (q; q)_\infty}{(-q^2; q^2)_\infty (q; q)_\infty} \\
 &= \frac{1}{(q, -q^2; q^2)_\infty},
 \end{aligned}$$

which was given by Euler. This may be written as

$$(-q^2; q^2)_\infty (q^2; q^4)_\infty = 1. \quad (4.2.26)$$

By using (4.2.26) into (4.2.25) equation (4.2.25) can be rewritten as

$$\frac{(-q^3, -q, q^4, q^4; q^4)_\infty}{(q^3, q, -q^4, -q^4; q^4)_\infty} = \frac{(-q, q^2; q^2)_\infty (-q^2, q^4; q^4)_\infty}{(q, -q^2; q^2)_\infty (q^2, -q^4; q^4)_\infty}.$$

Substitution of this into (4.2.24) and making use (4.2.1) the proof of (4.2.7) is completed.

Proof of (4.2.8). Replace  $q$  with  $q^{3/2}$  in (4.2.2) and let  $a = -q^{1/2}$  to obtain

$$\frac{(q^2, q, q^3, q^3; q^3)_\infty}{(-q^2, -q, -q^3, -q^3; q^3)_\infty} = 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n (q^n + q^{2n})}{1 + q^{3n}}.$$

We simplify the left hand side and manipulate the right hand side as follows:

$$\begin{aligned}
 &\frac{(q; q)_\infty (q^3; q^3)_\infty}{(-q; q)_\infty (-q^3; q^3)_\infty} \\
 &= 1 + 2 \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} (-1)^n (-1)^{j-1} q^{3jn-2n} + 2 \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} (-1)^n (-1)^{j-1} q^{3jn-n} \\
 &= 1 + 2 \sum_{j=1}^{\infty} (-1)^{j-1} \sum_{n=1}^{\infty} (-q^{3j-2})^n + 2 \sum_{j=1}^{\infty} (-1)^{j-1} \sum_{n=1}^{\infty} (-q^{3j-1})^n \\
 &= 1 + 2 \sum_{j=1}^{\infty} \frac{(-1)^j q^{3j-2}}{1 + q^{3j-2}} + 2 \sum_{j=1}^{\infty} \frac{(-1)^j q^{3j-1}}{1 + q^{3j-1}}.
 \end{aligned}$$

Replace  $q$  with  $-q$  to obtain

$$\begin{aligned}
 & \frac{(-q, q^2; q^2)_\infty (-q^3, q^6; q^6)_\infty}{(q, -q^2; q^2)_\infty (q^3, -q^6; q^6)_\infty} \\
 &= 1 + 2 \sum_{j=1}^{\infty} \frac{(-1)^j (-q)^{3j-2}}{1 + (-q)^{3j-2}} + 2 \sum_{j=1}^{\infty} \frac{(-1)^j (-q)^{3j-1}}{1 + (-q)^{3j-1}} \\
 &= 1 + 2 \sum_{j=1}^{\infty} \left[ \frac{q^{6j-2}}{1 + q^{6j-2}} + \frac{q^{6j-5}}{1 - q^{6j-5}} - \frac{q^{6j-1}}{1 - q^{6j-1}} - \frac{q^{6j-4}}{1 + q^{6j-4}} \right].
 \end{aligned}$$

Using (4.2.1) in the left hand side and applying the identity  $\frac{x}{1+x} = \frac{x}{1-x} - \frac{2x^2}{1-x^2}$  on the right hand side completes the proof of (4.2.8).

Proof of (4.2.9). By replacing  $q$  with  $q^7$  and putting  $a = -q^2, b = -q^4, t = -q^8$  in (2.6.14) we obtain

$$\begin{aligned}
 & \frac{(q^2; q^2)_\infty (q^{14}; q^{14})_\infty}{2 (-q^2; q^2)_\infty (-q^{14}; q^{14})_\infty} \\
 &= \frac{1}{2} \left( \frac{1 - q^2}{1 + q^2} + \frac{1 - q^4}{1 + q^4} + \frac{1 - q^8}{1 + q^8} - \frac{1 - q^{14}}{1 + q^{14}} \right) \\
 &+ \sum_{j=1}^{\infty} \left( \frac{q^{14j-2}}{1 + q^{14j-2}} + \frac{q^{14j-4}}{1 + q^{14j-4}} + \frac{q^{14j-8}}{1 + q^{14j-8}} - \frac{q^{14j-14}}{1 + q^{14j-14}} \right) \\
 &- \sum_{j=1}^{\infty} \left( \frac{q^{14j+2}}{1 + q^{14j+2}} + \frac{q^{14j+4}}{1 + q^{14j+4}} + \frac{q^{14j+8}}{1 + q^{14j+8}} - \frac{q^{14j+14}}{1 + q^{14j+14}} \right).
 \end{aligned}$$

Replacing  $q^2$  with  $-q$  and using (4.2.1) we arrive at

$$\begin{aligned}
 & \left( \sum_{m=-\infty}^{\infty} q^{m^2} \right) \left( \sum_{k=-\infty}^{\infty} q^{7k^2} \right) \\
 &= 1 - 2 \sum_{j=1}^{\infty} \frac{(-q)^{7j-6}}{1 + (-q)^{7j-6}} - 2 \sum_{j=1}^{\infty} \frac{(-q)^{7j-5}}{1 + (-q)^{7j-5}} + 2 \sum_{j=1}^{\infty} \frac{(-q)^{7j-4}}{1 + (-q)^{7j-4}} \\
 &- 2 \sum_{j=1}^{\infty} \frac{(-q)^{7j-3}}{1 + (-q)^{7j-3}} + 2 \sum_{j=1}^{\infty} \frac{(-q)^{7j-2}}{1 + (-q)^{7j-2}} + 2 \sum_{j=1}^{\infty} \frac{(-q)^{7j-1}}{1 + (-q)^{7j-1}},
 \end{aligned}$$

which may be written as

$$\left( \sum_{m=-\infty}^{\infty} q^{m^2} \right) \left( \sum_{k=-\infty}^{\infty} q^{7k^2} \right) = 1 - 2 \sum_{j=1}^{\infty} \frac{\chi_{(1,7)}(j) (-q)^j}{1 + (-q)^j},$$

where

$$\chi_{(1,7)}(j) = \begin{cases} 1 & \text{if } j \equiv 1, 2 \text{ or } 4 \pmod{7}, \\ -1 & \text{if } j \equiv 3, 5 \text{ or } 6 \pmod{7}, \\ 0 & \text{if } j \equiv 0 \pmod{7}. \end{cases}$$

This completes the proof of (4.2.9).

Proof of (4.2.11). By letting  $\alpha = \pi/3$  and  $\theta = \pi$  in (2.7.17) we have

$$F(e^{i\pi/3}, e^{i\pi}) F(e^{-i\pi/3}, e^{i\pi}) = \wp\left(\frac{\pi}{3}\right) - \wp(\pi). \quad (4.2.27)$$

If we apply (2.4.2) on the left hand side and use (2.7.5) on the right hand side of (4.2.27)

then we find that

$$\begin{aligned} \frac{(q; q)_{\infty}^2 (q^3; q^3)_{\infty}^2}{(-q; q)_{\infty}^2 (-q^3; q^3)_{\infty}^2} &= 1 - 4 \sum_{n=1}^{\infty} \frac{(6n-5)q^{6n-5}}{1 - q^{6n-5}} + 4 \sum_{n=1}^{\infty} \frac{(6n-4)q^{6n-4}}{1 - q^{6n-4}} \\ &\quad + 4 \sum_{n=1}^{\infty} \frac{(6n-2)q^{6n-2}}{1 - q^{6n-2}} - 4 \sum_{n=1}^{\infty} \frac{(6n-1)q^{6n-1}}{1 - q^{6n-1}}. \end{aligned} \quad (4.2.28)$$

Now replace  $q$  with  $-q$  and use (4.2.1) so that equation (4.2.28) may be represented as

$$\left( \sum_{m=-\infty}^{\infty} q^{m^2} \right)^2 \left( \sum_{k=-\infty}^{\infty} q^{3k^2} \right)^2 = 1 + 4 \sum_{\substack{j=1 \\ 3 \nmid j}}^{\infty} \frac{j q^j}{1 - (-q)^j}.$$

This completes the proof of (4.2.11).

The proof of Theorem 4.2.2 is now completed. ■

We remark that (4.2.3) can be obtained directly from Ramanujan's summation formula by replacing  $q$  by  $q^2$  and then setting  $x = q$ ,  $a = -1$ , and  $b = -q^2$  in (2.3.1).

Next we give an arithmetic interpretation of Theorem 4.2.2 as follows.

**Corollary 4.2.3** For  $n \geq 1$ ,

$$r_2(n) = 4 \left[ \sum_{\substack{d|n \\ d \equiv 1 \pmod{4}}} 1 - \sum_{\substack{d|n \\ d \equiv 3 \pmod{4}}} 1 \right], \quad (4.2.29)$$

$$r_4(n) = 8 \sum_{\substack{d|n, 4 \nmid d}} d, \quad (4.2.30)$$

$$r_6(n) = 4 \sum_{\substack{d|n \\ d \text{ odd}}} (-1)^{(d-1)/2} \left[ \frac{4n^2}{d^2} - d^2 \right], \quad (4.2.31)$$

$$r_8(n) = 16(-1)^n \sum_{d|n} (-1)^d d^3, \quad (4.2.32)$$

$$r_{(1,2)}(n) = 2 \left[ \sum_{\substack{d|n \\ d \equiv 1 \pmod{8}}} 1 + \sum_{\substack{d|n \\ d \equiv 3 \pmod{8}}} 1 - \sum_{\substack{d|n \\ d \equiv 5 \pmod{8}}} 1 - \sum_{\substack{d|n \\ d \equiv 7 \pmod{8}}} 1 \right], \quad (4.2.33)$$

$$\begin{aligned} r_{(1,3)}(n) &= 2 \left( \sum_{\substack{d|n \\ d \equiv 1 \pmod{3}}} 1 - \sum_{\substack{d|n \\ d \equiv 2 \pmod{3}}} 1 \right) \\ &\quad + 4 \left( \sum_{\substack{d|n \\ d \equiv 4 \pmod{12}}} 1 - \sum_{\substack{d|n \\ d \equiv 8 \pmod{12}}} 1 \right), \end{aligned} \quad (4.2.34)$$

$$r_{(1,7)}(n) = 2(-1)^n \sum_{d|n} (-1)^{\frac{n}{d}} \chi_{(1,7)}(d), \quad (4.2.35)$$

$$r_{(1,1,3,3)}(n) = 4(-1)^n \sum_{\substack{d|n \\ 3 \nmid d}} (-1)^d d, \quad (4.2.36)$$

where we use  $d|n$  to denote  $d$  is a divisor of  $n$  and  $\chi_{(1,7)}(d)$  is defined in (4.2.10).

**Proof** We give complete details for formulae (4.2.29) and (4.2.31); the other formulae can be proved in a similar way.

Proof of (4.2.29). If we use (4.1.3) in (4.2.3) and expand the right hand side using geomet-



ric series then

$$\begin{aligned}
 \sum_{n=0}^{\infty} r_2(n) q^n &= 1 + 4 \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m-1} q^{j(2m-1)} \\
 &= 1 + 4 \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} q^{j(4m-3)} - 4 \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} q^{j(4m-1)} \\
 &= 1 + 4 \sum_{n=1}^{\infty} \left[ \sum_{j(4m-3)=n} 1 - \sum_{j(4m-1)=n} 1 \right] q^n \\
 &= 1 + 4 \sum_{n=1}^{\infty} \left[ \sum_{\substack{d|n \\ d \equiv 1 \pmod{4}}} 1 - \sum_{\substack{d|n \\ d \equiv 3 \pmod{4}}} 1 \right] q^n.
 \end{aligned}$$

By comparing the coefficients of  $q^n$  on both sides we prove that (4.2.29) holds.

Proof of (4.2.31). If we use (4.2.1) in (4.2.5) and expand the right hand side using geometric series then

$$\begin{aligned}
 &\sum_{n=0}^{\infty} r_6(n) q^n \\
 &= 1 + 4 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (-1)^j (2j-1)^2 q^{(2j-1)k} \\
 &\quad + 16 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (-1)^{k-1} j^2 q^{j(2k-1)} \\
 &= 1 + 4 \sum_{n=1}^{\infty} \left[ \sum_{(2j-1)k=n} (-1)^j (2j-1)^2 + 4 \sum_{j(2k-1)=n} (-1)^{k-1} j^2 \right] q^n \\
 &= 1 + 4 \sum_{n=1}^{\infty} \left[ \sum_{\substack{d|n \\ d \text{ odd}}} (-1)^{(d+1)/2} d^2 + (-1)^{(d-1)/2} \frac{4n^2}{d^2} \right] q^n \\
 &= 1 + 4 \sum_{n=1}^{\infty} \sum_{\substack{d|n \\ d \text{ odd}}} (-1)^{(d-1)/2} \left[ \frac{4n^2}{d^2} - d^2 \right] q^n.
 \end{aligned}$$

We compare coefficients of  $q^n$  on both sides to complete the proof of (4.2.31). ■

We remark that our proof of  $r_6(n)$  is the same as Berndt [22, p. 40].

A classical result in number theory can easily be deduced from (4.2.30).

**Corollary 4.2.4**

$$r_4(n) > 0.$$

**Proof**

$$r_4(n) = 8 \sum_{d|n, 4 \nmid d} d \geq 8,$$

since  $d = 1$  is a divisor of every positive integer  $n$ . ■

We remark that [48, pp. 276–279] L. Euler tried unsuccessfully over 40 years to prove  $r_4(n) > 0$ . J. L. Lagrange proved that every positive integer is the sum of four squares in 1770.

Now we let  $p$  be an odd prime and  $\prod_p$  denote a product over all odd primes  $p$ . Let  $n$  be a positive integer and denote the prime factorisation of  $n$  by

$$n = 2^{\lambda_2} \prod_p p^{\lambda_p}, \quad (4.2.37)$$

where  $\lambda_2$  and  $\lambda_p$  are all nonnegative integers.

**Corollary 4.2.5** *Let the prime factorization of  $n$  be given by (4.2.37). Then*

$$r_2(n) = 4 \prod_{p \equiv 1 \pmod{4}} (\lambda_p + 1) \prod_{p \equiv 3 \pmod{4}} \frac{1 + (-1)^{\lambda_p}}{2}, \quad (4.2.38)$$

$$r_4(n) = \begin{cases} 8 \prod_p \frac{p^{\lambda_p+1} - 1}{p - 1}, & : n \equiv 1 \pmod{2}, \\ 24 \prod_p \frac{p^{\lambda_p+1} - 1}{p - 1}, & : n \equiv 0 \pmod{2}, \end{cases} \quad (4.2.39)$$

$$r_6(n) = 4 \left( 2^{2\lambda_2+2} - \prod_p (-1)^{\lambda_p(p-1)/2} \right) \times \prod_p \frac{p^{2\lambda_p+2} - (-1)^{(\lambda_p+1)(p-1)/2}}{p^2 - (-1)^{(p-1)/2}}, \quad (4.2.40)$$

$$r_8(n) = (-1)^n \frac{16}{7} (2^{3\lambda_2+3} - 15) \prod_p \frac{p^{3\lambda_p+3} - 1}{p^3 - 1}, \quad (4.2.41)$$

$$r_{(1,2)}(n) = 2 \prod_{p \equiv 1 \text{ or } 3 \pmod{8}} (\lambda_p + 1) \prod_{p \equiv 5 \text{ or } 7 \pmod{8}} \frac{1 + (-1)^{\lambda_p}}{2}, \quad (4.2.42)$$

$$r_{(1,3)}(n) = k \prod_{p \equiv 1 \pmod{3}} (\lambda_p + 1) \prod_{p \equiv 2 \pmod{3}} \frac{1 + (-1)^{\lambda_p}}{2} \quad (4.2.43)$$

where

$$k = \begin{cases} 2 \left( 2^{1+(-1)^{\lambda_2}} - 1 \right) & : n \equiv 0 \pmod{2}, \\ 2 & : n \equiv 1 \pmod{2}. \end{cases}$$

$$r_{(1,7)}(n) = 2 (-1)^n (\lambda_2 - 1) \prod_{p \equiv 1, 2 \text{ or } 4 \pmod{7}} (\lambda_p + 1) \times \prod_{p \equiv 3, 5 \text{ or } 6 \pmod{7}} \frac{1 + (-1)^{\lambda_p}}{2}, \quad (4.2.44)$$

$$r_{(1,1,3,3)}(n) = 4 (-1)^n (2^{\lambda_2+1} - 3) \prod_{3 \nmid p} \frac{p^{\lambda_p+1} - 1}{p - 1}. \quad (4.2.45)$$

**Proof** The proofs of this corollary follow from the results in Corollary 4.2.3. First we express the divisors of  $n$  in terms of their prime factorization. Then we sum the resulting geometric series. We give complete details for  $r_6(n)$  only; the other formulae can be proved in a similar way.

Let  $n$  be a positive integer,  $s = 1, 2, 3, \dots$ , and  $t = 1, 2, 3, \dots$ . Then

$$n = 2^\alpha \prod_{p_s \equiv 1 \pmod{4}} p_s^{\beta_s} \prod_{q_t \equiv 3 \pmod{4}} q_t^{\gamma_t},$$

where  $p_s, q_t$  are odd primes and  $\beta_s, \gamma_t$  are nonnegative integers.

By formula (4.2.31) we observe that

$$\frac{d-1}{2} = \begin{cases} \text{even if } j_1 + j_2 + \dots + j_t \equiv 0 \pmod{2}, \\ \text{odd if } j_1 + j_2 + \dots + j_t \equiv 1 \pmod{2}, \end{cases}$$

where  $0 \leq j_1 \leq \gamma_1, 0 \leq j_2 \leq \gamma_2, \dots, 0 \leq j_t \leq \gamma_t$ . This implies that

$$(-1)^{\frac{d-1}{2}} = (-1)^{j_1 + j_2 + \dots + j_t}.$$

Therefore formula (4.2.31) can be rewritten as

$$\begin{aligned} r_6(n) &= 4 \sum_{\substack{0 \leq i_1 \leq \beta_1, 0 \leq j_1 \leq \gamma_1 \\ 0 \leq i_2 \leq \beta_2, 0 \leq j_2 \leq \gamma_2 \\ \vdots \\ 0 \leq i_s \leq \beta_s, 0 \leq j_t \leq \gamma_t}} (-1)^{j_1 + j_2 + \dots + j_t} \left[ 2^{2+2\alpha} p_1^{2(\beta_1 - i_1)} p_2^{2(\beta_2 - i_2)} \dots p_s^{2(\beta_s - i_s)} \right. \\ &\quad \times q_1^{2(\gamma_1 - j_1)} q_2^{2(\gamma_2 - j_2)} \dots q_t^{2(\gamma_t - j_t)} - p_1^{2i_1} p_2^{2i_2} \dots p_s^{2i_s} q_1^{2j_1} q_2^{2j_2} \dots q_t^{2j_t} \Big] \\ &= 4 \left[ 2^{2+2\alpha} \sum_{0 \leq i_1 \leq \beta_1} p_1^{2(\beta_1 - i_1)} \sum_{0 \leq i_2 \leq \beta_2} p_2^{2(\beta_2 - i_2)} \dots \sum_{0 \leq i_s \leq \beta_s} p_s^{2(\beta_s - i_s)} \right. \\ &\quad \times \sum_{0 \leq j_1 \leq \gamma_1} (-1)^{j_1} q_1^{2(\gamma_1 - j_1)} \sum_{0 \leq j_2 \leq \gamma_2} (-1)^{j_2} q_2^{2(\gamma_2 - j_2)} \dots \sum_{0 \leq j_t \leq \gamma_t} (-1)^{j_t} q_t^{2(\gamma_t - j_t)} \\ &\quad - \sum_{0 \leq i_1 \leq \beta_1} p_1^{2i_1} \sum_{0 \leq i_2 \leq \beta_2} p_2^{2i_2} \dots \sum_{0 \leq i_s \leq \beta_s} p_s^{2i_s} \\ &\quad \times \sum_{0 \leq j_1 \leq \gamma_1} q_1^{2j_1} \sum_{0 \leq j_2 \leq \gamma_2} q_2^{2j_2} \dots \sum_{0 \leq j_t \leq \gamma_t} q_t^{2j_t} \Big]. \end{aligned}$$

By summing the geometric series and simplifying the result, we obtain

$$r_6(n) = 4 \left[ 2^{2+2\alpha} - (-1)^{\gamma_1+\gamma_2+\dots+\gamma_t} \right] \\ \times \prod_{p_s \equiv 1 \pmod{4}} \frac{p_s^{2(\beta_s+1)} - 1}{p_s^2 - 1} \prod_{q_t \equiv 3 \pmod{4}} \frac{q_t^{2(\gamma_t+1)} + (-1)^{\gamma_t}}{q_t^2 + 1},$$

which is equivalent to formula (4.2.40). This completes the proof. ■

### 4.3 Sums of triangular numbers

By Jacobi's triple product identity [67, p. 90] we know that

$$\psi(q) = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}.$$

Together with (4.1.2) we have

$$\psi(q) = \sum_{j=0}^{\infty} q^{j(j+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}. \quad (4.3.1)$$

#### Theorem 4.3.1

$$\left( \sum_{k=0}^{\infty} q^{k(k+1)/2} \right)^2 = \sum_{j=-\infty}^{\infty} \frac{q^j}{1 - q^{4j+1}}, \quad (4.3.2)$$

$$\left( \sum_{k=0}^{\infty} q^{k(k+1)/2} \right)^4 = \sum_{j=0}^{\infty} \frac{(2j+1)q^j}{1 - q^{2j+1}}, \quad (4.3.3)$$

$$\left( \sum_{k=0}^{\infty} q^{k(k+1)/2} \right)^6 = \frac{1}{16} \sum_{j=1}^{\infty} \frac{(2j-1)^2 q^{\frac{j-2}{2}}}{1 + q^{j-1/2}} \\ + \frac{1}{16} \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1)^2 q^{\frac{j-2}{2}}}{1 - q^{j-1/2}}, \quad (4.3.4)$$

$$\left( \sum_{k=0}^{\infty} q^{k(k+1)/2} \right)^8 = \sum_{j=1}^{\infty} \frac{j^3 q^{j-1}}{1 - q^{2j}}, \quad (4.3.5)$$

$$\left( \sum_{k=0}^{\infty} q^{k(k+1)/2} \right) \left( \sum_{m=0}^{\infty} q^{m(m+1)} \right) = \sum_{j=-\infty}^{\infty} \frac{q^{3j}}{1 - q^{8j+1}}, \quad (4.3.6)$$

$$\left( \sum_{k=0}^{\infty} q^{k(k+1)/2} \right) \left( \sum_{m=0}^{\infty} q^{3m(m+1)/2} \right) = \sum_{j=-\infty}^{\infty} \frac{q^{3j}}{1 - q^{6j+1}}, \quad (4.3.7)$$

$$\left( \sum_{k=0}^{\infty} q^{k(k+1)/2} \right) \left( \sum_{m=0}^{\infty} q^{2m(m+1)} \right) = \sum_{j=-\infty}^{\infty} \frac{q^{5j}}{1 - q^{8j+1}}, \quad (4.3.8)$$

$$\left( \sum_{k=0}^{\infty} q^{k(k+1)/2} \right) \left( \sum_{m=0}^{\infty} q^{7m(m+1)/2} \right) = \sum_{j=1}^{\infty} \frac{q^{j-1} (1 - q^{2j}) (1 - q^{4j}) (1 - q^{6j})}{1 - q^{14j}}, \quad (4.3.9)$$

$$\left( \sum_{k=0}^{\infty} q^{k(k+1)/2} \right)^2 \left( \sum_{m=0}^{\infty} q^{m(m+1)} \right)^2 = \sum_{j=-\infty}^{\infty} \frac{j q^j}{1 - q^{4j+3}}, \quad (4.3.10)$$

$$\left( \sum_{k=0}^{\infty} q^{k(k+1)/2} \right)^4 \left( \sum_{m=0}^{\infty} q^{m(m+1)} \right)^2 = \sum_{j=1}^{\infty} \frac{j^2 q^{j-1}}{1 + q^{2j}}. \quad (4.3.11)$$

**Proof** Proofs of (4.3.2)–(4.3.11) are given one at a time.

Proof of (4.3.2). We replace  $q$  by  $q^2$ , set  $a, b = q$  into (2.4.1) to obtain

$$\frac{(q^2; q^2)_{\infty}^2}{(q; q^2)_{\infty}^2} = \sum_{j=-\infty}^{\infty} \frac{q^j}{1 - q^{4j+1}}. \quad (4.3.12)$$

By employing (4.3.12) on the right hand side of (4.3.1) we complete the proof of (4.3.2).

Proof of (4.3.3). If we divide the Jordan-Kronecker function (2.4.1) and (2.4.2) by  $1 - q^2/ab$

and set  $b = q$  we obtain

$$\frac{1}{(1 - \frac{q}{a})} \sum_{j=-\infty}^{\infty} \frac{a^j}{1 - q^{2j+1}} = \frac{(aq, a^{-1}q^3, q^2, q^2; q^2)_{\infty}}{(a, a^{-1}q^2, q, q; q^2)_{\infty}}. \quad (4.3.13)$$

Next we manipulate the left hand side of (4.3.13) and find that

$$\frac{1}{(1 - \frac{q}{a})} \sum_{j=-\infty}^{\infty} \frac{a^j}{1 - q^{2j+1}} = \frac{a}{a - q} \sum_{j=0}^{\infty} \frac{a^{2j+1} - q^{2j+1}}{a^{j+1} (1 - q^{2j+1})}. \quad (4.3.14)$$

If we let  $a \rightarrow q$  in (4.3.14) and use (4.3.1) then we obtain (4.3.3).

Proof of (4.3.4). The proof is similar to the proof of Theorem 4.2.6. From (2.8.14), (2.9.10), (2.9.11), (2.11.7), (2.11.8), and (2.14.1) we observe that

$$\frac{1}{8q^{\frac{3}{2}}} f_2''(\pi + \pi\tau) - \frac{i}{8q^{\frac{3}{2}}} f_3''(\pi\tau) = \frac{1}{64q^{\frac{3}{2}}} z^3 x^{\frac{3}{2}}. \quad (4.3.15)$$

By (2.11.7), (2.11.8), and (4.3.1) we also find that

$$\frac{1}{4q^{\frac{1}{2}}} z x^{\frac{1}{2}} = \psi^2(q^2) = \left( \sum_{k=0}^{\infty} q^{k(k+1)} \right)^2. \quad (4.3.16)$$

Substitution of (4.3.16) into (4.3.15) gives

$$\left( \sum_{k=0}^{\infty} q^{k(k+1)} \right)^6 = \frac{1}{8q^{\frac{3}{2}}} f_2''(\pi + \pi\tau) - \frac{i}{8q^{\frac{3}{2}}} f_3''(\pi\tau). \quad (4.3.17)$$

The Lambert series for  $f_2''(\pi + \pi\tau)$  and  $f_3''(\pi\tau)$  can be obtained by setting  $j = 1$  in (2.10.15) and (2.10.12). Then equation (4.3.17) becomes

$$\left( \sum_{k=0}^{\infty} q^{k(k+1)/2} \right)^6 = \frac{1}{16} \sum_{m=1}^{\infty} \frac{(2m-1)^2 q^{m-2}}{1 + q^{2m-1}} + \frac{1}{16} \sum_{m=1}^{\infty} \frac{(-1)^m (2m-1)^2 q^{m-2}}{1 - q^{2m-1}}.$$

By replacing  $q$  with  $q^{\frac{1}{2}}$  in both sides we complete the proof of (4.3.4).

Proof of (4.3.5). Substituting (4.2.21) into (4.2.18) we have

$$\frac{1}{16} z^4 x = \sum_{j=1}^{\infty} \frac{j^3 q^j}{1 - q^{2j}}. \quad (4.3.18)$$

By (2.11.7), (2.11.8), and (4.3.1) we observe that

$$\frac{1}{16} z^4 x = q \psi^8(q) = q \left( \sum_{k=0}^{\infty} q^{k(k+1)/2} \right)^8. \quad (4.3.19)$$

Using (4.3.19) into (4.3.18) we obtain (4.3.5).

Proof of (4.3.6). If we replace  $q$  with  $q^4$ , set  $a = q^3$  and  $b = q$  in (2.4.1) then we obtain

$$\frac{(q^4, q^4, q^8, q^8; q^8)_\infty}{(q, q^7, q^3, q^5; q^8)_\infty} = \sum_{j=-\infty}^{\infty} \frac{q^{3j}}{1 - q^{8j+1}}. \quad (4.3.20)$$

The left hand side of (4.3.20) can be rewritten as

$$\begin{aligned} \frac{(q^4, q^4, q^8, q^8; q^8)_\infty}{(q, q^7, q^3, q^5; q^8)_\infty} &= \frac{(q^4; q^4)_\infty^2}{(q; q^2)_\infty} \\ &= \frac{(q^4; q^4)_\infty (q^2; q^2)_\infty}{(q; q^2)_\infty (q^2; q^4)_\infty}. \end{aligned}$$

Substituting this into (4.3.20), using (4.3.1) on the left hand side, completes the proof of (4.3.6).

Proof of (4.3.7). Replace  $q$  with  $q^3$ , set  $a = q^3$  and  $b = q$  in (2.4.1) to obtain

$$\frac{(q^4, q^2, q^6, q^6; q^6)_\infty}{(q, q^5, q^3, q^3; q^6)_\infty} = \sum_{n=-\infty}^{\infty} \frac{q^{3j}}{1 - q^{6j+1}},$$

and then use (4.3.1) on the left hand side to complete the proof of (4.3.7).

Proof of (4.3.8). Replace  $q$  with  $q^4$ , set  $a = q^5$  and  $b = q$  in (2.4.1) to obtain

$$\frac{(q^6, q^2, q^8, q^8; q^8)_\infty}{(q, q^7, q^5, q^3; q^8)_\infty} = \sum_{j=-\infty}^{\infty} \frac{q^{5j}}{1 - q^{8j+1}},$$

and then use (4.3.1) on the left hand side to complete the proof of (4.3.8).

Proof of (4.3.9). Replace  $q$  with  $q^7$  and set  $u = q, v = q^2, w = q^3$  into (2.6.18) to get

$$\begin{aligned} &\sum_{j=1}^{\infty} \frac{(q^{-j} - q^j)(q^{-2j} - q^{2j})(q^{-3j} - q^{3j})}{(q^{-7j} - q^{7j})} \\ &= q \frac{(q^2, q^4, q^6, q^{12}, q^{10}, q^8, q^{14}, q^{14}; q^{14})_\infty}{(q^{13}, q^{11}, q^9, q^7, q^7, q^5, q^3, q; q^{14})_\infty}. \end{aligned}$$



Simplification and rearrangement enables us to obtain

$$\frac{(q; q^2)_\infty (q^7; q^{14})_\infty}{(q; q)_\infty (q^7; q^7)_\infty} = \sum_{j=1}^{\infty} \frac{q^{j-1} (1 - q^{2j}) (1 - q^{4j}) (1 - q^{6j})}{1 - q^{14j}}.$$

Application of (4.3.1) to the left hand side completes the proof of (4.3.9).

Proof of (4.3.10). Replace  $q$  with  $q^2$ , set  $b = q^3$  and apply  $a \frac{\partial}{\partial a}$  to (2.4.2) to give

$$\sum_{n=-\infty}^{\infty} \frac{na^n}{1 - q^{4n+3}} = \frac{q \left( aq^3, \frac{q^5}{a}, q^4, q^4; q^4 \right)_\infty}{a \left( q^3, q, a, \frac{q^4}{a}; q^4 \right)_\infty} + (a - q) \frac{\partial}{\partial a} \left[ \frac{\left( aq^3, \frac{q^5}{a}, q^4, q^4; q^4 \right)_\infty}{\left( q^3, q, a, \frac{q^4}{a}; q^4 \right)_\infty} \right].$$

On setting  $a = q$ , this becomes

$$\sum_{n=-\infty}^{\infty} \frac{nq^n}{1 - q^{4n+3}} = \frac{(q^4; q^4)_\infty^4}{(q, q^3; q^4)_\infty^2}. \quad (4.3.21)$$

Then we observe that

$$\frac{(q^4; q^4)_\infty^4}{(q, q^3; q^4)_\infty^2} = \frac{(q^2; q^2)_\infty^2 (q^4; q^4)_\infty^2}{(q; q^2)_\infty^2 (q^2; q^4)_\infty^2}. \quad (4.3.22)$$

Using (4.3.1), (4.3.21), and (4.3.22) we obtain (4.3.10).

Proof of (4.3.11). Let  $b = -1$  in (2.4.2) to give

$$\sum_{n=-\infty}^{\infty} \frac{a^n}{1 + q^{2n}} = \frac{(-a, -a^{-1}q^2, q^2, q^2; q^2)_\infty}{(a, a^{-1}q^2, -1, -q^2; q^2)_\infty}.$$

Apply  $a \frac{\partial}{\partial a}$  to both sides of the above to obtain

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \frac{na^n}{1 + q^{2n}} \\ &= \frac{a(-a, -a^{-1}q^2, q^2, q^2; q^2)_\infty}{(a, a^{-1}q^2, -1, -q^2; q^2)_\infty} \sum_{n=1}^{\infty} \left[ \frac{2q^{2n-2}}{1 - a^2q^{4n-4}} - \frac{2q^{2n}}{a^2(1 - a^{-2}q^{4n})} \right]. \end{aligned}$$

If we differentiate both sides with respect to  $a$  we obtain

$$\begin{aligned}
 & \sum_{n=-\infty}^{\infty} \frac{n^2 a^{n-1}}{1+q^{2n}} \\
 = & \frac{(-a, -a^{-1}q^2, q^2, q^2; q^2)_{\infty}}{(a, a^{-1}q^2, -1, -q^2; q^2)_{\infty}} \sum_{n=1}^{\infty} \left[ \frac{2aq^{2n-2}}{1-a^2q^{4n-4}} - \frac{2q^{2n}}{a(1-a^{-2}q^{4n})} \right] \\
 & \times \sum_{m=1}^{\infty} \left[ \frac{2q^{2m-2}}{1-a^2q^{4m-4}} - \frac{2q^{2m}}{a^2(1-a^{-2}q^{4m})} \right] \\
 & + \frac{(-a, -a^{-1}q^2, q^2, q^2; q^2)_{\infty}}{(a, a^{-1}q^2, -1, -q^2; q^2)_{\infty}} \sum_{n=1}^{\infty} \left[ \frac{2q^{2n-2}(1+a^2q^{4n-4})}{(1-a^2q^{4n-4})^2} + \frac{2q^{2n}(a^2+q^{4n})}{(a^2-q^{4n})^2} \right].
 \end{aligned}$$

Set  $a = q$  to arrive at

$$\sum_{n=1}^{\infty} \frac{n^2 q^{n-1}}{1+q^{2n}} = \frac{(-q, q^2; q^2)_{\infty}^2}{(-q^2, q; q^2)_{\infty}^2} \sum_{n=1}^{\infty} \frac{q^{2n-2}(1+q^{4n-2})}{(1-q^{4n-2})^2}. \quad (4.3.23)$$

Next observe that

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \frac{q^{2n-2}(1+q^{4n-2})}{(1-q^{4n-2})^2} \\
 = & \sum_{n=1}^{\infty} \frac{q^{2n-2}(1-q^{4n-2}+2q^{4n-2})}{(1-q^{4n-2})^2} \\
 = & \sum_{n=1}^{\infty} \frac{q^{2n-2}}{1-q^{4n-2}} + 2 \sum_{n=1}^{\infty} \frac{q^{6n-4}}{(1-q^{4n-2})^2} \\
 = & \sum_{n=1}^{\infty} \frac{q^{2n-2}}{1-q^{4n-2}} + 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} mq^{-2m-2}q^{(4m+2)n} \\
 = & \sum_{m=1}^{\infty} \frac{q^{2m-2}}{1-q^{4m-2}} + 2 \sum_{m=1}^{\infty} \frac{mq^{2m}}{1-q^{4m+2}} \\
 = & \sum_{m=1}^{\infty} \frac{q^{2m-2}}{1-q^{4m-2}} + 2 \sum_{m=1}^{\infty} \frac{mq^{2m-2}}{1-q^{4m-2}} - 2 \sum_{m=1}^{\infty} \frac{q^{2m-2}}{1-q^{4m-2}} \\
 = & \sum_{m=1}^{\infty} \frac{(2m-1)q^{2m-2}}{1-q^{4m-2}}.
 \end{aligned}$$

Replace  $q$  with  $q^2$  in (4.3.3) and use (4.3.1) to obtain

$$\sum_{n=1}^{\infty} \frac{q^{2n-2}(1+q^{4n-2})}{(1-q^{4n-2})^2} = \frac{(q^4; q^4)_{\infty}^4}{(q^2; q^4)_{\infty}^4}.$$

Substituting these in (4.3.23) gives

$$\sum_{n=1}^{\infty} \frac{n^2 q^{n-1}}{1 + q^{2n}} = \frac{(q^2; q^2)_{\infty}^4 (q^4; q^4)_{\infty}^2}{(q; q^2)_{\infty}^4 (q^2; q^4)_{\infty}^2}. \quad (4.3.24)$$

Using (4.3.1) in (4.3.24) and rearranging the result gives (4.3.11), this completing the proof of Theorem 4.3.1. ■

Now we will show an arithmetic interpretation of Theorem 4.3.1 as follows.

**Corollary 4.3.2** For  $n \geq 1$ ,

$$t_2(n) = \sum_{\substack{d|4n+1 \\ d \equiv 1 \pmod{4}}} 1 - \sum_{\substack{d|4n+1 \\ d \equiv 3 \pmod{4}}} 1, \quad (4.3.25)$$

$$t_4(n) = \sum_{d|2n+1} d, \quad (4.3.26)$$

$$t_6(n) = \frac{1}{8} \sum_{\substack{d|4n+3 \\ d \equiv 3 \pmod{4}}} d^2 - \frac{1}{8} \sum_{\substack{d|4n+3 \\ d \equiv 1 \pmod{4}}} d^2, \quad (4.3.27)$$

$$t_8(n) = \sum_{\substack{d|n+1 \\ d \text{ odd}}} \left( \frac{n+1}{d} \right)^3, \quad (4.3.28)$$

$$t_{(1,2)}(n) = \sum_{\substack{d|8n+3 \\ d \equiv 1 \pmod{8}}} 1 - \sum_{\substack{d|8n+3 \\ d \equiv 7 \pmod{8}}} 1, \quad (4.3.29)$$

$$t_{(1,3)}(n) = \sum_{\substack{d|2n+1 \\ d \equiv 1 \pmod{6}}} 1 - \sum_{\substack{d|2n+1 \\ d \equiv 5 \pmod{6}}} 1, \quad (4.3.30)$$

$$t_{(1,4)}(n) = \sum_{\substack{d|8n+5 \\ d \equiv 1 \pmod{8}}} 1 - \sum_{\substack{d|8n+5 \\ d \equiv 3 \pmod{8}}} 1, \quad (4.3.31)$$

$$t_{(1,7)}(n) = \sum_{\substack{d|n+1 \\ d \equiv 1, 9 \text{ or } 11 \pmod{14}}} 1 - \sum_{\substack{d|n+1 \\ d \equiv 3, 5 \text{ or } 13 \pmod{14}}} 1, \quad (4.3.32)$$

$$t_{(1,1,2,2)}(n) = \frac{1}{4} \sum_{d|4n+3} d, \quad (4.3.33)$$

$$t_{(1,1,1,1,2,2)}(n) = \sum_{\substack{d|n+1 \\ d \text{ odd}}} (-1)^{\frac{d-1}{2}} \left( \frac{n+1}{d} \right)^2. \quad (4.3.34)$$

**Proof** We give complete details for the proofs of  $t_6(n)$  and  $t_{(1,1,1,1,2,2)}(n)$  only. The other formulae can be derived in a similar way.

Proof of (4.3.27). If we use (4.1.8) on the left hand side and expand the right hand side using geometric series of (4.3.4), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} t_6(n) q^n &= \frac{1}{16} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (2j+1)^2 q^{\frac{j-1}{2}} \left( -q^{\frac{2j+1}{2}} \right)^k \\ &\quad - \frac{1}{16} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^j (2j+1)^2 q^{\frac{j-1}{2}} \left( q^{\frac{2j+1}{2}} \right)^k \\ &= \frac{1}{16} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left[ (-1)^k - (-1)^j \right] (2j+1)^2 q^{[(2j+1)(2k+1)-3]/4}. \end{aligned} \quad (4.3.35)$$

It is clear that when  $j$  and  $k$  are of the same parity then the terms in the sum (4.3.35) become zero. Therefore we have

$$\begin{aligned} &\sum_{n=0}^{\infty} t_6(n) q^n \\ &= \frac{1}{8} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (4j+3)^2 q^{[(4j+3)(4k+1)-3]/4} \\ &\quad - \frac{1}{8} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (4j+1)^2 q^{[(4j+1)(4k+3)-3]/4} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8} \sum_{n=0}^{\infty} \left[ \sum_{(4j+3)(4k+1)=4n+3} (4j+3)^2 - \sum_{(4j+3)(4k+1)=4n+3} (4j+3)^2 \right] q^n \\
&= \frac{1}{8} \sum_{n=0}^{\infty} \left[ \sum_{\substack{d|4n+3 \\ d \equiv 3 \pmod{4}}} d^2 - \sum_{\substack{d|4n+3 \\ d \equiv 1 \pmod{4}}} d^2 \right] q^n.
\end{aligned}$$

Now we equate the coefficient of  $q^n$  on both sides to complete the proof of (4.3.27).

Proof of (4.3.34). If we use (4.1.8) on the left hand side and expand the right hand side using geometric series of (4.3.11), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} t_{(1,1,1,1,2,2)}(n) q^n &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k (j+1)^2 q^{(j+1)(2k+1)-1} \\
&= \sum_{n=0}^{\infty} \left[ \sum_{(j+1)(2k+1)=n+1} (-1)^k (j+1)^2 \right] q^n \\
&= \sum_{n=0}^{\infty} \left[ \sum_{\substack{d|n+1 \\ d \text{ odd}}} (-1)^{\frac{d-1}{2}} \left( \frac{n+1}{d} \right)^2 \right] q^n.
\end{aligned}$$

By comparing coefficients of  $q^n$  on both sides we obtain (4.3.34). ■

**Corollary 4.3.3** *Let the prime factorization of  $n$  be given by (4.2.37). Then*

$$t_2(n) = \prod_{\substack{p|4n+1 \\ p \equiv 1 \pmod{4}}} (\lambda_p + 1) \prod_{\substack{p|4n+1 \\ p \equiv 3 \pmod{4}}} \frac{1 + (-1)^{\lambda_p}}{2}, \quad (4.3.36)$$

$$t_4(n) = \prod_{p|2n+1} \frac{p^{\lambda_p+1} - 1}{p - 1}, \quad (4.3.37)$$

$$t_6(n) = \frac{1}{8} \prod_{p|4n+3} \frac{p^{2\lambda_p+2} - (-1)^{(\lambda_p+1)(p-1)/2}}{p^2 - (-1)^{(p-1)/2}}, \quad (4.3.38)$$

$$t_8(n) = 2^{3\lambda_2} \prod_{p|n+1} \frac{p^{3\lambda_p+3} - 1}{p^3 - 1}, \quad (4.3.39)$$

$$t_{(1,2)}(n) = \frac{1}{2} \prod_{\substack{p|8n+3 \\ p \equiv 1 \text{ or } 3 \pmod{8}}} (\lambda_p + 1) \prod_{\substack{p|8n+3 \\ p \equiv 5 \text{ or } 7 \pmod{8}}} \frac{1 + (-1)^{\lambda_p}}{2}, \quad (4.3.40)$$

$$t_{(1,3)}(n) = \prod_{\substack{p|2n+1 \\ p \equiv 1 \pmod{6}}} (\lambda_p + 1) \prod_{\substack{p|2n+1 \\ p \equiv 5 \pmod{6}}} \frac{1 + (-1)^{\lambda_p}}{2}, \quad (4.3.41)$$

$$t_{(1,4)}(n) = \frac{1}{2} \prod_{\substack{p|8n+5 \\ p \equiv 1 \pmod{4}}} (\lambda_p + 1) \prod_{\substack{p|8n+5 \\ p \equiv 3 \pmod{4}}} \frac{1 + (-1)^{\lambda_p}}{2}, \quad (4.3.42)$$

$$t_{(1,7)}(n) = \prod_{\substack{p|n+1 \\ p \equiv 1, 9 \text{ or } 11 \pmod{14}}} (\lambda_p + 1) \prod_{\substack{p|n+1 \\ p \equiv 3, 5 \text{ or } 13 \pmod{14}}} \frac{1 + (-1)^{\lambda_p}}{2}, \quad (4.3.43)$$

$$t_{(1,1,2,2)}(n) = \frac{1}{4} \prod_{p|4n+3} \frac{p^{\lambda_p+1} - 1}{p - 1}, \quad (4.3.44)$$

$$t_{(1,1,1,1,2,2)}(n) = 2^{2\lambda_2} \prod_{p|n+1} \frac{p^{2\lambda_p+2} - (-1)^{(\lambda_p+1)(p-1)/2}}{p^2 - (-1)^{(p-1)/2}}. \quad (4.3.45)$$

**Proof** The details are similar to the proof of Corollary 4.2.5 . ■

## 4.4 Remarks

Formulae (4.2.3)–(4.2.6) are due to Jacobi [67, pp. 159–170]. Formulae (4.2.7) and (4.2.8) are due to P. G. L. Dirichlet [49] and L. Lorenz [87], respectively. Formulae (4.2.3), (4.2.4) and (4.2.6) can be found in Ramanujan [92, Chapter 17] and he [92, Chapter 17, Entries 8 (iii) and (iv)] also gave different formulae without proof of (4.2.7) and (4.2.8); and Berndt

has given proofs in [19, pp. 115–116]. Formulae (4.2.9) and (4.2.11) are due to Ramanujan [92, Chapter 19, Entry 17 (ii) and Entry 3 (ii)]. Fine [54, pp. 59–76] was the first person to use Ramanujan’s  ${}_1\psi_1$  summation formula to obtain formulae (4.2.3), (4.2.4), (4.2.6)–(4.2.8), and (4.2.11). Proofs of (4.2.3) and (4.2.6) also appeared in Andrews [7]. Bhargava and Adiga [16] and Askey [12] have used Ramanujan’s  ${}_1\psi_1$  summation formula to obtain formulae (4.2.3) or (4.2.4). Berndt [19], [22] has given proofs of (4.2.3)–(4.2.6), (4.2.9), and (4.2.11). Proofs of (4.2.3), (4.2.7)–(4.2.9) also appeared in K. S. Williams [102]. M. D. Hirschhorn [64] has given proofs of (4.2.3), (4.2.4), (4.2.7), and (4.2.8). G. E. Andrews, R. Lewis and Z. G. Liu [10] have employed Bailey’s  ${}_6\psi_6$  summation formula [13] to obtain (4.2.3)–(4.2.9). Cooper [39], Cooper and Lam [47], and Lam [71] have also employed Ramanujan’s  ${}_1\psi_1$  summation formula to obtain formulae (4.2.3)–(4.2.6). Chan [33] has also given a proof of (4.2.5). Cooper and Hirschhorn [44] have used Ramanujan’s  ${}_1\psi_1$  summation formula to obtain formulae (4.2.3), (4.2.7)–(4.2.9). Cooper [42] has also employed Ramanujan’s  ${}_1\psi_1$  summation formula to obtain formulae (4.2.3), (4.2.4), (4.2.6)–(4.2.9).

Formulae (4.2.29)–(4.2.32) were probably known to Jacobi. See [46] for some historical details. Because Ramanujan knew (4.2.7)–(4.2.9) and (4.2.11), he probably also knew (4.2.33)–(4.2.36). Formulae (4.2.33), (4.2.34), and (4.2.36) were given explicitly by Fine [54, pp. 72–76]. We have not been able to find (4.2.35) in print.

Formula (4.2.38) was given by C. F. Gauss [55, p. 149] in 1801. It was also given by Jacobi [67], Ramanujan [93, p. 281] and Hardy and Wright [62, p. 242]. Formulae

(4.2.39)–(4.2.41) were given by Ramanujan [93, pp. 305–307].

Formulae (4.3.2)–(4.3.5) are due to Jacobi [67, pp. 159–170]. Formulae (4.3.2), (4.3.3), and (4.3.5) can be found in Ramanujan [92, Chapter 17]. Ramanujan [92, Chapter 19, Entry 3 (i) and Entry 17 (i)] also gave (4.3.7) and (4.3.9). Fine [54, pp. 73–77] was the first person to use Ramanujan's  ${}_1\psi_1$  summation formula to obtain formulae (4.3.2), (4.3.3), (4.3.6)–(4.3.8), and (4.3.10). Berndt [19], [22] has given proofs of (4.3.2)–(4.3.5), (4.3.7), and (4.3.9). Adiga has given proofs of (4.3.2) and (4.3.3). Cooper and Hirschhorn [44] have employed Ramanujan's  ${}_1\psi_1$  summation formula to obtain formulae (4.3.2), (4.3.7), and (4.3.9). Cooper [42] has also used Ramanujan's  ${}_1\psi_1$  summation formula to obtain formulae (4.3.2), (4.3.3), and (4.3.5). Williams [102] has given proofs of (4.3.7) and (4.3.9). Cooper [39], Cooper and Lam [47], and Lam [71] have also employed Ramanujan's  ${}_1\psi_1$  summation formula to obtain formulae (4.3.2)–(4.3.5).

See [23, pp. 79–84] for some historical comments about (4.3.25)–(4.3.28). It is unclear who first explicitly gave (4.3.29)–(4.3.33). Fine [54, pp. 73–77] gave formulae which are equivalent to (4.3.29)–(4.3.31), and (4.3.33). We have not been able to find (4.3.32) in the literature although undoubtedly it was known to Ramanujan [19, p. 302]. Williams [103] proved the equivalent formula for (4.3.33) but the proof we have given here is simpler and Cooper [40] mentioned that the result can be proved directly using Ramanujan's  ${}_1\psi_1$  summation formula. Hirschhorn [65] has given proofs of (4.3.25), (4.3.26), (4.3.29), and (4.3.31). We think (4.3.34) is new.

Alternative proofs of formulae (4.2.3)–(4.2.6), (4.3.2)–(4.3.5), (4.3.8), (4.3.10), and (4.3.11)



will be presented in Section 3 of Chapter 6. Additional findings on sums of squares and triangular numbers and several results involving both sums of squares and triangular numbers will be derived in Sections 3 and 4 of Chapter 6.

# Chapter 5

## Sixteen series theorems

### 5.1 Introduction

Ramanujan [92, Chapter 17, Entries 13–17] gave several families of identities. For example in [92, Chapter 17, Entry 14 (i)–(iv)], he gave

$$1 + 8 \sum_{k=1}^{\infty} \frac{(-1)^k k q^k}{1 + q^k} = z^2 (1 - x), \quad (5.1.1)$$

$$1 - 16 \sum_{k=1}^{\infty} \frac{(-1)^k k^3 q^k}{1 + q^k} = z^4 (1 - x^2), \quad (5.1.2)$$

$$1 + 8 \sum_{k=1}^{\infty} \frac{(-1)^k k^5 q^k}{1 + q^k} = z^6 (1 - x) (1 - x + x^2), \quad (5.1.3)$$

$$17 - 32 \sum_{k=1}^{\infty} \frac{(-1)^k k^7 q^k}{1 + q^k} = z^8 (1 - x^2) (17 - 32x + 17x^2). \quad (5.1.4)$$

We will show that in general, for  $n \geq 1$ ,

$$\frac{(-1)^n (2^{2n} - 1) B_{2n}}{2n} + 2 (-1)^n \sum_{k=1}^{\infty} \frac{(-1)^k k^{2n-1} q^k}{1 + q^k} = z^{2n} (1 - x) p_{n-1}(x), \quad (5.1.5)$$

where  $p_{n-1}(x)$  is a polynomial in  $x$  with rational coefficients of degree  $n-1$ . We also show that  $p_{n-1}(-1) = 0$  if  $n$  is even and a conjecture is presented for the case  $p_{n-1}(e^{i\pi/3}) = 0$  if  $n$  is a multiple of 3. Ramanujan's results (5.1.1)–(5.1.4) are the special cases  $n = 1, 2, 3, 4$ , respectively, of (5.1.5). He gave fourteen families; in each case giving only the first few examples. We give a total of sixteen infinite families which contain all of Ramanujan's examples. The sixteen families are those of Section 10 of Chapter 2, which arise by

considering derivatives of  $f_0, f_1, f_2, f_3$  at  $0, \pi, \pi\tau, \pi + \pi\tau$ . For example

$$f_1^{(2n-1)}\left(\pi; q^{\frac{1}{2}}\right) = \frac{(-1)^n (2^{2n} - 1) B_{2n}}{2n} + 2(-1)^n \sum_{k=1}^{\infty} \frac{(-1)^k k^{2n-1} q^k}{1 + q^k}.$$

We prove that the functions  $e_1, e_2, e_3$  and Ramanujan's Eisenstein series, namely  $P, Q, R$ , can be expressed in terms of  $z, x$ , and  $dz/dx$ . We also show the function

$$\sum_{\substack{\sigma \in S_3 \\ \sigma = (\sigma_1, \sigma_2, \sigma_3) \\ \{\sigma_1, \sigma_2, \sigma_3\} = \{1, 2, 3\}}} e_{\sigma_1}^{\lambda_1} e_{\sigma_2}^{\lambda_2} e_{\sigma_3}^{\lambda_3},$$

where  $\lambda_1, \lambda_2, \lambda_3 \geq 0$ , is a polynomial in  $Q$  and  $R$  with rational coefficients.

## 5.2 Sixteen recurrence formulae

This section initially establishes recurrence formulae for the sixteen Lambert series by employing the differential equations of  $\wp(\theta)$ ,  $f_1(\theta)$ ,  $f_2(\theta)$ , and  $f_3(\theta)$  in (2.9.21), (2.9.9)–(2.9.11), respectively.

**Theorem 5.2.1** For  $n = 2, 3, 4, \dots$ ,

$$\dot{f}_0^{(2n+3)}(0) = -\frac{6(n+1)(2n+1)}{(n-1)(2n+5)} \sum_{j=1}^{n-1} \binom{2n}{2j} \dot{f}_0^{(2j+1)}(0) \dot{f}_0^{(2n+1-2j)}(0), \quad (5.2.1)$$

$$f_0^{(2n+3)}(\omega) = 12\wp(\omega) f_0^{(2n+1)}(\omega) - 6 \sum_{j=1}^{n-1} \binom{2n}{2j} f_0^{(2j+1)}(\omega) f_0^{(2n+1-2j)}(\omega), \quad (5.2.2)$$

$$\begin{aligned}
\dot{f}_g^{(2n+1)}(0) = & \frac{(2n+1)!}{(2n+3)(n-1)} \left[ \frac{-3\dot{f}_g^{(1)}(0)\dot{f}_g^{(2n-1)}(0)}{(2n-1)!} \right. \\
& + 3 \sum_{\substack{j+k=n+1 \\ j,k \geq 1}} \frac{\dot{f}_g^{(2j-1)}(0)\dot{f}_g^{(2k-1)}(0)}{(2j-1)!(2k-1)!} \\
& \left. + \sum_{\substack{j+k+l=n+1 \\ j,k,l \geq 1}} \frac{\dot{f}_g^{(2j-1)}(0)\dot{f}_g^{(2k-1)}(0)\dot{f}_g^{(2l-1)}(0)}{(2j-1)!(2k-1)!(2l-1)!} \right], \quad (5.2.3)
\end{aligned}$$

where  $\omega = \pi, \pi\tau$  or  $\pi + \pi\tau$ ,  $g = 1, 2$  or  $3$ .

The values of  $f_0^{(1)}, f_0^{(3)}, f_0^{(5)}$  at 0 and  $\omega$  in (5.2.1) and (5.2.2) are

$$\dot{f}_0^{(1)}(0) = -\frac{P}{12}, \quad (5.2.4)$$

$$f_0^{(1)}(\omega) = -\frac{P}{12} - \wp(\omega), \quad (5.2.5)$$

$$\dot{f}_0^{(3)}(0) = -\frac{Q}{120}, \quad (5.2.6)$$

$$f_0^{(3)}(\omega) = -6\wp^2(\omega) + \frac{Q}{24}, \quad (5.2.7)$$

$$\dot{f}_0^{(5)}(0) = -\frac{R}{252}, \quad (5.2.8)$$

$$f_0^{(5)}(\omega) = 12\wp(\omega)f_0^{(3)}(\omega). \quad (5.2.9)$$

For  $n = 0$  and 1 in (5.2.3) we have

$$\dot{f}_g^{(1)}(0) = -\frac{1}{2}e_g, \quad (5.2.10)$$

$$\dot{f}_1^{(3)}(0) = -\frac{1}{120} + 2 \sum_{m=1}^{\infty} \frac{m^3 q^{2m}}{1 + q^{2m}}, \quad (5.2.11)$$

$$\dot{f}_2^{(3)}(0) = \frac{7}{960} - \frac{1}{4} \sum_{m=1}^{\infty} \frac{(2m-1)^3 q^{2m-1}}{1 - q^{2m-1}}, \quad (5.2.12)$$

$$\dot{f}_3^{(3)}(0) = \frac{7}{960} + \frac{1}{4} \sum_{m=1}^{\infty} \frac{(2m-1)^3 q^{2m-1}}{1 + q^{2m-1}}. \quad (5.2.13)$$

**Theorem 5.2.2** For  $n = 3, 4, 5, \dots$ , the recurrence relation formulae for  $f_1^{(2n-1)}(\pi)$ ,  $f_2^{(2n-1)}(\pi\tau)$ , and  $f_3^{(2n-1)}(\pi + \pi\tau)$  are of the form

$$\begin{aligned} f_g^{(2n-1)}(\omega) &= 3e_g f_g^{(2n-3)}(\omega) \\ &+ 2(2n-3)! \sum_{\substack{j+k+l=n \\ 1 \leq j, k, l \leq n-2}} \frac{f_g^{(2j-1)}(\omega) f_g^{(2k-1)}(\omega) f_g^{(2l-1)}(\omega)}{(2j-1)!(2k-1)!(2l-1)!}. \end{aligned} \quad (5.2.14)$$

For  $n = 1$  and  $2$  in (5.2.14), we have

$$f_1^{(1)}(\pi) = -\frac{1}{4} - 2 \sum_{m=1}^{\infty} \frac{(-1)^m m q^{2m}}{1 + q^{2m}}, \quad (5.2.15)$$

$$f_2^{(1)}(\pi\tau) = \sum_{m=1}^{\infty} \frac{(2m-1) q^{m-1/2}}{1 - q^{2m-1}}, \quad (5.2.16)$$

$$f_3^{(1)}(\pi + \pi\tau) = -i \sum_{m=1}^{\infty} \frac{(-1)^m (2m-1) q^{m-1/2}}{1 + q^{2m-1}}, \quad (5.2.17)$$

$$f_g^{(3)}(\omega) = 3e_g f_g^{(1)}(\omega). \quad (5.2.18)$$

**Theorem 5.2.3** For  $n = 3, 4, 5, \dots$ ,

$$\begin{aligned} &f_1^{(2n-2)}(\omega) \\ &= \left[ 3e_1 + 6 \left( \frac{1}{2i} + f_1^{(0)}(\omega) \right)^2 \right] f_1^{(2n-4)}(\omega) \\ &+ 6(2n-4)! \left( \frac{1}{2i} + f_1^{(0)}(\omega) \right) \sum_{\substack{j+k=n-2 \\ 1 \leq j, k \leq n-3}} \frac{f_1^{(2j)}(\omega) f_1^{(2k)}(\omega)}{(2j)!(2k)!} \\ &+ 2(2n-4)! \sum_{\substack{j+k+l=n-2 \\ 1 \leq j, k, l \leq n-4}} \frac{f_1^{(2j)}(\omega) f_1^{(2k)}(\omega) f_1^{(2l)}(\omega)}{(2j)!(2k)!(2l)!}, \end{aligned} \quad (5.2.19)$$

where  $\omega = \pi\tau$  or  $\pi + \pi\tau$ .

For  $n = 1$  and  $2$  in (5.2.19) we have

$$f_1^{(0)}(\pi\tau) = -2i \sum_{m=1}^{\infty} \frac{mq^m}{1+q^{2m}}, \quad (5.2.20)$$

$$f_1^{(0)}(\pi + \pi\tau) = -2i \sum_{m=1}^{\infty} \frac{(-1)^m mq^m}{1+q^{2m}}, \quad (5.2.21)$$

$$f_1^{(2)}(\omega) = 2 \left( \frac{1}{2i} + f_1^{(0)}(\omega) \right)^3 + 3e_1 \left( \frac{1}{2i} + f_1^{(0)}(\omega) \right). \quad (5.2.22)$$

**Theorem 5.2.4** For  $n = 3, 4, 5, \dots$ , the recurrence relation formulae for  $f_2^{(2n-2)}(\pi)$ ,  $f_2^{(2n-2)}(\pi + \pi\tau)$ ,  $f_3^{(2n-2)}(\pi)$ , and  $f_3^{(2n-2)}(\pi\tau)$  are of the form

$$f_g^{(2n-2)}(\omega) = 3e_g f_g^{(2n-4)}(\omega) + 2(2n-4)! \sum_{\substack{j+k+l=n-2 \\ 0 \leq j,k,l \leq n-2}} \frac{f_g^{(2j)}(\omega) f_g^{(2k)}(\omega) f_g^{(2l)}(\omega)}{(2j)!(2k)!(2l)!}. \quad (5.2.23)$$

Setting  $n = 1$  and  $2$  in (5.2.23) gives

$$f_2^{(0)}(\pi) = \frac{1}{2} - 2 \sum_{m=1}^{\infty} \frac{(-1)^m q^{2m-1}}{1 - q^{2m-1}}, \quad (5.2.24)$$

$$f_2^{(0)}(\pi + \pi\tau) = -2 \sum_{m=1}^{\infty} \frac{(-1)^m q^{m-1/2}}{1 - q^{2m-1}}, \quad (5.2.25)$$

$$f_3^{(0)}(\pi) = \frac{1}{2} + 2 \sum_{m=1}^{\infty} \frac{(-1)^m q^{2m-1}}{1 + q^{2m-1}}, \quad (5.2.26)$$

$$f_3^{(0)}(\pi\tau) = -2i \sum_{m=1}^{\infty} \frac{q^{m-1/2}}{1 + q^{2m-1}}, \quad (5.2.27)$$

$$f_g^{(2)}(\omega) = 3e_g f_g^{(0)}(\omega) + 2 [f_g^{(0)}(\omega)]^3. \quad (5.2.28)$$

**Proof** The proofs of Theorems 5.2.1–5.2.4 follow from the differential equations of  $\wp(\theta)$ ,  $f_1(\theta)$ ,  $f_2(\theta)$ , and  $f_3(\theta)$  in (2.9.21), (2.9.9)–(2.9.11), respectively. First we expand both sides of (2.9.21), (2.9.9)–(2.9.11) in powers of  $\theta$ , equate coefficients of  $\theta^{2n-1}$  on both sides and then simplify the result to obtain the recurrence relation formulae for  $\dot{f}_0^{(2n+3)}(0)$ ,  $\dot{f}_1^{(2n+3)}(0)$ ,  $\dot{f}_2^{(2n+3)}(0)$ ,  $\dot{f}_3^{(2n+3)}(0)$ , respectively, where  $n \geq 2$ . We obtain another twelve recurrence relations formulae in a similar way by replacing  $\theta$  with  $\theta + \pi$ ,  $\theta + \pi\tau$ ,  $\theta + \pi + \pi\tau$  in (2.9.21), (2.9.9)–(2.9.11), respectively.

Next we give complete details for the proof of (5.2.3). The other formulae can be derived in a similar manner.

All of (5.2.10)–(5.2.13) follow from (2.10.2)–(2.10.4). By expanding both sides of (2.9.9) in powers of  $\theta$  and then using (5.2.10) we find that

$$\begin{aligned} & \frac{2}{\theta^3} + \sum_{n=1}^{\infty} \dot{f}_g^{(2n+1)}(0) \frac{\theta^{2n-1}}{(2n-1)!} \\ &= 2 \left[ \frac{1}{\theta} + \sum_{n=1}^{\infty} \dot{f}_g^{(2n-1)}(0) \frac{\theta^{2n-1}}{(2n-1)!} \right]^3 \\ & \quad - 6 \dot{f}_g^{(1)}(0) \left[ \frac{1}{\theta} + \sum_{n=1}^{\infty} \dot{f}_g^{(2n-1)}(0) \frac{\theta^{2n-1}}{(2n-1)!} \right]. \end{aligned} \quad (5.2.29)$$

We observe that

$$\begin{aligned} & \left[ \frac{1}{\theta} + \sum_{m=1}^{\infty} \dot{f}_g^{(2m-1)}(0) \frac{\theta^{2m-1}}{(2m-1)!} \right]^3 \\ &= \theta^{-3} + \frac{3 \dot{f}_g^{(1)}(0)}{1!} \theta^{-1} + \sum_{m=1}^{\infty} \left\{ 3 \frac{\dot{f}_g^{(2m+1)}(0)}{(2m+1)!} + 3 \sum_{\substack{j+k=m+1 \\ j,k \geq 1}} \frac{\dot{f}_g^{(2j-1)}(0) \dot{f}_g^{(2k-1)}(0)}{(2j-1)! (2k-1)!} \right. \\ & \quad \left. + \sum_{\substack{j+k+l=m+1 \\ j,k,l \geq 1}} \frac{\dot{f}_g^{(2j-1)}(0) \dot{f}_g^{(2k-1)}(0) \dot{f}_g^{(2l-1)}(0)}{(2j-1)! (2k-1)! (2l-1)!} \right\} \theta^{2m-1}. \end{aligned} \quad (5.2.30)$$

Now by substituting (5.2.30) into (5.2.29), equating coefficients of  $\theta^{2m-1}$  on both sides and then simplifying the result, the proof of (5.2.3) is completed.

This completes the proofs of Theorems 5.2.1–5.2.4. ■

Ramanujan [62, p. 140] gave a formula equivalent to (5.2.1).

**Corollary 5.2.5** For  $n = 1, 2, 3, \dots$ ,

$$\dot{f}_0^{(2n+1)}(0) = \sum_{\substack{4r+6s=2n+2 \\ r,s \geq 0}} \mathcal{A}_{r,s} \left[ \dot{f}_0^{(3)}(0) \right]^r \left[ \dot{f}_0^{(5)}(0) \right]^s, \quad (5.2.31)$$

$$\dot{f}_g^{(2n-1)}(0) = \sum_{\substack{2r+4s=2n \\ r,s \geq 0}} \mathcal{B}_{r,s} \left[ \dot{f}_g^{(1)}(0) \right]^r \left[ \dot{f}_g^{(3)}(0) \right]^s, \quad (5.2.32)$$

$$f_0^{(2n+3)}(\omega) = \sum_{i=1}^{[(n+2)/2]} \mathcal{C}_{n,i} \left[ f_0^{(3)}(\omega) \right]^i [\wp(\omega)]^{n+2-2i}, \quad (5.2.33)$$

$$f_1^{(2n-1)}(\pi) = \sum_{i=1}^{[(n+1)/2]} \mathcal{D}_{n,i} \left[ f_1^{(1)}(\pi) \right]^{2i-1} e_1^{n+1-2i}, \quad (5.2.34)$$

$$f_2^{(2n-1)}(\pi\tau) = \sum_{i=1}^{[(n+1)/2]} \mathcal{E}_{n,i} \left[ f_2^{(1)}(\pi\tau) \right]^{2i-1} e_2^{n+1-2i}, \quad (5.2.35)$$

$$f_3^{(2n-1)}(\pi + \pi\tau) = \sum_{i=1}^{[(n+1)/2]} \mathcal{F}_{n,i} \left[ f_3^{(1)}(\pi + \pi\tau) \right]^{2i-1} e_3^{n+1-2i}, \quad (5.2.36)$$

$$f_1^{(2n)}(\omega) = \sum_{i=1}^{n+1} \mathcal{G}_{n,i} \left[ \frac{1}{2i} + f_1^{(0)}(\omega) \right]^{2i-1} e_1^{n+1-i}, \quad (5.2.37)$$

$$f_2^{(2n)}(\pi) = \sum_{i=1}^{n+1} \mathcal{H}_{n,i} \left[ f_2^{(0)}(\pi) \right]^{2i-1} e_2^{n+1-i}, \quad (5.2.38)$$

$$f_2^{(2n)}(\pi + \pi\tau) = \sum_{i=1}^{n+1} \mathcal{I}_{n,i} \left[ f_2^{(0)}(\pi + \pi\tau) \right]^{2i-1} e_2^{n+1-i}, \quad (5.2.39)$$

$$f_3^{(2n)}(\pi) = \sum_{i=1}^{n+1} \mathcal{J}_{n,i} \left[ f_3^{(0)}(\pi) \right]^{2i-1} e_3^{n+1-i}, \quad (5.2.40)$$



$$f_3^{(2n)}(\pi\tau) = \sum_{i=1}^{n+1} \mathcal{K}_{n,i} \left[ f_3^{(0)}(\pi\tau) \right]^{2i-1} e_3^{n+1-i}, \quad (5.2.41)$$

where  $g = 1, 2$  or  $3$ ;  $\omega = \pi, \pi\tau$  or  $\pi + \pi\tau$ ;  $\mathcal{A}_{r,s}, \mathcal{B}_{r,s}, \mathcal{C}_{n,i}, \mathcal{D}_{n,i}, \mathcal{E}_{n,i}, \mathcal{F}_{n,i}, \mathcal{G}_{n,i}, \mathcal{H}_{n,i}, \mathcal{I}_{n,i}, \mathcal{J}_{n,i}$ , and  $\mathcal{K}_{n,i}$  are constants.

**Proof** We give details for the proof of (5.2.34) only. The other formulae can be proved in a similar way.

(5.2.34) is true for  $n = 1$  and, by (5.2.14), it is true for  $n = 2$ . Suppose it is true for all the values  $n = 1, 2, 3, \dots, m$ . We show it is true for  $n = m + 1$  as well. Then

$$\begin{aligned} f_1^{(2m+1)}(\pi) &= 3f_1^{(2m-1)}(\pi)e_1 \\ &\quad + 2(2m-1)! \sum_{\substack{j+k+l=m+1 \\ j,k,l \geq 1}} \frac{f_1^{(2j-1)}(\pi) f_1^{(2k-1)}(\pi) f_1^{(2l-1)}(\pi)}{(2j-1)!(2k-1)!(2l-1)!}. \end{aligned} \quad (5.2.42)$$

By the induction hypothesis, we observe that

$$\begin{aligned} f_1^{(2j-1)}(\pi) f_1^{(2k-1)}(\pi) f_1^{(2l-1)}(\pi) &= \left( \sum_{1 \leq 2i_1-1 \leq j} \mathcal{D}_{j,i_1} \left[ f_1^{(1)}(\pi) \right]^{2i_1-1} e_1^{j+1-2i_1} \right) \\ &\quad \times \left( \sum_{1 \leq 2i_2-1 \leq k} \mathcal{D}_{k,i_2} \left[ f_1^{(1)}(\pi) \right]^{2i_2-1} e_1^{k+1-2i_2} \right) \\ &\quad \times \left( \sum_{1 \leq 2i_3-1 \leq l} \mathcal{D}_{l,i_3} \left[ f_1^{(1)}(\pi) \right]^{2i_3-1} e_1^{l+1-2i_3} \right), \end{aligned}$$

where  $j + k + l = n + 1$ . This is a linear combination of terms of the form

$$\left[ f_1^{(1)}(\pi) \right]^{2(i_1+i_2+i_3-1)-1} e^{(j+k+l+1)+1-2(i_1+i_2+i_3-1)},$$

that is  $\left[ f_1^{(1)}(\pi) \right]^{2i-1} e^{n+2-2i}$ , where  $i = i_1 + i_2 + i_3 - 1$ . Multiply by 2 and rearrange

$2i - 1 = (2i_1 - 1) + (2i_2 - 1) + (2i_3 - 1)$ , implying  $3 \leq 2i - 1 \leq m + 1$ .

Similarly  $3f_1^{(2m-1)}(\pi) e_1$  involves  $\left[f_1^{(1)}(\pi)\right]^{2i-1} e^{n+2-2i}$ ,  $1 \leq 2i-1 \leq m$ . Hence (5.2.42)

follows from (5.2.34).

Hence if the statement is true for  $n = m$  then it is true for  $n = m + 1$ . But it is true for  $n = 3$  therefore it is true for  $n = 3, 4$ , and so on. Therefore it is true for  $n = 3, 4, 5, \dots$

This completes the proof of Corollary 5.2.5 by induction. ■

### 5.3 The first few values of sixteen Lambert series in terms of $z$ , $x$ , and $dz/dx$

The aim of this section is to rewrite equations (a) (5.2.4)–(5.2.9); (b) (5.2.10)–(5.2.13); (c) (5.2.15)–(5.2.18); (d) (5.2.24)–(5.2.28); (e)  $e_1, e_2$ , and  $e_3$ ; (f)  $P, Q$ , and  $R$  in terms of  $z, x$ , and  $dz/dx$ .

The following lemmas are very important for later computations.

#### Lemma 5.3.1

$$f_1^{(1)}(\pi) = -\frac{1}{4}\varphi^4(-q^2), \quad (5.3.1)$$

$$f_1^{(0)}(\pi\tau) = \frac{i}{2}[1 - \varphi^2(q)], \quad (5.3.2)$$

$$f_1^{(0)}(\pi + \pi\tau) = f_1^{(0)}(\pi\tau; -q), \quad (5.3.3)$$

$$f_2^{(0)}(\pi) = \frac{1}{2}\varphi^2(q), \quad (5.3.4)$$

$$f_2^{(1)}(\pi\tau) = q^{\frac{1}{2}}\psi^4(q), \quad (5.3.5)$$

$$f_2^{(0)}(\pi + \pi\tau) = 2q^{\frac{1}{2}}\psi^2(q^2), \quad (5.3.6)$$

$$f_3^{(0)}(\pi) = f_2^{(0)}(\pi; -q), \quad (5.3.7)$$

$$f_3^{(0)}(\pi\tau) = -f_2^{(0)}(\pi + \pi\tau; -q), \quad (5.3.8)$$

$$f_3^{(1)}(\pi + \pi\tau) = -f_2^{(1)}(\pi\tau; -q). \quad (5.3.9)$$

**Proof** For the proofs of (5.3.1), (5.3.2), (5.3.4)–(5.3.6), we use (4.2.4), (4.2.6), (4.3.3), and (4.3.5).

By replacing  $q$  with  $-q$  in (5.2.20), (5.2.24), (5.2.25) and (5.2.16), we have (5.3.3), (5.3.7)–(5.3.9), respectively. ■

Using (3.5.1), (4.2.17), and (4.3.16), Lemma 5.3.1 can be rewritten as

$$f_1^{(1)}(\pi) = -\frac{1}{4}z^2\sqrt{1-x}, \quad (5.3.10)$$

$$f_1^{(0)}(\pi\tau) = \frac{i}{2}(1-z), \quad (5.3.11)$$

$$f_1^{(0)}(\pi + \pi\tau) = \frac{i}{2}[1 - z\sqrt{1-x}], \quad (5.3.12)$$

$$f_2^{(0)}(\pi) = \frac{1}{2}z, \quad (5.3.13)$$

$$f_2^{(1)}(\pi\tau) = \frac{1}{4}z^2\sqrt{x}, \quad (5.3.14)$$

$$f_2^{(0)}(\pi + \pi\tau) = \frac{1}{2}z\sqrt{x}, \quad (5.3.15)$$

$$f_3^{(0)}(\pi) = \frac{1}{2}z\sqrt{1-x}, \quad (5.3.16)$$

$$f_3^{(0)}(\pi\tau) = -\frac{i}{2}z\sqrt{x}, \quad (5.3.17)$$

$$f_3^{(1)}(\pi + \pi\tau) = -\frac{i}{4}z^2\sqrt{x(1-x)}. \quad (5.3.18)$$

### Lemma 5.3.2

$$e_1 = \frac{1}{12}z^2(2-x), \quad (5.3.19)$$

$$e_2 = \frac{-1}{12} z^2 (1 + x), \quad (5.3.20)$$

$$e_3 = \frac{-1}{12} z^2 (1 - 2x). \quad (5.3.21)$$

**Proof** By using (4.2.17) and (4.3.16) in the right hand side of equation (5.3.19), we have

$$\frac{1}{12} z^2 (2 - x) = \frac{1}{6} \phi^4(q) - \frac{4q}{3} \psi^4(q^2). \quad (5.3.22)$$

If we use (4.2.4) and replace  $q$  with  $q^2$  in (4.3.3) then the right hand side of (5.3.22) becomes

$$\begin{aligned} \frac{1}{6} \phi^4(q) - \frac{4q}{3} \psi^4(q^2) &= \frac{1}{6} + \frac{4}{3} \sum_{m=1}^{\infty} \frac{mq^m}{1 + (-q)^m} - \frac{4}{3} \sum_{m=1}^{\infty} \frac{(2m-1)q^{2m-1}}{1 - q^{4m-2}} \\ &= \frac{1}{6} + \frac{4}{3} \sum_{m=1}^{\infty} \frac{(2m-1)q^{2m-1}}{1 - q^{2m-1}} + \frac{4}{3} \sum_{m=1}^{\infty} \frac{2mq^{2m}}{1 + q^{2m}} \\ &\quad - \frac{4}{3} \sum_{m=1}^{\infty} \frac{mq^m}{1 - q^{2m}} + \frac{4}{3} \sum_{m=1}^{\infty} \frac{2mq^{2m}}{1 - q^{4m}}. \end{aligned}$$

By applying the trivial identity

$$\frac{nq^n}{1 - q^n} + \frac{nq^n}{1 + q^n} = \frac{2nq^n}{1 - q^{2n}}, \quad (5.3.23)$$

we have

$$\begin{aligned} \frac{1}{6} \phi^4(q) - \frac{4q}{3} \psi^4(q^2) &= \frac{1}{6} + \frac{2}{3} \sum_{m=1}^{\infty} \frac{mq^m}{1 - q^m} - \frac{2}{3} \sum_{m=1}^{\infty} \frac{mq^m}{1 + q^m} \\ &\quad + \frac{4}{3} \sum_{m=1}^{\infty} \frac{mq^{2m}}{1 - q^{2m}} + 4 \sum_{m=1}^{\infty} \frac{mq^{2m}}{1 + q^{2m}} \\ &= \frac{1}{6} + 4 \sum_{m=1}^{\infty} \frac{mq^{2m}}{1 + q^{2m}}. \end{aligned} \quad (5.3.24)$$

Substitution of (5.3.24) into (5.3.22) and use of (2.7.11) proves (5.3.19). The proofs of (5.3.20) and (5.3.21) can be derived in a similar manner, using the results of (4.2.17) and

(4.3.16) on the right hand side of (5.3.20) and (5.3.21), and finally by manipulating the series in them. This completes the proof. ■

Numerous identities can be derived in terms of  $e_1, e_2, e_3$ , serving as a translation between  $e_1, e_2, e_3$ , and  $z, x$ .

Next, by employing Lemma 5.3.1 and Lemma 5.3.2, (5.2.10), (5.2.18), (5.2.22), and (5.2.28) may be rewritten as

$$\dot{f}_1^{(1)}(0) = -\frac{1}{24}z^2(2-x), \quad (5.3.25)$$

$$f_1^{(3)}(\pi) = -\frac{1}{16}z^4\sqrt{1-x}(2-x), \quad (5.3.26)$$

$$f_1^{(2)}(\pi\tau) = \frac{i}{8}z^3x, \quad (5.3.27)$$

$$f_1^{(2)}(\pi + \pi\tau) = -\frac{i}{8}z^3x\sqrt{1-x}; \quad (5.3.28)$$

$$\dot{f}_2^{(1)}(0) = \frac{1}{24}z^2(1+x), \quad (5.3.29)$$

$$f_2^{(2)}(\pi) = \frac{1}{8}z^3(1-x), \quad (5.3.30)$$

$$f_2^{(3)}(\pi\tau) = -\frac{1}{16}z^4\sqrt{x}(1+x), \quad (5.3.31)$$

$$f_2^{(2)}(\pi + \pi\tau) = -\frac{1}{8}z^3\sqrt{x}(1-x); \quad (5.3.32)$$

$$\dot{f}_3^{(1)}(0) = \frac{1}{24}z^2(1-2x), \quad (5.3.33)$$

$$f_3^{(2)}(\pi) = \frac{1}{8}z^3\sqrt{1-x}, \quad (5.3.34)$$

$$f_3^{(2)}(\pi\tau) = \frac{i}{8}z^3\sqrt{x}, \quad (5.3.35)$$

$$f_3^{(3)}(\pi + \pi\tau) = \frac{i}{16}z^4\sqrt{x(1-x)}(1-2x). \quad (5.3.36)$$

Ramanujan has given three classical Eisenstein series, namely  $P, Q$ , and  $R$  in terms of  $E, K, z, x$ , and  $dz/dx$ ; and Lambert series expansion in his paper [94, p. 140]. The proofs are presented in the following three lemmas.

**Lemma 5.3.3**

$$1 - 24 \sum_{m=1}^{\infty} \frac{mq^{2m}}{1 - q^{2m}} = z^2 \left( \frac{3E}{K} + x - 2 \right). \quad (5.3.37)$$

**Proof** Using the results of (2.11.25) and (5.3.24) the right hand side of (5.3.37) becomes

$$z^2 \left( \frac{3E}{K} + x - 2 \right) = 1 - 24 \sum_{m=1}^{\infty} \frac{(-1)^m mq^{2m}}{1 - q^{2m}} - 48 \sum_{m=1}^{\infty} \frac{mq^{2m}}{1 + q^{2m}}. \quad (5.3.38)$$

Using (5.3.23), equation (5.3.38) becomes

$$\begin{aligned} z^2 \left( \frac{3E}{K} + x - 2 \right) &= 1 - 24 \sum_{m=1}^{\infty} \frac{2mq^{4m}}{1 - q^{4m}} + 24 \sum_{m=1}^{\infty} \frac{(2m-1)q^{4m-2}}{1 - q^{4m-2}} \\ &\quad - 48 \sum_{m=1}^{\infty} \frac{2mq^{2m}}{1 - q^{4m}} + 48 \sum_{m=1}^{\infty} \frac{mq^{2m}}{1 - q^{2m}} \\ &= 1 - 48 \sum_{m=1}^{\infty} \frac{2mq^{4m}}{1 - q^{4m}} + 24 \sum_{m=1}^{\infty} \frac{mq^{2m}}{1 - q^{2m}} \\ &\quad - 48 \sum_{m=1}^{\infty} \frac{2mq^{2m}}{1 - q^{4m}} + 48 \sum_{m=1}^{\infty} \frac{mq^{2m}}{1 - q^{2m}} \\ &= 1 - 96 \sum_{m=1}^{\infty} \frac{mq^{2m}(1 + q^{2m})}{1 - q^{4m}} + 72 \sum_{m=1}^{\infty} \frac{mq^{2m}}{1 - q^{2m}} \\ &= 1 - 24 \sum_{m=1}^{\infty} \frac{mq^{2m}}{1 - q^{2m}}, \end{aligned}$$

which completes the proof. ■

Note that by using (2.11.27), equation (5.3.37) may be represented as

$$P = (1 - 2x) z^2 + 6x(1 - x) z \frac{dz}{dx}. \quad (5.3.39)$$

Ramanujan has given (5.3.39) in [92, Chapter 17, Entry 9 (iv)] and Berndt [19, pp. 121–122] has given a proof of (5.3.39).

If we use Lemma 5.3.2 and (5.3.39), equations (5.2.4) and (5.2.5) can be rewritten as

$$f_0^{(1)}(0) = \frac{(1 - 2x) z^2}{12} + \frac{1}{2} x(1 - x) z \frac{dz}{dx}, \quad (5.3.40)$$

$$f_0^{(1)}(\pi) = -\frac{(1 - x) z^2}{4} - \frac{1}{2} x(1 - x) z \frac{dz}{dx}, \quad (5.3.41)$$

$$f_0^{(1)}(\pi\tau) = \frac{z^2 x}{4} - \frac{1}{2} x(1 - x) z \frac{dz}{dx}, \quad (5.3.42)$$

$$f_0^{(1)}(\pi + \pi\tau) = -\frac{1}{2} x(1 - x) z \frac{dz}{dx}. \quad (5.3.43)$$

#### Lemma 5.3.4

$$1 + 240 \sum_{m=1}^{\infty} \frac{m^3 q^{2m}}{1 - q^{2m}} = z^4 (1 - x + x^2), \quad (5.3.44)$$

$$1 - 240 \sum_{m=1}^{\infty} \frac{m^3 q^{2m}}{1 + q^{2m}} = \frac{1}{8} z^4 (8 - 8x - 7x^2), \quad (5.3.45)$$

$$7 - 240 \sum_{m=1}^{\infty} \frac{(2m - 1)^3 q^{2m-1}}{1 - q^{2m-1}} = z^4 (7 - 22x + 7x^2), \quad (5.3.46)$$

$$7 + 240 \sum_{m=1}^{\infty} \frac{(2m - 1)^3 q^{2m-1}}{1 + q^{2m-1}} = z^4 (7 + 8x - 8x^2). \quad (5.3.47)$$

**Proof** Observe that

$$\begin{aligned}
 & \varphi^8(-q^2) + 256q^2\psi^8(q^2) \\
 = & 1 - 16 \sum_{m=1}^{\infty} \frac{m^3 q^{2m} (-1)^{m-1}}{1 - q^{2m}} + 256 \sum_{m=1}^{\infty} \frac{m^3 q^{2m}}{1 - q^{4m}} \\
 = & 1 + 128 \sum_{m=1}^{\infty} \frac{m^3 q^{4m}}{1 - q^{4m}} - 16 \sum_{m=1}^{\infty} \frac{(2m-1)^3 q^{4m-2}}{1 - q^{4m-2}} + 256 \sum_{m=1}^{\infty} \frac{m^3 q^{2j}(1 + q^{2m} - q^{2m})}{1 - q^{4m}} \\
 = & 1 - 16 \sum_{m=1}^{\infty} \frac{(2m-1)^3 q^{4m-2}}{1 - q^{4m-2}} - 16 \sum_{m=1}^{\infty} \frac{(2m)^3 q^{4m}}{1 - q^{4m}} + 256 \sum_{m=1}^{\infty} \frac{m^3 q^{2m}}{1 - q^{2m}} \\
 = & 1 - 16 \sum_{m=1}^{\infty} \frac{m^3 q^{2m}}{1 - q^{2m}} + 256 \sum_{m=1}^{\infty} \frac{m^3 q^{2m}}{1 - q^{2m}} \\
 = & 1 + 240 \sum_{m=1}^{\infty} \frac{m^3 q^{2m}}{1 - q^{2m}}. \tag{5.3.48}
 \end{aligned}$$

Using (3.5.1) and (3.5.12) we see that

$$\varphi^8(-q^2) = z^4(1-x), \tag{5.3.49}$$

$$256q^2\psi^8(q^2) = z^4x^2. \tag{5.3.50}$$

Substitution of (5.3.49) and (5.3.50) into (5.3.48), gives (5.3.44).

The proof of (5.3.45) follows the same procedure by expressing

$$\varphi^8(-q^2) - 224q^2\psi^8(q^2),$$

as a Lambert series and expressing in terms of  $z$  and  $x$ .

From the definition (2.9.17), (5.3.44) can be represented as

$$Q = 1 + 240 \sum_{m=1}^{\infty} \frac{m^3 q^{2m}}{1 - q^{2m}} = z^4(1 - x + x^2). \tag{5.3.51}$$



Next we observe that

$$\begin{aligned} 8Q - Q\left(q^{\frac{1}{2}}\right) &= 7 + 240 \sum_{m=1}^{\infty} \frac{(2m)^3 q^{2m}}{1 - q^{2m}} - 240 \sum_{m=1}^{\infty} \frac{m^3 q^m}{1 - q^m} \\ &= 7 - 240 \sum_{m=1}^{\infty} \frac{(2m-1)^3 q^{2m-1}}{1 - q^{2m-1}}. \end{aligned} \quad (5.3.52)$$

Using (3.5.1) and (5.3.44), we have

$$8Q - Q\left(q^{\frac{1}{2}}\right) = z^4 (7 - 22x + 7x^2). \quad (5.3.53)$$

Substitution of (5.3.53) into (5.3.52), completes the proof of (5.3.46).

The proof of (5.3.47) leads by replacing  $q$  with  $q^{\frac{1}{2}}$  in the left hand side of (5.3.46) and using (3.5.1) for the right hand side. ■

Using Lemmas 5.3.2 and 5.3.4, equations (5.2.6), (5.2.7), and (5.2.11)–(5.2.13) may be rewritten as

$$\dot{f}_0^{(3)}(0) = -\frac{1}{120} z^4 (1 - x + x^2), \quad (5.3.54)$$

$$f_0^{(3)}(\pi) = -\frac{1}{8} z^4 (1 - x), \quad (5.3.55)$$

$$f_0^{(3)}(\pi\tau) = -\frac{1}{8} z^4 x, \quad (5.3.56)$$

$$f_0^{(3)}(\pi + \pi\tau) = \frac{1}{8} z^4 x (1 - x), \quad (5.3.57)$$

$$\dot{f}_1^{(3)}(0) = -\frac{1}{960} z^4 (8 - 8x - 7x^2), \quad (5.3.58)$$

$$\dot{f}_2^{(3)}(0) = \frac{1}{960} z^4 (7 - 22x + 7x^2), \quad (5.3.59)$$

$$\dot{f}_3^{(3)}(0) = \frac{1}{960} z^4 (7 + 8x - 8x^2). \quad (5.3.60)$$

Using Lemma 5.3.2, equations (5.3.55)–(5.3.57), equation (5.2.9) can be rewritten as

$$f_0^{(5)}(\pi) = -\frac{1}{8}z^6(1-x)(2-x), \quad (5.3.61)$$

$$f_0^{(5)}(\pi\tau) = \frac{1}{8}z^6x(1+x), \quad (5.3.62)$$

$$f_0^{(5)}(\pi + \pi\tau) = -\frac{1}{8}z^6x(1-x)(1-2x). \quad (5.3.63)$$

**Lemma 5.3.5**

$$1 - 504 \sum_{m=1}^{\infty} \frac{m^5 q^{2m}}{1 - q^{2m}} = z^6(1+x)(1-2x) \left(1 - \frac{x}{2}\right). \quad (5.3.64)$$

**Proof** Setting  $j = 3$  into (2.10.18) gives

$$f_0^{(5)}(\pi) + f_0^{(5)}(\pi\tau) + f_0^{(5)}(\pi + \pi\tau) = 63f_0^{(5)}(0). \quad (5.3.65)$$

By substituting (5.3.61)–(5.3.63) into the left hand side of (5.3.65) and using (2.10.1) to rewrite the right hand side as a Lambert series and then rearranging, we complete the proof. ■

By Lemma 5.3.5, equation (5.3.65) may be rewritten as

$$f_0^{(5)}(0) = -\frac{1}{252}z^6(1+x)(1-2x) \left(1 - \frac{x}{2}\right). \quad (5.3.66)$$

We summarise the results of functions  $P$ ,  $Q$ , and  $R$ . By (2.7.4), (2.9.23), (5.3.37), (5.3.39), (5.3.51), and (5.3.64), we have shown that

$$P = 1 - 24 \sum_{m=1}^{\infty} \frac{mq^{2m}}{1 - q^{2m}} \quad (5.3.67)$$

$$= z^2 \left( \frac{3E}{K} + x - 2 \right) \quad (5.3.68)$$

$$= (1 - 2x) z^2 + 6x(1 - x) z \frac{dz}{dx}, \quad (5.3.69)$$

$$Q = 1 + 240 \sum_{m=1}^{\infty} \frac{m^3 q^{2m}}{1 - q^{2m}} \quad (5.3.70)$$

$$= z^4 (1 - x + x^2), \quad (5.3.71)$$

and

$$R = 1 - 504 \sum_{m=1}^{\infty} \frac{m^5 q^{2m}}{1 - q^{2m}} \quad (5.3.72)$$

$$= z^6 (1 + x)(1 - 2x) \left( 1 - \frac{x}{2} \right). \quad (5.3.73)$$

Next we let

$$\sum_{\substack{\sigma \in S_3 \\ \sigma = (\sigma_1, \sigma_2, \sigma_3) \\ \{\sigma_1, \sigma_2, \sigma_3\} = \{1, 2, 3\}}} e_{\sigma_1}^a e_{\sigma_2}^b e_{\sigma_3}^c := f_{a,b,c}(e_1, e_2, e_3), \quad a, b, c \geq 0. \quad (5.3.74)$$

For example from (5.3.19)–(5.3.21), (5.3.71), and (5.3.73), it is easy to show that

$$f_{2,0,0}(e_1, e_2, e_3) = 2(e_1^2 + e_2^2 + e_3^2) = \frac{Q}{12}, \quad (5.3.75)$$

$$f_{1,1,0}(e_1, e_2, e_3) = 2(e_1 e_2 + e_1 e_3 + e_2 e_3) = -96Q, \quad (5.3.76)$$

$$f_{1,1,1}(e_1, e_2, e_3) = 6e_1 e_2 e_3 = \frac{R}{144}. \quad (5.3.77)$$

The following two lemmas are needed to prove Theorem 5.3.8.

**Lemma 5.3.6** For  $n \geq 1$ ,

$$f_{n,0,0}(e_1, e_2, e_3) = \sum_{\substack{2j+3k=n \\ j,k \geq 0}} c_{j,k} Q^j R^k, \quad (5.3.78)$$

where  $c_{j,k}$  is a rational number.

**Proof** The statement is true for  $n = 1$  and  $2$  by (2.7.14) and (5.3.75), respectively. Suppose it is true for all values  $n = 1, 2, 3, 4, \dots, m$ . We show it is true for  $n = m + 1$ . It can be shown that

$$\begin{aligned} & 3f_{1,1,0}(e_1, e_2, e_3) f_{m-1,0,0}(e_1, e_2, e_3) \\ &= -6f_{m+1,0,0}(e_1, e_2, e_3) + f_{1,1,1}(e_1, e_2, e_3) f_{m-2,0,0}(e_1, e_2, e_3). \end{aligned} \quad (5.3.79)$$

By (5.3.76) and (5.3.77), equation (5.3.79) can be represented as

$$f_{m+1,0,0}(e_1, e_2, e_3) = 48Q f_{m-1,0,0}(e_1, e_2, e_3) + \frac{R}{864} f_{m-2,0,0}(e_1, e_2, e_3).$$

Using the induction hypothesis, we obtain

$$\begin{aligned} f_{m+1,0,0}(e_1, e_2, e_3) &= 48Q \sum_{\substack{2j+3k=m-1 \\ j,k \geq 0}} c'_{j,k} Q^j R^k + \frac{R}{864} \sum_{\substack{2j+3k=m-2 \\ j,k \geq 0}} c''_{j,k} Q^j R^k \\ &= \sum_{\substack{2j+3k=m+1 \\ j,k \geq 0}} c_{j,k} Q^j R^k. \end{aligned}$$

Hence the statement of (5.3.78) is true for  $n = m + 1$ . ■

### Lemma 5.3.7

$$\begin{aligned} f_{\lambda_1, \lambda_2, 0}(e_1, e_2, e_3) &= \frac{1}{4} f_{\lambda_1, 0, 0}(e_1, e_2, e_3) f_{\lambda_2, 0, 0}(e_1, e_2, e_3) \\ &\quad - \frac{1}{2} f_{\lambda_1 + \lambda_2, 0, 0}(e_1, e_2, e_3), \end{aligned} \quad (5.3.80)$$

$$\begin{aligned} \frac{1}{2} f_{\lambda_1, \lambda_2, 0}(e_1, e_2, e_3) f_{\lambda_3, 0, 0}(e_1, e_2, e_3) &= f_{\lambda_1, \lambda_2 + \lambda_3, 0}(e_1, e_2, e_3) \\ &\quad + f_{\lambda_2, \lambda_1 + \lambda_3, 0}(e_1, e_2, e_3) + f_{\lambda_1, \lambda_2, \lambda_3}(e_1, e_2, e_3). \end{aligned} \quad (5.3.81)$$

**Proof** By (5.3.74), it is straightforward to check that

$$f_{\lambda_1,0,0}(e_1, e_2, e_3) f_{\lambda_2,0,0}(e_1, e_2, e_3) = 2f_{\lambda_1+\lambda_2,0,0}(e_1, e_2, e_3) + 4f_{\lambda_1,\lambda_2,0}(e_1, e_2, e_3).$$

Rearrangement proves the first part.

The second part follows from the first part. ■

The following theorem can now be proven.

**Theorem 5.3.8** For  $\lambda_1 + \lambda_2 + \lambda_3 \geq 1$ ,

$$f_{\lambda_1,\lambda_2,\lambda_3}(e_1, e_2, e_3) = \sum_{\substack{2j+3k=\lambda_1+\lambda_2+\lambda_3 \\ j,k \geq 0}} c_{j,k} Q^j R^k, \quad (5.3.82)$$

where  $c_{j,k}$  is a rational number.

**Proof** Apply Lemma 5.3.6 to equation (5.3.81) and simplify the result. ■

## 5.4 Sixteen Lambert series in terms of $z$ , $x$ , and $dz/dx$

In this section we show that the sixteen Lambert series can be represented as various polynomials in terms of  $z$ ,  $x$ , and  $dz/dx$ .

**Theorem 5.4.1** For  $n = 1, 2, \dots$ ,

$$\dot{f}_0^{(2n+1)}(0) = z^{2n+2} p_{n+1}(x), \quad (5.4.1)$$

$$f_0^{(2n+1)}(\pi) = z^{2n+2} (1-x) p_{n-1}(x), \quad (5.4.2)$$

$$f_0^{(2n+1)}(\pi\tau) = z^{2n+2} x p_{n-1}(x), \quad (5.4.3)$$

$$f_0^{(2n+1)}(\pi + \pi\tau) = z^{2n+2} x(1-x) p_{n-1}(x), \quad (5.4.4)$$

$$\dot{f}_g^{(2n-1)}(0) = z^{2n} p_n(x), \quad (5.4.5)$$

$$f_1^{(2n-1)}(\pi) = z^{2n} \sqrt{1-x} p_{n-1}(x), \quad (5.4.6)$$

$$f_1^{(2n)}(\pi\tau) = iz^{2n+1} x p_{n-1}(x), \quad (5.4.7)$$

$$f_1^{(2n)}(\pi + \pi\tau) = iz^{2n+1} x \sqrt{1-x} p_{n-1}(x); \quad (5.4.8)$$

$$f_2^{(2n)}(\pi) = z^{2n+1} (1-x) p_{n-1}(x), \quad (5.4.9)$$

$$f_2^{(2n-1)}(\pi\tau) = z^{2n} \sqrt{x} p_{n-1}(x), \quad (5.4.10)$$

$$f_2^{(2n)}(\pi + \pi\tau) = z^{2n+1} (1-x) \sqrt{x} p_{n-1}(x); \quad (5.4.11)$$

$$f_3^{(2n)}(\pi) = z^{2n+1} \sqrt{1-x} p_{n-1}(x), \quad (5.4.12)$$

$$f_3^{(2n)}(\pi\tau) = iz^{2n+1} \sqrt{x} p_{n-1}(x), \quad (5.4.13)$$

$$f_3^{(2n-1)}(\pi + \pi\tau) = iz^{2n} \sqrt{x(1-x)} p_{n-1}(x); \quad (5.4.14)$$

where  $g = 1, 2, 3$  and  $p_n(x)$  is a polynomial in  $x$  with rational coefficients of degree  $n$ .

For  $n = 0$  in (5.4.1)–(5.4.4), (5.4.7)–(5.4.9), and (5.4.11)–(5.4.13), we have

$$\dot{f}_0^{(1)}(0) = \frac{(1-2x)z^2}{12} + \frac{1}{2}x(1-x)z\frac{dz}{dx}, \quad (5.4.15)$$

$$f_0^{(1)}(\pi) = -\frac{(1-x)z^2}{4} - \frac{1}{2}x(1-x)z\frac{dz}{dx}, \quad (5.4.16)$$

$$f_0^{(1)}(\pi\tau) = \frac{z^2x}{4} - \frac{1}{2}x(1-x)z\frac{dz}{dx}, \quad (5.4.17)$$

$$f_0^{(1)}(\pi + \pi\tau) = -\frac{1}{2}x(1-x)z\frac{dz}{dx}, \quad (5.4.18)$$

$$f_1^{(0)}(\pi\tau) = \frac{i}{2} - \frac{i}{2}z, \quad (5.4.19)$$

$$f_1^{(0)}(\pi + \pi\tau) = \frac{i}{2} - \frac{i}{2}z\sqrt{1-x}, \quad (5.4.20)$$

$$f_2^{(0)}(\pi) = \frac{1}{2}z, \quad (5.4.21)$$

$$f_2^{(0)}(\pi + \pi\tau) = \frac{1}{2}z\sqrt{x}, \quad (5.4.22)$$

$$f_3^{(0)}(\pi) = \frac{1}{2}z\sqrt{1-x}, \quad (5.4.23)$$

$$f_3^{(0)}(\pi\tau) = -\frac{i}{2}z\sqrt{x}. \quad (5.4.24)$$

Note that  $p_{n-1}$  in (5.4.2)–(5.4.4) and (5.4.6)–(5.4.14) are different polynomials.

**Proof** The proofs of this theorem follow from the results in Corollary 5.2.5 and Section 3. We give complete details for the proof of (5.4.6). The other formulae can be proved in a similar way.

Substitution of (5.3.10) and (5.3.19) into (5.2.34) implies for  $n \geq 1$ ,

$$\begin{aligned} f_1^{(2n-1)}(\pi) &= \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} \mathcal{D}_{n,i} \left[ -\frac{1}{4}z^2\sqrt{1-x} \right]^{2i-1} \left[ \frac{1}{12}z^2(2-x) \right]^{n+1-2i} \\ &= \mathcal{D}_{n,1}z^{2n}\sqrt{1-x}p_{n-1}(x) + \mathcal{D}_{n,2}z^{2n}\sqrt{1-x}p_{n-2}(x) + \cdots \\ &\quad + \mathcal{D}_{n,\lfloor (n+1)/2 \rfloor}z^{2n}\sqrt{1-x}p_{\lfloor (n+1)/2 \rfloor}(x) \\ &= z^{2n}\sqrt{1-x}p_{n-1}(x), \end{aligned}$$

where  $p_{n-1}(x)$  is a polynomial in  $x$  with rational coefficients of degree  $n-1$ , completing the proof of (5.4.6).

This completes the proof of Theorem 5.4.1. ■

Ramanujan [62, p. 141] has given a result equivalent to (5.4.1). Cooper [39] has given different proofs and equivalent results to (5.4.2)–(5.4.4), (5.4.7), (5.4.9), (5.4.11), and (5.4.13).

**Theorem 5.4.2** For  $n \geq 1$ ,

$$f_0^{(6n-3)}(0) = z^{6n-2}(1-x+x^2)p_{3n-3}(x), \quad (5.4.25)$$

$$f_0^{(4n+1)}(0) = z^{4n+2}(1+x)(1-2x)\left(1-\frac{x}{2}\right)p_{2n-2}(x), \quad (5.4.26)$$

$$f_0^{(6n+1)}(0) = z^{6n+2}(1-x+x^2)^2p_{3n-3}(x), \quad (5.4.27)$$

$$f_0^{(4n+1)}(\pi) = z^{4n+2}(1-x)[1+(1-x)]p_{2n-2}(x), \quad (5.4.28)$$

$$f_0^{(4n+1)}(\pi\tau) = z^{4n+2}x(1+x)p_{2n-2}(x), \quad (5.4.29)$$

$$f_0^{(4n+1)}(\pi+\pi\tau) = z^{4n+2}x(1-x)[(1-x)-x]p_{2n-2}(x), \quad (5.4.30)$$

$$f_1^{(4n-3)}(0) = z^{4n-2}[1+(1-x)]p_{2n-2}(x), \quad (5.4.31)$$

$$f_2^{(4n-3)}(0) = z^{4n-2}(1+x)p_{2n-2}(x), \quad (5.4.32)$$

$$f_3^{(4n-3)}(0) = z^{4n-2}[(1-x)-x]p_{2n-2}(x), \quad (5.4.33)$$

$$f_1^{(4n-1)}(\pi) = z^{4n}(2-x)\sqrt{1-x}p_{2n-2}(x), \quad (5.4.34)$$

$$f_2^{(4n-1)}(\pi\tau) = z^{4n}(1+x)\sqrt{x}p_{2n-2}(x), \quad (5.4.35)$$

$$f_3^{(4n-1)}(\pi+\pi\tau) = iz^{4n}[(1-x)-x]\sqrt{x(1-x)}p_{2n-2}(x), \quad (5.4.36)$$

where  $p_n(x)$  is a polynomial in  $x$  with rational coefficients of degree  $n$ .

**Proof** We give complete details of the proof of (5.4.34) and the other formulae can be proved in a similar way.

Using the result of (5.2.34) we can rewrite the left hand side of (5.4.34) as

$$\begin{aligned} f_1^{(4n-1)}(\pi) &= \sum_{i=1}^n \mathcal{D}_{n,i} \left[ f_1^{(1)}(\pi) \right]^{2i-1} e_1^{2n+1-2i} \\ &= f_1^{(1)}(\pi) e_1 \sum_{i=0}^{n-1} \mathcal{D}_{n,i} \left[ f_1^{(1)}(\pi) \right]^{2i} e_1^{2n-2-2i}. \end{aligned} \quad (5.4.37)$$



Lastly, by substituting (5.3.10) and (5.3.19) into (5.4.37) and then simplifying the result, we arrive at (5.4.34), which completes the proof of Theorem 5.4.2. ■

**Conjecture 5.4.3** For  $n \geq 1$ ,

$$f_1^{(6n-1)}(\pi) = z^{6n} [1 + 14(1-x) + (1-x)^2] \sqrt{1-x} p_{3n-3}(x), \quad (5.4.38)$$

$$f_2^{(6n-1)}(\pi\tau) = z^{6n} (1 + 14x + x^2) \sqrt{x} p_{3n-3}(x), \quad (5.4.39)$$

$$f_3^{(6n-1)}(\pi + \pi\tau) = iz^{6n} [(1-x)^2 - 14x(1-x) + x^2] \sqrt{x(1-x)} p_{3n-3}(x), \quad (5.4.40)$$

where  $p_n(x)$  is a polynomial in  $x$  with rational coefficients of degree  $n$ .

We remark that formulae (5.4.38)–(5.4.40) have been verified for  $1 \leq n \leq 100$  by using a Maple program.

The next three diagrams show the first five examples in each sixteen families of identities and also show the interconnections between the various polynomials.

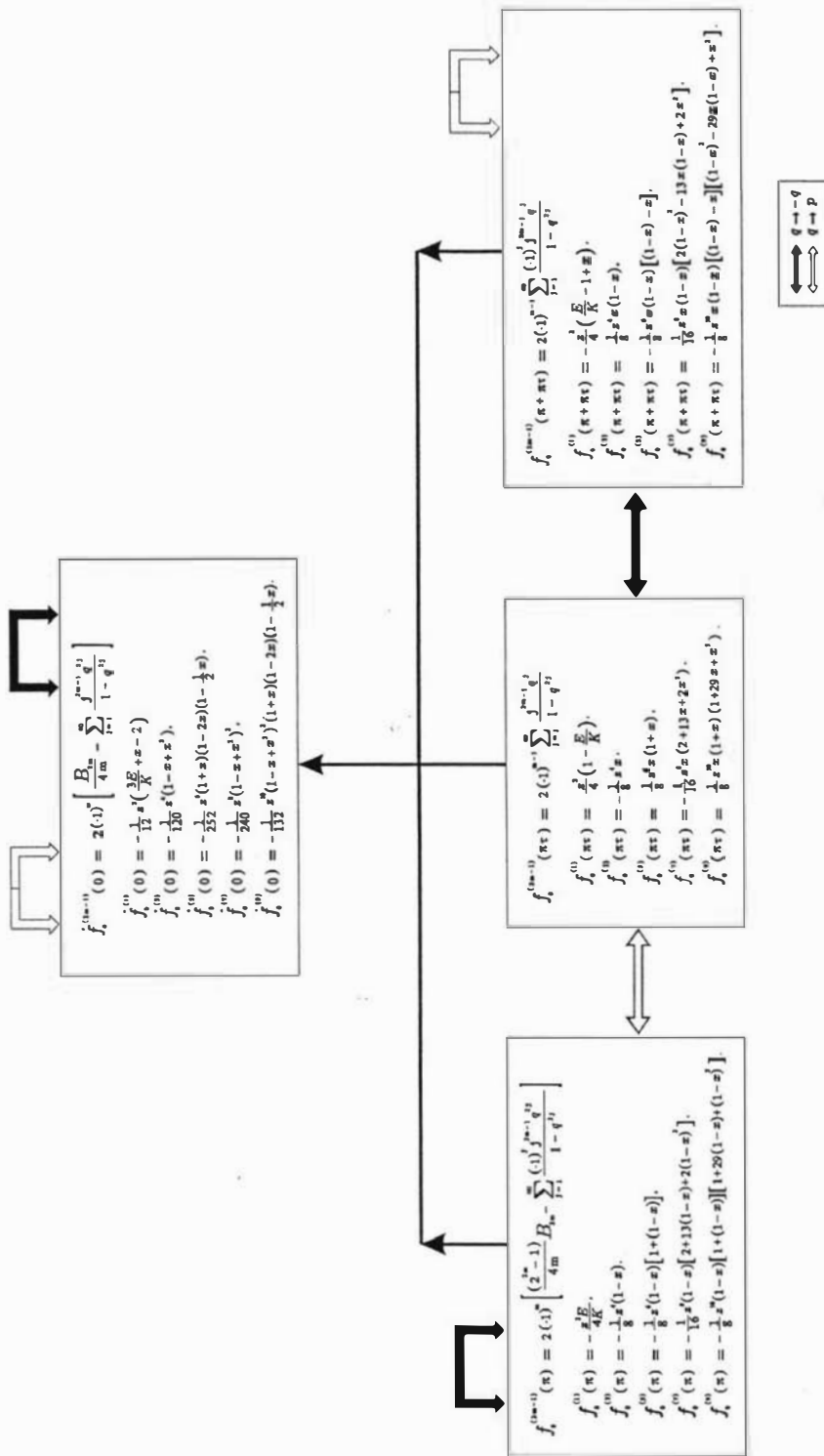


Figure 5.1. The first few values of  $f_0^{(2m-1)}(0)$ ,  $f_0^{(2m-1)}(\pi)$ ,  $f_0^{(2m-1)}(\pi\tau)$ , and  $f_0^{(2m-1)}(\pi+\pi\tau)$ .

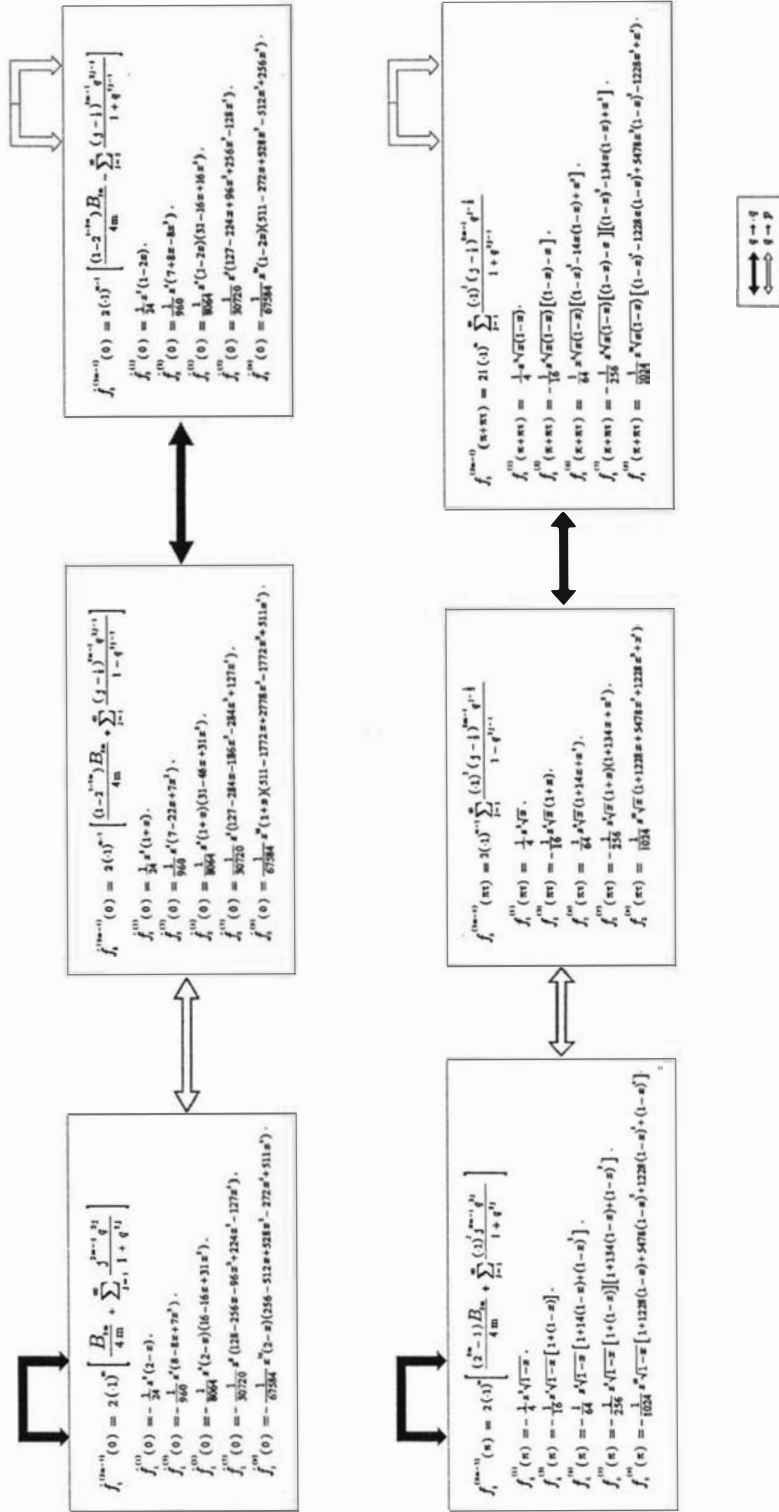


Figure 5.2. The first few values of  $f_1^{(2m-1)}(0)$ ,  $f_2^{(2m-1)}(0)$ ,  $f_3^{(2m-1)}(0)$ ,  $f_1^{(2m-1)}(\pi)$ ,  $f_2^{(2m-1)}(\pi\tau)$ , and  $f_3^{(2m-1)}(\pi + \pi\tau)$ .

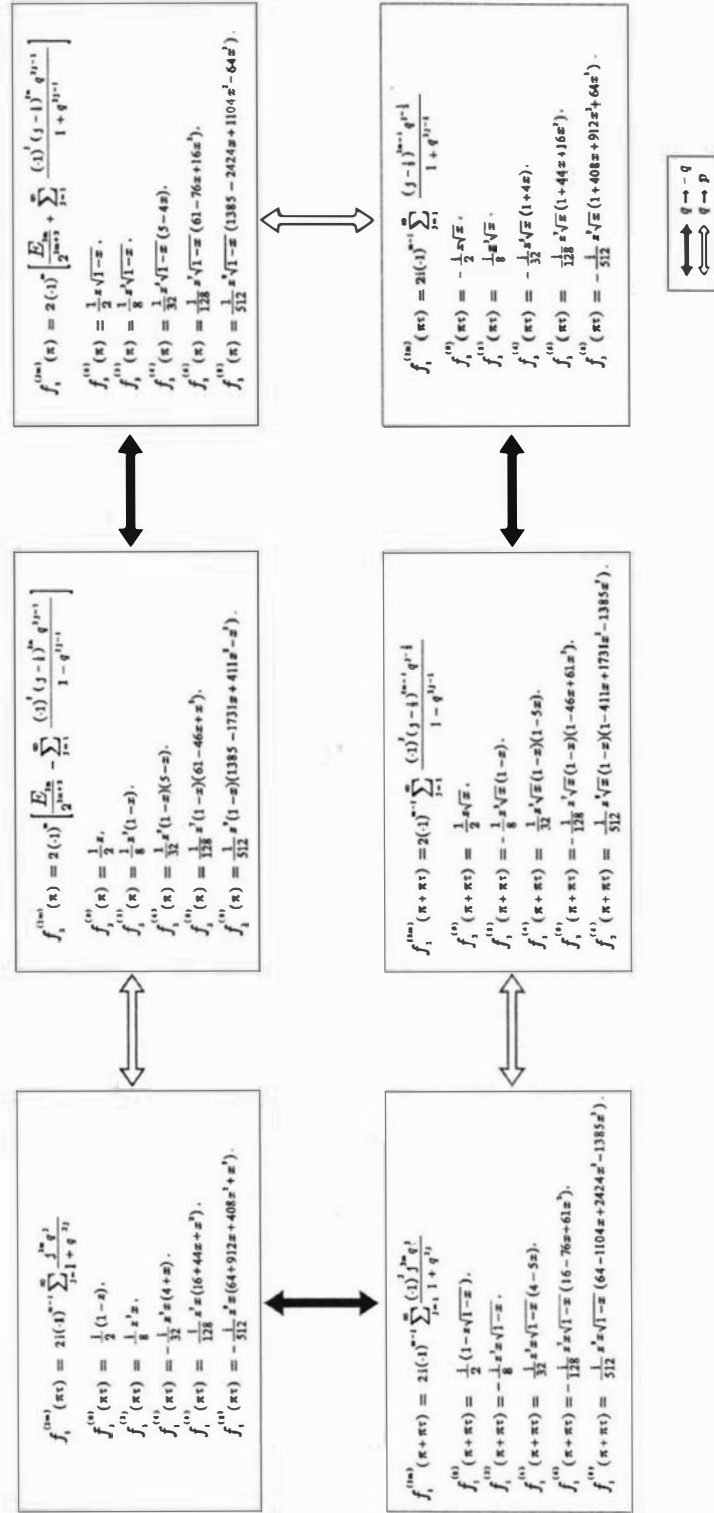


Figure 5.3. The first few values of  $f_1^{(2m)}(\pi\tau)$ ,  $f_1^{(2m)}(\pi+\pi\tau)$ ,  $f_2^{(2m)}(\pi)$ ,  $f_2^{(2m)}(\pi+\pi\tau)$ ,  $f_3^{(2m)}(\pi)$ , and  $f_3^{(2m)}(\pi\tau)$ .

## 5.5 Proofs of some of Ramanujan's results

As we mentioned earlier, Ramanujan gave several families of identities in [92, Chapter 17, Entries 13–17] and Berndt has given proofs in [19, pp. 126–138]. The aim of this section is to give a simple proof by using some transformations from Chapter 3 and Figures 5.1–5.3.

For Entry 13: Proofs of (i) and (ii). We use (5.3.54) and (5.3.66), respectively.

Proofs of (iii) and (iv). Replace  $q$  with  $q^{\frac{1}{2}}$  into Figure 5.1 of  $\dot{f}_0^{(2m-1)}(0)$  and put  $m = 2$  and 3, respectively, then employ (3.5.1).

Proofs of (v) and (vi). Replace  $q$  with  $q^2$  into Figure 5.1 of  $\dot{f}_0^{(2m-1)}(0)$  and put  $m = 2$  and 3, respectively, then employ (3.5.1).

Proofs of (viii), (x), and (xi). Replace  $q$  with  $q^{\frac{1}{2}}$  into Figure 5.2 of  $\dot{f}_1^{(2m-1)}(0)$  and put  $m = 1, 2$ , and 3, respectively, then employ (3.5.1).

Proofs of (ix), (xii), and (xiii). Employ Figure 5.2 of  $\dot{f}_1^{(2m-1)}(0)$  and put  $m = 1, 2$ , and 3, respectively.

For Entry 14: Proofs of (i)–(iv). Replace  $q$  with  $q^{\frac{1}{2}}$  into Figure 5.2 of  $f_1^{(2m-1)}(\pi)$  and put  $m = 1$  to 4, respectively, then employ (3.5.1).

Proofs of (v)–(viii). Replace  $q$  with  $q^{\frac{1}{2}}$  into Figure 5.1 of  $f_0^{(2m-1)}(\pi)$  and put  $m = 2$  to 5, respectively, then employ (3.5.1).

Proofs of (ix)–(xi). Use Figure 5.1 of  $f_0^{(2m-1)}(\pi)$  for  $m = 2, 3$ , and 4, respectively.

For Entry 15: Proofs of (i)–(iv). Employ Figure 5.1 of  $f_0^{(2m-1)}(\pi\tau)$  and put  $m = 2$  to 5, respectively.

Proofs of (v)–(viii). Replace  $q$  with  $q^2$  into Figure 5.1 of  $f_0^{(2m-1)}(\pi\tau)$  and put  $m = 2$  to 5,

respectively, then employ (3.5.1).

Proofs of (ix)–(xii). Replace  $q$  with  $q^2$  into Figure 5.2 of  $f_2^{(2m-1)}(\pi\tau)$  and put  $m = 1$  to 4, respectively, then employ (3.5.1).

Proofs of (xiii)–(xvi). Use Figure 5.2 of  $f_2^{(2m-1)}(\pi\tau)$  and put  $m = 1$  to 4, respectively.

For Entry 16: Proofs of (i)–(v). Employ Figure 5.2 of  $f_3^{(2m-1)}(\pi + \pi\tau)$  and put  $m = 1$  to 5, respectively.

Proof of (vi). Set  $n = 6$ ,  $g = 3$ , and  $\omega = \pi + \pi\tau$  into (5.2.14), then employ the results of Figure 5.2 of  $f_3^{(2m-1)}(\pi + \pi\tau)$ .

Proofs of (ix)–(xiii). Use Figure 5.3 of  $f_3^{(2m)}(\pi\tau)$  and put  $m = 0$  to 4, respectively.

For Entry 17: Proofs of (i)–(v). Employ Figure 5.3 of  $f_1^{(2m)}(\pi\tau)$  and put  $m = 0$  to 4, respectively.

Proofs of (vi)–(ix). Use Figure 5.3 of  $f_2^{(2m)}(\pi)$  and put  $m = 0$  to 3, respectively.

This completes the proofs of Ramanujan's identities.

The results in this chapter will be used in Chapters 6 and 8.

# Chapter 6

## Applications

### 6.1 Introduction

The aim of this chapter is to use selected transformations in Chapter 3 and results from Chapter 5 to prove a large number of identities.

In Section 2, we prove ten Lambert series identities.

In Section 3, we study functions listed in the following table:

Sums of squares	Sums of triangular numbers
$r_2(n)$	$t_2(n)$
$r_4(n)$	$t_4(n)$
$r_6(n)$	$t_6(n)$
$r_8(n)$	$t_8(n)$
$r_{1,4}(n)$	$t_{(2,8)}(n)$
$r_{(1,1,1,4)}(n)$	$t_{(2,2,4,4)}(n)$
$r_{(1,1,2,2)}(n)$	$t_{(2,2,2,8)}(n)$
$r_{(1,1,4,4)}(n)$	$t_{(1,1,1,1,2,2)}(n)$
$r_{(1,2,2,4)}(n)$	
$r_{(1,4,4,4)}(n)$	
$r_{(1,1,1,1,2,2)}(n)$	
$r_{(1,1,2,2,2,2)}(n)$	

Note that the results of  $r_2(n)$ ,  $r_4(n)$ ,  $r_6(n)$ ,  $r_8(n)$ ,  $t_2(n)$ ,  $t_4(n)$ ,  $t_6(n)$ ,  $t_8(n)$ ,  $t_{(2,8)}(n)$ ,  $t_{(2,2,4,4)}(n)$ , and  $t_{(1,1,1,1,2,2)}(n)$  have been studied in Chapter 4. We give another way to achieve these results.

So far, there are no Lambert series for the representations of  $n$  by  $\varphi(q)\varphi(q^4)$ ,  $\varphi^4(q)\varphi^2(q^2)$ ,

and  $\varphi^2(q) \varphi^4(q^2)$  in the literature. The result is possibly new.

Let  $k$  and  $m$  be positive integers. Let  $\lambda_1, \lambda_2, \dots, \lambda_k$  and  $\mu_1, \mu_2, \dots, \mu_m$  be positive integers where  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k$  and  $\mu_1 \leq \mu_2 \leq \dots \leq \mu_m$ . The function

$$r(\lambda_1 \square + \lambda_2 \square + \dots + \lambda_k \square + \mu_1 \triangle + \mu_2 \triangle + \dots + \mu_m \triangle)(n)$$

will denote the number of solutions in integers of

$$\begin{aligned} & \lambda_1 x_1^2 + \lambda_2 x_2^2 + \dots + \lambda_k x_k^2 \\ & + \mu_1 \frac{y_1(y_1 + 1)}{2} + \mu_2 \frac{y_2(y_2 + 1)}{2} + \dots + \mu_m \frac{y_m(y_m + 1)}{2} \\ & = n, \end{aligned} \tag{6.1.1}$$

where  $n = 0, 1, 2, 3, \dots$

We also define  $r(\lambda_1 \square + \lambda_2 \square + \dots + \lambda_k \square + \mu_1 \triangle + \mu_2 \triangle + \dots + \mu_m \triangle)(0) = 1$ .

Then the generating function for  $r(\lambda_1 \square + \lambda_2 \square + \dots + \lambda_k \square + \mu_1 \triangle + \mu_2 \triangle + \dots + \mu_m \triangle)(n)$  is

$$\begin{aligned} & \sum_{n=0}^{\infty} r(\lambda_1 \square + \lambda_2 \square + \dots + \lambda_k \square + \mu_1 \triangle + \mu_2 \triangle + \dots + \mu_m \triangle)(n) q^n \\ & = \varphi(q^{\lambda_1}) \varphi(q^{\lambda_2}) \dots \varphi(q^{\lambda_k}) \psi(q^{\mu_1}) \psi(q^{\mu_2}) \dots \psi(q^{\mu_m}). \end{aligned} \tag{6.1.2}$$



In Section 4, we study functions listed in the following table:

Results involving both sums of squares and triangular numbers
$r(\square + 8\triangle)(n)$
$r(4\square + 2\triangle)(n)$
$r(\square + \square + \square + 8\triangle)(n)$
$r(\square + \square + \triangle + \triangle)(n)$
$r(\square + \square + 4\triangle + 4\triangle)(n)$
$r(\square + \square + 8\triangle + 8\triangle)(n)$
$r(\square + 8\triangle + 8\triangle + 8\triangle)(n)$
$r(4\square + 2\triangle + 2\triangle + 2\triangle)(n)$
$r(\square + 4\square + 4\triangle + 4\triangle)(n)$
$r(\square + 2\square + 2\square + 8\triangle)(n)$
$r(\square + 4\triangle + 4\triangle + 8\triangle)(n)$
$r(\square + \square + \triangle + \triangle + \triangle + \triangle)(n)$
$r(\square + \square + 4\triangle + 4\triangle + 4\triangle + 4\triangle)(n)$
$r(\square + \square + \square + \square + 4\triangle + 4\triangle)(n)$

So far, most of the results in this section were not found in the literature except for  $\varphi^2(q)\psi^2(q)$ ,  $\varphi^2(q)\psi^2(q^4)$ , and  $\varphi^2(q)\psi^4(q)$ , so they may be new.

Many people have invested a huge amount of time, only to yield one result at a time. We unify a diverse set of these results and develop a powerful tool (four versatile functions  $f_0, f_1, f_2$ , and  $f_3$ ) to collect them all in one.

## 6.2 Proofs of ten Lambert series identities

In this section we will prove ten Lambert series identities.

### Theorem 6.2.1

$$\sum_{n=1}^{\infty} \frac{(2n-1)^5 q^{2n}}{1-q^{4n-2}} = Q \sum_{n=1}^{\infty} \frac{(2n-1) q^{2n}}{1-q^{4n-2}}, \quad (6.2.1)$$

$$\sum_{n=1}^{\infty} \frac{n^9 q^n}{1 - q^{2n}} = q\psi^8(q) \left( 1 + 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 + q^n} \right); \quad (6.2.2)$$

$$1 - 8 \sum_{n=1}^{\infty} \frac{n^5 (-q)^n}{1 - q^n} = \left\{ 1 + 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 + q^n} \right\} \left\{ 1 + 16 \sum_{n=1}^{\infty} \frac{n^3 (-q)^n}{1 - q^n} \right\}, \quad (6.2.3)$$

$$\begin{aligned} 17 + 32 \sum_{n=1}^{\infty} \frac{n^7 (-q)^n}{1 - q^n} &= \left\{ 1 + 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 + q^n} \right\} \left\{ 1 - 8 \sum_{n=1}^{\infty} \frac{n^5 (-q)^n}{1 - q^n} \right\} \\ &\quad + 9 \left\{ 1 + 16 \sum_{n=1}^{\infty} \frac{n^3 (-q)^n}{1 - q^n} \right\}^2, \end{aligned} \quad (6.2.4)$$

$$\begin{aligned} 2 \left\{ 31 - 8 \sum_{n=1}^{\infty} \frac{n^9 (-q)^n}{1 - q^n} \right\} &= \left\{ 1 + 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 + q^n} \right\} \left\{ 17 + 32 \sum_{n=1}^{\infty} \frac{n^7 (-q)^n}{1 - q^n} \right\} \\ &\quad + 45 \left\{ 1 + 16 \sum_{n=1}^{\infty} \frac{n^3 (-q)^n}{1 - q^n} \right\} \left\{ 1 - 8 \sum_{n=1}^{\infty} \frac{n^5 (-q)^n}{1 - q^n} \right\}, \end{aligned} \quad (6.2.5)$$

$$\begin{aligned} 691 + 16 \sum_{n=1}^{\infty} \frac{n^{11} (-q)^n}{1 - q^n} &= \left\{ 1 - 8 \sum_{n=1}^{\infty} \frac{n^5 (-q)^n}{1 - q^n} \right\}^2 \\ &\quad + 4 \left\{ 1 + 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 + q^n} \right\} \left\{ 31 - 8 \sum_{n=1}^{\infty} \frac{n^9 (-q)^n}{1 - q^n} \right\} \\ &\quad + 21 \left\{ 1 + 16 \sum_{n=1}^{\infty} \frac{n^3 (-q)^n}{1 - q^n} \right\} \left\{ 17 + 32 \sum_{n=1}^{\infty} \frac{n^7 (-q)^n}{1 - q^n} \right\}; \end{aligned} \quad (6.2.6)$$

$$\begin{aligned} &16 \left[ \sum_{n=1}^{\infty} \frac{n^2 q^n}{1 + q^{2n}} \right]^2 + 16 \left[ \sum_{n=1}^{\infty} \frac{n^2 (-q)^n}{1 + q^{2n}} \right]^2 \\ &- 4 \left[ -\frac{1}{4} + \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)^2 q^{2n+1}}{1 - q^{2n+1}} \right]^2 \\ &- 4 \left[ -\frac{1}{4} + \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)^2 q^{2n+1}}{1 + q^{2n+1}} \right]^2 \\ &= -\frac{1}{18} \left[ 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^{2n}}{1 - q^{2n}} \right] - \frac{4}{9} \left[ 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^{4n}}{1 - q^{4n}} \right], \end{aligned} \quad (6.2.7)$$

$$\begin{aligned}
& 17 \left\{ 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} \right\} \left\{ 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 (-q)^n}{1 - (-q)^n} \right\} \\
&= 273 \left\{ 1 + 480 \sum_{n=1}^{\infty} \frac{n^7 q^{2n}}{1 - q^{2n}} \right\} - 256 \left\{ 1 + 480 \sum_{n=1}^{\infty} \frac{n^7 q^{4n}}{1 - q^{4n}} \right\} \\
&\quad - 1036800 q^2 (q^2; q^2)_{\infty}^8 (q^4; q^4)_{\infty}^8, \tag{6.2.8}
\end{aligned}$$

$$\begin{aligned}
& \left\{ \frac{5}{4} + \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)^4 q^{2n+1}}{1 - q^{2n+1}} \right\}^2 + \left\{ \frac{5}{4} - \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)^4 q^{2n+1}}{1 + q^{2n+1}} \right\}^2 \\
&+ 16 \left\{ \sum_{n=1}^{\infty} \frac{n^4 q^n}{1 + q^{2n}} \right\}^2 + 16 \left\{ \sum_{n=1}^{\infty} \frac{n^4 (-1)^n q^n}{1 + q^{2n}} \right\}^2 \\
&= \frac{25}{248} \left\{ 32 \left( 1 - 264 \sum_{n=1}^{\infty} \frac{n^9 q^{4n}}{1 - q^{4n}} \right) - \left( 1 - 264 \sum_{n=1}^{\infty} \frac{n^9 q^{2n}}{1 - q^{2n}} \right) \right\} \\
&+ \frac{192}{31} \left\{ \frac{q^2 (q^2; q^2)_{\infty}^{20}}{(-q; q^2)_{\infty}^8} + \frac{q^2 (q^2; q^2)_{\infty}^{20}}{(q; q^2)_{\infty}^8} \right\}; \tag{6.2.9}
\end{aligned}$$

$$\begin{aligned}
& \left( \sum_{n=1}^{\infty} \frac{n^7 q^n}{1 - q^{2n}} \right)^2 + 16 q^2 (q; q)_{\infty}^{16} (q^2; q^2)_{\infty}^{16} \\
&= 17 q^2 (q^2; q^2)_{\infty}^{24} Q + 2^6 \cdot 17^2 \left[ \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2/2} \right]^{32}. \tag{6.2.10}
\end{aligned}$$

**Proof** The proof is divided into three parts. First we prove identities (6.2.1) and (6.2.2).

Then identities (6.2.3)–(6.2.6) and lastly proofs of (6.2.7)–(6.2.10) will be given.

Proofs of (6.2.1) and (6.2.2). Using (3.5.1), Figure 5.1, and Figure 5.2,

$$f_2^{(1)}(\pi\tau; q^2) = \sum_{m=1}^{\infty} \frac{(2m-1) q^{2m-1}}{1 - q^{4m-2}} = \frac{1}{16} z^2 x, \tag{6.2.11}$$

$$f_2^{(5)}(\pi\tau; q^2) = \frac{1}{16} \sum_{m=1}^{\infty} \frac{(2m-1)^5 q^{2m-1}}{1 - q^{4m-2}} = \frac{1}{256} z^6 x (1 - x + x^2), \quad (6.2.12)$$

$$f_0^{(9)}(\pi\tau) = 2 \sum_{m=1}^{\infty} \frac{m^9 q^m}{1 - q^{2m}} = \frac{1}{8} z^{10} x (1 + x) (1 + 29x + x^2), \quad (6.2.13)$$

$$\dot{f}_1^{(5)}\left(0; q^{\frac{1}{2}}\right) = -\frac{1}{252} - 2 \sum_{m=1}^{\infty} \frac{m^5 q^m}{1 + q^m} = -\frac{1}{252} z^6 (1 + x) (1 + 29x + x^2). \quad (6.2.14)$$

From (4.3.19), (5.3.71), (6.2.11)–(6.2.14), we can express both sides of (6.2.1) and (6.2.2) as polynomials in  $z$  and  $x$ , completing the proofs of (6.2.1) and (6.2.2).

Proofs of (6.2.3)–(6.2.6). Replace  $q$  with  $q^{\frac{1}{2}}$ , set  $\omega = \pi$ ,  $n = 1, 2, 3$ , and  $4$  in (5.2.2), respectively, use (2.10.5) and rearrange. This completes the proofs of (6.2.3)–(6.2.6).

Proofs of (6.2.7)–(6.2.10). Using the results in Section 10 of Chapter 2, identities (6.2.7)–(6.2.10) may be rewritten as

$$\begin{aligned} & -2 \left[ f_1^{(2)}(\pi\tau) \right]^2 - 2 \left[ f_1^{(2)}(\pi\tau; -q) \right]^2 - 8 \left[ f_2^{(2)}(\pi) \right]^2 - 8 \left[ f_2^{(2)}(\pi; -q) \right]^2 \\ & = 7 \dot{f}_0^{(5)}(0) + 56 \dot{f}_0^{(5)}(0; q^2), \end{aligned} \quad (6.2.15)$$

$$\begin{aligned} & 1020 \dot{f}_0^{(3)}\left(0; q^{\frac{1}{2}}\right) \dot{f}_0^{(3)}\left(0; iq^{\frac{1}{2}}\right) \\ & = -273 \dot{f}_0^{(7)}(0) + 256 \dot{f}_0^{(7)}(0; q^2) - 4320 q^2 (q^2; q^2)_{\infty}^8 (q^4; q^4)_{\infty}^8, \end{aligned} \quad (6.2.16)$$

$$\begin{aligned}
& 64 \left[ f_2^{(4)}(\pi) \right]^2 + 64 \left[ f_3^{(4)}(\pi) \right]^2 - 4 \left[ f_1^{(4)}(\pi\tau) \right]^2 - 4 \left[ f_1^{(4)}(\pi + \pi\tau) \right]^2 \\
&= \frac{25}{62} \left[ 33 f_0^{(9)}(0) - 1056 f_0^{(9)}(0; q^2) \right] + \frac{192}{31} \left[ \frac{q^2 (q^2; q^2)_\infty^{20}}{(-q; q^2)_\infty^8} + \frac{q^2 (q^2; q^2)_\infty^{20}}{(q; q^2)_\infty^8} \right],
\end{aligned} \tag{6.2.17}$$

$$\begin{aligned}
& \left[ f_0^{(7)}(\pi\tau) \right]^2 + 64 q^2 (q; q)_\infty^{16} (q^2; q^2)_\infty^{16} \\
&= -8160 q^2 (q^2; q^2)_\infty^{24} f_0^{(3)}(0) + 4624 \left[ f_0^{(3)}(\pi\tau) \right]^4,
\end{aligned} \tag{6.2.18}$$

respectively. Next by using (2.11.7)–(2.11.9) and (4.2.26), we easily observe

$$q^2 (q^2; q^2)_\infty^8 (q^4; q^4)_\infty^8 = \frac{1}{256} z^8 x^2 (1 - x), \tag{6.2.19}$$

$$\frac{q^2 (q^2; q^2)_\infty^{20}}{(-q; q^2)_\infty^8} = \frac{1}{256} z^{10} x^2 (1 - x)^2, \tag{6.2.20}$$

$$\frac{q^2 (q^2; q^2)_\infty^{20}}{(q; q^2)_\infty^8} = \frac{1}{256} z^{10} x^2 (1 - x), \tag{6.2.21}$$

$$q^2 (q; q)_\infty^{16} (q^2; q^2)_\infty^{16} = \frac{1}{256} z^{16} x^2 (1 - x)^4, \tag{6.2.22}$$

$$q^2 (q^2; q^2)_\infty^{24} = \frac{1}{256} z^{12} x^2 (1 - x)^2. \tag{6.2.23}$$

Finally, by employing (3.5.1), (3.5.12), Figures 5.1–5.3, and (6.2.19)–(6.2.23), we express both sides of (6.2.15)–(6.2.18) as polynomials in  $z$  and  $x$ , completing the proofs of (6.2.7)–(6.2.10).

This completes the proof of Theorem 6.2.1. ■

Identities (6.2.1) and (6.2.2) were given by Ramanujan [94, p. 146], [92, Chapter 17, Entry 17 example (vi)]. Identity (6.2.7) was given by H. F. Sandham [95, p. 234]. Identities (6.2.8) and (6.2.9) appeared in Sandham [96, pp. 35–36]. Identities (6.2.3)–(6.2.6) were given by Liu [85, pp. 168–169]. Identity (6.2.10) was needed by Chan, Cooper

and W. C. Liaw in a preliminary version of [29]. Chan et. al used (6.2.10) to prove that  $S_{17}(p^2)/17 = S_{32}(8p)/32$  for odd primes  $p$ , where  $S_k(n)$  denotes the number of representations of  $n$  as a sum of  $k$  positive odd squares. The proofs we have given here are simpler.

### 6.3 Additional findings on sums of squares and triangular numbers

In this section we present twenty infinite products and their Lambert series expansions. From these results, we deduce the formulae for the number of representations of an integer  $n$  by twenty different quadratic forms in terms of divisor sums or of some products of integers which contain primes. The following lemma is required to prove Theorem 6.3.2.

#### Lemma 6.3.1

$$\varphi^2(q) = 2f_2^{(0)}(\pi), \quad (6.3.1)$$

$$\varphi(q)\varphi(q^4) = \frac{1}{2} + f_2^{(0)}(\pi) + if_1^{(0)}(\pi + \pi\tau; q^2); \quad (6.3.2)$$

$$\varphi^4(q) = -4f_1'(\pi; iq^{\frac{1}{2}}), \quad (6.3.3)$$

$$\begin{aligned} \varphi^3(q)\varphi(q^4) &= -2f_1'(\pi; iq^{\frac{1}{2}}) - 2f_1'(\pi; q^2) \\ &\quad - 2if_3'(\pi + \pi\tau; q^2), \end{aligned} \quad (6.3.4)$$

$$\varphi^2(q)\varphi^2(q^2) = -2f_1'(\pi; iq^{\frac{1}{2}}) - 2f_1'(\pi), \quad (6.3.5)$$

$$\begin{aligned} \varphi^2(q)\varphi^2(q^4) &= -f_1'(\pi; iq^{\frac{1}{2}}) - 2f_1'(\pi; q^2) \\ &\quad - 2if_3'(\pi + \pi\tau; q^2) - f_1'(\pi), \end{aligned} \quad (6.3.6)$$

$$\begin{aligned}\varphi(q)\varphi^3(q^4) &= -\frac{1}{2}f'_1\left(\pi; iq^{\frac{1}{2}}\right) - \frac{3}{2}f'_1(\pi) - 2f'_1(\pi; q^2) \\ &\quad - if'_3(\pi + \pi\tau; q^2),\end{aligned}\quad (6.3.7)$$

$$\varphi(q)\varphi^2(q^2)\varphi(q^4) = -f'_1\left(\pi; iq^{\frac{1}{2}}\right) - f'_1(\pi) - 2f'_1(\pi; q^2); \quad (6.3.8)$$

$$\varphi^6(q) = 8f_2^{(2)}(\pi) - 8if_1^{(2)}(\pi\tau), \quad (6.3.9)$$

$$\varphi^4(q)\varphi^2(q^2) = 4f_2^{(2)}(\pi) - 4if_1^{(2)}(\pi\tau) + 4f_3^{(2)}(\pi), \quad (6.3.10)$$

$$\varphi^2(q)\varphi^4(q^2) = 4f_2^{(2)}(\pi) - 2if_1^{(2)}(\pi\tau) + 4f_3^{(2)}(\pi), \quad (6.3.11)$$

$$\varphi^8(q) = -8f_0^{(3)}\left(\pi; iq^{\frac{1}{2}}\right); \quad (6.3.12)$$

$$\psi^2(q) = \frac{i}{2q^{\frac{1}{4}}}f_3^{(0)}\left(q; q^{\frac{1}{2}}\right), \quad (6.3.13)$$

$$\psi(q^2)\psi(q^8) = \frac{i}{8q^{\frac{5}{4}}}f_3^{(0)}\left(\pi\tau; q^{\frac{1}{2}}\right) - \frac{i^{\frac{1}{2}}}{8q^{\frac{5}{4}}}f_3^{(0)}\left(\pi\tau; iq^{\frac{1}{2}}\right); \quad (6.3.14)$$

$$\psi^4(q) = \frac{1}{q^{\frac{1}{2}}}f'_2(\pi\tau), \quad (6.3.15)$$

$$\psi^2(q^2)\psi^2(q^4) = \frac{1}{8q^{\frac{3}{2}}}f'_2(\pi\tau) + \frac{i}{8q^{\frac{3}{2}}}f'_3(\pi + \pi\tau); \quad (6.3.16)$$

$$\begin{aligned}\psi^3(q^2)\psi(q^8) &= \frac{1}{32q^{\frac{7}{4}}}f'_3\left(\pi + \pi\tau; -q^{\frac{1}{2}}\right) - \frac{i}{32q^{\frac{7}{4}}}f'_3\left(\pi + \pi\tau; q^{\frac{1}{2}}\right) \\ &\quad + \frac{(-1)^{\frac{-3}{4}}}{32q^{\frac{7}{4}}}f'_3\left(\pi + \pi\tau; -iq^{\frac{1}{2}}\right) \\ &\quad - \frac{(-1)^{\frac{-1}{4}}}{32q^{\frac{7}{4}}}f'_3\left(\pi + \pi\tau; iq^{\frac{1}{2}}\right); \end{aligned}\quad (6.3.17)$$

$$\psi^6(q) = \frac{1}{8q^{\frac{3}{4}}}f_2^{(2)}\left(\pi + \pi\tau; q^{\frac{1}{2}}\right) - \frac{i}{8q^{\frac{1}{2}}}f_3^{(2)}\left(\pi\tau; q^{\frac{1}{2}}\right), \quad (6.3.18)$$

$$\psi^4(q)\psi^2(q^2) = \frac{-i}{2q}f_1^{(2)}(\pi\tau), \quad (6.3.19)$$

$$\psi^8(q) = -\frac{1}{2q}f_0^{(3)}(\pi\tau). \quad (6.3.20)$$

**Proof** Using (3.5.1), (3.5.12), (4.2.17), (4.3.16), and Figures 5.1–5.3, we express both sides of (6.3.1)–(6.3.20) in terms of  $z$  and  $x$ . This completes the proof of Lemma 6.3.1. ■

We now rewrite the right hand side of Lemma 6.3.1 as a Lambert series expansion.

**Theorem 6.3.2**

$$\varphi(q) \varphi(q^4) = 1 + 2 \sum_{j=1}^{\infty} \frac{(-1)^j q^{2j}}{1 + q^{4j}} - 2 \sum_{j=1}^{\infty} \frac{(-1)^j q^{2j-1}}{1 - q^{2j-1}}; \quad (6.3.21)$$

$$\begin{aligned} \varphi^3(q) \varphi(q^4) &= 1 + 4 \sum_{j=1}^{\infty} \frac{j q^j}{1 + (-q)^j} + 4 \sum_{j=1}^{\infty} \frac{(-1)^j j q^{4j}}{1 + q^{4j}} \\ &\quad - 2 \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1) q^{2j-1}}{1 + q^{4j-2}}, \end{aligned} \quad (6.3.22)$$

$$\varphi^2(q) \varphi^2(q^2) = 1 + 4 \sum_{j=1}^{\infty} \frac{j q^j}{1 + (-q)^j} + 4 \sum_{j=1}^{\infty} \frac{(-1)^j j q^{2j}}{1 + q^{2j}}, \quad (6.3.23)$$

$$\begin{aligned} \varphi^2(q) \varphi^2(q^4) &= 1 + 2 \sum_{j=1}^{\infty} \frac{j q^j}{1 + (-q)^j} + 4 \sum_{j=1}^{\infty} \frac{(-1)^j j q^{4j}}{1 + q^{4j}} \\ &\quad - 2 \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1) q^{2j-1}}{1 + q^{4j-2}} + 2 \sum_{j=1}^{\infty} \frac{(-1)^j j q^{2j}}{1 + q^{2j}}, \end{aligned} \quad (6.3.24)$$

$$\begin{aligned} \varphi(q) \varphi^3(q^4) &= 1 + \sum_{j=1}^{\infty} \frac{j q^j}{1 + (-q)^j} + 3 \sum_{j=1}^{\infty} \frac{(-1)^j j q^{2j}}{1 + q^{2j}} \\ &\quad + 4 \sum_{j=1}^{\infty} \frac{(-1)^j j q^{4j}}{1 + q^{4j}} - \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1) q^{2j-1}}{1 + q^{4j-2}}, \end{aligned} \quad (6.3.25)$$

$$\begin{aligned} \varphi(q) \varphi^2(q^2) \varphi(q^4) &= 1 + 2 \sum_{j=1}^{\infty} \frac{j q^j}{1 + (-q)^j} + 2 \sum_{j=1}^{\infty} \frac{(-1)^j j q^{2j}}{1 + q^{2j}} \\ &\quad + 4 \sum_{j=1}^{\infty} \frac{(-1)^j j q^{4j}}{1 + q^{4j}}; \end{aligned} \quad (6.3.26)$$

$$\varphi^4(q) \varphi^2(q^2) = 1 + 2 \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1)^2 q^{2j-1}}{1 - q^{2j-1}} + 8 \sum_{j=1}^{\infty} \frac{j^2 q^j}{1 + q^{2j}}$$



$$-2 \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1)^2 q^{2j-1}}{1+q^{2j-1}}, \quad (6.3.27)$$

$$\begin{aligned} \varphi^2(q) \varphi^4(q^2) &= 1 + 2 \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1)^2 q^{2j-1}}{1-q^{2j-1}} + 4 \sum_{j=1}^{\infty} \frac{j^2 q^j}{1+q^{2j}} \\ &\quad - 2 \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1)^2 q^{2j-1}}{1+q^{2j-1}}, \end{aligned} \quad (6.3.28)$$

$$\begin{aligned} \psi^3(q^2) \psi(q^8) &= \frac{1}{32q} \sum_{j=1}^{\infty} \frac{(2j-1) q^{\frac{j-2}{2}}}{1-q^{j-1/2}} + \frac{1}{32q} \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1) q^{\frac{j-2}{2}}}{1+q^{j-1/2}} \\ &\quad - \frac{1}{32q} \sum_{j=1}^{\infty} \frac{(2j-1) (-q)^{\frac{j-2}{2}}}{1-(-q)^{j-1/2}} - \frac{1}{32q} \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1) (-q)^{\frac{j-2}{2}}}{1+(-q)^{j-1/2}}. \end{aligned} \quad (6.3.29)$$

**Proof** Using the series expansions of the sixteen Lambert series in Section 10 of Chapter 2, the right hand sides of the results in Lemma 6.3.1 can be represented explicitly as Lambert series. This completes the proof. ■

Note that the Lambert series expansions for (6.3.1), (6.3.3), (6.3.9), (6.3.12)–(6.3.16), (6.3.18)–(6.3.20) are the same as the ones given in Chapter 4. Therefore they are omitted here.

We use the above theorem to establish an arithmetic interpretation of the following corollary.

**Corollary 6.3.3** For  $n \geq 1$ ,

$$r_{(1,4)}(n) = k \left[ \sum_{\substack{d|n \\ d \equiv 1 \pmod{4}}} 1 - \sum_{\substack{d|n \\ d \equiv 3 \pmod{4}}} 1 \right] \quad (6.3.30)$$

where

$$k = \begin{cases} 4 & : n \equiv 0 \pmod{4}, \\ 2 & : n \equiv 1 \pmod{4}, \\ 0 & : n \equiv 2, 3 \pmod{4}. \end{cases}$$

$$r_4(n) = k \sum_{\substack{d|n \\ d \text{ odd}}} d, \quad (6.3.31)$$

where

$$k = \begin{cases} 8 & : n \equiv 1 \pmod{2}, \\ 24 & : n \equiv 0 \pmod{2}. \end{cases}$$

$$r_{(1,1,1,4)}(n) = k \sum_{\substack{d|n \\ d \text{ odd}}} d \quad (6.3.32)$$

where

$$k = \begin{cases} 6 & : n \equiv 1 \pmod{4}, \\ 12 & : n \equiv 2 \pmod{4}, \\ 2 & : n \equiv 3 \pmod{4}, \\ 8 & : n \equiv 4 \pmod{8}, \\ 24 & : n \equiv 0 \pmod{8}. \end{cases}$$

$$r_{(1,1,2,2)}(n) = k \sum_{\substack{d|n \\ d \text{ odd}}} d \quad (6.3.33)$$

where

$$k = \begin{cases} 4 & : n \equiv 1 \pmod{2}, \\ 8 & : n \equiv 2 \pmod{4}, \\ 24 & : n \equiv 0 \pmod{4}. \end{cases}$$

$$r_{(1,1,4,4)}(n) = k \sum_{\substack{d|n \\ d \text{ odd}}} d \quad (6.3.34)$$

where

$$k = \begin{cases} 4 & : n \equiv 1, 2 \pmod{4}, \\ 0 & : n \equiv 3 \pmod{4}, \\ 8 & : n \equiv 4 \pmod{8}, \\ 24 & : n \equiv 0 \pmod{8}. \end{cases}$$

$$r_{(1,2,2,4)}(n) = k \sum_{\substack{d|n \\ d \text{ odd}}} d \quad (6.3.35)$$

where

$$k = \begin{cases} 2 & : n \equiv 1 \pmod{2}, \\ 4 & : n \equiv 2 \pmod{4}, \\ 8 & : n \equiv 4 \pmod{8}, \\ 24 & : n \equiv 0 \pmod{8}. \end{cases}$$

$$r_{(1,4,4,4)}(n) = k \sum_{\substack{d|n \\ d \text{ odd}}} d \quad (6.3.36)$$

where

$$k = \begin{cases} 0 & : n \equiv 2, 3 \pmod{4}, \\ 2 & : n \equiv 1 \pmod{4}, \\ 8 & : n \equiv 4 \pmod{8}, \\ 24 & : n \equiv 0 \pmod{8}. \end{cases}$$

$$r_{(1,1,1,1,2,2)}(n) = 4 \sum_{\substack{d|n \\ d \text{ odd}, n \text{ even}}} (-1)^{\frac{d+1}{2}} d^2 - 8 \sum_{\substack{d|n \\ d \text{ odd}}} (-1)^{\frac{d+1}{2}} \left(\frac{n}{d}\right)^2, \quad (6.3.37)$$

$$r_{(1,1,2,2,2,2)}(n) = 4 \sum_{\substack{d|n \\ d \text{ odd}, n \text{ even}}} (-1)^{\frac{d+1}{2}} d^2 - 4 \sum_{\substack{d|n \\ d \text{ odd}}} (-1)^{\frac{d+1}{2}} \left(\frac{n}{d}\right)^2, \quad (6.3.38)$$

$$t_{(2,2,2,8)}(n) = \begin{cases} \frac{1}{8} \sum_{d|4n+7} d, & : n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \quad (6.3.39)$$

Note the abbreviation  $k(n) = k$ .

**Proof** The details of (6.3.30), (6.3.37)–(6.3.39) are similar to the proof of Corollary 4.3.2.

We do give complete details for the proof of (6.3.32). Then equations (6.3.31), (6.3.33)–(6.3.36) can be proved in a similar way.

First use (4.1.7) and expand the right hand side using the geometric series in (6.3.22), so

that  $r_{(1,1,1,4)}(n)$  can be represented as

$$\begin{aligned} r_{(1,1,1,4)}(n) = & 4 \sum_{\substack{d|n \\ d \text{ odd}}} d + 2 \sum_{\substack{d|n, n \text{ odd} \\ d \equiv 1 \pmod{4}}} (-1)^{\frac{n+2+d}{2d}} d - 2 \sum_{\substack{d|n, n \text{ odd} \\ d \equiv 3 \pmod{4}}} (-1)^{\frac{n+2+d}{2d}} d \\ & + 4 \sum_{\substack{d|n \\ d \text{ even}}} d - 5 \sum_{\substack{d|n \\ 4|d}} d + 3 \sum_{\substack{d|n \\ 8|d}} d - 2 \sum_{\substack{d|n \\ 16|d}} d. \end{aligned}$$

If  $n \equiv 1 \pmod{4}$  then  $n = 4k + 1$  and so

$$\begin{aligned} r_{(1,1,1,4)}(n) &= 4 \sum_{\substack{d|n \\ d \text{ odd}}} d + 2 \sum_{\substack{d|4k+1 \\ d \equiv 1 \pmod{4}}} d + 2 \sum_{\substack{d|4k+1 \\ d \equiv 3 \pmod{4}}} d \\ &= 6 \sum_{\substack{d|n \\ d \text{ odd}}} d. \end{aligned} \tag{6.3.40}$$

Similarly, if  $n \equiv 3 \pmod{4}$  then  $n = 4k + 3$  and so

$$\begin{aligned} r_{(1,1,1,4)}(n) &= 4 \sum_{\substack{d|n \\ d \text{ odd}}} d - 2 \sum_{\substack{d|4k+3 \\ d \equiv 1 \pmod{4}}} d - 2 \sum_{\substack{d|4k+3 \\ d \equiv 3 \pmod{4}}} d \\ &= 2 \sum_{\substack{d|n \\ d \text{ odd}}} d. \end{aligned} \tag{6.3.41}$$

If  $n \equiv 2 \pmod{4}$  then  $n = 4k + 2$  and so

$$\begin{aligned} r_{(1,1,1,4)}(n) &= 4 \sum_{\substack{d|4k+2 \\ d \text{ odd}}} d + 4 \sum_{\substack{d|4k+2 \\ d \text{ even}}} d \\ &= 4 \sum_{\substack{d|n \\ d \text{ odd}}} d + 4 \sum_{\substack{d|n \\ d \text{ odd}}} 2d \\ &= 12 \sum_{\substack{d|n \\ d \text{ odd}}} d. \end{aligned} \tag{6.3.42}$$

If  $n \equiv 4 \pmod{8}$  then  $n = 8k + 4$  and so

$$\begin{aligned}
 r_{(1,1,1,4)}(n) &= 4 \sum_{\substack{d|8k+4 \\ d \text{ odd}}} d + 4 \sum_{\substack{d|8k+4 \\ d \text{ even}}} d - 5 \sum_{\substack{d|8k+4 \\ 4|d}} d \\
 &= 4 \sum_{\substack{d|n \\ d \text{ odd}}} d + 4 \sum_{\substack{d|n \\ d \text{ odd}}} (2d + 4d) - 5 \sum_{\substack{d|n \\ d \text{ odd}}} 4d \\
 &= 8 \sum_{\substack{d|n \\ d \text{ odd}}} d.
 \end{aligned} \tag{6.3.43}$$

If  $n \equiv 0 \pmod{8}$  then  $n = 8k + 8$  and so

$$\begin{aligned}
 r_{(1,1,1,4)}(n) &= 4 \sum_{\substack{d|8k+8 \\ d \text{ odd}}} d + 4 \sum_{\substack{d|8k+8 \\ d \text{ even}}} d - 5 \sum_{\substack{d|8k+8 \\ 4|d}} d + 3 \sum_{\substack{d|8k+8 \\ 8|d}} d \\
 &= 4 \sum_{\substack{d|n \\ d \text{ odd}}} d + 4 \sum_{\substack{d|n \\ d \text{ odd}}} (2d + 4d + 8d) - 5 \sum_{\substack{d|n \\ d \text{ odd}}} (4d + 8d) + 3 \sum_{\substack{d|n \\ d \text{ odd}}} 8d \\
 &= 24 \sum_{\substack{d|n \\ d \text{ odd}}} d.
 \end{aligned} \tag{6.3.44}$$

If  $n \equiv 2^j \pmod{16k+16}$  where  $j, k = 0, 1, 2, 3, \dots$ , then

$$\begin{aligned}
 r_{(1,1,1,4)}(n) &= 4 \sum_{\substack{d|2^j(16k+16) \\ d \text{ odd}}} d + 4 \sum_{\substack{d|2^j(16k+16) \\ d \text{ even}}} d - 5 \sum_{\substack{d|2^j(16k+16) \\ 4|d}} d + 3 \sum_{\substack{d|2^j(16k+16) \\ 8|d}} d \\
 &\quad - 2 \sum_{\substack{d|2^j(16k+16) \\ 16|d}} d \\
 &= 4 \sum_{\substack{d|n \\ d \text{ odd}}} d + 4 \sum_{\substack{d|n \\ d \text{ odd}}} (2 + 4 + 8 + \dots + 2^{j+4}) d - 5 \sum_{\substack{d|n \\ d \text{ odd}}} (4 + 8 + \dots + 2^{j+4}) d \\
 &\quad + 3 \sum_{\substack{d|n \\ d \text{ odd}}} (8 + 16 + \dots + 2^{j+4}) d - 2 \sum_{\substack{d|n \\ d \text{ odd}}} (16 + 32 + \dots + 2^{j+4}) d \\
 &= 24 \sum_{\substack{d|n \\ d \text{ odd}}} d.
 \end{aligned} \tag{6.3.45}$$

Combining (6.3.40)–(6.3.45), we arrive at (6.3.32).

This completes the proof of Corollary 6.3.3. ■

Formulae (6.3.31)–(6.3.36) can be found in Liouville [76]–[82]. Identity (6.3.23) was given by Ramanujan [93]; it also appeared in Fine [54, p. 89] and Chan [36, p. 620]. Fine [54] gave equivalent results of (6.3.31)–(6.3.33). Adiga, Cooper, and Han [4] gave a remarkable formula which can lead to the results of (4.3.25)–(4.3.27), (4.3.29)–(4.3.31), (6.3.39) and (4.3.33) directly from (4.2.29)–(4.2.31), (4.2.33), (4.2.34), (6.3.30)–(6.3.33), respectively. Cooper [41] mentioned that Theorem 6.3.2 may be proved directly from Ramanujan's  ${}_1\psi_1$  summation formula.

**Corollary 6.3.4** *Let the prime factorization of  $n$  be given by (4.2.37). Then*

$$r_{(1,4)}(n) = k \prod_{p \equiv 1 \pmod{4}} (\lambda_p + 1) \prod_{p \equiv 3 \pmod{4}} \frac{1 + (-1)^{\lambda_p}}{2} \quad (6.3.46)$$

where

$$k = \begin{cases} 4 & : n \equiv 0 \pmod{4}, \\ 2 & : n \equiv 1 \pmod{4}, \\ 0 & : n \equiv 2, 3 \pmod{4}. \end{cases}$$

$$r_{(1,1,1,4)}(n) = k \prod_p \frac{p^{\lambda_p+1} - 1}{p - 1} \quad (6.3.47)$$

where

$$k = \begin{cases} 6 & : n \equiv 1 \pmod{4}, \\ 12 & : n \equiv 2 \pmod{4}, \\ 2 & : n \equiv 3 \pmod{4}, \\ 8 & : n \equiv 4 \pmod{8}, \\ 24 & : n \equiv 0 \pmod{8}. \end{cases}$$

$$r_{(1,1,2,2)}(n) = k \prod_p \frac{p^{\lambda_p+1} - 1}{p - 1} \quad (6.3.48)$$

where

$$k = \begin{cases} 4 & : n \equiv 1 \pmod{2}, \\ 8 & : n \equiv 2 \pmod{4}, \\ 24 & : n \equiv 0 \pmod{4}. \end{cases}$$

$$r_{(1,1,4,4)}(n) = k \prod_p \frac{p^{\lambda_p+1} - 1}{p - 1} \quad (6.3.49)$$

where

$$k = \begin{cases} 4 & : n \equiv 1, 2 \pmod{4}, \\ 0 & : n \equiv 3 \pmod{4}, \\ 8 & : n \equiv 4 \pmod{8}, \\ 24 & : n \equiv 0 \pmod{8}. \end{cases}$$

$$r_{(1,2,2,4)}(n) = k \prod_p \frac{p^{\lambda_p+1} - 1}{p - 1} \quad (6.3.50)$$

where

$$k = \begin{cases} 2 & : n \equiv 1 \pmod{2}, \\ 4 & : n \equiv 2 \pmod{4}, \\ 8 & : n \equiv 4 \pmod{8}, \\ 24 & : n \equiv 0 \pmod{8}. \end{cases}$$

$$r_{(1,4,4,4)}(n) = k \prod_p \frac{p^{\lambda_p+1} - 1}{p - 1} \quad (6.3.51)$$

where

$$k = \begin{cases} 0 & : n \equiv 2, 3 \pmod{4}, \\ 2 & : n \equiv 1 \pmod{4}, \\ 8 & : n \equiv 4 \pmod{8}, \\ 24 & : n \equiv 0 \pmod{8}. \end{cases}$$

$$r_{(1,1,1,1,2,2)}(n) = k \prod_p \frac{p^{2\lambda_p+2} - (-1)^{(\lambda_p+1)(p-1)/2}}{p^2 - (-1)^{(p-1)/2}} \quad (6.3.52)$$

where

$$k = \begin{cases} 8 & : n \equiv 1 \pmod{2}, \\ 4(2^{2\lambda_2+1} - 1) & : n \equiv 0 \pmod{2}. \end{cases}$$

$$r_{(1,1,2,2,2,2)}(n) = k \prod_p \frac{p^{2\lambda_p+2} - (-1)^{(\lambda_p+1)(p-1)/2}}{p^2 - (-1)^{(p-1)/2}} \quad (6.3.53)$$

where

$$k = \begin{cases} 4 & : n \equiv 1 \pmod{2}, \\ 4(2^{2\lambda_2} - 1) & : n \equiv 0 \pmod{2}. \end{cases}$$

$$t_{(1,1,1,4)}(n) = \frac{1}{8} \prod_{p|8n+7} \frac{p^{\lambda_p+1} - 1}{p - 1}. \quad (6.3.54)$$

**Proof** The details are similar to those of the proof of Corollary 4.2.5. ■

We end this section by presenting the formulae for  $\varphi^{2n}(q)$  and  $\psi^{2n}(q)$  where  $n \geq 2$ , which were given by Cooper [39].

**Theorem 6.3.5** [39] For  $n \geq 1$ ,

$$\begin{aligned} |E_{2n}| \varphi^{4n+2}(q) &= |E_{2n}| - 4(-1)^n \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1)^{2n} q^{2j-1}}{1 - q^{2j-1}} \\ &\quad + 2^{2n+2} \sum_{j=1}^{\infty} \frac{j^{2n} q^j}{1 + q^{2j}} + \frac{(-q; -q)_{\infty}^{8n+4}}{(q^4; q^4)_{\infty}^{4n+2}} \\ &\quad \times \sum_{j=1}^{\lfloor n/2 \rfloor} d_j 16^j q^j \frac{(q^2; q^2)_{\infty}^{24j}}{(-q; -q)_{\infty}^{24j}}, \end{aligned} \quad (6.3.55)$$

$$\begin{aligned} \frac{2^{2n} (2^{2n} - 1) |B_{2n}|}{2n} \varphi^{4n}(q) &= \frac{2^{2n} (2^{2n} - 1) |B_{2n}|}{2n} + 2^{2n+1} \sum_{j=1}^{\infty} \frac{j^{2n-1} q^j}{1 - (-1)^{n+j} q^j} \\ &\quad + \frac{(-q; -q)_{\infty}^{8n}}{(q^2; q^2)_{\infty}^{4n}} \sum_{j=1}^{\lfloor (n-1)/2 \rfloor} d_j 16^j q^j \frac{(q^2; q^2)_{\infty}^{24j}}{(-q; -q)_{\infty}^{24j}}, \end{aligned} \quad (6.3.56)$$

$$\begin{aligned} |E_{2n}| \psi^{4n+2}(q^2) &= \frac{1}{2^{4n}} \sum_{j=1}^{\infty} (2j-1)^{2n} q^{j-n-1} \left\{ \frac{1}{1 + q^{2j-1}} - \frac{(-1)^{n+j}}{1 - q^{2j-1}} \right\} \\ &\quad + \frac{(q^4; q^4)_{\infty}^{8n+4}}{(q^2; q^2)_{\infty}^{4n+2}} \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{(-1)^j d_j (q^2; q^2)_{\infty}^{24j}}{(256q^2)^j (q^4; q^4)_{\infty}^{24j}}, \end{aligned} \quad (6.3.57)$$

$$\begin{aligned} \frac{2^{2n} (2^{2n} - 1) |B_{2n}|}{2n} \psi^{4n}(q^2) &= \frac{1}{4^{n-1}} \sum_{j=1}^{\infty} \left( \frac{1 + (-1)^{n+j}}{2} \right) \frac{j^{2n-1} q^{j-n}}{1 - q^{2j}} \\ &\quad + \frac{(q^4; q^4)_{\infty}^{8n}}{(q^2; q^2)_{\infty}^{4n}} \sum_{j=1}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^j d_j (q^2; q^2)_{\infty}^{24j}}{(256q^2)^j (q^4; q^4)_{\infty}^{24j}}, \end{aligned} \quad (6.3.58)$$

for some integers  $d_j$ .



Liouville [83], [84] gave formulae for  $\varphi^{10}(q)$  and  $\varphi^{12}(q)$ . Glaisher [58] gave formulae for  $\varphi^{2k}(q)$  up to  $2k = 18$ . In 1916 Ramanujan [94] stated a general formula for  $\varphi^{2k}(q)$ , proved by L. J. Mordell [89] in 1917. Note that these formulae will involve cusp forms for  $\varphi^{2k}(q)$ , where  $k > 4$ . An excellent source of information can be found in Dickson's book [48].

Theorem 6.3.5 is equivalent to Ramanujan's formulae [94, pp. 158, 159, and 191]. Cooper [39] employed equations (2.9.5)–(2.9.7), (2.10.7), and (2.10.10) to obtain (6.3.55). Similarly, using (2.9.5)–(2.9.7) with (2.10.5) and (2.10.9); (2.10.12) and (2.10.15); (2.10.9) and (2.10.13); we obtain (6.3.56)–(6.3.58), respectively.

## 6.4 Several results involving both sums of squares and triangular numbers

In this section we derive fourteen interesting infinite products and their Lambert series expansions. From these, we deduce formulae for the number of representations of an integer  $n$  by fourteen different quadratic form in terms of divisor sum or of some products of integers which contain primes. The following lemma is now proven.

### Lemma 6.4.1

$$\varphi(q) \psi(q^8) = \frac{1}{2q} \left[ -\frac{1}{2} + f_2^{(0)}(\pi\tau) - i f_1^{(0)}(\pi + \pi\tau; q^2) \right], \quad (6.4.1)$$

$$\varphi(q^4) \psi(q^2) = \frac{1}{4q^{\frac{1}{4}}} \left[ f_2^{(0)}\left(\pi + \pi\tau; q^{\frac{1}{2}}\right) + \frac{1}{i^{\frac{1}{2}}} f_2^{(0)}\left(\pi + \pi\tau; i q^{\frac{1}{2}}\right) \right]; \quad (6.4.2)$$

$$\varphi^3(q) \psi(q^8) = -\frac{1}{q} f_1'(\pi; i q^{\frac{1}{2}}) + \frac{1}{q} f_1'(\pi; q^2) + \frac{i}{q} f_3'(\pi + \pi\tau; q^2), \quad (6.4.3)$$

$$\varphi^2(q) \psi^2(q) = \frac{1}{2q^{\frac{1}{2}}} \left[ f_2' \left( \pi\tau; q^{\frac{1}{2}} \right) - i f_3' \left( \pi + \pi\tau; q^{\frac{1}{2}} \right) \right], \quad (6.4.4)$$

$$\varphi^2(q) \psi^2(q^4) = \frac{-1}{2q} f_1' \left( \pi; iq^{\frac{1}{2}} \right) + \frac{1}{2q} f_1'(\pi), \quad (6.4.5)$$

$$\begin{aligned} \varphi^2(q) \psi^2(q^8) &= \frac{-1}{4q^2} f_1' \left( \pi; iq^{\frac{1}{2}} \right) + \frac{1}{2q^2} f_1'(\pi; q^2) + \frac{1}{2q^2} i f_3'(\pi + \pi\tau; q^2) \\ &\quad - \frac{1}{4q^2} f_1'(\pi), \end{aligned} \quad (6.4.6)$$

$$\begin{aligned} \varphi(q) \psi^3(q^8) &= -\frac{1}{16q^3} f_1' \left( \pi; iq^{\frac{1}{2}} \right) - \frac{3}{16q^3} f_1'(\pi) + \frac{1}{4q^3} f_1'(\pi; q^2) \\ &\quad + \frac{i}{8q^3} f_3'(\pi + \pi\tau; q^2), \end{aligned} \quad (6.4.7)$$

$$\begin{aligned} \varphi(q^4) \psi^3(q^2) &= \frac{1}{16q^{\frac{3}{4}}} f_2' \left( \pi\tau; q^{\frac{1}{2}} \right) + \frac{i}{16q^{\frac{3}{4}}} f_2' \left( \pi\tau; -q^{\frac{1}{2}} \right) \\ &\quad - \frac{i^{\frac{1}{2}}}{16q^{\frac{3}{4}}} f_2' \left( \pi\tau; iq^{\frac{1}{2}} \right) + \frac{1}{16i^{\frac{1}{2}}q^{\frac{3}{4}}} f_3' \left( \pi + \pi\tau; iq^{\frac{1}{2}} \right), \end{aligned} \quad (6.4.8)$$

$$\varphi(q) \psi^2(q^4) \phi(q^4) = -\frac{1}{4q} f_1' \left( \pi; iq^{\frac{1}{2}} \right) + \frac{1}{4q} f_1'(\pi) - \frac{i}{2q} f_3'(\pi + \pi\tau; q^2), \quad (6.4.9)$$

$$\varphi(q) \varphi^2(q^2) \psi(q^8) = -\frac{1}{2q} f_1' \left( \pi; iq^{\frac{1}{2}} \right) - \frac{1}{2q} f_1'(\pi) + \frac{1}{q} f_1'(\pi; q^2), \quad (6.4.10)$$

$$\begin{aligned} \varphi(q) \psi^2(q^4) \psi(q^8) &= -\frac{1}{8q^2} f_1' \left( \pi; iq^{\frac{1}{2}} \right) + \frac{1}{8q^2} f_1'(\pi) + \frac{i}{4q^2} f_3'(\pi + \pi\tau; q^2); \end{aligned} \quad (6.4.11)$$

$$\varphi^2(q) \psi^4(q) = -\frac{2i}{q^{\frac{1}{2}}} f_3^{(2)}(\pi\tau), \quad (6.4.12)$$

$$\varphi^2(q) \psi^4(q^4) = \frac{1}{4q^2} f_2^{(2)}(\pi) - \frac{i}{4q^2} f_1^{(2)}(\pi\tau) - \frac{1}{4q^2} f_3^{(2)}(\pi), \quad (6.4.13)$$

$$\varphi^4(q) \psi^2(q^4) = \frac{1}{q} f_2^{(2)}(\pi) - \frac{i}{q} f_1^{(2)}(\pi\tau) - \frac{1}{q} f_3^{(2)}(\pi). \quad (6.4.14)$$

**Proof** The details are similar to the proof of Lemma 6.3.1. ■

We can now rewrite the right hand side of Lemma 6.4.1 as Lambert series expansions.

**Theorem 6.4.2**

$$\varphi(q) \psi(q^8) = - \sum_{j=1}^{\infty} \frac{(-1)^j q^{2j-2}}{1 - q^{2j-1}} - \sum_{j=1}^{\infty} \frac{(-1)^j q^{2j-1}}{1 + q^{4j}}, \quad (6.4.15)$$

$$\varphi(q^4) \psi(q^2) = -\frac{1}{2} \sum_{j=1}^{\infty} \frac{(-1)^j q^{\frac{j-1}{2}}}{1 - q^{\frac{2j-1}{2}}} - \frac{1}{2} \sum_{j=1}^{\infty} \frac{(-1)^j (-q)^{\frac{j-1}{2}}}{1 - (-q)^{\frac{2j-1}{2}}}, \quad (6.4.16)$$

$$\begin{aligned} \varphi^3(q) \psi(q^8) &= 2 \sum_{j=1}^{\infty} \frac{j q^{j-1}}{1 + (-q)^j} - 2 \sum_{j=1}^{\infty} \frac{(-1)^j j q^{4j-1}}{1 + q^{4j}} \\ &\quad + \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1) q^{2j-2}}{1 + q^{4j-2}}, \end{aligned} \quad (6.4.17)$$

$$\varphi^2(q) \psi^2(q) = \frac{1}{2} \sum_{j=1}^{\infty} \frac{(2j-1) q^{\frac{j-1}{2}}}{1 - q^{\frac{2j-1}{2}}} - \frac{1}{2} \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1) q^{\frac{j-1}{2}}}{1 + q^{\frac{2j-1}{2}}}, \quad (6.4.18)$$

$$\varphi^2(q) \psi^2(q^4) = \sum_{j=1}^{\infty} \frac{j q^{j-1}}{1 + (-q)^j} - \sum_{j=1}^{\infty} \frac{(-1)^j j q^{2j-1}}{1 + q^{2j}}, \quad (6.4.19)$$

$$\begin{aligned} \varphi^2(q) \psi^2(q^8) &= \frac{1}{2} \sum_{j=1}^{\infty} \frac{j q^{j-2}}{1 + (-q)^j} - \sum_{j=1}^{\infty} \frac{(-1)^j j q^{4j-2}}{1 + q^{4j}} \\ &\quad + \frac{1}{2} \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1) q^{2j-3}}{1 + q^{4j-2}} + \frac{1}{2} \sum_{j=1}^{\infty} \frac{(-1)^j j q^{2j-2}}{1 + q^{2j}}, \end{aligned} \quad (6.4.20)$$

$$\begin{aligned} \varphi(q) \psi^3(q^8) &= \frac{1}{8} \sum_{j=1}^{\infty} \frac{j q^{j-3}}{1 + (-q)^j} + \frac{3}{8} \sum_{j=1}^{\infty} \frac{(-1)^j j q^{2j-3}}{1 + q^{2j}} \\ &\quad - \frac{1}{2} \sum_{j=1}^{\infty} \frac{(-1)^j j q^{4j-3}}{1 + q^{4j}} + \frac{1}{8} \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1) q^{2j-4}}{1 + q^{4j-2}}, \end{aligned} \quad (6.4.21)$$

$$\begin{aligned} \varphi(q^4) \psi^3(q^2) &= \frac{1}{16} \sum_{j=1}^{\infty} \frac{(2j-1) q^{\frac{j-2}{2}}}{1 - q^{\frac{2j-1}{2}}} + \frac{1}{16} \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1) q^{\frac{j-2}{2}}}{1 - (-q^{\frac{1}{2}})^{2j-1}} \\ &\quad - \frac{1}{16} \sum_{j=1}^{\infty} \frac{(-1)^{\frac{j}{2}} (2j-1) q^{\frac{j-2}{2}}}{1 - (-q)^{\frac{2j-1}{2}}} - \frac{1}{16} \sum_{j=1}^{\infty} \frac{(-1)^{\frac{3j}{2}} (2j-1) q^{\frac{j-2}{2}}}{1 + (-q)^{\frac{2j-1}{2}}}, \end{aligned} \quad (6.4.22)$$

$$\begin{aligned}\varphi(q) \varphi(q^4) \psi^2(q^4) &= \frac{1}{2} \sum_{j=1}^{\infty} \frac{j q^{j-1}}{1 + (-q)^j} - \frac{1}{2} \sum_{j=1}^{\infty} \frac{(-1)^j j q^{2j-1}}{1 + q^{2j}} \\ &\quad - \frac{1}{2} \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1) q^{2j-2}}{1 + q^{4j-2}},\end{aligned}\quad (6.4.23)$$

$$\begin{aligned}\varphi(q) \varphi^2(q^2) \psi(q^8) &= \sum_{j=1}^{\infty} \frac{j q^{j-1}}{1 + (-q)^j} + \sum_{j=1}^{\infty} \frac{(-1)^j j q^{2j-1}}{1 + q^{2j}} - 2 \sum_{j=1}^{\infty} \frac{(-1)^j j q^{4j-1}}{1 + q^{4j}},\end{aligned}\quad (6.4.24)$$

$$\begin{aligned}\varphi(q) \psi^2(q^4) \psi(q^8) &= \frac{1}{4} \sum_{j=1}^{\infty} \frac{j q^{j-2}}{1 + (-q)^j} - \frac{1}{4} \sum_{j=1}^{\infty} \frac{(-1)^j j q^{2j-2}}{1 + q^{2j}} \\ &\quad + \frac{1}{4} \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1) q^{2j-3}}{1 + q^{4j-2}};\end{aligned}\quad (6.4.25)$$

$$\varphi^2(q) \psi^4(q) = \sum_{j=1}^{\infty} \frac{(2j-1)^2 q^{j-1}}{1 + q^{2j-1}},\quad (6.4.26)$$

$$\varphi^2(q) \psi^4(q^4) = \frac{1}{4} \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1)^2 q^{2j-3}}{1 - q^{4j-2}} + \frac{1}{4} \sum_{j=1}^{\infty} \frac{j^2 q^{j-2}}{1 + q^{2j}},\quad (6.4.27)$$

$$\varphi^4(q) \psi^2(q^4) = \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1)^2 q^{2j-2}}{1 - q^{4j-2}} + 2 \sum_{j=1}^{\infty} \frac{j^2 q^{j-1}}{1 + q^{2j}}.\quad (6.4.28)$$

**Proof** Using the series expansions of the sixteen Lambert series in Section 10 of Chapter 2, the right hand sides of the results in Lemma 6.4.1 can be represented explicitly as Lambert series. This completes the proof. ■

Next we use Theorem 6.4.2 to establish an arithmetic interpretation of the following corollary.

**Corollary 6.4.3** For  $n \geq 1$ ,

$$r(\square + 8\Delta)(n) = \sum_{\substack{d|n+1 \\ d \text{ odd}}} (-1)^{\frac{d-1}{2}} - \sum_{\substack{d|n+1 \\ n, d \text{ odd}}} (-1)^{\frac{n+d}{2}}, \quad (6.4.29)$$

$$r(4\square + 2\Delta)(n) = \sum_{\substack{d|4n+1 \\ d \equiv 1 \pmod{4}}} 1 - \sum_{\substack{d|4n+1 \\ d \equiv 3 \pmod{4}}} 1, \quad (6.4.30)$$

$$r(\square + \square + \square + 8\Delta)(n) = k \sum_{\substack{d|n+1 \\ d \text{ odd}}} d, \quad (6.4.31)$$

where

$$k = \begin{cases} 6 & : n \equiv 1 \pmod{4}, \\ 3 & : n \equiv 2 \pmod{4}, \\ 8 & : n \equiv 3 \pmod{8}, \\ 1 & : n \equiv 0 \pmod{4}, \\ 0 & : n \equiv 7 \pmod{8}. \end{cases}$$

$$r(\square + \square + \Delta + \Delta)(n) = \sum_{d|4n+1} d, \quad (6.4.32)$$

$$r(\square + \square + 2\Delta + 2\Delta)(n) = \sum_{d|2n+1} d, \quad (6.4.33)$$

$$r(\square + \square + 4\Delta + 4\Delta)(n) = k \sum_{\substack{d|n+1 \\ d \text{ odd}}} d, \quad (6.4.34)$$

where

$$k = \begin{cases} 4 & : n \equiv 1 \pmod{4}, \\ 0 & : n \equiv 3 \pmod{4}, \\ 1 & : n \equiv 0 \pmod{2}. \end{cases}$$

$$r(\square + \square + 8\Delta + 8\Delta)(n) = k \sum_{\substack{d|n+2 \\ d \text{ odd}}} d, \quad (6.4.35)$$

where

$$k = \begin{cases} 1 & : n \equiv 0, 1 \pmod{4}, \\ 0 & : n \equiv 3 \pmod{4}, \\ 4 & : n \equiv 2 \pmod{8}, \\ 0 & : n \equiv 6 \pmod{8}. \end{cases}$$

$$r(\square + 8\Delta + 8\Delta + 8\Delta)(n) = k \sum_{\substack{d|n+3 \\ d \text{ odd}}} d, \quad (6.4.36)$$

where

$$k = \begin{cases} 2 & : n \equiv 1 \pmod{8}, \\ \frac{1}{4} & : n \equiv 0 \pmod{4}, \\ 0 & : \text{otherwise.} \end{cases}$$

$$r(4\square + 2\Delta + 2\Delta + 2\Delta)(n) = k \sum_{d|4n+3} d, \quad (6.4.37)$$

where

$$k = \begin{cases} 0 & : n \equiv 1 \pmod{2}, \\ \frac{1}{4} & : n \equiv 0 \pmod{2}. \end{cases}$$

$$r(\square + 4\square + 4\Delta + 4\Delta)(n) = k \sum_{\substack{d|n+1 \\ d \text{ odd}}} d, \quad (6.4.38)$$

where

$$k = \begin{cases} 2 & : n \equiv 1 \pmod{4}, \\ 0 & : n \equiv 2, 3 \pmod{4}, \\ 1 & : n \equiv 0 \pmod{4}. \end{cases}$$

$$r(\square + 2\square + 2\square + 8\Delta)(n) = k \sum_{\substack{d|n+1 \\ d \text{ odd}}} d, \quad (6.4.39)$$

where

$$k = \begin{cases} 2 & : n \equiv 1 \pmod{4}, \\ 1 & : n \equiv 0 \pmod{2}, \\ 8 & : n \equiv 3 \pmod{8}, \\ 0 & : n \equiv 7 \pmod{8}. \end{cases}$$

$$r(\square + 4\Delta + 4\Delta + 8\Delta)(n) = k \sum_{\substack{d|n+2 \\ d \text{ odd}}} d, \quad (6.4.40)$$

where

$$k = \begin{cases} \frac{1}{2} & : n \equiv 1 \pmod{4}, \\ 0 & : n \equiv 2, 3 \pmod{4}, \\ 1 & : n \equiv 0 \pmod{4}. \end{cases}$$

$$r(\square + \square + \triangle + \triangle + \triangle + \triangle)(n)$$

$$= \sum_{\substack{d|2n+1 \\ d \text{ odd}}} (-1)^{\frac{d-1}{2}} \left( \frac{2n+1}{d} \right)^2. \quad (6.4.41)$$

$$r(\square + \square + 4\triangle + 4\triangle + 4\triangle + 4\triangle)(n)$$

$$= k \sum_{\substack{d|n+2 \\ d \text{ odd}}} (-1)^{\frac{d-1}{2}} \left( \frac{n+2}{d} \right)^2, \quad (6.4.42)$$

where

$$k = \begin{cases} \frac{1}{4} & : n \equiv 0 \pmod{2}, \\ \frac{1}{2} & : n \equiv 1 \pmod{4}, \\ 0 & : n \equiv 3 \pmod{4}. \end{cases}$$

$$r(\square + \square + \square + \square + 4\triangle + 4\triangle)(n)$$

$$= k \sum_{\substack{d|n+1 \\ d \text{ odd}}} (-1)^{\frac{d-1}{2}} \left( \frac{n+1}{d} \right)^2, \quad (6.4.43)$$

where

$$k = \begin{cases} 1 & : n \equiv 0 \pmod{4}, \\ 2 & : n \equiv 1 \pmod{2}, \\ 3 & : n \equiv 2 \pmod{4}. \end{cases}$$

**Proof** The details are similar to the proof of Corollary 6.3.3. ■

We remark that since equation (6.1.1) is equivalent to

$$\begin{aligned} & 2\lambda_1 x_1^2 + 2\lambda_2 x_2^2 + \cdots + 2\lambda_k x_k^2 \\ & + \mu_1 \left(y_1 + \frac{1}{2}\right)^2 + \mu_2 \left(y_2 + \frac{1}{2}\right)^2 + \cdots + \mu_m \left(y_m + \frac{1}{2}\right)^2 \\ & = 2n + \frac{m}{4}. \end{aligned}$$

Then geometrically,  $2^m r(\lambda_1 \square + \lambda_2 \square + \cdots + \lambda_k \square + \mu_1 \triangle + \mu_2 \triangle + \cdots + \mu_m \triangle)(n)$  counts the number of lattice points on the  $k + m$  dimensional ellipsoid centred at  $(0, 0, \dots, 0, -\frac{1}{2}, -\frac{1}{2}, \dots, -\frac{1}{2})$ , the point whose first  $k$  coordinates are 0 and remaining  $m$  coordinates are  $-\frac{1}{2}$ , with radius  $\sqrt{2n + \frac{m}{4}}$ .

Identities (6.4.16) and (6.4.26) were given by Ramanujan [92, Chapter 17]. Berndt [19, p. 140] has given a proof of (6.4.26). In [35, p. 71] and [36, p. 620], Chan gave proofs of equivalent identities to (6.4.18), (6.4.19), and (6.4.26).

**Corollary 6.4.4** *Let the prime factorization of  $n$  be given by (4.2.37). Then*

$$r(\square + 8\triangle)(n) = k \prod_{\substack{p|n+1 \\ p \equiv 1 \pmod{4}}} (\lambda_p + 1) \prod_{\substack{p|n+1 \\ p \equiv 3 \pmod{4}}} \frac{1 + (-1)^{\lambda_p}}{2}, \quad (6.4.44)$$

where

$$k = \begin{cases} 1 & : n \equiv 0 \pmod{2}, \\ 2 & : n \equiv 1 \pmod{4}, \\ 0 & : n \equiv 3 \pmod{4}. \end{cases}$$

$$r(2\square + \triangle)(n) = \prod_{\substack{p|8n+1 \\ p \equiv 1 \pmod{4}}} (\lambda_p + 1) \prod_{\substack{p|8n+1 \\ p \equiv 3 \pmod{4}}} \frac{1 + (-1)^{\lambda_p}}{2}, \quad (6.4.45)$$

$$r(\square + \square + \square + 8\triangle)(n) = k \prod_{p|n+1} \frac{p^{\lambda_p+1} - 1}{p - 1} \quad (6.4.46)$$



where

$$k = \begin{cases} 6 & : n \equiv 1 \pmod{4}, \\ 3 & : n \equiv 2 \pmod{4}, \\ 8 & : n \equiv 3 \pmod{8}, \\ 1 & : n \equiv 0 \pmod{4}, \\ 0 & : n \equiv 7 \pmod{8}. \end{cases}$$

$$r(\square + \square + \triangle + \triangle)(n) = \prod_{p|4n+1} \frac{p^{\lambda_p+1} - 1}{p - 1}, \quad (6.4.47)$$

$$r(\square + \square + 2\triangle + 2\triangle)(n) = \prod_{p|2n+1} \frac{p^{\lambda_p+1} - 1}{p - 1}, \quad (6.4.48)$$

$$r(\square + \square + 4\triangle + 4\triangle)(n) = k \prod_{p|n+1} \frac{p^{\lambda_p+1} - 1}{p - 1} \quad (6.4.49)$$

where

$$k = \begin{cases} 4 & : n \equiv 1 \pmod{4}, \\ 0 & : n \equiv 3 \pmod{4}, \\ 1 & : n \equiv 0 \pmod{2}. \end{cases}$$

$$r(\square + \square + 8\triangle + 8\triangle)(n) = k \prod_{p|n+2} \frac{p^{\lambda_p+1} - 1}{p - 1} \quad (6.4.50)$$

where

$$k = \begin{cases} 1 & : n \equiv 0, 1 \pmod{4}, \\ 4 & : n \equiv 2 \pmod{8}, \\ 0 & : n \equiv 3 \pmod{4}, \\ 0 & : n \equiv 6 \pmod{8}. \end{cases}$$

$$r(\square + 8\triangle + 8\triangle + 8\triangle)(n) = k \prod_{p|n+3} \frac{p^{\lambda_p+1} - 1}{p - 1} \quad (6.4.51)$$

where

$$k = \begin{cases} 2 & : n \equiv 1 \pmod{8}, \\ \frac{1}{4} & : n \equiv 0 \pmod{4}, \\ 0 & : \text{otherwise.} \end{cases}$$

$$r(2\Box + \Delta + \Delta + \Delta)(n) = \frac{1}{4} \prod_{p|8n+3} \frac{p^{\lambda_p+1} - 1}{p - 1}, \quad (6.4.52)$$

$$r(\Box + 4\Box + 4\Delta + 4\Delta)(n) = k \prod_{p|n+1} \frac{p^{\lambda_p+1} - 1}{p - 1} \quad (6.4.53)$$

where

$$k = \begin{cases} 2 & : n \equiv 1 \pmod{4}, \\ 0 & : n \equiv 2, 3 \pmod{4}, \\ 1 & : n \equiv 0 \pmod{4}. \end{cases}$$

$$r(\Box + 2\Box + 2\Box + 8\Delta)(n) = k \prod_{p|n+1} \frac{p^{\lambda_p+1} - 1}{p - 1} \quad (6.4.54)$$

where

$$k = \begin{cases} 2 & : n \equiv 1 \pmod{4}, \\ 1 & : n \equiv 0 \pmod{2}, \\ 8 & : n \equiv 3 \pmod{8}, \\ 0 & : n \equiv 7 \pmod{8}. \end{cases}$$

$$r(\Box + 4\Delta + 4\Delta + 8\Delta)(n) = k \prod_{p|n+2} \frac{p^{\lambda_p+1} - 1}{p - 1} \quad (6.4.55)$$

where

$$k = \begin{cases} \frac{1}{2} & : n \equiv 1 \pmod{4}, \\ 0 & : n \equiv 2, 3 \pmod{4}, \\ 1 & : n \equiv 0 \pmod{4}. \end{cases}$$

$$r(\Box + \Box + \Delta + \Delta + \Delta + \Delta)(n) = \prod_{p|2n+1} \frac{p^{2\lambda_p+2} - (-1)^{(\lambda_p+1)(p-1)/2}}{p^2 - (-1)^{(p-1)/2}}, \quad (6.4.56)$$

$$r(\Box + \Box + 4\Delta + 4\Delta + 4\Delta + 4\Delta)(n) = k \prod_{p|n+2} \frac{p^{2\lambda_p+2} - (-1)^{(\lambda_p+1)(p-1)/2}}{p^2 - (-1)^{(p-1)/2}} \quad (6.4.57)$$

where

$$k = \begin{cases} 2^{2\lambda_2-2} & : n \equiv 0 \pmod{2}, \\ \frac{1}{2} & : n \equiv 1 \pmod{4}, \\ 0 & : n \equiv 3 \pmod{4}. \end{cases}$$

$$r(\square + \square + \square + \square + 4\triangle + 4\triangle)(n) = k \prod_{p|n+1} \frac{p^{2\lambda_p+2} - (-1)^{(\lambda_p+1)(p-1)/2}}{p^2 - (-1)^{(p-1)/2}} \quad (6.4.58)$$

where

$$k = \begin{cases} 2^{2\lambda_2+1} & : n \equiv 0 \pmod{2}, \\ 1 & : n \equiv 1 \pmod{4}, \\ 3 & : n \equiv 3 \pmod{4}. \end{cases}$$

**Proof** The details are similar to the proof of Corollary 4.2.5. ■

I thank Professor Michael D. Hirschhorn for the references [45] and [65].

# Chapter 7

## Eisenstein series

### 7.1 Introduction

The aim of this chapter is to rewrite the sixteen Lambert series in the form of Eisenstein series. First we rewrite the sixteen Lambert series in the form of trigonometric functions, which then are presented in the form of  $\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{\theta+m+n\tau}$ . From these, we obtain the sixteen Eisenstein series. The significant point is that the sixteen Eisenstein series all originate from one source, namely Ramanujan's  ${}_1\psi_1$  summation formula.

We remark that all of the bilateral sums in this chapter are to be interpreted as their Cauchy principal value. That is,

$$\sum_{n=-\infty}^{\infty} \frac{1}{\theta+n} := \lim_{N \rightarrow \infty} \sum_{n=-N}^N \frac{1}{\theta+n}$$

and similarly for

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{\theta+m+n\tau} := \lim_{N \rightarrow \infty} \sum_{n=-\infty}^{\infty} \sum_{m=-N}^N \frac{1}{\theta+m+n\tau}.$$

### 7.2 Sixteen series in the form of trigonometry

The method used in this section is similar to the one used by Glaisher [57] with the exception that our notation is simpler. We can rewrite the results of (2.5.1)–(2.5.4) in terms of trigonometric functions, namely the cotangent and the cosecant, as follows

**Theorem 7.2.1**

$$f_0(\theta) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \cot\left(\frac{\theta}{2} + n\pi\tau\right), \quad (7.2.1)$$

$$f_1(\theta) = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n \cot\left(\frac{\theta}{2} + n\pi\tau\right), \quad (7.2.2)$$

$$f_2(\theta) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \csc\left(\frac{\theta}{2} + n\pi\tau\right), \quad (7.2.3)$$

$$f_3(\theta) = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n \csc\left(\frac{\theta}{2} + n\pi\tau\right). \quad (7.2.4)$$

**Proof** Beginning from the right hand side of equation (7.2.1), we have

$$\begin{aligned} & \frac{1}{2} \sum_{n=-\infty}^{\infty} \cot\left(\frac{\theta}{2} + n\pi\tau\right) \\ &= \frac{1}{2} \cot \frac{\theta}{2} + \frac{1}{2} \lim_{N \rightarrow \infty} \left[ \sum_{n=1}^N \cot\left(\frac{\theta}{2} + n\pi\tau\right) + \sum_{n=1}^N \cot\left(\frac{\theta}{2} - n\pi\tau\right) \right] \\ &= \frac{1}{2} \cot \frac{\theta}{2} + \frac{i}{2} \sum_{n=1}^{\infty} \left( \frac{e^{i\theta} q^{2n} + 1}{e^{i\theta} q^{2n} - 1} + \frac{1 + e^{-i\theta} q^{2n}}{1 - e^{-i\theta} q^{2n}} \right) \\ &= \frac{1}{2} \cot \frac{\theta}{2} + \frac{i}{2} \sum_{n=1}^{\infty} \left( \frac{-2e^{i\theta} q^{2n}}{1 - e^{i\theta} q^{2n}} + \frac{2e^{-i\theta} q^{2n}}{1 - e^{-i\theta} q^{2n}} \right). \end{aligned} \quad (7.2.5)$$

By using (2.5.26) in (7.2.5) we obtain (7.2.1). Similar methods can be used to prove (7.2.2)–(7.2.4). ■

### 7.3 Sixteen series in the form of $\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{\theta+m+n\tau}$

The aim of this section is to rewrite the sixteen series in the form of  $\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{\theta+m+n\tau}$ .

Recall from G. E. Andrews, Askey and R. Roy [8, p. 11] that

$$\cot \theta = \lim_{N \rightarrow \infty} \sum_{m=-N}^N \frac{1}{\theta + m\pi}, \quad (7.3.1)$$

$$\csc \theta = \lim_{N \rightarrow \infty} \sum_{m=-N}^N (-1)^m \frac{1}{\theta + m\pi}. \quad (7.3.2)$$

By applying these formulae in (7.2.1)–(7.2.4) and simplifying we find that

$$f_0(\theta) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{\frac{\theta}{2} + n\pi\tau + m\pi}, \quad (7.3.3)$$

$$f_1(\theta) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^n}{\frac{\theta}{2} + n\pi\tau + m\pi}, \quad (7.3.4)$$

$$f_2(\theta) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{\frac{\theta}{2} + n\pi\tau + m\pi}, \quad (7.3.5)$$

$$f_3(\theta) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{\frac{\theta}{2} + n\pi\tau + m\pi}. \quad (7.3.6)$$

Similarly, replacing  $\theta$  with  $\theta + \pi$ ,  $\theta + \pi\tau$ , and  $\theta + \pi\tau\pi$  in (7.3.3)–(7.3.6), respectively, and simplifying, gives

$$f_0(\theta + \pi) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{\frac{\theta}{2} + n\pi\tau + (m - \frac{1}{2})\pi}, \quad (7.3.7)$$

$$f_1(\theta + \pi) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^n}{\frac{\theta}{2} + n\pi\tau + (m - \frac{1}{2})\pi}, \quad (7.3.8)$$

$$f_2(\theta + \pi) = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{\frac{\theta}{2} + n\pi\tau + (m - \frac{1}{2})\pi}, \quad (7.3.9)$$

$$f_3(\theta + \pi) = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{\frac{\theta}{2} + n\pi\tau + (m - \frac{1}{2})\pi}; \quad (7.3.10)$$

$$f_0(\theta + \pi\tau) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{\frac{\theta}{2} + (n - \frac{1}{2})\pi\tau + m\pi}, \quad (7.3.11)$$

$$f_1(\theta + \pi\tau) = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^n}{\frac{\theta}{2} + (n - \frac{1}{2})\pi\tau + m\pi}, \quad (7.3.12)$$

$$f_2(\theta + \pi\tau) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{\frac{\theta}{2} + (n - \frac{1}{2})\pi\tau + m\pi}, \quad (7.3.13)$$

$$f_3(\theta + \pi\tau) = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{\frac{\theta}{2} + (n - \frac{1}{2})\pi\tau + m\pi}; \quad (7.3.14)$$

$$f_0(\theta + \pi + \pi\tau) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{\frac{\theta}{2} + (n - \frac{1}{2})\pi\tau + (m - \frac{1}{2})\pi}, \quad (7.3.15)$$

$$f_1(\theta + \pi + \pi\tau) = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^n}{\frac{\theta}{2} + (n - \frac{1}{2})\pi\tau + (m - \frac{1}{2})\pi}, \quad (7.3.16)$$

$$f_2(\theta + \pi + \pi\tau) = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{\frac{\theta}{2} + (n - \frac{1}{2})\pi\tau + (m - \frac{1}{2})\pi}, \quad (7.3.17)$$

$$f_3(\theta + \pi + \pi\tau) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{\frac{\theta}{2} + (n - \frac{1}{2})\pi\tau + (m - \frac{1}{2})\pi}. \quad (7.3.18)$$

A meromorphic function [6, p. 158] is a function  $f(z)$  of the form

$$f(z) = \frac{g(z)}{h(z)}$$

where  $g(z)$  and  $h(z)$  are entire functions with  $h(z) \not\equiv 0$ .

Equations (7.3.3)–(7.3.18) are examples of meromorphic function.

We remark that by considering the denominators of (7.3.3)–(7.3.6), the results in (2.5.27) are easily obtained.

## 7.4 Sixteen series in the form of Eisenstein series

In this section, we obtain the sixteen series in the form of the Eisenstein series. The Eisenstein series are defined by F. G. M. Eisenstein [51, p. 376],

$$E_{2k}(\tau) = \sum_{(n,m) \neq (0,0)} \frac{1}{(m + n\tau)^{2k}}, \quad (7.4.1)$$

where  $k = 2, 3, 4, \dots$

### Definition 7.4.1 [45]

A holomorphic function  $f$  satisfying the condition  $f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$  and  $ad - bc = 1$ , where  $a, b, c, d \in \mathbb{Z}$ , is called a modular form of weight  $k$ .

By using the above definition, it is easy to see that the function  $E_{2k}(\tau)$  is a modular form of weight  $2k$ .

### Theorem 7.4.2

$$\sum_{(n,m) \neq (0,0)} \frac{1}{(2m + 2n\tau)^{2k}} = -\dot{f}_0^{(2k-1)}(0) \frac{\pi^{2k}}{(2k-1)!}, \quad (7.4.2)$$

$$\sum_{(n,m) \neq (0,0)} \frac{(-1)^n}{(2m + 2n\tau)^{2k}} = -\dot{f}_1^{(2k-1)}(0) \frac{\pi^{2k}}{(2k-1)!}, \quad (7.4.3)$$

$$\sum_{(n,m) \neq (0,0)} \frac{(-1)^m}{(2m + 2n\tau)^{2k}} = -\dot{f}_2^{(2k-1)}(0) \frac{\pi^{2k}}{(2k-1)!}, \quad (7.4.4)$$

$$\sum_{(n,m) \neq (0,0)} \frac{(-1)^{m+n}}{(2m + 2n\tau)^{2k}} = -\dot{f}_3^{(2k-1)}(0) \frac{\pi^{2k}}{(2k-1)!}; \quad (7.4.5)$$



$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{((2m-1) + 2n\tau)^{2k}} = -f_0^{(2k-1)}(\pi) \frac{\pi^{2k}}{(2k-1)!}, \quad (7.4.6)$$

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^n}{((2m-1) + 2n\tau)^{2k}} = -f_1^{(2k-1)}(\pi) \frac{\pi^{2k}}{(2k-1)!}, \quad (7.4.7)$$

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{((2m-1) + 2n\tau)^{2k+1}} = -f_2^{(2k)}(\pi) \frac{\pi^{2k+1}}{(2k)!}, \quad (7.4.8)$$

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{((2m-1) + 2n\tau)^{2k+1}} = -f_3^{(2k)}(\pi) \frac{\pi^{2k+1}}{(2k)!}; \quad (7.4.9)$$

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(2m + (2n-1)\tau)^{2k}} = -f_0^{(2k-1)}(\pi\tau) \frac{\pi^{2k}}{(2k-1)!}, \quad (7.4.10)$$

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^n}{(2m + (2n-1)\tau)^{2k+1}} = -f_1^{(2k)}(\pi\tau) \frac{\pi^{2k+1}}{(2k)!}, \quad (7.4.11)$$

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{(2m + (2n-1)\tau)^{2k}} = -f_2^{(2k-1)}(\pi\tau) \frac{\pi^{2k}}{(2k-1)!}, \quad (7.4.12)$$

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{(2m + (2n-1)\tau)^{2k+1}} = -f_3^{(2k)}(\pi\tau) \frac{\pi^{2k+1}}{(2k)!}; \quad (7.4.13)$$

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{((2m-1) + (2n-1)\tau)^{2k}} = -f_0^{(2k-1)}(\pi + \pi\tau) \frac{\pi^{2k}}{(2k-1)!}, \quad (7.4.14)$$

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^n}{((2m-1) + (2n-1)\tau)^{2k+1}} = -f_1^{(2k)}(\pi + \pi\tau) \frac{\pi^{2k+1}}{(2k)!}, \quad (7.4.15)$$

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{((2m-1) + (2n-1)\tau)^{2k+1}} = -f_2^{(2k)}(\pi + \pi\tau) \frac{\pi^{2k+1}}{(2k)!}, \quad (7.4.16)$$

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{((2m-1) + (2n-1)\tau)^{2k}} = -f_3^{(2k-1)}(\pi + \pi\tau) \frac{\pi^{2k}}{(2k-1)!}. \quad (7.4.17)$$

Equations (7.4.2)–(7.4.7), (7.4.10), (7.4.12), (7.4.14), and (7.4.17) hold for  $k \geq 2$ . Equations (7.4.8), (7.4.9), (7.4.11), (7.4.13), (7.4.15), and (7.4.16) hold for  $k \geq 1$ .

**Proof** By expanding both sides of the expression (7.3.3) in ascending powers of  $\theta$  we find that

$$\begin{aligned}
 & \frac{1}{\theta} + 2 \sum_{t=1}^{\infty} \left[ \frac{B_{2t}}{4t} - \sum_{j=1}^{\infty} \frac{j^{2t-1} q^{2j}}{1 - q^{2j}} \right] \frac{(-1)^t \theta^{2t-1}}{(2t-1)!} \\
 &= \frac{1}{\theta} + \frac{1}{2} \sum_{(n,m) \neq (0,0)} \frac{1}{n\pi\tau + m\pi} \left( 1 + \frac{\theta/2}{n\pi\tau + m\pi} \right)^{-1} \\
 &= \frac{1}{\theta} + \frac{1}{2} \sum_{(n,m) \neq (0,0)} \frac{1}{n\pi\tau + m\pi} \left[ \sum_{t=0}^{\infty} \frac{(-1)^t \theta^t}{2^t (n\pi\tau + m\pi)^t} \right] \\
 &= \frac{1}{\theta} + \sum_{t=0}^{\infty} \sum_{(n,m) \neq (0,0)} \frac{(-1)^t \theta^t}{2^{t+1} (n\pi\tau + m\pi)^{t+1}}.
 \end{aligned}$$

We observe that

$$\begin{aligned}
 & \sum_{k=1}^{\infty} \sum_{(n,m) \neq (0,0)} \frac{1}{(m + n\tau)^{2k+1}} \\
 &= \sum_{k=1}^{\infty} \left[ \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(m + n\tau)^{2k+1}} + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(m - n\tau)^{2k+1}} \right] \\
 &= \sum_{k=1}^{\infty} \left[ \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(m + n\tau)^{2k+1}} - \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(m + n\tau)^{2k+1}} \right] \\
 &= 0.
 \end{aligned}$$

Therefore

$$\frac{1}{\theta} - \sum_{k=1}^{\infty} \sum_{(n,m) \neq (0,0)} \frac{\theta^{2k-1}}{(2n\pi\tau + 2m\pi)^{2k}} = \frac{1}{\theta} + 2 \sum_{k=1}^{\infty} \left[ \frac{B_{2k}}{4k} - \sum_{j=1}^{\infty} \frac{j^{2k-1} q^{2j}}{1 - q^{2j}} \right] \frac{(-1)^k \theta^{2k-1}}{(2k-1)!}.$$

By equating the coefficients of  $\theta^{2k-1}$  and simplifying we find that, if  $k > 1$ , then

$$\sum_{(n,m) \neq (0,0)} \frac{1}{(2m + 2n\tau)^{2k}} = (-1)^{k-1} \left[ \frac{B_{2k}}{2k} - 2 \sum_{j=1}^{\infty} \frac{j^{2k-1} q^{2j}}{1 - q^{2j}} \right] \frac{\pi^{2k}}{(2k-1)!}.$$

Using (2.10.1), this proves (7.4.2).

Equations (7.4.3)–(7.4.17) can be proved similarly. ■

We remark that by applying the modular transformation ( $\tau$  to  $-1/\tau$ ) into Theorem 7.4.2 and then simplifying the results, the results of Corollary 3.4.2 are immediately obtained.

Zucker [105] mentioned without proof that the sixteen Lambert series can only be found in closed form for either even or odd, but never for both. In Section 10 of Chapter 2, we show how the sixteen Lambert series arise from the Ramanujan's  ${}_1\psi_1$  summation formula. From the above theorem an alternative path is shown that the sixteen series form one system. The numerator is either  $1$ ,  $(-1)^n$ ,  $(-1)^m$ , or  $(-1)^{m+n}$ ; and the denominator contains one of the following combinations: both even numbers, one odd and one even number, or both odd numbers. We now have a better understanding of how the sixteen Lambert series arise from the way we derived the sixteen Lambert series.

In the next chapter, we will show how some of the sixteen Eisenstein series in this chapter can be used to obtain the transformation  $\tau$  to  $\frac{-1}{\tau} + 1$ .

# Chapter 8

## Conjectures

### 8.1 Introduction

In Chan and K. S. Chua's paper [28], two new explicit formulae were derived for sums of thirty-two squares and triangular numbers. They also gave five conjectures which are sums of  $8t + 2$ ,  $8t + 4$ ,  $8t + 6$ ,  $8t + 8$  squares and sums of  $8t + 8$  triangular numbers for  $t \geq 1$ . For example an equivalent conjecture for sums of  $8t + 8$  triangular numbers was presented as follows.

For  $k \geq 1$ , let

$$\mathcal{A}'_{2k-1}(q) = \frac{4k}{(2^{2k} - 1) B_{2k}} \sum_{j=1}^{\infty} \frac{j^{2k-1} q^{2j}}{1 - q^{4j}}.$$

Then for any positive integer  $t \geq 1$ ,

$$q^{2t+2} \psi^{8t+8}(q^2) = \sum_{l=1}^t b_l \mathcal{A}'_{2l+1}(q) \mathcal{A}'_{4t+1-2l}(q), \quad (8.1.1)$$

where  $b_l \in \mathbb{Q}$ .

Surprisingly, Chan and Chua did not give conjectures for sums of  $8t + 2$ ,  $8t + 4$ , and  $8t + 6$  triangular numbers for  $t \geq 1$ . Therefore we present the conjectures for sums of  $8t + 2$ ,  $8t + 4$ ,  $8t + 6$  triangular numbers where  $t \geq 1$ . We also derive further conjectures for sums of  $4t + 2$ ,  $4t + 4$ ,  $4t + 6$ ,  $8t$ ,  $8t + 4$  squares and triangular numbers where  $t \geq 1$ . Lastly proofs that are true for the case  $1 \leq t \leq 5$  are given for all the conjectures in this chapter.

## 8.2 Conjectures for sums of $2t$ squares and triangular numbers

In this section we present eighteen conjectures that lead to sum of  $2t$  squares and triangular numbers for  $t \geq 3$ .

For  $k \geq 1$ , let

$$\mathcal{A}_{2k-1}(q) = 1 - \frac{4k}{(2^{2k} - 1) B_{2k}} \sum_{j=1}^{\infty} \frac{j^{2k-1} q^j}{1 - (-q)^j}, \quad (8.2.1)$$

$$\mathcal{A}'_{2k-1}(q) = \frac{4k}{(2^{2k} - 1) B_{2k}} \sum_{j=1}^{\infty} \frac{j^{2k-1} q^{2j}}{1 - q^{4j}}, \quad (8.2.2)$$

$$\mathcal{B}_{2k-1}(q) = 1 + \frac{4k}{(2^{2k} - 1) B_{2k}} \sum_{j=1}^{\infty} \frac{j^{2k-1} q^j}{1 + (-q)^j}, \quad (8.2.3)$$

$$\mathcal{B}'_{2k-1}(q) = -\frac{8k}{(2^{4k} - 2^{2k}) B_{2k}} \sum_{j=1}^{\infty} \frac{(2j-1)^{2k-1} q^{2j-1}}{1 - q^{4j-2}}, \quad (8.2.4)$$

where  $B_{2k}$  are the Bernoulli numbers defined in equation (2.4.8).

For  $k \geq 0$ , let

$$\mathcal{C}_{2k}(q) = 1 - \frac{4}{E_{2k}} \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1)^{2k} q^{2j-1}}{1 - q^{2j-1}} - \frac{2^{2k+2}}{E_{2k}} \sum_{j=1}^{\infty} \frac{j^{2k} q^j}{1 + q^{2j}}, \quad (8.2.5)$$

$$\mathcal{C}'_{2k}(q) = -\frac{4i}{E_{2k}} \left[ \sum_{j=1}^{\infty} \frac{(2j-1)^{2k} q^{j-1/2}}{1 + q^{2j-1}} + \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1)^{2k} q^{j-1/2}}{1 - q^{2j-1}} \right], \quad (8.2.6)$$

where  $E_{2k}$  are the Euler numbers defined in equation (2.4.9).

**Conjecture 8.2.1** For any positive integer  $t \geq 1$ ,

$$\varphi^{8t+8}(q) = \sum_{l=1}^t a_l \mathcal{A}_{2l+1}(q) \mathcal{A}_{4t+1-2l}(q), \quad (8.2.7)$$

$$q^{2t+2} \psi^{8t+8}(q^2) = \sum_{l=1}^t b_l \mathcal{A}'_{2l+1}(q) \mathcal{A}'_{4t+1-2l}(q); \quad (8.2.8)$$

$$\varphi^{8t+4}(q) = \sum_{l=1}^t c_l \mathcal{B}_{2l-1}(q) \mathcal{A}_{4t+1-2l}(q), \quad (8.2.9)$$

$$q^{2t+1} \psi^{8t+4}(q^2) = \sum_{l=1}^t d_l \mathcal{B}'_{2l-1}(q) \mathcal{A}'_{4t+1-2l}(q); \quad (8.2.10)$$

$$\varphi^{8t+2}(q) = \sum_{l=1}^t g_l \mathcal{C}_{2l}(q) \mathcal{B}_{4t-1-2l}(q), \quad (8.2.11)$$

$$q^{2t+\frac{1}{2}} \psi^{8t+2}(q^2) = i \sum_{l=1}^t h_l \mathcal{C}'_{2l}(q) \mathcal{B}'_{4t-1-2l}(q); \quad (8.2.12)$$

$$\varphi^{8t+6}(q) = \sum_{l=1}^t i_l \mathcal{C}_{2l}(q) \mathcal{A}_{4t+1-2l}(q), \quad (8.2.13)$$

$$q^{2t+\frac{3}{2}} \psi^{8t+6}(q^2) = i \sum_{l=1}^t j_l \mathcal{C}'_{2l}(q) \mathcal{A}'_{4t+1-2l}(q), \quad (8.2.14)$$

where  $a_l, b_l, c_l, d_l, g_l, h_l, i_l, j_l \in \mathbb{Q}$ .

Note that

$$f_0^{(2k-1)}\left(\pi \middle| \frac{\tau+1}{2}\right) = \frac{(-1)^{k-1} (2^{2k}-1) B_{2k}}{2k} \mathcal{A}_{2k-1}(q) \quad (8.2.15)$$

$$= -\frac{(2k-1)!}{\pi^{2k}} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{((2m-1)+n(\tau+1))^{2k}}, \quad (8.2.16)$$

$$f_0^{(2k-1)}(\pi\tau|2\tau) = \frac{(-1)^{k-1} (2^{2k}-1) B_{2k}}{2k} \mathcal{A}'_{2k-1}(q) \quad (8.2.17)$$

$$= -\frac{(2k-1)!}{\pi^{2k}} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(2m+2(2n-1)\tau)^{2k}}, \quad (8.2.18)$$

$$f_1^{(2k-1)}\left(\pi\left|\frac{\tau+1}{2}\right.\right) = \frac{(-1)^k (2^{2k}-1) B_{2k}}{2k} \mathcal{B}_{2k-1}(q) \quad (8.2.19)$$

$$= -\frac{(2k-1)!}{\pi^{2k}} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^n}{((2m-1)+n(\tau+1))^{2k}}, \quad (8.2.20)$$

$$f_2^{(2k-1)}(\pi\tau|2\tau) = \frac{(-1)^k (2^{2k}-1) B_{2k}}{2k} \mathcal{B}'_{2k-1}(q) \quad (8.2.21)$$

$$= -\frac{(2k-1)!}{\pi^{2k}} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{(2m+2(2n-1)\tau)^{2k}}, \quad (8.2.22)$$

$$f_2^{(2k)}(\pi) - i f_1^{(2k)}(\pi\tau) = \frac{(-1)^k E_{2k}}{2^{2k+1}} \mathcal{C}_{2k}(q) \quad (8.2.23)$$

$$= -\frac{(2k)!}{\pi^{2k+1}} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{((2m-1)+2n\tau)^{2k+1}} \\ + i \frac{(2k)!}{\pi^{2k+1}} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^n}{(2m+(2n-1)\tau)^{2k+1}}, \quad (8.2.24)$$

$$f_3^{(2k)}(\pi\tau) + i f_2^{(2k)}(\pi+\pi\tau) = \frac{(-1)^k E_{2k}}{2^{2k+1}} \mathcal{C}'_{2k}(q) \quad (8.2.25)$$

$$= -\frac{(2k)!}{\pi^{2k+1}} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{(2m+(2n-1)\tau)^{2k+1}} \\ + i \frac{(2k)!}{\pi^{2k+1}} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{((2m-1)+(2n-1)\tau)^{2k+1}}. \quad (8.2.26)$$

Chan and Chua [28] gave conjectures equivalent to (8.2.7)–(8.2.9), (8.2.11), and (8.2.13).

For  $k \geq 0$ , let

$$\mathcal{D}_{2k}(q) = 1 - \frac{4}{E_{2k}} \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1)^{2k} q^{2j-1}}{1 - q^{2j-1}}, \quad (8.2.27)$$

$$\mathcal{D}'_{2k}(q) = \frac{-4i}{E_{2k}} \sum_{j=1}^{\infty} \frac{(2j-1)^{2k} q^{j-\frac{1}{2}}}{1 + q^{2j-1}}, \quad (8.2.28)$$

where  $E_{2k}$  are the Euler numbers.

**Conjecture 8.2.2** For any positive integer  $t \geq 1$ ,

$$\phi^{4t+6}(q) = i \sum_{l=1}^t k_l \mathcal{A}_{2l+1}(q) \mathcal{D}_{2t-2l}(q), \quad (8.2.29)$$

$$q^{t+\frac{3}{2}} \psi^{4t+6}(q^2) = \sum_{l=1}^t l_l \mathcal{A}'_{2l+1}(q) \mathcal{D}'_{2t-2l}(q); \quad (8.2.30)$$

$$\varphi^{8t}(q) = \sum_{l=1}^t m_l \mathcal{B}_{2l-1}(q) \mathcal{B}_{4t-1-2l}(q), \quad (8.2.31)$$

$$q^{2t} \psi^{8t}(q^2) = \sum_{l=1}^t n_l \mathcal{B}'_{2l-1}(q) \mathcal{B}'_{4t-1-2l}(q); \quad (8.2.32)$$

$$\varphi^{8t+4}(q) = \sum_{l=1}^t r_l \mathcal{C}_{2l}(q) \mathcal{C}_{4t-2l}(q), \quad (8.2.33)$$

$$q^{2t+1} \psi^{8t+4}(q^2) = \sum_{l=1}^t s_l \mathcal{C}'_{2l}(q) \mathcal{C}'_{4t-2l}(q); \quad (8.2.34)$$

$$\varphi^{4t+2}(q) = \sum_{l=1}^t u_l \mathcal{B}_{2l-1}(q) \mathcal{D}_{2t-2l}(q), \quad (8.2.35)$$

$$q^{t+\frac{1}{2}} \psi^{4t+2}(q^2) = i \sum_{l=1}^t v_l \mathcal{B}'_{2l-1}(q) \mathcal{D}'_{2t-2l}(q); \quad (8.2.36)$$

$$\phi^{4t+4}(q) = \sum_{l=1}^t w_l \mathcal{C}_{2l}(q) \mathcal{D}_{2t-2l}(q), \quad (8.2.37)$$

$$q^{t+1} \psi^{4t+4}(q^2) = \sum_{l=1}^t y_l \mathcal{C}'_{2l}(q) \mathcal{D}'_{2t-2l}(q); \quad (8.2.38)$$



where  $k_l, l_l, m_l, n_l, r_l, s_l, u_l, v_l, w_l, y_l \in \mathbb{Q}$ .

Note that

$$f_2^{(2k)}(\pi) = \frac{(-1)^k E_{2k}}{2^{2k+1}} \mathcal{D}_{2k}(q) \quad (8.2.39)$$

$$= -\frac{(2k)!}{\pi^{2k+1}} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{((2m-1) + 2n\tau)^{2k+1}}, \quad (8.2.40)$$

$$f_3^{(2k)}(\pi\tau) = \frac{(-1)^k E_{2k}}{2^{2k+1}} \mathcal{D}'_{2k}(q) \quad (8.2.41)$$

$$= -\frac{(2k)!}{\pi^{2k+1}} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{(2m + (2n-1)\tau)^{2k+1}}. \quad (8.2.42)$$

### 8.3 The first few values

In this section the first few values of (8.2.1)–(8.2.6), (8.2.27), and (8.2.28) will be given by using the results of (3.5.1), (3.5.12), and Figures 5.1–5.3 in (8.2.15), (8.2.19), (8.2.23), and (8.2.39). We find that the first few results for  $\mathcal{A}_{2k-1}(q)$ ,  $\mathcal{B}_{2k-1}(q)$ ,  $\mathcal{C}_{2k}(q)$ , and  $\mathcal{D}_{2k}(q)$  are

$$\mathcal{A}_1(q) = z^2(1-2x) + 4zx(1-x)\frac{dz}{dx},$$

$$\mathcal{A}_3(q) = z^4,$$

$$\mathcal{A}_5(q) = z^6(1-2x),$$

$$\mathcal{A}_7(q) = \frac{1}{17}z^8(17-32x+32x^2),$$

$$\mathcal{A}_9(q) = \frac{1}{31}z^{10}(1-2x)(31-16x+16x^2);$$

(8.3.1)

$$\begin{aligned}
\mathcal{B}_1(q) &= z^2, \\
\mathcal{B}_3(q) &= z^4(1-2x), \\
\mathcal{B}_5(q) &= z^6(1-x+x^2), \\
\mathcal{B}_7(q) &= \frac{1}{17}z^8(1-2x)(17-2x+2x^2), \\
\mathcal{B}_9(q) &= \frac{1}{31}z^{10}(31-77x+78x^2-2x^3+x^4);
\end{aligned}
\tag{8.3.2}$$

$$\begin{aligned}
\mathcal{C}_0(q) &= 1, \\
\mathcal{C}_2(q) &= z^3, \\
\mathcal{C}_4(q) &= z^5(1-2x), \\
\mathcal{C}_6(q) &= \frac{1}{61}z^7(61-91x+91x^2), \\
\mathcal{C}_8(q) &= \frac{1}{277}z^9(1-2x)(277-82x+82x^2);
\end{aligned}
\tag{8.3.3}$$

and

$$\begin{aligned}
\mathcal{D}_0(q) &= z, \\
\mathcal{D}_2(q) &= z^3(1-x), \\
\mathcal{D}_4(q) &= \frac{1}{5}z^5(1-x)(5-x), \\
\mathcal{D}_6(q) &= \frac{1}{61}z^7(1-x)(61-46x+x^2), \\
\mathcal{D}_8(q) &= \frac{1}{1385}z^9(1-x)(1385-1731x+411x^2-x^3).
\end{aligned}
\tag{8.3.4}$$

**Lemma 8.3.1** *If  $\alpha\beta = \pi$  then [19, p. 43]*

$$2\sqrt{\alpha}\psi\left(e^{-2\alpha^2}\right) = \sqrt{\beta}e^{\alpha^2/4}\varphi\left(e^{-\beta^2}\right). \quad (8.3.5)$$

Recall  $p = e^{\frac{-i\pi}{\tau}}$  from Section 4 of Chapter 3. Now if we substitute  $\alpha = \sqrt{-i\pi\tau}$ ,  $\beta = \frac{\pi}{\sqrt{-i\pi\tau}}$  into (8.3.5), simplify and rearrange, then we obtain

$$\varphi(-p) = 2\sqrt{i\tau}q^{\frac{1}{4}}\psi(q^2). \quad (8.3.6)$$

**Lemma 8.3.2**

$$\mathcal{A}_{2k-1}(-p) = (2\tau)^{2k} \mathcal{A}'_{2k-1}(q), \quad (8.3.7)$$

$$\mathcal{B}_{2k-1}(-p) = (2\tau)^{2k} \mathcal{B}'_{2k-1}(q), \quad (8.3.8)$$

$$\mathcal{C}_{2k}(-p) = \tau^{2k+1} \mathcal{C}'_{2k}(q), \quad (8.3.9)$$

$$\mathcal{D}_{2k}(-p) = \tau^{2k+1} \mathcal{D}'_{2k}(q). \quad (8.3.10)$$

**Proof** If we combine (8.2.15) and (8.2.16) then we can rewrite  $\mathcal{A}_{2k-1}(q)$  to become

$$\mathcal{A}_{2k-1}(q) = \frac{(2k-1)!2k(-1)^k}{(2^{2k}-1)B_{2k}\pi^{2k}} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{((2m-1)+n(\tau+1))^{2k}}.$$

By changing  $q$  to  $-p$ , which implies changing  $\tau$  to  $\frac{-1}{\tau} + 1$ , we find that

$$\begin{aligned} \mathcal{A}_{2k-1}(-p) &= \frac{(2k-1)!2k(-1)^k}{(2^{2k}-1)B_{2k}\pi^{2k}} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{\left((2m-1)+n\left(\frac{-1}{\tau}+2\right)\right)^{2k}} \\ &= \frac{(2k-1)!2k(-1)^k}{(2^{2k}-1)B_{2k}\pi^{2k}} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\tau^{2k}}{((2(m+n)-1)\tau-n)^{2k}} \\ &= \frac{(2k-1)!2k(-1)^k}{(2^{2k}-1)B_{2k}\pi^{2k}} \sum_{n=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \frac{(2\tau)^{2k}}{(2(2j-1)\tau+2n)^{2k}}. \end{aligned}$$

Using equations (8.2.17) and (8.2.18), we obtain

$$\mathcal{A}_{2k-1}(-p) = (2\tau)^{2k} \mathcal{A}'_{2k-1}(q).$$

This proves (8.3.7).

Equations (8.3.8)–(8.3.10) can be proved similarly. ■

Using (3.5.1), (3.5.12), and (8.3.1)–(8.3.4), the first few results of  $\mathcal{A}'_{2k-1}(q)$ ,  $\mathcal{B}'_{2k-1}(q)$ ,  $\mathcal{C}'_{2k}(q)$ , and  $\mathcal{D}'_{2k}(q)$  are as follows:

$$\begin{aligned} \mathcal{A}'_1(q) &= \frac{1}{8}zx - xx' \frac{dz}{dx}, \\ \mathcal{A}'_3(q) &= -\frac{1}{16}z^4x^2, \\ \mathcal{A}'_5(q) &= \frac{1}{64}z^6x^2(1+x'), \\ \mathcal{A}'_7(q) &= -\frac{1}{4352}z^8x^2(17-2x'+17x'^2), \\ \mathcal{A}'_9(q) &= \frac{1}{31744}z^{10}x^2(1+x')(31-46x'+31x'^2); \end{aligned} \tag{8.3.11}$$

$$\begin{aligned} \mathcal{B}'_1(q) &= -\frac{1}{4}z^2x, \\ \mathcal{B}'_3(q) &= \frac{1}{16}z^4x(1+x'), \\ \mathcal{B}'_5(q) &= -\frac{1}{64}z^6x(1-x'+x'^2), \\ \mathcal{B}'_7(q) &= \frac{1}{4352}z^8x(1+x')(17-32x'+17x'^2), \\ \mathcal{B}'_9(q) &= -\frac{1}{31744}z^{10}x(31-47x'+33x'^2-47x'^3+31x'^4); \end{aligned} \tag{8.3.12}$$

$$\mathcal{C}'_0(q) = 0,$$

$$\mathcal{C}'_2(q) = iz^3x^{\frac{3}{2}},$$

$$\begin{aligned}
C'_4(q) &= -iz^5 x^{\frac{3}{2}} (1 + x'), \\
C'_6(q) &= \frac{i}{61} z^7 x^{\frac{3}{2}} (61 - 31x' + 61x'^2), \\
C'_8(q) &= -\frac{i}{277} z^9 x^{\frac{3}{2}} \sqrt{x} (1 + x') (277 - 472x' + 277x'^2);
\end{aligned} \tag{8.3.13}$$

and

$$\begin{aligned}
\mathcal{D}'_0(q) &= -iz\sqrt{x}, \\
\mathcal{D}'_2(q) &= iz^3\sqrt{x}, \\
\mathcal{D}'_4(q) &= -\frac{i}{5} z^5\sqrt{x} (1 + 4x), \\
\mathcal{D}'_6(q) &= \frac{i}{61} z^7\sqrt{x} (1 + 44x + 16x^2), \\
\mathcal{D}'_8(q) &= -\frac{i}{1385} z^9\sqrt{x} (1 + 408x + 912x^2 + 64x^3).
\end{aligned} \tag{8.3.14}$$

The results in this section will be used to give proofs that Conjectures 8.2.1 and 8.2.2 are true for the first few cases.

## 8.4 Proof of the first few cases of the conjectures

In this section we show Conjectures 8.2.1 and 8.2.2 are true for  $1 \leq t \leq 5$ .

Formula (8.2.7) is true for  $1 \leq t \leq 5$  by using the results of (8.3.1) and verifying polynomial identities.

The first few results are as follows:

$$\varphi^{16}(q) = \mathcal{A}_3^2(q),$$

$$\varphi^{24}(q) = \frac{1}{9} [17\mathcal{A}_3(q) \mathcal{A}_7(q) - 8\mathcal{A}_5^2(q)].$$

$t$	$8t + 8$	$a_t$
1	16	1
2	24	$\frac{1}{9} (17, -8)$
3	32	$\frac{1}{4725} (11056, -12400, 6069)$
4	40	$\frac{1}{3274425} (8366121, -12079732, 14472304, -7484268)$
5	48	$\frac{1}{219009916125} \begin{pmatrix} 631170572364, -978299900500, \\ 1381991161869, -1752923659500, \\ 937071741892 \end{pmatrix}$

(8.4.1)

For the proof of (8.2.8) we first employ (4.3.19) to rewrite the left hand side of (8.2.8) to become

$$q^{2t} \psi^{8t}(q^2) = \frac{z^{4t} x^{2t}}{4^{4t}}. \quad (8.4.2)$$

Then it is clear that formula (8.2.8) is true for  $1 \leq t \leq 5$  by using the results of (8.4.2) and 8.3.11) to verify polynomial identities.

The first few results are as follows:

$$\begin{aligned} q^4 \psi^{16}(q^2) &= \frac{1}{8} [\mathcal{A}'_3(q)]^2, \\ q^6 \psi^{24}(q^2) &= \frac{1}{36} (17\mathcal{A}'_3(q) \mathcal{A}'_7(q) - 8[\mathcal{A}'_5(q)]^2), \end{aligned}$$

$t$	$8t + 8$	$b_t$
1	16	$\frac{1}{256}$
2	24	$\frac{1}{36864} (17, -8)$
3	32	$\frac{1}{309657600} (11056, -12400, 6069)$
4	40	$\frac{1}{3433483468800} (8366121, -12079732, 14472304, -7484268)$
5	48	$\frac{1}{3674376668971008000} \begin{pmatrix} 631170572364, -978299900500, \\ 1381991161869, -1752923659500, \\ 937071741892 \end{pmatrix}$

(8.4.3)

The proofs of (8.2.9)–(8.2.14) and (8.2.29)–(8.2.38) are similarly straightforward by verifying polynomial identities.

We remark that Conjectures (8.2.7)–(8.2.14), (8.2.29)–(8.2.38) are true for  $t$  from 1 up to 85 by using a Maple algorithm to verify polynomial identities.

# Chapter 9

## Summary and conclusions

The aim of this thesis was to investigate and explore applications of Ramanujan's  ${}_1\psi_1$  summation formula to number theory and combinatorics. We employed Ramanujan's  ${}_1\psi_1$  summation formula to consider some important classical results on elliptic functions and gave proofs of these results using methods which could have been used by Ramanujan.

### 9.1 Summary

In Chapter 2 we introduced and gave a proof of Ramanujan's  ${}_1\psi_1$  summation formula. Then we constructed sixteen functions arising from Ramanujan's  ${}_1\psi_1$  summation formula. We have obtained (a) Fourier series expansions; (b) Fourier expansions of their squares; (c) four special values; (d) addition formulae of the four functions  $f_0, f_1, f_2$ , and  $f_3$ . We have also given the infinite products expansion, the reciprocals and quotients of the functions  $f_1, f_2$ , and  $f_3$ . We demonstrated that  $f_0$  is not an elliptic function but its derivative  $f'_0$  is. We have also shown that  $f_1, f_2, f_3$  are elliptic functions. We obtained sixteen Lambert series by expansions of the four functions  $f_0, f_1, f_2, f_3$  at four points  $0, \pi, \pi\tau$ , and  $\pi + \pi\tau$ . We also obtained a connection between the twelve functions and Jacobian elliptic functions.

We examined selected transformations to functions  $f_0, f_1, f_2, f_3$ , as well as  $z, x, 1 - x, dz/dx$ , and  $E$  in Chapter 3. In Chapter 4 we have shown that eighteen problems in the area of sums of squares and triangular numbers can be proved by using Ramanujan's  ${}_1\psi_1$  summation formula and the fundamental multiplicative identity. From these eighteen prob-



lems, we deduced formulae for the number of representations of an integer  $n$  by eighteen different quadratic forms in terms of their divisor sums. From these results, we further deduced formulae for the number of representations of an integer  $n$  by eighteen different quadratic forms using some product of integers which contain primes. We found a new Lambert series for the representation of  $n$  by  $\psi^4(q) \psi^2(q^2)$ .

We mentioned that Ramanujan recorded fourteen families of identities and gave only the first few examples in each case. In Chapter 5 we developed a powerful tool (sixteen Lambert series) to collect them together. We presented the sixteen Lambert series as various polynomials in terms of  $z$ ,  $x$ , and  $dz/dx$  and then used these sixteen families of identities to prove all of Ramanujan's examples. We also have proved that functions  $P$ ,  $Q$ ,  $R$ ,  $e_1$ ,  $e_2$ , and  $e_3$  can be expressed as in terms of  $z$ ,  $x$ , and  $dz/dx$ ; and that the function  $\sum_{\substack{\sigma \in S_3 \\ \sigma=(\sigma_1, \sigma_2, \sigma_3) \\ \{\sigma_1, \sigma_2, \sigma_3\}=\{1,2,3\}}} e_{\sigma_1}^{\lambda_1} e_{\sigma_2}^{\lambda_2} e_{\sigma_3}^{\lambda_3}$ , where  $\lambda_1, \lambda_2, \lambda_3 \geq 0$ , is a polynomial in  $Q$  and  $R$  with rational coefficients.

In Chapter 6 we employed results from Chapters 3 and 5 to prove forty-four identities which consist of ten Lambert series identities and thirty-four infinite products and their Lambert series expansions. From these thirty-four identities, we deduced formulae for the number of representations of an integer  $n$  by thirty-four different quadratic forms, in terms of divisor sums or of some product of integers which contain primes. We also found fourteen new Lambert series for the representation of  $n$  by  $\varphi(q) \varphi(q^4)$ ,  $\varphi(q) \psi(q^8)$ ,  $\varphi(q^4) \psi(q^2)$ ,  $\varphi^3(q) \psi(q^8)$ ,  $\varphi^2(q) \varphi^4(q^2)$ ,  $\varphi^2(q) \psi^2(q^8)$ ,  $\varphi(q) \psi^3(q^8)$ ,  $\varphi(q^4) \psi^3(q^2)$ ,  $\varphi(q) \psi^2(q^4) \phi(q^4)$ ,  $\varphi(q) \varphi^2(q^2) \psi(q^8)$ ,  $\varphi(q) \psi^2(q^4) \psi(q^8)$ ,  $\varphi^2(q) \psi^4(q^4)$ ,  $\varphi^4(q) \varphi^2(q^2)$ ,

and  $\varphi^4(q) \psi^2(q^4)$ .

In Chapter 7 we presented the sixteen Lambert series in the form of Eisenstein series. These gave us another way to see how the sixteen series form one system. In Chapter 8 we employed selected sixteen Lambert series to construct eighteen conjectures that leads to sums of  $2k$  squares and triangular numbers for  $k \geq 3$ . We proved that all eighteen conjectures were true for the first five cases. By using a Maple program to verify polynomial identities, all eighteen conjectures were also found to be true for the first eighty-five cases.

## 9.2 Future research

Further research will be to carry on a deep investigation on the following three problems. The first one is to find a formula for the zeros of  $f_0$ . Zeros of the Weierstrass  $\wp$  function are known [50] but the formula is complicated. A connection between  $f'_0$  and the Weierstrass function was given in Section 7 of Chapter 2. There is unlikely to be a simple formula for the location of the zeros of  $f_0$ , apart from the ones at  $(2m+1)\pi$  where  $m$  is any integer. The second problem is to give proofs of the conjectures given in Chapter 5. More investigation is needed to achieve this goal. The third problem is to give proofs of the conjectures in Chapter 8. This problem has resisted attempts by many people to date.

Another area to investigate is the cubic, quintic, and septic elliptic and theta functions. Some published papers on those problems can be found in [24], [30]–[32], [43], and [86]. To achieve these, the development of ideas related to Ramanujan's  ${}_1\psi_1$  summation formula may be required. This approach could be used to construct new proofs of some recent

problems in elliptic function theory. This will increase and deepen our understanding of the work of Ramanujan himself.

# Appendix A

## Fourier series

Replacing  $\theta$  with  $\theta + \pi$ ,  $\theta + \pi\tau$ , and  $\theta + \pi + \pi\tau$  in (2.5.5), (2.5.7)–(2.5.9), respectively, and simplifying, we attain

$$f_0(\theta + \pi) = -\frac{1}{2} \tan \frac{\theta}{2} + 2 \sum_{m=1}^{\infty} \frac{(-1)^m q^{2m}}{1 - q^{2m}} \sin m\theta, \quad (\text{A.1})$$

$$f_0(\theta + \pi\tau) = \frac{1}{2i} - 2 \sum_{m=1}^{\infty} \frac{q^m}{1 - q^{2m}} \sin m\theta, \quad (\text{A.2})$$

$$f_0(\theta + \pi + \pi\tau) = \frac{1}{2i} - 2 \sum_{m=1}^{\infty} \frac{(-1)^m q^m}{1 - q^{2m}} \sin m\theta, \quad (\text{A.3})$$

$$f_1(\theta + \pi) = -\frac{1}{2} \tan \frac{\theta}{2} - 2 \sum_{m=1}^{\infty} \frac{(-1)^m q^{2m}}{1 + q^{2m}} \sin m\theta, \quad (\text{A.4})$$

$$f_1(\theta + \pi\tau) = \frac{1}{i} \left[ \frac{1}{2} + 2 \sum_{m=1}^{\infty} \frac{q^m}{1 + q^{2m}} \cos m\theta \right], \quad (\text{A.5})$$

$$f_1(\theta + \pi + \pi\tau) = \frac{1}{i} \left[ \frac{1}{2} + 2 \sum_{m=1}^{\infty} \frac{(-1)^m q^m}{1 + q^{2m}} \cos m\theta \right], \quad (\text{A.6})$$

$$f_2(\theta + \pi) = \frac{1}{2} \sec \frac{\theta}{2} - 2 \sum_{m=1}^{\infty} \frac{(-1)^m q^{2m-1}}{1 - q^{2m-1}} \cos \left( m - \frac{1}{2} \right) \theta, \quad (\text{A.7})$$

$$f_2(\theta + \pi\tau) = 2 \sum_{m=1}^{\infty} \frac{q^{m-\frac{1}{2}}}{1 - q^{2m-1}} \sin \left( m - \frac{1}{2} \right) \theta, \quad (\text{A.8})$$

$$f_2(\theta + \pi + \pi\tau) = -2 \sum_{m=1}^{\infty} \frac{(-1)^m q^{m-\frac{1}{2}}}{1 - q^{2m-1}} \cos \left( m - \frac{1}{2} \right) \theta, \quad (\text{A.9})$$

$$f_3(\theta + \pi) = \frac{1}{2} \sec \frac{\theta}{2} + 2 \sum_{m=1}^{\infty} \frac{(-1)^m q^{2m-1}}{1 + q^{2m-1}} \cos \left( m - \frac{1}{2} \right) \theta, \quad (\text{A.10})$$

$$f_3(\theta + \pi\tau) = -2i \sum_{m=1}^{\infty} \frac{q^{m-\frac{1}{2}}}{1 + q^{2m-1}} \cos \left( m - \frac{1}{2} \right) \theta, \quad (\text{A.11})$$

$$f_3(\theta + \pi + \pi\tau) = -2i \sum_{m=1}^{\infty} \frac{(-1)^m q^{m-\frac{1}{2}}}{1 + q^{2m-1}} \sin \left( m - \frac{1}{2} \right) \theta. \quad (\text{A.12})$$

# Appendix B

## Infinite products

Replacing  $\theta$  by  $\theta + \pi$ ,  $\theta + \pi\tau$ ,  $\theta + \pi + \pi\tau$ , in (2.5.10), (2.5.12), and (2.5.14), respectively, we find

$$f_1(\theta + \pi) = \frac{(e^{i\theta}, q^2 e^{-i\theta}, q^2, q^2; q^2)_\infty}{i(-e^{i\theta}, -q^2 e^{-i\theta}, -1, -q^2; q^2)_\infty} \quad (\text{B.1})$$

$$= \frac{-(q^2; q^2)_\infty^2}{(-q^2; q^2)_\infty^2} \tan \frac{\theta}{2} \prod_{n=1}^{\infty} \frac{(1 - 2q^{2n} \cos \theta + q^{4n})}{(1 + 2q^{2n} \cos \theta + q^{4n})}, \quad (\text{B.2})$$

$$f_2(\theta + \pi) = \frac{e^{\frac{i\theta}{2}} (-qe^{i\theta}, -qe^{-i\theta}, q^2, q^2; q^2)_\infty}{(-e^{i\theta}, -q^2 e^{-i\theta}, q, q; q^2)_\infty} \quad (\text{B.3})$$

$$= \frac{1}{2} \frac{(q^2; q^2)_\infty^2}{(q; q^2)_\infty^2} \sec \frac{\theta}{2} \prod_{n=1}^{\infty} \frac{(1 + 2q^{2n-1} \cos \theta + q^{4n-2})}{(1 + 2q^{2n} \cos \theta + q^{4n})}, \quad (\text{B.4})$$

$$f_3(\theta + \pi) = \frac{e^{\frac{i\theta}{2}} (qe^{i\theta}, qe^{-i\theta}, q^2, q^2; q^2)_\infty}{(-e^{i\theta}, -q^2 e^{-i\theta}, -q, -q; q^2)_\infty} \quad (\text{B.5})$$

$$= \frac{1}{2} \frac{(q^2; q^2)_\infty^2}{(-q; q^2)_\infty^2} \sec \frac{\theta}{2} \prod_{n=1}^{\infty} \frac{(1 - 2q^{2n-1} \cos \theta + q^{4n-2})}{(1 + 2q^{2n} \cos \theta + q^{4n})}, \quad (\text{B.6})$$

$$f_1(\theta + \pi\tau) = \frac{(-qe^{i\theta}, -qe^{-i\theta}, q^2, q^2; q^2)_\infty}{i(qe^{i\theta}, qe^{-i\theta}, -1, -q^2; q^2)_\infty} \quad (\text{B.7})$$

$$= \frac{1}{2i} \frac{(q^2; q^2)_\infty^2}{(-q^2; q^2)_\infty^2} \prod_{n=1}^{\infty} \frac{(1 + 2q^{2n-1} \cos \theta + q^{4n-2})}{(1 - 2q^{2n-1} \cos \theta + q^{4n-2})}, \quad (\text{B.8})$$

$$f_2(\theta + \pi\tau) = \frac{q^{\frac{1}{2}} e^{\frac{i\theta}{2}} (q^2 e^{i\theta}, e^{-i\theta}, q^2, q^2; q^2)_\infty}{i(qe^{i\theta}, qe^{-i\theta}, q, q; q^2)_\infty} \quad (\text{B.9})$$

$$= 2q^{\frac{1}{2}} \frac{(q^2; q^2)_\infty^2}{(q; q^2)_\infty^2} \sin \frac{\theta}{2} \prod_{n=1}^{\infty} \frac{(1 - 2q^{2n} \cos \theta + q^{4n})}{(1 - 2q^{2n-1} \cos \theta + q^{4n-2})}, \quad (\text{B.10})$$

$$f_3(\theta + \pi\tau) = \frac{q^{\frac{1}{2}} e^{\frac{i\theta}{2}} (-q^2 e^{i\theta}, -e^{-i\theta}, q^2, q^2; q^2)_\infty}{i(qe^{i\theta}, qe^{-i\theta}, -q, -q; q^2)_\infty} \quad (\text{B.11})$$

$$= \frac{2q^{\frac{1}{2}} (q^2; q^2)_\infty^2}{i(-q; q^2)_\infty^2} \cos \frac{\theta}{2} \prod_{n=1}^{\infty} \frac{(1 + 2q^{2n} \cos \theta + q^{4n})}{(1 - 2q^{2n-1} \cos \theta + q^{4n-2})}; \quad (\text{B.12})$$

$$f_1(\theta + \pi + \pi\tau) = \frac{(qe^{i\theta}, qe^{-i\theta}, q^2, q^2; q^2)_\infty}{i(-qe^{i\theta}, -qe^{-i\theta}, -1, -q^2; q^2)_\infty} \quad (\text{B.13})$$

$$= \frac{1}{2i} \frac{(q^2; q^2)_\infty^2}{(-q^2; q^2)_\infty^2} \prod_{n=1}^{\infty} \frac{(1 - 2q^{2n-1} \cos \theta + q^{4n-2})}{(1 + 2q^{2n-1} \cos \theta + q^{4n-2})}, \quad (\text{B.14})$$

$$f_2(\theta + \pi + \pi\tau) = q^{\frac{1}{2}} e^{\frac{i\theta}{2}} \frac{(-q^2 e^{i\theta}, -e^{-i\theta}, q^2, q^2; q^2)_\infty}{(-qe^{i\theta}, -qe^{-i\theta}, q, q; q^2)_\infty} \quad (\text{B.15})$$

$$= 2q^{\frac{1}{2}} \frac{(q^2; q^2)_\infty^2}{(q; q^2)_\infty^2} \cos \frac{\theta}{2} \prod_{n=1}^{\infty} \frac{(1 + 2q^{2n} \cos \theta + q^{4n})}{(1 + 2q^{2n-1} \cos \theta + q^{4n-2})}, \quad (\text{B.16})$$

$$f_3(\theta + \pi + \pi\tau) = \frac{q^{\frac{1}{2}} e^{\frac{i\theta}{2}} (q^2 e^{i\theta}, e^{-i\theta}, q^2, q^2; q^2)_\infty}{(-qe^{i\theta}, -qe^{-i\theta}, -q, -q; q^2)_\infty} \quad (\text{B.17})$$

$$= 2iq^{\frac{1}{2}} \frac{(q^2; q^2)_\infty^2}{(-q; q^2)_\infty^2} \sin \frac{\theta}{2} \prod_{n=1}^{\infty} \frac{(1 - 2q^{2n} \cos \theta + q^{4n})}{(1 + 2q^{2n-1} \cos \theta + q^{4n-2})}. \quad (\text{B.18})$$

## Appendix C

### Squares functions

Successively changing  $\theta$  to  $\theta + \pi$ ,  $\theta + \pi\tau$ , and  $\theta + \pi + \pi\tau$  respectively in (2.8.6), (2.8.16)–(2.8.18) we have

$$\begin{aligned} f_0^2(\theta + \pi) &= \frac{1}{4} \tan^2 \frac{\theta}{2} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{2n}}{(1 - q^{2n})^2} \cos n\theta \\ &\quad + 2 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} (1 - (-1)^n \cos n\theta), \end{aligned} \quad (\text{C.1})$$

$$f_1^2(\theta + \pi) = -\frac{1}{4} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n nq^{2n}}{1 - q^{2n}} + \frac{1}{4} \sec^2 \frac{\theta}{2} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n nq^{2n}}{1 - q^{2n}} \cos n\theta, \quad (\text{C.2})$$

$$f_2^2(\theta + \pi) = 2 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} + \frac{1}{4} \sec^2 \frac{\theta}{2} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n nq^{2n}}{1 - q^{2n}} \cos n\theta, \quad (\text{C.3})$$

$$f_3^2(\theta + \pi) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n nq^n}{1 - q^{2n}} + \frac{1}{4} \sec^2 \frac{\theta}{2} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n nq^{2n}}{1 - q^{2n}} \cos n\theta; \quad (\text{C.4})$$

$$f_0^2(\theta + \pi\tau) = 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{(1 - q^{2n})^2} \cos n\theta + 2 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} (1 - \cos n\theta), \quad (\text{C.5})$$

$$f_1^2(\theta + \pi\tau) = -\frac{1}{4} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n nq^{2n}}{1 - q^{2n}} - 2 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} \cos n\theta, \quad (\text{C.6})$$

$$f_2^2(\theta + \pi\tau) = 2 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} - 2 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} \cos n\theta, \quad (\text{C.7})$$

$$f_3^2(\theta + \pi\tau) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n nq^n}{1 - q^{2n}} - 2 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} \cos n\theta; \quad (\text{C.8})$$

$$f_0^2(\theta + \pi + \pi\tau) = 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{2n}}{(1 - q^{2n})^2} \cos n\theta + 2 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} (1 - (-1)^n \cos n\theta), \quad (\text{C.9})$$

$$f_1^2(\theta + \pi + \pi\tau) = -\frac{1}{4} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n nq^{2n}}{1 - q^{2n}} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n nq^n}{1 - q^{2n}} \cos n\theta, \quad (\text{C.10})$$

$$f_2^2(\theta + \pi + \pi\tau) = 2 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n nq^n}{1 - q^{2n}} \cos n\theta, \quad (\text{C.11})$$

$$f_3^2(\theta + \pi + \pi\tau) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n nq^n}{1 - q^{2n}} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n nq^n}{1 - q^{2n}} \cos n\theta. \quad (\text{C.12})$$



## Appendix D

### Jacobian elliptic functions

Upon comparing the Fourier series in Appendices A and C, with those in [88, pp. 12–13, eqn. (2.14)–(2.16), (2.18)–(2.28)] gives

$$\frac{4}{xz^2} f_2^2 \left( \frac{2u}{z} + \pi\tau \right) = \operatorname{sn}^2(u, k), \quad (\text{D.1})$$

$$\frac{4}{x'z^2} f_1^2 \left( \frac{2u}{z} + \pi \right) = \operatorname{sc}^2(u, k), \quad (\text{D.2})$$

$$\frac{-4}{xx'z^2} f_3^2 \left( \frac{2u}{z} + \pi + \pi\tau \right) = \operatorname{sd}^2(u, k); \quad (\text{D.3})$$

$$\left( \frac{2i}{z\sqrt{x}} f_3 \left( \frac{2u}{z} + \pi\tau \right) \right)' = \operatorname{sn}(u, k) \operatorname{dn}(u, k), \quad (\text{D.4})$$

$$\left( \frac{2i}{z} f_1 \left( \frac{2u}{z} + \pi\tau \right) \right)' = \operatorname{sn}(u, k) \operatorname{cn}(u, k), \quad (\text{D.5})$$

$$\left( \frac{2}{z\sqrt{x}} f_2 \left( \frac{2u}{z} + \pi + \pi\tau \right) \right)' = \frac{\operatorname{sn}(u, k)}{\operatorname{dn}^2(u, k)}, \quad (\text{D.6})$$

$$\left( \frac{2i}{z\sqrt{x'}} f_1 \left( \frac{2u}{z} + \pi + \pi\tau \right) \right)' = \frac{\operatorname{sn}(u, k) \operatorname{cn}(u, k)}{\operatorname{dn}^2(u, k)}, \quad (\text{D.7})$$

$$\left( \frac{2}{z} f_2 \left( \frac{2u}{z} + \pi \right) \right)' = \frac{\operatorname{sn}(u, k)}{\operatorname{cn}^2(u, k)}, \quad (\text{D.8})$$

$$\left( \frac{2}{z\sqrt{x'}} f_3 \left( \frac{2u}{z} + \pi \right) \right)' = \frac{\operatorname{sn}(u, k) \operatorname{dn}(u, k)}{\operatorname{cn}^2(u, k)}; \quad (\text{D.9})$$

$$\frac{4}{z^2 x^2} f_2^2 \left( \frac{u}{z} + \pi\tau | 2\tau \right) = \frac{\operatorname{sn}^2(u, k) \operatorname{cn}^2(u, k)}{\operatorname{dn}^2(u, k)}, \quad (\text{D.10})$$

$$-\frac{2}{zx'} f_1 \left( \frac{2u}{z} + \pi | \frac{\tau}{2} \right) = \frac{\operatorname{sn}(u, k)}{\operatorname{cn}(u, k) \operatorname{dn}(u, k)}, \quad (\text{D.11})$$

$$-\frac{2}{z} f_1 \left( \frac{2u}{z} + \pi | \frac{\tau+1}{2} \right) = \frac{\operatorname{sn}(u, k) \operatorname{dn}(u, k)}{\operatorname{cn}(u, k)}, \quad (\text{D.12})$$

$$\frac{4}{z^2 x'^2} f_1^2 \left( \frac{2u}{z} + \pi | \frac{\tau}{2} \right) = \frac{\operatorname{sn}^2(u, k)}{\operatorname{cn}^2(u, k) \operatorname{dn}^2(u, k)}, \quad (\text{D.13})$$

$$\frac{4}{z^2} f_1^2 \left( \frac{2u}{z} + \pi | \frac{\tau+1}{2} \right) = \frac{\operatorname{sn}^2(u, k) \operatorname{dn}^2(u, k)}{\operatorname{cn}^2(u, k)}. \quad (\text{D.14})$$

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