Comparison of Oxygen Transfer in Three Types of Gas Exchangers

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Abstract

The avian respiratory system is very different from that of mammals. In birds, air flows through the gas exchange area in one direction while in mammals it flows in two directions - in and out of the gas exchange area. It has been hypothesised that gas exchange occurs more efficiently in avian lungs than in mammalian lungs due to the difference in structure. We test this hypothesis by comparing oxygen exchange in three types of gas exchangers. First, we examine how gas exchange occurs in mammalian lungs, using a well-mixed stationary container of air with blood flowing along one axis. Next, we investigate the case where air flows in the opposite direction to the blood, which is similar to the gas exchange mechanism seen in fish. We then analyse how gas exchange occurs in the avian respiratory system, where the blood flow is perpendicular to the airflow. We compare the results under normal and extreme conditions and conclude that the avian respiratory system exhibits significantly higher gas exchange efficiency compared to mammals, ultimately enabling birds to live in environments where mammals could not survive.

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Chapter 1

Introduction

Mammals and birds are both warm blooded vertebrates, they move quickly from low to high metabolic rate and they have a similar metabolic scope (number of times by which the resting metabolic rate increases) [11]. However, birds have developed a very different lung structure than mammals. In mammalian lungs, gas exchange happens in tiny air sacs called alveoli at the end of the airways. In avian lungs, air capillaries and blood capillaries form a grid shape, and the airflow can be considered as being perpendicular to the blood flow [5]. Another key structural difference between mammalian and avian lungs is the shape of the airways. Unlike the tree-shaped airways in mammalian lungs, birds have air sacs that direct the airflow in one direction through the gas exchange area during both inspiratrion and expiration [11].

It is widely accepted that birds have the most complex lung structure among all known living vertebranes, and it is considered the main reason that birds can adapt to hypoxic environment (i.e. an environment with less oxygen) more easily than humans [11] [14]. The hypoxic envrionment is commonly seen at high altitude, where mammals can hardly survive. In such environment, birds can still take enough oxygen to meet the huge demand of flying, and thus it is hypothesised that their lungs are more efficent than those of mammals.

The anatomical structure of avian lungs has been observed in early research [2] [5] [16] and has been conclusively established in the past decades through various ways including radiographic methods[8] and physiological measurements[17][22]. Duck is the most common bird used in previous experiments, the others are pigeon and goose. Although bird species are highly diverse, several studies show that most birds share common features in structure [6] [10][20]. Therefore the species difference is omitted in this thesis.

Computational models of gas exchange in mammals [1] [18] [9], fish [12], and birds [19][21] [4] have been developed. It is widely accepted that oxygen moves from the lungs to the blood capillaries through diffusion across the surface of the alveoli (in the case of mammals), gills (in the case of fish) or airways (in the case of birds). However, not all models take hemoglobin into account in the fish gill and avian lung models, while hemoglobin is commonly seen in mammalian lung models. Hemoglobin is the oxygen-binding protein that carries oxygen in the blood, it can be found in almost all vertebrates [24]. Oxygen binding to hemoglobin is a key process in gas exchange as well as oxygen diffusion. Due to the commonality of these two key gas exchange processes through out vertebrates, comparisons between mammals, fish and birds are possible under normal biological and environmental conditions or extreme conditions such as high-altitude environment or exercise. Some previous research compared the avian lungs with mammalian lungs or fish gills [3][13], but the comparison was based on experiments instead of modelling.

This thesis aims to find out how the lung structures influence the efficiency of oxygen transfer. For simplicity, carbon dioxide and other gases are not considered. Furthermore, the membrane difference between mammals and birds is omitted. An existing human lung model [1] is simplified and modified to represent the mammalian lung. New models are then developed to represent a counter direction lung structure, which is similar to fish gills, and a cross-current model of avian lungs. The efficiency of the different models are compared under both general and extreme conditions.

Chapter 2

Mathematical modelling of gas exchangers

In this chapter we develop mathematical models for gas exchange processes which occur in the respiratory systems of mammals, fish and birds. There are three models - flexible for mammals, counter-directional for fish, and cross-directional for birds. The flexible model is modified and simplified from a human lung model [1], the other two are developed in this thesis. All models share two key processes specific to gas exchange: oxygen diffusing from the air-side of the lungs to the capillaries, and oxygen binding to hemoglobin.

2.1 Key processes of gas exchange

As outlined above, diffusion and oxygen binding to hemoglobin are two key processes of gas exchange in respiratory systems.

Diffusion

Oxygen is transfered from the lungs to the capillaries via diffusion. The diffusion process can be described by [9]:

$$q = D_o(p_a - p_b) \tag{2.1}$$

where q is the flux of oxygen, D_o is the diffusion capacity of oxygen, p_a and p_b are the partial pressures of oxygen in the alveoli and in the blood respectively. Oxygen diffuses from the alveoli to the capillaries when the flux is positive, and in the opposite direction when the flux is negative. Diffusion occurs due to the difference between the partial pressures in the alveoli and the capillaries.

To calculate p_a, we first assume that gas in the air-side of the lungs obeys the ideal gas law $P = \frac{n}{V}kT$, where P is the gas pressure, V is the volume of gas, n is the number of moles of gas, k is the ideal gas constant and T is the temperature. $\frac{n}{V}$ is called molarity (molar concentration) and can be divided into $\frac{n_1}{V} + \frac{n_2}{V} + \frac{n_3}{V} + ...$ for different gases with number of moles n_1, n_2, n_3...and total volume V. Thus we can predict the partial pressure of a particular gas in mixed gases:

$$\begin{aligned} p_1 &= \frac{n_1}{V}kT \\ &= \frac{n_1}{n}\frac{n}{V}kT \\ &= \frac{n_1}{n}P \end{aligned} \tag{2.2}$$

where $\frac{n_1}{n}$ is the mole fraction of the particular gas in mixed gases (referred to as the concentration of oxygen in the air).

The total pressure and the number of moles of gases is increased by the vapour pressure inside the lungs. The number of moles of gases is $n + n_w$ where n_w is the number of moles of the vapour gas. $P = \frac{n_1+n_2+n_3+...+n_w}{V}kT$. To calculate the partial pressure of a particular gas inside the lungs, we need to consider the vapour pressure:

$$\begin{aligned} p_1 &= \frac{n_1}{n}\frac{nkT}{V} \\ &= \frac{n_1}{n}(n + n_w - n_w)\frac{kT}{V} \\ &= \frac{n_1}{n}((n + n_w)\frac{kT}{V} - n_w\frac{kT}{V}) \\ &= \frac{n_1}{n}(P - p_w) \end{aligned} \tag{2.3}$$

where p_w is the vapour pressure at 37 °C.

Under ideal gas conditions, we have [1]:

$$p_a = f_{ao}(P_A - p_w) \tag{2.4}$$

where f_{ao} is the concentration (mole fraction) of oxygen in the alveoli and P_A is the total pressure of air in the alveoli.

The partial pressure of oxygen in the blood is given by [9]:

$$p_b = \frac{f_o}{\sigma} \tag{2.5}$$

where σ is the solubility and f_o is the molarity of oxygen in the blood (referred to as the concentration of dissolved oxygen in the blood).

Note that f_{ao} has no units while the unit of f_o is $moll^{-1}$. Also note that q in Eq. 2.1 could have a unit of ls^{-1} or $mols^{-1}$, depending on the unit of D_o. To distinguish between these two cases we take D_o as having the unit $ls^{-1}mmHg^{-1}$ and use the expression D_o/C_u to represent a change in unit to $mols^{-1}mmHg^{-1}$, where C_u is a unit conversion factor.

Oxygen binding to hemoglobin

Hemoglobin is a protein molecule in red blood cells that carries oxygen. Once oxygen is diffused into the blood, it binds to hemoglobin in red blood cells, thus reducing the concentration of oxygen in the blood. In general, only 1.4% of diffused oxygen is dissolved in plasma, whereas the rest is carried by hemoglobin [23]. Each hemoglobin molecule can bind up to four oxygen molecules, which means that the amount of oxygen being carried by the blood is limited by the number of hemoglobin molecules. By assuming that hemoglobin is completey saturated with oxygen molecules and that the binding process happens quickly (dissolved oxygen molecules attach to the empty hemoglobin instantaneously), the concentration of oxygen molecules that bind to hemoglobin can be calculated by $4T_h \cdot S$ where T_h is the concentration of hemoglobin and S is the saturation of hemoglobin . The saturation function used in this thesis is [18]:

$$S = \frac{p_b^m}{p_b^m + K_p^m} \tag{2.6}$$

where $m = 2.5$ and $K_p = 26\ mmHg$. The derivative of S with respect to p_b which will be used later is:

$$\frac{dS}{dp_b} = \frac{mp_b^{m-1}K_p^m}{(p_b^m + K_p^m)^2} \tag{2.7}$$

Although hemoglobin is also involved in transporting other gases (for example, carbon dioxide), here it is assumed that the ability to bind with oxygen is not affected by other gases.

2.2 Mammlian lungs: flexible container model

In the lungs of mammals, gas exchange takes place in blind-ended alveoli while the air flows in and out periodically. The alveoli are wrapped by blood capillaries, and the oxygen diffuses across the surface of the capillaries as blood passes through. For simplificity, the lung is considered as a single flexible container, representing all the alveoli lumped together (see Figure 2.1). A blood vessel, representing all the capillaries lumped together, is attached to the flexible container. When blood passes through the surface of the alveoli, oxygen diffuses from the air-side to the blood-side. The dissovled oxygen in the blood bind to the hemoglobin, thus increases the concentration of oxygen in the blood.

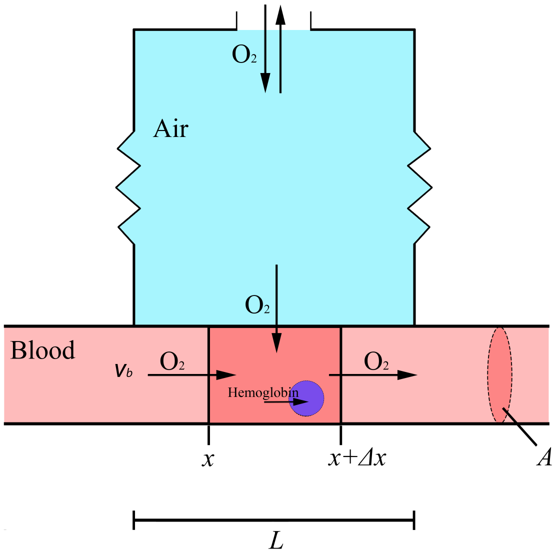

Fig. 2.1: A simplified model of mammalian lungs. Air moves in and out of the flexible container. Blood moves from left to right, gathering additional oxygen as it passes through the gas exchange area. Some dissolved oxygen binds to hemoglobin inside blood cells. L is the length of the gas exchange area. v_b is the velocity of blood. A is the total cross-sectional area of the capillaries.

We further assume that the gas in the container is well mixed and the concentration of oxygen is even at every point of contact with the capillaries. Under this assumption, the gas exchange between the alveoli and the capillaries can be represented by a "conveyor" model (Figure 2.2) [1]. In the conveyor model, the blood in the capillaries, is lumped together between heart beats. Dissolved gas is represented by a small container and saturated hemoglobin is represented by a larger container. The initial concentration of oxygen in each container is the concentration in the blood when it enters the lungs. Each set of containers moves forward after one heart beat. Thus we can describe the rate of change of dissolved oxygen as:

$$\left\{\begin{array}{c}\text{Rate of}\\\text{change of}\\\text{dissolved oxygen}\end{array}\right\} = \left\{\begin{array}{c}\text{Oxygen}\\\text{diffusion}\\\text{to blood}\end{array}\right\} - \left\{\begin{array}{c}\text{Oxygen}\\\text{binding to}\\\text{hemoglobin}\end{array}\right\}$$

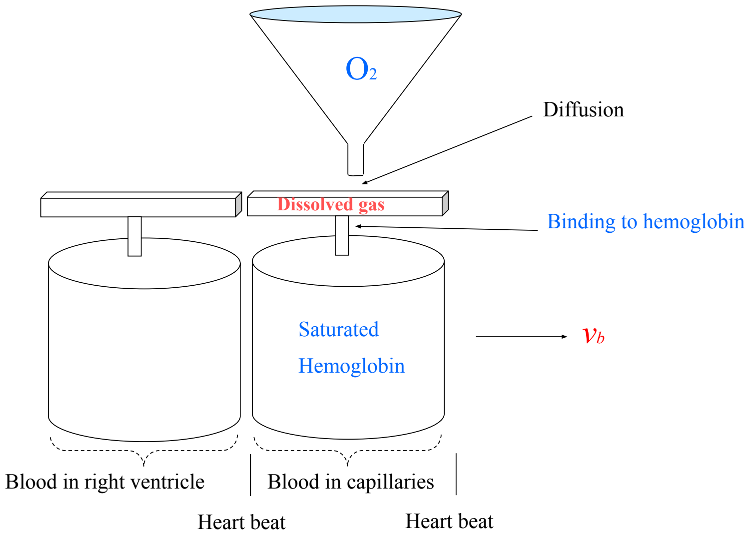

Fig. 2.2: The conveyor model adapted from [1]. A small container representing dissolved oxygen and a large container representing saturated hemoglobin are being pushed to the lung with every heart beat.

The number of moles of oxygen in the blood is $f_o \cdot V_c$ where f_o is the molarity of oxygen in the blood, and V_c is the total volume of the capillaries being engaged in gas exchange (within length L). We have:

$$\text{The rate of change of dissolved oxygen} = \frac{df_o}{dt} V_c \tag{2.8}$$

The rate of change of oxygen due to diffusion from the alveoli to the blood is given by Eq. 2.1 with q taken as $mol s^{-1}$.

The rate of change of oxygen due to binding to hemoglobin is:

$$\text{Rate of oxygen binding to hemoglobin} = 4T_h \frac{dS}{dt} V_c \tag{2.9}$$

Putting together Eqs. 2.1, 2.8 and 2.9 yields:

$$\frac{df_o}{dt} V_c = \frac{D_o}{C_u}(p_a - p_b) - 4T_h \frac{dS}{dt} V_c \tag{2.10}$$

Dividing both sides of Eq. 2.10 by V_c and substituting in Eq. 2.5 gives the rate of change of partial pressure of oxygen in the blood:

$$\frac{dp_b}{dt} = \frac{D_o}{C_u V_c \sigma}(p_a - p_b) - \frac{4T_h}{\sigma}\frac{dS}{dt} \tag{2.11}$$

The saturation function S is a function of p_b (Eq. 2.6), so:

$$\frac{dS}{dt} = \frac{dS}{dp_b}\frac{dp_b}{dt} \tag{2.12}$$

Substituting Eq. 2.12 into 2.11 and rearranging gives [1]:

$$\frac{dp_b}{dt} = \frac{D_o}{C_u V_c \sigma}(1 + \frac{4T_h}{\sigma}\frac{dS}{dp_b})^{-1}(p_a - p_b) \tag{2.13}$$

where $\frac{dS}{dp_b}$ is calculated by Eq. 2.7.

To find the rate of change of oxygen concentration in the alveoli we recall that f_{ao} can be described as $\frac{n_o}{n}$ where n_o is the number of moles of oxygen and n is the number of moles of all gases in the alveoli. By the ideal gas law, we know that $\frac{n_o}{n} = \frac{V_o}{V_A}$ where V_o is the volume of oxygen in the alveoli, and V_A is the total alveolar volume. The derivative of f_{ao} with respect to time is [1]:

$$\begin{aligned}\frac{df_{ao}}{dt} &= \frac{d}{dt}(\frac{n_o}{n}) \\ &= \frac{d}{dt}(\frac{V_o}{V_A}) \\ &= \frac{1}{V_A}(\frac{dV_o}{dt} - f_{ao}\frac{dV_A}{dt})\end{aligned} \tag{2.14}$$

Since respiration of mammals occurs through inspiratrion and expiratrion, let q_i be the inspired airflow and q_e be the expired airflow. Then the rate of change of the volume of oxygen is [1]:

$$\frac{dV_o}{dt} = -D_o(p_a - p_b) + f_{oi}q_i - f_{ao}q_e \tag{2.15}$$

where f_{oi} is the concentration of oxygen in the inspired air. The rate of change of alveolar volume is estimated by adding the inspired airflow, q_i, and subtracting the expired airflow out, q_e, (here we assume that the diffusion of oxygen into the blood cancels out with the diffusion of carbon dioxide into the lungs) so we can assume that:

$$\frac{dV_A}{dt} = q_i - q_e \tag{2.16}$$

Substituting Eqs. 2.15 and 2.16 into 2.14 and rearranging yields:

$$\frac{df_{ao}}{dt} = -\frac{1}{V_A}(D_o(p_a - p_b) + (f_{oi} - f_{ao})q_i) \tag{2.17}$$

For simplicity, we let f_{oi} equal to the concentration of oxygen in the mouth, that is, $f_{oi} = f_{om}$.

The airflow can be expressed as a periodic function with the inspired airflow given by [1]:

$$q_i = \begin{cases} \frac{V_T}{2}\omega sin(\omega t) & 0 < t \leq \frac{\pi}{\omega} \\ 0 & \frac{\pi}{\omega} < t \leq \frac{2\pi}{\omega} \end{cases} \tag{2.18}$$

where V_T is the tidal volume of the lung (the amount of inspired air in one single breath) and ω is the frequency of respiration.

Converting the concentration of oxygen f_{ao} in Eq. 2.17 to partial pressure p_a by using Eq. 2.4 gives:

$$\frac{dp_a}{dt} = \frac{1}{V_A}(-D_o(p_a - p_b)(P_A - p_w) + f_{om}(P_A - p_w)q_i - p_a q_i) \tag{2.19}$$

The flexible container model is described by Eqs. 2.13 and 2.19. To compare with the other models, we assume that V_A is constant.

2.3 Fish-like lungs: counter direction model

Fish use gills to breathe instead of lungs. When swimming forward in water, the water constantly flows into the mouth and then passes through the gills in opposite direction to the blood flow in the capillaries [12]. Figure 2.3 shows a schematic description of a counter direction gas exchanger. The carrier of oxygen is air instead of water so that the same parameters can be used in the model for comparison with the other models. The air vessel has a total cross-sectional area A_a and the capillaries have a total cross-sectional area A_b; the velocity of airflow and blood flow are v_a and v_b respectively.

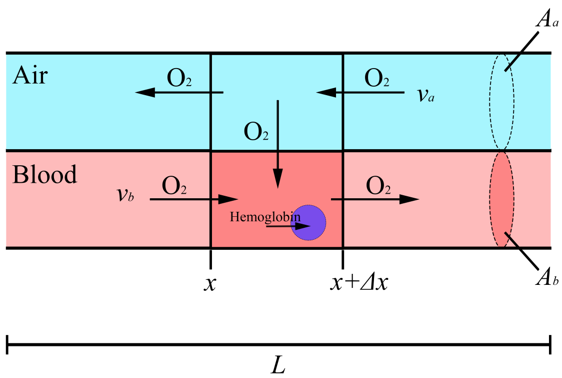

Fig. 2.3: Schematic description of a counter direction gas exchanger. Air flows from right to left while blood flows from left to right. Oxygen moves from the air-side to the blood-side along the surface of the vessels while air and blood keep flowing. Some dissolved oxygen binds to hemoglobin in the blood. L is the total length of the gas exchange area. v_a and v_b are the velocities of air and blood respectively. A_a and A_b are the cross-sectional areas of the air and blood vessels respectively.

Let Δx be a small distance along the total length of the capillaries L. The volume of this section is $\Delta x \cdot A_b$. As blood passes through the section in one direction, the rate of change of oxygen in the blood within the volume $\Delta x \cdot A_b$ can be described as:

$$\left\{\begin{array}{c}\text{Rate of change}\\ \text{of oxygen}\\ \text{in blood}\end{array}\right\} = \left\{\begin{array}{c}\text{Oxygen}\\ \text{inward}\\ \text{flow}\end{array}\right\} - \left\{\begin{array}{c}\text{Oxygen}\\ \text{outward}\\ \text{flow}\end{array}\right\} + \left\{\begin{array}{c}\text{Oxygen}\\ \text{diffusion}\\ \text{to blood}\end{array}\right\} - \left\{\begin{array}{c}\text{Oxygen}\\ \text{binding to}\\ \text{hemoglobin}\end{array}\right\}$$

A reasonable approximation of the amount of oxygen in the chosen volume is:

$$f_o(x+\frac{\Delta x}{2},t)\cdot\Delta x\cdot A_b$$

where f_o is the concentration of oxygen in the blood. The change of oxygen with respect to time is:

$$\text{Rate of change of oxygen in blood} = \frac{\partial f_o(x+\frac{\Delta x}{2},t)}{\partial t}\cdot\Delta x\cdot A_b \tag{2.20}$$

The inward and outward flow of oxygen are given by:

$$\text{Oxygen inward flow} = f_o(x,t)\cdot v_b\cdot A_b \tag{2.21}$$

$$\text{Oxygen outward flow} = f_o(x+\Delta x,t)\cdot v_b\cdot A_b \tag{2.22}$$

The amount of oxygen gained by the alveoli through diffusion depends on the surface area of the blood vessel. Eq. 2.1 gives the diffusion flux per unit area. Let p be the perimeter of capillaries, and Lp the total area of diffusion. Therefore in the space $\Delta x\cdot A_b$ the amount of diffused oxygen is:

$$\text{Oxygen diffusion} = \frac{D_o}{C_u}(p_a-p_b)\frac{\Delta x p}{Lp} \tag{2.23}$$

The rate of change of oxygen binding to hemoglobin is affected by the saturation function S. The rate of oxygen binding to hemoglobin is estimated by the difference between the flow of oxygen bound to hemoglobin when it enters the chosen volume and when it leaves it (see also Fig. 2.4) under the assumption that the process of oxygen binding to hemoglobin happens very fast:

$$\text{Oxygen binding to hemoglobin} = 4T_h(S(x+\Delta x)-S(x))\cdot v_b\cdot A_b \tag{2.24}$$

Putting together Eqs. 2.20, 2.21, 2.22, 2.23 and 2.24 we get:

$$\frac{\partial f_o(x+\frac{\Delta x}{2},t)}{\partial t}\Delta x A_b = (f_o(x,t)-f_o(x+\Delta x,t))v_bA_b+\frac{D_o}{C_u}(p_a-p_b)\frac{\Delta x}{L} \\ -4T_h(S(x+\Delta x,t)-S(x,t))v_bA_b \tag{2.25}$$

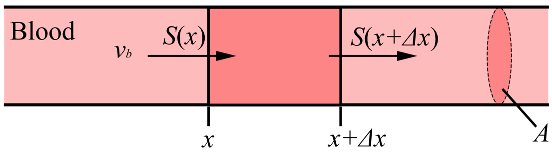

Fig. 2.4: The saturation of hemoglobin S at x and $x+\Delta x$ in the capillaries are different due to oxygen binding to hemoglobin. Dividing the difference by Δx gives the rate of change of hemoglobin with respect to distance.

Dividing both sides of Eq. 2.25 by $\Delta x A_b$ and taking the limit as $\Delta x \to 0$ gives:

$$\frac{\partial f_o}{\partial t} = -\frac{\partial f_o}{\partial x} v_b + \frac{D_o}{C_u}(p_a - p_b)\frac{1}{A_b L} - 4T_h v_b \frac{\partial S}{\partial x} \tag{2.26}$$

Since the volume of capillaries $V_c = A_b \cdot L$, converting concentration to partial pressure by substituting Eq. 2.5 into Eq. 2.25 gives:

$$\frac{\partial p_b}{\partial t} = -\frac{\partial p_b}{\partial x} v_b + \frac{D_o}{C_u V_c \sigma}(p_a - p_b) - \frac{4T_h}{\sigma}\frac{\partial S}{\partial x} v_b \tag{2.27}$$

Noting that S represents saturation as a function of p_b (Eq. 2.6),

$$\frac{\partial S}{\partial x} = \frac{\partial S}{\partial p_b}\frac{\partial p_b}{\partial x}. \tag{2.28}$$

Substituting Eq. 2.28 into 2.27 and rearranging gives:

$$\frac{\partial p_b}{\partial t} = -(1 + \frac{4T_h}{\sigma}\frac{\partial S}{\partial p_b})\frac{\partial p_b}{\partial x} v_b + \frac{D_o}{C_u V_c \sigma}(p_a - p_b) \tag{2.29}$$

where $\frac{\partial S}{\partial p_b}$ is computed by Eq. 2.7.

Comparing Eqs. 2.11 and 2.27, we see that the diffusion expressions are the same but $\frac{dS}{dt}$ in the flexible model is replaced by two expressions involving $\frac{\partial p_b}{\partial x}$ and $\frac{\partial S}{\partial x}$ in the counter direction model.

To find the equation for the rate of change of oxygen within the airflow in the counter direction model, we follow a process similar to that used for the blood-side, where oxygen is now lost in the diffusion process. Considering airflow in the opposite direction, the rate of change of oxygen is:

$$\left\{\begin{array}{c}\text{Rate of change}\\\text{of oxygen}\\\text{in the air}\end{array}\right\} = \left\{\begin{array}{c}\text{Oxygen}\\\text{inward}\\\text{flow}\end{array}\right\} - \left\{\begin{array}{c}\text{Oxygen}\\\text{outward}\\\text{flow}\end{array}\right\} - \left\{\begin{array}{c}\text{Oxygen}\\\text{diffusion}\\\text{to blood}\end{array}\right\}$$

The volume of air along Δx is $\Delta x \cdot A_a$. The approximated amount of oxygen in the chosen volume is

$$f_{ao}(x + \frac{\Delta x}{2}, t) \cdot \Delta x \cdot A_a.$$

The oxygen flow difference in the counter direction with velocity v_a is:

$$\text{Oxygen inward flow} = f_{ao}(x + \Delta x, t) \cdot v_a \cdot A_a \tag{2.30}$$

$$\text{Oxygen outward flow} = f_{ao}(x, t) \cdot v_a \cdot A_a \tag{2.31}$$

The oxygen lost through diffusion is the same as the oxygen gained by the blood with D_o:

$$\text{Oxygen diffusion} = D_o(p_a - p_b)\frac{\Delta xp}{Lp} \tag{2.32}$$

So the rate of change of oxygen in the air is:

$$\frac{\partial f_{ao}(x + \frac{\Delta x}{2}, t)}{\partial t}\Delta x A_a = ((f_{ao}(x + \Delta x, t) - f_{ao}(x, t)))\, v_a A_a - D_o(p_a - p_b)\frac{\Delta xp}{Lp} \tag{2.33}$$

Dividing both sides of Eq. 2.33 by $\Delta x A_a$ and taking the limit as $\Delta x \to 0$ gives:

$$\frac{\partial f_{ao}}{\partial t} = \frac{\partial f_a}{\partial x} v_a - D_o(p_a - p_b)\frac{1}{A_a L} \tag{2.34}$$

For comparison with the flexible lung model, we assume that $A_a L = V_A$. Converting concentration to partial pressure by substituting Eqs. 2.4 and 2.5 into Eq. 2.34 gives:

$$\frac{\partial p_a}{\partial t} = \frac{\partial p_a}{\partial x} v_a - \frac{D_o}{V_A}(P_A - P_w)(p_a - p_b) \tag{2.35}$$

The counter direction gas exchange model is described by Eqs. 2.29 and 2.35. In this model, p_b enters the gas exchange area at $x = 0$ and p_a enters at $x = L$. The model equations need to satisfy the boundary conditions: $p_b(0,t) = 40mmHg$, $p_a(L,t) = f_{om}(P_A - p_w)$.

2.4 Avian lungs: cross-current direction

The respiratory system of birds comprises the gas exchange area and several sets of air sacs. The gas exchange process is thus separated from the ventilation process. The air sacs do not play a direct role in gas exchange, however they keep air flow through the lungs constantly in one direction. The inspired air is first stored in the back air sacs, and then pass through the lungs to the front air sacs. In bird's lungs air flows cross-currently to the blood flow and for simplicity we assume here that the airflow is perpendicular to the blood flow within the gas exchange area. Similar to the case of counter direction, fresh air keeps flowing through the airways. In this case both axes are considered in the model: the blood gains oxygen along the x-axis, while the air loses oxygen along the y-axis. p_a and p_b are therefore functions of x, y and t. The rate of change of oxygen in the blood is the same as in Eq. 2.29

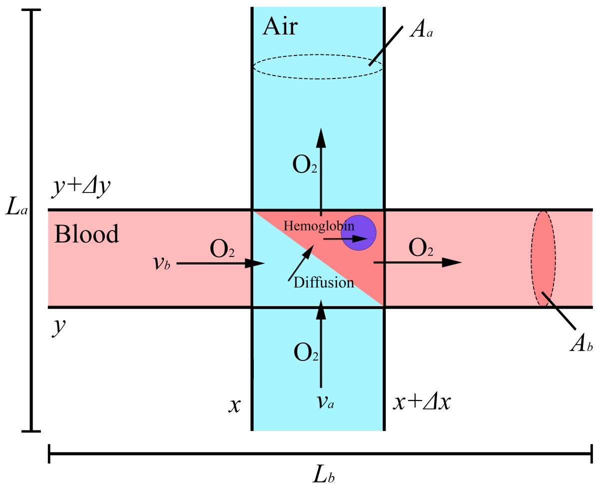

Fig. 2.5: The hypothesized cross-current gas exchanger in which the airflow is perpendicular to the blood flow. The oxygen moves from the air-side to the blood-side along the surface of the vessels while air and blood keep flowing. Some dissolved oxygen binds to hemoglobin in the blood cells. L_a is the total length of the air vessel and L_b is the total length of the capillaries. v_a and v_b are the velocities of each flow. A_a and A_b are the cross-sectional areas of each vessel.

for the counter direction model. With y-axis involved, the equation can now be

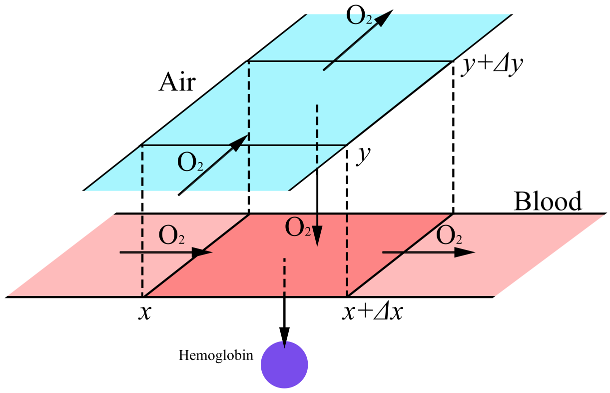

Fig. 2.6: Another view of Fig. 2.5

written as:

$$\frac{\partial p_b(x,y,t)}{\partial t} = -(1+\frac{4T_h}{\sigma}\frac{\partial S}{\partial p_b})\frac{\partial p_b}{\partial x}v_b + \frac{D_o}{C_u V_c \sigma}(p_a - p_b) \tag{2.36}$$

The rate of change of oxygen in the air is slightly different from Eq. 2.35 due to the directional change of the airflow. Take a small distance Δy along the air vessel, the volume chosen is $\Delta y \cdot A_a$. The approximated amount of oxygen in the chosen volume is:

$$f_{ao}(y+\frac{\Delta y}{2},t)\cdot\Delta x\cdot A_a$$

Then the rate of change of oxygen with respect to time in the air is:

$$\text{Rate of change of oxygen in air} = \frac{\partial f_{ao}(x,y+\frac{\Delta y}{2},t)}{\partial t}\cdot\Delta y\cdot A_a \tag{2.37}$$

The oxygen flow difference in the y direction with velocity v_a is:

$$\text{Oxygen inward flow} = f_{ao}(y,t)\cdot v_a\cdot A_a \tag{2.38}$$

$$\text{Oxygen outward flow} = f_{ao}(y+\Delta y,t)\cdot v_a\cdot A_a \tag{2.39}$$

The oxygen lost via diffusion is:

$$D_o(p_a - p_b)\frac{\Delta y p}{Lp} \tag{2.40}$$

So the rate of change of oxygen in the air vessel is:

$$\frac{\partial f_{ao}(x, y+\frac{\Delta y}{2}, t)}{\partial t}\Delta y A_a = (f_{ao}(x, y, t) - f_{ao}(x, y+\Delta y, t))v_a A_a - D_o(p_a - p_b)\frac{\Delta y}{L_a} \tag{2.41}$$

Dividing both sides of Eq. 2.41 by $\Delta y A_a$ and taking the limit as $\Delta y \to 0$ gives:

$$\frac{\partial f_{ao}}{\partial t} = -\frac{\partial f_{ao}}{\partial y}v_a - D_o(p_a - p_b)\frac{1}{L_a A_a} \tag{2.42}$$

For comparisons with the other models we take $A_a \cdot L_a$ as the lung volume V_A. Converting concentration to partial pressure by substituting Eqs. 2.4 and 2.5 into Eq. 2.42 yields:

$$\frac{\partial p_a(x, y, t)}{\partial t} = -\frac{\partial p_a}{\partial y}v_a - \frac{D_o}{V_A}(P_A - p_w)(p_a - p_b) \tag{2.43}$$

The cross-current model is the set of Eqs. 2.36 and 2.43. In this model, p_b enters the gas exchange area at $x = 0$ and p_a enters at $y = 0$. The model equations need to satisfy the boundary conditions: $p_b(0, y, t) = 40mmHg$, $p_a(x, 0, t) = f_{om}(P_A - p_w)$.

Chapter 3

Numerical methods

This chapter outlines numerical methods used for the models that were discussed previously.

3.1 Flexible model

The flexible container model (Eqs. 2.13 and 2.19) is solved in Matlab using the subroutine function ODE45, which is based on the Runge-Kutta method. Additionally, p_b is re-initialized to its initial value every T_L seconds according to the assumption of the "conveyor" model (Fig. 3.1).

3.2 Counter direction model

The finite-difference approximation method is used for the counter direction model. Let i be the distance along L with time step k. The rate of change of the partial pressure of oxygen in the air with respect to time is:

$$\frac{\partial p_a}{\partial t} = \frac{p_a(i,k+1) - p_a(i,k)}{\Delta t} \tag{3.1}$$

and in the blood:

$$\frac{\partial p_b}{\partial t} = \frac{p_b(i,k+1) - p_b(i,k)}{\Delta t} \tag{3.2}$$

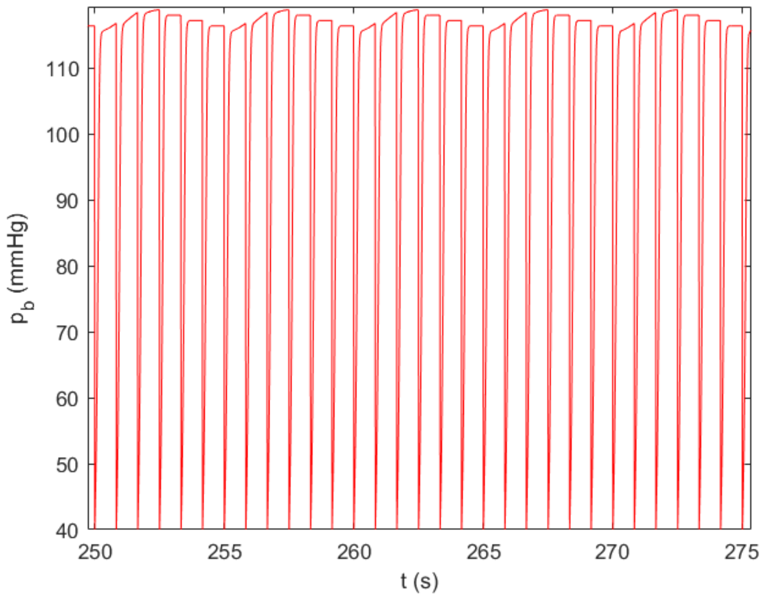

Fig. 3.1: The value of the partial pressure of oxygen in the blood, p_b, is reset to 40 mmHg every heart beat.

The rate of change of the partial pressure of oxygen in the air with respect to distance is:

$$\frac{\partial p_a}{\partial x} = \frac{p_a(i+1,k) - p_a(i,k)}{\Delta x} \tag{3.3}$$

and in the blood:

$$\frac{\partial p_b}{\partial x} = \frac{p_b(i+1,k) - p_b(i,k)}{\Delta x} \tag{3.4}$$

Substituting Eqs. 3.1 and 3.3 into the Eq. 2.35 gives the rate of change of p_a with respect to time in the counter direction model:

$$\begin{aligned}\frac{p_a(i,k+1) - p_a(i,k)}{\Delta t} = &\frac{p_a(i+1,k) - p_a(i,k)}{\Delta x} v_a \\ &- \frac{D_o}{V_A}(P_A - p_w)(p_a(i,k) - p_b(i,k)).\end{aligned} \tag{3.5}$$

Rearranging Eq. 3.5,

$$
\begin{aligned}
p_a(i,k+1) = &\frac{\Delta t}{\Delta x} v_a(p_a(i+1,k) - p_a(i,k)) \\
&- \Delta t \frac{D_o}{V_A}(P_A - p_w)(p_a(i,k) - p_b(i,k)) + p_a(i,k)
\end{aligned} \tag{3.6}
$$

Substituting Eqs. 3.2 and 3.4 into the Eqs. 2.29 gives the rate of change of p_b with respect to time in the counter direction model:

$$
\begin{aligned}
\frac{p_b(i,k+1) - p_b(i,k)}{\Delta t} = &-(1 + \frac{4T_h}{\sigma}\frac{\partial S}{\partial p_b})\frac{(p_b(i+1,k) - p_b(i,k))}{\Delta x} v_b \\
&+ \frac{D_o}{V_c \sigma}(p_a(i,k) - p_b(i,k))
\end{aligned} \tag{3.7}
$$

Rearranging Eq. 3.7:

$$
\begin{aligned}
p_b(i,k+1) = &-\frac{\Delta t}{\Delta x} v_b (1 + \frac{4T_h}{\sigma}\frac{\partial S}{\partial p_b})(p_b(i+1,k) - p_b(i,k)) \\
&+ \Delta t \frac{D_o}{V_c \sigma}(p_a(i,k) - p_b(i,k)) + p_b(i,k).
\end{aligned} \tag{3.8}
$$

The difference equations for the counter direction model are given by Eqs. 3.6 and 3.8, and $\frac{\partial S}{\partial p_b}$ is computed by Eq. 2.7. The system of difference equations is solved in Matlab with boundary conditions $p_b(0,t) = 40\ mmHg$ and $p_a(L,t) = f_{om}(P_A - p_w)$. $\frac{\Delta t}{\Delta x}$ is set to be 0.001. The velocity of blood flow is calculated using:

$$
v_b = \frac{L}{T_L} \tag{3.9}
$$

where T_L is the time between heart beats. The inspired air flow in the flexible model is periodic, but is constant in the counter and cross-directional models. To compare between the models, we compute the average flow in one single breath taking T as the period of breathing, T_i as the inspiration time and using Eq. 2.18:

$$
\begin{aligned}
W &= \frac{1}{T}\int_0^{T_i} q_i dt \\
&= \frac{\omega}{2\pi}\int_0^{\frac{\pi}{\omega}} \frac{V_T}{2}\omega \sin(\omega t) dt \\
&= \frac{V_T \omega}{2\pi}
\end{aligned} \tag{3.10}
$$

The average airflow W is the volume of the inspired air per second. The velocity of airflow v_a can now be calculated using the cross-sectional area A_a:

$$v_a = \frac{W}{A_a} = \frac{WL_a}{V_A} \tag{3.11}$$

3.3 Cross-current model

The finite-difference approximation method is also used for this model (see Fig. 2.5). Let i be the distance along L_b (x-axis), and let j be the distance along L_a (y-axis), with time step k. Then the avian lungs are represented by a rigid structure (i, j), where both the x and y directions will influence the pair (p_a, p_b) at every point (i, j). The rate of change of the partial pressure of oxygen in the air with respect to time is:

$$\frac{\partial p_a}{\partial t} = \frac{p_a(i,j,k+1) - p_a(i,j,k)}{\Delta t} \tag{3.12}$$

and in the blood:

$$\frac{\partial p_b}{\partial t} = \frac{p_b(i,j,k+1) - p_b(i,j,k)}{\Delta t} \tag{3.13}$$

The rate of change of the partial pressure of oxygen in the air with respect to distance is:

$$\frac{\partial p_a}{\partial y} = \frac{p_a(i,j+1,k) - p_a(i,j,k)}{\Delta y} \tag{3.14}$$

and in the blood:

$$\frac{\partial p_b}{\partial x} = \frac{p_b(i+1,j,k) - p_b(i,j,k)}{\Delta x} \tag{3.15}$$

Substituting Eqs. 3.12 and 3.14 into the Eq. 2.43 gives the rate of change of p_a in the cross-current model:

$$\begin{aligned}\frac{p_a(i,j,k+1) - p_a(i,j,k)}{\Delta t} &= \frac{p_a(i,j+1,k) - p_a(i,j,k)}{\Delta y} v_a \\ &\quad - \frac{D_{ao}}{V_A}(P_A - p_w)(p_a(i,j,k) - p_b(i,j,k))\end{aligned} \tag{3.16}$$

Rearranging Eq. 3.16:

$$p_a(i,j,k+1) = -\frac{\Delta t}{\Delta y} v_a(p_a(i,j+1,k) - p_a(i,j,k)) \\ - \Delta t \frac{D_{ao}}{V_A}(P_A - p_w)(p_a(i,j,k) - p_b(i,j,k)) + p_a(i,j,k) \qquad (3.17)$$

Substituting Eqs. 3.13 and 3.15 into the Eq. 2.36 gives the rate of change of p_b in the cross-current model:

$$\frac{p_b(i,j,k+1) - p_b(i,j,k)}{\Delta t} = -(1 + \frac{4T_h}{\sigma}\frac{\partial S}{\partial p_b})\frac{(p_b(i+1,j,k) - p_b(i,j,k))}{\Delta x} v_b \\ + \frac{D_o}{V_c \sigma}(p_a(i,j,k) - p_b(i,j,k)) \qquad (3.18)$$

Rearranging Eq. 3.18:

$$p_b(i,j,k+1) = -\frac{\Delta t}{\Delta x} v_b(1 + \frac{4T_h}{\sigma}\frac{\partial S}{\partial p_b})(p_b(i+1,j,k) - p_b(i,j,k)) \\ + \Delta t \frac{D_o}{V_c \sigma}(p_a(i,j,k) - p_b(i,j,k)) + p_b(i,j,k). \qquad (3.19)$$

The difference equations for the cross-current model are given by Eqs. 3.17 and 3.19, and $\frac{\partial S}{\partial p_b}$ is computed by Eq. 2.7. The boundary conditions are $p_b(0,y,t) = 40$ $mmHg$ and $p_a(x,0,t) = f_{om}(P_A - p_w)$. The velocities v_a and v_b are given by Eqs. 3.11 and 3.9 from the counter direction model.

Chapter 4

Results and comparisons

The three models were simulated under three sets of conditions: (1) normal conditions in which p_a and p_b can reach a high saturation level of oxygen at the end of the capillaries, (2) high altitude conditions (lack of oxygen) in which p_a and p_b have relatively lower saturation levels, and (3) extreme conditions (lack of oxygen and exercise) in which p_a and p_b are even lower than in (2). The various conditions are mimicked in our models by changes to some of the parameters. In order to compare the efficiency of the three models, we calculate the average partial pressure of oxygen at the end of the capillaries for each of the three models.

4.1 Normal conditions

Normal conditions refer to average body features, at sea level, at rest. The concentration of oxygen in the mouth, f_{om}, is 21%, which is the concentration of oxygen in the atmosphere. The body temperature is taken as 37 °C, which is the human body temperature; this affects the value of p_w (vapour pressure). The total pressure of the alveoli, P_A, is taken as one atmosphere, as it is at sea level, and is assumed to be constant in all the models. The length of the air vessels and capillaries L, L_a and L_b is chosen to be 1 m. The frequency of respiration, ω, is 0.4π $rads^{-1}$.

Fig. 4.1 shows the solution of p_a and p_b in the flexible lung model under normal conditions for a given set of initial conditions. The estimated average p_a at steady state is 117.65 $mmHg$. The estimated average value of p_b at points that are just before reinitialization (which is the end of capillaries) is 117.62 $mmHg$. Fig. 4.2 shows that p_a and p_b are very close together at steady state.

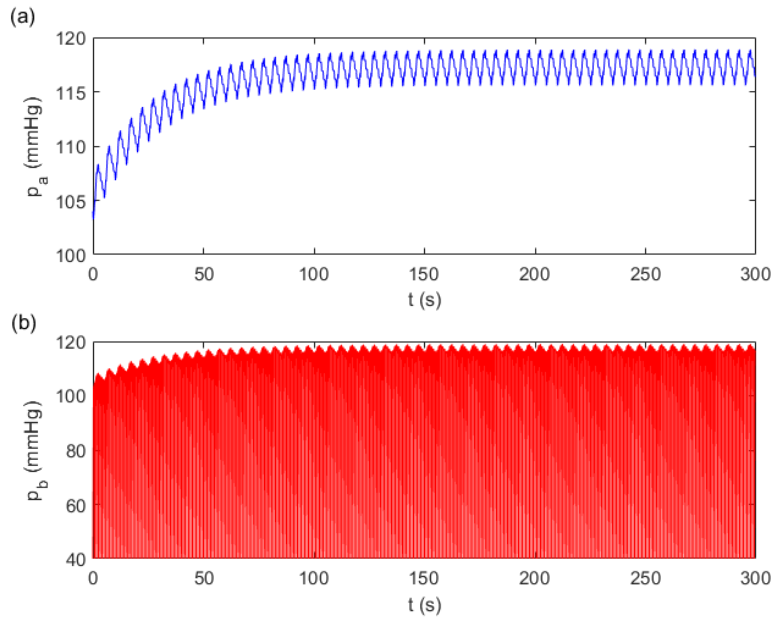

Fig. 4.1: Solution of the flexible lung model for given initial conditions. (a) Partial pressure of oxygen in the alveoli, p_a, with initial value of 104 mmHg. (b) Partial pressure of oxygen in the capillaries, p_b, with initial value of 40 mmHg. p_b is reinitialized every T_L seconds where $T_L = 60/72$ s is the time between heart beats.

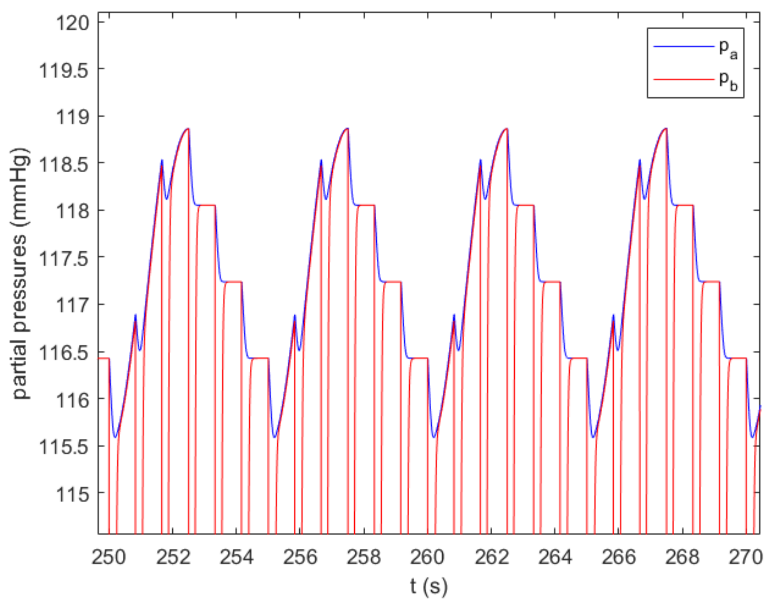

Fig. 4.2: Solution of the flexible lung model at steady state (enlargement of Fig. 4.1).

Fig. 4.3 shows the solution of p_a and p_b in the counter direction model under normal conditions for given boundary conditions. Recall that air flows from right ($x = 1$) to left ($x = 0$) while blood flows from left ($x = 0$) to right ($x = 1$). $p_a = 149.72$ $mmHg$ at $x = 1$, and p_a=138.92 $mmHg$ at $x = 0$. The value of p_b at the end of the capillaries ($x = 1$) is 149.72 $mmHg$.

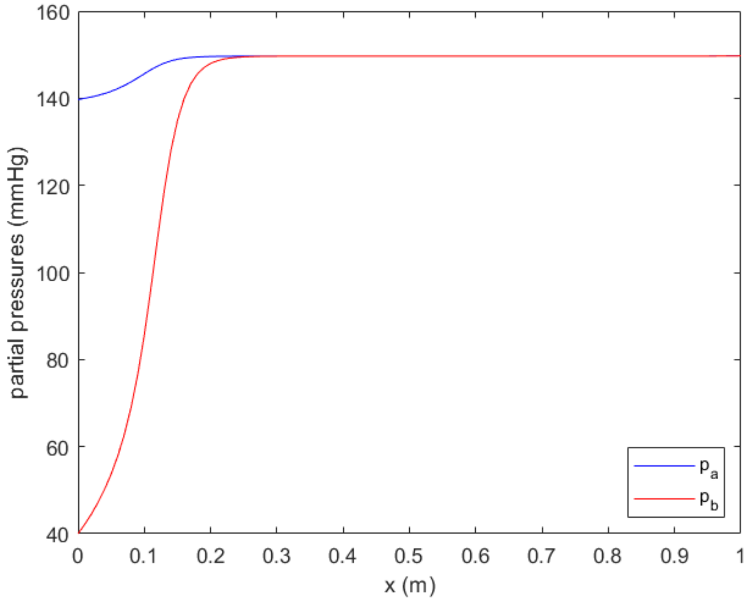

Fig. 4.3: Solution of the counter direction model (blood flows from left to right and air flows from right to left). Blue line: partial pressure of oxygen in the air vessel, p_a, with boundary value of 149.72 mmHg at $x = 1$. Red line: partial pressure of oxygen in the capillaries, p_b, with boundary value of 40 mmHg at $x = 0$.

Fig. 4.4 and 4.5 shows the solution of p_a and p_b in the cross-current model under normal conditions for given boundary conditions. Recall that air flows along the y-axis (from $y = 0$ to $y = 1$) while blood flows along the x-axis (from $x = 0$ to $x = 1$). The average value of p_a is 149.72 $mmHg$ at $y = 0$, and 149.66 $mmHg$ at $y = 1$. The average value of p_b at the end of the capillaries ($x = 1$) is 149.66 $mmHg$.

Under normal conditions, p_b is very close to p_a at steady state in all three models. The estimated average p_b at the end of the capillaries can reach a high saturation level of oxygen. With the same body features and environment, p_b at steady state in the flexible model is lower than in the counter direction and cross-current models, while p_b in the counter direction model is slightly higer than in the cross-current model.

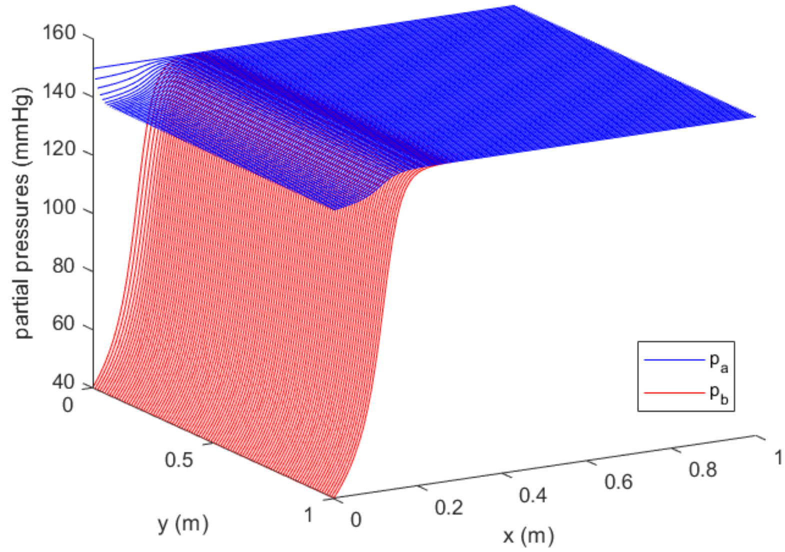

Fig. 4.4: Solution of the cross-current model (the blood flows from $x = 0$ to $x = 1$ and the air flows from $y = 0$ to $y = 1$). Blue line: partial pressure of oxygen in the air vessel, p_a, with boundary value of 149.72 mmHg at $y = 0$. Red line: partial pressure of oxygen in capillaries, p_b, with boundary value of 40 mmHg at $x = 0$.

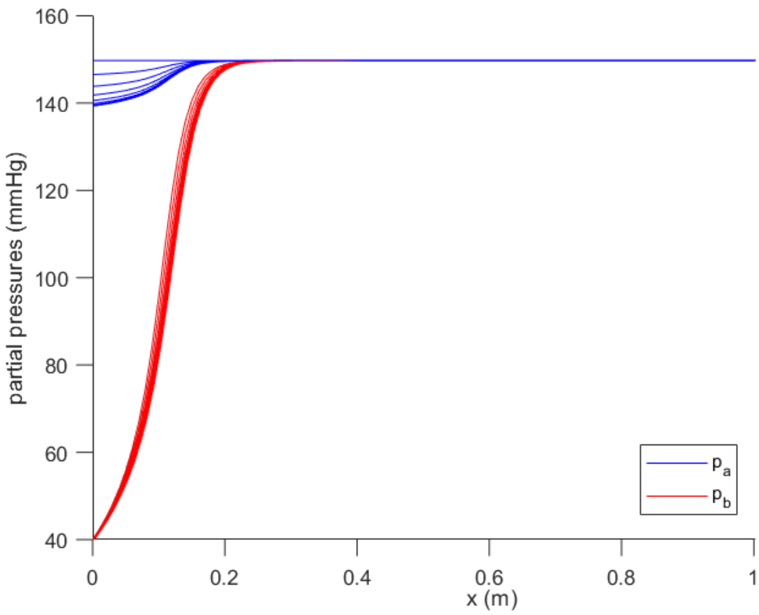

Fig. 4.5: Solution of the cross-current model (side section of Fig. 4.4)

4.2 High altitude conditions

In high altitude conditions, we assume that the body features are the same as in normal conditions. We take an altitude of 5500 m above sea level, where the atmospheric pressure (P_A in our model) is only 50% of its value at sea level [15]. Although the concentration of oxygen in the natural environment is the same, reduction in P_A leads to a lower value of p_a [13].

Fig. 4.6 shows the solution of p_a and p_b in the flexible model under high altitude conditions for given initial conditions. The average p_b is 60.5 $mmHg$. The average p_a is 60.73 $mmHg$. In contrast to the normal case where p_a and p_b are close together, here a gap between p_a and p_b appears.

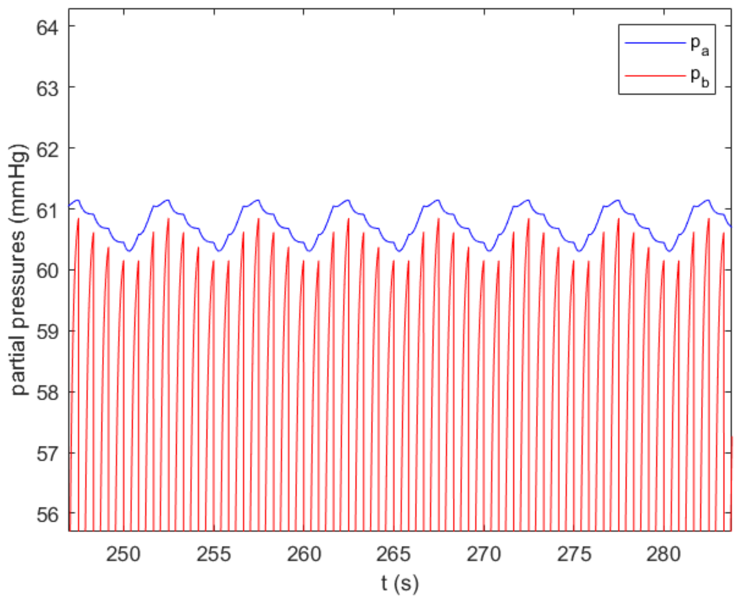

Fig. 4.6: Solution of the flexible lung model under high altitude conditions for given initial conditions. Blue line: partial pressure of oxygen in the alveoli, p_a, with initial value 104 mmHg. Red line: partial pressure of oxygen in the capillaries, p_b, with initial value 40 mmHg. p_b is reinitialized every T_L seconds where $T_L = 60/72s$ is the time between heart beats. The total lung pressure P_A=380 mmHg.

Fig. 4.7 shows the solution of p_a and p_b in the counter direction model under high altitude conditions for given boundary conditions. The air flows from right ($x = 1$) to left ($x = 0$) while blood flows from left ($x = 0$) to right ($x = 1$). $p_a = 69.93$ $mmHg$ at $x = 1$, and $p_a = 68.50$mmHg at $x = 0$. The value of p_b at $x = 1$ is 69.84 $mmHg$.

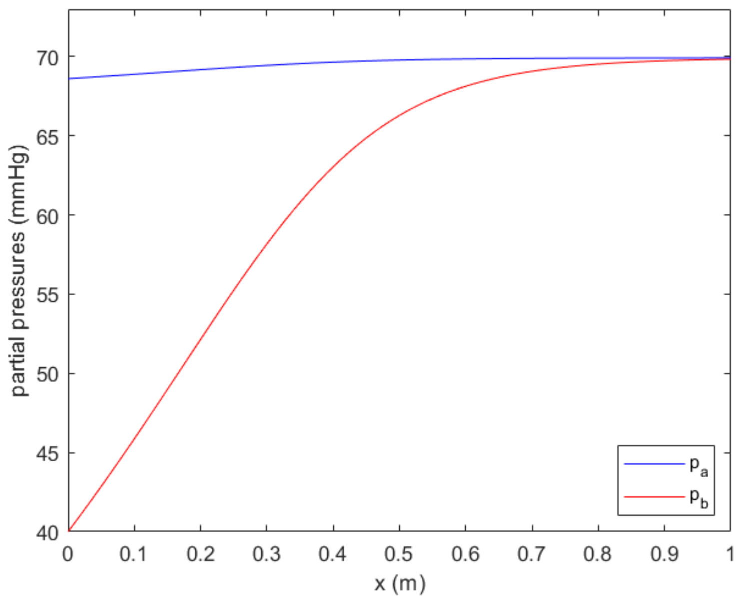

Fig. 4.7: Solution of the counter direction model (blood flows from left to right and air flows from right to left). Blue line: partial pressure of oxygen in the air vessel, p_a, with boundary value of 69.93 mmHg at $x = 1$. Red line: partial pressure of oxygen in the capillaries, p_b, with boundary value of 40 mmHg at $x = 0$. The total lung pressure P_A=380 mmHg.

Fig. 4.8 shows the solution of p_a and p_b in the cross-current model under high altitude conditions for given boundary conditions. The air flows along the y-axis ($y = 0$ to $y = 1$) while blood flows along the x-axis ($x = 0$ to $x = 1$). The average value of p_a is 69.93 $mmHg$ at $y = 0$, and 69.92 $mmHg$ at $y = 1$. The average value of p_b at the end of the capillaries ($x = 1$) is 69.84 $mmHg$.

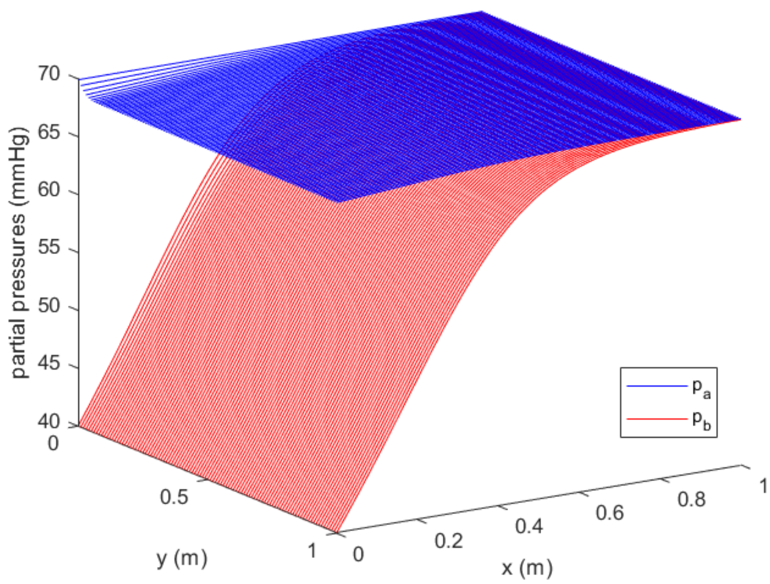

Fig. 4.8: Solution of the cross-current model (the blood flow is perpendicular to the air flow). Blue line: partial pressure of oxygen in the air vessel, p_a, with boundary value of 69.93mmHg at $y = 0$. Red line: partial pressure of oxygen in the capillaries, p_b, with boundary value of 40 mmHg at $x = 0$. The total lung pressure P_A=380 mmHg.

At high altitude, the average p_b at the end of the capillaries in each model drops significantly since the inspired p_a is reduced. p_b is still close to p_a but the gap is larger than under normal conditions. With the same body features and environment, p_b in the flexible model is lower than in the counter direction and cross-current models (which in this example perform the same).

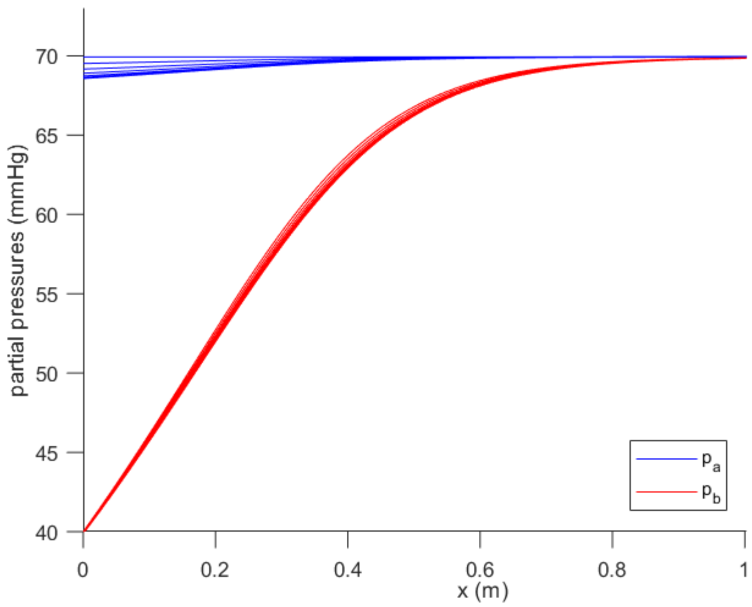

Fig. 4.9: Solution of the cross-current model (side section of Fig. 4.8)

4.3 Extreme conditions

Extreme conditions refer to average body features, at high altitude, in exercise. Exercise increases the heart rate as well as breathing rate. In this case, the faster heart rate and breathing take place to enable the body to get more oxygen despite the decrease in p_b. To set the extreme conditions, we double the heart beat and respiratory rate. The body features are the same as of normal and high altitude conditions; the altitude is 5500 m.

Fig. 4.10 shows the solution of p_a and p_b in the flexible model under extreme conditions for given initial conditions. The average p_b is 57.30 $mmHg$. The average p_a is 61.03 $mmHg$.

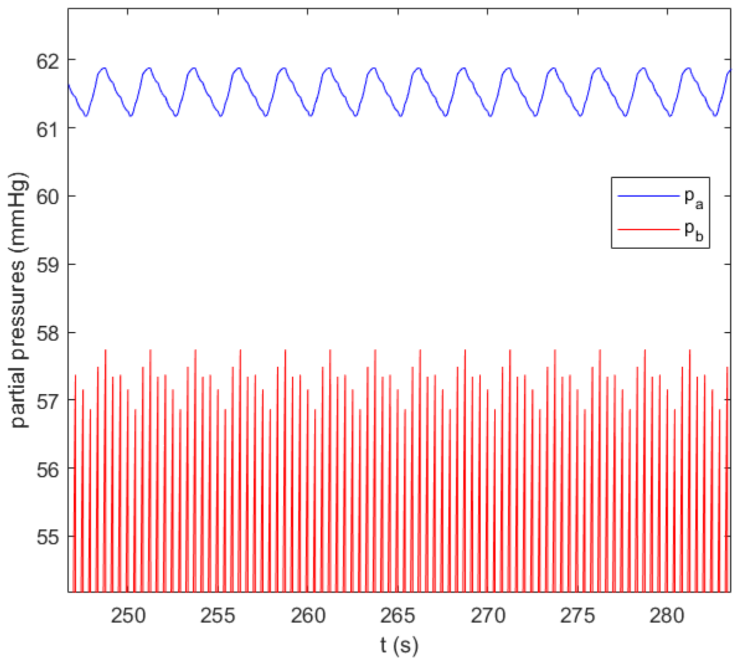

Fig. 4.10: Solution of the flexible lung model under extreme conditions for given initial conditions. Blue line: partial pressure of oxygen in the alveoli, p_a, with initial value 104mmHg. Red line: partial pressure of oxygen in the capillaries, p_b, with initial value 40 mmHg. p_b is reinitialized every T_L seconds where $T_L = 30/72s$ is the time between heart beats. The total lung pressure P_A=380 mmHg. The breathing frequency $\omega = 0.8\pi$ rad/s

Fig. 4.11 shows the solution of p_a and p_b in the counter direction model under extreme conditions for given boundary conditions. The air flows from right $(x = 1)$

to left ($x = 0$) while blood flows from left ($x = 0$) to right ($x = 1$). $p_a = 69.93$mmHg at $x = 1$, and $p_a = 68.50$ $mmHg$ at $x = 0$. The value of p_b at $x = 1$ is 66.32 $mmHg$.

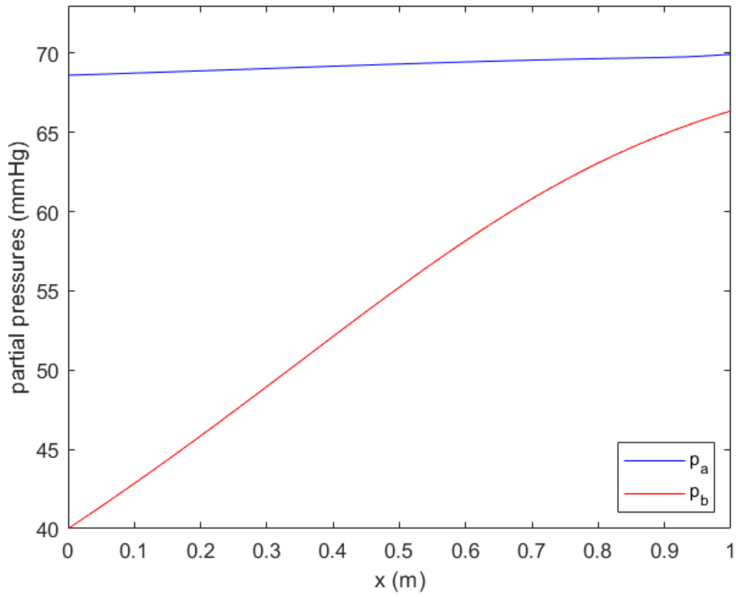

Fig. 4.11: Solution of the counter direction model (blood flows from left to right and air flows from right to left). Blue line: partial pressure of oxygen in the air vessel, p_a, with boundary value of 69.93 mmHg at $x = 1$. Red line: partial pressure of oxygen in the capillaries, p_b, with boundary value of 40 mmHg at $x = 0$. The total lung pressure P_A=380 mmHg. The breathing frequency $\omega = 0.8\pi$ rad/s.

Fig. 4.12 shows the solution of p_a and p_b in the cross-current model under extreme conditions for given boundary conditions. The air flows along the y-axis ($y = 0$ to $y = 1$) while blood flows along the x-axis ($x = 0$ to $x = 1$). The average value of p_a is 69.93 $mmHg$ at $y = 0$, and 69.76 $mmHg$ at $y = 1$. The average value of p_b at the end of the capillaries ($x = 1$) is 66.28 $mmHg$.

Under the extreme conditions, p_b in the flexible model decreases more than in the other two models. A clear gap now exists between p_a and p_b. With the same body features and environment, p_b in the flexible model is much lower than in the counter direction and cross-current models which perform similarly.

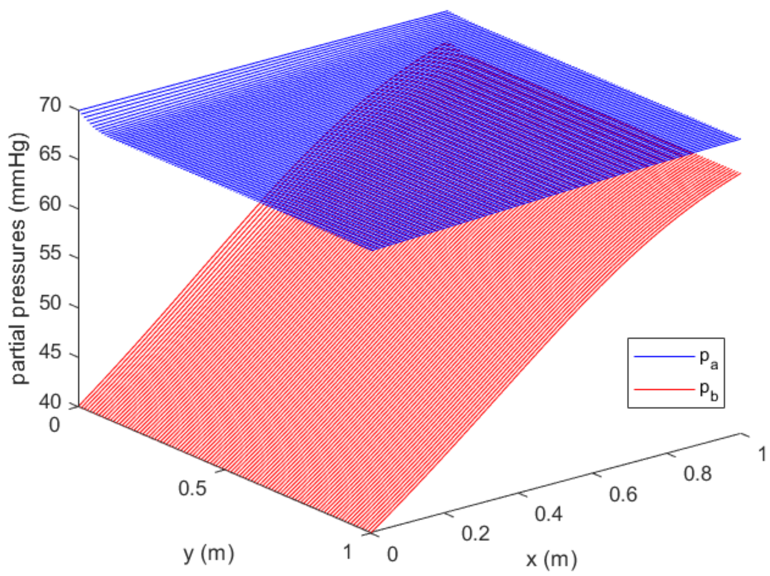

Fig. 4.12: Solution of the cross-current model (the blood flow is perpendicular to the air flow). Blue line: partial pressure of oxygen in air vessel p_a with boundary value of 69.93mmHg at $y = 0$. Red line: partial pressure of oxygen in the capillaries, p_b, with boundary value of $40mmHg$ at $x = 0$. The total lung pressure P_A=760 mmHg. The breathing frequency $\omega = 0.8\pi$ rad/s

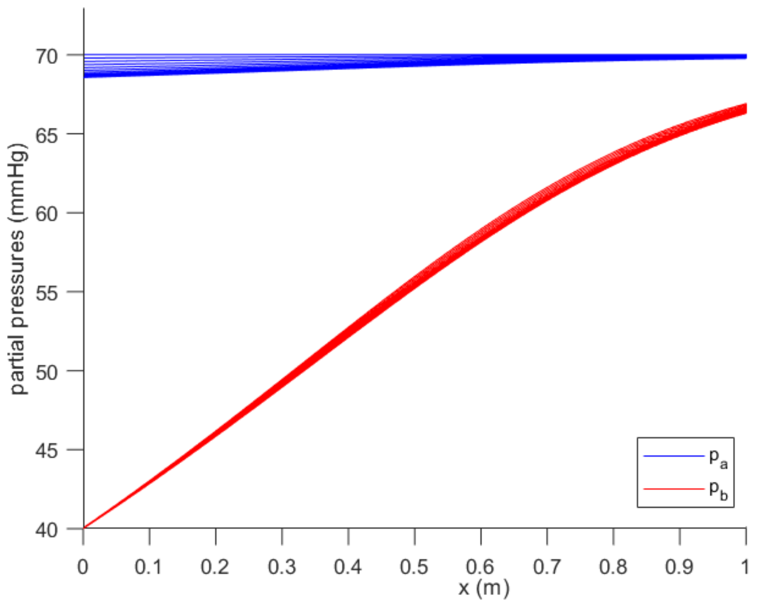

Fig. 4.13: Solution of the cross-current model (side section of Fig. 4.12)

Chapter 5

Conclusions

In this thesis we have developed oxygen exchange models of three types of gas exchangers within the respiratory systems of mammals, fish and birds. The purpose of this work was to study and compare the efficiency of mammalian and avian lung structures regardless of the species.

The mathematical models of the gas exchangers are based on the two key processes - diffusion and oxygen binding to hemoglobin (Chapter 2). The flexible model is a simplification of a published model [1]. The counter direction model is obtained by changing the periodical airflows in the flexible model to the constant counter directional airflow. The cross-current model is an extension of the counter direction model, with the airflow taken as perpendicular to the blood flow. These three models are solved by numerical methods which have been described in Chapter 3.

The outputs of all three models are compared in Chapter 4. Change of environment, or strenuous exercise may significantly reduce the concentration of oxygen in the blood and therefore impair the function of the lungs. We compared the blood partial pressure of oxygen in three models under normal conditions, high altitude, and extreme conditions (high altitude and exercise). The value of the blood partial pressure of oxygen in the counter direction and cross-current model were higher than in the flexible model, showing that the flexible model is less efficient. In other words, the lung structures of fish and birds ensure that more oxygen can be transfered into the blood even in harsh environment.

In constructing the three models, we assumed that the lung volume V_A and the total alveolar pressure P_A are constant. The dead space was removed from the flexible and avian models for comparison with the fish-like model. We also assumed that the airflow is constant in the avian lungs while in reality it could change with time [7]. These assumptions could be relaxed in future work. The gas exchanger models in this thesis are only involving the exchange of oxygen. The exchange of carbon dioxide may be added to the models in the future.

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Appendix A

Variables and Parameters

Table A.1: Variables

Symbol	Meaning	Unit
f_{ao}	Concentration of O_2 in the alveoli/air vessel	
f_o	Concentration of O_2 in the blood	$moll^{-1}$
p_a	Partial pressure of O_2 in the alveoli/air vessel	$mmHg$
p_b	Partial pressure of O_2 in blood	$mmHg$

Table A.2: Parameters

Symbol	Meaning	Value	Unit
D_o	Diffusion Capacity	3.5×10^{-4}	$ls^{-1}mmHg^{-1}$
C_u	Unit conversion factor	22.436	$lmol^{-1}$
σ	Solubility of O_2 in blood	1.4×10^{-6}	$moll^{-1}mmHg^{-1}$
ω	Frequency of respiration	0.4π	$rads^{-1}$
L	Length of capilaries and air vessels	1	m
f_{om}	Concentration of O_2 in mouth	0.21	
f_{oi}	Concentration of O_2 inspired	f_{om}	
T_h	Concentration of hemoglobin molecule	2×10^{-3}	$moll^{-1}$
P_m	Air pressure in mouth	760	$mmHg$
P_A	Total pressure in lung	P_m	$mmHg$
p_w	Vapour pressure in air at 37°C	47	$mmHg$
V_A	Total volume of lung	2.5	l
V_T	Lung tidal volume	0.4	l
V_c	Total volume of capillaries	0.07	l
T_L	Time between heart beats	$60/72$	s
v_a	Velocity of air	WL/V_A	ms^{-1}
v_b	Velocity of blood	L/T_L	ms^{-1}
n	Constant in saturation function	2.5	
k_p	Constant in saturation function	26	$mmHg$
T	Time period of one whole breathe	$2\pi/\omega$	s
T_i	Time period of insipiration	π/ω	s

The value of parameters are taken from [1].

Appendix B

Matlab code

Flexible model

```
%% Flexible Oxygen Exchange Model
%This is a simulation of how O2 exchange in respiratory system of mammalian
%lungs.All data based on 'Simplified Models for Gas Exchange in the Human
%Lungs' by Alona Ben-Tal.
%Function file: flexible_gas_exchange_function.m
clear

global omega a b c d V_T V_A V_D
%Parameters
D_ao = 3.5e-4;    %ls^(-1)mmHg^(-1)   O2 diffusion capacity in alveoli
D_o = 1.56e-5;    %mols^(-1)mmHg^(-1) O2 diffusion Capacity in blood
omega = 0.4*pi;   %rad s^(-1)         Frequency of respiration
s = 1.4e-06;      %moll^(-1)mmHg^(-1) Sigma-Solubility of O2 in plasma
T_h = 2e-03;      %moll^(-1)          Concentration of hemoglobin molecule
P_A = 760;        %mmHg               Alveolar total pressure
p_w = 47;         %mmHg               Vapor pressure of water at 37C
V_T = 0.4;        %l                  Lung tidal volume
V_A = 2.5-V_T/2;  %l                  Volume of air in lung
V_c = 0.07;       %l                  Volume of capillaries
f_om = 0.21;      %(no unit)          Concentration of O2 in mouth
T_L = 60/72;      %s                  Time between heart beats

%
a = D_ao * (P_A - p_w);           %s^(-1)
b = f_om*(1-V_D/V_T)*(P_A-p_w);   %l^(-1)
c = D_o / (V_c * s);              %s^(-1)
d = 4*T_h/s;                      %(no unit)

%Initial values
t0 = 0.0;    %Initial time
tf = 300.0;  %Final time
y0 = [104, 40]; %Initial partial pressure of oxygen p_a & p_b
```

```
%Error control
options = odeset('RelTol',1e-9, 'AbsTol',1e-9, 'MaxStep', 0.01);

%p_b reinitialization
N_HR=floor(tf/T_L);

for i = 1:N_HR
    t1 = i*T_L;
    [t,y] = ode45('flexible_gas_exchange_function', [t0:0.01:t1], y0, options);
    t0 = t(end)+0.01;
    y0(1) = y(end,1);
    y0(2)=40;
    if i == 1
        t2 = t;
        y1 = y;
    else
        t2 = [t2;t];
        y1 = [y1;y];
    end
end

subplot(2,1,1)
plot(t2,y1(:,1),'-b')
xlabel('t (s)')
ylabel('p_a (mmHg)')
subplot(2,1,2)
plot(t2,y1(:,2),'-r')
xlabel('t (s)')
ylabel('p_b (mmHg)')
hold on
```

```
%% Flexible Oxygen Exchange Model - DE System
%--------------------time function------------------------------
% q = V_T/2 * omega * sin(omega*t) if 0 <= t < pi/omega
% q = 0                            if pi/omega <= t < 2pi/omega
%---------------------------------------------------------------
function dydt = flexible_gas_exchange_function(t, y)

global a b c d omega V_T V_A V_D

%Constant in saturation function
m=2.5;
kp=26;

%Condition setting for time period
t_p = mod(t, 2*pi/omega);
if (t_p >= 0) && (t_p < pi/omega)
    qi = V_T/2*omega*sin(omega*t);
    V_A= -V_T/2*cos(omega*t)+2.5;
else
    qi = 0;
end

dydt=zeros(2,1);
```

```
%y(1)=p_a, y(2)=p_b
dydt(1) = (-a*(y(1)-y(2))+(y(1)*V_D/V_T+b-y(1))*qi)/V_A ;
dydt(2) = c*(1+d*m*y(2)^(m-1)*kp^m/((y(2)^m+kp^m)^2))^(-1)*(y(1)-y(2));
```

Counter direction model

```
%% Counter Direction Oxygen Exchange Model
%This is a simulation of how O2 exchange in fish gills (with the assumption
%that water is replaced by air). All data based %on 'Simplified Models for
%Gas Exchange in the Human Lungs' by Alona Ben-Tal.
clear

%parameters
D_ao = 3.5e-4;    %ls^(-1)mmHg^(-1)   O2 diffusion capacity in alveoli
D_o = 1.56e-5;    %mols^(-1)mmHg^(-1) O2 diffusion Capacity in blood
omega = 0.4*pi;   %rad s^(-1)         Frequency of respiration
s = 1.4e-06;      %moll^(-1)mmHg^(-1) Sigma-Solubility of O2 in plasma
T_h = 2e-03;      %moll^(-1)          Concentration of hemoglobin molecule
P_A = 760;        %mmHg               Alveolar total pressure
p_w = 47;         %mmHg               Vapor pressure of water at 37C
V_T = 0.4;        %l                  Lung tidal volume
V_A = 2.5-V_T/2;  %l                  Volume of air in lung
V_c = 0.07;       %l                  Volume of capillaries
f_om = 0.21;      %(no unit)          Concentration of O2 in mouth
T_L = 60/72;      %s                  Time between heart beats
W = V_T*omega/(2*pi); %ls^(-1)        Volume of inspired air per second
L=1;              %m                  Length of the gas exchange area of both vessels
v_a = W*L/V_A;    %ms^(-1)            Velocity of air
v_b = L/T_L;      %m^(-1)             Velocity of blood
m=2.5;
kp=26;            %mmHg

%
alpha = D_o/(V_c*s);        %s^(-1)
gama = D_ao*(P_A-p_w)/V_A;  %s^(-1)
n = 100;
dx = L/n;                   %m
beta = 0.001;               %beta=dt/dx
dt = beta*dx;

maxtime=1.0;      %s
maxsteps=maxtime/dt;

%Initial values
P_ao = f_om*(P_A-p_w);
P_bo = 40;

%The initial partial pressure along the length at step 1
for i=2:(n+1)
    P_a(i-1)=P_ao;
    P_b(i)=P_bo;
```

```
end
P_a(n+1)=P_ao;
P_b(1)=P_bo;

for t=1:maxsteps
 for i=2:(n+1)
     %Partial pressure change of O2 in air
        P_a_new(i-1)=beta*v_a*(P_a(i)-P_a(i-1))-dt*gama*(P_a(i-1)-P_b(i-1))+P_a(i-1);
     %Saturation rate
        dSdp=m*P_b(i)^(m-1)*kp^m/((P_b(i)^m+kp^m)^2);
     %Partial pressure change of O2 in blood
        P_b_new(i)=-beta*v_b*(1+4*T_h/s*dSdp)* ...
                   (P_b(i)-P_b(i-1))+dt*alpha*(P_a(i)-P_b(i))+P_b(i);
 end
 P_a=P_a_new;
 P_b=P_b_new;
 P_a(n+1)=P_ao;
 P_b(1)=P_bo;

end

%The average p_b at the end of the capillaries
mean(P_b(end))


x=[0:dx:L];

plot(x,P_a,'b',x,P_b,'r')
xlabel('x (m)')
ylabel('partial pressures (mmHg)')
hold on
```

Cross-current model

```
%% Counter Direction Oxygen Exchange Model
%This is a simulation of how O2 exchange in respiratory system of birds.
%All data based %on 'Simplified Models for Gas Exchange in the Human Lungs'
%by Alona Ben-Tal.
clear

%parameters
D_ao = 3.5e-4;   %ls^(-1)mmHg^(-1)    O2 diffusion capacity in alveoli
D_o = 1.56e-5;   %mols^(-1)mmHg^(-1)  O2 diffusion Capacity in blood
omega = 0.4*pi;  %rad s^(-1)          Frequency of respiration
s = 1.4e-06;     %moll^(-1)mmHg^(-1)  Sigma-Solubility of O2 in plasma
T_h = 2e-03;     %moll^(-1)           Concentration of hemoglobin molecule
P_A = 760;       %mmHg                Alveolar total pressure
p_w = 47;        %mmHg                Vapor pressure of water at 37C
V_T = 0.4;       %l                   Lung tidal volume
V_A = 2.5-V_T/2; %l                   Volume of air in lung
V_c = 0.07;      %l                   Volume of capillaries
f_om = 0.21;     %(no unit)           Concentration of O2 in mouth
```

```
T_L = 60/72;      %s                  Time between heart beats
W = V_T*omega/(2*pi); %ls^(-1)        Volume of inspired air per second
L=1;              %m                  Length of the gas exchange area of both vessels
v_a = W*L/V_A;    %ms^(-1)            Velocity of air
v_b = L/T_L;      %m^(-1)             Velocity of blood
m=2.5;
kp=26;            %mmHg

%
alpha = D_o/(V_c*s);         %s^(-1)
gama = D_ao*(P_A-p_w)/V_A;  %s^(-1)
n = 100;
dx = L/n;                    %m
beta = 0.001;                %beta=dt/dx
dt = beta*dx;

maxtime=1;        %s
maxsteps=maxtime/dt;

%initial values
P_ao = f_om*(P_A-p_w);
P_bo = 40;

%The initial partial pressure along the length at step 1
for j=1:(n+1)
    for i=1:(n+1)
    P_a(i,j)=P_ao;
    P_b(i,j)=P_bo;
    end
end
P_b(1,1)=P_bo;
P_a(1,1)=P_ao;
P_a_new=P_a;
P_b_new=P_b;

for t=1:maxsteps
 j=1;
 for i=2:(n+1)
     dSdp=m*P_b(i,j)^(m-1)*kp^m/((P_b(i,j)^m+kp^m)^2);
     P_b_new(i,j)=-beta*v_b*(1+4*T_h/s*dSdp)* ...
                  (P_b(i,j)-P_b(i-1,j))+dt*alpha*(P_a(i,j)-P_b(i,j))+P_b(i,j);
 end
 for j=2:n+1
     P_a_new(1,j)=-beta*v_a*(P_a(1,j)-P_a(1,j-1))-dt*gama*(P_a(1,j)-P_b(1,j))+P_a(1,j);
   for i=2:(n+1)
     %Partial pressure change of O2 in air
        P_a_new(i,j)=-beta*v_a*(P_a(i,j)-P_a(i,j-1))-dt*gama*(P_a(i,j)-P_b(i,j))+P_a(i,j);
     %Saturation rate
        dSdp=m*P_b(i,j)^(m-1)*kp^m/((P_b(i,j)^m+kp^m)^2);
     %Partial pressure change of O2 in blood
        P_b_new(i,j)=-beta*v_b*(1+4*T_h/s*dSdp)* ...
                  (P_b(i,j)-P_b(i-1,j))+dt*alpha*(P_a(i,j)-P_b(i,j))+P_b(i,j);
    P_a_new(i,1)=P_ao;
```

```
    end

    P_b_new(1,j)=P_b0;

  end

  P_a_new(1,1)=P_a0;
  P_b_new(1,1)=P_b0;
  P_a=P_a_new;
  P_b=P_b_new;
  t/maxsteps*100 %progress check
end

%The average p_a at the end of the air vessels
mean(P_a(end,:))
%The average p_b at the end of the capillaries
mean(P_b(end,:))

x=[0:dx:L];
[X,Y] = meshgrid(x);

plot3(X,Y,P_b,'r', X,Y,P_a,'b')
xlabel('y(m)')
ylabel('x(m)')
zlabel('p_a(blue), p_b(red)(mmHg)')
hold on
```