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TRANSFER OF THREE DIMENSIONAL SPATIAL VISUALISATION TO TWO DIMENSIONS

AN INVESTIGATION INTO THE EFFECT OF USING MANIPULATIVES ON THE TRANSFER OF THREE DIMENSIONAL SPATIAL VISUALISATION

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requirements for the degree in**

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ABSTRACT

This investigation attempted to determine the effect of using manipulatives on the transfer of spatial visualisation of three dimensions to its two dimensional representation.

The research was carried out in three sections investigating the three topics of the School Certificate prescription that require the visualisation of three dimensions.

These topics were 1. Volume of prisms

2. Isometric drawings

3. Three dimensional trigonometry

The method used was to qualitatively assess students' three dimensional spatial visualisation before and after using appropriate manipulatives.

The theoretical basis of the study was linked to the van Hiele levels as it was believed that these are a valuable tool for teachers to evaluate students mathematical thinking and to develop appropriate teaching programmes. The lack of appropriate descriptors and assessments for determining levels of spatial visualisation made it difficult to carry out quantitative testing. As a consequence of the study suitable descriptors were proposed which should enable teachers to determine their students thinking processes.

The study was carried out in a secondary school with approximately 75 fifth formers and the worksheets formed part of the normal teaching programme for the three topics being investigated. This caused the least disruption for the students but led to some difficulties with administration and collection of their responses.

The results are inconclusive except for the visualisation of angles between lines and lines and between lines and planes in the third topic, this showed an improvement after constructing three dimensional models. This may have been due to the fact that identifying angles in three dimensions is a new concept to students and therefore affected to a greater extent by the use of manipulatives. There was not the same improvement shown for angles between planes, teachers comments supported the conclusion that the manipulatives used were not as appropriate for reinforcing this concept.

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CHAPTER 1 INTRODUCTION

As a teacher of 5th form mathematics classes I became concerned with the difficulty that my students had in interpreting diagrams used in three dimensional trigonometry problems. They had mastered the mathematics of using Pythagoras' theorem to solve for the length of sides and trigonometry to solve for angles and sides but experienced difficulty in determining the relationship between the relevant sides and angles.

I attempted to remedy this by using various objects within the classroom and then tried getting the students to build their own models of cuboids and pyramids using flexi straws. This activity was useful for identifying the quantity to be calculated, reinforcing the concept of height, but the students preferred to work with their model in front of them.

Similarly when working with interlocking blocks and drawing them isometrically students often asked if they were able to take the blocks into the examination with them. When I had to reply in the negative I began to consider whether the time spent working with blocks had been of benefit when the students had to interpret two dimensional diagrams in the examination.

The students I taught were girls and during readings for earlier studies I found that there was a lot of evidence that females have poorer spatial ability than males. Other research linked spatial ability with mathematical achievement and this gender issue is also of prime concern. It seemed important to determine whether improving spatial ability could improve mathematics achievement.

1. Geometry in the New Zealand Mathematics Curriculum

A new mathematics curriculum was introduced in New Zealand in 1984, incorporating 5 strands, Algebra, Geometry, Measurement, Number and Statistics and 8 levels of achievement. The curriculum document states (MINZC, p 13)

"The importance of the use of apparatus to help students form mathematical concepts is well established. Using apparatus provides a foundation of practical experience on which students can build abstract ideas. It encourages them to be inventive, helps to develop their confidence, and encourages independence. "

For the geometry strand it states (MINZC, p 91)

" The mathematics curriculum intended by this statement will provide opportunities for students to:

...gain a knowledge of geometrical relations in two and three dimensions, and recognise and appreciate their environment;

...develop spatial awareness and the ability to recognise and make use of the geometrical properties and symmetries of everyday objects

... develop the ability to use geometrical models as aids to solving practical problems in time and space"

The second of these objectives is of particular interest in that it expects students to develop "spatial awareness" and later in the document various learning experiences are suggested to help develop this ability. One of these suggested learning experiences for Level 4 is the use of interlocking cubes to aid in the drawing of different views of a three dimensional object. Another suggested learning experience is the drawing of nets and using them to construct a three dimensional figure. These techniques require students to use spatial visualisation and it was this particular aspect of spatial ability that I decided to study.

The emphasis that the suggested learning experiences of the new curriculum place on the use of manipulatives such as cubes, nets etc may assist in spatial visualisation but is this ability transferred to visualising objects when they are drawn in 2 dimensions? This is of particular interest at 5th form level when the formal external examination is of pen and paper type and therefore tests the interpretation of 2 dimensional representations of three dimensional objects.

The three topics in the school certificate prescription that involved three dimensions are

- volume
- isometric drawing
- three dimensional trigonometry

This study incorporates the use of manipulatives in each of these topics and qualitatively assesses their effectiveness when students are required to interpret two dimensional representations of three dimensional objects.

The curriculum document also outlines the lower participation rate of girls and Maori in higher level mathematics, and suggests that their confidence in their abilities must be enhanced by providing suitable learning experiences.

It is not only in the new N.Z. mathematics curriculum that the importance of geometry is realised. Reviewers in other countries have also commented on its relevance, Leiva (1992) states that in the American mathematics curriculum there is an emphasis on the study of two and three dimensional geometry. The curriculum also outlines that teachers should encourage and use materials such as building blocks.

This international awareness of the importance of geometry also extends to an increased emphasis on spatial skills, and the use of manipulatives within these curricula. However the link, between the use of these tools and the changes in spatial ability required to interpret two-dimensional representations, needs greater investigation.

Despite this changing emphasis the assessment of spatial awareness is still normally conducted in a formal situation by using two-dimensional diagrams in pen and paper mode. Clements and Ellerton (1995) showed in their research that correct answers given by students in this type of assessment does not always indicate an understanding of the required mathematical concepts or relationships that the tests were designed for.

Clements and Battista (1983, p 457) provide an excellent review of spatial reasoning and in their conclusion state "Research is needed to identify the specific cognitive constructions that children make at all levels, especially in the context of supportive environments (for example, those including manipulatives, computer tools, and engaging tasks.)"

Another aspect of this instruction is how specific the training program needs to be in order to be effective. Does the training in a specific spatial skill affect other spatial abilities? Bishop (1980) suggests that research into the transfer effect of training in spatial tasks is necessary.

Bishop (1980) also proposes that there are two different directions of research into teaching spatial abilities. Firstly those studies which focus on individualised teaching programs so that students can benefit from the abilities that they do possess. An alternative approach is to identify and then teach specific spatial abilities. This seems to Bishop to be the more rewarding approach and a considerable amount of research has been successful with this. The latter is the format followed by this investigation, whereby the specific skills required for each concept were identified and taught.

It may also be beneficial to consider investigating different forms of testing that have greater validity than the formal pen and paper assessments used currently. Usiskin and Senk (1990) suggest that there is a need for a geometry test that is consistent with a theoretical model of learning, has both descriptive and predictive power, satisfies psychometric concerns and is easy to administer. This investigation did not develop such a test but suggests some sample problems that may be developed into such a resource.

2. The van Hiele levels

The van Hiele levels of thinking have received considerable recognition as a tool for identifying students' thinking and hence improving their learning. The sequential and hierarchical aspects of the theory have been rigourously studied but most of the research that has been conducted on them has been with polygons.

Despite the dominance of polygons in previous research it was decided to use the van Hiele levels as a theoretical basis for this research. In order to improve students' spatial visualisation their teachers needed a clearer understanding of the processes involved in the development of the skill. To enable this to occur there is a need for descriptors for the skill at each level and also sample problems. These were not available in the literature but were developed as a consequence of the investigation.

Other reviewers have recognised the need for descriptors and assessments for evaluating and improving students' learning in other mathematical concepts. Senk (1989) suggests that content specific tests should be used in research that is attempting to differentiate between the thinking processes that are characteristic of the van Hiele levels.

The emphasis of this investigation was to study techniques that will improve the spatial ability of students and it was felt that this would be assisted by a greater awareness and understanding of their thinking using the van Hiele levels .

The specific question that this study seeks to answer is

Does the use of manipulatives assist in the spatial visualisation of three dimensional objects in their two dimensional representation?

CHAPTER 2 LITERATURE REVIEW

1. GENERAL

(i) Geometry

The importance of geometry is recognised in the mathematics curricula of many countries including the United States. "In grades 9-12, the mathematics curriculum should include the continued study of the geometry of two and three dimensions so that all students can interpret and draw three-dimensional objects". (NCTM 1989, 157)

Geometry has historically been an important component in the study of mathematics, however the type of geometry studied has varied at different times..

Usiskin (1987) quoted in Clements and Battista (1992, p 420) has described four dimensions of geometry

- "a. visualisation, drawing, and construction of figure
- b. study of the spatial aspects of the physical world
- c. use as a vehicle for representing nonvisual mathematical concepts and relationships
- d. representation as a formal mathematical system."

These four dimensions have each assumed importance at different times in the evolution of mathematics.

The first three of these dimensions require the use of spatial reasoning but the purpose of this investigation is the relationship between the first two listed.

The use of diagrams as a representation of three dimensional space is an important tool and skill which is not always easy to acquire, the importance of this is recognised by Bishop when he states "Geometry is the mathematics of space.....Mathematics educators, therefore, are concerned with helping pupils gain knowledge and skills in the mathematical interpretation of space." Bishop (1983 p 175).

The study of geometry is important not only in mathematics but also for the insights and interpretations of other disciplines that it provides. Clements and Battista (1983) reinforce this attribute and stress its importance for the study of the physical environment in science.

The use of geometry as a formal deductive system has become less important in the current NZ mathematics curriculum. The decline of this aspect is supported by many authors, including Egsgard (1970) who believes that geometry is no longer the study of proof but rather the study of relationships in three dimensions and on two dimensional planes within this space.

The Netherlands, and the Freudenthal Institute in particular, have been pioneers in the teaching and learning of geometry. Freudenthal believed that too great an emphasis on plane geometry in grades 7-10 often prevented an understanding of space geometry when it was introduced. He helped instigate the introduction of geometry through space rather than by the plane.

(ii) Spatial ability

Despite the current emphasis on spatial ability, it is not a recent phenomenon, but has been historically of great importance too. Fruchter (1954 p 387) states that "spatial abilities have been considered to be an important segment of the general area of aptitudes".

Considerable research has been carried out in primary and lower secondary mathematics classes to investigate different aspects of spatial ability but the ability is of great benefit in more advanced mathematics classes. Lean and Clements (1981) compiled a list of ten major topics in senior secondary and lower tertiary courses which require spatial ability. It is useful for educators to be aware of these links which may help explain the difficulties that some students have with some concepts associated with calculus that may not appear to be dependent on spatial ability.

(a) Definitions

In order to investigate spatial ability it is necessary to define the terminology used and evaluate the usefulness of the definitions.

The term “spatial ability” indicates that it is considered as an aspect of a holistic make up of human intelligence. This is supported by Gardner (1983) who considers it one of the components of human intelligence.

A general term such as spatial sense has been defined as “an intuitive feel for one’s surroundings and the objects in them.” ([NCTM 1989, p 49). Further clarification of spatial sense is provided by Gardner (1983) who states that it is dependent on visual spatial intelligence, which is the ability to form a mental model of a spatial world and to be able to manipulate this model.

This concept of manipulating a mental model is also used by Connor and Serbin (1985) who label this as visual-spatial ability and that it is one of the cognitive skills.

An important researcher in the development of perception was Jean Piaget who studied the child’s mental transformations from real space to their representation of it. His theory sought to explain the changes in these mental transformations. He clarified an important difference between perception, which is the knowledge of objects as a result of direct contact with them, and representation which is the perceptual knowledge of objects not actually perceived.

In Piaget’s words “Perception is the knowledge of objects resulting from direct contact with them. As against this, representation or imagination involves the evocation of objects in their absence or, when it runs parallel to perception, in their presence. It completes perceptual knowledge by reference to objects not actually perceived.” Piaget & Inhelder (1967 p 17) in Herschowitz (1990 p 72).

An important ability is the awareness that the appearance of an object is related to the position from which it is viewed which Piaget and Inhelder (1956) have termed "co-ordination of perspectives". This co-ordination of perspectives also refers to the *ability* to determine what that appearance will be for any specific viewing position.

Fishbein et al (1972) studied this "co-ordination of perspectives" and concluded that two development factors, social and cognitive, are involved in the child's ability to co-ordinate perspective's. The social factors deal with the child's understanding of the relationships between his perceptions and the perceptions of others. The cognitive factors reflect the child's ability to deal with the projective internal relationships between three objects.

One of the important considerations is whether spatial ability comprises one or many factors, Fruchter (1954) identified the importance of determining whether spatial ability is unitary or whether several fundamental types of spatial abilities can be distinguished.

This conflict was also discussed by Eliot and Hauptman (1981) who found that the term spatial ability was often used ambiguously term to describe a wide range of global to specific behaviours. Eliot and Hauptman (1981, p 46) believe that

"we need to consider spatial ability **both** as an intrinsic aspect of thinking **and** as a set of operations for solving spatial problems"

This investigation is designed to study the second of these two aspects. They further define this particular skill as, Eliot and Hauptman (1981, p 46)

"a set of mental operations for solving certain types of problems, spatial ability is manifest in the ways we respond to tasks which require the estimation or prediction of rotated objects, and to tasks which require judgements about object arrangements when imagined from different perspectives."

It therefore becomes necessary to define the components of spatial ability and in particular the term "spatial visualisation" as used in this investigation. Many authors have done this by dividing spatial ability into two main components.

Bishop (1983) classified spatial abilities into two main factors

1. a visualisation factor
2. an orientation factor

This form of classification was also used by McGee (1979, p893) who proposed two distinct spatial skills..

1. spatial visualisation, "the ability to mentally manipulate, rotate, twist or invert a pictorially presented object"
2. spatial orientation, "the comprehension of the arrangement of elements within a visual stimulus pattern and the aptitude to remain unconfused by the changing orientations in which a spatial configuration may be presented"

Connor and Serbin (1980) supported McGee's distinction but Linn and Peterson (1985) proposed a different classification

1. spatial perception : which requires an understanding of the spatial relationship of an object without being distracted by unimportant perceptual clues
2. mental rotation: which is the ability to rotate either two- or three- dimensional figures accurately
3. spatial visualisation which they defined as "those spatial tasks which involve complicated multi-step manipulations of spatially presented information". They distinguish it from the other two categories "by the possibility of multiple solution strategies. (p1484).

This investigation will use the definitions suggested by McGee (1979) which proposes that spatial orientation tasks do not require mentally moving an object, but that spatial visualisation does require this movement. The necessity for multi-step manipulations specified by Linn and Peterson (1985) is not taken into consideration.

Connor and Serbin (1985) state that visual-spatial ability is a cognitive skill involving the ability to perceive spatial relationships and to manipulate visual material mentally. This ability is not uni-dimensional.

There are more important questions than defining the terms used, Herschowitz (1990) feels that it is more important to determine what abilities are needed to learn geometry and how can these abilities best be fostered. This is the purpose of this investigation, to determine whether the use of manipulatives will improve spatial visualisation.

Bishop (1983) states that "It is clear to a mathematics educator at least, that there can never be a "true" definition of spatial ability; we must seek definitions and descriptions of abilities and processes that help us to solve our own particular problems.

(b) Spatial ability and mathematical ability

An aspect of spatial ability that is of critical importance for educators is the possibility of a link between it and mathematical ability. If this link does in fact exist is mathematical ability predetermined or, could an improvement in spatial ability also lead to an improvement in mathematical ability?

Many researchers have found a positive correlation between spatial ability and mathematics achievement (Fennema & Sherman, 1977; Fennema & Sherman, 1978) Fennema and Sherman (1977) found significant correlations between math achievement and a spatial-visualisation measure in their study of 4 secondary schools.

Other researchers , Bishop (1980) for example, have found that the relationship between spatial ability and mathematical ability differs from study to study. One of the benefits of the research carried out has been the development of better teaching resources and strategies which can assist both the classroom teacher and their pupils.

Some studies have differentiated between the different spatial abilities and investigated them separately. Connor and Serbin's (1985) study results suggest that visualisation skill and spatial orientation skill are somewhat distinct and both contribute to predicting mathematics achievement.

It is also possible that the link between spatial and mathematical ability is two way and that an improvement in understanding geometric concepts will lead to an increase in spatial ability. del Grande (1990) states that the improvement in spatial abilities and the learning of geometry are interdependent, and that an improvement in one leads to an improvement in the other.

(c) Spatial ability and gender

Equity issues are being addressed in mathematics curricula in many ways including the use of appropriate resources and language. This is welcomed by most educators but another gender issue of concern is whether the increased emphasis on three-dimensional activities is penalising some students, who have poor spatial ability.

One group that this may affect is females because some research has shown that they have poorer spatial ability. There are two main areas of concern for researchers

1. Is gender a factor in spatial ability?
2. What are the causes of any gender differences?

A great deal of research has been carried out in attempt to answer the first question, and Clements and Battista (1992) provide a comprehensive summary of the results. Most research has shown that boys perform better on tests of spatial ability than girls do, however there is a great deal of variation in the results obtained and the test and criteria used. Some of the results obtained may in fact be an indication of the type of test used rather than the ability being tested.

Another factor that needs to be taken into account is time, Gallagher and Johnson (1992) studied the effect of time constraints on a mental rotation test for girls and boys. The results of their study demonstrate that the reason for differences in spatial ability may have more to do with speed of performance than ability. Their research suggests that opportunities should be provided for girls to improve their spatial skills through one of many techniques. Practice with spatial activities (including spatially orientated sports and games like jigsaw puzzles), additional time on mathematics, and practice taking tests under timed conditions could all help girls to develop increased facility with their spatial skills.

It is also necessary to clarify which type of spatial ability is being investigated, "The conflicting evidence on gender differences in spatial ability and its relationship with mathematics achievement may, in part, be explained by conflicting evidence on gender differences in spatial ability per se and in particular the type of spatial ability being tested." (Mathematics performance of NZ Form 2 and Form 3 students, p23.) Not all testing clearly differentiates between the different spatial abilities being investigated.

With particular reference to spatial visualisation ability the research is again contradictory. Sherman (1979) reported that spatial visualisation was an important predictor of females' problem-solving performance, but Armstrong (1981) found that sex-related differences in problem-solving performances could not be attributed to differences in spatial visualisation scores.

In their summary of research into spatial visualisation Fennema and Tarte (1985, p 205) state "All differences found between girls and boys were small, and the intrasex differences were larger than the intersex differences". This is an important statement because it is crucial to remember that not all girls are less able than all boys . It is more appropriate to focus on the sub-groups of high or low achieving girls and boys to determine their characteristics.

Fennema and Sherman (1977) support the view that gender differences in spatial ability do not explain differences in mathematical and science achievement.

The other consideration is if gender differences in spatial ability do exist, then what is the cause? Newcombe and Bandura (1983) have reviewed the literature on this question including studies which report on spatial ability having a genetic basis, being determined by a recessive X-linked gene.

Other possible causes are experiential, girls having had different experience with activities that could help develop spatial ability, these activities include Lego, mechano and other toys requiring manipulation.

Another potential attribute is the effect of the age of onset of puberty on brain development and its effect on spatial ability. Newcombe and Bandura (1983) studied this phenomena and their results were inconclusive.

Notwithstanding the contradictory results reported, this investigation seeks to study the use of manipulatives on spatial visualisation. Although the intrasex differences may be larger than the intersex, there is still a commonly held view that girls have poorer spatial visualisation than boys. Although gender had originally been a consideration it was not one of the variables studied in this investigation.

(d) Spatial ability and van Hiele levels

The van Hiele levels, discussed later in this literature review, are an important learning theory especially for geometry. The theory is characterised by different levels of thinking and it is advantageous to study the role of spatial visualisation at these different levels. Battista (1990) hypothesises that spatial visualisation may be a more important factor in geometric problem solving for students who are at the visual, rather than higher levels, of geometric thinking.

The type of reasoning that students use is determined by the level at which they are operating, and van Hiele (1986) reports that students at the visual level are much more reliant on observation than those at the descriptive/analytical level. The students at this higher level use a network of relations for their reasoning. One of the characteristics of the level theory is that it is hierarchical and students will not progress to a higher level without mastering the concepts required at a lower one. The visualisation required at lower levels is therefore of prime importance.

(e) Visualisation

Visualisation of objects is bi-directional, the ability to interpret two-dimensional representations of three dimensional objects and the ability to represent three dimensional objects by two dimensional diagrams. Formal assessments use two-dimensional diagrams to represent three-dimensional objects and students need the ability to visualise the object that the diagram represents in order to complete the assessment.

Various learning strategies are used, including the use of manipulatives, to develop this visualisation skill and some research has been carried out to investigate the effectiveness of strategies used, Izard (1990) found that many students are unable to identify how an object would look if viewed from another perspective, but he reported that activities with solids can increase the proportion of students who are able to visualise these objects.

One feature of spatial visualisation that is important is the environment within which the object or its representation actually exist. Educators often use diagrams that can represent different objects depending on within which environment the student believes them to exist.

Diezmann (1966) provides a table of Types of geometric environments and their characteristics

Characteristics of Geometric Environments

a *three dimensional sensory environment* in which objects physically exist.

a *pseudo three-dimensional visual environment* where three-dimensional objects are represented using conventions, such as shading and oblique lines.

a *three-dimensional imaginal environment* where objects exist in the mind as constructions of the physical world.

a *two-dimensional visual environment* that depicts two-dimensional shapes through drawing.

a *two-dimensional imaginal environment* where plane shapes, such as squares, exist.

The existence of these environments is important because the interpretation of a representation is dependent on the identification of the spatial environment in which it exists. e.g. figure of cube or hexagon

If the figure is considered within a pseudo three-dimensional environment it is a cube, however within a two-dimensional visual environment it is a hexagon. Therefore, the environment provides the code for interpretation. Students problems with visualisation may therefore be a function of this interpretation of environment.

One of the frequently used teaching tools is matching pairs of drawings from different perspectives representing the same three-dimensional object. Shepard and Metzler (1971) found that the time required to recognise that two perspective drawings portray objects of the same three dimensional shape was:

1. a linearly increasing function of the angular difference in the portrayed orientations of the two objects
2. no shorter for differences corresponding simply to a rigid rotation of one of the two dimensional drawings in its own picture plane than for differences corresponding to a rotation of the three-dimensional object in depth.

Another aspect of spatial visualisation that has received the attention of researchers is its role in solving non-visual problems Frandsen and Holder (1969) studied the role of spatial visualisation in solving complex verbal problems. They found that the successful solutions are related to spatial-visualisation aptitude. They also found that deficiency in spatial-visualisation can be compensated for by instruction in diagrammatic techniques for representing the data and conditions of the problems.

Pallascio et al (1993) state that two types of mental images seem to exist: one pertains to the immediate intake of data (*perception*, which relates to analytic competencies), while the other deals with the purely mental reconstruction of objects (*representation*, which relates to synthetic competencies). The first type of mental image involves the perception of three dimensional objects only. With respect to two dimensional images of three dimensional objects the mechanisms are not the same, as such images constitute a type of representation which must be processed in some manner to be understood.

(iii) Language and verbalisation

The van Hiele levels are an important descriptive tool for teaching geometry. An essential characteristic of this levels theory is the use of language at each level. This investigation will ask the students to explain concepts in their own words so that there is an indication of the van Hiele level at which they are operating.

The imperativeness of using appropriate language was emphasised by van Hiele "Each level has its own linguistic symbols and its own system of relations connecting these symbols. A relation which is "correct" at one level can reveal itself to be incorrect at another. Think, for example, of a relation between a square and a rectangle. Two people who reason at different levels cannot understand each other. Neither can manage to follow the thought processes of the other." (van Hiele, 1985 p. 246)

(iv) Diagrams

Diagrams are used as a form of communication by teachers, text books, students and examiners but the information is not always being interpreted accurately. This can be for a number of reasons, one of which is the limitations of the drawing themselves. Zykova (1969) refers to the rigidity associated with geometric drawings in textbooks and the inability of pupils to generalise from these drawings. They point out that it is impossible to draw a generalised diagram and that students try to read superfluous properties from diagrams.

Teachers often presume that a diagram is imparting the information that they intend to their students but this is not always so. Diezmann (1994) found that there is an intrinsic expectation in the use of a geometric diagram as a form of communication by the diagram encoders (teachers, text illustrators and researchers) that diagram decoders (children) share a common meaning. Children who are not diagrammatically literate are essentially denied access to the geometric communication that is being used.

Some teachers become aware of this conflict when they find students referring to a parallelogram as a pushed over rectangle, or isosceles triangles as triangles that point upwards

The flexibility of graphic communication can lead to difficulty in interpreting geometric diagrams. The difficulty occurs when diagrams of two and three-dimensional shapes are interpreted pictorially. If diagrams of three-dimensional shapes are interpreted pictorially, there may be some confusion between the characteristics of the diagram and the characteristics of the referent.

With the greater flexibility in the representation of three-dimensional objects that is encouraged in the new curriculum students are expected to become familiar with different representation formats. These are not all equally accessible for all students, Cooper & Sweller (1989) studied secondary school students' ability to interpret various representations of simple three dimensional objects. Their results suggest that verbal descriptions, perspective drawings, and actual solids are more easily interpreted than layer plans, coded plans, elevations, co-ordinate descriptions and verbal quasi-coordinate descriptions. i.e. real life representations of structures of the type studied are more easily interpreted than orthogonal two-dimensional representations and descriptions based on coordinates. Also, perspective drawings and actual solids are more easily interpreted than verbal descriptions.

If diagrams are used to represent three-dimensional objects then teachers need to be aware of information that is hidden or lost to the students, they can fail to count faces or blocks on a hidden side. Parzysz (1988) states that the loss of information caused by the reduction of a three-dimensional referent to a two-dimensional diagram, can result in a conflict between what is known about a shape and what is not known. Students that are operating at a visual level can be severely limited in this way.

This processing of information presumes an experience and knowledge of three-dimensional objects, Bishop (1983) showed that two-dimensional images of three-dimensional objects require a type of representation which must be processed in some manner to be understood. This representational processing is not always expected in reverse, students often believe that they can draw an explicit diagram which cannot be misrepresented. Parzysz (1988) investigated diagrams of 3 dimensional objects and found that pupils frequently have the illusion that they can make a representation of a geometrical object without any ambiguity at all.

Implicit in the use of the concrete materials for drawing objects in three dimensions is the expectation that their use is advantageous in acquiring the concepts required. This is borne out by research carried out by many researchers. Ben-Chaim et al (1985) found that students have difficulty relating pictures of three dimensional objects to solid objects, and also that in teaching students to "read" such representations, instruction may need to be complemented by introducing concrete experiences with solid objects, even in the middle school.

Mitchelmore (1992 a) states that early activities such as making models from common shapes using drinking straws, colouring components (such as opposite sides) of drawn shapes, and finding the number of triangles in a pattern of intersecting lines might enhance the development of disembedding skills. These disembedding skills enable a student to see geometrical figures as consisting of sides and vertices which can be treated in isolation and have to be learnt.

With the expectation that diagrams are useful, studies have also been carried out to investigate the effect of training students in the drawing of diagrams. Bartolini Bussi (1996) describes a long term study of drawing in primary school children in Italy. She found that it was possible to train students in spatial visualisation. Frandsen and Holder's (1969) study showed the positive effects of diagrammatic training with pupils of low spatial ability.

The alternation by students between concrete experiences and abstractions (drawings) was studied by Lappan et al (1984) who found it an effective method to improve their representation of three-dimensional objects. This approach has also been followed in this investigation with the alternation between building and drawing the solids.

It is recognised that geometry is an important aspect of mathematics and that spatial ability is necessary for both the learning of geometry and other subjects. There are equity issues with respect to both gender and ability but the purpose of this investigation is to help improve the classroom teaching of spatial visualisation for all students

2.THEORIES

(i) Learning theories

The constructivism theory of learning presumes a sequential build up of learning based on previous experience, and most mathematics educators support this model of learning.

The learning of the concepts of three-dimensions can be described by this theory. Cobb et al (1992) state that this is accomplished by a process in which students gradually construct mental representations that accurately mirror the mathematical features of external representations.

The knowledge structure that is developed as a result of this process can be an important determining factor in achievement. Chinnappan and Lawson (1995) proposed that the superior solution attempts of high achieving students appear to be driven by a geometric knowledge base that is better structured and more extensive in nature than that of low achieving students.

Spatial visualisation also plays a role in geometry learning by providing superior mental models. This view is supported by Battista (1990) who proposes a spatial like character of mental models.

Another advance in the curricula of New Zealand schools has been the introduction of the technology curriculum. Schools can implement this either as a separate subject area or incorporate it into existing subjects. The latter approach offers a rewarding challenge to mathematics teachers to teach mathematics by a technological format. This method is supported by Fletcher (1971) who believes that the "teaching by technique then application" method should be replaced by a more contextual approach. Following this method allows the student to develop an understanding by making a sequence of conceptual reorientations. The challenge for teachers is to establish learning situations that encourage this reorientation. The Freudenthal institute has developed resources for developing spatial abilities using this approach.

There are two main areas of study for mathematics educators , the process of learning and the appropriate resources. Gagatsis and Patronis (1990) state that the models used in mathematics education can be divided into two categories:

1. models of the learners mind or of the learning process
2. models of the intuitive processes involved in the subject-matter which is to be taught

Further investigation of the learning process shows that there is a differentiation into two main types, Skemp (1971) distinguishes these as:

1. intuitive; via senses, of data from the external environment, which is automatically classified and related to other data, but without the learners' awareness of the processes involved.
2. reflective; the learner is aware of the mental activities involved.

This latter process, "reflective thinking" receives a great deal of attention from researchers and educators. Gagatsis and Patronis (1990, p133) describe the main stages of a process of reflective thinking in mathematical activity as a five stage process as follows:

Stage 0: Initial thoughts, primary intuitions and conceptions on a subject matter or a problem; starting the work with "local" success or failure; unclear and fragmentary" mental images; making observations "at random".

Stage 1: Reflecting on the subject and trying to understand, i.e. to organise the new experience into previously existing intuitive structures; classifying observations, analysing wholes into parts, reflecting on them, recalling other similar examples, finding counter examples, questioning former beliefs and conceptions.

Stage 2: Discovery and (partial) understanding; finding and/or justifying a rule; finding an explanation for some error; recombining parts of a decomposed whole into new wholes; interpretation and (partial) reorganisation of the new facts according to previous structures; "completion" of mental images and deriving a plan of the solution or of the proof; intuitive feeling of certainty for the success of the plan.

Stage 3: Introspection; trying to see "what it is all about" i.e. reflecting on the process of the solution, the logic of the proof and one's own mental structures and processes; checking or testing one's own results (or conclusions) into other problems or fields; examining analogies and setting up new questions; analysing the whole situation again, but at a higher level; questioning again; questioning "questioning"; epistemological dispute.

Stage 4: Full awareness; understanding the underlying logic; illumination of the whole subject; becoming aware of one's own mental structures and processes; widening the old structures, transforming or fully rejecting them (and constructing new ones); radical reorganisation of ideas, possibly on new foundations; making sound generalisations and extensions; constructing and formulating new theories".

These stages are recursive and represent an interaction with the physical world necessary for spatial cognition .

(ii) The van Hiele levels

"Tracing of levels of thinking that play a part in Geometry is not a simple affair, for the levels are situated not in the subject matter but in the thinking of man.

van Hiele, 1986 p 41

The van Hiele levels are a model for learning in geometry proposed by Pierre Marie van Hiele and his wife Dina van Hiele-Geldof. In the thirty years since they were first postulated they have been the subject of a great deal of research but their relevance for this study is reflected in the quotation above. They are an important theoretical base for researchers and educators to explore students' thinking and thereby improve the teaching process.

(a) Describing the levels

The theory proposes that there are 5 discrete levels of thinking about geometrical concepts. In van Hiele's original numbering scheme these were numbered 0 to 4 but Wirszip and Hoffer who have been responsible for bringing the levels to the attention of American researchers have modified the numbering system to 1 to 5. Later writings of van Hiele have also followed this numbering system. Using this latter system the levels, with Hoffer's descriptors in brackets, can be described as

Level 1 (recognition)

Students recognise figures by their global appearance. They can say triangle, square, cube etc. but they do not explicitly identify properties of figures.

Level 2 (analysis) (descriptive-analytical)

Students analyse properties of figures: "rectangles have equal diagonals" and "a rhombus has all sides equal" but they do not explicitly interrelate figures or properties

Level 3 (ordering)

Students relate figures and their properties, "every square is a rectangle", but they do not organise sequences of statements to justify observations.

This is a transitional stage from informal to formal geometry, knowledge at this level is derived from short chains of reasoning about properties of a figure that are derived from thinking at the lower levels.

Level 4 (deduction)

Students develop sequences of statements to deduce one statement from another, such as showing how the parallel postulate implies that the angle sum of a triangle is 180° . However they do not recognise the need for rigour nor do they understand relationships between other deductive systems.

Level 5 (rigour)

Students analyse various deductive systems with a high degree of rigour comparable to Hilbert's approach to the foundations of geometry. They understand such properties of a deductive system as consistency, independence, and completeness of the postulates.

van Hiele also asserted that students passed through the levels in consecutive order but at differing rates. In any classroom therefore students, teachers and resources can be at different levels. The model also asserts that two people operating at different levels may not understand each other.

An earlier level, Level 0 Pre-recognition, has been postulated by Clements and Battista (1992) because many (9-34%) of secondary students failed to demonstrate thinking characteristic of even the visual level. The existence of level 0 is controversial, van Hiele believes that all students enter at level 1 (new numbering system) because all students have the ability to identify common geometric figures by sight. van Hiele believes that all students need to be classified at level 1 because they could be brought up to level 1 with a little instruction.

There has also been controversy about the descriptions of some levels, particularly level 2, Pegg (1997) modifies level 2 as:

Level 2A Figures are identified in terms of a single property (usually sides)

Level 2B Figures are identified in terms of properties which are seen as independent
 of one another

He explains the reason for this as many students, especially up to the first or second year of secondary education respond at level 2 when asked to describe a figure and yet level 2A only requires the acquisition of a single concept or property. It is an interface between level 1 and the multi-concepts/properties required at level 2B.

Another aspect of the theory that has been studied is the link between it and the SOLO taxonomy. Pegg and Woolley (1994) point out that Level 4 could be coded within the Formal mode of SOLO taxonomy of Biggs and Collis (1991). Their study provides new evidence of the quality or depth of thinking that indicates the early stages of van Hiele's Level 4. It also highlights the futility of seeing Level 4 thinking as an either/ or situation.

Pegg (1992) provides an excellent summary of the van Hiele levels. He compares the theories of Piaget and van Hiele. The important differences are that the van Hiele theory:

- (i) places great importance on the role of language in moving through the levels
- (ii) concentrates on learning rather than development, hence the focus is how to help develop student understanding
- (iii) postulates that ideas at a higher level result from the study of the structure at the lower level.

(b) Testing the theory

Many researchers have attempted to determine whether the levels do in fact exist but Hoffer (1983) states that it is the fact that they are being used as a description that is important and not whether they exist or not. It is not the labelling of students by a certain level that is of interest but how to use them as a guide for instruction.

Usiskin and Senk (1990) were concerned that the van Hiele levels were being accepted without having been tested scientifically. They attempted to answer two questions that are critical for any scientific theory

1. is the theory descriptive, i.e. can a unique level be assigned to each student?
2. is the theory predictive, i.e. will a students level predict their performance in a course?

Mayberry (1983) tested two hypotheses

H1. For each geometric concept, a student at level N will answer all questions at a level below N to criterion but will not meet the criterion on questions above level N

H2. A student will meet the criterion at the same level on all geometric concepts tested.

These correspond to two implications of van Hiele's theory (1959):

- (a) A student cannot function adequately at a level without having had experiences that enable the student to think intuitively at each preceding level.
- (b) If the language of instruction is at a higher level than the student's thought processes are, the student will not understand the instruction

Both hypotheses were supported by the research. But they identified the need to design appropriate experiences for the student and that adequate assessment procedures are required to establish students' level

Pegg (1997) links the use of the van Hiele levels with the SOLO taxonomy, this is useful for judging the quality of instructional dependent tasks. It evaluates the quality of students' responses to various stimulus items. i.e. the levels describe responses not people.

Thus for level 2 the characteristic of thinking in terms of independent properties can be interpreted within SOLO as an aspect of a broader thinking category in which concepts are addressed in isolation.

At level 3 the characteristic of thinking concerns the acceptance and use of relationships between properties and figures, associated with the ability to have an overview of relevant elements and to form, on this basis, appropriate generalisations.

Pegg and Woolley (1994) studied senior secondary school students' answers to parallelogram questions and found new evidence of the quality or depth of thinking that indicates the early stages of van Hiele's Level 4. Their work highlights the futility of seeing level 4 thinking as an either/or situation. They used the SOLO taxonomy to identify a pattern of growth and a means of distinguishing between the quality of responses at this level.

The Cognitive Development and Achievement in Secondary School Geometry project administered a test to all the students in a 10th grade geometry course in 13 schools representative of the US. Based on this test Usiskin (1982) concluded that

1. In the form given by the van Hieles, level 5 either does not exist or is not testable. All other levels are testable.
2. Over two-thirds and perhaps as many as nine-tenths of students respond to test items in ways that make it easy to assign them a van Hiele level.
3. Arbitrary decisions regarding the number of correct responses necessary to attain a level can affect the level assigned to many students. (Usiskin, 1982, pp 79-80)

Currie and Pegg (1997) studied class inclusion, which van Hiele described as Level 3 thinking and implied that it did not develop until all Level 2 properties became a totality. They found that student responses provide evidence which reveals that class inclusion is not a "hit or miss" notion. There seems to be a gradual development path which enables students to acquire this skill.

Gutierrez et al (1991) studied the levels and three dimensional geometry and found that students can develop two consecutive levels of reasoning at the same time, although the acquisition of the lower level is usually more complete. Some students used several levels at the same time.

Senk (1989) investigated the van Hiele levels and students ability in writing geometry proofs. She found that the high students achievement in writing geometry proofs is positively related to van Hiele levels of geometric thought and to achievement on standard non proof geometry content.

(c) Other characteristics of the van Hiele levels

There are other important features of the theory

1. The learning process is discontinuous, there are jumps between the levels
2. The levels are sequential and hierarchical , to progress to the next level students must have mastered the lower level.
3. Concepts implicitly understood at one level become explicitly understood at the next level.
4. Each level has its own language, language structure is a critical factor in the movement through the levels.

There is considerable questioning of the discontinuity aspect. Burger and Shaughnessy (1986) had difficulty deciding between levels and considered this as evidence to question the discreteness of the levels. Some students are in transition between levels. They stated that the levels appear to be dynamic rather than static and more continuous than discrete. They also found there was a great deal of imprecise language used by students and that it was an important factor in progressing through the levels. They stressed the necessity for teachers to appreciate that students' understanding of geometrical concepts may be vastly different to the teachers' own.

Gutierrez et al (1991) carried out research to support their theory that the van Hiele levels are dynamic rather than discrete and are continuous rather than static. They observed that although most students show a dominant level of thinking, a response frequently displays some reflection typical of another level.

Currie and Pegg (1997) investigated the class inclusion aspect of level 3. their results indicate that class inclusion is not a "hit or miss " notion. There appears to be a developmental path which enables students to gradually acquire this skill.

One way to improve the teaching process is recognising and removing obstacles to learning. Hoffer (1983) states that one of these obstacles is vocabulary. If students who are thinking at a lower level confront a problem that requires vocabulary, concepts, or thinking at a higher level, the consequences can be frustration, anxiety, and even anger.

(d) Levels across different geometry topics

Of considerable interest to educators is the cross level consistency, if this could be presumed then teaching programs would be easier to design and administer. Research has tended to show that students can be at different levels for different topics in geometry.

Burger and Shaughnessy (1986), Fuys, Geddes and Tischler (1985), and Mayberry (1983) have concluded that students do not necessarily think at the same van Hiele level in different topics of geometry.

Pegg (1992) states that students may be on different levels for different concepts. The multiple choice test used by Usiskin (1982) and others has a basic flaw because it assumes that students will be at the same level for different concepts. This is contrary to the van Hiele theory and also Mayberry's (1981) research.

Pegg (1997) found that the current level descriptors are narrow and not easily generalisable to a range of question types common in school geometry. This is of concern to educators wishing to use them across a wide range of topics, as was the case for this investigation. Descriptors for spatial visualisation are not listed in the literature.

(e) Instruction Process

The van Hiele model is not only a description of students' thought, it also provides guidelines on the teaching process. The students' achievement is directly controlled by the teacher and the curriculum. (Senk 1989)

The model does not support an "absorption theory" model of learning and teaching. A suitable choice of exercises are needed for students to achieve at a higher level. These exercises should be structured and incorporated into the phases of instruction that are listed below:

Phase 1: Information - students become familiar with the content through discussion with the teacher.

Phase 2: Guided Orientation - students are actively engaged in exploring objects, in a structured sequential manner.

Phase 3: Explication - students become explicitly aware of geometrical concepts.

Phase 4: Free Orientation - students solve problems using concepts and relations previously learnt.

Phase 5: Integration - students integrate their knowledge into a coherent network.

The structure of the worksheets and associated teaching programme of this investigation endeavoured to utilise these phases. Clements and Battista (1992) stated that the phases of instruction are potentially more important for education than the levels of thinking but that very little research other than the van Hiele's has been carried out to study the phases.

(iii) Instruction

(a) General

The traditional method used for teaching geometry has relied heavily on the use of diagrams of figures but these do not always convey the required concepts. Battista, Clements and Wheatley (1991) suggest that when teaching geometry teachers should not focus solely on properties of figures and relationships among them. Teachers should also help students develop vivid images and coordinate these images with their conceptual knowledge.

They make two recommendations for teaching

1. Teachers should recognise that many students use visual imagery to reason and they should help students use it to analyse and convince themselves of the truth of geometric ideas.
2. Teachers should discuss visual reasoning with students and they should ask questions that might help students incorporate conceptual knowledge into their visual-reasoning processes.

A further use of diagrams to reinforce concepts is to use non-examples, Charles (1980) found that an appropriate use of examples and non-examples can have a strong effect on concept formation.

Teachers are more aware of different learning styles of students within their classes, Presmeg (1986) studied the effect of teaching styles on students and found non-visual teaching had an inhibitory effect on the learning of visual students and led them to believe that success in mathematics depends on rote memorisation of rules and formulae.

There have been a number of studies carried out to investigate the effect of training on spatial ability and also on the long term effects of this training the results of these studies have been that there is a positive effect on spatial ability after instruction was given. There is a need for the long term effects of this training to be evaluated.

Cox (1978) found that the scores of students trained in perspective ability were significantly higher than those of controls in all the perspective tasks at the post test: moreover, tests given seven weeks and seven months after training had ended showed that this difference was maintained. Similarly Smith and Schroeder (1979) found that fourth grade students, regardless of sex, profited strikingly from instruction involving spatial ability, using tangrams, as measured immediately following instruction. However, they also suggest that further study is needed to determine whether the effect of this spatial visualisation instruction persists over time and whether both sexes are affected comparably over time.

Wheatley and Wheatley (1979) conducted an experiment with low achieving 14 year olds. For one month they were engaged in spatial processing activities, these were hands on activities with concrete rather than abstract representations. The result was a marked improvement on a spatial ability test (used tangrams, tessellations and polyominoes)

Shubber and Al-Mudaifa (1991) investigated the effect of using slides demonstrating the effect of rotation around different axes on diagrams of simplified molecules. They had reported that many senior science students had difficulty visualising the effects of performing essential spatial operations (i.e. rotations, reflections and inversions) on diagrams of three dimensional structures such as molecules. They found that slides of a sequential rotation were effective in improving students ability.

(b) Apparatus

The emphasis on the use of apparatus in learning mathematics is not a recent phenomenon , Freudental (1971, p 414) states that " Modern educators are likely to subscribe to a variation of Comenius' device. Not

"The best way to teach an activity, is to show it" but rather "The best way to learn an activity, is to perform it." It is a mere shift of stress but an important one".

The curricula of many countries are emphasising the use of such resources, the United States is one example. (NCTM 1991, p 52) states:

“The teacher of mathematics , in order to enhance discourse, should encourage and accept the use of... concrete materials used as models”.

The use of concrete materials is also recommended in the New Zealand curricula, (MINZC 1992, p13)

“students are capable of solving quite difficult problems when they are free to use concrete apparatus to help them think the problems through.”

However it is necessary for these materials to be both appropriate and flexible in use so that a process of reflective thinking is encouraged. Bishop (1973) found a relationship between student's earlier use of apparatus and the later ability to think and perform in contexts where spatial relationships are involved. This relationship was not dependent on the type of apparatus used. A practical implication is that remedial work using structural apparatus could well help pupils who have spatial difficulties.

Connor and Serbin (1985) found that 30 minutes of exposure to visual-spatial training materials resulted in significant increases in performance on a visual-spatial test for two of the five sets of material developed. Their results support the conclusion that visual-spatial skills are teachable in a classroom setting with junior high school students. There is a difficulty with designing effective materials as well as the difficulty in having an impact on skills in brief training periods. Spatial visualisation may be a more difficult skill to teach than orientation.

Leeson (1994) developed a set of activities using interlocking cubes and polygonal shapes with the aim of improving the students' spatial sense with respect to three-dimensional shapes.. He states that "the hands-on activities described in this article involve the use of a number of senses, sight, touch and hearing. Such activities need to be experienced on a regular basis by students in fifth and sixth grades to aid in the development of their spatial sense."

Leiva et al (1992) related building structures with blocks to developing concepts of functions. Their objective was " students will make connections among the concrete, pictorial, and abstract representations of a three-dimensional figure and will investigate relations and functions by:

1. representing a structure from pictorial representations
2. building one or more structures that correspond to a given pictorial representation, map or number of blocks.

In the conclusion of Owens' (1994) research on visual imagery she states that it is through actively engaging in problem-solving situations that students use perceptions, ideas, and images to reflect on the properties of shapes and existing abstractions. The materials are important for the students to be able to make and create new shapes and, in so doing, they began to think mathematically and to construct meaning. This supports the van Hiele phases of instruction as a learning process.

The use of concrete materials to enhance thinking processes has been investigated by Owens (1992 a) who stated in the conclusion of her study that children use a wide range of thinking processes in solving spatial problems and that classroom spatial activities improve and enrich students' spatial thinking processes. Of particular importance was the effect of interactions with both the materials and other people in encouraging students' responsiveness and their development of spatial and other thinking processes. This supports the van Hiele concept of interactions in the process of instruction.

Traversi (1979), as quoted in Pallascio et al (1993, p8), " there is much more to be gained in terms of understanding if we attribute to the teaching of geometry an operational and experimental character where knowledge is transformed into know how through tasks which are centred on the manipulation, construction, and comparison of geometric models."

Post (1989) states that " researchers in mathematics education are in the process of accumulating a persuasive body of evidence that supports the use of manipulative materials in the mathematics classroom." But little work has examined the use of instructional aids and their effects on student achievement.

Although there have been a number of studies that have reported a positive correlation between the use of concrete materials and achievement there may be other factors that need investigating. Raphael and Wahlstrom (1989) studied the use of instructional aids by teachers and related this to their students achievement. They found that greater use of aids was related to greater topic coverage and that the use of aids was, for the most part, related to student achievement as part of the association between content coverage and student achievement. Related to this greater content coverage is the greater amount of time that is spent when using materials. Some teachers perceive this as a disadvantage. Bishop (1983) recommends that studies of training are needed including the optimum time for training.

Another contra-indication is classroom control. Post (1980) reported that one of the reasons that manipulatives are not used more is that they are more difficult for the teacher to administer, and that students need to develop self-control for classroom management.

The manipulatives used need to be appropriate Ben-Chaim et al (1988) reviewing the research - quoted Mitchelmore (1975) p172 " The greatest need is for the development of practical geometric and spatial teaching programs and for their experimental testing" Also Sherman (1979) p26-27 " methods for achieving this [improving spatial skill] need to be devised, and their feasibility and advisability evaluated". Their investigation shows that the effect of instruction to increase spatial visualisation skills provides evidence to support the notion that these skills are teachable and can be learned. The specific skills involved in the manifestation of the spatial visualisation ability improve with practice. When these spatial visualisation skills have been attained, they last and even continue to develop over time.

Pallascio et al (1993) stress the importance of using tasks which require the manipulation, construction and comparison of geometric models to transfer knowledge into know how. They investigated the effect of alternating analytic (perception) and synthetic (representation) activities. They found that "in the creation or generation of spatial representations, hands-on work with physical media is certainly important, but it is also important not to create new obstacles to learning. The materials used must therefore be chosen very carefully.

Ben-Chaim et al (1985) state that there have been few studies of training programs to improve spatial visualisation reported. Their results after students studied a 3 week spatial visualisation unit suggest that concrete experiences with cubes, building, representing in 2-dimensional drawings, and reading such drawings are helpful in improving students' performance.

Herschowitz (1990) clarified the difference between 3-d to 2-d and vice versa

The interpretation of 2-d drawings back to 3-d has been studied less. Many children have difficulty in communicating 3-d information

The 3-d to 2-d research raises the following questions,

1. what are the factors that influence the description and interpretation of drawings of 3d shapes?
2. can these visualisation abilities be acquired or improved by specific training?
3. if yes, what should be included in the curriculum, and how could it be taught?

Piaget believed that children learned best from concrete activities and concrete operations are the highest level that children can operate at until 11 or 12. Many educators support the use of such activities when teaching younger pupils but this investigation was designed to evaluate their use by older pupils.

This review has summarised the information that is available in the literature. The van Hiele levels are still the subject of intensive study but their usefulness to the learning process ensured their relevance for this investigation. More information is needed however in order to utilise them in other curriculum areas. The use of apparatus is strongly recommended by educators but research into their most effective use is continuing.

CHAPTER 3 METHODOLOGY

1. Research Question

This study investigates the question

Does the use of manipulatives improve the spatial visualisation of three dimensional objects when presented in a two dimensional form?

The use of manipulatives is an important feature of the new New Zealand mathematics curriculum, the curriculum document states (MINZC, p 13)

“ students are capable of solving quite difficult problems when they are free to use concrete apparatus to help them think the problem through. Such an approach is equally valid with older students and should be used whenever possible.”

The use of apparatus is now an integral part of secondary school mathematics in those schools which have embraced the spirit of the new curriculum. It was therefore not possible or desirable to conduct a study with a control and experimental group as this would have unfairly disadvantaged a group of students.

Many teachers support the use of manipulatives in the primary and lower secondary school classes but their use is less common in senior classes. The use of concrete materials does pose some problems for classroom teachers, these include

- (i) choosing the appropriate materials
- (ii) management of student use
- (iii) most effective use of time

The issue for administrators is the cost of these resources and their overall management. Management of students' use of manipulatives tends to be a more important issue with junior classes, whereas time management is of more concern to teachers of senior and especially examination classes. Teachers wish to be assured that this type of activity is of direct benefit to their students' performance in the examinations.

The first formal external assessment in New Zealand is at the fifth form level and the examination is of the pen and paper variety. Students are required to interpret pictorial representations of three dimensional objects without the assistance of concrete materials.

2. Overview

This investigation was carried out with students in three school certificate mathematics classes in order to study the use of manipulatives in the three topics that require students to use spatial visualisation to solve problems. The students had the nature of the research explained to them at the start and parental/ guardian consent to participate was obtained.

The three topics concerned, in their teaching order, were

1. volume
2. drawing and visualising three dimensional solids
3. measurements in three dimensions

The research was designed to be as unobtrusive as possible so that class time would not be used in activities that were not associated with the syllabus. Each topic was evaluated for the spatial skills required and worksheets designed to assess spatial ability before and after activities using manipulatives. These manipulatives were also carefully evaluated.

The worksheets were incorporated at the appropriate time in the normal teaching schedule. In the case of topics 1 and 2 the two worksheets were used before and after the practical activities. The worksheet for topic 2 comprised a number of skills and activities associated with isometric drawing and was designed to be a self contained student centered resource. These worksheets were designed specifically for the research and did not incorporate other materials. Particularly in the case of worksheet 2 this necessitated careful design in order to cover all aspects required by the curriculum.

The three classes involved are streamed for ability so they completed the worksheets at different times so this did not cause difficulty with sharing of resources. The teachers were responsible for administering the worksheets at the appropriate time and their collection. There was constant feedback between them and the researcher on the progress of the classes.

3. Terminology

The term manipulatives was used to describe any object that students could use to build a three dimensional object and manipulate it in order to view it from different view points. There are different forms of manipulatives that can be used, these can be classified into two types (i) those that students construct themselves from paper, card, straws, pipe cleaners etc.

- (ii) those that are already formed into some geometrical shape, but have the potential for further construction. These include solid polygonal shapes, Geo-shapes, interlocking cubes etc.

The advantages of type (i) are that the students are more involved with the construction process, including accurate drawing of geometrical shapes. Their disadvantages are that they are time consuming and students with poor practical skills may be disadvantaged when using them.

The advantages of type (ii) are that they are quick and relatively easy to use. They are also visually stimulating and can be used in other curriculum areas. The disadvantages are their expense and resource management.

The definition of spatial visualisation used was that of McGee (1979, p 893)

“ the ability to mentally manipulate, rotate, twist or invert a pictorially presented stimulus object”

This differs from spatial orientation which only requires seeing an object from a different perspective but not mentally moving the object.

4.Experimental Considerations

A. School Studied

The school studied is an independent girls school with a roll of approximately 500 pupils. The principal and board of trustees were extremely co-operative when approached. There were 75 students in the fifth form with a small group participating in an alternative program, Maths Applied 1. The school certificate classes are streamed into three different ability groups .

B. Mathematics Department

The mathematics department is fully committed to the new curriculum and well organised with respect to schemes and resources. The fifth form classes studied have been accustomed to using many of the manipulatives before and their teachers are also accustomed to them. The teachers involved are all qualified, and highly motivated.

C. Participation

An information sheet describing the research was distributed to the students and parental/guardian consent was sought as outlined by the Massey University Ethics Committee. The format of the study was also explained verbally to the pupils and questions answered. There was a high level of co-operation due to the fact that the worksheets used were an integral part of the normal teaching programme and also many pupils expressed a desire to assist for personal reasons.

The completion rate for each of the three topics were

1. $61/70 = 87.14\%$

2. $45/70 = 64.29\%$

3. $57/70 = 81.43\%$

The completion rate for topic 2 was considerably lower than the others because the worksheet took much longer to complete and it was therefore more difficult to administer and collect.

D Manipulatives.

The manipulatives used were carefully evaluated to be the most appropriate for the topic concerned.

For topic 1 the concrete materials used were Geo-Shapes. These are coloured polygons of different shapes made out of plastic wire frames that can be linked together to form nets and/or polyhedrons. The shapes used were squares, triangles, pentagons and hexagons.

For topic 2 interlocking plastic cubes were used

For topic 3 Geo-shapes were used again, and coloured pipe cleaners used to form the diagonals and heights.

5. Experimental Design

There are three topics in the school certificate prescription that require the use of three dimensional objects, these are

1. volume
2. drawing and visualising three dimensional solids
3. measurements in three dimensions

It was decided to investigate the use of manipulatives in the teaching of each of these topics. The design of the study had to be such that the research was as unobtrusive as possible because of the pressure of time constraint caused by an external examination at the end of the year.

The format used was a worksheet designed for each topic so that spatial visualisation was investigated before and after using appropriate manipulatives for that topic.

Worksheet Formats

1. Volume

This topic has been taught at previous levels but most teachers accept that it needs to be taught again at the fifth form level. The greatest difficulty for pupils occurs in deciding which shape is the appropriate cross section to use in a non-rectangular prism.

This topic was introduced by an introductory lesson on revising the volume of cuboids with appropriate exercises for practice. The worksheets were presented in consecutive lessons

The first worksheet design was

- (i) a selection of prisms with students shading in, and naming their cross sections
- (ii) using Geo shapes, constructing prisms of given cross section shape.
- (iii) sketching these constructions and shading required cross section
- (iv) giving a definition of “cross section” in their own words

The second sheet comprised

- (i) Diagrams of twenty two prisms to be matched with six given ones.
- (ii) A question asking which were the hardest ones to match

2. Drawing and Visualising Three Dimensional Solids

This topic also appears at lower levels in the curriculum, but at this level students are required to

- (i) visualise the effect of removing/ adding cubes to an object and drawing the resultant object
- (ii) drawing the object as it would appear from a different perspective
- (iii) draw an object from a plan, or coded view

The worksheets were collected in by the teachers who made relevant comments on worksheet content and design. Student responses were analysed and used to develop suitable descriptors for van Hiele levels of spatial visualisation.

CHAPTER 4 RESULTS

The results are presented in three sections corresponding to the three topics investigated.

TOPIC 1 Volume

Worksheet 1 Shading and defining cross sections

Worksheet 1 is given in appendix 1

This required the students to shade in, and name the cross sections of 8 diagrams of prisms.



Figure 1: Two of the prisms for shading and naming

TABLE 1

Shading of cross sections

All cross sections shaded correctly	59
Cross sections not shaded	2
Total number worksheets collected	61

The shading of all cross sections was completed successfully by all the students who attempted it. Two did not complete any shading. This high success rate is supported by teacher comments that students found this exercise easy.

TABLE 2
Naming of cross Sections

Cross sections named correctly	25
Cross sections not named	2
Cross sections named incorrectly	34

Naming of the cross sections was attempted by the same 59 students who had completed the shading exercise. Of these students 34 made some error in the naming of the cylinders' cross sections. These are analysed in table 3.

TABLE 3
Incorrect cylinder responses

Response	Frequency
Both named as cylinder	2
Both named as oval	3
Vertical named as oval, horizontal as ellipse	1
Vertical named as circle, horizontal as oval	28
Total number of incorrect cylinder responses	34

The large number of students naming the vertical cylinder's cross section as a circle may be a consequence of their previous teachers having drawn cylinders in the vertical position. Less exposure to diagrams of horizontal cylinders may have made these students less able to visualise their cross sections as circles.

TABLE 4

Level 1 responses to “in your own words describe what cross section means”

Response	Frequency
Shape which goes right through the prism	7
Slices are still the same shape	9
The side of a prism that can be repeated through it	1
Shape can be cut evenly so it looks the same	1
The front piece stays the same	2
It’s the main shape	1
It represents the shape	1
A shape that fits into the shape it is	1
Somewhere that can be cut so that everything is the same	1
The shape can be cut into parts and it stays the same shape and size	1
When you cut it up, it’s the shape that is always on the bit You cut	1

Level 2 Responses to “in your own words describe what cross section means”

Slices stay as prisms and the same shape and size	5
Cross section is the prisms original shape	2

These definitions may reflect the wording at the top of the first worksheet, “need to identify the cross section that remains the same through the solid”. The concept of slicing through a prism so that the slices remain the same shape, similar to slices of bread is a common example used by teachers.

Some of the definitions given show evidence of greater complexity, for example the concept of the slices being prisms that retain the original's size and shape reflects this level of analysis

In terms of van Hiele levels for the verbal skill, level 1 is indicated by interpreting sentences that describe figures. Responses such as the first eleven listed in table 4 could be evidence of this level and therefore 26 of the 33 students that responded are thinking at this level

Level 2 of the verbal skill, analysis, requires accurate description of various properties of a figure. The awareness that the slices retain the properties of the **original** prism reflects this awareness and are shown in the last two responses in the table. These definitions were given by 7 out of the 33 students who responded and could therefore be considered to be at level 2.

Teachers have commented that the more able students did not find it necessary to build the models to be able to identify the correct cross section. Less able students however did find it beneficial.

Worksheet 2: Matching Prisms

This worksheet is given in Appendix 2

This worksheet required the students to match the diagrams of 22 prisms to the diagrams of 6 given prisms. The cross sections of the given prisms were

- A. right angled triangle
- B. rectangle
- C. circle
- D. trapezium
- E. equilateral triangle
- F. square or rectangle

12. a semi-cylinder

17. a pentagonal prism

22. a three-quarter cylinder

TABLE 5
Frequency of correct matching of prisms

Prism	Matching prisms	Transformation	Frequency
A	1	Reflection	58
	6	None	60
	15	Rotation	56
B	5	Rotation	60
	18	Rotation	59
	19	Reflection	59
C	4	Rotation	60
	10	Skew	57
	16	None	61
D	3	Rotation	59
	11	Reflection	59
	14	Rotation	56
E	2	Rotation	59
	7	Rotation	58
	13	Rotation	59
	21	Rotation	58
F	8	None	58
	9	Reflection	58
	20	Reflection	57

This exercise was also completed successfully by most of the students. The success rates vary from 91.80 % for matching number 15 with A , (quarter turn rotation of right angle triangle) to 100 % for matching 16 with C (same view)

TABLE 6
Other responses

Response	Frequency
12, 22 same as C	8
17 put down as both D, and E	2
22 put down as both C, and E	1
12 put down as E	1
Evidence of shading	6
Incorrect shading	1

7 students carried out shading as a strategy to assist them in the matching exercise, of these, 1 student shaded incorrectly.

The most common error is including 12, the semi-circular cylinder and 22 the three quarter cylinder as matching C the given cylinder. This may in fact be a consequence of students’ belief that all the prisms were intended to match one of the given ones. They then selected those that were the closest match, having some form of circular cross section.

This may be evidence of thinking at van Hiele level 1, at the recognition level of the visual skill, recognising the curved surface as a visual clue. To demonstrate the analysis required by level 2 of the visual skill students would need to identify the curved surface as part of a larger figure which did not have a complete circle for its cross section.

TABLE 7
Responses to “which were the hardest to match”

Response	Frequency
12, 17, 22	14
17	2
1	1
10	1
1, 10	1
A	2
A, D	2
A, D, F	1
A, E	2
B, F	1
D	6

The responses to “which were the hardest to match” question also reflect the fact three of the prisms did not match as shown. This is evidenced by 14 responses of “12, 17 and 22”. Of the given prisms D was named a total of 9 times, of these 6 named D only. D is a trapezium which may be a shape that students have less experience of as a cross section of a prism. Another explanation may be that its shape was not one of the prisms built by the students at the end of worksheet 1.

Interpretation of results

1. almost all students shaded and named the cross sections correctly except for the discrepancies with the cylinder
2. most students matched the prisms successfully, non-matching ones caused problems, students may have believed that all were intended to match
3. most students showed evidence of level 1 verbal skills when defining a cross section

TOPIC 2 Drawing and Visualising Three Dimensional Solids

This worksheet is given in Appendix 3

This worksheet comprised 6 sub sections, corresponding to the 6 skill areas required by the prescription, followed by a review.

1. Drawing Solids made of 5 Cubes

TABLE 8

Frequency of solids drawn correctly after building

Result	Frequency
Solids drawn correctly	40
Solids drawn incorrectly	5
Number answered	45

TABLE 9

Frequency of solids drawn correctly without building

Result	Frequency
Solid drawn correctly	41
Solid drawn incorrectly	3
Not attempted	1
Total number worksheets	45

This was a comparatively simple exercise demonstrating the method involved in drawing a simple 5 cube solid on isometric paper.

There is an insignificant difference between the results for drawing the solids with building them and without building them.

2. Smoothing out lines

This required the students to decide which surfaces could be “smoothed and painted” so that the lines between the cubes disappear. It necessitates the differentiation of one surface from another.

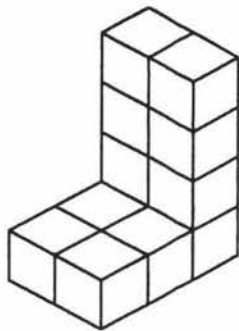


Figure 2: Solid 2 for smoothing

TABLE 10
Drawing and smoothing out lines on Solid 1

Result		Frequency
Solid drawn correctly	Smoothing correct	40
	Smoothing incorrect	3
Solid drawn incorrectly	Smoothing correct	1
	Smoothing incorrect	1
Total number worksheets		45

Solid 1 was similar to the 4 cube solid that was smoothed in the given example and was drawn and smoothed correctly by 88.89 % of the students.

Solid 2 was quite different to the given example in that it was more than one block high, and visualising the two surfaces that met at the hinge caused difficulty for two of the students. Four students who drew the solid correctly failed to smooth it correctly. There were another eight students who erred in their smoothing either by not visualising the position of the “hinge” or by not smoothing the sides. These results are summarised below

TABLE 11
Drawing and smoothing out lines on Solid 2

Result			Frequency
Solid drawn correctly	Smoothing correct		30
	Smoothing incorrect		4
	Smoothing partially Correct	Sides	6
		Hinge missing	2
Solid drawn incorrectly	Smoothing correct		3
	Smoothing incorrect		0
Total completed			45

The incorrect and partially correct shading could be interpreted as thinking at level 2 of the visual skill.

3. Matching 4 cube solids

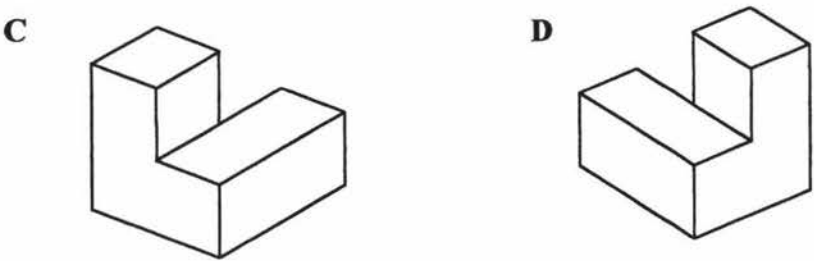


Figure 3: Two of the 4 cube solids for matching

This component required the students to match 16 diagrams of smoothed 4 cube solids into pairs and then answer the question “were any solids particularly difficult to match, and if so which one(s)”

The frequency of each number of correctly matched pairs is shown below.

TABLE 12
Number of correctly matched pairs

Number	8	7	6	5	4	3	2	1	0	Total
Frequency	34	0	10	1	0	0	0	0	0	45

The results indicate this was completed successfully by 75.56 % of the students. Of the unsuccessful students 90.9 % incorrectly matched two pairs which effectively means one error.

The analysis of the incorrect matching is given in table 13

TABLE 13
Breakdown of incorrect matching of pairs

Response	C=D, P=K	C=K, D=P	C=D,D=K No match for P	Total
Frequency	7	3	1	11

All incorrect matching involved C, D, K, and P. All these solids comprised two 2 cube columns rotated at 90 degrees. The matching of them required spatial visualisation in order to visualise what they would look like after various rotations. 24.44 % of students did not achieve this successfully and these could be interpreted as thinking at level 2 of the visual skill.

The responses to the question “Were any pairs difficult to match?” are given below.

TABLE 14

Responses to “were any pairs difficult to match?”

Response	Yes	No	No response	Total
Frequency	28	12	5	45

More students replied “Yes” (28) than the number of students who matched the pairs incorrectly (11) indicating that although they had found the task difficult they had been able to solve it.

An analysis of responses to the question “which pairs were difficult to match” is given in table 15

TABLE 15

Responses to “which pairs were difficult to match”

Response	CDKP	EG,CD	EOP	OL,CDKP
Frequency	23	2	1	2

These responses show a very high percentage (82.14 %) naming C, D, K, and P as the solids causing difficulty. The students were therefore aware of the problem caused by the similarity of these solids which required a high level of spatial visualisation to differentiate between them.

Although not specifically asked for the reason for their difficulties in matching pairs of the solids, 6 students provided a reason, these are listed below

TABLE 16
Reasons given for difficulty of matching

Reason	Frequency
They are similar but facing in opposite direction	1
If the picture was turned around they would look the same	5

Both of these reasons acknowledge the rotational aspect of the difficulty in matching them.

4. Adding cubes to solids

This exercise showed a diagram of a 5 cube solid and a further diagram showing the result of adding a cube on top of it in **one** of 4 possible positions. Students were asked to draw the 3 different solids that would result by adding a cube in the other 3 possible positions. Unfortunately the instructions were not understood by one of the teachers who was then unable to explain them to her class. This resulted in a large number of inappropriate answers. The number of solids drawn correctly are shown in table 17

TABLE 17
Drawing solids after building

Number of solids drawn correctly	Frequency
3	17
2	0
1	12

The students were then asked to build 3 different solids, add 2 cubes to each of them in designated positions and then draw the resultant solid.

TABLE 18
Frequency of drawing solids correctly with added cubes after building

Number of solids drawn correctly	Frequency
3	31
2	10
1	1
0	2
Incomplete	16

The next task was to draw resultant solid after adding cubes but without having built them, the frequencies of correct diagrams are given in table 19

TABLE 19
Frequency of drawing solids with added cubes without building

Result of drawing solid without building them					
A		B		C	
Correct	Incorrect	Correct	Incorrect	Correct	Incorrect
40	4	39	5	20	24

Most students were able to complete this for solid A (90.91 %) and solid B (86.34 %) but fewer were successful with solid C (45.45 %).

Solid C differed from solids A and B because the new cubes were being added onto different sides of the **same** cube. This appeared to cause the students difficulty.

TABLE 20

Frequency of responses to “was it harder to draw them without building them first?”

Response	Yes	No	No comment
Frequency	9	31	4

The students response (70.45 %) that it was not more difficult to draw the added onto solids without building them first was surprising given their lower success rate. This lower rate could have been caused by the fact that it was more difficult to visualise the cubes added onto C than in any of the solids in the previous exercise.

The next question asked the students if any of the solids were particularly to draw after adding on the two cubes.. The results are given below in table 21.

TABLE 21

Responses to “were any of these solids particularly difficult to draw?”

Response	Frequency
Yes	8
No	33

80.49 % of those who responded did so in the negative. This is despite the fact that 45.45 % had drawn the added onto solid C incorrectly. They had not recognised the difficulty they had in visualising how to draw the new solid.

The students were then asked which solid, if any, had caused them difficulty. There were 10 responses to this question despite only 8 replying in the positive to the previous question.

TABLE 22
Frequency of responses to “which one”

Solid	Frequency
A	3
B	3
C	4

There is an insignificant difference in the frequency of responses to which solids caused difficulty when drawing “added onto” solids without having built them first. This is despite the fact that only 45.45% of students drew solid C correctly.

TABLE 23
Responses given to “why?”

Solid	Reason
A	Because its on a different angle
	It took longer to visualise
B	Got confused where blocks went
C	Couldn’t get block on top
	Had to think where to put them

The final task in this section asked the students to explain “ how you drew these added onto solids”. The results are summarised in table 24.

TABLE 24
Explanations given for “how you drew these added onto solids”

Explanation	Frequency
Visualised it in my mind	12
Visualised block drawn on, drew it from my mind	1
Drew solids first, and added blocks on	10
Added block where shading was and copied	1
Don't know	1

Of the 25 responses 13, (52 %) used some concept of visualisation whereas 11, (44 %) concentrated on the drawing aspect 1 student (4 %) replied “ don't know”.
A significant number did not have to rely on the first diagram to draw the new solid.

Using Hoffer's (1981 p15) table of basic skills in geometry most of these students could be placed in level 2 or 3. Those students reliant on using the existing drawing could be presumed to be operating at level 2 “ uses given properties of figures to draw or construct the figures.” The students who did not rely on drawing onto the given diagram could be presumed to be operating at level 3 “ given certain figures, is able to construct other figures related to the given ones”.

5. Subtracting cubes from solids

The next task required the students to build 6 different solids, remove the shaded cube from each one and then draw the resultant solid.

TABLE 25
Result of drawing solids with cubes subtracted after building

Result	Frequency
All solids drawn correctly	32
Not all solids drawn correctly	12

Of the 44 students who completed this task, 32 (72.73 %) drew all 6 resultant solids correctly. An analysis of the incorrectly drawn solids is given in table 26

TABLE 26
Frequency of incorrectly drawn solids

Solid	1	2	3	4	5	6
Frequency Incorrect	3	6	1	3	2	2

Solid 2 was drawn incorrectly by 6 students, of the 6 incorrect responses for no. 2, 4 had presumed there was no block under the one removed. This is not an error in visualisation but in logic.

The students were then asked to repeat this task but without building the solid from which the cubes were removed.

TABLE 27
Result of drawing solids with cubes subtracted without building

Result of drawing solid without building them first							
A		B		C		D	
Correct	Incorrect	Correct	Incorrect	Correct	Incorrect	Correct	Incorrect
40	3	41	2	41	2	42	1

The success rate for this task is high, ranging from 93.02 % for solid A to 97.67 % for solid D. This high success rate is reflected in the responses to the question “ was it harder to draw them without building them first?” The frequency of each response is shown in table 28.

TABLE 28
Frequency of response to “was it harder to draw them without building them first?”

Response	Frequency
Yes	7
No	32
No response	4

Of the 39 students who responded 32 (82.05 %) did so in the negative. This supports the high success rate in drawing these solids without building them first.

TABLE 29

Response to “were any of these solids particularly difficult to draw?”

Response	Frequency
Yes	7
No	31
No response	5

Similarly, of the 38 students who responded to the question “were any of the solids particularly difficult to draw”, 31 (81.58 %) replied in the negative.

TABLE 30

Frequency of responses to “which one?”

Solid	A	B	C	D
Number of Responses	6	4	1	0

Of the 11 responses to “which one” was particularly difficult to draw, A or B were named by 10 of the 11 (90.91 %) . These solids differed from C and D, because the cube to be removed was sandwiched between 2 lateral cubes. It is therefore more difficult to visualise the resultant faces once the required cube has been removed.

Students were then asked “why” any of them were particularly difficult to draw and their responses are shown in table 31.

TABLE 31
Frequency of responses to “why”

Solid/s	Responses
A	Couldn't see it in my mind
A,C	Couldn't see them in my mind
A,B	Difficult to line up
B	Had a lot of blocks

Of the 4 explanations given for the difficulties experienced, 2 use the concept of their problem of “seeing them”.

TABLE 32
Explanations given for “how would you explain how you drew these new solids”

Explanation	Frequency
Drew the solid first and then removed block	11
Visualised missing piece and drew from my mind	15

Of the 26 students who responded to the question “how would you explain how you drew these solids?” 11 (42.31 %) used the technique of drawing the solid and then removing the solid. These students appear to be unable to visualise the resultant solid after removing a cube. However 15 students (57.69 %) gave the explanation that they were able to visualise the resultant solid. This compares with the 52 % of students who used some explanation of visualisation in the equivalent “adding cubes on” exercise.

6. Other ways of drawing solids

Students had difficulty with this exercise because of an error in the diagram of the “birds eye view” where the numbers representing the heights of the cubes were omitted. Most did not attempt to draw a birds eye view of the solids represented. These are not included in the results. Only 33 out of 45 students (73.33 %) attempted to draw the required isometric drawings. The analysis of their drawings is given below in table 33

TABLE 33
Frequency of solids drawn correctly

Solid	Drawn correctly	Drawn incorrectly	Not attempted
1	22	11	12
2	21	12	12

The success rate for those that attempted the task is 66.67 % for solid 1, and 63.64 % for solid 2. This comparatively low rate may be an indication of students’ inability to translate one representational form to another.

Review

The review comprised two tasks, drawing two “smoothed out” solids and matching 10 solids made of 4 or 5 cubes.

The results of the smoothing out task are shown in table 34.

TABLE 34
Frequency of Correct Drawing of Smoothed Solids

Solid	Smoothed correctly	Smoothed incorrectly	Not attempted
1	24	9	12
2	16	17	12

Of the 33 students who completed the task 24 (72.73 %) drew solid 1 correctly, whereas only 16 (48.48 %) drew solid 2 correctly. Both solids had three faces that required smoothing but they were less successful with solid 2. This may have been caused by the fact that solid 2 had two indented corners with a cube missing from each. Students could have found these difficult to visualise.

The matching exercise was not straightforward because two of the solids D and H did not match any of the other solids. The results of the matching exercise are shown in table 35.

TABLE 35
Results of matching solids

Result	Frequency
All pairs matched correctly	4
Not all pairs matched correctly	33
Not attempted	8

Only 4 students of the 37 (10.81 %) who attempted this task completed it successfully. 33 out 37 (89.19 %) did not match all pairs correctly. Of the 33 incorrect responses 19 students (57.58 %) matched D=H. This indicates that students tried to match all solids although some of them may have recognised that D and H did not really match.

This awareness is shown in their responses to “ were any solids difficult to match, and if so which one(s)?” The responses to the second part of the question are shown in table 36.

TABLE 36
Responses to “which solids were particularly difficult to match?”

Solids	Frequency of responses
DH	14
BEH	1
CDEH	4
DE	1
ABCIJ	1
DFGH	1

Of the 22 students who responded 14 (63.64 %) named D and H. D and / or H was named by 21 out of the 22 (95.45 %) respondents.

This task was in fact harder than the equivalent in part 3, matching 4 cube solids.

Interpretation of results

1. there was no difference in success rate for drawing the 5 cube solid with and without building first.
2. most of the smoothing tasks were completed successfully.
3. all errors in matching the cube solids involved solids which required a high level of spatial visualisation.
4. students did not believe that building the solids first helped them draw the “added onto” solids, however they also did not recognise the difficulty they had completing the task. 50 % used some concept of visualisation in their explanation of how they completed the task.
5. similarly students did not believe that building the solids first helped them draw the “subtracted from” solids. 60% of those who were incorrect with solid 2 made a logic error.
6. there was a poor success rate with different ways of representing a solid, this was partly caused by an error on the worksheet and by the teachers’ inability to correct it. However students still experienced difficulty with another part of the task that was not affected.
7. there was a low success rate with matching solids in the review, this may have been caused by the task being more difficult than the equivalent task earlier in the worksheet.

It is not possible to determine whether the students’ spatial visualisation had improved as a consequence of using the interlocking cubes to build and manipulate solids. The students did not believe that using the cubes made the tasks easier, many of them felt that they could have completed the exercises without using the cubes.

This type of activity has been completed in previous school levels so the use of manipulatives may be less appropriate at this later stage.

TOPIC 3 : Measurements in Three Dimensions

Worksheet 1

This worksheet is given in Appendix 4

This worksheet was comprised of three sections each of which required the students to name specified angles, and then a question asking if any were particularly difficult to identify, and how would they explain how they found the required angles. The three sections were,

- 1. 10 questions on angles between lines
- 2. 7 questions on angles between lines and planes
- 3. 6 questions on angles between planes

1. Angles between lines and planes

TABLE 37

Frequency of correct responses of angles between lines

Number of correct responses /10	Frequency
10	26
9	12
8	3
7	6
6	3
5	3
4	0
3	1
2	0
1	0
0	0
Not attempted	3
Total	57

The mean number of correct responses = 8.87 , which is 88.70 %

This high success rate indicates that these tasks were relatively easy for the students, in fact 26 (49.06 %) named them all correctly.

1. Angles between lines and planes

TABLE 38

Frequency of correct responses for angles between lines and planes

Number of correct responses / 7	Frequency
7	1
6	4
5	7
4	8
3	7
2	8
1	8
0	1
Not attempted	13
Total	57

The mean number of correct responses = 3.23, which is 46.43 % . Students found this task more difficult. This is evidenced by the lower success rate and also the larger number of students who did not attempt any questions in the section.

2. Angles between 2 planes

TABLE 39

Frequency of correct responses for angles between planes

Number of correct responses / 6	Frequency
6	6
5	4
4	5
3	4
2	7
1	4
0	0
Not attempted	27
Total	57

The mean number of correct responses is 3.53, which is 58.89 %

The students were then asked if they had found any of the tasks particularly difficult, and if so which one(s).

TABLE 40

Frequency of responses to “were any of these difficult to identify?”

Response	Frequency
Yes	16
No	7
No response	33

Of the 23 students who responded to this question 16 (69.57 %) replied Yes, and 7 students (30.43 %) replied in the negative.

The students were then asked which angles had been difficult to identify, the analysis of their responses is given in table 41

TABLE 41
Responses to “which ones”

Response	Frequency
2	3
2,3	2
2A, 3B	2
3	3
3A	1
3B	1

Of the 12 students who responded to this question, none named question 1. 7 students (58.33 %) responded with some combination of question 2 and 9 students (75 %) responded with question 3 in some combination.

They were then asked to explain how they found the required angles and their responses are given in table 42

TABLE 42
Responses to “how would you explain how you found the angles?”

I drew in where the planes were and found the angles
I drew the planes in to work it out
I coloured them in
I drew in lines, and shaded in planes
By using one side of the plane
I looked at angles in between lines and planes
I first identified the planes and lines and then thought about it

These explanations show a heavy reliance on drawing the lines or planes that would assist them on the given diagram.

After completing this part of the worksheet the students were then instructed to build various models using geo-shapes and pipe cleaners where appropriate. These models represented

1. Intersecting rectangular planes
2. Intersecting triangular planes
3. A cube with three dimensional diagonal
4. A pyramid with angle between base and sloping face
5. A pyramid with angle between sloping edge and base

Teachers' comments indicate that this was an enjoyable and worthwhile exercise. It had been intended that this, and the preceding questions would be completed in one lesson, but the practical exercise took longer than anticipated. Although students were familiar and competent with the use of Geo-shapes the pipe cleaners were a novelty and more difficult to work with. They were however a successful technique for forming diagonals and heights. The following lesson they completed worksheet 2

Worksheet 2

This worksheet is given in Appendix 5

This followed a similar format to worksheet 1,

1. 10 questions on angles between lines, in a cuboid and a pyramid
2. 8 questions on angles between lines and planes, in a cuboid and a wedge
3. 4 questions on angles between planes, in a cuboid and a wedge

1. Angles between lines

The students were asked to name the angles between given pairs of lines, the frequency of correct responses is given in table 43

TABLE 43

Frequency of correct responses for angles between lines

Number of correct responses / 10	Frequency
10	28
9	13
8	5
7	4
6	3
5	0
4	0
3	0
2	0
1	0
0	0
Not attempted	2
Total	55

The mean number of correct responses = 9.11, which is 91.13 %, this high success rate indicates a familiarity with finding angles between lines in two dimensions.

2..Angles between lines and planes

The students were asked to name the angles between given lines and planes, the frequency of correct responses is given in table 44

TABLE 44

Frequency of correct responses for angles between lines and planes

Number of correct responses / 8	Frequency
8	2
7	3
6	4
5	2
4	6
3	8
2	12
1	6
0	3
Not attempted	9
Total	55

The mean number of correct responses = 3.24, which is 40.49 %, this lower success rate is an indication of the students' unfamiliarity with finding angles associated with planes in three dimensions.

2. Angles between 2 planes

The third part of the worksheet required students to name angles between planes, the frequency of correct responses is given in table 45

TABLE 45

Frequency of correct responses for angles between planes

Number of correct responses / 4	Frequency
4	6
3	8
2	9
1	10
0	9
Not attempted	13
Total	55

The mean number of correct responses is 1.81, which is 45.24 %, this low success rate is further indication of students difficulty with visualising planes.

The students were then asked if any of the angles had been difficult to identify, their responses are shown in table 46

TABLE 46

Frequency of responses to “were any of these difficult to identify”

Response	Frequency
Yes	32
No	10
No response	13

Of the 42 students who responded to this question 32 (76.1 %) replied Yes and 10 students (23.81 %) replied in the negative.

The students were then asked which angles had been difficult to identify, and their responses are shown in table 47

TABLE 47
Frequency of responses to “which ones”

Response	Frequency
2,3	17
2	3
3	3
1B, 3B	1
3A	2
All of them	2

Only one student stated that they found part of question 1 (angles between lines) difficult, whereas 24 responded with some combination of question 3. This supports the lower success rate of correct answers for question 3 but it is interesting that the students recognised their difficulty.

The next question asked the students how they found the angles, and their responses are given in table 48

TABLE 48
Frequency of responses to “ how you found the angles”

Response	Frequency
By shading in and sketching on the angles	3
By sketching on the shapes	2
I pictured them in my head	1

There were only 6 responses, and 5 (83.33%) of these indicate using the diagram to identify the required angle. Only one student stated that she could visualise the angle. This indicates a heavy reliance on the diagram.

The students were then asked if it was easier to find the angles than in the previous exercise before they had built the figures, their responses are shown in table 49

TABLE 49

Frequency of responses to

“is it easier to find the angles than it was before you built the figures?”

Response	Frequency
Yes	16
No	15
No response	23

Of the 31 students who responded to this question, 16 (51.61 %) replied in the affirmative and 15 (48.39 %) in the negative.

Of the 16 who replied that it was easier to identify the angles after building the models only 8 replied to the follow up question “if YES, how did building the figures help?” their replies are listed in table 50

TABLE 50

Frequency of responses to “if YES, how did building the figures help?”

Response	Frequency
You could see the real shapes	4
You could see in three dimensions	2
It gives you a real idea of what each line or angle looks like	1
You could picture it in your head	1

All the responses to the question “how did building the figures help” include some reference to “seeing” the solid. There is therefore some awareness of increased spatial visualisation.

TABLE 51
Responses to “if NO, why not?”

I didn’t really understand how to find the angle
I find that way confusing
I couldn’t see them even though I built them

One student commented on her continuing inability to “see” the angle. The teachers’ commented that some students still needed the model in front of them in order to determine the requisite angle.

In order to draw some conclusions the success rates achieved in worksheet 1 and 2 are listed in table 52

TABLE 52
Success rates for worksheets 1 and 2

Task		Success rate %
Worksheet 1	Section 1	88.70
	Section 2	46.43
	Section 3	58.89
Worksheet 2	Section 1	91.13
	Section 2	40.49
	Section 3	45.24

The success rates from worksheet 1 to worksheet 2 have only increased for section 1 and not sections 2 and 3. This is despite the model building exercises that they had carried out. These success rates are calculated using only the number of students that attempted the task, but the number of students that did so varied from section to section. Using the total number of students completing the particular worksheet yields different results. These are shown in table 53.

TABLE 53

Success rates for worksheets 1 and 2 taking into account the number of students not attempting the section

Task		Success rate %
Worksheet 1	Section 1	84.03
	Section 2	25.09
	Section 3	18.60
Worksheet 2	Section 1	87.82
	Section 2	27.10
	Section 3	13.82

The success rates have increased from worksheet 1 to worksheet 2 for sections 1 and 2 but not for section 3. This lack of improvement is not altogether surprising when the responses to the question “was it easier to find the angles than it was before you built the figures” are also considered. Only 51.61 % responded “Yes”

Interpretation of results

There was an increase in the success rate of correct responses for (i) angles between lines and (ii) angles between lines and planes but not for angles between planes. For this last task it may have been more appropriate to use solid planes for students. This latter task was also the most novel for the students and therefore needed the most appropriate manipulatives and structured phases of instruction.

Teachers comments

Whole programme

- whole unit very positive,
- will use again next year
- was good to do something practical
- students very positive

Worksheet 1

- will use this for revision at end of year
- good activity for girls to find volume (3)
- many girls thought they could do the shading and drawing of cross sections without the geo shapes (1 & 2)
- could be more useful for more junior classes

Worksheet 2

- all classes really enjoyed the activities
- class felt that they could do activities without the blocks (1)
- well received, needed little teacher guidance (2)
- quite long, some students bored at the end
- girls needed the blocks, found the drawing tricky and needed other examples of isometric drawings to guide them

Worksheet 3

- all classes found this hard
- took 4 lessons rather than 2 as planned
- “planes” of cardboard may have helped visualisation, needed solid boxes at times
- use of geo shapes more accurate and stronger than students own constructions
- would have been beneficial to have stickers of letters to place on vertices
- some students still unable to do the trigonometry without the shapes in front of them, but once the materials were given to them they were fine.

Summary

This investigation was designed to evaluate the effectiveness of using manipulatives to improve spatial visualisation. The research was carried out within the normal teaching programme of the classes and this put constraints on the design of the worksheets and also affected the amount of data collected.

It was decided to incorporate the van Hiele levels of thinking into the research as a method of identifying spatial ability levels. However there are no descriptors available in the literature for these skills so it was not possible to design the worksheets specifically to evaluate the levels. This therefore also limited the conclusions that could be drawn from the data.

As a consequence of evaluating students' responses it has been possible to draw up a list of suggested descriptors and these are given in table 54. This investigation could therefore be treated as a pilot study to develop these descriptors to be used in more quantitative research into the use of manipulatives and other aspects of spatial ability.

It is felt that the van Hiele are a useful tool for evaluating learning and teaching and should be incorporated into more mathematical topics. Teachers should be aware of their relevance for use in the classroom.

Aspects of this investigation that were difficult were

- designing the worksheets so that they formed part of the normal teaching programme and provided the data required
- lack of information on using the van Hiele levels for spatial ability, and particularly the absence of descriptors
- low retrieval rate of the worksheets

Aspects of this investigation that were easy were

- dealing with a supportive principal, board and pupils
- using a well resourced and up to date mathematics department
- working with enthusiastic, motivated and professional mathematics teachers

CHAPTER 5 DISCUSSION

The purpose of this investigation was to evaluate whether the use of manipulatives improves the spatial visualisation of three dimensional objects when drawn in two dimensions.

Experience of Classroom Research

The principal and board of trustees were extremely supportive of this research and had allowed me to apply for the study award which enabled me to complete it in one year. My colleagues in the mathematics department that taught the 5th form level were also enthusiastic and supportive.

The teachers involved were experienced and competent with the subject areas being studied and the manipulatives being used. There was no difficulty with the allocation of resources because the three classes were of different ability levels and consequently would be ready to start the topic at different times. The students were also familiar with the manipulatives and on the whole there was a high standard of co-operation and behaviour by them.

However the realities of classroom teaching meant that there were some problems with the administration of the materials. The worksheets were completed as part of the normal curriculum, at the appropriate time within the department's scheme. They were designed to be a learning resource for the students as well as a research tool. The completion of them by the students was not strictly enforced and there were varying rates of completion due to normal variations in work habits as well as absences. This was especially evident in the second worksheet which was designed to cover quite an extensive topic area and therefore took a longer period of time.

The desire to have the worksheets as part of the normal teaching programme meant that if a teacher was absent then a reliever would be left to supervise. This reduced the appropriate completion of one set of worksheets but is a normal part of classroom teaching. The worksheets were meant to be self explanatory but there were parts that required greater clarification by the teachers. This need was commented on at the appropriate parts of the worksheet but in one instance was caused by a misinterpretation of the instructions by the teacher.

The students had had the nature of the research explained to them at the beginning of the year and many of them continued to make supportive comments and express a desire to be co-operative. However there was not an ongoing reminder of the research's purpose as it was felt that the students' desire to be co-operative may influence their responses.

The main difficulty of classroom research was that the classroom was not a controlled environment and this was compounded in this case by the desire to have it as part of the normal learning and teaching programme. Pre and post testing would have enabled greater links with the van Hiele levels to be established.

Usefulness of the van Hiele levels in evaluating students' spatial ability

The van Hiele levels are a reflection of students thinking and not of their achievement, the greatest benefit of using the levels is therefore to understand students' levels of thinking in order to improve their learning. The levels can be used to advantage to determine the most effective teaching programme. Most of the investigations into the levels have used polygons and particularly quadrilaterals for testing and describing the levels.

Although the relevance of these levels is recognised and I believe of benefit to teachers for their understanding of students learning there are some appropriate concerns. These include the belief, supported by many researchers that students can be at different levels for

different topics and presumably even within topics. Progression to a higher level is dependent on appropriate mathematical experience **and** the student's internalisation of concepts within their own knowledge structure. Exposure to appropriate resources and activities does not ensure progress to a higher level of thinking.

The reality of classroom teaching means that it is not feasible for teachers to assess at which van Hiele level each of their students is operating for each topic within the curriculum. The value of the levels to classroom teachers is to make them aware that such levels do exist and to have a global expectation of the students' levels within their classes. This awareness should prepare them better for misinterpretations and misunderstandings that do occur. The other benefit is that the teacher should become more conscious of the appropriateness of mathematical language used in the classroom, both by themselves and by their students.

Despite the awareness of the usefulness of the van Hiele levels for teaching geometry there were difficulties in applying them in this investigation. This is due to the fact that there are no descriptors available for spatial visualisation. The worksheets were written without descriptors and it was therefore difficult to assign students to a particular level as a consequence of their responses.

However as the result of interpreting students' responses a table of descriptors was developed and used to assign students to particular levels. The verbal responses were used in preference to quantitative analysis of specific questions because these had not been designed with descriptors in mind. The majority of the students in this study of fifth formers were found to be at level 1 or 2.

In order for the van Hiele levels to be used to interpret any improvement in spatial visualisation the testing method used must be designed so that the responses can be analysed using appropriate descriptors.

The van Hiele levels should be used by teachers to help ascertain at which spatial visualisation level their pupils are operating. This would assist them in selecting the most appropriate resources and teaching strategies. In order for this to occur there needs to be descriptors for each level and a set of sample problems.

Potential descriptors

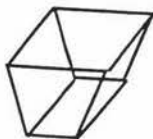
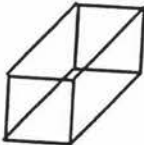
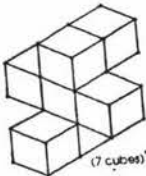
Most of the reviews of the van Hiele levels use polygons for the descriptors and sample problems. Descriptors for verbal and drawing skills for quadrilaterals have been used where appropriate in the results section. Descriptors for spatial visualisation were not available for this investigation but have been developed as a consequence of analysing students' responses. A table of suggested descriptors is given below

Table 54
Basic skills in spatial visualisation

Level	Skill
1. Recognition	Recognizes views of the same three dimensional object
2. Analysis	Identifies a view as representative of a three dimensional object
3. Ordering	Can sort figures into different types according to their three dimensional Properties
4. Deduction	Can deduce another property of a three dimensional object from a diagram of it.
5. Rigor	Recognizes unjustified assumptions made by using information from three dimensional figures

In order to use the van Hiele levels to evaluate students’ spatial visualisation it would also be advantageous to have a series of appropriate problems. These problems need to be designed for each aspect of spatial visualisation, a selection of ones suitable for cross sections are listed below in table 55.

Table 55
Sample problems for spatial visualisation

Level	Sample problem
1	What shape is the cross section of this solid? 
2	How many different cross sections that could be used for evaluating volume does this cuboid have? 
3	Can you find the cross section of a tetrahedron that has the shape of a rectangle?
4	How many axes of symmetry does this solid have? 
5	Most secondary school students do not operate at this level so a sample problem is not appropriate.

It would be possible to design a set of problems to enable teachers to evaluate at which level of spatial visualisation their students are operating. These could be included with problems to evaluate other skill areas to provide a diagnostic assessment for each student. There is not enough classroom time for teachers to administer an assessment for each new topic in mathematics.

The sample problem given for level 1 in table 55 could be changed to a different shaped cross section or more than one example given. The results of worksheet 1 of topic 1 of this investigation indicate that most students are capable of successfully answering this type of problem. This indicates that they have mastered the concepts required at level 1

The sample problem for level 2 in table 55 requires students to differentiate between different cross sections of the same solid. This specific problem was not used in topic 1 but the matching exercise of worksheet 2 is equivalent to it. The results of worksheet 2 showed that students who attempted to match all solids with a curved cross section may not be operating at level 2 for this skill. 55.74 % of students also incorrectly labelled the cross section of the cylinder indicating further difficulty with curved cross sections.

There was not an equivalent question for level 3 used in the cross section topic of this worksheet. The matching of solids exercise in the review section required level 3 thinking for four of the examples. These four solids caused considerable difficulty for the students and 89.19 % were unable to match them correctly.

There were no problems used for levels 4 or 5 in this investigation because these skills are not required by the school certificate prescription and the research was intended to be as content specific as possible. Level 4 thinking for cross sections is required for finding volumes of rotation in the Bursary calculus prescription. It would be beneficial to investigate how many of those students are operating at that level.

Phases of instruction

The other component of the van Hiele theory is the “ phases of instruction”, these have not been investigated comprehensively and yet they have important implications for classroom teachers. These phases are (Clements and Battista, 1992, p 431)

1. Information
2. Guided orientation
3. Explication
4. Free orientation
5. Integration

This investigation suggests that the teacher’s role for each phase could be described as:

1. To discuss appropriate materials, and learn how students interpret the language used.
2. To direct student’s activity and to provide structured, sequential tasks.
3. To lead discussion of geometrical concepts in their own language and then introduce appropriate mathematical terminology.
4. To provide appropriate materials and problems that allow students to reflect on the problems and the processes they have used.
5. To encourage students to reflect on and consolidate their geometric knowledge.

At the completion of this final phase students should have attained a higher thought level.

Although these phases list teacher and student roles, some of the teacher’s roles have been fulfilled in this investigation by the structured nature of the worksheets. It would be appropriate to use manipulatives during phase 1, 2 and 4. A suggested sequence for teaching cross sections could be

1. provide suitable objects and discuss the concept of cross section. The definitions of the polygons involved would be reviewed.
2. provide materials that allow students to build objects with required cross sections

3. allow students to examine the objects they have constructed and develop their own definition of cross section
4. provide other materials so that different types of cross sections can be investigated. These could include the different results obtained by making oblique cuts in an ice cream cone, or requiring the students to design the net for a house that could be built on a slope.
5. allow students to reflect on the different types of cross sections that are possible in three dimensions.

These three phases are often used by educators but the reflective phases 3 and 5 are neglected

Worksheet content and format

The three school certificate mathematics classes are streamed according to ability and the some of the comments made by the teachers reflect the differences in their class' ability. Numbers in brackets following comments indicate the relevant class level (1 being highest)

Whole programme

- whole unit very positive,
- will use again next year
- was good to do something practical
- students very positive

The format of the investigation was designed to be used within the normal teaching programme and to be an appropriate resource. It was encouraging that the teachers felt that the worksheets had fulfilled their purpose and also that the students had maintained a positive attitude towards the worksheets.

Worksheet 1

- will use this for revision at end of year
- good activity for girls to find volume (3)
- many girls thought they could do the shading and drawing of cross sections without the geo shapes (1 & 2)
- could be more useful for more junior classes

This topic (volume) incorporated revision of earlier work but this worksheet had been designed to assist in the identification of cross sections for calculating the volumes of prisms. The more able students and their teachers felt that they did not need this activity at this level. Students had greater difficulty identifying the prism with a trapezoidal cross section and this may have been due to less use of these by teachers in examples or because it was not one of the prisms they built during the practical session.

Worksheet 2

- all classes really enjoyed the activities
- class felt that they could do activities without the blocks (1)
- well received, needed little teacher guidance (2)
- quite long, some students bored at the end
- girls needed the blocks, found the drawing tricky and needed other examples of isometric drawings to guide them

There is a discrepancy again between the response of different class levels. The more able students felt that they did not need to use the blocks provided for the activities. The less able students continued to experience difficulty with visualising the objects and needed the blocks. The comment regarding the length of the unit is of interest because it relates to the need to investigate the optimum length of time to work with manipulatives. This worksheet was split into sections requiring different skills and techniques and yet some students found it too long. This is despite other comments which described the activities as enjoyable and worthwhile.

Worksheet 3

- all classes found this hard
- took 4 lessons rather than 2 as planned
- “planes” of cardboard may have helped visualisation, needed solid boxes at times
- use of geo shapes more accurate and stronger than students own constructions
- would have been beneficial to have stickers of letters to place on vertices
- some students still unable to do the trigonometry without the shapes in front of them, but once the materials were given to them they were fine.

These activities were designed to reinforce totally new concepts at this level, angles in three dimensions. The comments reflect this, the students found these worksheets difficult. Agreement with the use of geo shapes rather than students own constructions reinforces the belief that these are more appropriate at this level. The use of solid planes to study angles between lanes is probably more appropriate than the “hollow” solids formed from geo shapes.

Teachers also commented on students’ inability to visualise the three dimensional shapes without the models in front of them, these students still had difficulty interpreting the diagrams despite practice with the models.

Effect of using manipulatives

This study was specifically designed to investigate the use of manipulatives and whether this had any effect on students’ spatial visualisation. This was not carried out using pre and post tests but relied on the students’ responses to questions about whether using the manipulatives had been of benefit in completing tasks requiring spatial visualisation. .

The results obtained are inconclusive, there was insufficient evidence of improvement in tasks in topics 1 or 2. Students did not report that tasks were easier after using manipulatives and results were affected by the issue of students attempting to match all solids into pairs. Of the three tasks studied in topic 3 there was a higher success rate for identifying angles between lines and between lines and planes, but not for angles between planes.

This lack of improvement for angles between planes could be due to the lower suitability of the manipulatives used for visualising these angles. It may be more appropriate to use solid planes rather than the hollow geo shapes. This highlights the importance of ensuring that the manipulatives used are the most appropriate for the specific topic being taught.

The efficacy of manipulatives could be improved by greater integration of them into the five phases of instruction proposed by van Hiele. This then requires the particular materials used to be carefully chosen so that they can fulfil the needs of both the guided orientation and free orientation phases.

The effectiveness of the manipulatives was assessed by the students' responses to questions on whether they found tasks easier after using them. Most students said they did not find them beneficial and yet they described better visualisation after their use. When the assessment of spatial visualisation is carried out by the use of diagrams it may well be diagram interpretation skills that are being assessed rather than spatial visualisation.

The improvement in students' spatial ability in the first two tasks of worksheet 3 suggests that the use of manipulatives may have the greatest benefit for concepts that are being introduced for the first time.

Teachers' comments on the use of manipulatives also suggest that they may also be more beneficial for students of lower mathematical ability. The spatial ability of these students may improve and may also enhance their achievement in other areas of mathematics.

This investigation did not show conclusively whether the use of manipulatives improves spatial visualisation when interpreting two dimensional diagrams but it has shown the importance of using appropriate resources and their incorporation into a suitably structured teaching programme.

Suggestions for further research;

This investigation has suggested other research questions for future study, these questions include

- what is the optimum length of time that should be spent working with manipulatives, and how should this time be shared between guided and free orientation.
- whether any improvement in spatial visualisation is specific to the materials used or whether it is transferable to other resources.
- whether specific training in diagram interpretation is more appropriate for interpreting representations of three dimensional objects than using manipulatives.
- whether there is a correlation between the pupils achievement in spatial visualisation questions in examinations and their overall achievement.

CHAPTER 6 CONCLUSION

Summary

This investigation set out to determine whether the use of manipulatives improves the spatial visualisation of students when interpreting two dimensional diagrams.

The new New Zealand mathematics curriculum places a heavy emphasis on practical activities and the use of manipulatives. The use of these resources can cause difficulties for classroom teachers in terms of their cost, time, distribution and collection. Teachers and others involved in mathematics education must be confident that these resources are efficacious. The particular type of resources used and their incorporation into the learning programme must be carefully monitored.

The research was carried out as part of a normal teaching programme and the advantage of this approach was that the resources developed had to be closely linked to the curriculum. The consequent disadvantage was that the administration, completion and collection of completed worksheets was not always strictly enforced.

Theoretical background

The literature reviewed for this study had linked the spatial ability of students and their mathematical ability and this provides a strong justification for developing ways of improving spatial ability. Many reviews also suggested a link between gender and spatial ability with females showing a lower ability in some studies. This implication also provides a strong motivation to investigate methods to improve the spatial ability of all students.

The importance of spatial ability is recognised for its use in other areas of mathematics, especially problem solving and a few studies have been carried out in the use of training to improve it. There are different definitions of spatial ability, and of the skills which it incorporates. Spatial visualisation is considered by many to be one these skills. The definition of spatial visualisation used by this investigation is that of McGee (1979, p 893)

“the ability to mentally manipulate, rotate, twist or invert a pictorially presented object”.

Not all of the tasks in these worksheets necessitated the use of spatial visualisation, those that did included

- Matching of prisms in topic 1
- Matching and drawing of solids in topic 2
- Drawing other views of solids in topic 2
- Various questions in topic 3

The other spatial ability is spatial orientation, this is defined by McGee (1979, p 893) as

“the comprehension of the arrangement of elements within a visual stimulus pattern and the aptitude to remain unconfused by the changing orientations in which a spatial configuration may be presented”.

The “smoothing” of solids section in topic 2 was a task that required spatial orientation.

The difficulty that students experienced with matching solids that had been rotated is an indication of poor spatial visualisation ability.

A theory of learning that has been the subject of a great deal of study by mathematics educators is the van Hiele levels theory. These levels are of benefit to teachers because they provide a framework on which to interpret students thinking and language. They do not provide a tool for assessing achievement but tests have been devised to assess the levels at which students are operating.

The important implications of the theory are that it is sequential and hierarchical and an awareness of these aspects can assist teachers in developing their teaching programmes. Students are unable to progress to a higher level until the concepts of a lower level have been internalised.

An integral part of this investigation was to utilise the van Hiele levels to interpret students thinking. Questions requiring students to reply in their own words were incorporated into the worksheets in order to analyse their responses in terms of verbal skills using Hoffer's (1981) table of skills. It was not possible to design other parts of the worksheets in order to evaluate their spatial visualisation ability because the literature does not have descriptors for this skill.

As a consequence of the experience gained by interpreting the responses of students a list of such descriptors has been drawn up and are shown in table 53 in the discussion chapter. These descriptors and sample problems could be used for more quantitative evaluation of students' spatial visualisation as well as by teachers in their teaching programmes.

Educational implications

In order for these descriptors to be used by teachers there needs to a series of sample problems developed for each area of the curriculum for which spatial visualisation is an essential skill. These problems do not necessarily have to be used in a formal assessment but teachers could use them in an informal format to interpret their students level of thinking.

An acknowledgement of the thinking and language used by students at each level can allow teachers to evaluate their progress in the classroom setting without having to wait for formal assessments. These formal assessments may not provide appropriate information on the students' levels of achievement unless they are very carefully designed. These assessments are frequently of the pen and paper format and therefore rely on the

interpretation of diagrams rather than on spatial ability. An awareness of the van Hiele levels when conducting assessments could be beneficial in utilising alternative forms of assessment in order to evaluate the understanding of students.

If in fact diagrams continue to be used to assess the understanding of three dimensional concepts then it may be beneficial to increase the time allocated to the interpretation of the diagrams and particularly the conventions of their use. The different formats used to represent three dimensional objects in worksheet 2 were less familiar to the students and caused them the most difficulty. They had had less practice at drawing the solids represented by these alternative representations.

Another component of the van Hiele theory with important implications for teaching mathematics is their “phases of instruction” which propose a sequence of interaction between the teacher and pupil. If these are combined with a teacher’s appreciation of their students’ levels then a teaching programme can be more successful. Appropriate resources including manipulatives can also be utilised more efficiently.

This investigation is not able to conclude that manipulatives do improve spatial visualisation of the two dimensional representation of three dimensional objects. However it does highlight some important factors for teaching this ability. The van Hiele levels can be used to develop a more appropriate teaching programme and the manipulatives used must be carefully chosen and administered.

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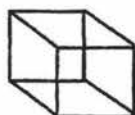
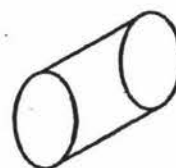
APPENDIX 1

NAME _____

CROSS-SECTIONS

To find the volume of a right prism we need to identify the cross-section that remains the same through the solid.

For each right prism below shade in the cross-section and name its shape.



Using the Geo shapes provided, construct right prisms with the following cross-sections

- (a) triangle
- (b) square
- (c) hexagon
- (d) pentagon

On the back of this sheet, sketch each of your right prisms, shading in the required cross-section and in your own words describe what cross-section means.

APPENDIX 2

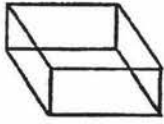
NAME _____

CROSS-SECTIONS(CONT)

Below each right prism labelled A - F write the number of any numbered prism below that has the same cross-section



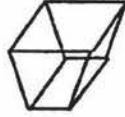
A.



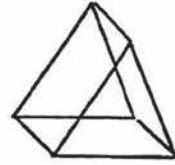
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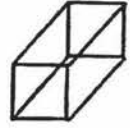
C.



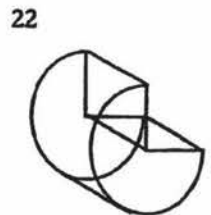
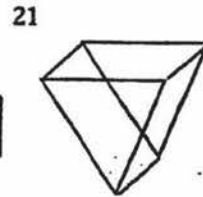
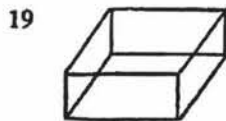
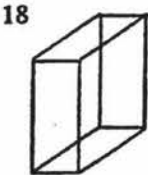
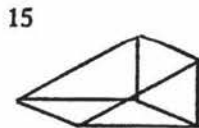
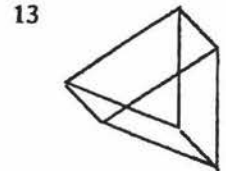
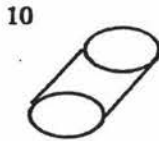
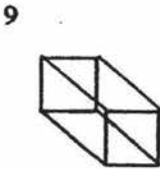
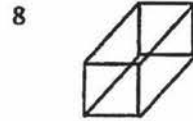
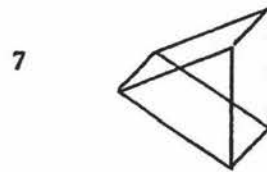
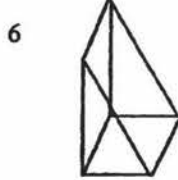
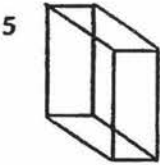
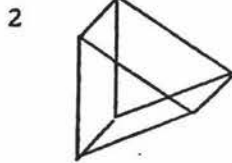
D.



E.



F.



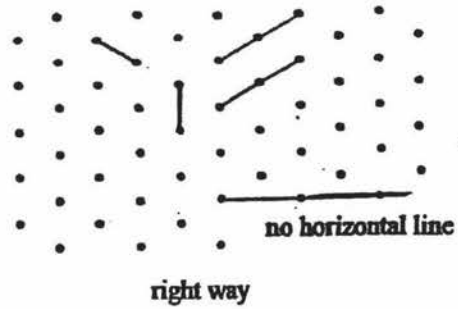
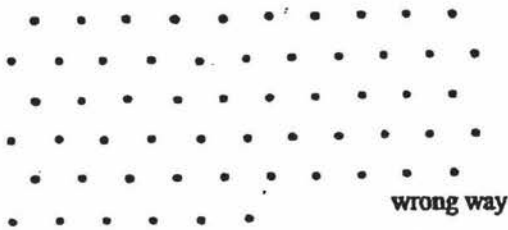
Which were the hardest to match?

APPENDIX 3

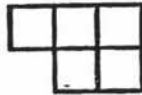
DRAWING AND VISUALISING 3-D SOLIDS

Isometric paper is often used to draw solids in three dimensions, there are some rules for using it,

1. All corners are on dots
2. All lines join adjacent dots
3. No horizontal lines are drawn
4. Make sure your isometric paper is round the right way

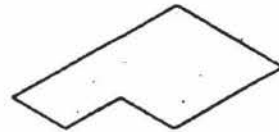
**1. Drawing Solids made of 5 Cubes**

1. make this solid using 5 cubes

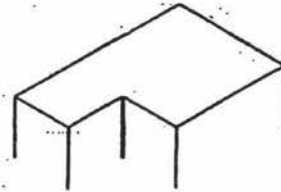


top view

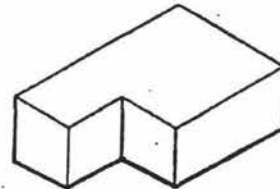
2. draw the top surface looked at from the side



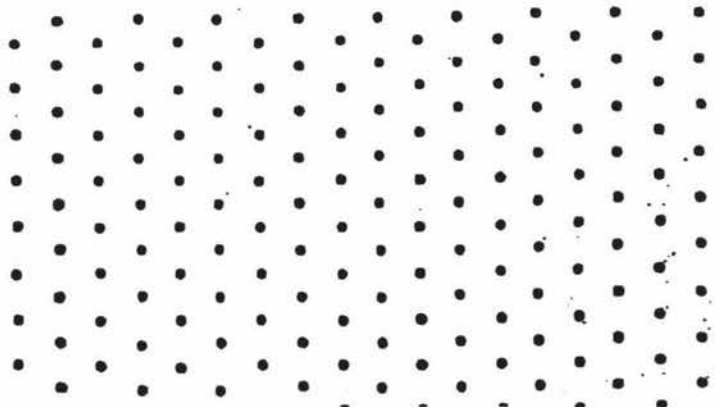
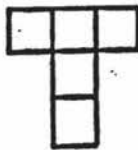
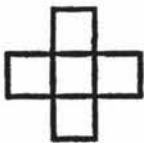
3. draw a line down each front vertical side



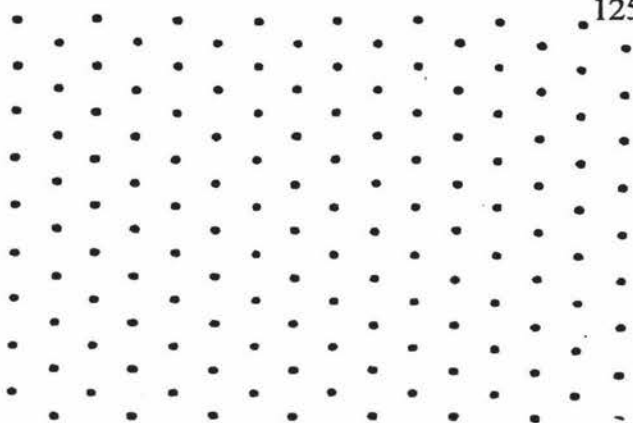
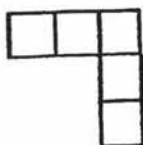
4. draw the lines along the bottom of the solid



5. draw these solids using the same method

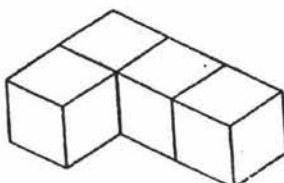


6. draw this 5 cube solid,
without building it first

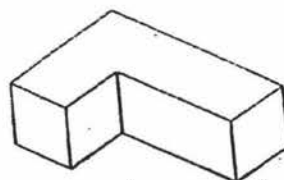


2. Smoothing out Lines

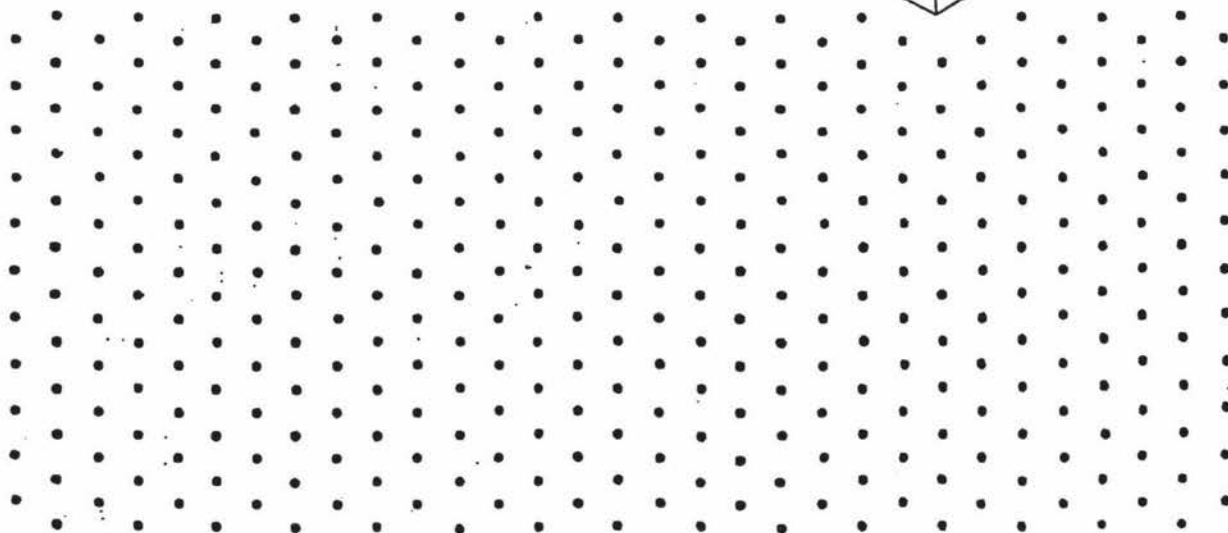
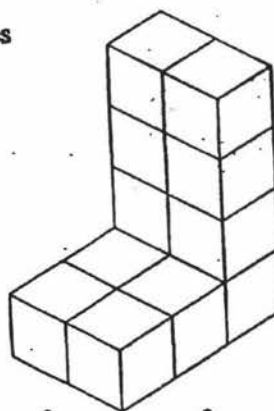
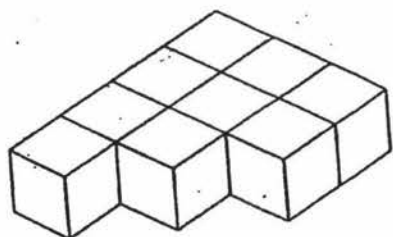
Four cubes can be joined together like this



If the solid is then smoothed and painted,
the lines between the cubes disappear, and
the solid looks like this

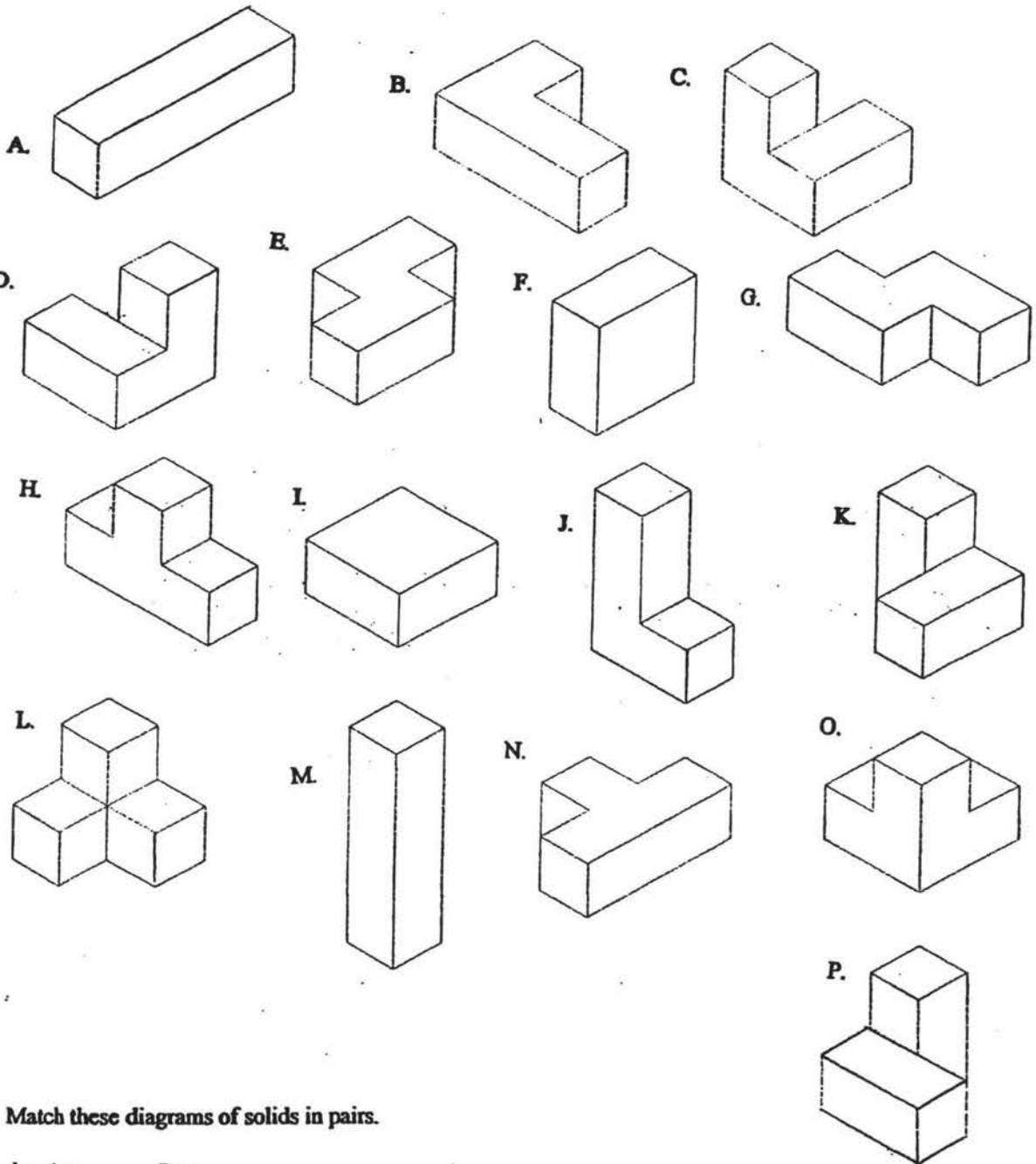


Draw isometric diagrams of these solids, leaving out the "extra" lines



3. Matching 4 - Cube Solids

Each of these "smoothed" solids is made out of 4 cubes



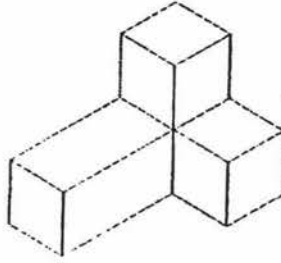
Match these diagrams of solids in pairs.

i.e. A = , B = ,

Were any solids particularly difficult to match, and if so which one(s).....

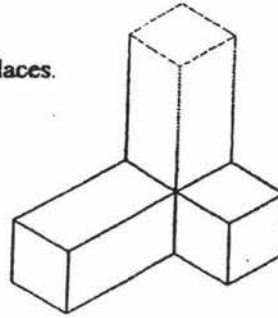
4. Adding Cubes to Solids

Build this solid using 5 cubes



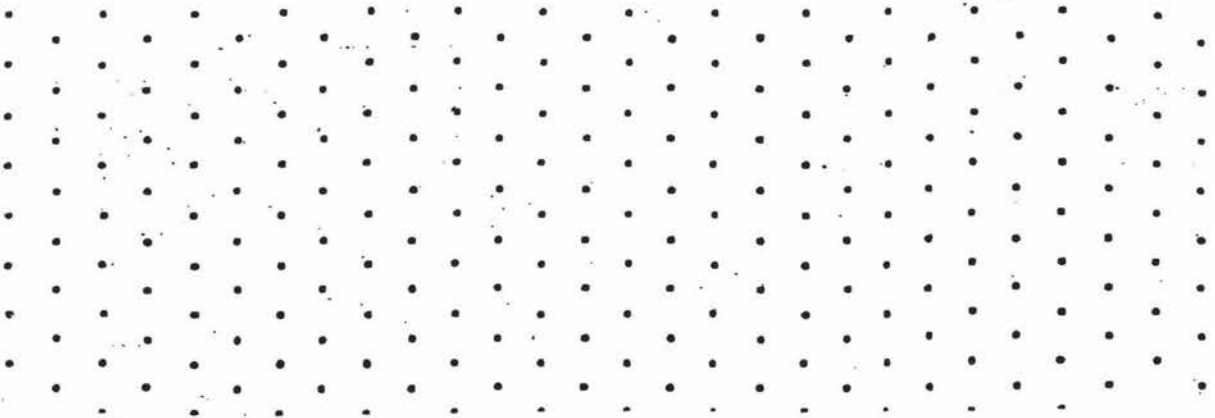
An extra cube can be added on top of this solid in 4 different places.

One way of doing this gives this solid.

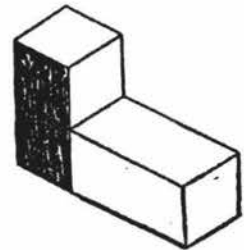
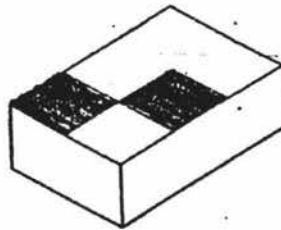
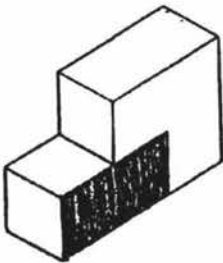


Build the 3 different solids you would get by adding a cube in the other 3 possible positions.

Draw isometric sketches of the 3 solids you have built.

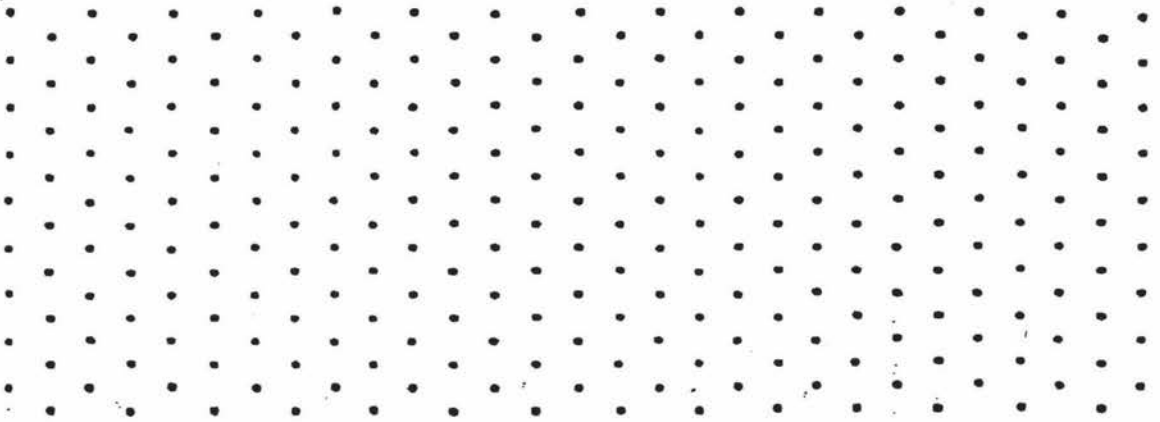


Build these solids,

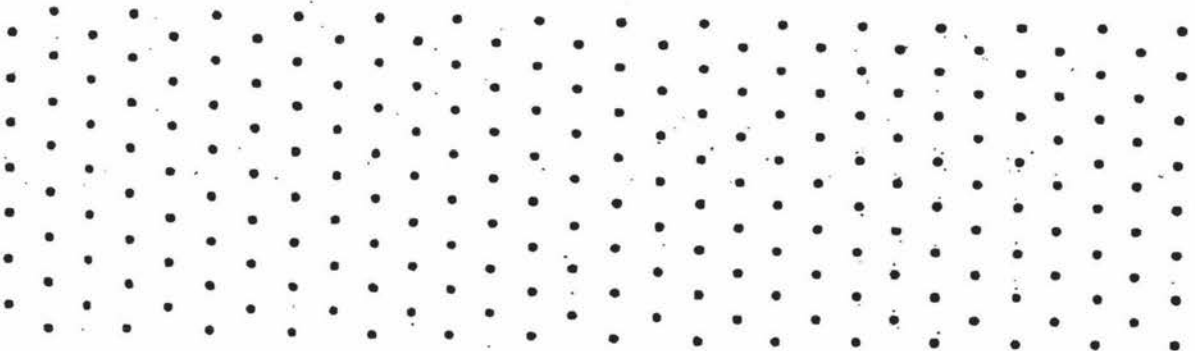
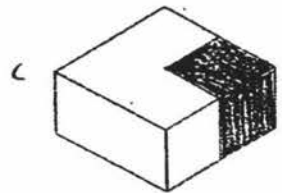
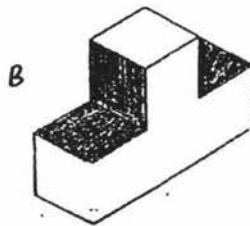
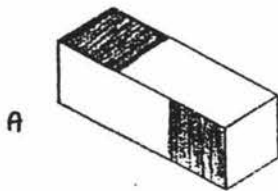


and then add 2 more cubes to each solid at the shaded face

Draw the solids that you have built,



Without building them, draw the solids that would be built if 2 cubes were added to the shaded faces of these solids



Was it harder to draw them without building them first?.....

Were any of these solids particularly difficult to draw?

and if so, which one?.....

why?.....

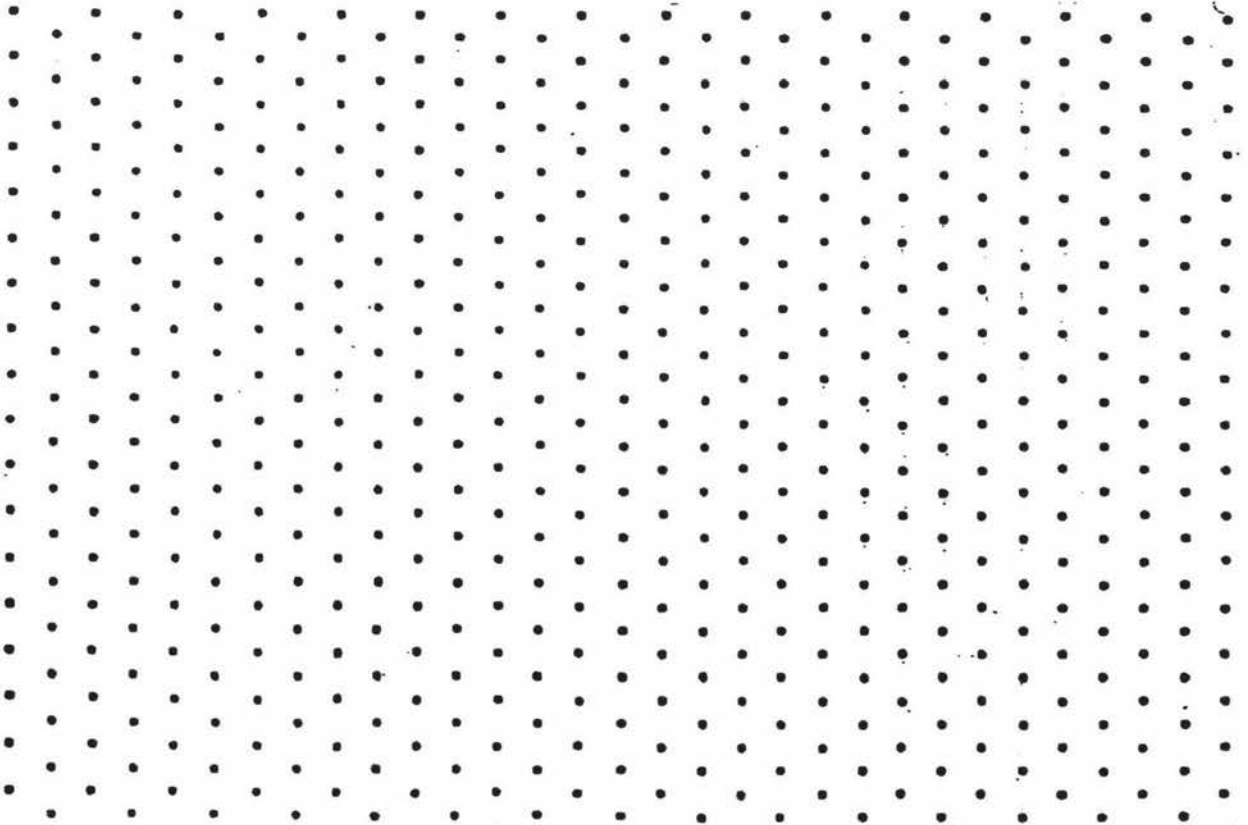
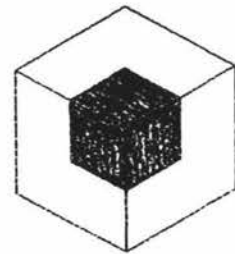
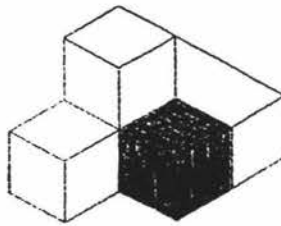
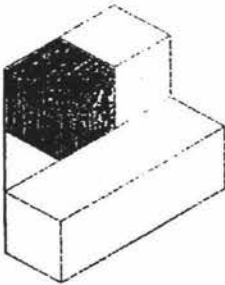
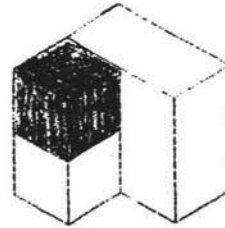
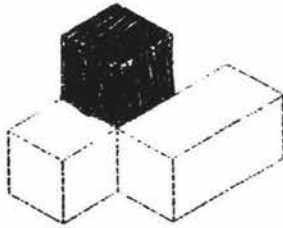
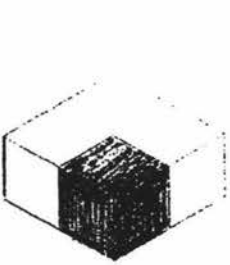
How would you explain how you drew these added onto solids.....

.....

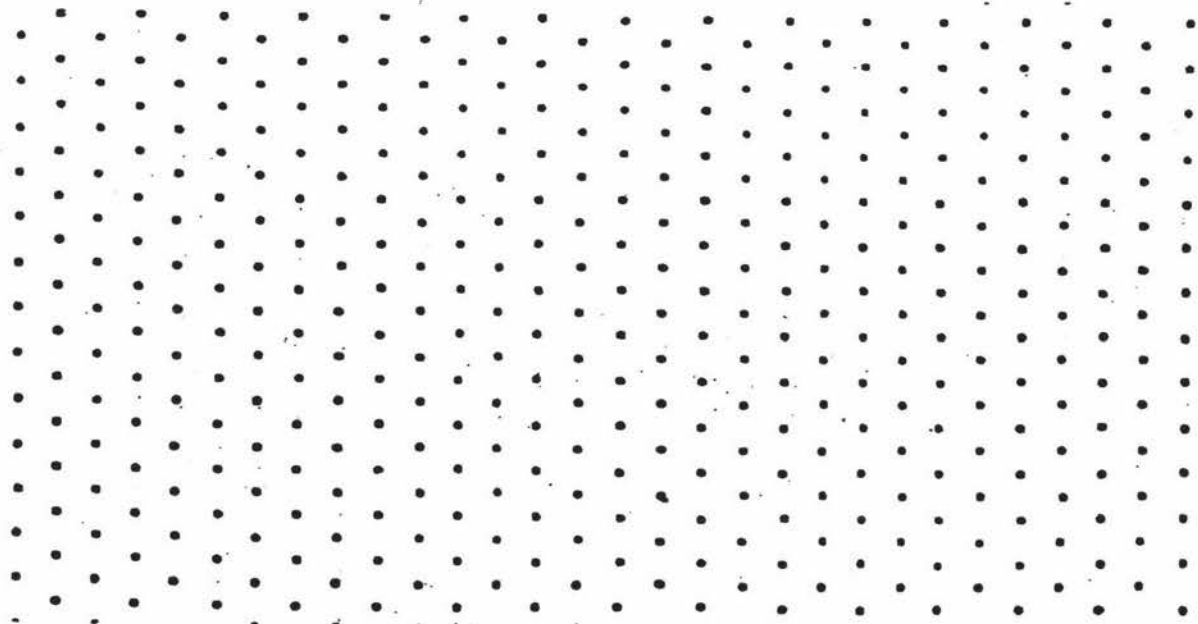
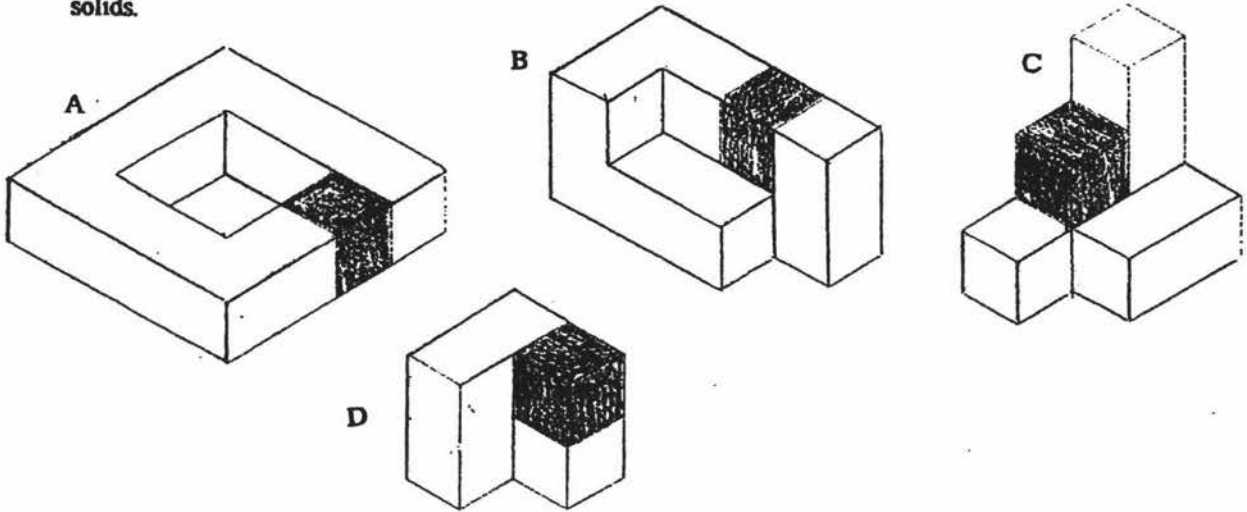
.....

5. Subtracting Cubes from Solids

Build these solids, remove the shaded cube from each and draw the solids that remain



Without building them, draw the solid that remains after removing the shaded cube from each of these solids.



Was it harder to draw them without building them first?.....

Were any of these solids particularly difficult to draw?

and if so, which one?.....

why?.....

How would you explain how you drew these new solids.....

.....

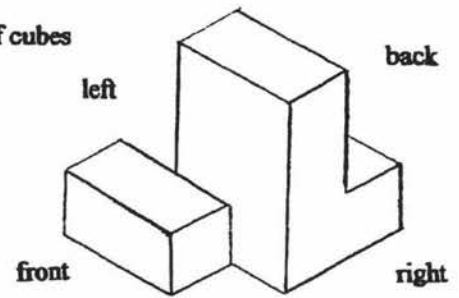
.....

6. Other Ways of Drawing Solids

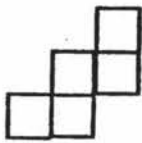
There are other ways of drawing solids

1. Its "birds eye" view, with the numbers representing the height of cubes
2. Its front, right, back and left views

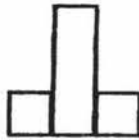
This solid



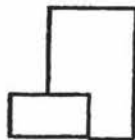
would be drawn as



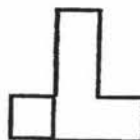
birds eye



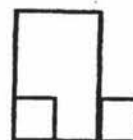
left



front

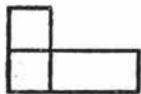


right



back

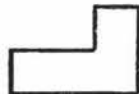
1. Build the solids represented by these views
2. Draw the "birds eye" view
3. Draw the solid on the isometric paper



front



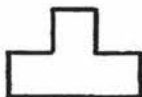
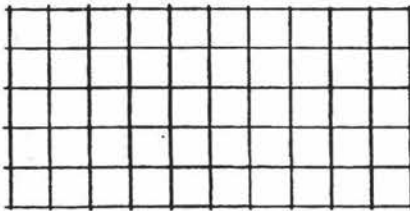
left



back



right



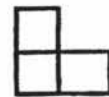
front



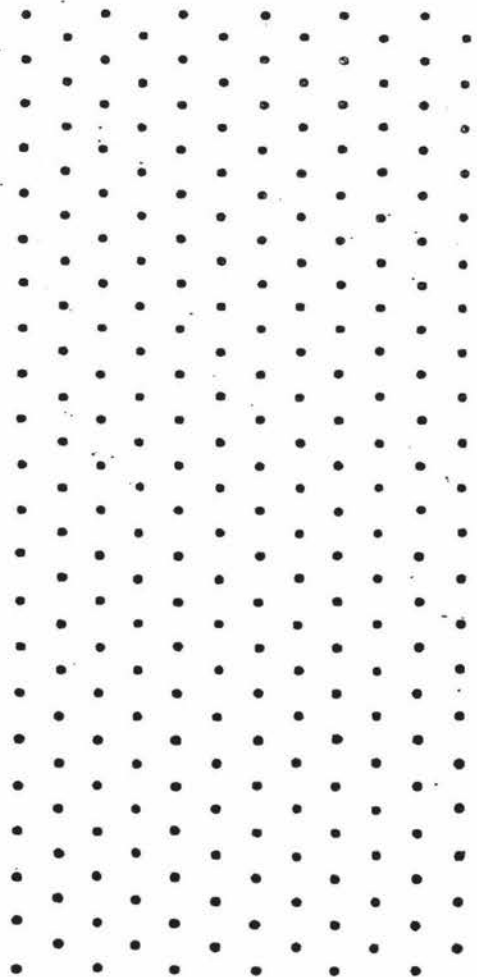
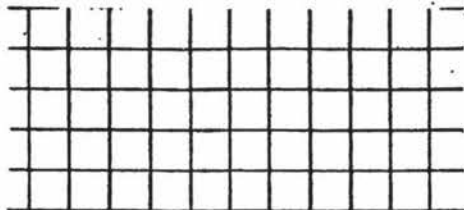
left



back

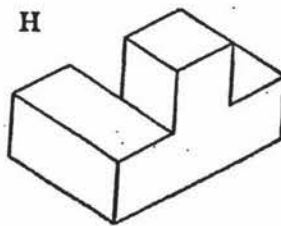
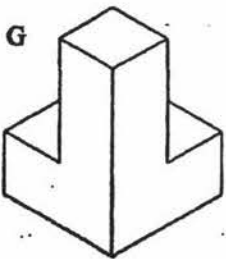
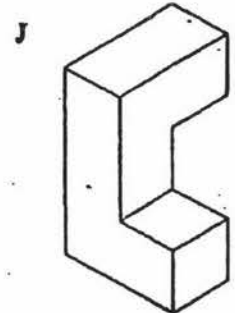
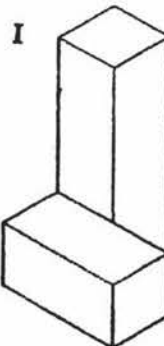
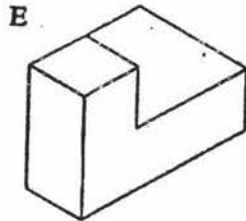
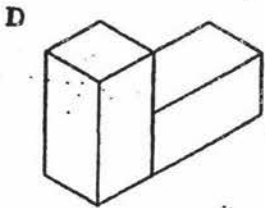
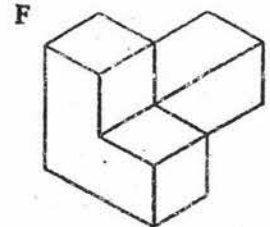
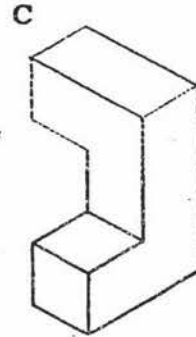
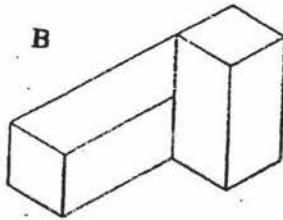
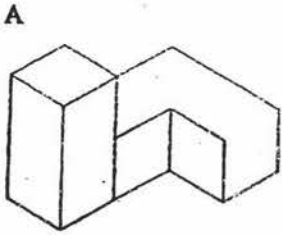
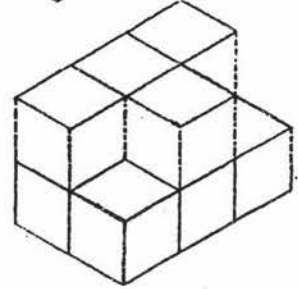
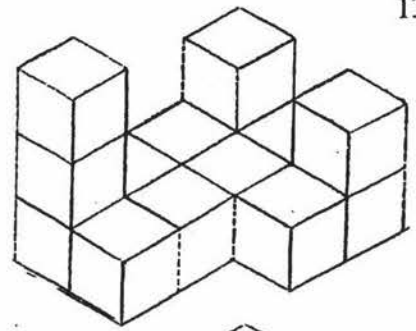
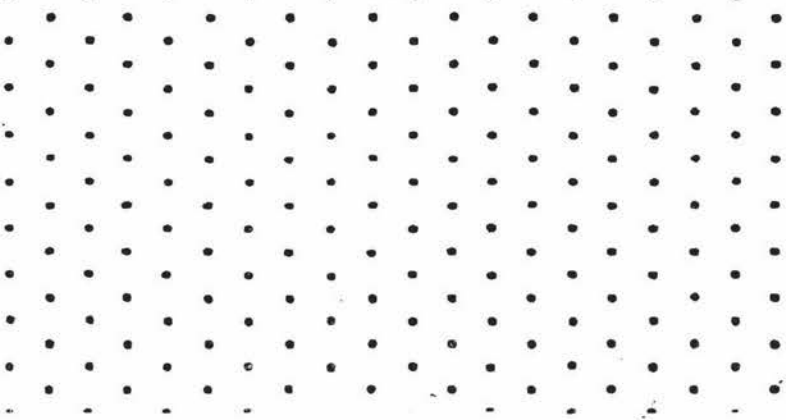


right



Review

Draw these solids on isometric paper, smoothing out the lines



Match these diagrams of solids in pairs.

i.e. A = , B = ,

Were any solids particularly difficult to match, and if so which one(s).....

APPENDIX 4

Name.....

MEASUREMENTS IN 3 DIMENSIONS

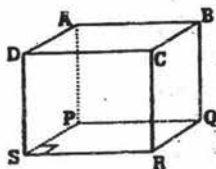
In order to make calculations of 3 dimensional objects it is necessary to be able to identify

1. angles between lines and lines
2. angles between lines and planes
3. angles between two planes
4. line required

1. Angles between lines and lines

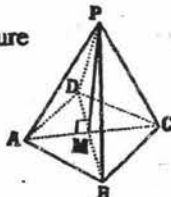
A. Name the angle between the following pairs of lines in this figure

- (a) DC and RC.....
- (b) CR and QR.....
- (c) PR and PQ.....
- (d) AR and PR.....
- (e) SP and RP.....
- (f) AR and QR.....



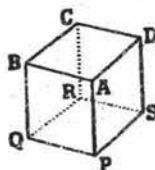
B. Name the angle between the following pairs of lines in this figure

- (a) AD and AC.....
- (b) PB and PC.....
- (c) AC and PC.....
- (d) AP and AD.....

**2. Angles between lines and planes**

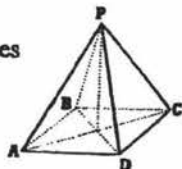
A. Name the angle between the following lines and planes

- (a) line BS and plane CDSR.....
- (b) line CP and plane PQRS.....
- (c) line QD and plane ABCD.....
- (d) line AR and plane ADSF.....
- (e) line AQ and plane BCRQ.....



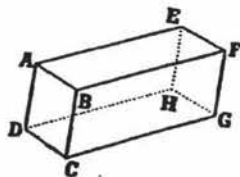
B. Name the angle between the following lines and planes

- (a) line PA and plane ABCD.....
- (b) line PH and plane ABCD.....

**3. Angles between two planes**

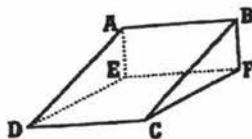
A. Name the angle between the following pairs of planes

- (a) plane ABFE and plane ABCD.....
- (b) plane ABGH and plane DCGH.....
- (c) plane BCHE and plane ADHE.....



B. Name the angle between the following pairs of planes

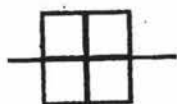
- (a) plane ABCD and plane ABFE.....
- (b) plane ABCD and plane EFCD.....
- (c) plane AED and plane EFCD.....



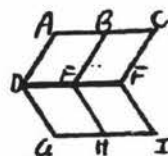
Were any of these difficult to identify?..... If YES, which one(s).....
How would you explain how you identified the angles?.....

Using the Geo-shapes or shapes made from flexi-straws make the following figures

1. Join 4 squares,



bend them along the middle to form the following

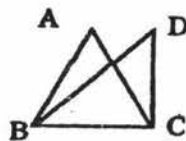
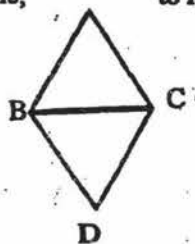


The angle between the 2 rectangular planes is the angle between the sides that have been bent
 $\angle ADG = \angle BEH = \angle CFI$

2. Join 2 triangles, bend them along the middle,

A

to form the following



To find the angle between the sides it is necessary to construct a line at right angles to the join. Take a length of pipe cleaner and attach at A wrap around bend at right angles and then attach to D.

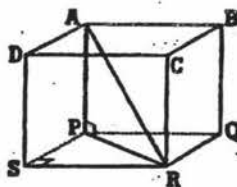


The angle between the pieces pipe cleaner is the angle between the planes ABC and DBC

3. Make a cube from 6 squares

Using a length of pipe cleaner form a diagonal from A to R and then to P.

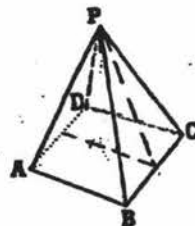
The angle between the pieces of pipe cleaner is the angle between the line AR and the plane PQRS



4. Make a pyramid using 4 triangles and a square base

Attach a length of pipe cleaner to P then to mid point of BC and then to mid point of AD.

The angle between the pieces of pipe cleaner is the angle between plane PBC and plane ABCD

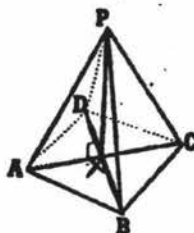


5. Using the same pyramid remove the pipe cleaner.

Take smaller pieces and use them to form the diagonals of the square base.

Attach another piece to the intersection of the diagonals X, attach the other end to P.

This is the height of the pyramid. The angle between PA and the base is the angle between PA and diagonal AC.

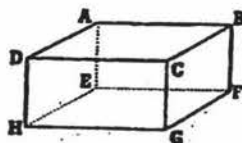


APPENDIX 5

MEASUREMENTS IN 3 DIMENSIONS (CONT)**1. Angles between lines and lines**

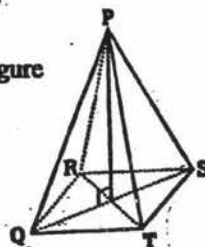
A. Name the angle between the following pairs of lines in this figure

- (a) HD and CD.....
 (b) EF and GF.....
 (c) HA and AE.....
 (d) HB and FB.....
 (e) GE and HE.....
 (f) DF and HF.....



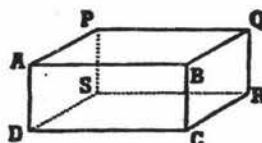
B. Name the angle between the following pairs of lines in this figure

- (a) ST and QS.....
 (b) TP and QT.....
 (c) PR and RT.....
 (d) QP and TP.....

**2. Angles between lines and planes**

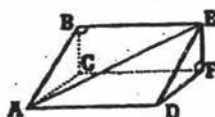
A. Name the angle between the following lines and planes

- (a) BR and SRCD.....
 (b) AR and SRCD.....
 (c) QD and APQB.....
 (d) PC and ABCD.....



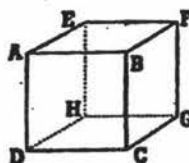
B. Name the angle between the following lines and planes

- (a) AB and plane BCFE.....
 (b) BC and plane ACFD.....
 (c) AE and plane BCFE.....
 (d) AE and plane ACFD.....

**3. Angles between two planes**

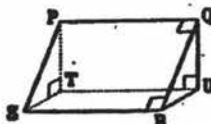
A. Name the angle between the following pairs of planes

- (a) plane ABFE and plane DBFH.....
 (b) plane ABCD and plane EBCH.....



B. Name the angle between the following pairs of planes

- (a) plane PQRS and plane STUR.....
 (b) plane PQRS and plane PQUT.....



Were any of these difficult to identify?.....If YES, which ones.....

How would you explain how you found the angles?.....

Is it easier to find the angles than it was before you built the figures YES/ NO

If YES, how did building the figures help?.....

If NO why not?.....