Students' Perspective of a Mathematics Extension Programme Designed with Special Interest in History

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ABSTRACT

The current Mathematics in the New Zealand Curriculum (Ministry of Education, 1992) includes the development of mathematical talent as a major aim of mathematics education. In catering for the individual needs of all students, the document emphasizes that students with exceptional ability in mathematics must be extended and are not expected to repeat the work they have already mastered. Talented students should be exposed to broader, richer, and more challenging mathematical experiences, should be allowed to investigate whole new topics, and work at a higher conceptual level.

Despite a growing awareness among secondary school teachers of the needs of mathematically gifted and talented students in the New Zealand secondary school classrooms, there are few exemplars of how mathematics programmes can be adapted for class groups of talented students. This study involves an investigation based on student perceptions of a mathematics programme that build on specific interest of a whole class group of students.

The aim of this qualitative exploratory case study, undertaken in an urban secondary school for girls, was to seek students' views on a Year 10 mathematics extension programme. As part of their Year 10 general extension programme, they participated in mathematics extension and studied history as their chosen option. While all students in this class were academically talented and high achievers in their core subject areas, not all of them were equally talented, or equally interested in mathematics. The mathematics extension programme, designed by their mathematics teacher (the researcher), specifically integrated their interest in history.

Data was generated from student self-evaluation questionnaires at the beginning of the course, and student questionnaires and focus-group interviews at the end of the course. Students' written and verbal responses were analyzed and then conclusions drawn. The findings suggested that by approaching mathematics from a historical point of view and thereby building on their common interest, the programme of study facilitated the development of mathematical talent and supported students in developing interest and a positive disposition towards mathematics.

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CHAPTER ONE: Introduction

1.1 Background to study

The curriculum document, Mathematics in the New Zealand Curriculum (1992), states that all students have the right to achieve to the maximum of their personal potential. Students should experience a range of mathematics which is appropriate to their level, interests and capabilities. In recommending the provision for the gifted and talented, the document states that students with exceptional mathematical abilities should be extended. In addition, the document highlights the fact that improved access is needed for girls in the field of mathematics, as the participation rate of female students in mathematics is still lower than that of male students. In regard to teaching mathematics to girls, suggested learning experiences include strategies that utilize girls' strengths and interests. This includes setting mathematics in relevant social contexts, involving students in co-operative learning tasks and in extended investigations.

The vision for education in New Zealand, as described in the New Zealand Curriculum *(draft)* (2006), is that our young people will be confident, connected, lifelong learners, and actively involved in a range of contexts. The aim is to build an education system where every student is stimulated to learn. As the proposed curriculum will allow greater flexibility for teachers to develop new and innovative teaching approaches, it is expected that all students will have the opportunity to engage in rich and authentic learning experiences. There are five key competencies identified in the curriculum draft. These are: managing self; relating to others; participating and contributing; thinking; using language, symbols, and texts. In terms of recommended effective pedagogy, the draft suggests: encourage reflective thought and action; make connections; provide multiple opportunities to learn; facilitate shared learning; enhance the relevance of new learning; create a supportive learning environment. The extension programme, offered to the Year 10 extension class of this research, was developed in the spirit of these curriculum guidelines.

1.2 The Gifted and Talented

Within the education literature the two terms, "gifted and talented", tend to be used together to describe a certain group of students, who share some characteristics. Both concepts of 'gifted' and 'talented refer to human abilities, both are normative and target individuals who differ from the norm or average, both refer to individuals who are 'above' the norm because of their outstanding behaviours.
However, some researchers in the literature offer clear distinctions for the terms gifted and talented. Gagne's (1997) Differential Model of Giftedness and Talent (DMGT) can be used to define the distinction between the two concepts:

- ***Giftedness*** defines the possession and use of untrained and spontaneously expressed natural abilities (called aptitudes or gifts), in at least one ability domain, to a degree that places an individual at least among the top 10 percent of age peers.
- ***Talent*** defines the superior mastery of systematically developed abilities (or skills) and knowledge in at least one field of human activity to a degree that places an individual at least among the top 10 percent of age peers.

Accordingly, 'giftedness', as natural abilities, can be observed through the various tasks that confront children in the course of their development. High aptitudes or giftedness can be observed more easily in young children because environmental influences and systematic learning have only played a limited part in their development. High aptitudes or giftedness can manifest themselves in older children and in adults as well, through the facility and speed these individuals acquire new skills in any field of human activity. The easier and faster the learning process, the greater the natural ability.

Talents result from the transformation of high aptitudes into well trained or systematically developed skills particular to a special field of human activity. Measuring talent, in a school situation, is relatively simple. It corresponds to outstanding performance in the specific skills of any occupational field. During the developmental phase of any talent there are many opportunities for normative assessments: school exams, tests, competitions, scholarship. When normative assessments are not available, in an out-of-school situation, the assessment of talent mostly relies on peers' or superiors' ratings, therefore their validity can be questionable. As talent is a developmental construct, the level of achievement can change as learning progresses.

In a school situation, students, who are identified as academically talented, usually obtain grades within the top 10 percent in their class in their first years at school. It is possible that for some reason their progress may slow down and they are removed from the talented group. The reverse is also possible. However, because of high correlation between yearly achievements, most talented students maintain their label during their formal schooling.

The concept of giftedness has increasingly become broader in scope from the beginning of the century to today, reflecting the diversity of abilities in society (McAlpine, 1996). In the identification of gifted students the transition from a single category concept (intelligence quotient) to a multi-category is a significant change. This multi-category approach toward the identification of giftedness is illustrated by works of educationalists, such as Renzulli and Gardner. Renzulli's Three Ring model (1986) identified above average ability, task commitment, and creativity as the three components that are necessary and sufficient to demonstrate giftedness.

Within Renzulli's model above general ability is characterised by high levels of abstract thinking, verbal and numerical reasoning, spatial relations, memory and word fluency. Specific ability requires the application of general abilities to specialized areas of knowledge and skills. Task commitment is one way of conceptualising motivation. It describes an ability for sustained motivation, and dedication that leads to excellence in the development of ideas and products. Creativity describes fluency, flexibility and originality of thought, and the ability to produce novel and effective solutions to problems and to create clever and unique products.

The multi-category approach is extended further in Gardner's (1983) Theory of Multiple Intelligences. Gardner identified nine intelligences, challenging the traditional view that emphasised scholastic intelligence alone in assessing giftedness.

These nine intelligences are:

1. Linguistic (mastery, sensitivity, desire to explore, and love of spoken words, spoken and written languages);
2. Logical-Mathematical (confront, logically analyse, assess and empirically investigate objects, abstractions and problems, discern relations and underlying principles, carry out mathematical operations, handle long chains of reasoning);
3. Musical (skill in producing, composing, performing, listening, discerning and sensitivity to the components of music and sound);
4. Spatial (accurately perceive, recognise, manipulate, modify and transform shape, form and pattern);
5. Bodily- Kinaesthetic (orchestrate and control body motions and handle objects skilfully, to perform tasks or fashion products);
6. Interpersonal (sensitive to, can accurately assess and understand others' actions, motivations, moods, feelings, and others' mental states and act productively on the basis of that knowledge);
7. Intrapersonal (ability to accurately assess, understand and recognise one's own motivations, feelings, and act productively on the basis of that knowledge);
8. Naturalistic (expertise in recognition and classification of natural objects, i.e., flora and fauna, or artefacts, i.e., cars, coins or stamps);
9. Existential (capturing and pondering the fundamental questions of existence, an interest and concern with 'ultimate' issues).

According to Gardner's Multiple Intelligences Theory, each type of intelligence is a relatively autonomous intellectual potential capable of functioning independently of others. However, it is not suggested that normally functioning individuals demonstrate intelligences that work completely independently of one another, in fact intelligences, in most cases, work in harmony with one another. What differs among individuals is their profile of intelligences at any given time. Individuals exhibit a more jagged profile of intelligences, indicating various strength and weaknesses. It cannot be assumed that an individual who displays exceptional linguistic and logical-mathematical skills will also demonstrate exceptional ability or even interest in, for example, kinaesthetic intelligence.

While acknowledging the importance of other essential factors in the identification of gifted and talented students, Fraser (1996) argued that the key characteristic is creativity. Gardner (1993) supported this notion, by stating that creativity can occur in any of the nine domains of intelligences, and that creativity represents the highest level of functioning in each domain.

Schiever and Maker (2003) offer a more academically oriented approach to giftedness and point toward some specific requirements of the mathematically gifted. They believed that the key concept in giftedness is the ability (as well as interest and willingness) to solve complex problems. Gifted people solve these complex problems in the most efficient, effective, ethical, elegant or economical way. Schiever and Maker identified a further key element: the enjoyment of challenges and complexity.

<u>Identification of Mathematically Gifted</u>

While the general characteristics of academically gifted students also apply to the mathematically gifted, Krutetskii (1976) and Greenes (1981) specify essential abilities that characterise mathematical thinking. These include the ability to generalize, to eliminate intermediate steps, to transfer ideas, to think flexibly, the desire to search for alternative solutions, the tendency to deal in the abstract, and to view the world mathematically, and the originality of interpretation.

However, the 'mathematically gifted' term by no means describes a homogenous group of students. Chang (1984) argued that essentially three different groups of students are identified here:

1) Students who learn the prescribed content well, perform accurately, but have difficulty when taught at a faster pace or at a deeper conceptual level.
2) Students who can learn more content and at a deeper level, who can reason well, and are able to solve more complex problems than their average peers.
3) Students who are highly talented, capable of performing well with little or no formal instruction at a higher level than their peers, and able to work expertly on complex, difficult mathematical problems.

Within school settings the identification of mathematically gifted students does not follow a well-defined path. From data analysis of a nation-wide survey, Winsley (2000) concluded that a consistent approach, toward the identification of the mathematically gifted students was lacking in New Zealand. Identification was largely left to individual teachers and to concerned parents. At the time of the study, there were no definitive tests available, however, several tests were recommended in official ministry guidelines.

In a recent nation-wide research study of identification of and provisions for gifted and talented students Bicknell and Riley (2005) found that the most often used methods for identification were teacher observation (94.1%) and achievement tests (89.7%). While teacher observation is a strongly supported method, Bicknell and Riley raised concerns regarding the effectiveness of this method, on the grounds of possible teacher bias, stereotyping, lack of sufficient knowledge about the mathematically gifted and the possibility of teachers focusing on a narrow set of skills only. Access is another dimension of the identification process for these gifted and talented students, as the selection process into programmes is strongly influenced by the provisions available at the particular school.

1.3 Research objectives

As Riley and Bicknell (2005) argued, there is no 'one-size-fits-all' solution to provisions for gifted and talented students. Gifted and talented students display differences in their intellectual, social, emotional or cultural profile. Therefore flexibility and variety is needed in the ways the schools meet the needs of the individual gifted and talented students. The authors warn, however, that flexibility could result in inconsistent and scattered approaches, so schools need to decide (based on their school culture) what strategies to provide and how to implement these in each classroom. Riley and Bicknell recommended further research, both, quantitative and qualitative to be undertaken in order to examine the cognitive and affective effects of strategies upon gifted and talented students; the ease and difficulty involved in the implementation of the strategies; the impact of using these strategies on teaching and on the perception of giftedness; student, teacher and parent perceptions on the usefulness, appropriateness and enjoyment of these strategies. As part of this research agenda, this study seeks to examine one niche programme in Year 10 mathematics that involves deliberate integration of the history of mathematics.

The objective of this research was to explore students' perceptions of the mathematics extension programme they participated in during their Year 10 study. Specific research questions sought students' perceptions on the differences between the extension class and the mainstream class; on changes, due to their participation in the extension programme, in their view of mathematics and mathematics learning; on benefits gained from their participation in the programme; and on the bearing this extension programme had on their future plans.

The research setting

The location of this research was a large, urban secondary school for girls. The students, the subjects of this research, were members of the Year 10 extension class, the history option class. I, the researcher, was also the mathematics teacher of these students. I have been teaching the Year 10 mathematics extension class for a number of years. The composition of these classes varied from time to time, due to the variation of the selection method. During the year of this research, students were selected into this class based on their overall academic ability, and according to their interest, they all choose the history option. The academic achievement score was calculated as the average percentage score, at the end of Year 9, of the four core subjects: English, mathematics, science and social studies. Two extension classes were created: a history option and a geography option class. Students of these extension classes were mathematically able, but not all of them were mathematically gifted and talented.

The selection process influenced the type of extension programme I put together for these students. This combination, of high academic ability and interest in history, presented an opportunity for me to create an extension programme that integrated the history of mathematics into the Year 10 teaching programme. My goal was to cater for the needs of all my students and, while ensuring they achieved excellent grades, also to build on their common interest: history, allowing them to gain a greater and deeper understanding and appreciation of mathematics.

1.4 Outline of thesis

Chapter 2 reviews the literature from both an international and a New Zealand perspective and provides a theoretical background to the thesis. It describes the two main forms of provision for the gifted and talented: enrichment and acceleration. Well-known enrichment models, used in secondary schools, are discussed, and the issue of homogeneous grouping is addressed. As this research is built on the students' perspective, a review of literature is included on the value of student voice. The chapter continues with summarizing relevant research findings on the role of motivation and interest within teaching. Theoretical and practical aspects of curriculum integration follow, with specific reference to curriculum integration in mathematics. The enrichment based extension programme, used with Year 10 participants of this research, was designed to incorporate students' interest of history. A discussion on the key elements of integrating the history of mathematics in mathematics teaching and methods of integrating the history of mathematics in mathematics teaching conclude this chapter.

Chapter 3 outlines research methodology used for this research. Data gathering techniques are presented and justification is provided for the choice of particular techniques. A discussion is included on quality criteria such as reliability, validity, generalizability, triangulation, and ethical issues.

Chapter 4 describes the research process along with participants and settings of the research. The mathematics extension programme, used with the Year 10 extension class, is outlined, and data processing techniques are identified.

Chapter 5 provides details on research findings, based on students' responses to questionnaires and focus group interview discussions. Students' responses to questionnaires and focus group interview discussions were collated into broad categories and analyzed. Students' perceptions to the following aspects form the major outcome of this research:

:

(a) Historical focus of the mathematics extension programme: on including the history of mathematics in the teaching of mathematics and on integrating humanities and sciences;
(b) Learning in the mathematics extension class: on pace of learning, on academic focus, and on social and emotional aspects;
(c) Long-term outcomes.

Chapter 6 discusses the results with emphasis on the major findings in the areas of: classroom situation; historical focus of the mathematics extension programme; learning in the mathematics extension class; long-term outcomes. Recommendations are made regarding further research opportunities.

Concluding thoughts, the final section, closes the study with my personal reflections as a teacher, regarding programme design, outcomes, and implications for the future.

CHAPTER TWO: Literature Review

Introduction

The chapter begins with literature-based review of provisions for the gifted and talented in schools. As the research setting is an enrichment based extension programme, four enrichment models are described that were incorporated in the programme design.

Motivation, motivating factors and interest are central aspects of learning. Through an overview of literature, general aspects of motivation and interest are discussed. As integrated learning enhances student achievement, following the review of literature on motivating factors influencing learning, is a review of research on general aspects of curriculum integration. This section is followed by discussing students' benefits relating to curriculum integration. Through a review of literature recommended approaches are included on how to achieve integration in learning, in particular, in mathematics.

Incorporating the history of mathematics in the teaching of mathematics is the central theme of this research. History was the uniting aspect that connected all students in the mathematics extension class. This interest was the fundamental stimulus for the design of the extension programme. Through an overview of literature, the rationale supporting the inclusion of the history of mathematics in the mathematics teaching programme is identified, followed by a discussion on the methods of integrating the history of mathematics.

2.1 Provisions for the Gifted and Talented

When designing a differentiated curriculum, educators need to select a suitable curriculum model. They need to decide whether a model should be used as stand-alone or in conjunction with others, and select the model by considering appropriateness to the situation, comprehensiveness, flexibility and adaptability, practicality, and validity (Maker & Nielson 1995). There are essentially two approaches available in a school situation to meet the needs of the gifted and talented (Townsend, 1996). These can be used exclusively or in combination:

1) enrichment; and
2) acceleration.

Enrichment is a term that is used to refer to curriculum as well as programme delivery services. Enriched curriculum refers to richer, more varied educational experience that provides the challenge and offers growth in the area of students' special abilities. There are a number of enrichment models appropriate for secondary school students. The four enrichment models, selected for the literature review, are the ones that provided a basis for the instructional programme design used in this research.

1) The most well-known curriculum enrichment model is Renzulli's (1977) Enrichment Triad Model. This model aims to encourage the creative productivity of students by exposing them to various topics, areas of interest, and fields of study; train them to apply advanced content, process-training skills, and methodology training to self-selected areas of interest. Specifically, the Enrichment Triad Model includes three types of enrichment:

1) *Type I enrichment* is designed to expose students to a wide variety of disciplines, topics, occupations, hobbies, persons, places, and events that normally would not be covered in the regular curriculum.
2) *Type II enrichment* consists of materials and methods designed to promote the development of thinking and feeling processes. This type of enrichment programme includes the development of creative thinking and problem solving, critical thinking and affective processes, wide variety of specific learning-how-to-learn skills, skills in the appropriate use of advanced level reference materials, written, oral and visual communication skills. It can also involve advanced instruction in an interest area selected by the student.
3) *Type III enrichment* occurs when students become interested in pursuing a self-selected area and are willing to commit time necessary for advanced content acquisition and process training, taking on the role of a first-hand inquirer. This type of enrichment provides opportunities for applying interests, knowledge, creative ideas, and task commitment to a self-selected problem or

area of study. It allows students to acquire advanced-level understanding of the knowledge (content) and methodology (process) that are used within particular disciplines, artistic areas of expression, and interdisciplinary studies. The development of authentic products is primarily directed toward bringing about a desired impact on a specified audience. This type of enrichment aims at developing self-directed learning skills in the areas of planning, organisation, resource utilisation, time management, decision making and self-evaluation, task commitment, self-confidence, and feelings of creative accomplishment.

2) In another enrichment model, the Purdue Secondary Model for Gifted and Talented Youth (Feldhusen & Robinson, 1986), attention is given to talent in intellectual and academic areas as well as in the arts and vocational areas. There are an extensive range of options available to students to meet their individual needs and characteristics. The model is both functional (offering counselling, vocational programmes, cultural experiences) and deliverable (through extra-school instruction, seminars). Students with special abilities receive growth plans, as their tools to establish simple links between their needs and the appropriate services.

3) The enrichment model, Treffinger's Model for Increasing Self-Directedness (Treffinger, 1975), uses a four-step model:

1) First step: command style, involves teacher-directed curriculum decisions and instructions.
2) Second step: task style, the teacher creates learning options for the students to choose from.
3) Third step: the teacher involves students in the option-creating process, so students begin to take more responsibility regarding their learning experiences.
4) Fourth step: self-directed style, students have the chance to select and complete an activity of their choice.

4) Another model for enrichment is the Autonomous Learner Model (Betts, 1985). This model provides a variety of learning options. It aims at students developing positive self-concepts and social skills; increasing their knowledge; understanding their giftedness; and becoming responsible for their learning.

This model is divided into five major dimensions:

1) Orientation (students, their teachers and parents are provided experiences, activities, discussions that help them learn more about the concept of giftedness, group-building, self-understanding).
2) Individual development (students learn skills, concepts, and attitudes for life-long autonomous learning).
3) Enrichment activities (students decide what they want to study through explorations, investigations, cultural activities, trips, and they become more aware of what is out there and find their passions).
4) Seminars (in small groups students produce ideas and topics to research, carry out the research, then plan and implement it, present information, participate in group discussion).
5) In-depth study (involves the development of an in-depth study contract, which outlines the study, states objectives and activities, questions, time line, list of human and material resources, and a plan for on-going and final presentations).

Every subsequent level builds on the learning experiences, skills, and the increasing level of independence and self-directedness gained in the previous level.

There are some concepts, ideas that are common to all above-described models and each model has a strong potential to provide valuable enrichment opportunities to secondary students. However, there is a level of concern, relating to the time and effort required in implementing these models. Some of the models require the reorganization of the whole school, some require special training for staff, while others rely on extra personnel.

Acceleration is the other basic approach available in a school situation to meet the needs of the gifted and talented. Acceleration, as a curriculum model, involves speeding up the pace the material is presented at and is expected to be mastered. While often used in our schools, acceleration is criticised as being responsible for social maladjustment or for creating skill gaps in core areas (Holton & Daniel, 1996).

In practice enrichment and acceleration are often combined in order to meet the needs of gifted students. In their area of giftedness students need to be taught abstract and complex concepts (enrichment) and they need to proceed at a more rapid pace than the average learner (acceleration). Townsend (1996) argued that there is a clear need to integrate enrichment and acceleration in a seamless fashion.

Both, enrichment and acceleration, can take place in various forms of student grouping arrangements.

Homogeneous grouping involves putting same-ability students together, so teachers can target the appropriate level of instruction needed for student learning. It is suggested that while most students may learn best in a mixed ability situation, students with special abilities learn best when they are working together (Cathcart, 1996). Other educators also support homogenous groupings, arguing that gifted and talented students who need the stimulation and challenge that come from advanced, enriched instruction, highly knowledgeable teachers and equally talented peers, are best supported in groupings on a subject specific basis, and not on a general tracking plan (Feldhusen, 1997). However, the author added that some youth are multi-talented and for them the entire programme should be advanced in all subjects, and they should be in the company of other talented students.

Homogeneous grouping is advantageous to mathematical achievement (Bulgar & Tarlow, 1999; Hersberger, 1995). One of the reasons while mathematics teachers tend to support ability grouping is due to the hierarchical nature of the discipline, where concepts build on previous concepts (Ruthven, 1987).

<u>The New Zealand situation</u>

In New Zealand, a model for establishing programmes for gifted and talented students was designed by Rosemary Cathcart. This model has four key concepts: generating a high level of interest, developing the 'tools of thought', developing intellectual and creative potential, and fostering emotional, social, and ethical development (Cathcart, 1994). Cathcart (1996) added that a high quality programme, offered to students with high abilities, should include provision for genuine challenge, for choice and decision-making; recognition and respect for prior knowledge; opportunity for creative responses and to pursue individual interests, to engage in higher order thinking; exposure to creative approaches; autonomy and independence; and the teaching of independent learning skills.

Bicknell and Riley (2005), in their research on provisions and identification of gifted and talented students in New Zealand, found that the majority of schools reported a preference for a combination of acceleration and enrichment approaches. All New Zealand schools were invited to participate in this research and there was a 48% response rate, of 1285 schools nationwide. Enrichment was preferred by the schools not opting for a combined approach. Classroom-based provisions were implemented by the majority of schools. These included ability grouping, independent study, teacher planning, learning centres, individual education plans, curriculum compacting (acceleration), and the use of a consulting specialist teacher. The most preferred option was ability grouping. Bicknell and Riley emphasised that the curriculum should be rich in depth and should proceed at a pace commensurate with students' abilities. It should also address social and emotional needs as well as meeting academic needs.

Provisions for the mathematically gifted and talented

Provisions for the mathematically gifted and talented, in terms of pedagogical concepts, are similar to the provisions generally identified for students gifted and talented in different areas. However, there are specific concepts, which relate to the mathematically gifted and talented. Stanley (1980) emphasised creativity in the mathematics programme appropriate for gifted learners. Stanley's goals include the provision of appropriate context for students to learn as much as possible about mathematical concepts, ideas and skills, and to enable them to appreciate the beauty of mathematics; and the preparation for creative and independent thinking.

In designing programmes for mathematics, Johnson (1994) recommended a combination approach by using three curriculum models: the content model, the process/product model, and the concept model.

The Content Model aims at maximising learning about mathematical concepts, skills and ideas. High-ability learners can be accelerated (linear shift) or the content of a course can be modified through enrichment. Enrichment can include additional topics or in-depth study of course concepts based on curriculum materials designed to enhance the learning experience. Effective materials should avoid the emphasis on rote learning and memory recall questions, and should emphasise higher-order thinking skills, such as analysis, synthesis, and evaluation.

The Process / Product Model aims at promoting creativity and independent thinking. This model emphasises problem finding and solving, and product development. It also helps the development of organisational and research skills. This model is well-suited for implementing enrichment topics that supplement the core content. Students can work in small groups, individually or under the direction of a mentor. Project ideas may include, for example: researching the history of a mathematical idea, learning to make Escher-type drawings, computer modelling of a student generated problem, analysing and inventing mathematical games and puzzles, studying conic section applications in astronomy, to name a few.

The Concept Model aims at encouraging the appreciation of mathematics. This model uses themes or key ideas to organise areas of study. The emphasis is on abstract concepts and modes of thinking. The interdisciplinary orientation helps students appreciate the workings of mathematical ideas. Some themes or key ideas may include proof (can be interdisciplinary: algebra, geometry, number theory, topology, calculus, or can include the history of famous theorems); patterns in nature (spirals in shells and pine cones, planetary motion, flower petal arrangements, weather patterns); space (tessellations, Archimedean solids); mathematical models; systems (ancient number systems, number systems with bases other than ten); and problem solving.

Johnson (1994) argued that a key element included in the mathematics curriculum for the mathematically gifted is interdisciplinary connections. Practices which are more commonly observed in science need to be used in mathematics. Connections can be made through the process of observing, forming a hypothesis, or experimentation. These are common practice in science and this can be done in relation to mathematical ideas.

While science is an obvious motivator to the development of mathematical ideas, other disciplines also offer rich connections to mathematics. Another opportunity for forging connections is through the inclusion of historical perspectives on the development of ideas over time and geography, and the usage of biographies of famous mathematicians and scientists. These can serve as role models to students with those particular talents. Stories from the history of mathematics can be inspirational and can be appreciated especially by those who have interest in and understanding of mathematical ideas. The applications of mathematics to other disciplines could include exploring geometric ideas in art (tessellations, golden proportion, symmetry, proportional drawing). In social studies the rates of change in populations can be graphed and studied and predictions can be made about the future. A strong connection is formed between language and mathematics by the need to communicate mathematical ideas precisely and clearly, and to use correct mathematical terminology. Oral and written communication skills are essential in all learning areas.

In the more recent study, Bicknell and Riley (2005) identified three crucial aspects of providing suitable programmes for the mathematically gifted and talented. These are:

- The curriculum should be rich in depth and breadth;
- Should proceed at a pace commensurate with students' abilities; and
- Should address social and emotional needs.

International literature draws attention to the issue of underachievement among mathematically gifted girls. When planning suitable programmes for gifted and talented girls, as was the case in this research, the importance of identification and provision for the gifted and talented is significant. Despite a dramatic increase in the percentage of women involved in science, the continued under-representation of women in most fields of science is a proof of under-achievement amongst gifted women (Davis & Rimm, 1998; Kelly, 1993). Kerr (1994) claimed that girls continue to have less confidence in their mathematical abilities than boys, and see less relevance of mathematics to their own lives. Fox and Tobin (1988) suggested the provision of programmes that encourage the development of positive self-concept and attitude towards mathematics, and increase awareness and information on careers involving mathematics.

2.2 Motivation

Maehr and Midgley (1991) argued that the best laid plans, aiming at improving education, will fail unless the motivational states of students are taken into consideration in the design of instruction. Ability and motivation are two main factors that determine students' willingness to seek challenges, to persist when encountering difficulty, and to use their skills effectively (Dweck, 1986).

Motivation can be extrinsic or intrinsic (Henley, 2006), or a combination of both. Proponents of extrinsic motivation claim that individual motivation is a response to environmental stimuli that reinforce behaviour (Canter & Canter, 1992; Skinner, 1978). Examples of extrinsic motivation include the use of tangible incentives, such as stickers, grades, food, and social incentives, such as praise, positive feedback. These can be given to students as reward for good work, however, extrinsic rewards may undermine intrinsic motivation.

Developmentalists argue that the intrinsic needs of an individual are central to motivation (Hall, Lindsey & Cambell, 1998; Maslow, 1971), and educationalists need to take advantage of those incentives that occur naturally. Interest, success, and enjoyment motivate students to strive, and as these are intrinsic reinforcement factors, they result in a long-lasting impact (Henley, 2006). Some of the best practices of intrinsic motivation involve the use of fun activities, increase student decision making, the usage of meaningful and relevant activities, and the provision of activities for socialization and affiliation.

Social learning theories incorporate ideas from extrinsic and intrinsic theories, and introduce the element of individual perception into the concept of motivation (Bandura, 1977; Kelly, 1955). There is a reciprocal exchange between the environment and the individual, which allows people to control their environment and also to be shaped by their environment. For example, positive learning experiences may build self-confidence and negative learning experiences may undermine self-efficacy and the will to persevere. According to Pajares (2000) students are motivated to participate in tasks they can succeed in and tend to avoid the tasks that make them feel incompetent.

Murphy and Alexander (2006) strongly emphasize the goal-directed characteristic of motivation. For students, academic goals include mastery and performance goals. Mastery goals relate to increased knowledge and understanding, while performance goals correspond to the desire to do well in a task, achieve recognition and avoid failure. According to Murphy and Alexander, students' academic goals can be significantly influenced by social factors within the school setting. Students are generally keen to understand and comply with the social codes of conduct in the classroom.

Students' goals in academic situations are related to the way they think about their intelligence. If according to students' view, intelligence is fixed (entity theory) their goal orientation is performance related. If they have confidence in their present ability, their behaviour pattern is mastery-oriented, they seek challenge and display high persistence. If their confidence in their present ability is low, their behaviour pattern displays helplessness in the form of challenge avoidance and low persistence. If according to their view, intelligence is malleable (incremental theory) their learning goal is to increase competence. Their confidence in their present ability may be high or low, but their behaviour pattern is mastery-oriented, they seek challenge that fosters learning and display high persistence.

Students are not typically concerned with identifying factors that are motivating to them, therefore teachers need to use appropriate strategies to uncover their students' beliefs, goals, perceptions and their values (Pintrich, 2003). By matching the curriculum to students' interests, both in and out of school, teachers can significantly increase students' motivation in the classroom. Through a review of literature, different approaches to increasing student motivation were recommended by the following educationalists:

1) *Challenge students' minds in meaningful ways.*

Students' time should not be wasted by demeaning and worthless tasks. They should be engaged in suitably challenging activities that must carry personal value (Csikszentmihalyi, 1990).

2) *Acknowledge students' efforts and accomplishments appropriately.*

Students must feel that their individual characteristics and abilities are recognized and appreciated through tasks assessed on jointly established criteria or not graded at all

(Eccles, Wigfield, & Schiefele, 1998). Students' personal progress can be acknowledged even if they do not reach exceptional levels on the normative scale.

3) *Maintain high expectations for students.*

Teachers' expectations influence student motivation and achievement (Eggen & Kauchak, 2006). As expectations tend to be self-fulfilling, when teachers believe their students can learn and they promote learning, over time their students' motivation level will increase.

4) *Use feedback that highlights student effort and control.*

Effective feedback should offer clear information about students' competence in specific tasks, in order to enhance students' performance (Kohn, 1993). Constructive feedback allows students to feel in greater control of their learning and be motivated to work hard.

5) *Conceptualize motivation as a continuous, multifaceted, and developing process.*

Students are not equally motivated in all academic domains, nor are they equally motivated with all school tasks (Pintrich & Schunk, 2001). This fluctuation is inevitable, and by using the strengths and interests of all students in the class, there is opportunity to promote variety and excitement, resulting in increased student motivation.

As discussed by educationalists, motivation is one of the key factors determining student achievement. Lack of motivation can result in underachievement, and can affect all students, regardless of ability. The source of underachievement among gifted students is likely to be either environmental or psychological (McNabb, 1997). In certain situations, family problems and / or the school environment may not allow the full expression of giftedness. In a school situation exceptional performance may not be encouraged or may be even discouraged through lack of suitable courses or negative classroom interactions. As the result of certain psychological factors, student may exhibit academic behaviours, such as low effort, challenge avoidance, unreasonably low or unreasonably high self-expectations, low persistence at difficult tasks, or lack of joy in learning. The three main factors of intrinsic motivation are: challenge seeking, persistence, and task commitment. These factors are considered adaptive academic behaviours, and their opposites: challenge avoidance, giving up, and lack of enjoyment are considered maladaptive academic behaviours. Motivation should be looked upon as being situation-specific, students are neither motivated nor unmotivated as a trait, although they may not be motivated to meet the teacher's or the school's academic goals, they are motivated to meet their own psychological needs. Adaptive achievement behaviours and maladaptive achievement behaviours have more to do with students' classroom goals than with differences in their academic abilities (Dweck, 1986).

Bandura (2000) argued that by identifying students' strengths and interests, there is a way to accentuate their ability and build mastery. Using other people's examples can inspire effort. If the observer is able to identify with the model, there is a positive effect on self-efficacy. Verbal encouragement, coupled with successful experiences, has a positive effect on student achievement.

Student interest is also found to be another important characteristic of motivation (Murphy & Alexander, 2000). In creating learning environments that spark motivation, while incorporating students' interests in their learning, is a proven way of increasing their attention and motivation.

2.3 Interest

Interest, identified as a subset of motivation (Murphy & Alexander, 2000), is frequently defined in different ways within literature. Athanashou (1998) described interest as an actualised state, featuring emotional components, such as happiness, effort, enthusiasm, enjoyment and desire. Hidi (2000) described interest as a psychological state, featuring focused attention, increased cognitive functioning, persistence and affective involvement. In this thesis I adopt a combination of Athanasou's and Hidi's definition, of the actualised and psychological state, where the emotional components, listed above, are desired factors in creating an atmosphere conducive to effective learning.

Interest can also be discussed from a personal and from a situational context. Deci (1992) highlighted the relationship between individual and situational interest, by stating: "To design interesting tasks, one must understand the characteristics of the people who will engage in them; and to assess people's interest, one must do so with respect to activities" (p.47).

Researchers (Hidi, 1990; Mayer, 1998; Schiefele, 1991; Shirey & Reynolds, 1988) discussed the influence of interest on student performance. They argued that students' interest level affects their selection and processing of information, the degree of persistence at the task, the level of retention, and their problem solving transfer abilities. Students who are interested in the particular topic are more likely to use deep-level learning strategies and have a positive emotional experience. Their orientation towards the topic is more likely to be learning based, rather than performance based (Schiefele, 1991).

2.4 Curriculum integration

In education, integration means the simultaneous consideration of different aspects of knowledge (Usiskin, 2003). Huber and Hutchings (2004) claimed that one of the greater challenges of higher education is to foster students' abilities to integrate their learning over time. By developing the integrative capacity of their mind they are better prepared to make informed decisions in their personal, professional and civic life. The difficulty can be found in the traditional structures of academic life that encourage students to see their courses as isolated entities.

Increasingly, integrated curriculums are promoted as one way of ensuring an inclusive classroom (Drake, 1998). When the curriculum is relevant it is more motivating to all students and is more likely to ensure active participation by all students.

Integrated curriculum can be defined, as many educationalists believe, as a continuum along which progressively more and more connections are made (Drake, 1998). This continuum begins with the traditional method and moves through stages, culminating in the transdisciplinary approach, as seen below:

Traditional – Fusion – Within one subject – Multidisciplinary – Interdisciplinary - Transdisciplinary

The traditional teaching method teaches the material through one discipline. Fusion allows the topic to be inserted into several subject areas. Subdisciplines can be integrated within one subject. The multidisciplinary approach allows the disciplines to be connected through a theme or issue that is studied in the same timeframe. The interdisciplinary curriculum allows the subjects to be interconnected via a common theme or issue. In the transdisciplinary approach the planning begins from a real-life context. The disciplines are embedded in the learning, and while cross-disciplinary outcomes are expected, there is a strong emphasis on personal growth and social responsibility.

An integrated approach toward learning is regarded as beneficial to students. Hargraves, Earl, and Ryan (1996) argued that the traditional subject-based curriculum does not result in optimum student engagement in learning, as some of the material taught may seem irrelevant and provides little challenge. Teachers, when teaching different subjects, may present knowledge in a fragmented way. In contrast, interdisciplinary curriculum is associated with real-life problems, which do not naturally occur as fragmented subject areas. A real-life context gives students a reason to learn (Drake, 1998). In terms of academic benefits, Vars (1995, 1996), based on his review of more than 100 studies that took place between 1956 and 1995, concluded that students in integrated programmes do as well as, and often better than students in conventional programmes. The interdisciplinary programmes also enhance learning for the gifted and talented (Clark, 1986), as it allows them to participate in challenging, real-life problem solving.

An integrated curriculum approach that incorporates connections, drawn between content areas, helps to develop greater understanding of concepts across disciplines (Still & Bobis, 2005). Boyer (1990) and Suedfeld, Tetlock, and Streufert (1992) concluded that classroom activities, whether focused on discovery and creativity, integrating and interpreting knowledge from different disciplines, applying knowledge through real world engagements, or communicating with the public, all require taking account of different dimensions of a problem, looking at it from different perspectives and making conceptual links between them. In their research, on the effect of integration of mathematics and music in the primary school classroom, Still and Bobis (2005) found that this method had a positive impact on the students. The students demonstrated positive engagement with the stories, relating to 'time, mathematics, and music' and solving mathematical problems derived from the stories.

In addition to intellectual appeal, integrated learning has emotional appeal. In Still and Bobis's research (2005) the researchers observed that the excitement about the characters in the stories flowed into excitement about solving mathematical problems.

Curriculum integration promotes intuitive thinking (Drake, 1998). It is recognized that there may be more than one right answer, and there is a possibility that the different solution methods cross disciplinary boundaries. Intuitive thinking provides room for creativity and encourages students to move outside the boundaries.

Research evidence, based on a five year study that collected feedback from over 700 grade six, seven and eight students, identified that students enjoyed learning, found the topics interesting, learned from working in groups, and expressed high quality in their work (Davis, 1992). The successful interdisciplinary units, used in the above research, included relevant topics, clear goals and objectives, variety in activities and groupings, choice, adequate time, development of products, skill development woven into topics, organized group work, field trips, sharing results with others, and community involvement.

Research has also noted that there are numerous non-academic benefits for students in integrated programmes: improvement in inter-racial relationships and enthusiasm for school and teachers (Arhar, Johnston, & Markle, 1989); increased student self-confidence, cooperation, mutual respect (Budzinsky, 1995); and students enjoying learning and discovering that learning is fun (Cole, 1994).

2.4.1 Achieving integration in learning

Dynamical knowledge is the result of learners constructing their own meanings through their own experiences (Caine & Caine, 1997). They argued that learning must go beyond surface knowledge to become dynamical or perceptual knowledge. This is created by generating deep meaning, through connecting to the values and goals of the students, and by establishing emotional connection.

Educators can help students to develop a capacity to make connections themselves. The development of 'intentional learning' is the key to integrative learning (Bereiter

& Scardamalia, 1989). Intentional learners have a sense of purpose. They feel the need to connect fragmentary learning experiences, they understand their own processes and goals as learners, and they make choices that promote connections and depth of understanding. They make the most of their study time, practise new skills and ask probing questions. They see learning as a goal, not as an incidental outcome. While teaching students how to learn, educators need to provide clear guidelines that identify to students how to be a better student, how to conduct inquiry and construct knowledge in certain disciplines or fields, and how to be a self-directing learner (Fink, 2003).

Intentional learners are strategic in their approach to learning. They use metacognitive knowledge about themselves as learners; about different types of academic tasks; about strategies and methods for acquiring, integrating, thinking about, and using new knowledge; about how prior content knowledge can be applied, and about knowledge of present and future contexts in which new information could be useful (Weinstein, 1996). Reflection is another key component of intentional learning. Schon (1983) claims that through reflection the relationship between thought and action is highlighted, as doing and thinking are complementary.

There are a variety of approaches that support a more scholarly, intentional approach to classroom work, and integrating the history of mathematics into the teaching of mathematics is a good opportunity to form connections within the discipline and across disciplines.

Integrative assessment, like teaching and learning, is a complex process. Integrative assessment is a fairly underdeveloped area of assessment and raises a number of issues, such as what kinds of connections are to be assessed, in what contexts, and in what ways could this be demonstrated. Integrative assessment implies more focus on student self-assessment. Loacker (2002) argues that this kind of approach carries intentional learning to its logical conclusion. Student portfolios are a good way of fostering integrative abilities and can be used for assessment. Students' perceptions of the programme can provide valuable information, and identify the changes needed.

2.4.2 Curriculum integration in mathematics

The rationale for combining subject areas and / or different subjects, is based on the notion that knowledge cannot be divided into categories: the separation of learning by disciplines is artificial (Fiscella & Kimmel, 1999). An integrated mathematics programme is a holistic mathematics curriculum. Usiskin's (2003) description of curriculum integration is applicable to mathematics as well. In mathematical context, curriculum integration is generally applied to the many ways different areas of knowledge can be brought together. Using unifying concepts, merging different areas into broader areas, removing distinctions between areas of mathematics, teaching different strands each year, and interdisciplinary integration of mathematics with other subjects, are some of the ways integration can be achieved.

Based on findings from a national survey Lott and Reeves (1991) define an integrated mathematics programme as one that blends a wide variety of topics from different fields of mathematics to emphasize the connections between those fields, and emphasize the relationship between mathematics and other disciplines. It includes topics at levels appropriate to students' abilities, while providing multiple contexts for students to learn mathematical concepts. It is problem centered and application based, with a strong focus on problem solving and mathematical reasoning. It makes appropriate use of technology.

2.5 Integrating the history of mathematics in school mathematics education

Mathematics is a living subject, not merely a set of established rules and facts to be memorised and practised for educational purposes. As long ago as the beginning of the 20th century, educators have advocated the value of the history of mathematics in teaching (Barwell, 1913). Today the inclusion of history of mathematics is strongly supported by international organizations, such as the National Council of Teachers of Mathematics, the Mathematical Association of America, is discussed by international study groups and is the subject of conferences, papers and Internet-groups (Fried, 2001).

Hayes (1988) noted that it is a "grave mistake and error of strategy to attempt to teach mathematics without reference to its cultural, social, philosophical, and historical background." Likewise, Fried (2001) claimed that introducing history of mathematics into the mathematics classroom humanizes mathematics. Mathematics must be attached to real human beings (thinking, making, doing mathematics) and it must be attached to real human circumstances in social and cultural contexts. Furinghetti (2000) added another justification, arguing that the history of mathematics is a good vehicle for promoting flexibility and open-mindedness in mathematics education.

The most significant justifications for the inclusion of the history of mathematics in the teaching of mathematics are attributed to the bearing the history of mathematics has on the development of one's view of mathematics, and the fact that it provides the opportunity for a better, deeper understanding of concepts and theories (Barbin, 2000). The history of mathematics humanizes mathematics, it makes mathematics more interesting, more understandable and more approachable, and it gives insights into concepts, problems and problem-solving.

The history of mathematics forms a natural connection between mathematics and other subjects taught at school. The famous historian of mathematics, Gino Loria (1899), advocated the usage of the history of mathematics to cross disciplines. Loria promoted, through the study of the history of mathematics, flexibility in ways of approaching mathematics and approaching teaching.

While mathematics is usually taught in a deductively oriented organization, Freudenthal (1983) argued that "No mathematical idea has ever been published in the way it was discovered". The integration of the history of mathematics into mathematics education can help to uncover how "our mathematical concepts, structures, ideas have been invented as tools to organize the phenomena of the physical, social and mental world" (Freudenthal, 1983).

Through a review of literature, it is noted that educationalists identified five different reasons to incorporate the history of mathematics in classroom mathematics.

1) History is a rich resource, it can provide a wide range of relevant questions and problems. Historically inspired exercises can stimulate student interest and contribute

to curricular enhancement. Through these exercises, aspects of the historical development of the subject become a working knowledge for students and an integral part of mathematics (Tzanakis, 1997).
History is a bridge between mathematics and other subjects. It exposes the interrelations between different mathematical domains and between mathematics and other disciplines.
History of mathematics allows students to engage in historically oriented projects. This presents a wider educational value, not only mathematical development, but the development of a range of skills, such as reading, writing, locating resources, documenting, discussing, analyzing, and talking about mathematics, not just 'doing' mathematics (Ransom, Arcavi, Barbin, & Fowler, 1991).

2) Incorporating the history of mathematics in the content of mathematics and mathematical activity allows students to learn that mistakes, heuristic arguments, uncertainties, doubts, intuitive arguments, controversies, and alternative approaches to problems are legitimate and integral part of mathematics in the making (Arcavi, 1987). History highlights the evolutionary nature of mathematical knowledge. Mathematics is evolving in its form as well as in its content. Notation, terminology, computational methods, modes of expression and representations changed, developed over time. Students, through history, become aware of the advantages and disadvantages of modern forms of mathematics.

3) By studying the historical development of specific mathematical topics teachers can identify the motivations behind the introduction of new mathematical knowledge. Students become aware of the difficulties that appeared in history and may reappear in the classroom. The nature of their mathematical activity tends towards the creative process of 'doing mathematics', and their didactical repertoire is enriched with explanations, examples, and alternative approaches to presenting a subject area or to solve problems.

4) History can provide role models of human activity. It highlights the fact that mathematics is an evolving and human subject rather than a system of rigid truth. Mathematics is the result of human endeavour that requires intellectual effort. When studying the work of the most prominent mathematicians, students learn the value of

persisting with ideas, of attempting to undertake lines of inquiry, of posing questions, and of attempting to develop creative or idiosyncratic ways of thought. They learn not to get discouraged by failure, mistakes, uncertainties, or misunderstandings (Ransom, Arcavi, Barbin, & Fowler, 1991).

5) History shows how the internal development of mathematics has been influenced, sometimes determined by social and cultural factors. Mathematics, in its modern form, is mostly viewed as a product of a particular western culture. Nouet (1996) argued that through the study of history of mathematics one becomes more aware of the other, less known approaches to mathematics that appeared in other cultures. These cultural aspects play an important role in the multi-ethnic classroom, by valuing local cultural heritage, and by developing tolerance and respect among students. In addition, students learn to appreciate that mathematics is not always driven by utilitarian reasons, but also developed for its own sake (Hallez, 1990), and motivated by intellectual curiosity, aesthetic criteria, challenge, pleasure and recreational purposes (Chandrasekhar, 1987).

There are, however, some dilemmas concerning the incorporation of history of mathematics in mathematics education. If students have a limited knowledge of history, an erratic sense of the past, then the historical contextualization of mathematics may prove impossible without teaching general history (Fauvel, 1991). Many students do not like history and find it boring. For these students the progress in mathematics should be focusing on tackling more difficult problems and not looking back. For many teachers, concerns about integration are of a practical nature, such as lack of time, lack of resources, lack of expertise, and lack of assessment.

Fried (2001) outlines two strategies that schools commonly employ to overcome these dilemmas from both, philosophical and practical viewpoints. The first strategy is by introducing historical anecdotes, short biographies, isolated problems. This, he called the 'strategy of addition', as it does not alter the curriculum, except by simply enlarging it. Time is a factor to be considered when using this strategy. Suggested solutions, to overcome possible time constraint, can include homework assignments, 'problem of the week' activities that support the task worked on in class, or by replacing an ordinary problem with one referring to the same material but having a historical context (Avital, 1995, Swetz, 1995).

The second strategy actually changes how the material is presented, for example by using an historical development in the presentation of a technique or idea, or organizing the subject matter according to a historical scheme. This strategy is called the 'strategy of accommodation', as it accommodates the curriculum to historical circumstances or to an historical model.

While the problem, relating to the need for extra time, can be partially solved, one that is not so easy to solve is the problem of relevance. Integrating the history of mathematics into a mathematics curriculum, forces educators to take the role as 'editor' of history, accepting (based on the prescribed curriculum) of what is relevant and what is not.

Another dilemma that educators face is that they are committed to teaching modern mathematics, teaching the kind of mathematics students need for their later study of mathematics, science, economics, or psychology. In this context the history of mathematics becomes something that is used to justify, enhance, explain and encourage modern subjects and practices, and not something that is studied for its own right.

2.5.1 Methods of integrating the history of mathematics in mathematics education

There are three broad areas where the integration of the history of mathematics can be accomplished. These are:

1) *Learning history by the provision of direct historical information.*

   The emphasis here is more on resourcing history than on learning mathematics. There is a wide range of opportunities here, from the provision of isolated factual information, such as names, dates, time charts, biographies, famous problems and questions, to full courses or books on the history of mathematics, or a history of conceptual developments.

2) *Learning mathematical topics by following a teaching and learning approach inspired by history.*

According to Toeplitz (1963) and Edwards (1977) this approach has several advantages that motivate student learning. These are:

i. Reconstruction of examples to identify the motivation for the introduction of new concepts, theory, method, or proof.
ii. Learner and teacher take on the role of researchers.
iii. Reveals the interrelationship between different mathematical and non-mathematical domains.
iv. Opportunity to compare modern mathematics with its form in the past.
v. Interest is naturally induced by historically important and mathematically fruitful questions in order to improve students' mathematical knowledge.

3) *Developing deeper awareness of mathematics itself and the social and cultural context in which mathematics has been done.*

The learner becomes aware of the evolving nature of mathematics, in content and in form. Students learn of the role of general conceptual frameworks and of associated motivations, questions and problems that led to the developments of particular mathematical domains. An appreciation can develop regarding the role of doubts, paradoxes, contradictions, intuitions, difficulties in the context of specific questions and problems, and the motivations for generalising, abstracting and formalising in such a context (Lakatos, 1976).

History may illustrate that mathematics is not disconnected from social and cultural influences. It is closely related to philosophical questions and problems, the arts, sciences and humanities (Montesinos Sierra, 1996; Perez, 1996). Mathematics is an integral part of the cultural heritage and practices of different civilizations, nations and ethnic groups (Horng, 1996).

There is a wide range of possible practical ways the history of mathematics can be implemented in the teaching of mathematics in a classroom situation (Arcavi, 1987):

Historical snippets: photographs, books, biographies, anecdotes, dates, chronologies, architectural, artistic, or cultural designs, origins an devolution of an idea, problems of historical origin, ancient methods of calculation, different ways of noting and representing ideas as opposed to modern ones.

Research projects based on history texts: projects considering the nature and structure of mathematics with particular regard to its methods, theories and organization, in order to examine philosophical issues, historical developments, and the social role of mathematics.

Primary sources: excerpts from original mathematical documents.

Worksheets: structured around short historical extracts, accompanied by historical information to describe their context, followed by questions aimed at supporting the understanding of the contexts, then discussion of mathematical concepts involved, and comparison between the old and current mathematical approach.

Historical packages: collection of materials narrowly focused on a small topic, with strong ties to the curriculum, suitable for two or three class periods, ready for use by teachers in their classrooms.

Taking advantage of errors, alternative conceptions, change of perspective, revision of implicit assumptions, intuitive arguments

Historical problems: problems with no solution, famous problems still unsolved or solved with great difficulty, problems having clever, alternative or exemplary solutions, problems that motivated and/or anticipated the development of a whole domain, recreational problems.

Mechanical instruments: instruments used to measure time, mass.

Experiential mathematical activities: re-living arguments, notations, methods, games and other ways of doing mathematics in the past.

Plays: dramatising aspects of famous mathematicians' lives, historical events.

Film and other visual means: movies, posters, portraits of famous mathematicians, facsimiles of famous works, time charts with chronological or thematic historical developments.

Outdoors experience: identification of form and shape, patterns in nature, in architecture and in art, exploring historical outdoor instruments for navigation, surveying, visit to museums, science exhibits.

The World Wide Web: used as a resource and as a means of communication.

CHAPTER THREE: Research Design

Introduction

Within schools, the design and implementation of any innovative programme by teachers is based on a multitude of factors: the teachers' prior experience in instructional design, their reflection on what works and what does not, attention to students' prior knowledge and interests, the provision of appropriate challenge, and the curriculum objectives, to mention a few. Teachers also become adept at reflecting on the programme implementation in terms of evaluating the success or otherwise of the implementation, formatively assessing the impact on students, and using summative outcomes to inform the overall success of the programme. However, less frequently do teachers have the opportunity to seek detailed feedback from all of the students, and less frequently do teachers have the opportunity to reflect on that feedback. Recent research suggests that students can provide informed and informative feedback about programmes and instruction. Student voice is a valuable tool that allows the researcher to form a more balanced, informed view of an educational programme.

Student perspective

Successful learning requires that students use a wide variety of learning processes in a flexible fashion. However, greater understanding is needed regarding students' perspectives of learning and teaching. Brown (2002) argued that teachers and students in secondary schools continue to 'talk past each other in terms of their conceptions of learning'. Delpit (1988) suggested that the 'teacher can not be the only expert in the classroom', and Fullan (1991) observed the absence of students' voices in research about learning and schooling, and advised to treat the student as someone whose opinion mattered.

Educational research can benefit greatly from consulting students on their learning experiences. Through a review of literature, Taylor, Hawera, and Young-Loveridge (2005) concluded that students' voices are important for understanding their schooling experiences. Students can hold strong views, and often have great awareness of the social and organizational matters that affect their learning. They can clearly articulate the physical and cognitive aspects of their environment that support their learning. They can identify a variety of teaching strategies that are effective for their learning.

The authors argued that, as teaching context can influence students' learning in many ways, understanding their perception of the learning process is crucial to creating an effective learning environment. There is developing tradition to attempt to listen to what students have to say and take note of their own interpretations of their lives at school. However, students' views and experiences are sometimes absent from educational studies. Vaughan (2003) argued that in New Zealand there is a lack of research that aims to focus on the views of young people. In order to make a difference in education, it is necessary to put young people's perspectives in the centre of research and policy development. Within this context a New Zealand research project: *Making Sense of Learning at Secondary School* was developed, aiming at involving students to improve teaching practice. The two-year project was conducted in three New Zealand secondary schools in 2004 and 2005. Kane and Maw (2005), researchers involved in the project, argued that "good practice must necessarily be informed by students' needs and therefore by consultation with students".

In order to seek students' perceptions about their learning of mathematics using an integrated approach incorporating historical aspects of mathematics, qualitative research methods were the primary means of data generation. Qualitative research explores the meaning, concepts, definitions, characteristics and description of things (Berg, 2001). It seeks to find out how people make sense of their world, through observing and talking to people in order to share their understandings and perceptions. Taylor and Bogdan (1998) pointed out that, as there is not one single construct of reality, in qualitative research the importance is placed on participant's perspectives and on how they construct their realities and understand their world. By contrast, quantitative research aims at employing an objective, systematic investigation and data analysis in order to establish what the case actually is, through numerically quantifying findings (Burns, 1997). Quantitative research is better suited to research that does not rely on the individual participants' ability to interpret and reflect on their own experiences.

Qualitative methodology was chosen for this research. In order to understand what impact the extension programme had in terms of students' views of mathematics; their attitudes towards mathematics; and learning mathematics, it was necessary to gain their perspectives and interpretations of their experiences. In some instances the qualitative data was supported by some quantitative data, such as the frequency of students' responses.

3.1 Case Study

This qualitative study takes the form of a case study. According to Yin (1994) a case study is an empirical inquiry that investigates a contemporary phenomenon within its real-life context. Merriam (1988) defines case study as an in-depth examination of a specific programme, an event, a person, a process, an institution, or a social group. The in-depth nature of a case study investigation allows the exploration of relationships among the factors affecting participants' behaviours. The focus on in-depth analyses makes this method particularly suitable for smaller groups (Verma & Beard, 1981).

In terms of research goals, case study research can be predominantly exploratory, descriptive or explanatory (Massey University, 2004). The research question posed in this study suggest that for the most part the investigation of student perspectives will be exploratory in the first instance, but insights from the students may well provide some explanations as how utilization of student interest in history affects student motivation for learning and perceptions of mathematics. In the evaluative sense the case study may also be used to summarize and assess the main benefits of the programme in terms of student outcomes.

Although case study methodology appears to be well-suited to the research objectives, the methodology is not without limitations. A significant limitation of the case study methodology concerns the risk of observer bias (Gay, 1987). Observer bias acknowledges the possibility that as the teacher/researcher I "see" what I want to see, I "hear" what I want to hear. To minimize observation bias, attention in the data collection strategies is given to seeking and incorporating multiple sources of evidence which ultimately confirm or disconfirm the same set of facts or findings or chain of evidence (Gillham, 2000; Yin, 1994). In this current research, evidence was collected from three different sources: student self-evaluation questionnaire, programme-evaluation questionnaire, and focus group interviews.

3.2 Data generation techniques

There are three main data generating techniques in qualitative research: observation, interview and document analysis. Other techniques, such as questionnaires are also used. As this research is concerned with students' perspective, the teacher/researcher observation was used only in an historical sense, specifically, relating to the informed interpretation of student responses. As the researcher was also the mathematics teacher of these students, the possibility of teacher/researcher bias can not be completely dismissed, however, triangulation of evidence aimed at eliminating this problem. Document analysis, as a data collecting technique was not relevant in this situation either. The researcher chose to use questionnaires and focus group interviews. Both of these techniques are suitable to gain insight into authentic student activity, behaviour, and perceptions.

3.2.1 Questionnaire

Educationalists have a divided view relating to the usage and usefulness of questionnaires in educational research. Southworth and Conner (1999) claimed that questionnaires are an efficient use of time, they allow anonymity for those completing them and they are seen as being reliable because the same questions are used for everyone. They argued that a high rate of return can be ensured and questionnaires can be used effectively with adults and children. Disadvantages, according to Southworth and Conner (1999) relate to the factual nature of the information collected. Respondents tend to describe what is happening rather than offering an explanation as to why something is happening.

Another disadvantage, discussed by Gillham (2000) is the inadequate completion of questionnaires by respondents who often have to fill out the questionnaires without assistance. Despite these concerns the nature of this research was such that this data collecting technique proved appropriate. The students involved in this study were very motivated to participate and although help was available regarding the interpretation of questions, none was requested.

The questionnaire (see Appendix 2) consisted of open-ended response type questions allowing students to express their views relating to their learning experiences. Many students chose to illustrate their judgments and ratings with actual classroom examples, and instances of both positive and negative perceptions were forthcoming. Appreciation that their views were considered important for the teacher, as to be the focus of her research and reflections on her teaching, ensured a high level of student interest and involvement.

An important benefit of questionnaires, for a study involving teacher as a researcher, is anonymity. This factor enabled students to respond with complete honesty, without being influenced by peer, parental, or teacher/assessment factors.

3.2.2 Focus group interviews

Gillham (2000) argued that interviews of one kind or another are indispensable in a case study research. They provide a very direct way of finding out the beliefs and opinions of participants (Frey & Fontana, 1993). Interviews are especially useful when: a small number of people are involved; they are accessible; they are the key subjects of the research; the questions asked are mainly open and require extended responses involving prompts and probes for clarification purposes; and if the material is of a sensitive nature that people may prefer to discuss in a face-to-face interview. The negative side of conducting interviews is the time involved in the transcription and analysis.

The form of interviews can range across a continuum from the highly structured to the unstructured. Gillham (2000) provided the following scale of verbal data dimension:

Unstructured ⟵⟶ Structured

Listening to other people's conversation; a kind of verbal observation	Using 'natural' conversation to ask research questions	'Open-ended' interviews; just a few key open questions	Semi-structured interviews; open and closed questions	Recording schedules; in effect, verbally administered questionnaires	Semi-structured questionnaires: multiple choice of open and closed questions	Structured questionnaires : simple, specific, closed questions

In a school situation researchers tend to employ one of the three forms of interviewing techniques: structured, standardized interviews; focused interviews, or conversational interviews (Southworth & Conner, 1999). Structured, standardized interviews use a set of predetermined questions in a prescribed order. The analysis allows for clear comparison between respondents. This type of interview is typically used in the early stages of a school-based inquiry in order to establish a base line of information. In focused interviews attention is directed towards the particular topic or theme to be investigated. Interviewees may come prepared for the interview, by having been exposed to some pre-prepared and previously negotiated questions.

In contrast, conversational interviews are carried out in a relaxed environment where students can offer insights and opinions about the topic that may be difficult or impossible to access by other means. However, there are ethical issues involved in this form of interviewing that need to be considered carefully. As Morgan (1988) argued, the success of the interview depends on thorough preparation and clear understanding of the topic. To ensure that the conversation will have flow and a variety of pace, the interviewer will have to compile individual questions prior to the interview to guide the conversation.

The overwhelming strength of the face-to-face interview is the likelihood of gaining very rich information. As the participants of this small-scale research were in the same mathematics extension class, and the researcher was the mathematics teacher of this group, a face-to-face interview in the form of focus groups was chosen as the appropriate mode.

Vaughn, Shay Schumm, and Sinagub (1996) argued that focus group interviews are superior to individual interviews. Offering a higher degree of anonymity for participants and the security of being among others, this interview style is conducive in obtaining people's perceptions, experiences and beliefs. Focus-group interviews allowed and encouraged the students, involved in this research, to talk at length and express their personal views of the topic. Interviews were conducted in four sessions with seven-eight students being involved in each session. The size of these focus-groups proved to be ideal. They were big enough to allow for discussions and sometimes facilitate a healthy debate among students, but were not too big to allow the forming of dominant groups that could inhibit others.

In this research the conversational interview format was used, because it allowed students in the focus group to not only express their opinion regarding certain aspects of the topic, but also to engage in a discussion or debate relating to the topic. Within the focus group students were able to describe their views in a 'story format', often supporting their claims with examples and anecdotes that related to specific learning experiences.

3.3 Ethical Considerations

The two main issues that dominate guidelines of ethics in research with human participants are: informed consent and the protection of participants from harm. Informed consent ensures that participation is voluntarily and participants understand the nature of the study, the obligations and potential dangers that may be involved. Protection of participants from potential harm ensures that they are not exposed to risks that are greater than the benefits they might obtain due to the result of the research (Bogdan & Biklen, 1992).

This research has followed the guidelines outlined by the Massey University Human Ethics Committee (MUHEC). A successful application was made to the Massey University Human Ethics Committee. All students of the extension class received written information regarding the research and the nature of their involvement in the research. Participants were made aware of their rights to decline participation, refuse to answer any particular question, withdraw from the study at any time, and to ask any questions relating to the study during the time of participation. As the researcher was the mathematics teacher of the student participants, students were assured that their participation, or lack of it, had absolutely no bearing on their mathematics achievement grades.

Following written approval from the Chairman of Board of Trustees of the research school, informed consent was obtained from all 33 students and their respective parents / guardians. Separate written consent was received from all focus-group interview participants to allow the interviews to be audio-taped. Students were able to participate in their choice of focus group and select the day that was suitable for them. These interviews were held at school over a period of one week during lunchtime.

The protection of students' right to privacy was ensured by the non-disclosure of names and other personal details. Student questionnaires were assigned a numerical code, the same code that has been used in the findings section to identify student responses. Assurances were given that neither the school, nor the students involved would be intentionally identified.

3.4 Validity and reliability

External readers of research judge the quality of a research study according to a number of criteria. Yin (1994) proposed the following criteria pertinent to case study: internal validity, external validity and reliability. To construct validity the researcher used multiple sources of evidence. These included student self-evaluations, questionnaires and focus group interviews.

Internal validity generally concerns casual, explanatory case studies, using pattern-matching, explanation-building, and time-series analysis. While this research is essentially an exploratory case-study, there is a clear opportunity here to identify and follow through a number of common themes arising through the various responses given by individual students.

External validity relates to the study's findings in terms of how generalizable these findings are beyond the immediate case study. Berg (2004) addressed the need for objectivity in the research process, by claiming that objectivity is measured by the ability of the researcher to articulate clearly what areas have been investigated and through what means. Berg (2004) claims that, dependent on the nature of the case study, generalization is possible to some extent. Given the complexity of the teaching process and the uniqueness of the learning situation, it is not expected that the findings will be generalizable. However, it is hoped that teacher-readers will find that the findings of the study point to the benefits of listening to student voice, and they are prompted to consider investigation of similar extension programmes within their mathematics programmes.

Reliability addresses the extent to which the research findings can be replicated. A detailed, well described documentation of the research process is essential to allow another researcher to replicate the original study. However, as human behaviour is not static, there can be many interpretations of what is happening. Therefore the main concern with qualitative research should not be whether the same findings will be found again, but whether the results are consistent with the data collected. To ensure that the results are dependable, I used triangulation – multiple methods in data collection and analysis. A detailed account of this process is given in the following sections: 4.3 and 4.4.

CHAPTER FOUR: Research Programme

4.1 Participants and settings

The research was undertaken in a large, urban secondary school for girls. The following situation applied during the school-year this research was carried out.

Junior school

Students were grouped in form classes and studied all the four core subjects in their form groups. Students were grouped together with a different peer group for their selected option subjects. At Year 9 level students were in mixed ability classes. There was one music class, not streamed, that allowed students with the music option to work together. At Year 10 level there were two extension classes, complementing 11 mixed ability classes. Here students studied all their core subjects in their form classes as well as their main chosen option. Two option choices were available to create the two extension classes: history and geography.

In the two extension classes the type of extension varied between subjects: acceleration in science and history/geography, and enrichment in English and mathematics. Student participants of the research were of the history option class. This research is situated in the year 10 mathematics enrichment programme, over a one year period.

Senior school

At Year 11-13 levels students studied their chosen subjects in specific option classes, not in their form classes. The subject-specific programme, included an acceleration option, was available to students. The school offered both, NCEA and Cambridge examinations in a number of subjects at all 3 levels.

Year 10 extension

As the above description indicates, there was only one level, the Year 10, where students stayed together, as a form class, and shared a common interest, identified by their choice of option. Students were selected for the Year 10 extension classes based on their overall academic results in Year 9. Two classes of 33 students each were created, taking the 66 top-ranking students from the Year 9 cohort. The four core subjects that formed the basis of this ranking process were English, mathematics, science and social studies. This selection process established class groups that exhibited overall high academic ability. From my previous experience working with extension Year 10 classes, these top achievers typically exhibit a strong personal drive to gain top grades. The disadvantages of this selection process were due to the fact that students were in a 'high-ability' class for every core subject. Here the pace and expectations were considerably higher than in the mainstream, yet not every student excelled in each core subject or even had the same interest in every core subject. However, uniting the students around a common choice of option – in this instance a shared interest - had the potential to compensate for the shortcomings of the selection process.

Teacher as the researcher

The researcher was also the mathematics teacher of the Year 10 history option class, where the research was carried out. As the teacher, I have an extensive formal and informal educational and experience based background in humanities, in history, and in art history. Coupled with my experience as a mathematics teacher, I was ready to take on the challenge of incorporating the history of mathematics in the mathematics extension programme, in order to provide a course that was beneficial for this academically talented group of students.

The following measures were put in place to avoid any possible disadvantage of the situation where the researcher was also the teacher of the participants in the research.

1) Ethical considerations were carefully adhered to, based on MUHEC guidelines, and full written undertaking of co-operation was obtained from all parties involved.

2) Student evaluation was sought regarding the extension programme, which was not formally assessed during the course of study, and had no bearing on students' grades, that were obtained through the assessment of core material only.

3) Participation in the research study was strictly on voluntary basis.

4) Questionnaires were used to aim for non-bias responses and to ensure anonymity.

5) Focus-group interview data was coded, in a simple numerical fashion, to ensure students' right to privacy.

The student participants and the researcher enjoyed a very positive working relationship during the year of this research. This was clearly evident in the students' overwhelmingly positive response regarding participation in the research. All 33 students completed the self-evaluation and answered the questions presented in the questionnaire in detail. Initially one or two focus-group interviews were planned, comprising 4-5 students in each. However, due to a very keen interest (31 students were eager to participate) the researcher decided to carry out four focus-group interviews to accommodate all these students. This decision was made out of respect for all participants. While the length of interviews had a considerable bearing on time involved in transcribing the audio-taped material, it certainly had a positive effect on the richness of responses. The quality of responses was backed by the quantity, ensuring that the whole class was represented and everyone's opinion was heard.

4.2 Programme design

Enrichment and acceleration are the two main approaches used in a school situation to meet the needs of gifted and talented students. In my Year 10 extension class enrichment was used as the form of extension.

The review of the literature highlighted several well known enrichment models that provided a rich resource in my programme design. These were: Renzulli's Enrichment Triad Model; Purdue Secondary Model for Gifted and Talented Youth; Treffinger's Model for Increasing Self-Directedness; and The Autonomous Learner Model. These models have several common elements. I chose to use these in 'spirit' and in conjunction to one another, rather than just focusing on one specific model. Through a variety of learning options I intended to increase my students' mathematical knowledge, skills, and awareness. With the enrichment based extension programme I aimed at developing students' self-directed learning skills. In order to facilitate the development of self-directed learning skills, in my programme design I focused on incorporating the following elements:

- Expose students to various topics through their areas of interest;
- Use advanced instruction in their selected area of interest;
- Engage students in in-depth study of some content areas;

- Provide opportunities for the development of creative thinking and problem solving;
- Expose students to a wide variety of occupations where mathematics is used;
- Use students' affective ability to appreciate the work of famous mathematicians and to raise social and cultural awareness;
- Provide students with the chance to select and complete research / study of their choice;
- Allow students to produce and present their research in small group situations;
- Use 'fun', 'interest' in order to increase motivation, task commitment;
- Provide diverse activities that ensure all students experience success in the class and build self-confidence.

I applied the strategy of addition (Fried, 2001) in most topic areas in the research context, as this strategy did not alter the curriculum, rather it allowed for the extension of the existing curriculum. By introducing historical anecdotes, biographies, famous historical problems through all topic areas, students' interest levels were constantly raised and they were motivated to learn. While still added on to the main Year 10 syllabus, the 'history of number' extension topic allowed the class to experience the 'the strategy of accommodation', when the whole topic was presented through using a historical development formula. Appendices 4-9 describe some of the favourite (selected by students) enrichment based extension topics / activities.

4.3 Research process

At the beginning of the year students completed a short, self-evaluation questionnaire (Appendix 1). The data was collected in order to create the learning profile of each student. This information supported the researcher's decision to design an extension programme that provided a good balance between mathematics and humanities, catering for students' interests, strengths and preferences.

Students were given a comprehensive questionnaire towards the end of their course to describe their experiences while participating in the extension programme (Appendix 2). They had the opportunity to discuss, at length and anonymously, their opinion of the course. By doing this they highlighted any personal benefits, and the overall impact this course had on them. They were also invited to make recommendations for future courses.

Having completed the questionnaire, students were offered the opportunity to participate in focus group interviews. These interviews were carried out over four successive lunchtime sessions, of approximately 40 minutes each, with a different group of 7-8 students. Students were free to make up their own groups and choose the suitable time. Each student involved in the focus group interviews participated in one interview session only and the content of each discussion centred around similar focus questions. The purpose of these interviews was to further investigate responses given in the questionnaire, debate, clarify certain issues, and allow for additional information to surface in the course of a pleasant lunch-time discussion. These interviews were carried out in a relaxed, friendly environment where students were keen to participate, express their viewpoint and engage in a positive discussion. The discussion was carefully guided by the researcher to allow for new information to arise, but controlled, so students were focusing on the main issues. This was achieved by using open questions that focused on the key points of the research (Appendix 3).

4.4 Data analysis

While the majority of data analysis was carried out towards the end of the research, the information collated from students' self-evaluations was used during the course of the programme, and had some bearing on the nature of work presented to students. Student self-evaluations identified their strengths in humanities and placed emotional intelligences in the forefront. This information prompted the teacher/researcher to present extension material that built on students' strengths and interests.

To ensure credibility and validity in the research, data was coded numerically for identification of individual responses and also colour-coded for cross-matching purposes. In student questionnaires a simple coding system was used to refer to students' responses. Each self-assessment questionnaire was coded from student 1-33 as SES1 (self-assessment student 1), SES2 (self-assessment student 2). Each end-of year questionnaire was coded from student 1-33 as S1 (student 1), S2 (student 2). Useful quotes were highlighted and collected, based on student code numbers and appearance, relative to the question number.

The audio taped focus group interview results were transcribed on separate, colour-coded (black, red, blue, green) sheets (based on which group presented that particular response), under question-type subheadings. This system allowed the researcher to assess frequency of responses in certain areas. In a conversation situation students were identified as FG1, FG2 (focus group interview student 1, 2), however this numbering referred to each conversation only, students were not assigned a code number for the duration of the focus group interview. When only one particular response (provided by a member of a focus group) was referred to in the results section, it was simply coded as FG. If a point was raised in more than one focus group discussion, the situation was identified by the number of groups inserted in brackets (FG3). Useful quotes, conversations were identified within the subheadings.

The results section retains the above-described coding of SES1, SES2...; S1, S2...; FG; FG1, FG2...; (FG2). Additionally, the number of similar observations, made by different students, as responses to questionnaires, are identified with the number of students shown in brackets (6).

At the next stage data was analyzed from different perspectives to provide the most holistic view obtainable. During this stage of the analysis, it became more apparent that some of the data could be coded differently and that some of the responses overlapped. Therefore all data were checked for patterns and recurring responses. Responses were collated from the questionnaire and from the focus group interviews under emerging theme headings, using existing codes in order to establish frequency of responses and identify trends. Qualitative data was also backed up with quantitative information. The researcher took particular care that no identifying information, relating to participants of this research, was disclosed in the written results.

4.5 Limitations

This research involved one extension mathematics class at Year 10 level. As other extension classes in different secondary school settings may operate in different ways, in terms of identification, selection, provision, the results obtained in this research may not be generalizable to them. However, it is likely that the reader of this research can identify what is applicable in his or her own situation and what is not. In a case-to-case transfer situation, findings based on this exploratory case study can potentially be applicable, if determined by the teacher of the extension class.

CHAPTER FIVE: Results and Discussion

Introduction

The first section of this chapter contains the results of students' self-evaluation, collated at the beginning of their course. This section provides insight into:

- Students' perceptions of themselves as mathematical learners;
- Students' approaches towards studying mathematics;
- Students' perception of their own intelligences and their view of intelligences in general.

The second section contains information collated from the student questionnaire and focus group interviews completed at the end of the course. These findings are presented under three themes:

- Students' perceptions of learning in the mathematics extension class;
- Students' perceptions of the historical focus of the mathematics extension programme;
- Students' perceptions of the mathematical aspects of the extension programme.

5.1 Students' self-evaluation at the beginning of the programme

5.1.1 Students' perceptions of themselves as mathematical learners

At the beginning of the school year students were asked to describe themselves as students of mathematics, and their approach towards studying mathematics. They were asked to describe their perception of their own intelligences and their view of intelligences in general. Enabling a better understanding of students, this information provided a valuable resource for setting the context of the extension programme.

In response to the request to describe their mathematical ability, student responses grouped broadly into the descriptors: excellent, good, OK, not too great, no good.

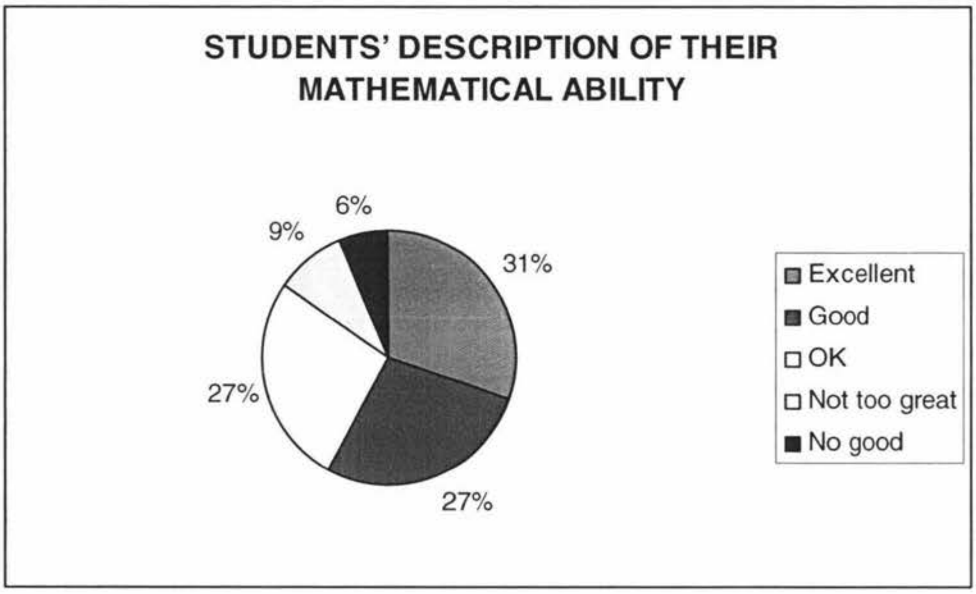


Figure 1

The majority of students described their ability as OK to excellent. 15% responded that they perceived their mathematical ability to be less than OK (see Figure 1). Excellent students typically described themselves as students who enjoy mathematics very much, have a genuine interest in mathematics, learn and understand it easily, do mathematics with enthusiasm, and find it fun. For the remaining students, their responses related more closely to their mathematical ranking in the classroom. They looked at mathematics as a school subject and graded themselves based on their assessment results. Their comments ranged from: hard worker with satisfactory results; capable but panic easily and makes silly mistakes; just scraping past by effort; to the one who claimed that she needed lots of explaining, felt she was always behind, and did not fit into the programme.

5.1.2 Students' approaches towards studying mathematics

At the beginning of the year a large proportion of students identified approaches to learning mathematics that fitted traditional, routine-based methods:

- get the topics explained well, need teacher's explanation first (8)
- take notes (11)
- complete exercises and homework to reinforce the concept (11)

However, a number of students referred to specific learning strategies, such as: 'try to understand the concepts first' (2), 'discuss with other students' (7), 'solve problems in many different ways (oral, visual, kinaesthetic hands-on, individually, group-work)' (5), 'through investigations and research' (2), 'learn visually through images' (5), 'solve longer complex problems'.

Student comments indicated awareness of a range of activities and preference for specific working arrangements: 'learn more working individually, but enjoy group work' (3), 'use worked examples', 'complete advanced exercises from higher level texts', 'only pay attention if interested in it'. In identifying grouping arrangements approximately 1/3 of the students indicated a preference for working alone, with the majority preferring to work in pairs or in small group arrangements.

5.1.3 Students' perception of their own intelligences and their view of intelligences in general

When describing their strengths, most students' perceptions can be characterised as consistent with Gardner's (1983) definition of intrapersonal intelligence. Intrapersonal intelligence was identified by the majority of students as their strength. They felt that they had ability to accurately assess, understand and recognize their own feelings, actions and motivations. Linguistic intelligence was rated as second, closely followed by interpersonal intelligence. Mathematical intelligence was identified as the fifth in the line of nine intelligences, described in Gardner's Theory of Multiple Intelligences. Students identified naturalistic intelligence as their weakest point.

In evaluating their mathematics ability most students thought that intelligence was not fixed.

Students' views of intelligence	Yes	No	In some areas
Intelligence is fixed	9	22	2
Possible to increase intelligence / competence	28	1	4

The majority of students believed that it is possible to increase competence even in areas where the person is not naturally intelligent. In addition to hard work and effort, students recognized the effect of ones upbringing. They noted that upbringing and environment may influence the extent of increasing one's intelligence. A small number of students believed that, while it is possible to increase intelligence/competence in different areas, it is possible only to a certain point, claiming that practice is the way to increase knowledge/skills.

> I believe intelligence is hard to define, as it means so many different things to different people. Different people have different talents in specific areas of life. I believe a person is born with these forthcoming talents, and it is his/her responsibility to develop these God-given talents. But people without these talents can be taught the skills needed for it, to a certain level, thus increasing their capability in this talent area. (SES 5)

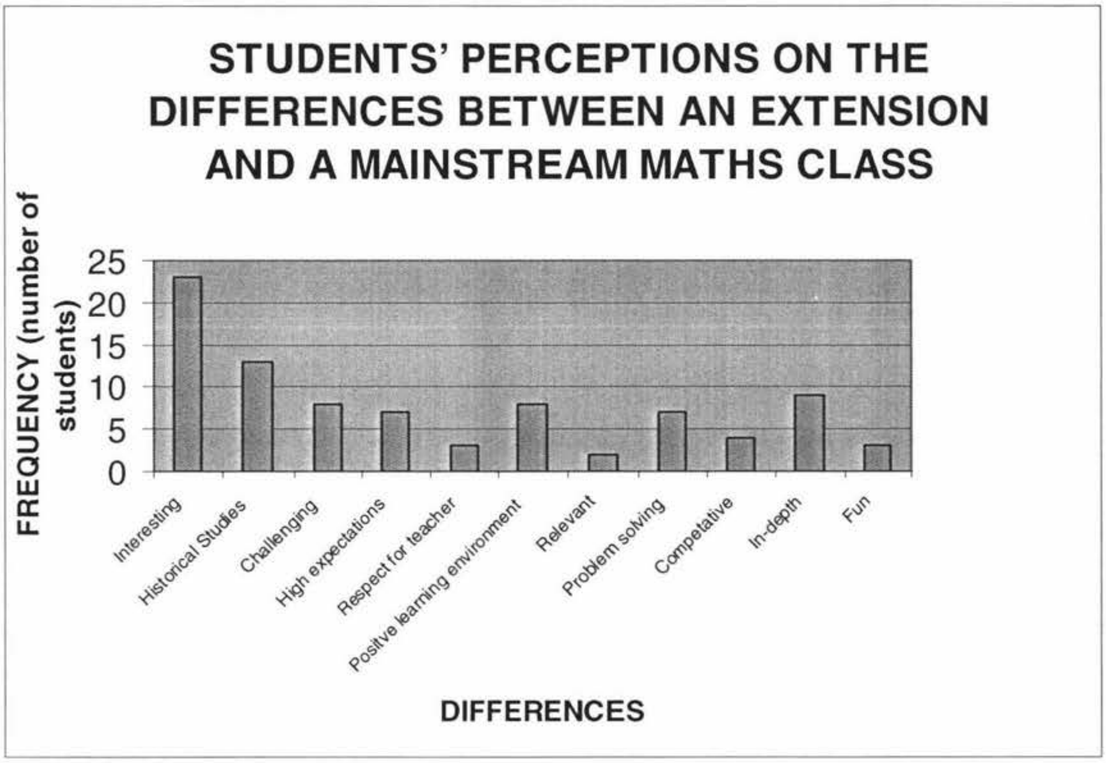
5.2 Students' responses to questionnaire and to focus-group interview discussions

The following section discusses students' responses from the questionnaire and focus group interviews, completed during the second half of Term Four. These responses relate to experiences in the mathematics extension class. The discussion is structured around the following themes:

- Students' perceptions of learning in the mathematics extension class;
- Students' perceptions of the historical focus of the mathematics extension programme;
- Students' perceptions of the mathematical aspects of the extension programme.

5.2.1 Students' perceptions of learning in the mathematics extension class

When asked to identity the differences in learning mathematics in the extension programme from learning in their mainstream class in Year 9, a wide range of features related to the learning environment and task expectations were mentioned. The graph below provides an overview of those areas in which students noted differences between extension mathematics class and their previous mainstream experience.



When reporting on differences between learning mathematics in the extension class and in the mainstream class, the most striking difference, identified by the majority of students, was that learning was *interesting* in the extension class. Students noted that historical studies increased their interest in the subject and helped them gain background information.

While describing their learning of mathematics in the extension class, student responses focused on the pace of learning; the depth of learning; and the social, cultural and emotional needs of the learner. Topics were studied more in-depth and required a high level of understanding. Students participated in more class discussions, completed projects and practical activities. Higher expectations, a more competitive environment and much more respect for the mathematics teacher, also featured as significant differences. Material studied was relevant to everyday life and students were able to contribute their own ideas, make choices in content and mode of study. These views, supported by quotes, are discussed in the following section, centred around the following topic areas:

- The pace of learning;
- The depth of learning;

- Emotional aspects of learning;
- Effectiveness of extension activities.

The final part of this section looks at the extension activities students engaged in during the year, aiming at identifying the effective approaches that enhanced their learning.

5.2.1.1 The pace of learning

Students identified favourably with being part of a learning environment involving people of similar ability who were enthusiastic about learning, wanted to learn, and were focused on the task at hand. Students commented on the fact that if the pace is too slow, the class becomes boring and students switch off. The faster pace that this special grouping allowed, ensured that they were not bored and were not held back by slow students.

> The pace is faster and we do more in-depth historic studies and we study faster than the normal class. The mood is much more focused. (S32)
>
> People are a lot more advanced and we are able to move at a quicker pace, which is good.
>
> People want to learn and work. (S31)
>
> It's good to be in a class where all want to learn and work at a high level, and the pace is very stimulating compared to how I know the other classes work. (S25)
>
> I get bored if we are on the same thing for ages. (S27)

Students appreciated that by completing the core material at a faster pace, they were able to study material included in the extension programme.

> The good thing about the extension programme is the exposure to new things, not only in mathematics, but in history, architecture, general everyday life. It also challenges and broadens your mind more. (S1)

Most students were happy with the rate of learning in the extension class, and with how the extension material fitted in with the core curriculum. They appreciated the fact that they did not have to complete routine exercises on material they already knew, as they had to do in previous years. The pace kept them motivated, interested, and on task.

> In a mainstream class you have to wait ages for people to catch up, and you always do the same thing. (S8)

One student expressed relief that her initial concerns, that there was not going to be enough time to complete both core work and extension work, were unfounded.

> I worried at first that we had to do everything that mainstream did, and a whole lot more. Will we have time? During the year I realized that it ended up making the other stuff easy and helped in preparation for the exams. (FG)

While most students in the class were able to keep up with the pace, a few lower attaining students felt pressured to keep up with their peers.

> In the extension class there is a pressure to do well, which is good, if you can keep up, but not so good if you struggle. But we still got to enjoy the history side, and sometimes we solved problems before the others, just a fluke, I suppose. (FG)

5.2.1.2 The depth of learning

Students expressed awareness that the extension programme was rich in depth.

> ...we can think about it, discuss it, take it home, solve it different ways, show our different solutions in class, discuss it, and if we had famous problems, we could also see how our solution compared with how famous mathematicians solved it (FG)

While it was very important to many students to gain high grades, they appeared to become more appreciative that high grades could be attained through a combination of both routine practice and problem solving approaches.

> The challenging problems made you think about them, not just do routine work. It motivated us, we wanted to do them. And at the end everyone benefited, even the very competitive students who only wanted to focus on core material, to get top grades, improved their problem solving skills and this also helped them achieve top grades. (FG)

> We were doing stuff that was good for us. The extension activities gave you that thought process, exercised your mind. We learned how to work out problems an easier, shorter way from these really hard problems. (FG)

Some students also noted that they were pleased to extend both their mathematical knowledge and their general knowledge through exploration of the relationship between mathematics and other subjects. There was strong agreement that the integration of history with mathematics enhanced their enjoyment and allowed them to gain a deeper insight into mathematics. The responses referred to the human aspect of mathematics—to see that people created it, discovered it, it is a result of human endeavour that is marked by success, failure, perseverance.

> We learnt about the evolution of mathematics, it is actually developed to what it is now. (FG)

> The history motivates you, makes you get the problem done, makes you feel good as you are solving famous problems of famous mathematicians, you are as good as they are. (FG)

In addition, students reported that the investigative activities enabled them to learn how extensively mathematics is used in different jobs.

> We got to realize that mathematics is part of most things. Architects are a bit like mathematicians and when you explore different jobs, you see that mathematics is part of most jobs. (FG)

5.2.1.3 Emotional aspects of learning

Students noted that mathematics became enjoyable through the interesting learning activities.

> Mathematics has now become more interesting (S22)

> It is much more interesting and detailed. I love the enthusiastic atmosphere. (S3)

The cultural aspect of the programme was viewed positively by the class. Students representing ethnicities from European, Asian, and Arabic background reported that it was important for them to see how their own culture contributed to the global mathematical knowledge, and that they could supply some unique knowledge originating from their cultural background. It became clear to students that the mathematics we know today is not just of Western European origin, its development can be traced back to different parts of the world over different historical periods.

> The research activities allowed us to look at other countries and in different historical era, and see how different countries contributed to the knowledge we have today and give them credit for it. We can respect people's achievements, and you respect different cultures more as you get to know their achievements and their contribution to mathematics. (FG)

> I'm from Egypt and I was very interested in what mathematics was like in ancient Egypt, and I can appreciate the non- European contribution to the development of mathematics as well. (FG)

Emotional aspects of the programme were frequently highlighted in students' responses. They appreciated the positive classroom atmosphere where learning was enjoyable, fun and interesting. The combination of challenging and interesting problems involving historical aspects, motivated them to work hard.

> I like the history of mathematics, and I'm pleased when I've solved a problem. This motivates you to do more and look for other ones as well. (FG)

A small number of students identified negative aspects of being in the extension class. Increased stress, more competition and more pressure to succeed at a high level, were voiced. Other comments identified a desire for acceleration instead of enrichment, and a wish for less homework. Lower attaining students were concerned by the challenging aspects of the programme.

> "...the class was catering for the able students and not for the dumb", "there is too much problem solving", "a bit crammed", and "too much extension activities" (FG)

5.2.1.4 Extension activities

Students identified a range of factors associated with extension activities that supported their learning, such as:

- ✓ working with fun and challenging activities (S16, S26, S27)
- ✓ variety (individual / group; bookwork / projects / research) (S25)
- ✓ allowing choice in the extension activity (S7, S19)
- ✓ incorporating kinaesthetic activities (S7, S28, S30, S32, S38)
- ✓ participating in discussions (converse and argue details, see how others worked it out) (S2, S3, S13, S22, S28)
- ✓ usage of lots of visual information, diagrams.

The opportunity for independent work, and working on something new and creative were also identified by students as positive factors.

> Make activity fun and challenging. Something that I had never done before and to perform myself and be creative. (S26)

An equal number of students stated preference for group work and for independent work. In expressing preference for different types of extension activities historical activities were identified as the most favoured by all students. Other activities noted included complex problems, discussions, investigations, research and presentations.

Whilst extension activities were designed to complement most core topic areas studied, students identified some specific activities they have enjoyed. The distribution of these activities between the topic areas, based on student preference:

- Number: 19 students (History of number: 12 students,
  number sequences: 7 students)
- Algebra: 14 students
- Geometry: 20 students (Practical geometry, transformations: 17 students,
  Trigonometry: 3 students
- General: 3 students (research investigations)

<u>Number</u>

- History of Number (Appendix 4)

Students who chose the study on history of number, as significant for their learning, noted that they gained extra knowledge while learning about ancient times and origins of number. These students reported an increased appreciation of the many years of work that went into the mathematics we use today.

> I mainly enjoyed the ancient mathematics with the different number systems.
> It was challenging but it opened my view on how much mathematics has achieved.
> It amazed me how logical and how well everything seem to fit in and work. (S27)

- Number sequences

Students identified specific activities, such as the "Rich Aunt" problem, Fibonacci (Appendix 5) and the use of ICT .

> I enjoyed the Fibonacci sequence study and the Golden Ratio because it related to this book I was reading, called the Da Vinci Code. It made me understand the terminology of the book. (S22)

> I learned the importance of Phi in nature and also how the Fibonacci sequence is used in unexpected places. I found both of them very fascinating. (S3)

Algebra

In this section the "King Arthur" problem (Appendix 6) was identified as a clear favourite.

> The King Arthur problem was one of my favourite. This was because the activity sounded interesting because it had an interesting story to it. Basically what makes something interesting is the way it is presented. (S33)

This was an algebraically challenging problem that successfully motivated students through the novel presentation and solution methods. Students enjoyed the fact that they could be engaged in a role play while acting out the problem. It was a fun activity and they were working all together as a group.

> I had a lot of fun working through the problem, and the role play at the beginning increased my interest in it. I found that once I started on the problem, I couldn't stop until I figured it out. I think this activity was great and it increased my problem solving skills (S33)

> I really liked King Arthur, the way everyone was really involved and we could get up and do something that was fun. It was good because it had an interesting historical story behind it. (S25)

Students noted that this problem also broadened their view of algebra and algebraic problem solving. They found that sometimes collecting a lot of data could help finding a pattern. Students commented on learning a variety of approaches, presented by different students, and learning to describe their different solutions to others.

> I had so much fun that I really enjoyed myself. I even found out that algebra can be fun. (S29)

> I thought it was a very clever problem and it was a way to make mathematics fun. It improved my perception of algebra! (S19)

> I've learnt that sometimes patterns only appear after you have collected a lot of data. I think this will help me next time I have to solve such a problem. (S12)

> After I found out that no one can understand my way of solving the problem, I learnt that it is very important to make rules understandable for others. (S15)

Geometry

Students reported that the opportunity to study geometry at a greater depth provided links to real life. Geometry was not just a school subject any more, without relevance to the outside world, they begun to see it everywhere (Appendix 7).

> I used to think that geometry was of no use, but after the activities I learnt that geometry is used everywhere, every time and is very important (S8)

> I enjoyed the geometry investigation and now I see everyday things in a different light. Buildings, art and designs that use features like symmetry, I now find attractive and I can appreciate the thought, work and effort put into them. (S9)

Students commented on the fact that their general knowledge has increased through investigations and research. They were able to use their creativity and discovered relationships between mathematics and many other subject areas. It was important to many students that they were able to make choices and investigate what they were interested in the topic area.

> I learnt more about tessellations drawing my own designs. I learnt which shapes can tessellate and which ones can't. I enjoyed researching about people who used geometry in their works. (S6)

> I liked learning about how mathematics is applied in architecture and art which also generated more interest for mathematics in me. (S10)

> I liked the geometry project on the use of geometry in art and architecture, as I was given the chance and time to investigate aspects of mathematics of my own interest. (S24)

> I enjoyed the geometry project we did because it made me look for something interesting (geometry-wise) to present about. This made me begin to see things differently. (S30)

> Looking at transformations in art and architecture I was able to see just how often people apply mathematics to architecture. (S13)

> I have developed an interest in distinguishing geometric qualities in ordinary objects. I have a clearer view of the theoretical aspects. (S11)

> Geometry used to be my least favourite subject, but through research and presentations I really enjoy it now. Geometry is everywhere. (S29)

- Trigonometry

Students described their enjoyment of the research work carried out on Pythagoras and the usage of trigonometry through history.

> My favourite extension activity was right-angled triangles, Pythagoras, his life-work, and trigonometry. Learning about his life and things in older times and learning how they have changed through time. (S1)

They commented on the value of solving famous problems, like the Two Towers (Appendix 8).

> I enjoyed solving the Two Towers problem my way, using trigonometry, and then trying to understand how Fibonacci did it a different way. (FG)

General research

Students acknowledged the benefits gained from carrying out background research on famous mathematicians. They listed increased general knowledge, deeper understanding of the content studied, and generating further interest in the particular subject area, as some of these benefits (Appendix 9).

Students commented on the level, and the mathematical content of extension activities. 94% of students identified the activities as manageable, yet challenging. Extension activities were identified as:

- ✓ fun
- ✓ at a perfect level
- ✓ great
- ✓ not overwhelmingly hard but made you think a lot.

However, there were some concerns regarding the extension activities by those few students who were mathematically less able. They found the level of problem solving a bit too hard, and they struggled with some extension activities as they took longer time to understand the concept. One student identified algebra as her weak point and therefore she was not able to progress as well as the others.

As well as reporting on the activities that they found stimulating, students were also asked to suggest extension activities they would include in the extension programme. Responses identified many activities that were part of the extension programme already. These students wanted more of the same. They wished that time and core syllabus did not play a dominant part in programme planning and that they could spend much more time on extension activities.

More of the same activities	New, different activities
- more historically based activities as I found them really interesting (S2, S23, S25, S27)	- include some practical activities eg: observing nature, real life, like snails' shells for Fibonacci (S4)
- more fun, thought stimulating activities like King Arthur (S5)	- drawing (S6)
- fun activities involving the whole class like King Arthur (S7, S9, S14)	- 'create the question' activity to match an existing equation (S8)
- lots of problem solving: Mathex (S10, S15, S30)	- outdoor activity (S14, S16, S17, S19, S25)
- research (S10, S14, S21) - small projects (S21, S24)	- going deeper into curriculum, learning next year's curriculum (S28)
- research (S10, S14, S21)	- competitive activities eg: first person gets a prize (S28)
- independent research (S24)	- kinaesthetic activities (S31)
- group work (S10, S16, S30)	- look at what mathematics is needed in certain jobs (S33)
- more of the same as we do now (S12, S13, S22, S29, S33)	
- fun activities, creative fun activities (S26)	
- presentations, projects (S16)	
- hands-on activities (S17, S33)	
- lots of class discussions (S18)	
- individual work, enjoy doing things myself more (S18, S24)	
- spend more time on Golden Ratio, mathematics and architecture (S20)	

5.2.2 Students' perceptions of the historical focus of the mathematics extension programme

Students identified several benefits of studying the history of mathematics, and completing historically inspired exercises and projects. They noted that engagement in activities, involving a historical focus, were interesting (4FG); fun (3FG); challenging, hands-on (2FG); real and relevant (3FG); exciting, enjoyable (2FG); presented new skills, built awareness and appreciation (2FG); increased general knowledge and understanding.

Students' responses regarding the reasons why the history of mathematics should be included in the teaching of mathematics, grouped into three themes:

- The history of mathematics humanizes mathematics.
- The history of mathematics is motivating, it makes mathematics more interesting, and much more enjoyable.
- The history of mathematics makes mathematics more understandable, approachable, and it gives insights into concepts, problems and problem solving.

5.2.2.1 The history of mathematics humanizes mathematics

Students, in their focus group interview discussions, commented that the history of mathematics allowed them to see that people actually thought about the concepts, ideas they learn about in the mathematics class and that mathematics is not an external entity.

> I take a lot of things in mathematics for granted, but learning about history made me appreciate the work people put into it. I am interested in history in general, so it changed my interest in mathematics as well. (S28)

Students claimed that mathematics is generally thought of as a cold and dry school subject, but the history of mathematics showed them the human aspects of mathematics. They gained a greater understanding of why things were discovered.

> Mathematics became an interesting subject through the history of mathematics. It is not just a formula, eg: $a^2 + b^2 = c^2$, but it is about Pythagoras and other people who discovered them, or it is not just tessellations, but it is about Escher and other famous mathematicians who used tessellations in their works. (FG)

> Learning about famous mathematicians and discoveries has thrown mathematics into a new light (S11)

> I was fascinated with the fact that all those famous mathematicians found their rules from their surroundings (eg: Archimedes solved the problem while he was having a bath and Pythagoras from tiling). It captured my interest and made my lessons more enjoyable. (S15)

Responses indicated that students began to identify with the emotions of mathematicians through their journey of discoveries, became aware of their joy and despair, and learned to see things from their perspective. Generally, they could feel empathy with people who had made mathematical discoveries.

> How smart those famous mathematicians were, and all the effort that went into those important discoveries! (FG)

Through research activities they found out about famous mathematicians and their achievements (9 students).

> It was interesting to find out how different mathematicians discovered all the things we just take for granted. Through biographies we found out how they grew up, the effort that went into their work, and the influence their environment had on their achievements. (FG)

The evolutionary aspect of mathematics was also discussed by student groups during focus-group interviews. The history of mathematics allowed them to see that mathematics has changed, evolved to what it is now and that mathematics can be different in different cultures. They were able to appreciate the contribution of different cultures to our mathematics of today. The history of mathematics extended students' knowledge of ancient civilizations and increased their cultural and social awareness, as claimed by three students.

> For example, look at measurement, like time, length, and mass … it was measured differently in some countries in different times. (FG)

> Also, mathematics has, (strangely), increased my interest in Asian & Arabic cultures – ever since we did that history of mathematics in different cultures and learnt how to write numbers and calculate in Ancient Chinese and Egyptian. I have often found that I am more interested in the history we learn in mathematics than the history we learn in the history class. (S18)

Students also became acquainted with the utilitarian aspects of mathematics, while identifying the direct relationship between the need to solve a particular real-life problem and the mathematical discovery, and also learnt that mathematics can also be seen as a recreational activity.

5.2.2.2 Making mathematics more interesting and much more enjoyable

In the questionnaire responses, 82% of the students reported a positive change in their interest in mathematics. A small group (3) stated that they have always liked mathematics very much, therefore they did not experience any change, and the remaining two students reported that they still did not like mathematics very much. One student did not respond.

Whilst positive changes were reported in motivation, interest, and enjoyment, ten students stated that they became interested in the mathematics problem or mathematics project when it had a historical content or aspect.

> Mathematics has never been more interesting, it has become about great men with brilliant discoveries. It introduced me to the logic of mathematics, the sheer complicity and I do, definitely enjoy mathematics now. (S22)

> The extension activities that relate to history have attracted me to mathematics even further (S23)

> This started my interest in architecture. I would have never thought about it in relation to Geometry. It was interesting to see that Geometry is all around us. I was able to see more for myself, I recognized and noticed more things. (FG)

> I'm a 'non-mathematical' student, who loves history, very much into humanities, and never paid much attention to mathematics. Our study on Fibonacci helped me understand the book 'Da Vinci Code' better. (I was reading it at the time) This made more interested in mathematics. (S32)

> The history increased my interest in the subject, I can now relate it to things outside the classroom (S29)

For some activities, they noted that being given a choice resulted in studying what they were interested in.

> I like the investigations as I can choose to do something interesting that I enjoy (S9)

As noted in the literature (Murphy & Alexander, 2000), student interest was strongly linked to motivation. Students discussed the fact that when they were presented with interesting problems, they were more motivated to solve them.

> Without history of mathematics, classroom mathematics is just equations to work out and other, not too interesting questions to answer. But now it is fun. It is fun to do projects and fun to watch other students presenting their projects as well. (FG)

Fun was also strongly linked to motivation. Students stated that they were much more motivated to work on these fun problems and projects than if they just had boring work to complete.

> I always thought mathematics was boring, now it's fun, I really like mathematics now. (S8)

> I found a fun way of learning mathematics by doing extension activities and assignments. (S33)

> I enjoyed the interesting activities that weren't bookwork, and not being bored in class. (S27)

The challenging problems proved to be strong motivators as well.

> It was great, all students got involved, we worked in groups, sometimes independently, we discussed, debated our ideas, and compared our solution methods with the great mathematicians. (FG)

However, for one student, the positive image of the mathematics extension programme existed separately to the somewhat negative image of the conventional mathematics class.

> History changed my interest in mathematics, however I still see repetitive exercises the same way. (S10)

Not every student in the class was able to transfer their interest in history to mathematics through the history of mathematics.

> I like art history, art – I think of them differently, my practical view of mathematics didn't change. (S20)

> I don't like mathematics but I like the history of mathematics. (FG)

One student commented on the 'feel good factor', one of the emotional benefits relating to the history of mathematics.

> It made me feel good, special, when I solved famous historical problems. It is if I'm as good, or as clever as Gauss or Fibonacci, well, almost, ha-ha. (FG)

Students commented that the historically inspired problems and projects allowed for out-of-class work, as some of them were very much hands-on activities. These problems were relevant in terms of context, interest and mathematical learning.

While the real problems related to everyday life, the story, the history behind them gave these problems another dimension. Students found these real problems very motivating.

> It motivates you, gives you a reason to solve them. (FG)

They found it exciting to discover solutions to famous historical problems and learn new skills that students would not normally learn in a mainstream mathematics class.

> Like in the Two Towers, we had to first deicide for ourselves what method we could use to solve the problem, like trigonometry, algebra, or geometry, and then had to explain it so everyone could understand how we did it, and then we even had to try to figure out how Fibonacci did it, just by working through his solutions. (FG)

Students commented on the positive learning environment, which allowed for a choice of activities, for group work, for open-ended discussions, and provided opportunities to learn a lot from each other.

> FG 1: We had to think about what we were doing.
>
> FG 2: Yeah, we weren't just told how to do it 'the right way', we could explore different ways, plan our own (*solution method*) and talk about it.
>
> FG 1: Ideas were just bouncing off each other and everyone could contribute in their own way.
>
> FG 2: With the history of mathematics even …(*some of the weaker students*) had a chance to be first to solve a problem.

5.2.2.3 Making mathematics more understandable and approachable

Students discussed that the challenging problems allowed them to use different methods, sometimes a combination of methods, to work with.

> We didn't have to use that method we were learning at the time in the core topic area, we could approach the problem in different ways. (FG)

> And we got to see their thinking processes and found out how different people figured things out in different ways. (FG)

Students' responses indicated that the history of mathematics allowed them to see more mathematical relationships in everyday life, with architecture and artwork mentioned most often.

Overall, students summarized the following benefits of incorporating the history of mathematics.

> FG 1: The history of mathematics let us gain a deeper insight into mathematics. Mathematics is about people, about how they made their discoveries, figured things out.
>
> FG 2: And we got to see their thinking processes and found out how different people figured things out in different ways.
>
> FG 3: Also remember how mathematics helped people in the old days, in Egypt, and that, ... there is a point to learning mathematics.
>
> FG 4: All that extra knowledge, background information, it ended up making the core mathematics easy. We learnt how to solve all those challenging problems, the other stuff was easy.

The flow-on effect, the inter-relationship between academic, socio-cultural and emotional benefits of the extension programme was well-articulated by Students 23, 1, and 14, stating:

> I realized that mathematics has a very interesting historical background. This has made me even more interested in mathematics and is making me strive to achieve better. (S23)

> I have gained knowledge about famous mathematicians and architects and different areas of their works. It has broadened my mind and made these things more interesting. It also made me interested in other stuff, sparked interest in art history and architecture. (S1)

> I found the history of mathematics very interesting. It helped understand some mathematical ideas and concepts better. It also made me realize that mathematics is not something that was always there, made for the classroom. (S14)

5.2.3 Students' perceptions of the mathematical aspects of the extension programme

Students' responses indicated that, as a result of the extension programme, they recognised increases in their mathematical knowledge and in their general knowledge in relation to specific aspects of school and real life. The following table provides detail in terms of the actual gains identified.

School gain	Real life gain
- Deeper understanding I can understand better why we do certain things in mathematics (S28) I became better at mathematics both from deeper insight to help understand concepts and also from the more challenging activities (S25) - Application I can put the theory into different contexts and find different ways of solving things (S31) - Higher level learning I had the opportunity to go further in mathematics than my peers (S28) pushed my abilities (S19) - Improved grades Became better at mathematics (S6) - Improved problem solving It increased my problem solving ability (S6) I became better at solving problems, mathematics is now simple to me, very logical (S22) - Motivation It made me think, I felt challenged mathematically (S3) Made me try harder to learn different concepts when I know that I may use them when I leave school (S12) The more competitive atmosphere made me work hard, so I don't stick out, I would get lazy in a mainstream class (S20) - Skills I learnt a lot of new skills, like thinking, learning, organizing (S17) - Critical thinking I think differently now, textbooks just tell you what to do and you don't get a chance to use your imagination (S15) I learnt to see things from different perspective	- Mathematics is part of life I see how much more mathematics is there in life than I expected (S4, S8, S11, S33) Appreciate mathematics more as I can recognize it in everyday life (S2) - Awareness It has changed the way I look at stuff, normally I would just walk past and ignore it (S30) I understand why mathematics is so important in society (S12) - Logical thinking I benefited from a kind of logic sequence that has taught me to read between things (S22) - Improved communication I am able to explain things more confidently to my friends in other classes (S18) - Skills I noticed improvement in my researching, presenting, lateral thinking skills. (FG) - History of mathematics I could put a face, a picture to the mathematics (Pythagoras, Fibonacci) (S30) Mathematics means more, not just a number pattern or a rule (S11) It expanded my knowledge, gave background information (S8) I've learnt a lot about ancient Civilizations (S8, S10) See how mathematics evolved through the ages, the differences in different cultures (S7, S24) - Added value I value general knowledge tremendously because it is something that sets you apart from other people (S22)

and thinking on different levels. When we had a problem, we could use multiple methods and we realized that there are many ways to reach a solution. (FG) - Appreciation of aesthetics of Mathematics I can see the beauty in mathematics. I learnt to appreciate how different mathematicians approached their tasks, and how to see beauty in design, form, simplicity, and in logical, clear solutions. (FG)	

Students' responses regarding their experiences in the mathematics extension class, from a mathematical perspective, can be grouped into the following themes:

- Increased mathematical knowledge;
- Increased mathematical awareness;
- Change in students' views of mathematics.

5.2.3.1 Increased mathematical knowledge

All groups highlighted the fact that, as the extension programme was interesting and enjoyable, they had learnt more and their mathematical knowledge increased greatly.

> The extension programme helped with the core mathematics. It made us think, trained our minds, and helped with problem solving skills. We also gained more general knowledge. (FG)

> The ideas and concepts I've learnt expanded my knowledge of mathematics in everyday life. I've learnt more than I expected, which is not only good for my mathematical knowledge, but also increased my general knowledge of mathematics in everyday life. (S33).

The notion that mathematics need not be boring and repetitive in order to be effective, was articulated by students during the focus group sessions. For some, the integration of history into the mathematics class was a surprise.

> The mathematics extension we had was fun and when it is fun it makes you learn more, you do more mathematics than you think. (FG)

> I was surprised to learn the history of mathematics, thought that we would just do hard algebra and things like that in the mathematics extension class. (FG)

Students claimed that the extension programme helped change some students' attitude towards mathematics, as a small group of students initially did not like mathematics and were not very interested in learning it.

> Your attitude towards mathematics changes through the history of mathematics, you have fun, learn more, you enjoy it, get passionate about it, and get caught up in it. (FG)

Students discussed the effect the mathematics extension programme had on their general approach to problems. They became accustomed to follow a logical thinking process, used in mathematics, in other problem situations as well.

> I guess the fact mathematics is really logical and that has affected me because now when I have a problem, that is not necessarily mathematics, I can sit down and work through it logically after doing so much practice in the classroom. (S25)

> I began to realize that mathematics applies to everyday life and it is an essential part of being. It helps solve problems, helps your logical thinking, and helps work out non-mathematical problems as well. (S26)

> It makes me see everything in a mathematician's view and I approach a lot of things in a mathematical way every day. (S14)

Students appeared to be readily aware of the transfer of mathematical concepts to other subjects. Several students reported approaching problems from a mathematical perspective, for example: sciences, social sciences, technology, design, and art. This approach, as students stated, became second nature, not necessarily a conscious decision every time, due to the practice they had in solving complex problems in the mathematics extension class.

> I use practical mathematics skills in art, drawing, graphics. I use it for decision making, to analyze things. (FG)

> Logical thinking is needed in science, social sciences, even for structuring essays in history and in English, the step-by-step logical approach is very good. (FG)

> It's not just how we actually solve a problem, it is that we understand connections, quotes, remarks, participate in conversation, explain things to others, we have an increased background knowledge, and greater general knowledge. (FG)

> The history of mathematics changed my approach to mathematics, I didn't like mathematics before, but I like it now. I'm generally interested in humanities, not sciences, and the history of mathematics formed a link, so I guess, I use mathematics in problem solving, even if I don't think about it. (FG)

5.2.3.2 Increased mathematical awareness

The extension programme resulted in increased awareness of mathematical ideas and concepts in everyday life.

> They made me realize that mathematics is not just in the classroom, but all around us ALWAYS. (S14)

> It makes me so much more aware. I now look at things and actually think about them where I used to just walk past. (S28)

Through increased awareness, students discovered areas of knowledge that they were not exposed to before. While this heightened awareness was noticed by students in more familiar areas, such as science, some students developed a keen interest in architecture.

> They have made me aware that mathematics exists in literally everything and anything can be a mathematics problem to solve if you look at it the right way. It also interests me that a certain order exists in nature as well. (S11)

> Phi makes me look at nature and think. (S16)

> They definitely made me more aware of mathematics in everyday life. I look at things differently now, I notice mathematics more in architecture, art work. (S2)

> Extension activities, such as the geometry research have made me more aware of mathematical ideas and concepts in everyday life. Now, when I pass buildings I look for geometry in them, and when I look at things, such as the plughole, I also notice geometry in that. (S23)

When referring to personal benefits deriving from increased mathematical awareness, some students were commenting on aesthetical aspects.

> I've found more beauty in everyday life. I appreciate the symmetry in buildings, tessellations in jewellery and fabrics, geometry in clothes design. (S29)

Increased mathematical awareness was reported by each focus group. As the result of the mathematics extension problem students became aware of mathematics being an integral part of life and began to see mathematics in virtually everything.

> Mathematics is in everything: fabric technology, food technology, art, design, graphics, science, especially physics. (FG)
>
> FG 1: Mathematics is not just notes, exercises and homework any more
>
> FG 2: Yeah, it is much broader, and is not just for the classroom
>
> FG 1: We got to see how mathematics relates to everyday life, so we are much more aware of mathematics around us

Students' responses indicate their awareness of the transferable nature of the mathematical skills they have learnt and the everyday benefit these skills represent.

> FG 1: Mathematics trains your mind to follow a logical set of steps that can help solve any problem. In all sciences, like bio, physics, chemistry, in art, even in poetry, remember the rhyming patterns?
>
> FG 2: And then in architecture and even in history, like BC and AD.
>
> FG 1: I notice myself thinking about everyday objects mathematically. I see it everywhere I look now in daily life.
>
> FG 3: Yeah, mathematics is in everything and anything can be a mathematics problem to solve if you look at it that way.
>
> FG 2: Like the Golden Ratio, when look at buildings, architecture, I appreciate their beauty. It has changed the way I look at things. I would have never looked at buildings like that before. I would have ignored them, but now I subconsciously analyze, compare architectural styles in buildings.
>
> FG 3: I do see more mathematical relationships in everyday things
>
> FG 1: I like it how it makes you feel special when you get that famous concept, idea of famous mathematicians.
>
> FG 2: Even history is better, I liked the classical history, we learnt about in mathematics much better than the history we learn in the history class, that's only about Germany and wars.

5.2.3.3 Change in students' views of mathematics

Students reported that their view of mathematics changed due to an increased awareness of mathematics being an integral part of everyday life. Through extension activities and research, students discovered mathematics in nature, in art and in areas they had never looked at before from a mathematical point of view. They became more receptive and were able to find aesthetics in mathematics._Students' responses, regarding their view of mathematics, incorporate three general concepts:

- Mathematics is recognised as a very important school subject that is fundamental in other subject areas as well.
- Mathematics is seen as an essential part of everyday life.
- Mathematics is problem solving.

School subject	Part of everyday life	Problem solving
- a very important, essential subject for later studies (3) - a challenging but necessary subject (4) - a necessary subject at school; in real life (5) Learning mathematics is not just about gaining achievement standards, but about knowledge. (FG) - an interesting and easy subject (2) Mathematics is not like it used to be any more. It is not a boring, cold, routine subject. (FG) We learnt about interesting things, like the Golden Ratio, 'Phi' in art, and architecture, and that was cool. (FG) - a mental challenge and an opportunity to learn (2) - a very logical subject that you don't have to study for	- helps understand things around us – understand life (7) Mathematics is very logical and it can give meaning to lots of things that happen in the world (S12) - it is the study of numbers, but it is actually applicable to everyday life (10) - mathematics is essential to our way of living Mathematics is essential to our way of living, without it our society would crumble. I don't get too excited by mathematics, but I must know it to survive (S5) - an important, basic skill, we need it every day like reading and writing (2) - no need for more advanced mathematics in everyday life (2)	- problem solving (4) The extension programme helped integrating all learnt topic areas. We had to apply all the things we've learnt, and had to devise a solution. (FG) When we were solving real problems, long word problems, we had time to work it out, discussed it in class and learnt a lot more mathematics through that than if we had just used the book. (FG) - logical steps that allow us to find a solution (3) Mathematics is a way of solving problems, investigating things and finding out more (S13)

Students reported that the extension programme influenced their view of mathematics. Through a more in-depth exposure to the application of mathematics in everyday life they noticed that mathematics has a strong bearing on our preconceived ideals.

> Mathematics is embedded in our views of how things should be, like what we think of as nice, like symmetry (S24)

They also noted that through a heightened mathematical awareness their view of mathematics has changed from the school subject only concept to that of an essential everyday skill.

> Mathematics plays a large part in practical and theoretical music. Music is a very important part of my life, and mathematics has also helped me with science. (S25)
>
> Through the history of mathematics I see mathematics outside the textbook. (S25)
>
> Good to see mathematics in things that you wouldn't have looked at twice before. (S27)
>
> I've noticed myself thinking about random, everyday objects mathematically, like the symmetry of a cushion. (S30)

Some comments identified a positive change in their view regarding the emotive aspect.

> It is rewarding when you solve a really tricky problem (S13)

The evolutionary nature of mathematics is articulated in one student's view of mathematics:

> As I look at mathematics now I see it as an endless intelligence (S26)

Two contributing factors were identified in focus-group discussions that had a bearing on the change of students' view of mathematics. These were:

- excellent academic achievement
- increased subject knowledge and increased general knowledge

> We had a good balance, focused on core mathematics around assessment times, so we got good grades, and gained extra knowledge when we had time for extension activities. (FG)

Regarding the non-assessed nature of the extension programme, some students found that they did not need to work as hard while studying extension topics, while a few other, less mathematically able students felt they were under pressure to keep up with the more able students in the extension class.

> Not everyone is mathematically able here, some people are lazy, don't want to do extension work. If there is no assessment, they can slack off. (FG)

> There is a lot of pressure to keep up and some students need to spend extra time on extension mathematics at home. (FG)

Even those students who at first were concerned regarding time spent on extension activities, realized the value of the mathematics extension programme in terms of benefits in their academic results.

> I was concerned at first, I thought we won't have time to do the core stuff and all that extra extension work, but it worked out well. The extra work ended up making the core easy, it helped with our thought process. Jogged your mind. And showed you that there are lots of different ways of solving problems and that they can all be good. (FG)

Another indication of the change of students' view of mathematics was their readiness to carry out further work and to consider career options involving mathematics.

In response to describing any further work carried out in their own time, students stated that the extension activities inspired some of them to carry out research and investigations in certain topic areas. A lot of their work involved finding out more about famous mathematicians (Fibonacci, Pythagoras), using the internet and the library. In some instances their research sparked interest in different areas of learning.

> Because of the study of Raphael's painting 'The School of Athens', I have now become very interested in Classical Studies. I am fascinated by the secret mathematics societies and the evolutionary ideas of many Greek mathematicians. This is something that only happened because of the extension course. (S22)

> I found a lot out about Phi, in nature, in architecture (S15)

Many continued on from their geometry investigation and looked at famous artists, buildings, mathematicians well-known for their interest and work in geometry.

> I continued to research Luis Tiffany (geometry project) and his works. I have looked at some of the background information on him and his works. (S2)

> I researched more about the man who made complex geometrical shapes from paper plates. (S5)

Some students carried out more in-depth research into the history of number and trigonometry.

> I was interested by the different ways of multiplication carried out in different countries. This inspired me to research more countries and find out how they used to (or even still do) multiply numbers. I researched quite a few countries and now I sometimes use those methods when I don't have a calculator with me. (S23)

> I was quite interested in the calculations in different bases and transformations in everyday life. For calculations in different bases and for the binary system I tried to work out lots of normal numbers to other bases to see what they came out as. Also for the transformations I identified many transformations around the house that I have never realized was there before. (S7)

There was also a small group of students, who didn't carry out extra work, however, they would have, if time had permitted.

> Guess I've looked deeper into science and music to find mathematics, however this extension year has been far too busy for me to do more mathematics extension in my own time, when I had to study for IGCSE and for NCEA as well in other subjects. (S25)

As expected with this Year 10 extension class, a number of students identified future plans that involved mathematics, some plans were somewhat uncertain, but with a desire to continue with mathematical studies, and others did not include mathematics. When asked whether their experiences of learning mathematics this year had affected their career aspirations, 12 students reported that their plans had changed as a direct result of their positive experiences of the mathematics extension programme.

Most notably, students reported increased interest in mathematics, in the history of mathematics and in ancient history, interest in subject areas that involved mathematics, interest in the teaching profession, and the desire to study mathematics at tertiary level.

Future plans involve mathematics	Future plans don't involve mathematics	Plans are not definite (may involve mathematics)
- accountancy, financial work (x3) Hoping to do something in business, mathematics extension has changed my view of mathematics. I want to do something with numbers. (S1) - GP, paediatrician, medical school (x4) Need mathematics to go to med. school & the extension programme has taught me that mathematics isn't boring and not too hard (S8) Beforehand I aspired to study law, but now my focus has changed to sciences. Mathematics has now become a stepping stone to my ultimate life goal – to study medicine. (S22) - architect (x4) The idea of studying architecture has become more appealing (S11) - tertiary study (involving mathematics) Due to the extension programme my mathematics has improved heaps. It challenged and pushed me and made me get better at mathematics. It has made me think that I should do something involving mathematics at uni (S31) I am history-classics oriented, but the mathematics extension programme broadened my knowledge of ancient civilizations	- lawyer / journalist I play by my strengths: English and History (S32) I'll most likely become a lawyer, however I much appreciated learning about mathematics this year, I still continue to fulfil my previous aspirations (S24) - high-school teacher (x2) (history, PE) The extension programme has shown me that if you are a good teacher and find common interest activities and present them in the right way, you can generate interest within the subject, even if it is usually seen as boring and pointless (S10) - army / navy / actress / author /(S21) - wedding planner (S29) - fashion designer(S6) - sport I like to work in sport, no my Y10 mathematics experience probably hasn't changed my aspirations for later life (S30)	I haven't made up my mind yet, but these activities influenced a lot how I see mathematics now. I understand more of its history and origins. Perhaps when I do things later I would be able to use this knowledge.(S7) I don't know, but mathematics extension helped me to keep my options open (S19) I am still unsure, but it would probably be something to do with mathematics. As the extension programme showed us, many different aspects of life, such as architecture, involves mathematical concepts. (S23) Maybe bio-engineering, but I'm a lot more interested in archaeology and ancient history now (S27) I really enjoyed learning some mathematical history. Maybe I continue it later. (S33)

from a mathematical perspective. This encouraged me to continue learning mathematics (both present and the history of mathematics) (S12)		

CHAPTER SIX: Discussion and Conclusion

Introduction

This chapter reviews the classroom setting and programme design features and then, based on the research findings (Chapter 5), discusses how the programme of enrichment with history appeared to impact on students' learning and perceptions of mathematics.

The research study was carried out to explore students' perspectives of a mathematics extension programme that was designed to cater for the needs of this Year 10 class of academically able and talented students. The researcher experienced positive attitudes and willingness on the students' behalf towards participation in the research; similar to that identified by Kane and Maw (2005) in their *Making Sense of Learning at Secondary School Project.* The students were willing to contribute, as the focus of the research had relevance to them as secondary school students and they had a well-established positive relationship with their mathematics teacher- researcher and with each other.

The selection process into this mathematics extension programme resulted in a unique situation. Students were selected into this class based on their overall academic ability, however, not all girls in the group saw themselves as mathematically able and talented students. Designed to unify this otherwise diverse group based upon their common interest (history), this mathematics extension programme approached mathematics from a historical perspective.

In this chapter, following a reflection on the classroom situation, students' experiences and their perceptions are discussed in three areas:

- Historical focus of the mathematics extension programme;
- Learning in the mathematics extension class;
- Long-term outcomes of the mathematics extension programme.

6.1 The classroom situation

My intention, as the mathematics teacher of the Year 10 extension class, was to create an extension programme that built on students' strengths and interests in order to provide a challenging and enjoyable course. The need for designing extension programmes, that suit specific situations, is supported by Riley and Bicknell (2005), as they argued that there is no 'one-size-fits-all' solution regarding provisions for the gifted and talented. They called for flexibility and variety in order to accommodate the differences in students' intellectual, social, emotional and cultural profiles.

In this extension class only 10 out of the 33 students regarded themselves as excellent students of mathematics. They had varied previous experiences in the mathematics classroom, had different opportunities to build information and skills and individual students had different mathematical preferences. Based on the students' description of their way of studying mathematics, about one-third of students in this extension class could not be identified as mathematically gifted. When characteristics of giftedness in mathematics are discussed, the preference for a routine, somewhat mechanical way of learning, is never included. Rather the opposite is the case, as supported by Schiever and Maker (2003) who claimed that giftedness is the ability to solve more complex problems. Gifted students tend to be bored with the conventional, they are divergent thinkers, prefer open-ended and more complex activities. However, by the end of the year, possibly as a result of their extension programme, the majority of students expressed interest and enjoyment while working on complex mathematical problems.

Even when students are identified as mathematically gifted, they do not necessarily form a completely homogeneous group (Holton & Daniel, 1996). Whilst students of this extension class were not all mathematically gifted, they were all highly motivated to gain excellent academic results. They all displayed task commitment – one of the characteristics identified by Renzulli (1977), as one of the components of gifted behaviour, and essential for high academic achievement.

At the end of the year, the summative assessment results provided clear evidence that almost all students in the class were mathematically able and capable of gaining very good grades in mathematics. Twenty-nine students received an invitation to a Year 11 accelerate class to study Cambridge mathematics (IGCSE) in conjunction with the NCEA Level 1 course. Their path to excellence, in some cases, led through the more conventional 'memorise-practice-repeat-apply' approach, while in some cases through a high level of understanding and ability to solve complex problems.

Despite excellent academic results, these students viewed their achievement with considerable modesty. One might ask, are these students too critical of themselves? Do they have exceptionally high expectations of themselves? The answer is, possibly, yes. Moltzen (1996) argued that gifted children not only think differently but also feel differently from their peers. He listed perfectionism and idealism among their emotional characteristics. Students in an extension class measure themselves against the very best in the subject. As the selection into the extension class was not subject-specific, but based on overall academic ability, some students found themselves under a lot of pressure to keep up with the mathematically very able students in the class.

6.2 Historical focus of the mathematics extension programme

In order to gain all students' interest in the mathematics extension programme, the obvious choice, in this class, was to approach mathematics from a historical perspective. Several educationalists support the inclusion of the history of mathematics in mathematics education. Freudenthal (1983), Tzanakis (1997), Ransom (1991), Arcavi (1987), Nouet (1996) are among the many who highlight the educational and emotional benefits. Fauvel (1991) however, raised a possible objection, that students may find history boring. This possible problem certainly did not apply to this class. In fact, history became the bridge that led students to mathematics.

When Fried (2001) discussed the inclusion of the history of mathematics in the teaching of mathematics, the time constraint was raised as a potential problem. I was very much aware of this problem while planning the extension programme. In my planning and teaching I placed strong importance on student-initiated learning. This approach, favoured by students, enabled them to make real decisions in what was studied in the programme, and allowed all of us to take ownership of the course.

While the study does not provide causation evidence as to why the students in the class gained high academic results at the end of the year, student feedback from the current research reports that they were engaged in meaningful, challenging learning activities, they were motivated, and gained exposure to complex problems. Specifically, the history of mathematics created an interest in mathematics. Many students noted that this engagement factor resulted in further learning, and in a positive change in their view of mathematics. There was ample opportunity to incorporate students' interests in the extension programme. This resulted in greater engagement, and enjoyment of the subject.

6.2.1 Students' perceptions on including the history of mathematics in their mathematics programme

According to students' reports, the history of mathematics helped them learn mathematics better. Through all collected data, there is a frequently recurring statement: 'the history of mathematics made mathematics much more interesting and it motivated us to learn'. Cathcart (1994) contends that 'generating a high level of interest' is at the top of the list of four essential components of a programme for gifted and talented students. Raising interest in the subject was an important aspect of this programme, and the history of mathematics did just that.

Specifically, students identified that the historically inspired exercises made mathematics more relevant, and gave a purpose to solving more difficult problems. Another reported outcome of integrating history into the mathematics activities was the change in student views about mathematics. Students frequently commented on the fact that the history of mathematics allowed them to see mathematics in a different light. Before the programme they were looking at mathematics as a predominantly cold, but essential school subject, with set rules and formulae that 'were always there'. Through the extension programme they began to see the nature of mathematics differently. Mathematics, in their eyes, became the result of human endeavour. History also highlighted the evolutionary nature of mathematics. Through their studies of famous mathematicians, students came to the realization that the road leading to mathematical discoveries and learning, is marked with successes, failures, doubts, arguments, and alternative approaches to problems.

In this multi-ethnic classroom, students greatly appreciated the cultural aspects of mathematics. The history of mathematics allowed them to get acquainted with the mathematics associated with other cultures and from different times in history. Students reported on a sense of pride when learning about mathematical discoveries belonging to their particular cultural heritage, and also of a greater appreciation and respect towards the contribution made by different cultures.

Students commented on the range of practical ways the history of mathematics was included in the classroom. The inclusion of different practical methods, when implementing the history of mathematics in the teaching of mathematics, was also endorsed by Arcavi (1987). In this extension class a range of historical snippets, research projects, historical problems, worksheets, experimental mathematical activities, films, visuals, and outdoor experiences, represented the different modes of implementation. This variety, as claimed by students, ensured interest, motivation, and allowed for accommodating students' different learning styles and preferences in learning.

Students identified numerous positive aspects of including history, historically inspired exercises, and the history of mathematics in each topic area studied. These included greater, deeper understanding of the content studied in the topic area; gaining background information, resulting in increased mathematical and general knowledge; positive attitude towards learning; appreciation of human endeavour in the development of mathematics; increased cultural awareness; and a significant change in how they view mathematics.

In terms of practical extension activities, students reported on:

- Gaining an insight into the life and work of many famous mathematicians;
- Greater understanding and appreciation regarding the evolutionary aspects of mathematics;
- Increased cultural awareness;
- Gaining an appreciation of the aesthetic and recreational side of mathematics;
- Finding enjoyment in mathematics;
- Discovering inter-relationships between mathematics and other subject areas.

6.2.2 Students' perceptions of integrating humanities and sciences

Integrating history and mathematics allowed for a holistic approach towards learning. Johnson (1994) supported the idea of interdisciplinary connections and claimed that this is a key element to be included in the mathematics curriculum for the mathematically gifted and talented students.

As a consequence of the integrated nature of the programme, students reported on discovering relationships between mathematics and other subject areas. Their accounts identified two broad lines where these connections were made:

- Between history and mathematics, and within mathematics itself;
- Between mathematics and the real world through all other disciplines.

First connections were made between history and mathematics, and within mathematics itself. Mathematics became a 'human' subject, inseparable from its history, its origins, and development. Mathematics came to life. It ceased to be a cold, dry subject of facts and figures, as it was often seen before, a subject made for the classroom.

The history of mathematics also allowed students to see mathematics more holistically as a body of knowledge. They began to appreciate the connections between different topic areas and were less inclined to view separate topic areas as fragmented units of information. By working on complex, relatively difficult problems, students commented that they had to devise their own strategies which often incorporated knowledge gained in different topic areas. They also valued the opportunity to discuss different solution methods, and to gain exposure to solutions to problems created by famous mathematicians.

The history of mathematics allowed students to see the two sides of mathematics, the utilitarian and the recreational side. They were able to identify that the need arising in a different social, cultural environment in a certain time in history, initiated a certain mathematical discovery. On the other side, they also gained exposure to purely recreational problems that conveyed mathematics also exists for fun, enjoyment, and for the sake of a challenging mental exercise.

The second broad line, where connections were made, was between mathematics and the real world through all other disciplines. By discovering that mathematics is in everywhere, students were able to appreciate and value the intellectual side of learning mathematics. They identified strong interconnections between mathematics and science, and felt that their mathematical knowledge proved very useful in all sciences.

As an outcome of the extension programme, mathematics, as a subject, also took on an emotional appeal. Students reported on feelings, such as enjoyment, fun, joy, excitement, and the ability to see beauty in mathematics. Naturally, there is a strong connection between learning and emotional appeal. When students were working on topics they were enthusiastic about, they reported on increased awareness and insight, indicating that integration was taking place. These research findings support Boyer's claim (1990), that integrative learning has intellectual and emotional appeal.

6.3 Learning in the mathematics extension class

Students discussed learning in the mathematics extension class from three perspectives: the pace of learning, the academic focus, and social and emotional aspects. Their reflections support the emphasis Riley and Bicknell (2005) placed on these three components of the curriculum for the gifted and talented, by claiming that the mathematics curriculum should be rich in depth, should proceed at a pace in line with students' abilities, and should address social and emotional needs.

6.3.1 Students' perceptions on pace of learning

Students' positive response, relating to the pace of learning in the extension mathematics class, was encouraging. The majority of students claimed that the pace was just right. It allowed them to complete all core material to a high standard and to pursue interesting extension topics and activities. As students found the extension activities especially enjoyable, some expressed a wish to be able to do more, even at the expense of core work. Their comments highlight the dilemma teachers face when choosing the appropriate strategy to introduce the history of mathematics in the mathematics curriculum.

Those small group of students, who identified themselves as mathematically not very able, felt the pace of learning was occasionally too fast for them. They felt their focus should be on the core curriculum in order to gain high grades. However, as the content studied in the extension programme was not part of any summative assessment, they were generally happy to participate in the extension activities, and found it beneficial and even enjoyable from time to time.

6.3.2 Students' perceptions on academic focus

There is a clear indication, based on students' responses, that maintaining a strong academic focus is absolutely essential in the extension mathematics class. Students placed great importance on achieving high grades in the subject and, to a certain extent, assessed the success of the extension programme in terms of their academic achievement. This approach to learning is discussed by Renzulli (1977), who linked task commitment to motivation, and claimed that task commitment was one of the components of gifted behaviour. As expected from these academically very able students, they displayed strong task commitment.

Even those students, who were reluctant at the start to engage in the extension activities, identified academic benefits that were a direct result of their participation in the extension programme. By focusing on mathematical concepts and ideas from a historical and interdisciplinary angle, they were able make considerable progress in mathematics.

As identified, the previous experiences of some students, regarding studying mathematics, were not always pleasant. Some students used to associate mathematical learning with completing copious amounts of boring, repetitive exercises from the textbook. In the extension class they were using interesting, motivating and complex problems, and they found that they were learning much more mathematics than they had realized at the time.

The extension topic areas studied were linked to the core topics and provided students with in-depth knowledge in the particular topic area. Students remarked on the fact that the extension activities enriched their mathematical learning by providing a greater and deeper insight into topics studied. Exposure to mathematical concepts and skills required them to attempt complex problems. This resulted in improved higher order thinking, and presented several opportunities to practise creativity and independent thinking. Their previously fragmented mathematical knowledge was starting to turn into a complex body of knowledge to call upon in different situations.

6.3.3 Students' perceptions on social and emotional aspects

Responses from both the questionnaires and focus group discussions confirmed that the majority of students identified positively with the learning environment. General comments, relating to classroom atmosphere and learning, described an environment where everyone benefited from the programme.

Competitiveness was raised as a concern by some, but a virtue by others. Some students expressed concern in terms of academic pressure, resulting from being in an extension class. This pressure applied in all subject areas to a mostly different group of students, as their academic ability varied from subject to subject. They were under more of an emotional-type pressure in the mathematics extension class, as their academic grades came from the core curriculum only. These students described feeling the pressure to come up with smart solutions to difficult problems in an acceptable time frame, so they do not loose face in the class. The history focus, however, allowed these students to shine from time to time and they very much enjoyed their success in certain topic areas. In fact, they claimed that the extension programme motivated them to work harder. Several of those students who did not think of themselves as mathematically talented, and did not have much interest in mathematics before, claimed that without the extension activities they would have become lazy and lost interest in class work.

Students reported on their enjoyment of working in groups. They benefited from group and whole class discussions and appreciated the opportunity to gain exposure to different perspectives. Being able to engage in activities that ensured creativity was regarded as highly rewarding and enjoyable. Stanley's (1980) and Fraser's (1996) claim, regarding creativity, as the key characteristic of gifted and talented students, supports the findings of this research. Students were delighted to be able to make choices and decisions affecting their learning, to pursue their individual interests, to be genuinely challenged, to gain exposure to creative approaches, and to be able to get their creative responses taken seriously and considered as appropriate solutions to problems. As they had listed their favourite extension activities, they identified opportunities to work independently, in pairs, as a small group, and highlighted the benefits they've gained from group discussions. The social-emotional gains derived from class discussions included pride in one's ability, experiencing the joy of success and approval, excitement of a challenge, and acceptance.

Responses relating to cultural awareness, appreciation, and acceptance were also indicators that the extension programme was reaching higher educational goals. The history of mathematics allowed students to gain insight into the origins and development of mathematics. Through their discoveries they came to the understanding that different cultures made different contributions to the mathematics of today. They displayed pride in their own culture and became appreciative of other cultures, resulting in a very positive, tolerant and appreciative classroom atmosphere.

6.4 Long-term outcomes of the mathematics extension programme

According to students who participated in this study, there are a number of long-term benefits gained from participating in the mathematics extension programme. In terms of school mathematics, students identified a strong positive relationship between their excellent academic achievement grades and the work they had completed in the extension programme.

By incorporating the history of mathematics and integrating other subject areas into the mathematics programme, students' mathematical awareness increased considerably. This manifested itself in different everyday situations where students reported seeing mathematical concepts and ideas in areas they had not previously expected to find anything mathematical. As indicated by educationalists (Chandrasekhar, 1987; Hallez, 1990; Nouet 1996), the students and the researcher also found that the research assignments and presentations students completed, the history of mathematics they have studied, the complex problems they have solved, all helped in generating increased mathematical awareness.

The history-based mathematics extension programme generated interest in the subject that resulted in increased motivation to work well and, in some cases to follow on with additional research and investigation. Through increased mathematical awareness students' views of mathematics changed. This is one of the great benefits of the programme, indicating a lasting change, especially when coupled with students' desire to include mathematics in their future learning. Positive affinity towards mathematics resulted in the change of the future plans in a number of cases. Students expressed their wish to pursue careers they began to see appealing through their exposure to 'interesting and enjoyable mathematics'.

Apart from the academic gains in mathematics, students saw reported benefits in other content areas. These areas included the sciences, history, art, English language and social sciences. Students reported on improved study and research skills, logical thinking, logical structuring, analyzing. These skills are readily transferable to other subject areas, and can be called on as essential tools to solve problems. Beyond the subject-specific gain, general academic gain was reported in increased knowledge and understanding regarding the history of mathematics, art and architecture.

The extension programme situated mathematics in real-life context. A frequently recurring statement, that 'history humanizes mathematics' speaks of students' changed view of mathematics. The history of mathematics introduced students to famous mathematicians, to their work and achievements, and allowed them to see mathematics as a constantly evolving body of knowledge through different cultures and different times. Students came to perceive mathematics as a very important subject that is part of everyday life. Their mathematical knowledge was used to solve real-life problems and helped them understand the world, 'to read between things'. Through the history of mathematics and by integrating different subject areas into mathematics, students were able to discover inter-relationships and developed a 'mathematical eye' for everyday situations. Mathematics ceased to be a subject only used at school; it became an essential pool of knowledge and skills that are used in everyday life.

6.5 Further research

While the findings of this single case study allowed the researcher to analyze the students' perceptions of this mathematics extension programme, further research is needed to look at similar and contrasting ways to provide enrichment programmes at Year 10 level. This line of further research is supported by Yin (1994) who claimed that evidence found from multiple case study designs are often considered to be more compelling than data obtained from single case study designs.

There are other areas, relating to this study, that warrant further research. This case study assessed students' perceptions on the cognitive and affective effects of the mathematics extension programme. Subsequent research could assess parents' perceptions of programmes that focus on enrichment/extension. Teachers' perceptions could be assessed regarding the relative ease and difficulty involved in the implementation of this programme.

A nation-wide survey of secondary schools could be carried out that aims at assessing the need for more varied provisions for academically talented students. Riley and Bicknell (2005) called for flexibility and variety on the schools' part, in order to meet the needs of the individual gifted and talented students. To improve the inconsistent and scattered approaches, currently employed (Riley & Bicknell, 2005), a nation-wide research could identify effective classroom-based enrichment provisions. The findings of this large-scale investigation could result in providing schools and teachers with more diverse, extensive and effective resource material to be adapted to specific student contexts in mathematics enrichment.

CONCLUDING THOUGHTS: My Reflections as a Teacher

At the beginning of the year the students in my Year 10 class did not know what to expect from their mathematics extension programme. As they had no previous experience with an extension programme, some of them thought they would do more problems, perhaps more difficult problems, but in general, just have the same type of experience in the mathematics class as they had before.

The extension programme approached mathematics differently to what these students have experienced before. Introducing the history of mathematics proved to be an effective approach. It was based on their common interest, history, and built on their existing knowledge in a subject area they all enjoyed. It also harnessed their intelligence in the linguistic and emotional domains.

Learning the history of mathematics featured as the favourite extension activity. The history of mathematics helped students see how mathematics evolved over time, gain deeper insight into the origin of the theory studied, enhanced understanding, and allowed them to see the human effort involved in each discovery. The history of mathematics increased their general knowledge and allowed for the integration of other subject areas. Students were able to apply and discover mathematics in art and architecture.

Students enjoyed the historic activities that involved the history of mathematics, studying the life of famous mathematicians, and solving famous problems. Completing research assignments, investigations, presentations that related to mathematics in everyday life was another favourite. There were opportunities for students to have their say in what they were learning, make recommendations and choices. Complex problems, which could be solved in many different ways, allowed students to engage in in-depth class discussions. Students also found intercurricular activities enjoyable. These incorporated history, art, architecture, science and generated further interest in those topic areas. Some activities allowed students to physically participate, 'act-out' the problem, and these were enjoyed by the many kinaesthetic type students in the class. Students appreciated the opportunities to engage in artistic, creative and fun activities.

By providing this extension programme to the Year 10 history option students for the whole school-year, I witnessed a win-win situation. Students who enjoyed mathematics before and considered themselves mathematically able progressed further. It broadened their horizon, created interest in different areas and developed a greater appreciation of mathematics. Students who have never liked mathematics and considered themselves as average or not very able in mathematics, also benefited greatly from the humanities focus of the extension programme. Portraying mathematics in a different light allowed these students to overcome their initial dislike towards mathematics. Once this barrier was broken, these very able students made good progress. They achieved higher grades than they expected in their NCEA assessments, mostly excellences and some merits, and most importantly their view of mathematics changed for the better.

Students' approaches towards studying mathematics have changed during the course of the year. While initially they identified with rather routine-based methods, the students realized that there are other effective ways of learning mathematics. They came to enjoy working on challenging problems, finding their own solution methods, discussing, analysing and debating a particular approach, comparing it to the methods used by famous mathematicians. Students were more willing to take an investigative approach, rather than relying passively on teacher explanations and notes. Routine practice exercises were gladly exchanged for complex, real-life problems. Students, initially reluctant to deviate from their well-practised, trusted methods, found that their fears were unfounded.

The level of enjoyment, displayed by students, in the mathematics class was very pleasing. As a teacher, I aim to inspire my students and impart a love of learning. My students responded very positively to this approach and they frequently remarked on how much they enjoy mathematics now.

> What really helps me learn maths is to have a passionate teacher who cares about what she is telling you. (S25)

This research, through the students' voice, offers evidence that the outcome of this extension programme is positive. For students, it resulted in excellent core achievement grades as well as increased mathematical knowledge and awareness.

Moreover, a positive change occurred in the students' view of mathematics. Classroom mathematics ceased to be a compulsory school subject only. Instead, it became a treasure trove of fascinating and very relevant knowledge as well as a source of enjoyment. The future of these types of extension programmes is promising, as interest and enjoyment of the subject are vital to real success. As the teacher of this extension programme, I am in agreement with my student's advice: "I definitely would encourage this course to continue." (S22)

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APPENDICES

APPENDIX 1: Student self-evaluation

1) Describe how you see yourself as a mathematics student.

2) How do you study mathematics?

3) Intelligence can be observed in different areas.
After reading the following section, rate your intelligence on the scale of 0 – 4 in each of these categories (0 being the lowest, 4 the highest).

1. Linguistic (mastery, sensitivity, desire to explore, and love of spoken words, spoken and written languages).
2. Logical-Mathematical (confront, logically analyse, assess and empirically investigate objects, abstractions and problems, discern relations and underlying principles, carry out mathematical operations, handle long chains of reasoning).
3. Musical (skill in producing, composing, performing, listening, discerning and sensitivity to the components of music and sound).
4. Spatial (accurately perceive, recognise, manipulate, modify and transform shape, form and pattern).
5. Bodily- Kinaesthetic (orchestrate and control body motions and handle objects skilfully, to perform tasks or fashion products).
6. Interpersonal (sensitive to, can accurately assess and understand other's actions, motivations, moods, feelings, and other's mental states and act productively on the basis of that knowledge).
7. Intrapersonal (ability to accurately assess, understand and recognise one's own motivations and productively act on the basis of that knowledge).
8. Naturalistic (expertise in recognition and classification of natural objects, i.e., flora & fauna, or artefacts, i.e., cars, coins or stamps).
9. Existential (capturing and pondering the fundamental questions of existence, an interest and concern with 'ultimate' issues).

4) In your opinion, is intelligence fixed or can it be increased / developed?

APPENDIX 2: Student questionnaire questions

1) How are things different in your extension mathematics class to a mainstream mathematics class?

2) What are the good things about the mathematics extension programme?

3) What type of extension activities do you enjoy?

4) What do you see as negative aspects of your Year 10 mathematics course (if any)?

5) As you are interested in history, (you chose it for your option) has the extension programme changed your interest in mathematics? Describe these changes.

6) Explain what mathematics means to you, describe your view of mathematics. What benefits did you gain from the extension activities?

7) List 1-2 specific extension activities, maybe your favourites, and describe the benefits you have gained from working through them.

8) What is your opinion of the level of the extension activities? Did you feel that the mathematical content was manageable but also challenging for a Year 10 extension class?

9) If you were the teacher what sort of extension activities would you include in the extension programme?

10) What is the most effective approach for you; that allowed you to get the most out of the extension activities?

11) Have some of the extension activities inspired you to carry out further work, investigation in a particular area? If so describe what you have done.

12) How do the extension activities affect your awareness of mathematical ideas and concepts in everyday life?

13) What are you hoping to do when you leave school? Have these aspirations changed because you participated in the mathematics extension programme?

APPENDIX 3: Summary of focus group interview guiding questions

1) A good extension programme is rich in depth and meets academic needs; proceeds at the right pace; and addresses social, emotional and cultural needs. In your experience, how did this extension programme address these issues?

2) There are many reasons to include the history of mathematics in the teaching of mathematics. Can you think of any reasons to include the history of mathematics in the mathematics extension programme?

3) What is the value of using historically inspired exercises? (Fibonacci's Rabbits, calculations in different number systems, Escher – type drawings, the Two Towers, King Arthur, Trig. Problem set in Ancient Egypt, Archimedes' volume problem)

4) What is the value of completing historically inspired, small projects? (biography research, ancient mathematicians, geometry poster, measurement research, speech)

5) Some of you started the year with not liking mathematics much. Has the extension programme changed any of this? (for you or friends in the class) How, why?

6) Do you approach problems from a mathematical perspective, in any other situation really, apart from the mathematics class?

7) The extension activities helped you develop some skills that are not necessarily seen as 'mathematics skills'. Can you describe some of these?

8) Someone said "mathematics exists in everything and anything can be a mathematics problem to solve if you look at it the right way". What do you think she meant?

9) A lot of you said that the extension programme increased your awareness of mathematics in everyday life. Can you describe your experiences?

10) At the beginning of the year some students were a bit apprehensive regarding the extra work the extension programme involved. How has the extension programme helped / hindered preparations for the assessments?

APPENDICES 4-9: Teaching instructions for units of extension work

APPENDIX 4: History of number

1) The concept of number

Origins: everyday life, magnitude and form; not a unique discovery, but gradual awareness from as early as 300000 years ago; contrast: one wolf and many, similarities: one sheep, one tree; early ancestors: one, two, many

Early number bases: used fingers on one, both hands, fingers and toes; stones piled in groups of five (hand); Aristotle: widespread use of decimal system is due to the anatomical fact – ten fingers, ten toes; recording: notches in a stick or a piece of bone

2) Early records

The rise of civilizations: Egypt, Mesopotamia, India, China
(the use of metal tools took place at first in the river valleys: Nile, Tigris and Euphrates, Indus, Yangtze)

Egypt:
Rosetta Stone (1799 – Napoleonic expedition - trilingual account in Hieroglyphic, Demotic, Greek – Champollion and Young)
Hieroglyphic numeration - 10-base system – as old as the pyramids, dating back 5000 years
* Use ancient Egyptian numbers
(1=stick, 10=heelbone, 100=coil of rope, 1000=lotus flower, 10000=bent finger, 100000= burbot fish, 1000000=kneeling figure, God))
Familiarity with large numbers – astonishing accuracy in counting and measuring, high degree of precision in construction and orientation – construction of great monuments

* Use Egyptian multiplication technique

Ahmes Papyrus – addition (our operations of multiplication and division were performed as successive doubling / halving: eg: 69x19: add 69 to itself to get 138, add this to itself to get 276, add again to get 552, then once more to get 1104, which is 16 times 69. As 19=16+2+1, just add 1104+138+69 to get 1331)

* Astronomy – observed that the annual flooding of the Nile took place shortly after the heliacal risings of Sirius on the East, that were separated by 365 days

Egyptian Solar calendar made up of 365 days: twelve months of 30 days each and 5 extra feast days

Used method of false position

Unit fractions (fractions with unit numerators) were used, apart from the special fraction: 2/3. To find one third of a number they first found two thirds, then divided it by two. They had a preference for fractions derived from 'natural' fractions 1/2, 1/3, 2/3, by successive halving.

Mesopotamia:

Cuneiform documents (wedge shape writing on clay tablets, date back 4000 to 5000 years)

Behistun Cliff (trilingual account in Persian, Elamitic, Babylonian; mathematical content analyzed by Thureau-Dangin & Neugebauer) Wedge-shaped marks, standing vertically or sideways, represented the units and tens.

Base 60 was used, probably as it allowed ten possible subdivisions (by 2, 3, 4, 5, 6, 10, 12, 15, 20, 30).

* Relative positional numeration: the symbol could be assigned any value, depending on the relative position in the representation of the number

(eg: *YY YY YY* – three groups of wedges could denote: $2(60)^2 +2(60)+2$

(or 7322 in our notation, however, they had no clear way of indicating an empty position, so *YY YY* could mean either $2(60)+2$ or $2(60)^2 +2$)

From the time of Alexander the Great a special sign, two small wedges placed obliquely, served as a placeholder when a numeral was missing

(eg: *YY // YY* meant $2(60)^2 + 0(60) +2$), however the sign was used for intermediate empty positions only, not in a terminal position.

They extended the principle of position to fractions as well, their fractional notation was the best in any civilization until the time of the Renaissance.

China:

Date back 3000 to 4000 years – Chou Pei Suan Ching document (considered to be the oldest of the mathematical classics, is concerned with astronomical calculations, properties of right-angle triangle, use of fractions)

Geometry was only an exercise in arithmetic or algebra – indications of the Pythagoras' theorem in the Chou Pei.

The Nine Chapters on the Mathematical Art (Chui-chang suan-shu) – 246 problems on surveying, agriculture, partnerships, engineering, taxation, calculation, solution to equations, properties of right angle triangles.

* Use 'rod numerals'

digits from one to nine:

the first nine multiples of ten:

These symbols were positioned alternately from left to right to represent any number.

The symbol for the empty position: O, appeared relatively late (XIIIth century).

* Magic squares (Chinese were fond of patterns, first record of magic squares)

- complete magic squares
- write your own magic square for others to solve

India:

Archeological excavations at Mohenjo Daro identified a highly cultured civilization in India during the era of Egyptian pyramid builders, however there are no Indian mathematical documents from that age.

The author of one of the oldest Indian mathematical texts, Aryabhata was born in 476 AD (year of the fall of Western Roman Empire).

* Hindu numerals

(development: Mohenjo Daro era – vertical strokes arranged in groups; by the 3rd Century BC Karosthi script – higher order symbols for four, ten, twenty and one hundred; then Brahmi characters, similar to the alphabetic Ionian system; Babylonian positional system led to the modification of the Brahmi system and the Chinese influence of rod numerals led to the reduction of nine ciphers)

3) New cultures along the shores of the Mediterranean Sea

Thalassic Age ('sea' age) 800BC to 800AD

Greece:

VIth Century BC – Thales and Pythagoras – imaginative spirit, travels in Egypt and Babylon – Greeks keen to take over elements from foreign cultures – a lower level of arithmetic or computation satisfied the needs of the vast majority (Hellens were keen traders and businessmen)

* Use symbols to write numbers in 'Greek'.

Two main systems of numeration: Attic and Ionian. Both systems are for integers, based on the ten-scale.

Attic (Herodianic) notation is the earlier, more primitive. Numbers one to four were represented by vertical strokes, for five the first letter was used: Π (or Γ of the word for five, pente), for six to nine the combination of Π and unit strokes, then the initial letters for the powers of ten were used: for 10, deka: Δ; for 100, hekaton: H; for 1000, khilioi: X; for 10000, myrioi: M.

Ionian (alphabetic) notation used 27 letters of the older Greek alphabet: nine for integers less than 10, nine for multiples of 10 that are less than 100, and nine for multiples of 100 that are less than 1000. (Characters in brackets are approximations of the archaic letters used)

α β γ δ ε (ϛ) ζ η θ (numerals 1-9)

ι κ λ μ ν ξ ο π (ϙ) (numerals 10-90)

ρ σ τ υ φ χ ψ ω (ϡ) (numerals 100-900)

Arabic Empires:

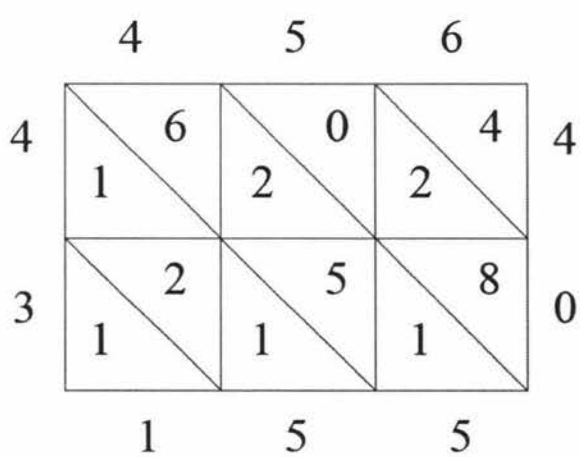
Prophet Mohammed (born in Mecca, 570AD) – Mohammedan state with centre at Mecca – strong influence on the development of mathematics – Mohammed dies in Medina, in 632 while planning to move against the Byzantine Empire - his followers rapidly overrun neighbouring territories (Damascus, Jerusalem, Mesopotamian Valley, Alexandria – mathematical center of the world conquered in 641)

Arabs were eager to absorb the learning of civilizations they had overrun. Scholars were called to Baghdad from Syria, Iran, Mesopotamia, and the Byzantine Empire; great works of ancient science and mathematics were translated to Arabic.

Mohammed ibn-Musa al-Khwarizmi – wrote two books on arithmetic and algebra. Only one of these survived in Latin translation (Concerning the Hindu Art of Reckoning). In this he gave full account of the Hindu numerals - translators / readers attributed not only the book but the numeration as well to Al-Khwarizmi (resulting in the false impression that our system is of Arabic origin). The new notation became known as al-Khwarizmi – algorismi - algorithm.

4) The Hindu- Arabic System:

Earliest reference to Hindu numerals: 662 – writings of Severus Sebokt
The symbol for zero - first occurrence in 876 (possibly originated in the Greek world – Alexandria, however also known by the Mayas of Yucatan, before Columbus)
Medieval Hindu numeration used ten numerals (round egg-shape for zero) and it is the combination of three basic principles:
1) decimal base (Egypt)
2) positional notation (Babylonia / China)
3) ciphered form for each of the ten numerals (Greece)
The Hindus were the first to link all three to form the modern system of numeration.
* Use Hindu multiplication
Lattice / gelosia multiplication (eg: multiply 456 by 34 – above the cells is 456, along the left is 34, as partial products are recorded in the cells, the diagonal rows are added and the answer is read off at the base).



5) Using different number bases

Converting between decimal, binary, octonary or other number bases

* Write 35 in the binary system (divide by 2 to the left and write the remainder below: 100011)

* Write 35 in the octonary system (divide by 8 to the left and write the remainder below: 43)

* Use binary addition: add 35 and 25 in the binary system (100011+11001=111100)

* Use the octonary addition table: add 35 and 245 in the octonary system (43+365=430)

APPENDIX 5: Fibonacci

* Students start with an investigation problem: 'Toni' tramping trip'.
"Toni enjoys tramping, especially trips that involve river crossings. She wants to work out how many different ways she can cross the river by using boulders in the water. She can jump from one boulder to the next boulder – short jump (S), or she can jump over just one boulder to land on the next one – long jump (L).
So if there is one boulder in the river, she can cross in 2 ways: SS or L.
If there are two boulders, she can cross 3 ways: SSS, LS or SL."
Investigate the number of ways to cross the river with 3, 4, 5, etc. number of boulders.
Summarize your results for the first 20 boulders.
Describe the rule for your pattern.
Have you seen this pattern before? – Fibonacci sequence
Research: Who was Fibonacci?

* Investigate the original problem – Fibonacci's Rabbits
- In Fibonacci's day mathematical competitions and challenges were common.
In 1225 Fibonacci participated in a tournament in Pisa, ordered by the emperor Frederick II.
- Fibonacci investigated the original rabbit problem in 1202, about how fast rabbits could breed in ideal circumstances.
"Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits. Suppose that or rabbits never die and that the female always produces one new pair (one male, one female) every month from the second month on. The puzzle Fibonacci posed was: How many pairs will there be in one year?"
- Students draw 'family tree' of rabbits and identify the number of rabbit pairs

* Draw Fibonacci Rectangles and Spirals
Starting with 2 small squares of 1 unit sides, next to each other, on top of these draw the square with 2 units sides, then touching the top of both squares of 1 and 2 unit sides draw the next one of 3 units side length, then touching the 2 and 3 sides draw

the next square with 5 units sides, etc. Each new square has the side-length of the sum of the two previous squares.

To create the spiral, start with a quarter circle in the 1 unit square and continue drawing quarter circles in each joining square to form the continuous spiral.

* Fibonacci in nature

Nautilus shell – Fibonacci spiral

Investigation: where else in nature can you find this pattern?

- Students are encouraged to bring in the objects (eg: flowers) or pictures to discuss the pattern

* Fibonacci and the Golden Ratio

- Investigate the ratio of two successive terms in the Fibonacci series – spreadsheet activity

1) column headings: term, Fibonacci number, ratio of Fibonacci number / preceding Fibonacci number – show that the ratio tends to 1.61803398874989

2) column headings: term, Fibonacci number, ratio of Fibonacci number / subsequent Fibonacci number – show that the ratio tends to 0.618033988

- The Golden Ratio – Phi (1.618…) – phi (0.618…)

- Greek mathematicians of Plato's time (400BC) recognized the significant value and Greek architects used the ratio 1: Phi as an integral part of their designs. The Greeks recognized the role the Golden Ratio played in the human body. They believed that humanity and their gods should belong to a higher universal order, therefore they used the same proportions for their temple structures. Renaissance architects also explored the Golden Ratio in their work, and it is also used in today's architecture.

- By using posters, plans, diagrams (OHT, photocopies) investigate the Golden Ratio:

1) in Leonardo da Vinci's drawing the Vitruvian Man

2) in famous buildings: the Parthenon in Athens; Bramante's Tempietto of S. Pietro in Montorio, Rome; Le Corbusier's World Musem project, Geneva.

* Measurement – spreadsheet activity

Is the ratio of our arm span to our height really equal to 1?

- students collect data from the whole class to discuss the answer to the question

* Research, investigation opportunity:

- Who is Leonardo da Vinci?

- Who was his 'Vitruvian man'? Why did Leonardo da Vinci choose him to name his work after? (Marcus Vitruvius Pollio, 1st Century BC Roman architect)

- Discover the Golden Ratio in a famous building

APPENDIX 6: King Arthur

* Introduce the King Arthur story – students receive a photocopy

Arthur was born in the 6th Century AD. His father was Uther Pendragon. After the Romans left, the British Isles were ruled by chieftains in small kingdoms. Arthur welded some of the kingdoms together to form England.
There are many legends of King Arthur and of his knights Sir Lancelot, Sir Galahad and also Merlin, the magician.
In one such legend Arthur is trying to marry off his daughter, Hypatia, who is a clever mathematician, and works out her own way to pick her husband. She said:
"Invite all the kings of the realm, and seat them at a round table, on numbered seats. Take your sword and say to number 1- you live. But to the next – cut off his head. To the next say – you live. But to the next – cut off his head. Carry on cutting off every second knight's head until only one remains alive. He will be my husband."
Hypatia wished to marry an intelligent man who could work out where to sit no matter how many knights arrive.
Can you work out where to sit?

* Act out the situation with different number of students participating each time (use plastic toy sword – students love it! – 'you live', 'you die, good bye')
* Record each time the number of knights participating and the number of the safe seat.
* When students had enough of the 'play' return desks / chairs and start simulating the situation.
* Try to establish a pattern / relationship between the number of knights and the safe seat.
* Explain your method of predicting the number of the safe seat.

(Solution:
Students, at this level, can identify that there are groups of numbers identifying the safe seat. These groups are based on the powers of 2.
Safe seat = 2 x the number of knights – the last number in that group)

APPENDIX 7: Geometry assignment

* Geometry project: aim to notice – identify – analyze geometrical transformations in everyday life and to understand its purpose / function.

Students work in groups of 4.

The assignment is presented on A2 poster-size cardboard. Essentially it has two parts: visual component and the mathematical analysis.

1) Visual component: photos, drawings, pictures, photocopies, models, etc.

2) Mathematical analysis of the identified transformations.

The oral presentation, of approx. 3-6 minutes, is performed by all members of the group, using their poster and, if required, enlarged sections on OHT.

* No specific directions are given to students regarding the source of their project, in order to allow for a diverse range of possibilities.

Students were exposed to certain aspects of transformations in everyday life through the learning of this topic.

Discussions and visual – mathematical analysis involved:

- works of Escher
- architecture through regions and eras (eg: Luxor Temple, Egypt; Acropolis in Athens; Pont du Gard in southern France; Alhambra in Granada; Dome of the Rock, Jerusalem; Cologne Cathedral; Christchurch Cathedral, NZ; Chrysler Tower, New York; Civilta Italica, EUR, Rome; terrace housing in London)

APPENDIX 8: Two Towers

Old problem – appeared in Leonardo Pisano's (Fibonacci) famous work Liber Abaci (The Book of Calculations) in 1202.

"Two towers, the heights of which are 30 paces and 40 paces, have a 50 paces distance. Between the two towers there is a font where two birds, flying down fro the two towers at the same speed, will arrive at the same time. What is the distance of the font from the two towers?"

* Students attempt on their own to sketch the situation and identify the given information on their diagrams.
* Picture of the situation is shown on OHT (this picture appeared in the Calandri manuscript in 1491)
* Student diagrams amended (if needed)
* Students describe their method of calculating the distance.
* Discussion of Fibonacci's solution – students receive photocopies of both strategies
1) First strategy – using the method of false position (used by the ancient Egyptians) – arithmetic solution.
2) Second strategy – geometric solution, based on the similarity of triangles.

(I found that students find it easier to relate to the second strategy)

APPENDIX 9: Famous mathematicians

Art – Painting

The School of Athens – Raphael Stanza – Vatican – Raphael (painting 1510-1511)
Show painting – identified sections enlarged on OHT – identifying the famous philosophers
Centre: Plato (red robe) and Aristotle (blue/brown robe)
Left back: Sokrates (green robe) engaging in argument with youth
Left front: Pythagoras (yellow top, red robe) demonstrates his proportion system on a slate
Middle: Diogenes (old man sprawling on the step)
the man in the front, leaning on a marble block, has features of Michaelangelo
Right: Ptolemy (black hat) contemplates a globe
next to Ptolemy are Raphael and Sodoma (dark face) by the right pillar
Euclid (red robe) bending down, describing a circle on a slate

* Who are these people? Do you recognize their names?
* When did they live? What are they famous for?

Imagine the two stanzas on opposite walls of a room in the Vatican Museum:
- On one wall is the third stanza, Raphael's first work, is the Dispute of the Holy Sacrament (1509). The subject of this fresco is the glory of the Eucharist, the exaltation of faith.
- On the opposite wall is the School of Athens, representing the celebration of reason.
* Why do you think Raphael included them together on his famous stanza?

* Research opportunities:
- Did these Greek philosophers know each other / know of each other / learn from each other? - compile of brief introduction of each of them

- Select one of these famous Greek philosophers that contributed to our knowledge of mathematics and through an introduction into the life and work of this person identify the mathematical contribution.
- Select a famous mathematician of your choice, and introduce through life, work the mathematical contribution made by your person.