MODELLING OF MEREOTOPOLOGICAL RELATIONSHIPS IN MULTIDIMENSIONAL SPACE

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To the memory of my angel mom who all that I am,

I owe to her.

Abstract

Inferences based on spatial knowledge play an important role in human lives. Humans are easily able to deal with spatial knowledge without any need to refer to numerical computation. The field of Qualitative Spatial Representation and Reasoning (QSRR) aims to model human common sense of space. Among the various types of qualitative relationship between spatial objects, connectivity (or topology) and parthood (or mereology) serve as the most basic underlying aspects.

Most current mereotopological theories are restricted to objects with the same dimension. However, sometimes spatial entities of different dimensions must be considered for many practical applications (e.g. map reading, spatial analysis). The inability of current theories to interact with entities of different dimensions has motivated the foundation of multidimensional spatial theories. However, these theories are less efficient in terms of reasoning power. Moreover, their set of introduced mereotopological relations has not been cognitively validated.

This research presents a multidimensional mereotopological theory using *part of* and *boundary part* as primitive concepts. We introduce a set of nine spatial relations with the jointly exhaustive and pairwise disjoint property based on these primitives. This property allows us to develop an efficient reasoning strategy (i.e. constraint-based reasoning) which makes our approach more practical than previous works.

We used automated theorem provers and finite model finders to aid the formal verification of the theory, proving its properties and generating the composition table for reasoning purposes. This work is the first multidimensional mereotopological theory that not only has properties that are verified by traditional logical deduction techniques (like the other multidimensional mereotopological theories), but that also it supports an efficient reasoning strategy that was not being available before.

Furthermore, we verified the cognitive adequacy of our proposed set of relations using human subjects experiments, applying clustering and thematic analyses to empirical data. Our study is the first to provide evidence for the cognitive plausibility of a *multidimensional* mereotopological theory (going beyond previous studies that have only shown cognitive adequacy for equidimensional mereotopological theories) supporting its closeness to human cognition. In addition, we demonstrate our multidimensional theory by applying it to a real-world scenario (i.e. a flood event).

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Chapter 1

Introduction

The world around us is, by its nature, spatial, and the ability to cope with this spatial environment has been an essential survival skill for humans over the centuries. A key aspect of human intelligence is so-called spatial intelligence, which is interpreted as our ability to conceptualize and reason about our physical surroundings. Much of this happens subconsciously in our everyday activities such as cooking, dressing, organizing our spaces, or navigating around our homes and communities.

An important feature of human spatial intelligence is that we use not only spatial scale information, such as distances (e.g. “do I need to step forward to reach this pot on the counter?”) and sizes (e.g. “do my souvenirs fit into my suitcase?”), but also various kinds of qualitative properties and relationships to describe and reason about space using aspects such as distance (e.g. “the bank is further away than the pharmacy”), orientation (e.g. “do I take the northbound or southbound highway?”), size (e.g. “my laptop is smaller than my suitcase”), connectivity (e.g. “this path goes through the park”), and parthood (e.g. “the playground is in the park”) of entities.

Modelling common-sense human understanding of space was the motivation of the sub-field of artificial intelligence called qualitative spatial representation and reasoning (QSRR). Qualitative approaches have been advocated as a foundation to connect intelligent agents and human cognition (Wolter and Wallgrün, 2013). Furthermore, they provide simplified representations of space, supporting complex decision tasks.

Among the different kinds of qualitative representation of spatial objects, characterising their properties relative to each other, connectivity (or topology) and parthood (or mereology), are considered to be among the most fundamental (Randell and Cohn, 1989; Varzi, 1994; Asher and Vieu, 1995; Eschenbach and Heydrich, 1995; Borgo and Masolo, 2010) and cognitively plausible (Knauff et al., 1995, 1997; Klippel et al., 2013). Topology is a study of invariants under changes of form. Intuitively, it is concerned with those properties of the drawn shape on a rubber sheet which are stable under twisting and stretching, but not folding and tearing. On the other hand, mereology is a study of parts and wholes. They together (known as mereotopology) are among the most studied aspects of qualitative space (Leśniewski, 1927; Whitehead, 1929; Egenhofer and Herring, 1989, 1991; Egenhofer et al., 1994; Randell et al., 1992a; Hahmann, 2013, 2018).

Mereotopology, as a framework for qualitative spatial representation, was introduced by Whitehead in 1929 (Whitehead, 1929) and formally developed by Clarke (Clarke, 1981, 1985) and others (Randell et al., 1992a; Asher and Vieu, 1995; Borgo et al., 1996; Gotts, 1996; Hahmann and Grüninger, 2012). In the artificial intelligence community, the Region Connection Calculus (RCC) (Randell et al., 1992a) was the first[1] logical formalism for mereotopology proposed to support automated and efficient spatial reasoning. The theories that are based on Whitehead's conceptualization of space take regions (or extended entities) as the spatial primitive, since they consider them most intuitive. These theories are known as region-based theories. Point-based theories, as region-based theories counterparts, take the more common approach in mathematical topology (the well-known point-based mereotopological theory is the 9-intersection model (9-IM) (Egenhofer and Herring, 1989, 1991; Egenhofer et al., 1994)).

The mereotopological theories that are based on Whitehead's ideas (known as Whiteheadian theories) are all restricted to entities that have the same dimensions and that are located in space of the same dimension. These theories are known as equidimensional theories. In these theories, there is no dimensional difference between the space and all the objects therein (e.g. a two-dimensional space containing only two dimensions regions). Consequently, objects of different dimensions (such as boundaries of regions as dependent entities or other independent lower-dimensional entities) are not permitted in these theories. However, sometimes objects of lower dimensions are necessary to account for certain basic intuitions, like describing the boundary between two adjacent pieces of land. Also, relations between independent entities of different dimensions, e.g. the relation between a road (which may be conceptualised as a one-dimensional area) and a park (which may be conceptualised as a two-dimensional line (Egenhofer and Mark, 1994)), are frequently referred to by humans, but equidimensional theories are unable to address these types of relations.

The equidimensional theories' limitations in dealing with regions of various dimensions has led to the introduction of theories of multidimensional space (Clementini and Cohn, 2014; Galton, 1996; Gotts, 1996; Hahmann, 2013, 2018; Smith, 1996; Smith et al., 2000). Almost all of these accounts closely follow the idea of generalizing the combination of mereological (part-whole) and topological (contact or connection) relations that have been developed in traditional mereotopological theories to describe spatial relations among entities of various dimensions. However, they differ in their representation languages: some of them use first-order logic (FOL) while others offer higher-order (HOL) formalizations.

The wide range of variables and quantifiers over individuals, relations and functions make the HOL multidimensional theories more expressive than the FOL ones, in which the variables and quantifiers only range over individuals. However, the more expressive the representation approach the less efficient it is in terms of reasoning power (Brachman and Levesque, 1985), and accordingly, the HOL theories are not as efficient as their FOL counterparts in terms of inference.

In addition, the FOL multidimensional theories are mostly based on theorem provers and finite model finders to meet the inference purposes. These techniques generate a very large search space to solve a given

[1]In addition to the RCC, another approach of defining mereotopological relations among spatial entities has been studied in the literature: *nine intersection model (9-IM)*. It generates matrices of the nine point-set intersection of interior, boundary and exterior of a pair of entities. Spatial relations were represented via a mathematical structure in 9-IM that differs from the logical representation in Whiteheadian methods (including RCC).

problem which is never guaranteed to terminate. This limits their reasoning mechanisms that can be applied to these theories. To our knowledge, only one multi-dimensional mereotopological FOL theory, RCC*-9 (Clementini and Cohn, 2014), has been supported an efficient reasoning mechanism (i.e. constraint-based reasoning) over a multidimensional space with the help of pre-computation of compositions of jointly exhaustive and pairwise disjoint (JEPD) relations among zero to two-dimensional entities. *Composition* is an algebraic operation in which a pair of (binary) relations over a common spatial entity combines in a way to infer a *new* (binary) relation between a pair of given entities. However, the correctness of the RCC*-9's composition table has not been supported by a formal proof. Also, its set of introduced relations have not been cognitively validated.

Furthermore, the existence of lower-dimensional entities was assumed to depend on those of higher-dimension (i.e. a surface cannot be available without referencing the solid that it bounds) in some of the proposed multidimensional mereotopological theories such as the INCH calculus (Gotts, 1996), Take-dim (Galton, 1996) and Smith's theory (Smith, 1996). These theories are not able to represent lower-dimensional entities (e.g. an independent road (as shown on a map) without considering it as the boundary part of a piece of land). However, some times there is no such higher-dimensional entity for the lower-dimensional entity to depend on, as with for instance a (part of) river passing through a parcel.

The purpose of this research is to:

- present a formalism that represents and reasons about objects of dimensions zero to three, including both independent lower-dimensional entities and those whose existence is dependent on higher dimensional entities (e.g. edges), in three-dimensional space;
- propose a set of JEPD relations among these objects which is cognitively valid, and
- develop an efficient reasoning strategy (i.e. constraint-based reasoning) over a multidimensional space with the help of pre-computation of compositions of JEPD relations among zero to three-dimensional entities.

The structure of this chapter is as follow: we explain the motivation for the research in Section 1.1. Then, in Section 1.2, we will reveal our challenges in proposing a new multidimensional mereotopological theory. After establishing the research structure, we will briefly point out our contributions in Section 1.3, and present the structure of this dissertation in Section 1.4.

1.1 Applications of Mereotopological Theories

The development of mereotopological theories for spatial representation is supported by a number of applications. Mereotopology allows the formalisation of spatial properties like parthood, connection, boundaries, interiors, holes and multi-pieces. It can also be applied as an instrument for qualitative spatial reasoning, with constraint calculi.

Among the most important applications of mereotopological theories are geographic information systems (GIS) (Clementini and Di Felice, 1997, 1998; Cohn, 1995; Egenhofer and Herring, 1989, 1991; Egenhofer, 1994; Frank, 1996; Hernández et al., 1995; Egenhofer and Mark, 1994; Gotts, 1996; Randell et al., 1992b; Hahmann, 2013, 2018; Clementini and Cohn, 2014; Smith, 1996), robot navigation (Kuipers and Levitt, 1988; Kuipers and Byun, 1991), biomedical ontologies (Bittner, 2009; Smith et al., 2005; Donnelly, 2004; Rosse and Mejino, 2003), and natural language processing (NLP) (Verhagen et al., 2005; Rashid et al., 1998; Chaudet, 2004; Wang and Li, 2013; Wang and Schwering, 2009).

GIS: GIS is the most important and oldest application area of mereotopological theories because formal understanding of the relations among spatial objects is an important concept in such systems. GIS users perform spatial queries such as "Retrieve all rivers *within* the Auckland area", or "Find all cities that *touch* Lake Taupo" where *within* and *touch* are mereotopological relations. The spatial relations could not be explicitly stored in a spatial database (due to the enormous amount of geo-data); thus, they must be extractable from the objects' properties. For instance, in order to answer the second query, we need to know whether the borders of a city and the lake share a common part. Therefore, the spatial databases require a standard procedure for extracting the relevant properties of the spatial objects and answering the queries. Since the geometries of spatial objects are defined based on their interior, boundary, and exterior parts in the Open Geographical Consortium (OGC) standard (Standard, 2006), a formal model can extract the mereotopological relations between spatial objects based on the intersection between these elements. The formal definitions of 9-IM (Egenhofer and Herring, 1989) are used to describe the operation of spatial querying operators in current GIS, applying to objects of different dimensions. However, neither 9-IM nor its variants enable reasoning. For instance, these theories could not infer any new information given "parcel A is disjoint from parcel B which (partially) contains river C".

Our work could be used to formalise query processing in geo-databases, since it supports an effective reasoning strategy for all entities in the presence of qualitative rather than geometric information. Furthermore, the set of introduced JEPD relations would optimize topological queries since it matches human interpretations of the relations (see Chapter 5).

Navigation: Robot navigation can benefit from mereotopological relations. Robots constantly map their surroundings via their vision (including cameras, sensors and radars). These maps consist of objects of various dimensions, including three-dimensional ones (if the robot has 3D-radar). Also, many robotic applications use topological maps (graph-based representations of space supplemented with robot orientation information) for navigation.

Some mobile robots make sensor snapshots equivalent to two-dimensional regions and apply RCC to characterize their spatial relations (Santos, 2007). However, the multidimensional theory we develop in this thesis may be a suitable replacement for graph-based representations (like Voronoi diagrams or connectivity graphs) for topological map learning. Our theory can thus provide a higher-level model of space for robots with an in-depth understanding of the search space (like determining the layout of a room and its location within a building). As a result, it can improve the operation of common robotic tasks such as collision avoidance, target seeking, and area exploring.

On the other hand, the feasibility of learning a topological map will be increased by the introduction of JEPD relations, reducing the search space since all the possible spatial configurations are limited to the cases represented by JEPD relations. Finally, our theory can aid in deciding the next “valid” path for a robot. Determining whether an existing path and all the given constraints (from the knowledge base) are satisfiable at the same time is a constraint satisfaction problem (CSP). Solving CSPs is straightforward, using the composition table for the set of JEPD relations that we provide in our theory.

Biomedical ontology: In recent decades, fields like biology, biomedical and medical research have addressed the representation of anatomical, genetic, spatial and spatio-temporal relations to support the interpretation of medical or biological documents. As a result, a repository, the Open Biomedical Ontology (OBO) (Smith et al., 2007), has been created to store relevant ontologies. Among these ontologies, mereotopological relations were found to be fundamental in describing the structure of living organisms (Schulz et al., 2006). Such relations have been explored by Smith and Bittner and refined with locational information (Smith et al., 2005; Bittner, 2009; Schulz et al., 2006). Furthermore, these relations are exploited to assess medical procedures and biological processes by the incorporation of their temporal aspects (Bittner and Goldberg, 2007).

However, the OBO's ontologies are expressed in description logic, which is less expressive than first-order logic. We are only aware of one theory (Cohn, 2001) in this domain which is presented in first-order logic. It combines RCC with formalized cell biology, but focuses on the representation aspect rather than extracting further concepts through reasoning. Since our proposed theory is in first-order logic and supports an efficient constraint-based reasoning, it provides a useful tool for multidimensional representation of biological space.

NLP: The interpretation of spatial terms in natural language is essential in artificial intelligence (Bateman, 2010; Bateman et al., 2010; Kordjamshidi et al., 2014; Hois and Kutz, 2008). Human languages use certain linguistic structures to describe spatial arrangements that are associated with the spatial situations in which they are used (Bateman, 2010). Spatial language describes spatial information mainly in qualitative, more specifically mereotopological, terms (Kordjamshidi et al., 2014). To model the flexibility of human spatial language, we need to map spatial terms into formalized mereotopological relations. Such mereotopological formalization is intended to capture the level of detail required to represent the collected linguistic features while at the same time preventing information overload. An example includes the mereotopological relation between “*Twin Coast Discovery Highway*” and “*Orewa*” expressed via the preposition “*in*” on “*Traffic congestion is reported on the Riverside Road in Orewa*”.

Moreover, mereotopological formalism has lead to more efficient reasoning strategies, i.e. mapping from spatial language to a mereotopological formal representation allows qualitative spatial reasoning, whereas spatial reasoning with human spatial language itself may not be possible (Bateman et al., 2010). Qualitative spatial reasoning emulates human spatial reasoning by mapping back from the mereotopological relations resulting from formal qualitative spatial reasoning into a spatial language term. For instance, by giving the spatial language expression “*The Riverside Road is partially flooded*” along with the previous example (after mapping to relevant formal relations) to the qualitative reasoning mechanism a new relation (i.e. “*Orewa is fully (or partially) flooded*”) will be generated which can be provided to the user after mapping back to human language terms.

Our multidimensional mereotopological theory has the capacity to provide an application to NLP that better reflects the nature of the real world as described in natural language, both in representation, by considering regions of various dimensions, and reasoning, by providing an automatic reasoning mechanism.

1.2 Research Hypothesis and Scope

To exploit the full potential of the mereotopological representation of multidimensional space and make it usable for efficient reasoning, we will center our research on the following research hypothesis:

I The introduction of a set of dimension-independent mereotopological relationships, which is cognitively plausible and supporting constraint-based reasoning to describe the relative locations of spatial entities of various dimensions (i.e. from zero to three dimensions), is possible in a three-dimensional space.

II We can identify a subset of jointly exhaustive and pairwise disjoint (JEPD) multidimensional relations that generate an algebra for constraint-based reasoning.

III Our proposed set of multidimensional mereotopological relations closely aligns with people's conceptual understanding of spatial relationships among objects of various dimensions (i.e. cognitively adequate).

IV Constraint-based reasoning over a set of multidimensional mereotopological relations is possible in a dimension-independent way.

To situate our new multidimensional theory that supports efficient constraint-based reasoning within the broader research context, we now explain our underlying assumptions and limitations.

Qualitative vs. quantitative representation of space

Before continuing, we clarify the difference between qualitative and quantitative representations of space. The focus of quantitative spatial representation is accurate metric spatial information. For example, quantitative systems represent the precise geometric area (with coordinates) of overlapping of a parcel and city zone, or the numerical distance between two locations. The quantitative approach has two problems: (1) information with the required level of accuracy is not always available; and (2) inferring new knowledge from this representation has a high computational cost. An alternative approach, which we use here, is a qualitative representation of space, in which we define a manageable number of relations to describe the space, and these relations are the basis for qualitative reasoning as well. For instance, instead of referring to the precise distance of two locations or the shared area of two pieces of land, we only say that two entities are disconnected, or overlap each other.

Finding an appropriate qualitative spatial representation depends on many facets: e.g. humans can only remember and use a limited number of relations, although this is not a barrier for machines. Similarly, some applications require more fine-grained spatial relations. For instance, it may not be enough to know whether two pieces of land overlap; we may need to know whether the overlapping area is only the object boundaries, or also includes interior parts. The required level of detail to be captured may affect the number of relations in a theory. For all those reasons, it is important to define the spatial relations in the light of the users' requirements.

Abstract vs. physical space

We assume that our domain of discourse is an abstract space against a well-known material space (consult (Hahmann, 2013, p. 5-6) for the differences between these two spaces). This space has not been developed based on the well-known physical rules about the material objects, though it has a relationship with the physical space. Material objects occupy abstract space entities, known as regions. According to this assumption, regions are abstract mathematical entities without any material properties. The presumption also gives us a chance to talk about properties such as interior, boundary, overlapping, intersection and dimension.

The abstract space provides models of the real world such as sketch-maps, navigation maps or GIS datasets in which regions might have different dimensionality (i.e. defined as a number of independent vectors which are the base elements of the corresponding vector space). Examples of objects that may be modelled as one-dimensional regions are pipelines, roads and rivers; common two-dimensional regions are land parcels and countries, and three-dimensional regions could be tunnels and buildings. The dimensionality of real world objects may vary according to different levels of granularity, or the task for which the dataset is being created. Thus, we have to decide on it before producing a map.

Since an abstract space containing regions of different dimensions is inherently more complex than the space of equidimensional regions (which is commonly used in traditional mereotopological theories), we enforce restrictions on every region consisting of a unified dimension (e.g. a single region could not contain both a line and a volume as its elements) though the space could contain a mixture of objects of different dimensions. Also, in our space, every region must satisfy two properties here[2]: interior connectivity and exterior connectivity. The former means that the regions must (topologically) connect, i.e. be a one-piece region. For instance, a two-dimensional region must not consist of several pieces (Egenhofer and Franzosa, 1991; Herring, 1991; Randell et al., 1992b; Clementini et al., 1993). Similarly, the connected exterior removes the existence of holes in the regions (Egenhofer and Franzosa, 1991; Herring, 1991; Randell et al., 1992b; Clementini et al., 1993). As a result of these properties, the region can't consist of disjoint parts, and can't have holes. The consideration of the multi-pieces regions and regions with holes allows the introduction of much finer-grained relations, for instance, both main islands of New Zealand can represent as a whole. Furthermore, the spatial relationships among the regions do not have any temporal aspect in our formalism, i.e. do not change over time. We focus only on the spatial aspect of relations.

In addition, the use of abstract space makes our multidimensional theory free of any of the philosophical criticisms raised by philosophers about the adequacy of models of physical space (Smith and Mark, 1998; Varzi, 1996). In this environment, we can easily talk about the abstract concepts forming in human spatial mental models. However, this does not mean that abstract space is equivalent to cognitive space.

[2]Note that *points* do not require to satisfy these propertied. We will discuss about it in Chapter 4

1.3 Summary of Contributions

The research presented in this thesis makes four contributions:

I As mentioned, the limitations of the traditional mereotopological (i.e. equidimensional) theories led to the introduction of multidimensional mereotopological theories. However, these theories have not yet been comprehensively compared with respect to their various properties. This research fills this gap by comparing the available multidimensional theories (see Subsection 2.1.2, Appendix A). A collection of prior works is woven together and their properties are reviewed and compared. Their comparison is not only be a tool for applying these theories but also it can be the basis for their development or even the introduction of a new multidimensional mereotopological theory with a combination of the capabilities of the theories reviewed.

II The existing multidimensional mereotopological theories, with one exception CODI (Hahmann, 2013), do not introduce a set of jointly exhaustive and pairwise disjoint (JEPD) relations and formally verify their properties. Even that one does not support effective qualitative spatial reasoning. This research introduces a new multidimensional mereotopological theory (see Chapter 4), which, unlike other multidimensional theories, has a set of (formally verified) JEPD relations. The set is also expressive enough to describe all of the possible spatial configurations between regions of different dimensions.

III Although the cognitive plausibility of the existing equidimensional theories has been investigated (Knauff et al., 1995; Renz et al., 2000), none of the multidimensional theories has been evaluated from this perspective. This research analyses the cognitive plausibility of the set of introduced JEPD relations in the proposed multidimensional mereotopological theory through human subjects experiments (see Chapter 5).

IV The functionality of a mereotopological theory depends on supporting an efficient reasoning system. This is the main reason for the success of well-known equidimensional mereotopological theories in which the efficient constraint-based reasoning mechanism assists their systems. By contrast, most of the current multidimensional mereotopological theories only used theorem provers and finite model finders that are typically less powerful and less expandable, though allow more expressive reasoning. Our multidimensional formalism with a set of introduced JEPD relations supports an efficient reasoning mechanism (i.e. constraint-based reasoning) to infer new spatial information (see Chapter 6).

1.4 Overview of Chapters

This dissertation is structured as illustrated in Figure 1.1. Chapter 2 presents previous work on equidimensional mereotopological theories and their intended spatial domain, which was the motivation of proposing multidimensional theories (Contribution I). We also provide a rigorous analysis of existing multidimensional theories in this chapter. Discussion about the importance of boundaries in the multidimensional

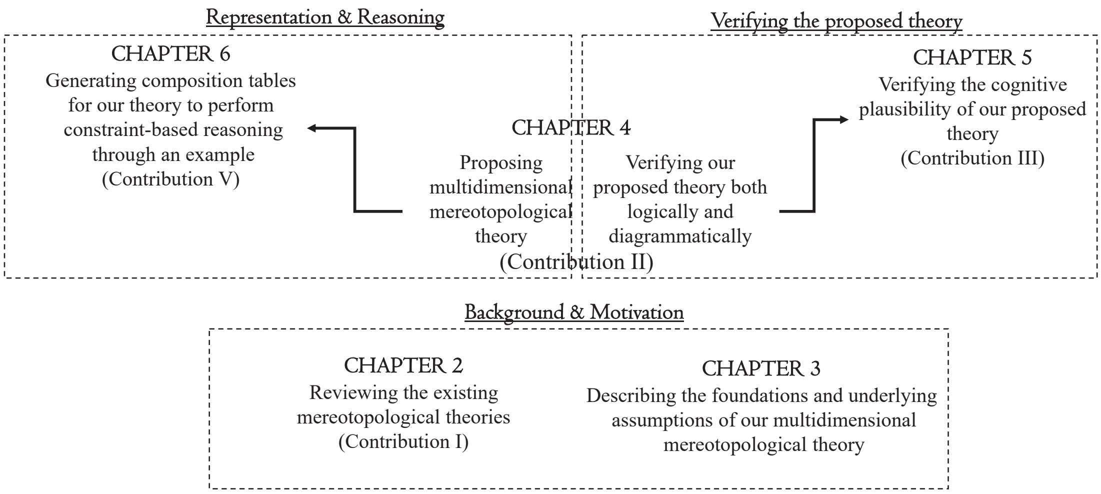


Figure 1.1: Thesis chapter synopsis and dependencies.

mereotopological theories is also given here. The analysis, along with the required ontological concepts, reveals the research gaps that our research addresses by identifying the lack of a set of mereotopological relations (with JEPD property) and associated constraint-based reasoning.

Before proposing our multidimensional mereotopological theory, the space of objects of different dimensions must be formally defined. In Chapter 3, we characterize the multidimensional domain of regions using the mathematical notions of *manifolds with boundaries* and developing more complicated spatial objects from them. This domain will then be described by our proposed theory in Chapter 4. We also give a concise summary of the necessary logical language (i.e. *first-order logic*) which specifies the ontology of the domain in our theory.

Chapter 4 includes the axiomatization of our multidimensional mereotopological theory (Contribution II). In addition, the correctness of the theory specification is checked, and we discuss how to use automated (mechanical) theorem proving to verify the theory. We also examine how well our theory deals with the range of different cases, including extreme cases, using diagrams.

The cognitive adequacy of our multidimensional mereotopological theory is evaluated in Chapter 5 (Contribution III). We describe our empirical investigations in which subjects had to group a set of given models and describe their categorizations. Implications of the collected results are discussed with respect to both a grouping task and description analysis in order to get an insight into human mental models of multidimensional space.

Our final technical chapter, Chapter 6, exploits the theory that we developed in Chapter 4 for reasoning purposes. In particular, we show how the various introduced spatial relations of the theory can be combined to help us deduce more knowledge about a particular spatial scene (Contribution IV). The chapter finishes with a use case scenario that demonstrates answering the questions of the rescue team when a flood occurs in a residential area using our proposed multidimensional theory.

Although each chapter includes a summary of its analysis and findings, we review the central insights of the entire dissertation, revisit our research questions, and give an outlook to further research in Chapter 7.

Chapter 2

Background: Mereotopological Theories

As mentioned earlier, the qualitative representation of space plays an important role in spatial information systems due to the closeness of this approach to human spatial modelling ability. It is also supported by other benefits such as less memory storage and lower computational costs. Hence, several theories have been introduced based on this idea, in which space is presented from several aspects, such as direction (Forbus et al., 1991; Frank, 1992; Chen et al., 2007), size (Zimmermann, 1995), distance (Cristani et al., 2000; De Laguna, 1922), parthood (mereology) and connection (topology). The combination of the last two aspects, known as mereotopology, is a fundamental aspect of qualitative spatial representation and reasoning since it can make sufficient qualitative distinctions to represent the space (Cohn and Renz, 2008, p. 558). Mereotopological characteristics of space are, therefore, widely studied in the literature (Tarski, 1956; Clarke, 1981; Allen, 1983; Randell et al., 1992b; Smith, 1996; Gotts, 1996; Clementini and Cohn, 2014; Hahmann, 2018).

Mereotopological properties may be represented via logical or algebraic frameworks. They provide justifiable assumptions and reliable reasoning. A logical formalism is a general tool to symbolically specify the properties of entities in a domain and supports logical relations (specifically the entailment relation) among these properties. The properties are represented in the form of logical statements being interpreted in terms of the entities and their properties. A satisfied interpretation, known as a *model* of a theory, is a collection of logical statements in which all the statements are true. Models play a crucial role in establishing the properties of a theory. The family of logical representations of mereotopological properties has been divided into Whiteheadian theories (or equidimensional mereotopological theories), mereogeometrical theories and multidimensional mereotopological theories (Hahmann and Grüninger, 2012). We discuss the mereogeometrical theories here, but the other approaches will be discussed in more detail later in Section 2.1. Mereogeometrical theories (De Laguna, 1922; Nicod, 1924; Tarski, 1956; Bennett et al., 2000; Bennett, 2001; Donnelly, 2001; Borgo and Masolo, 2010) are logical formalism (in first-order logic) aiming to formalize geometrical notions along with mereotopological properties. The geometrical notion is usually represented via a primitive called "being a region". This region might be a solid (Tarski, 1956), regular (Borgo and Masolo, 2010) or regular of finite diameter (De Laguna, 1922). This notion is not expressible with pure equidimensional and multidimensional mereotopological theories. With the help of this notion, mereogeometrical theories can fully represent Euclidean space. It thus increases their expressivity in comparison to other logical formalisms. An expressive representation is not practical as it loses its efficient reasoning capability.

Another strategy to represent mereotopological properties is using an algebraic structure. Algebraic mereotopological frameworks have a more specific approach than logical systems. They represent a *specific* domain of individuals and describe the structure of the domain in terms of a limited number of properties. To provide an algebraic representation of a mereotopological structure, we need a logical theory that has a subset of its models (i.e. models describing the specific intended domain) that can be described via the algebraic structure. Thus, the idea of a logical representation of mereotopological properties is a prerequisite for algebraic structures. In this thesis, we are interested in a general (i.e. a logical) representation of mereotopological relations to make our proposed structure applicable in various domains.

In this Chapter, we will introduce the mereotopological aspect of the space and look at the qualitative theories that have been developed for this feature (Section 2.1). Analysis of the properties of these theories motivated us to divide them into two categories named *equidimensional* and *multidimensional* mereotopological theories detailed in Subsection 2.1.1 and Subsection 2.1.2, respectively. The importance of multidimensional theories prompted us to examine and compare these theories in great detail. The full detail is in Appendix A, and just its key points are summarised here. We will also look at the role of the boundary of spatial entities in mereotopological theories in Section 2.2, which motivates the boundary representation in our proposed mereotopological theory. Finally, we will discuss the reasoning strategies over the multidimensional mereotopological theories in Section 2.3.

2.1 Mereotopological Theories

Mereotopology is a theory combining two distinct but related concepts of mereology and topology. While the former studies the relations among wholes, parts, and parts of parts, the latter studies the unchanged properties of geometries on stretching and bending, such as boundary and connectivity. This combination has been extensively studied in qualitative spatial reasoning to describe the spatial relations among the objects. In a mereotopological theory T, the intended interpretation of the mereotopological relations can be explained in terms of common point-set theoretic notions. It means that the *semantics* of T are explainable in terms of set-theoretic concepts (although this is not the aim of all mereotopological theories[1]); thus, we use the common concepts of point set-theory (i.e. topological space) in this work. Given a topological space X, we want to explore how the basic relations of various mereotopological theories are defined. We follow concepts of topological spaces like the ones provided in the relevant standard textbooks such as (Prasolov, 1995; Weeks, 2020). A topological space is defined as follow:

[1]The theories following the Whiteheadian approach (introduced in this Section) are presented in point free space, but their relations are interpretable in terms of points.

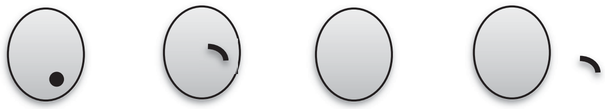


Figure 2.1: Example of non-regular regions showing a two-dimensional entity with an element of lower dimension.

Definition 2.1.1. *A* <u>*topological space*</u> $\langle \boldsymbol{X}, \tau \rangle$ *is a set* $\boldsymbol{X}$ *as a universe together with a topology* τ *on it which is a collection of all open subsets*[2] *of* $\boldsymbol{X}$.

Every subspace S of a topological space X is a topological space with the same topology. Every topological space has some topological properties determined via the topological operators. They include *interior*, *closure* and *boundary*:

Definition 2.1.2. *The* <u>*interior*</u> *of a topological (sub)space* S *is the union of all open subsets contained in it, shown by* $int(S)$. *In other words, it is the largest open set contained in* S.

Definition 2.1.3. *The* <u>*closure*</u> *of a topological (sub)space* S *is the intersection of all closed subsets contained in it, shown by* $cl(S)$. *In other words, it is the smallest closed set containing* S.

Definition 2.1.4. *The* <u>*boundary*</u> *of a topological (sub)space* S *is the difference between its closure and interior, shown by* $bdy(S)$.

Definition 2.1.5. *The* <u>*exterior*</u> *of a topological (sub)space* S *is the union of all open subsets of topological space* X, *which disjoints of* S, *shown by* $ext(S)$.

The same set-theoretic operators (`union` ($\cup$), `intersection` ($\cap$), `complement` ($\tilde{.}$) and `inclusion` ($\subseteq$)) are definable on topological spaces. A topological subspace (S) is *regular* if and only if $cl(S) = cl(int(S))$ and $int(S) = int(cl(S))$. Similarly, a subset is *regular closed* if $S = cl(S) = cl(int(S))$, and *regular open* if $S = int(S) = int(cl(S))$. Conceptually, regular regions are of uniform dimension. Examples of entities with spike of a lower-dimensional part are shown in Figure 2.1. The regular form of all them is a disk.

The mereotopological theories use two main approaches to describe spatial relations according to the literature. Both of them aim to represent common-sense geometry.

The first category is based on standard mathematical set theory. In this class of theories, space is defined as an infinite number of points. In this space, objects are the subsets of points whose mereotopological relations are defined by comparing the intersections of their topological components (i.e. interior, boundary and exterior). The most famous theory that has been presented based on this idea is the nine intersection model (9-IM) (Egenhofer and Herring, 1989; Egenhofer and Franzosa, 1991; Egenhofer and Mark, 1995a,b). In this theory, the existence or emptiness of the intersection between the topological components of the spatial objects is the criterion for determining their spatial relation. For example, if two objects **A** and **B**, do not have any common part (i.e. $int(\mathbf{A}) \cap int(\mathbf{B}) = \emptyset$, $bdy(\mathbf{A}) \cap bdy(\mathbf{B}) = \emptyset$ and $ext(\mathbf{A}) \cap ext(\mathbf{B}) = \emptyset$),

[2]Intuitively, an open subset is a subset that does not contain its boundary. The complement of an open set represents a closed set.

they are disconnected. Although 9-IM has been implemented in geographic information systems (GIS) (Richardson et al., 2017), it does not support any form of reasoning. Moreover, the specific extension of this method (i.e. the DE-9IM (Clementini et al., 1993)), which defines a set of mereotopological relations based on the dimension of the common part of the participated entities in a relation, has also been criticized (Clementini et al., 1993).The criticism is that the high number of spatial relations defined in this method (for example, 33 relations between two simple lines) makes it difficult to use.

The search for theories that accurately describe the geometry of space has led to introduction of the second group of theories named *point-free* or *region-based*. They are free of the criticisms mentioned above. This name does not mean that a point does not exist in these theories, but that points are not spatial primitives and are definable via other primitives. The ontological concept of the region is much less troublesome and more realistic than the point (Casati and Varzi, 1999). An important property of the region-based geometry is the fact that its space and objects are not always distributive sets of points. Distributive sets have the property that given a set X, usually elements of elements of X are not an element of X (Gruszczyński and Pietruszczak, 2009, p. 6). Indeed, the region-based theories describe the space and its elements via the mathematical notion of parthood relation. This relation formalizes the membership property between an object and a whole. This approach to sets is known as *mereology* and was introduced by Leśniewski (Leśniewski, 1927, 1928, 1929, 1930, 1931). In these mereological theories, all the spatial objects are parts of space. An *region* is a sub-space being occupied by an object (regardless of its dimensional type). Whitehead tried to explain topological concepts (including boundary and connectivity) within mereology. His idea of describing the space by integrating topology and mereology was improved and developed in region-based theories in the literature known as *mereotopological* theories (Clarke, 1981, 1985; Allen, 1983; Randell et al., 1992b; Asher and Vieu, 1995; Gotts, 1996; Galton, 1996; Hahmann, 2013).

Generally, we have adopted Whitehead's approach and used the notion of mereotopology to describe the space of regions, but our assumptions are not the same as the Whiteheadian theories (we will detail their differences Subsection 2.1.2).

2.1.1 Equidimensional Mereotopological Theories

Most region-based theories are defined over the regions of the same dimensions (known as *equidimensional* regions). This assumption leads us to hereafter refer to them as “equidimensional mereotopological theories”[3]. These regions must satisfy the following principles (Mormann, 1998):

- The dimension of all the regions are the same as the dimension of space they are located in,
- The regions can only include other regions as part of themselves and can also be part of other regions,
- A region can be interpreted as a point-set.

The satisfaction of the first and second principles leads to the equidimensionality between the regions and their embedding space. The third principle is further equating the region-based theories with the point-based theories, which is not the subject of our study. However, considering it, along with the first principle, leads to the regularity of the regions (see Section 2.1 where we defined regular regions).

[3]They are sometimes referred to as “boundary-free mereotopological theories” in the literature.

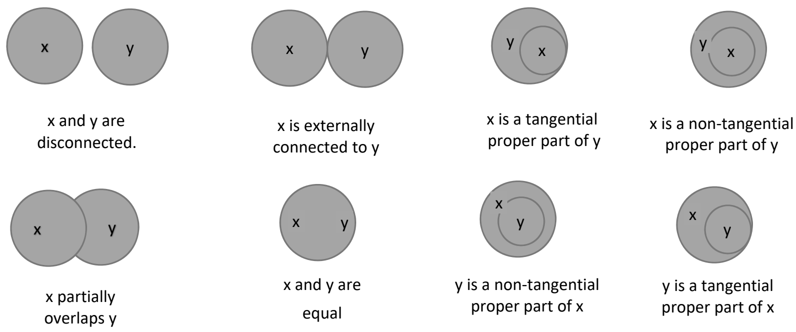


Figure 2.2: Set of eight JEPD relations in RCC-8.

The most well-known equidimensional mereotopological theories are the calculus of individuals (Clarke, 1981, 1985), interval calculus (Allen, 1983), region connection calculus (RCC) (Randell et al., 1992b; Cohn, 1995; Cohn and Gotts, 1996; Li and Ying, 2004; OuYang et al., 2007; Dong, 2008; Clementini and Cohn, 2014) and Asher and Vieu's theory (Asher and Vieu, 1995).

Among them, RCC or more specifically RCC-8 (Randell et al., 1992b), has received much attention in artificial intelligence. It follows Clarke's steps in the calculus of individuals (Clarke, 1981, 1985) and assumes all regions are exclusively of the same dimension (typically of dimension two or three) embedded in the space of the equivalent dimension. Furthermore, it restricts regions to the closed ones in the calculus of individuals (Clarke, 1981, 1985) and presents its formalism in first-order logic. It introduces a set of eight based (or JEPD) topological relations (shown in Figure 2.2) between equidimensional features based on the single *connection* relation. These relations are the same as those introduced by 9-IM between two-dimensional entities. The popularity of RCC-8 is due to the introduction of a small set of user-friendly and expressive spatial relations which is also the basis of an efficient and straightforward inference method (i.e. composition table).

As we have seen, the most common assumption among the equidimensional theories of mereotopology (i.e. the restriction to only entities of the same dimensionality) does not allow them to consider lower-dimensional entities necessary for the common-sense representation. People easily describe and reason over regions of different dimensions such as "the southbound highway (1-dimensional entity) goes through the protected area (2-dimensional entity)" or the boundary of the regions of the exact dimensions like "a common wall (2-dimensional entity) between two buildings (3-dimensional entities)" on digital maps. Furthermore, the existence of lower-dimensional entities breaks down some of the defined relations in the equidimensional theories; for example, there is no difference between the *overlapping* and *connection* relations defined in RCC-8 when applied to a pair of regions of different dimensions (consult (Galton, 2014; Clementini and Cohn, 2014) for further information). Moreover, the complementation notion in which if y is not part of x, there exists something that comprises exactly those parts of y that are disjoint from x (Varzi, 2019) will be violated by accepting lower-dimensional entities in theory.

Thus, the common-sense requirement to cater to objects of multiple dimensions has led to introducing the new version of the mereotopological theories representing the spatial relationships between regions of various dimensions known as multidimensional mereotopological theories.

2.1.2 Multidimensional Mereotopological Theories

A few multidimensional mereotopological theories have been proposed in the literature to reflect common-sense reasoning. Regions of different (including lower) dimensions can co-exist in these theories. However, the expressivity of the represented space differs among them due to the various representation languages. Some of these theories use *higher-order logic* (HOL) (including (Galton, 1996; Smith, 1996)) to represent the mereotopological relations between the spatial regions while others use the language of *first-order logic* (FOL) (including (Gotts, 1996; Hahmann, 2013, 2020; Clementini and Cohn, 2014; Baumann et al., 2016)). The former group is more expressive than the latter one. The range of variables is defined over individuals, relations and functions in the HOL theories, whereas it is limited to individuals in the FOL group. High expressivity of HOL theories comes at the cost of a less efficient reasoning mechanism; thus, a trade-off between expressive power and reasoning power is not balanced in these theories. The FOL theories do not suffer from the trade-off problem between expressivity and reasoning. Also, several reliable automated theorem provers and finite model finders developed for them have the potentiality of supporting efficient reasoning.

Furthermore, previous HOL theories (i.e. Take-dim and Smith's theory) define lower-dimensional regions as boundaries (or subsets of the boundary) of the higher-dimensional regions. In other words, the lower-dimensional regions are only considered as dependent entities in these theories. For instance, a one-dimensional region could only be described as a boundary of a two-dimensional region (like the parcel boundary). However, humans can refer to a one-dimensional region as a dependent (like a parcel boundary) or independent (like a river) object. Thus, none of the HOL theories is compatible with human perception.

Multidimensional mereotopological FOL theories do not suffer from this limitation. However, of the multidimensional mereotopological FOL theories, three (i.e. INCH (Gotts, 1996), CODI (Hahmann, 2013) and, GFO Space (Baumann et al., 2016)) use theorem-proving techniques for inference purposes which allows more expressive but less efficient reasoning. To achieve highly efficient reasoning, despite limiting expressivity of a theory, a pre-computation of the composition of the mereotopological relations is essential. Compositions of the relations are achievable via a special form of the qualitative spatial relations, i.e. binary relations. If the set of introduced binary mereotopological relations over a spatial domain satisfies the jointly exclusive and pairwise disjoint (JEPD) property, it will support reasoning via an algebraic operation known as composition operation (Renz and Ligozat, 2005). The search space is simplified for reasoning purposes because all possible spatial arrangements are limited to the situations represented by JEPD relations. In this framework, spatial knowledge can be expressed in the form of relational expressions like $R_i(x, y)$ (i.e. R_i is a JEPD relation) ($R_i(x, y)$), its converse form ($R_i^{-1}(x, y)$), or a disjunction of them ($R_1(x, y) \vee R_2(x, y) \vee ... \vee R_n(x, y)$) respectively. Meanwhile, the number of JEPD relations affects the computational cost of generating their compositions. The more relations, the higher the computational cost.

To our best of knowledge, only the region connection calculus*-9 (RCC*-9) (Clementini and Cohn, 2014) represents the space of multidimensional regions via the set of JEPD mereotopological relations, which makes it usable for efficient constraint-based reasoning. However, the hierarchical structure of the proposed relations is not verifiable according to the definitions provided in this theory. We have dealt with this issue and proposed a solution in our paper (Izadi et al., 2019) (see Appendix B). Nevertheless, even that fix does not overcome other limitations of the RCC*-9, especially not having a verified composition table and has not yet been tested for cognitive adequacy.

Therefore, we propose a theory that represents binary mereotopological relations between regions of various dimensions from zero to three using a set of JEPD relations to support constraint-based reasoning and use the language of first-order logic to define them. Our axiomatization applies to entities up to three-dimension. Throughout this dimension-independent method, rather than taking higher-dimensional regions as spatial primitives and making the lower-dimensional entities dependent on them, we axiomatize mereotopological relations between regions of different dimensions while considering them as first-class objects in the domain.

To develop our multidimensional mereotopological theory to meet the above requirements, we have compared the existing multidimensional mereotopological theories using several criteria to help the reader better understand their similarities and dissimilarities. In particular, we compare their main mereological and topological properties. This in-depth comparison of the existing theories is presented in Appendix A and prepared for publication. The comparison provides a strong foundation for us to develop our formalism by considering the requirements of common-sense representation and reasoning. The summary of this comparison is provided in Table 2.1, with the last column showing the properties of our theory.

2.2 Mereotopology and Boundaries

As the boundaries of objects are of interest to people in their descriptions of their surrounding space, they also play an essential role in theories of mereotopology. Different features of the boundaries have been mentioned in various multidimensional mereotopological theories. All of them imply the complexity of the boundary concept.

In Smith's work (Smith, 1996; Smith et al., 2000), the spatial domain is axiomatized to have the lower-dimensional entities as the higher-dimensional ones' boundaries. Thus, a three-dimensional spatial entity might have areal, linear, or even point entities as its boundary parts. However, there is no constraint on boundaries to be of lower dimension. For instance, a three-dimensional object might have a three-dimensional (or bulky) boundary.

While Smith's constraint on the existence of lower-dimensional entities has been lifted in CODIB (Hahmann, 2013), boundaries (and their parts) still can have any dimensionality. However, bulky and thin boundaries are axiomatized separately.

Take-dim (Galton, 1996), like Smith's work (Smith, 1996; Smith et al., 2000) and CODIB (Hahmann, 2013), considered the boundary of a higher-dimensional object consisting of a collection of entities of various dimensions. They then axiomatized it to be not of the same dimension as the higher-dimensional object.

The INCH calculus (Gotts, 1996) added more constraints to Take-dim's (Galton, 1996) assumptions to limit the boundary of a higher-dimensional object to just one dimension lower than it. It means that the boundary of a three-dimensional entity itself can not have lower-dimensional elements.

GFO Space (Baumann et al., 2016) follows the INCH calculus (Gotts, 1996) approach in dealing with boundaries, while it uses a spatial boundary relation to categorize the spatial entities into four pairwise disjoint classes, including three-dimensional, two-dimensional, one-dimensional and zero-dimensional entities and then defines different mereotopological relations for every category.

Among the multidimensional mereotopological theories, RCC*-9 (Clementini and Cohn, 2014) has a different approach in representing the boundaries of the entities. It assumed that boundaries are (tangential) proper parts of the bounded region. However, there is no explicitly or implicitly predicate to describe the boundaries' dimensionality.

Based on the above explanations, we have briefly learned about the importance and representation of the boundary in the mereotopological theories. We decided to (partially) follow CODIB's approach Hahmann2013a. Boundaries (and parts thereof) can have either the same or lower dimension than the bounded regions in CODIB. However, our theory limited the boundaries to the lower-dimensional regions bounding higher-dimensional ones. This choice has been made to improve the efficiency of the reasoning by restricting our theory to thin boundaries (i.e. lower-dimensional boundaries).

The claim of similarity between qualitative spatial representation and our everyday common-sense of spatial knowledge has also led to the validity checking of qualitative (especially mereotopological) theories by human empirical investigations (Mark and Egenhofer, 1995; Mark et al., 1995; Knauff et al., 1995, 1997; Renz et al., 2000; Klippel et al., 2013) in addition to the established formal verification methods. The theories are perceptually evaluated from the human point of view. These assessments are known as *cognitive adequacy* checking in the qualitative spatial reasoning (QSR) community. We will check the cognitive adequacy of our formalism in chapter 5.

2.3 Reasoning in Mereotopological Theories

A practical theory depends not only on representing spatial knowledge but also on a reasoning mechanism. The reasoning problem is a task of checking the consistency of a theory, i.e. given some spatial information, it is consistent or inconsistent. A consistent theory involves no logical contradiction. A logical contradiction is the conjunction of a statement ϕ and its negation $\neg\phi$.

FOL theories use deduction reasoning to verify the validity of a theory. Deduction is forming a conclusion based on the accepted facts or logical statements. It has an advantage that the truth of statements guarantees the truth of the conclusion. In contrast, the validity of HOL theories is verified by inductive reasoning, which involves probability in inference. This reasoning forms generalization based on what is known. Thus, there are no limits for premises. Neither of these modes of reasoning has any guarantee of termination, which make them impractical. Thus, they are not suitable candidates for checking the consistency of a theory. However, some techniques make the consistency problem decidable over the FOL theories, which are unavailable for HOL ones.

Researchers consider the structure of knowledge to create a decidable reasoning strategy over the FOL theories. Most of the mereotopological properties are represented in the form of binary relations in the reviewed theories. This style of representation (known as constraint-based reasoning) transforms the relations into constraints restricting the mereotopological properties of regions in a reasoning procedure. Now, the inference mechanism has a chance to be terminated (i.e. efficient). A constraint-based representation of mereotopological knowledge is presented in the form of an existentially quantified FOL expression: $\exists x_1 ... \exists x_n \bigwedge_{i,j} \bigvee_{R \in A} R(x_i, x_j)$ where $x_1, ..., x_n$ are variables over the regions, A is the set of JEPD relations, and $R(x_i, x_j)$ is a binary constraint restricting the possible assignments of x_i, x_j to R (Cohn and Renz, 2008). This formula is a constraint satisfaction problem (CSP). Although CSPs are undecidable, the problem would be decidable by computing transitivity over the most specified relations (i.e. JEPDs) and providing the results as pre-computed information to the CSP solver.

Our approach represents the mereotopological properties of multidimensional regions via binary relations in this thesis. We are then extracting a subset of them satisfying the JEPD properties. This subset would support efficient constraint-based reasoning.

2.4 Summary

In this Chapter, we introduced the concept of mereotopology and reviewed some of the most relevant mereotopological theories focusing on the ones proposed to represent spatial relations in a multidimensional space. We then provided a summary of our comprehensive analysis of these theories. Based on these investigations, we concluded that each theory should be designed according to its intended use. In summary, the application of a theory will affect the two dual factors of expressivity and efficient inference in inverse proportion. It means that increasing one will reduce the other. Since our goal is to propose a multidimensional theory supporting constraint-based reasoning, we will propose our theory using binary spatial relations in the language of first-order logic to ensure they are well defined. It would then be a basis for generating the so-called composition tables for constraint-based reasoning. The role of the boundary of spatial entities has also been investigated in this Chapter. Its importance led us to consider it one of the spatial primitives in our multidimensional theory.

Table 2.1: Comparison of different multidimensional mereotopological theories.

	Properties \ Multidimensional theories	**The INCH calculus**	**CODI /CODIB**	**RCC*-9**	**Take-dim**	**Smith's theory**
Foundation	Representation language	FOL	FOL	FOL	HOL	HOL
	Basis	Mereotopology	Mereotopology	Topology	Mereology	Mereology
	Supported Dimensionality	0D–2D regions	0D–3D regions	0D–2D regions	0D–2D regions	3D regions
	Regions	Closed	Closed	Closed	Open/closed	Open/closed
Mereological properties	Supplementation principle	SSP	SSP	Not available	SSP	WSP
	Unrestricted Fusion	N.A	N.A	N.A	Applicable	Applicable
	Mereological operators	Defined for regions of equal dimension	Defined for all regions	Not defined	Defined for regions of equal dimension	Defined for three-dimensional regions (if exists)
	Atomisim	Atomistic or Atom-tolerant	Atomistic or Atomless	Atomless	Atom-tolerant	Atomless
Topological properties	Self-connectedness	Potential to define	Defined	Not defined	Potential to define	Defined
	Topological operators	Not defined	Not defined	Not defined	Not defined	*interior*, *closure*, *boundary*
	Boundary predicate	Defined	Defined	Defined	Defined	Defined
Reasoning		FOL	FOL	Constraint-based	HOL	HOL
		Theorem proving	Theorem proving		Theorem proving	Theorem proving

Chapter 3

Foundation For Multidimensional Mereotopology

This Chapter sheds light on the facets required to propose our multidimensional mereotopological theory. We will specify first-order logic to enforce the properties of the space containing entities of various dimensions.

We discussed the limitations of equidimensional mereotopological theories in representing spatial relations among entities of various dimensions in Chapter 2. We also understood the advantages of using the language of first-order logic in formalizing the multidimensional mereotopological relations there.

This Chapter describes the spatial features that their mereotopological relations will represent in our multidimensional mereotopological theory. The domain consists of a container, and its contained by spatial regions of various dimensions from zero to three. These regions are subspaces of the container occupied by the zero to three-dimensional geometric objects. These abstract entities are required to represent adequate modelling of the geometries and their relations to space. In this thesis, every region is formally described by the mathematical concept of a *manifold with boundaries*. This concept allows us to define objects of lower dimensions that are not necessarily the boundary of objects of higher dimensions. In addition, if an object in question has a boundary, this concept allows us to recognize it. TAs fundamental spatial objects, the manifolds with boundaries can also be very useful in combining and defining more complex objects. We will formally define regions and their properties in Section 3.1.

Then in Section 3.2, we will review the fundamentals of first-order logic as our intended representation language, as it is a convenient and straightforward language to express the necessary information formally. Moreover, it has the potentiality to support efficient reasoning. We will provide some details about the structure (i.e. syntax) and meaning (i.e. semantic) of the sentences in this language. Also, we will review how inferring new knowledge is in this language. These logical terminologies and notations will then use in Chapter 4 where we propose and verify our multidimensional mereotopological theory and develop reasoning over it (in Chapter 6.

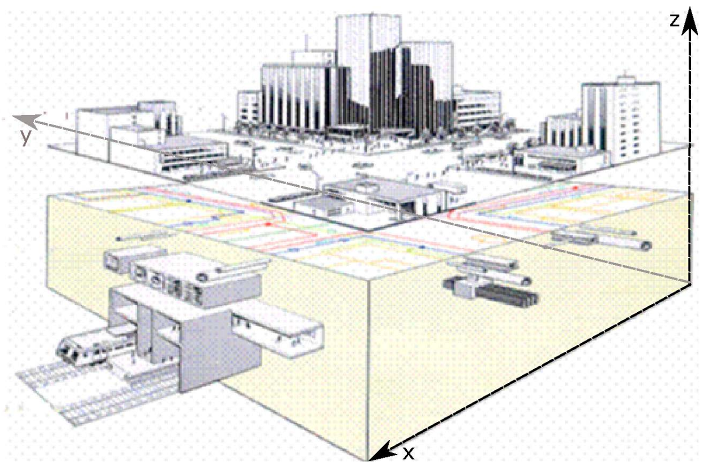


Figure 3.1: Three-dimensional space

3.1 Objects in Three-Dimensional Space

In this section, we will talk about our domain of discourse. Our intended space is a three-dimensional container holding spatial regions of any dimensions from zero to three. It would be something like the space shown in Figure 3.1 as represented on a digital map. This abstract space is a self-connected infinite spatial region containing regions of dimension one (such as power lines) and their zero-dimensional intersections, two (like land parcels), and three (like buildings).

To express the model for the spatial regions in our multidimensional theory, we refine the base class of spatial regions through the subclasses based on the dimension of the regions. Also, the regions (regardless of their dimensionality) might have either simple or complex structures.

A simple structure is made of a single connected regular closed region. Mathematically, the concept of m-manifold with boundaries ($0 \leq m \leq 3$) represents simple structures. It is a tool to define the distinction between entities of different dimensions. The m-manifold with boundaries has been defined in standard mathematical textbooks (M.Lee, 2011; Tu, 2011) as locally Euclidean spaces. Thus, it can be embedded in a higher-dimensional space ($m \leq n$) without any changes in its dimensional values.

Furthermore, the definition of manifolds supports dimensional unity over the region. Thus, an m-manifold with boundaries does not have any constituent part of the lower dimension.

Some topological properties such as *boundary* and *interior* points are easily definable on the m-manifold with boundaries. Moreover, the no-boundary concept like the boundary of a plane (i.e. a two-dimensional surface extending infinitely far) is explainable via the concept of the empty boundary in the structure of an m-manifold with boundaries.

The m-manifolds with boundaries, as simple fundamental elements of our space, do not have any self-intersection (figures in the first and second rows of Figure 3.2), weakly connected pieces (figures in the third row of Figure 3.2), or holes (figures in the last row of Figure 3.2).

Simple regions (i.e. m-manifolds with boundaries) are used to define complex structures. These structures are made of an aggregation of the simple entities of the same dimension, gluing them together on their boundaries. This idea was developed and detailed in (Hahmann, 2013) by introducing a composite

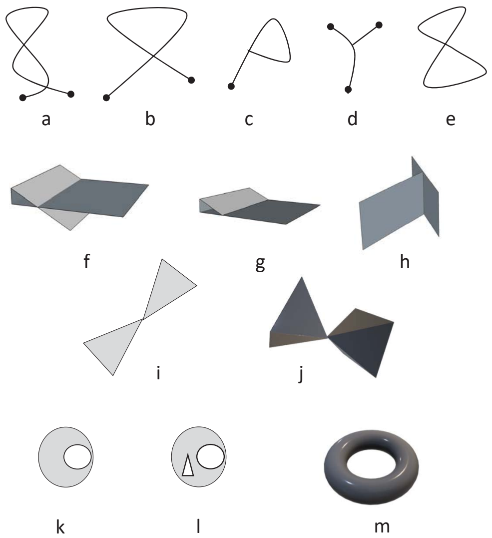


Figure 3.2: Examples of non-m-manifolds with boundaries.

m-manifold. The spatial entities shown in Figure 3.3 are samples of composite 1, 2, and 3-manifolds. The dimensionality of a composite m-manifold is defined by the dimensionality of its constituent m-manifolds with boundaries. Like the m-manifolds with boundaries, the interior and boundary points are definable on the composite m-manifold.

The taxonomy of accepted spatial regions in our multidimensional theory is based on the presented idea of simple and composite m-manifolds and is shown in Figure 3.4. It shows some further refinement according to the dimensionality of the regions and the presence or absence (i.e. closed m-manifold) of the boundary named as *bounded* regions and regions with *implicit boundaries* (i.e. having coincident boundaries) respectively in Chapter 4.

This classification of the spatial objects is compatible with the definitions of simple and complex geometric features as commonly described in geospatial data standards, such as the OGC GeoSparql standard (Open Geospatial Consortium, 2004), the OGC/ISO Geography Markup Language (GML) (Open

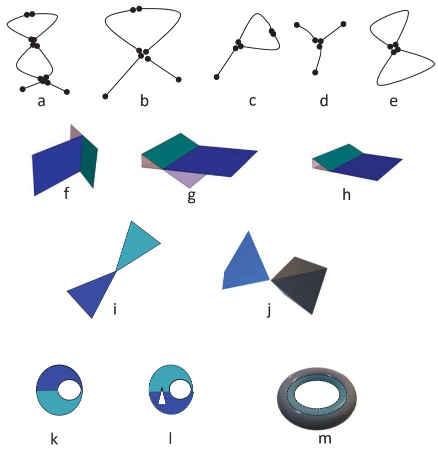


Figure 3.3: The examples from Figure 3.2 as composite m-manifolds.

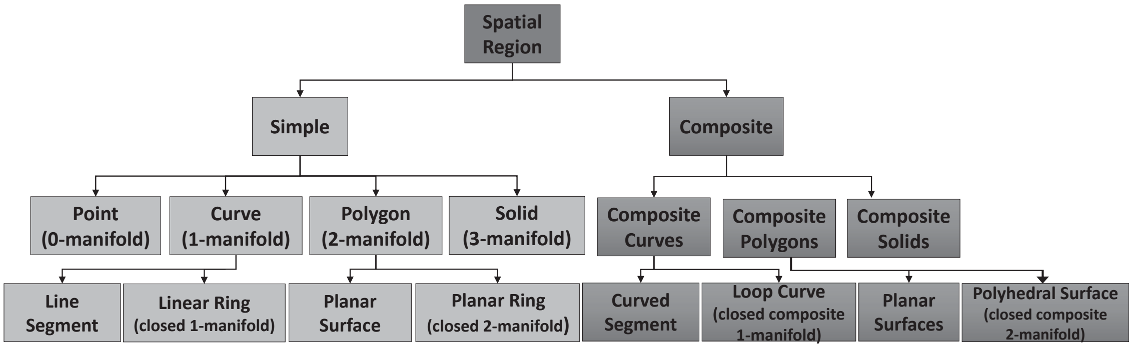


Figure 3.4: Taxonomy of spatial regions in our domain.

Geospatial Consortium, 2007), and OGC spatial schema (Open Geospatial Consortium, 2001).

3.2 First-order Logic as a Language for the Proposed Mereotopological Theories

This section briefly reviews first-order logic as a logical language to represent the relationships among the spatial regions introduced in the previous section. *First-order logic (FOL)* or *predicate logic* is a method for knowledge representation in artificial intelligence. It allows us to build complex expressions out of basic ones. It has been shown that this language is sufficiently expressive to formulate spatial concepts in a multidimensional space (discussed in Subsection 2.1.2). It also supports efficient reasoning[1].

Every logical formalism, including FOL, is composed of three sections, the syntax (the collection of symbols that can form a logical expression), the semantics (how logical expressions can be interpreted), and inference rules (how to deduce new logical expressions from given ones) (Brachman and Levesque, 2004, p. 16). We will review the three main aspects of the language of first-order logic in the following sections.

3.2.1 Syntax

The syntax of FOL determines which collection of symbols can be used to form a logical expression. The basic syntactic elements of first-order logic are symbols which are classified as follows (adopted from (Brachman and Levesque, 2004)):

- *logical symbols*:

  (i) *variables*: A set of symbols typically denoted by the lower case letters from ‘u’ to ‘z’.

  (ii) *Boolean connective*: A set of symbols: ‘$\neg$’ (not), ‘$\wedge$’ (and), ‘$\vee$’ (or), and meta-mathematical expressions: ‘$\rightarrow$’ (implies), ‘$\leftrightarrow$’ (if and only if).

  (iii) *quantifiers*: Two symbols to show “for all...” ($\forall$), and “for some...” ($\exists$).

[1]Although reasoning in propositional logic is more efficient than predicate logic, at least four prepositions are required to represent every predicate of arity two. Thus, we would end up with a large number of prepositions in a theory which losses its generalizability.

(iv) *punctuation*: A set of markers including paired square brackets '[' and ']', open brackets '(' and ')', and the comma ',' as a term separator.

- *non-logical symbols*:

(i) *functions*: A set of n-place symbols typically denoted by strings of lower case letters, e.g. `sum` or `complement`.

(ii) *predicates*: A set of n-place symbols denoted by either string of upper case letters, e.g. '***P***', '***D***', or by strings of lower case letters prefixed by an upper case letter, e.g. '***Point***'.

FOL describes some rules that determine how formulas can be validly formed from the language's syntax symbols. They define the terms and formulas of first-order logic (the following definitions of this Section are adopted from (Smullyan, 1971)).

Definition 3.2.1. *Terms are defined as follows:*

(i) an individual constant is a term,

(ii) an individual variable is a term,

(iii) if α is an n-place function symbol, and $x_l, ..., x_n$ terms, then $\alpha(x_l, ..., x_n)$ is a term.

Definition 3.2.2. *Atoms (i.e. atomic formulae) are defined as follows: if Φ is an n-place predicate symbol, and $x_l, ..., x_n$ terms, then $\Phi(x_l, ..., x_n)$ is an atom.*

Definition 3.2.3. *Formulas are defined as follows:*

(i) an atom is a formula,

(ii) if Φ is a formula, then $\neg\Phi$ is a formula,

(iii) if Φ and Ψ are formulas, then $[\Phi \vee \Psi]$, $[\Phi \wedge \Psi]$, $[\Phi \rightarrow \Psi]$ and $[\Phi \leftrightarrow \Psi]$ are formulas,

(iv) if Φ is a formulas and α a free variable in Φ, then $\forall\alpha[\Phi]$ and $\exists\alpha[\Phi]$ are formulas.

Definition 3.2.4. *If Φ is an atom, then both Φ and $\neg\Phi$ are literals.*

Definition 3.2.5. *Sentences are formulas with no free variables.*

Definition 3.2.6. *A clause is a finite disjunction of literals: the* empty *clause is a disjunction of zero literals, a* unit *clause a disjunction of one literal.*

A specific arrangement of clauses, known as conjunctive normal form, is helpful for automated theorem proving.

Definition 3.2.7. *Conjunctive normal form (CNF) expresses formulas as conjunctions of clauses with an $\wedge$ (and).*

3.2.2 Semantics

Syntactic questions deal with the formal structure of formulas. However, semantic questions concern the truth of a formula relative to some truth assignments. The truth value (whether or not it is true) of a first-order sentence depends on how we interpret the language's functions, constants, and predicates.

Definition 3.2.8. *An* <u>*interpretation*</u> *of $\mathcal{L}(\mathscr{T})$ is a pair $\Gamma = \langle D, \mathcal{I} \rangle$ that assigns meanings to all symbols in $\sigma(\mathscr{T})$. D denotes a non-empty set of objects, called a domain $\mathcal{I}$, mapping symbols in $\sigma(\mathscr{T})$ to functions and predicates over the domain. Every function (including constants) maps to an element of the domain, while every predicate maps to* `True`|`False`*, where* `True` *means the predicate holds and* `False` *means the predicate does not hold.*

On the other hand, the truth value depends on what we are quantifying over (i.e. domain). We require information that provides a precise meaning to the formulas to extract their truth values. Since providing all required information about a domain is usually impossible, further information can *follow* from the stated knowledge. This notion is imported to logic as a *logical consequence.*

Definition 3.2.9. *(adopted from (Audi, 2015)) A sentence σ is a* <u>*logical consequence*</u> *of a theory $\mathscr{T}$, $\mathscr{T} \vDash \sigma$, if and only if there is no interpretation in which all sentences of $\mathscr{T}$ are* `true` *and σ is* `false`*.*

The *supporting* information is a *theory* in logical consequence.

Definition 3.2.10. *A* <u>*theory*</u>*, denoted by $\mathscr{T}$, is a set of first-order sentences closed under logical consequences.*

If a sentence is a logical consequence of a theory, i.e. $\mathscr{T} \vDash \sigma$, then the sentence ($\sigma$) is a theorem for the theory.

The *signature* and *language* of a theory describe its non-logical symbols.

Definition 3.2.11. *The collection of non-logical symbols used in a theory ($\mathscr{T}$) is its* <u>*signature*</u> *shown by $\Sigma(\mathscr{T})$.*
The set of all first-order formulae in a theory ($\mathscr{T}$) only using symbols in $\Sigma(\mathscr{T})$ is the <u>*language of the theory*</u> *shown by $\mathcal{L}(\mathscr{T})$.*

All the information we require to determine the truth value of each sentence in a language is available when we have a domain and all interpretations of the symbols in a theory. This dual structure (the domain with the interpretations) is known as a *model.*

Definition 3.2.12. *An interpretation (Γ) is a* <u>*model*</u> *of a theory $\mathscr{T}$, if and only if all sentences of $\mathscr{T}$ (or their logical consequences) are satisfied, that is, if all sentences are evaluated to statements with the truth value* `True`*; we can write $\Gamma \vDash \mathscr{T}$ and read it as $\mathscr{T}$ is* `True` *at Γ.*

When there is a model for a theory, it is *consistent.*

Definition 3.2.13. *A sentence σ is a* <u>*logical equivalence*</u> *of a theory $\mathscr{T}$ ($\mathscr{T} \equiv \sigma$) if they have the same truth values in every model.*

3.2.3 Inference rules

Logic, here first-order logic, is the study of the forms of reasoning in arguments and the development of standards and criteria to evaluate arguments (Bezian, 2020). An *argument* in logic is:

Definition 3.2.14. *An argument is a sequence of sentences where the last one, the* conclusion*, purportedly follows from (i.e. is a logical consequence of) the sentences that precede it—the* assumptions*.*

The conclusions that follow deductively from the assumptions (or premises) represent a *valid* argument. If inferencing over `true` premises leads to `true` logical consequences, then the argument is *sound*. All sound arguments are valid.

The logical argument develops based on the inference rules. They are used to deduce new facts or sentences from existing ones. The most important rule is *modus ponens*, which is supported by some rules for handling quantifiers. It is mostly used for manual proving. However, its general form, known as *resolution*, is used for automated deductive reasoning. It takes two (parent) clauses and infers a new clause. It is defined as:

Definition 3.2.15. *(Brachman et al., 2004, p. 52) Given a clause of the form $c_1 \cup \{\Phi\}$ containing some literal Φ and another clause of the form $c_2 \cup \{\neg\Phi\}$ containing the negation of Φ, the resolution rule infers the clause $c_1 \cup c_2$ of those clauses. It resolves the literals (Φ and $\neg\Phi$) in the clauses and keeps the conjunction of the other statements.*

Clause $c_1 \cup c_2$ is known as *resolvent* with respect to Φ. Resolution is part of a refutation proof, where you add the negation of a new sentence (sometimes called goal or query sentence) to a theory and applying this rule on all pairs of complementary literals (including the queried sentence) and simplifies the resolvent clause in every iteration. Derivation of an empty clause proofs the theory inconsistency meaning that the negation of the queried sentence is `False` or similarly the queried sentence is `True`. The resolution is a sound argument.

3.3 Summary

In this Chapter, we gave a summary of accepted spatial entities, which were named *regions*. These abstract entities allow us to define mereotopological relations, without any bias, to represent the overlapping or coincidence of a pair of geometries. The spatial properties of these regions are defined by the mathematical concept of m-manifolds with boundaries in which m shows the dimensionality of the manifolds. This concept is also the basis for defining the complex structures, i.e. composite regions. They consist of a several m-manifolds with boundaries gluing in a network having similar properties as the m-manifolds with boundaries. This structure (i.e. m-manifold with boundaries or its composite form) locally captures the dimensionality of regions located in $\mathbb{R}^n$ where n might be greater or equal to m. Thus, the topological properties of the regions such as interior and boundary can be defined without any issues.

In addition, we reviewed the fundamental of first-order logic as our representation language. It not only has a powerful expressivity but allows us to verify the properties of our theory. This logical language also smooths the reasoning process (i.e. generating the composition tables) by allowing powerful automated FOL theorem provers and finite model finders.

After providing these preliminaries, we are ready to propose our multidimensional mereotopological theory in the following Chapter, in which we will define a set of mereotopological relations among spatial regions of different dimensions using the set of first-order logical statements.

Chapter 4

Multidimensional Mereotopological Theory

In Chapter 2, we identified the limitations of previous approaches in representing multidimensional mereotopological spatial relations and reasoning about them. This Chapter presents our multidimensional mereotopological theory to address these limitations, using the foundations presented in Chapter 3 as a basis.

In our multidimensional mereotopological theory, we follow Smith's (Smith, 1996) footsteps and begin our theory by laying out our view on part-whole properties of multidimensional spatial entities. We will define the entities' mereological spatial structures via the *parthood* relation. Since mereology alone cannot go far (Casati and Varzi, 1999), we integrate it with the topological aspect of the spatial entities by introducing a *boundary part* relation. This combination will introduce a set of mereotopological relations that form a hierarchy of relations. In this structure, the set of most specific relations has the JEPD property. It will be the basis of efficient qualitative spatial reasoning presented in Chapter 6. Our proposed theory of mereotopological relations and their structural lattice are logically and intuitively verified in this Chapter. However, we will evaluate the relations from the cognitive point of view in Chapter 5.

This Chapter is organised as follows: we propose our multidimensional mereotopological theory in Section 4.1, defining and axiomatizing our relations and presenting them in a hierarchical form (i.e. a lattice of the relations). In Section 4.3, we verify the formalism both logically (as an axiomatic system) and diagrammatically to evaluate of the robustness of the proposed relations.

4.1 Multidimensional Theory

Varzi accounted (Varzi, 1996) that there are various strategies for combining mereological and topological aspects of space. Among them, developing a mereotopological theory based on a single primitive relation (representing either the topological aspect of space and subsuming the mereological aspect, like the *connection* relation in RCC, or both aspects in a single primitive like the *includes a chunk of* relation in the INCH calculus) is customary when the goal is to provide a calculus supporting a machine to reason over the spatial entities. This approach is presented in the equidimensional mereotopological theory of Cohn and his associates (Randell et al., 1992b; Randell and Cohn, 1992; Cohn et al., 1997; Gotts et al., 1996) and previously presented in multidimensional mereotopological theories (Gotts, 1996; Hahmann, 2013).

However, developing a mereotopological system based on a single primitive is problematic (Smith, 1996; Varzi, 1996). The single primitive relation controls the topological aspects of space, which eventually controls of the mereological aspects. Thus, it is not easy to differentiate these two facets of the theory, which typically reduces its expressive powers. Moreover, the goal of the initial mereological theories (i.e. proposed in Leśniewski (Leśniewski, 1927, 1928, 1929, 1930) and his follower's works like Tarski (Tarski, 1972)) in proposing a formalism with a minimal set of non-logical primitives (or ideally constructing a theory based on a single primitive) are not helpful in practical applications (Smith, 1996), because, for example, a single ternary primitive relation (i.e. *betweenness*) in Tarski's mereological theory is less intuitive than binary primitive relations (i.e. *parthood* and *interior part*) in Smith's theory.

This limitation has been addressed in our formalism, in which distinct mereological and topological primitives are introduced. This approach is found in Smith's works (Smith, 1996). Like Smith, we use the *parthood* primitive relation to representing space's mereological aspect. However, our approach differs from his in that the topological aspect is represented by the *boundary part* primitive relation instead of the *interior part of* relation used in Smith's work. This change lets us describe a space involving multidimensional entities, which was restricted to only the dependent lower-dimensional entities in Smith's work[1]. These primitives together can define a small set of relations, a subset of which are JEPD mereotopological relations, accepting lower-dimensional entities both independently and as a boundary of the higher-dimensional entities.

Furthermore, our domain of discourse differs from Smith's, in which variables range over three-dimensional entities (i.e. *realia*). In our theory, variables can be assigned from the domain of objects of different dimensions from zero to three (either as independent or dependent entities). Our formalism also differs from Smith's in its representation language. He used higher-order logic while we kept our formalism purely in first-order logic (with identity) as it supports an efficient reasoning facility for our calculus. In addition, our work further differs from Smith's work in that we accommodate *null* values. The null entity is added to allow arbitrary Boolean combinations of regions to be expressed as functions in the formalism, in particular where two regions have no common part or nonexistence regions passed. Moreover, we introduce a set of JEPD mereotopological relations that supports constraint-based reasoning as an efficient inference mechanism.

Since the proposed ontology is for a spatial domain, the reader is advised to read the introduced relationships in the light of solely spatial interpretations.

4.1.1 Parthood

When we think about space, the first things that come to our mind are its spatial objects. Regions of space are described by the spatial objects occupying them. The objects have parts that are arranged based on a

[1] Both Smith's (Smith, 1996) and Take-dim (Galton, 1996) multidimensional formalisms constructed the lower-dimensional entities from the higher-dimensional ones.

structure. For example, a residential zone has parts like buildings, streets, and green spaces separate from each other. However, their parts, such as apartments, alleys, and trees, are also parts of the residential zone. This structure can be represented by expressing the arrangement of the spatial objects, their parts, and the relationships between them.

In this Section, we present a fundamental (i.e. mereological) aspect of our theory to represent the structure of space: a binary primitive relation, *parthood*. As our first primitive, it is defined as a partial order relation. The (partial) order relation lets us compare the properties of the spatial objects with respect to each other (for example, comparing a part of an entity to itself, another entity, or even the whole). The parthood relation is denoted by $\boldsymbol{P}(x, y)$ to indicate any portion of given entity x in another entity y and reads as "x is part of y." It requires the participation of non-empty entities (i.e. not **null**). Also, it applies even when x and y are identical. The parthood relation is generally defined to be a *reflexive*, *anti-symmetric* and *transitive* relation - a partial order[2].

Axiom 4.1.1. $x \neq \boldsymbol{null} \rightarrow \boldsymbol{P}(x, x)$ $\qquad$ ***P*** *(restricted) reflexive*

Axiom 4.1.2. $\boldsymbol{P}(x, y) \wedge \boldsymbol{P}(y, x) \rightarrow x = y$ $\qquad$ ***P*** *anti-symmetric*

Axiom 4.1.3. $\boldsymbol{P}(x, y) \wedge \boldsymbol{P}(y, z) \rightarrow \boldsymbol{P}(x, z)$ $\qquad$ ***P*** *transitive*

Note that we suppress all initial universal quantifiers in our axioms, definitions, and theorems statements. The first axiom (**Axiom 4.1.1**) says that every non-empty entity is part of itself. The second axiom (**Axiom 4.1.2**) says that for any pair of entities if x is part of y, y won't be part of x unless they are identical. **Axiom 4.1.3** states that for all elements x, y, and z of the domain of spatial entities, whenever x is part of y, and y is part of z, then x is also part of z.

According to **Axiom 4.1.1**, the empty entity (i.e. ***null***) is our next primitive. The '***null***' is defined as an individual not being part of anything and not having anything as its part:

Axiom 4.1.4. $x = \boldsymbol{null} \leftrightarrow \forall y\ (\neg \boldsymbol{P}(y, x) \wedge \neg \boldsymbol{P}(x, y))$

without the ***null*** entity, functions[3] are not always definable. For instance, the existence of the boundary (Definition 4.1.13) of a *ring* is not definable without the ***null*** entity.

Axiom 4.1.1-Axiom 4.1.4 guarantee the existence of the ***null*** entity in our domain:

Theorem 4.1.1. $\exists x \forall y\ (\neg \boldsymbol{P}(y, x) \wedge \neg \boldsymbol{P}(x, y))$

According to **Axiom 4.1.1-Axiom 4.1.4**, we can say that $\boldsymbol{P}(x, y)$ represents various types of inclusion between non-empty entities, from "just a small part of a boundary" to "full coverage of an entity by another". Figure 4.1 shows the intended meaning of $\boldsymbol{P}(x, y)$ over a pair of entities in which the relation holds for a pair of one and two-dimensional entities.

[2]One of the fundamental structures on which the mathematical subjects are founded is *order*. A set is said to be *(partially) ordered* when the order relation is established between its elements and renders them comparable.

Let S be a set. A partial order on S is a binary relation $\mathbf{R}$ on S such that, for every $x, y, z \in S$, $\mathbf{R}$ is reflexive, antisymmetric and transitive. The set S with an order relation, $\mathbf{R}$, is called a (non-strict) partially ordered set (or poset). A relation which is irreflexive, asymmetric and transitive is called *strict* partial order relation.

[3]We have a limited number of functions in our theory. However, it is possible to extend our theory by defining Boolean functions such as *sum*, *product*, *difference* and *complement* functions. They will allow us to describe new spatial regions. For instance, a common region between a pair of overlapped regions can be explicitly described by the *product* function.

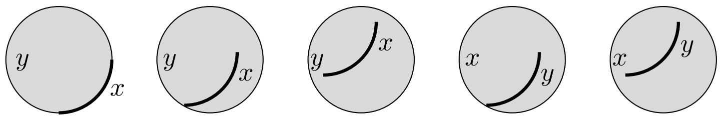


Figure 4.1: Range of satisfied parthood relations.

If we look abstractly at spatial objects and consider them as regions of space, we do not see any difference between spatial inclusion and parthood (Casati and Varzi, 1999). Thus, we define spatial relations between regions based on the parthood relation here and in the following Sections.

As an immediate consequence of the anti-symmetric property of the parthood relation, we can introduce the *equality* relation as a general form of the usual identity relation. This name is more in line with our preferred concept than identity. According to the definition of *equality* in set theory, two objects are equal if they have the same parts. However, we also want the objects that are part of the same set of regions to be equal. Furthermore, equality ($\boldsymbol{EQ}$) is a relation defined in terms of part-of relation $\boldsymbol{P}$ rather than a pre-assumed logical relation (like identity).

Definition 4.1.1. $\boldsymbol{EQ}(x, y) \leftrightarrow \boldsymbol{P}(x, y) \wedge \boldsymbol{P}(y, x)$ *x equals to y*

It should be pointed out that this relation can only hold between equidimensional regions since only then since only then the definition can be true. The equality relation proves to be reflexive, symmetric, and transitive (for non-null entities):

Theorem 4.1.2. $x \neq \boldsymbol{null} \rightarrow \boldsymbol{EQ}(x, x)$ ***EQ** (restricted) reflexive*

Theorem 4.1.3. $\boldsymbol{EQ}(x, y) \rightarrow \boldsymbol{EQ}(y, x)$ ***EQ** symmetric*

Theorem 4.1.4. $\boldsymbol{EQ}(x, y) \wedge \boldsymbol{EQ}(y, z) \rightarrow \boldsymbol{EQ}(x, z)$ ***EQ** transitive*

In addition, several mereological relations between two regions can be defined to describe more spatial configurations. These definitions are based on the 'parthood' relation (**Axiom 4.1.1-Axiom 4.1.4**):

Definition 4.1.2. $\boldsymbol{O}(x, y) \leftrightarrow \exists z\ (\boldsymbol{P}(z, x) \wedge \boldsymbol{P}(z, y))$ *x overlaps y*

Definition 4.1.3. $\boldsymbol{D}(x, y) \leftrightarrow \neg \boldsymbol{O}(x, y)$ *x discretes from y*

Definition 4.1.4. $\boldsymbol{PP}(x, y) \leftrightarrow \boldsymbol{P}(x, y) \wedge \neg \boldsymbol{P}(y, x)$ *x is a proper part of y*

Definition 4.1.5. $\boldsymbol{P}^{-1}(x, y) \leftrightarrow \boldsymbol{P}(y, x)$ *y has x as a part*

Definition 4.1.6. $\boldsymbol{PP}^{-1}(x, y) \leftrightarrow \boldsymbol{PP}(y, x)$ *y has x as a proper part*

The overlap relation holds when at least one common region exists between x and y. Note that this relation does not prohibit a full coverage of one of the participating regions by the other. Indeed, the intended meaning of the overlap relation is a pairwise incursion. Axioms and definitions (i.e. **Axiom 4.1.1**, **Definition 4.1.2**) imply that the overlap relation is reflexive for non-null entities and symmetric (from **Definition 4.1.2**) but not transitive.

Theorem 4.1.5. $x \neq \boldsymbol{null} \rightarrow \boldsymbol{O}(x, x)$ ***O** (restricted) reflexive*

Theorem 4.1.6. $\boldsymbol{O}(x,y) \rightarrow \boldsymbol{O}(y,x)$ *$\boldsymbol{O}$ symmetric*

The transitivity property of the parthood relation (**Axiom 4.1.3**) satisfies the following property:

Theorem 4.1.7. $\boldsymbol{P}(x,y) \leftrightarrow \forall z\ (\boldsymbol{O}(z,x) \rightarrow \boldsymbol{O}(z,y))$

This theorem states that the part of relation amounts to the inclusion of the overlapped regions. The discrete relation ($\boldsymbol{D}$) represents a case where there is no shared region between the participants. It is proved irreflexive[4] (from **Axiom 4.1.1** and **Definition 4.1.3**) and symmetric (from **Theorem 4.1.6** and **Definition 4.1.3**).

Theorem 4.1.8. $x \neq \boldsymbol{null} \rightarrow \neg\boldsymbol{D}(x,x)$ *$\boldsymbol{D}$ (restricted) irreflexive*

Theorem 4.1.9. $\boldsymbol{D}(x,y) \rightarrow \boldsymbol{D}(y,x)$ *$\boldsymbol{D}$ symmetric*

The proper part relation ($\boldsymbol{PP}$) excludes the equality case (or improper part of relation) from the parthood relation ($\boldsymbol{P}$). Contrary to the overlap relation, the proper part relation is transitive but irreflexive and asymmetric - a strict partial order. Verifying these properties is straightforward by **Definition 4.1.4** and axioms of $\boldsymbol{P}$ (i.e. **Axiom 4.1.1, Axiom 4.1.2**, and **Axiom 4.1.3**).

Theorem 4.1.10. $\neg\boldsymbol{PP}(x,x)$ *$\boldsymbol{PP}$ irreflexive*

Theorem 4.1.11. $\boldsymbol{PP}(x,y) \rightarrow \neg\boldsymbol{PP}(y,x)$ *$\boldsymbol{PP}$ asymmetric*

Theorem 4.1.12. $\boldsymbol{PP}(x,y) \wedge \boldsymbol{PP}(y,z) \rightarrow \boldsymbol{PP}(x,z)$ *$\boldsymbol{PP}$ transitive*

Moreover, a weaker form of the transitivity property of the proper part relation sufficed to capture the intuition that boundaries of the regions are part of them:

Theorem 4.1.13. $\boldsymbol{PP}(x,y) \wedge \boldsymbol{P}(y,z) \rightarrow \boldsymbol{PP}(x,z)$ *$\boldsymbol{PP}$ left weak transitive*

Theorem 4.1.14. $\boldsymbol{P}(x,y) \wedge \boldsymbol{PP}(y,z) \rightarrow \boldsymbol{PP}(x,z)$ *$\boldsymbol{PP}$ right weak transitive*

The non-symmetric properties of $\boldsymbol{P}$ and $\boldsymbol{PP}$ result in the introduction of their converse forms $\boldsymbol{P}^{-1}$ (**Definition 4.1.5**) and $\boldsymbol{PP}^{-1}$ (**Definition 4.1.6**), respectively. They have similar properties as $\boldsymbol{P}$ and **PP** relations.

To make our theory more compatible with the structure of spatial objects, we establish further restrictions to express that "*whenever an object has a proper part, it has more than one*" (Casati and Varzi, 1999, p. 38) (i.e. there exists some mereological distinction between a whole and its parts). It is enforced by an axiom saying that if a region has a proper part, that proper part region must be supplemented by another discrete proper part (supplementation principle):

Axiom 4.1.5. $\boldsymbol{PP}(x,y) \rightarrow \exists z\ (\boldsymbol{P}(z,y) \wedge \boldsymbol{D}(z,x))$

In opposition to the above axiom, we need another restriction on forming the whole from its parts. It should express how suitably related regions together represent a whole. The following axiom enforces this idea:

[4]The irreflexivity of the discrete relation is for non-null entities. From now on, we show theorems for non-null entities.

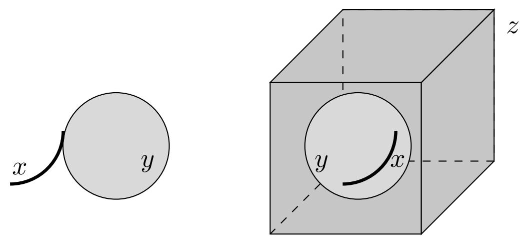


Figure 4.2: The universal region in various spatial configurations.

Axiom 4.1.6. $\exists y \forall x\ (x \neq \boldsymbol{null} \rightarrow \boldsymbol{P}(x, y))$

This axiom would lead to the introduction of an individual as an upper bound of the domain that has everything as its part (except the ***null*** entity), known as the *universal* region:

Definition 4.1.7. $x = \boldsymbol{Universal} \leftrightarrow \forall y\ (y \neq \boldsymbol{null} \rightarrow \boldsymbol{P}(y, x))$

Some spatial configurations and their relevant universal regions are shown in Figure 4.2. In this figure, the universal entity consists of $x + y$ in the left spatial arrangement and $x + y + z$ in the right arrangement. The universal region is not part of anything larger than itself.

4.1.2 Points

Some mereological theories do not clearly represent points (Clarke, 1985; Randell et al., 1992b). Mereotopologically, they deny the existence of points by assuming a proper part for all the spatial regions in their domain. However, points are necessary here since they represent zero-dimensional extended regions. They also clarify the chosen interpretations for mereotopological relations, for instance, by allowing the different types of overlap relation to be valid for two connected regions.

A *point* is defined as *an entity with a position but no dimensions* in Euclidean geometry. Previous researchers have formulated this concept in different fashions, including a nested definition (i.e. limiting cases of sets of nested regions (Gruszczyński and Pietruszczak, 2009)), an algebraic definition (i.e. using distributed lattice or Boolean rings (Stone, 1937; Tarski, 1938)), or what is known as a curious hybrid (Simons, 1987; Galton, 1996). We adopt the latter approach in which a point is defined to have nothing, except itself as a part:

Definition 4.1.8. $\boldsymbol{Point}(x) \leftrightarrow \forall y\ (\boldsymbol{P}(y, x) \rightarrow \boldsymbol{EQ}(x, y))$

Some theories use the term "atom" (Nicod, 1970; Simons, 1987) as an alternative to "point" because the application of those theories considered objects instead of regions. According to this definition (**Definition 4.1.8**), points are ontologically different from other spatial regions (i.e. lines, surfaces, and volumes). All other spatial regions (except points) can accept regions as their parts. However, a point *only* has a point as its part. This property can be more restricted by considering the definition of the proper part relation (**Definition 4.1.4**):

Theorem 4.1.15. $\boldsymbol{Point}(x) \rightarrow \neg\exists y\ (\boldsymbol{PP}(y, x))$

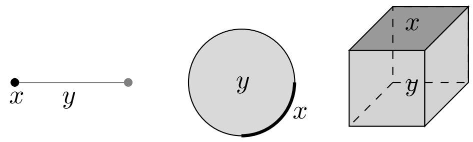


Figure 4.3: x is a boundary part of y.

It says that a point does not have any other proper part than itself.

According to the above-given information, the only spatial configurations between a pair of points is either equal ($\boldsymbol{EQ}$) or discrete ($\boldsymbol{D}$) (compare the definition of point in **Definition 4.1.8** with the definitions of these spatial relations in **Definition 4.1.1**, **Definition 4.1.3**).

4.1.3 Boundary Part

A complete representation of the structure of multidimensional space requires topology to go beyond mereology. Topology gives us a clear view of the structure of a *whole*. Mereology alone is not enough for this purpose. For instance, it could not formalize the configuration of two externally connected regions. The primary topological notion that meets our objectives is a *boundary part* relation. Boundaries are necessary to reflect ordinary intuition — to represent, for example, the walls of a building, the edges of the wall and vertices of the edge. As we said in Section 2.2, various notions of boundaries have been introduced within theories of qualitative space (Chisholm, 1983; Martinich and Stroll, 1991; Barnes et al., 1991). We adopt the definition of a boundary that allows lower-dimensional entities to exist alone - without depending on higher-dimensional entities. We axiomatize a boundary in a way that it is a proper part of the bounded region (see Section 3.1 for definition). Thus, our topological primitive is a *boundary part* ($\boldsymbol{B}(x, y)$), read as "x is a boundary part of y". It is like the *boundary containment* relation ($\boldsymbol{BCont}$) in CODIB (Hahmann, 2013). We say 'boundary part' and not 'boundary of' to permit boundaries that are not *maximal* (corner points, edges, surfaces)[5]. $\boldsymbol{B}(x, y)$ is stipulated as:

Axiom 4.1.7. $\boldsymbol{B}(x, y) \rightarrow \boldsymbol{PP}(x, y)$

Axiom 4.1.8. $\boldsymbol{B}(x, y) \wedge \boldsymbol{B}(y, z) \rightarrow \boldsymbol{B}(x, z)$ — ***B*** *transitivity*

Axiom 4.1.9. $\boldsymbol{P}(x, y) \wedge \boldsymbol{B}(y, z) \rightarrow \boldsymbol{B}(x, z)$ — ***B*** *right weak transitivity*

Axiom 4.1.10. $\boldsymbol{B}(x, y) \wedge \boldsymbol{P}(y, z) \rightarrow \boldsymbol{P}(x, z)$ — ***B*** *left weak transitivity*

Axiom 4.1.7 says that the boundary part is a proper part of a region, which is compatible with the assumption of closed regions. For instance, if y is a simple line (without any self-intersection), $\mathbf{B}(x, y)$ represents one of its endpoints in x. Figure 4.3 shows the boundary part of a volume, surface and line in x. Transitivity of the boundary parts is shown in **Axiom 4.1.8**, which says that any region that bounds a region that bounds z also bounds z. **Axiom 4.1.9** says that any part of a region bounding z is also a boundary of z. On the other hand, **Axiom 4.1.10** says that any boundary part of a region which itself is part of z is still part of z.

[5]We will define the maximal boundaries via $\texttt{bdy}(x)$ relation later.

Intuitively, the universal region, defined in **Definition 4.1.7**, and point, defined in **Definition 4.1.8**, must be unbounded. These properties are represented in **Axiom 4.1.11** and **Theorem 4.1.16**, respectively:

Axiom 4.1.11. $\neg\exists x\ (\boldsymbol{B}(x, \boldsymbol{Universal}))$

Theorem 4.1.16. $\boldsymbol{Point}(x) \rightarrow \neg\exists y\ (\boldsymbol{B}(y, x))$

Another important spatial relation between regions is a specialized form of the proper part relation defined via the boundary part relation. The relation is named *tangential proper part* relation ($\boldsymbol{TPP}$). It holds between a pair of regions x and y while x is part of y and touches its outside (like the cases in Figure 4.4). This relation is defined as:

Definition 4.1.9. $\boldsymbol{TPP}(x, y) \leftrightarrow \boldsymbol{PP}(x, y) \wedge (\exists z\ (\boldsymbol{P}(z, x) \wedge \boldsymbol{B}(z, y)))$

The tangential proper part relation is irreflexive, asymmetric and transitive:

Theorem 4.1.17. $\neg\boldsymbol{TPP}(x, x)$ — $\boldsymbol{TPP}$ *irreflexivity*

Theorem 4.1.18. $\boldsymbol{TPP}(x, y) \rightarrow \neg\boldsymbol{TPP}(y, x)$ — $\boldsymbol{TPP}$ *asymmetric*

Note that all boundary parts are tangential proper parts, but not vice versa. For instance, all xs in Figure 4.3 are tangential proper parts of ys. It is a provable property:

Theorem 4.1.19. $\boldsymbol{B}(x, y) \rightarrow \boldsymbol{TPP}(x, y)$

Because of the non-symmetric property of $\boldsymbol{TPP}$, it has a converse form which is defined:

Definition 4.1.10. $\boldsymbol{TPP}^{-1}(x, y) \leftrightarrow \boldsymbol{TPP}(y, x)$

with similar properties as $\boldsymbol{TPP}$.

Another refinement of the proper part relation would be described via the interior part relation, which will be introduced in the next Section.

4.1.4 Interior Part

As we said, boundaries are defined via the regions they bound. In conformity with this thesis, every boundary separates a region from its surrounding. For instance, a boundary (wall) separates a building's interior from its exterior. In this case, what remains of the region, excluding its boundary, is its interior. Mathematically, all regions have an interior part, but they might either have an explicit boundary like a disk, implicit (i.e. having coincident) boundaries like a circle, or without boundaries like a point or the universal region. So, another critical relationship between a pair of regions is the *interior part* relation ($\boldsymbol{I}(x, y)$), which is defined as:

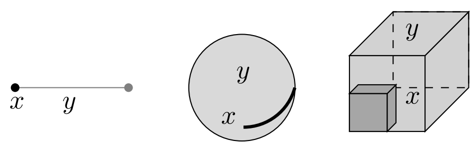


Figure 4.4: x is a tangential proper part of y.

Definition 4.1.11. $\boldsymbol{I}(x, y) \leftrightarrow \boldsymbol{P}(x, y) \wedge \forall z\ (\boldsymbol{B}(z, y) \rightarrow \boldsymbol{D}(z, x))$

This definition reads as "x is an interior part of y", when x is an interior part of y that is *away from* the boundary of y. In other words, an interior part of a region is its part but not its boundary part, and it has no common part with the boundary. The interior relation is irreflexive and asymmetric for the bounded regions.

Theorem 4.1.20. $\exists z\ (\boldsymbol{B}(z, x)) \rightarrow \neg \boldsymbol{I}(x, x)$ *$\boldsymbol{I}$ (restricted) irreflexivity*

Theorem 4.1.21. $\exists z\ (\boldsymbol{B}(z, x)) \rightarrow (\boldsymbol{I}(x, y) \rightarrow \neg \boldsymbol{I}(y, x))$ *$\boldsymbol{I}$ (restricted) asymmetric*

However, it is reflexive and anti-symmetric for regions with implicit boundaries like the universal region and a point:

Theorem 4.1.22. $\neg \exists z\ (\boldsymbol{B}(z, x)) \rightarrow \boldsymbol{I}(x, x)$ *$\boldsymbol{I}$ (restricted) reflexivity*

Theorem 4.1.23. $\neg \exists z\ (\boldsymbol{B}(z, x)) \rightarrow (\boldsymbol{I}(x, y) \wedge \boldsymbol{I}(y, x) \rightarrow x = y)$ *$\boldsymbol{I}$ (restricted) antisymmetric*

and transitive for all regions regardless of the existence/non-existence of the boundary:

Theorem 4.1.24. $\boldsymbol{I}(x, y) \wedge \boldsymbol{I}(y, z) \rightarrow \boldsymbol{I}(x, z)$ *$\boldsymbol{I}$ transitivity*

As can be seen, there is a close relationship between the interior part relation (**Definition 4.1.11**) and part of relation (**Axiom 4.1.1, Axiom 4.1.2, and Axiom 4.1.3**). The part of relation shows a weaker form of the transitivity of the interior part relation to capture the notion that the interior of any region is its part. Accordingly, all parts of a region y, which is entirely in the interior part of a region z are also interior parts of z (from **Axiom 4.1.3** and **Definition 4.1.11**):

Theorem 4.1.25. $\boldsymbol{I}(x, y) \wedge \boldsymbol{P}(y, z) \rightarrow \boldsymbol{I}(x, z)$ *$\boldsymbol{I}$ left weak transitivity*

Similarly, all interior parts of a region y, which is completely contained in a region z, are also part of z (from **Axiom 4.1.3** & **Definition 4.1.11**):

Theorem 4.1.26. $\boldsymbol{P}(x, y) \wedge \boldsymbol{I}(y, z) \rightarrow \boldsymbol{I}(x, z)$ *$\boldsymbol{I}$ right weak transitivity*

Definition 4.1.11 also yields that the interior and boundary parts of a bounded region are two separate parts of a region:

Theorem 4.1.27. $\exists z\ (\boldsymbol{B}(z, x)) \rightarrow \forall y\ (\boldsymbol{I}(y, x) \rightarrow \neg \boldsymbol{B}(y, x))$

Theorem 4.1.28. $\exists z\ (\boldsymbol{B}(z, x)) \rightarrow \forall y\ (\boldsymbol{B}(y, x) \rightarrow \neg \boldsymbol{I}(y, x))$

Similarly, this definition (**Definition 4.1.11**) says that the interior part of a bounded region is its proper part:

Theorem 4.1.29. $\exists z\ (\boldsymbol{B}(z, x)) \rightarrow \forall y\ (\boldsymbol{I}(x, y) \rightarrow \boldsymbol{PP}(x, y))$

According to this theorem, along with the definitions of the *tangential proper part* (**Definition 4.1.9**) and *interior part* (**Definition 4.1.11**) relations, we can see that every proper part of a bounded region is either its interior part or tangential proper part:

Theorem 4.1.30. $\exists z\ (\boldsymbol{B}(z,x)) \rightarrow \forall y\ (\boldsymbol{PP}(y,x) \rightarrow \boldsymbol{I}(y,x) \vee \boldsymbol{TPP}(y,x))$

Intuitively, we expect that an interior part of a bounded region does not have any intersections with the boundary part of the region. Formally, every region y satisfying the tangential proper part relation with respect to its boundary part leads to the disjointedness of the boundary part from the interior part relation in the bounded region:

Theorem 4.1.31. $\boldsymbol{TPP}(x,y) \rightarrow \neg \boldsymbol{I}(x,y)$

On the other hand, the universal region does not have any boundary part (**Axiom 4.1.11**). This means that it is an interior part of itself:

Theorem 4.1.32. ***I(Universal, Universal)***

Consequently, all the other spatial regions are interior parts of the universal region (from **Theorem 4.1.10**, **Definition 4.1.11** and **Axiom 4.1.11**):

Theorem 4.1.33. $\boldsymbol{I}(x, \boldsymbol{Universal})$

Because of the non-symmetric property of the interior part relation, it has a converse form:

Definition 4.1.12. $\boldsymbol{I}^{-1}(x,y) \leftrightarrow \boldsymbol{I}(y,x)$

This definition reads as "x has y as its interior part". It has the same properties as the interior relation (**I**) between bounded regions and regions with implicit boundaries.

Up to now, we have talked about parts of a region that are either interior part or boundary part of it, but we have not mentioned *the boundary of* (i.e. the whole boundary of) a region. Indeed, it is a region representing *all* the limits of the bounded region, so it is *maximal.* It is defined via a *boundary* operation (`bdy`) purely in terms of the $\boldsymbol{B}$ relation:

Definition 4.1.13. $\texttt{bdy}(x) = y \leftrightarrow (y = \boldsymbol{null} \vee (\boldsymbol{B}(y,x) \wedge (\forall z\ (\boldsymbol{B}(z,x) \rightarrow \boldsymbol{P}(z,y)))))$

in which $y = \boldsymbol{null}$ accounts for the special case where $\texttt{bdy}(x)$ is empty. The boundary is empty for points,

Theorem 4.1.34. $\forall x\ (\texttt{bdy}(x) = \boldsymbol{null} \rightarrow \boldsymbol{Point}(x))$

for the universal region

Theorem 4.1.35. $\forall x\ (\texttt{bdy}(x) = \boldsymbol{null} \rightarrow x = \boldsymbol{Universal})$

and for the ***null*** entity.

Theorem 4.1.36. $\forall x\ (\texttt{bdy}(x) = \boldsymbol{null} \rightarrow x = \boldsymbol{null})$

Also, enclosed regions such as rings (i.e. 1D closed regions) have an empty boundary.

Axiom 4.1.12. $\forall x\ (\texttt{bdy}(x) = \boldsymbol{null} \leftrightarrow (\forall z\ (\boldsymbol{PP}(z,x) \rightarrow \boldsymbol{I}(z,x))))$

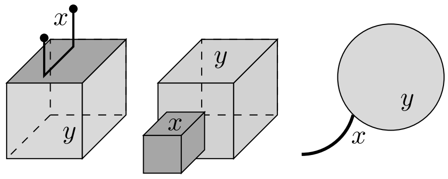


Figure 4.5: Left to right: x has interior contact with, overpasses, or is externally connected to y.

According to **Definition 4.1.13**, every contributory part of y is part of the maximal boundary. So, for instance, vertices of a polygon are also elements of a closed curve which is the (maximal) boundary for the polygon.

Since our theory is intended to describe spatial relations in a multidimensional space, we define relations representing different types of overlap in the spatial configuration. The *imbricate* ($\boldsymbol{IMB}$) relation covers all the overlap configurations excluding part of and its converse, meaning that some part of both objects must be outside the other object. Samples of the imbricate spatial configurations are shown in Figure 4.5. This relation can be refined further in three other relations. The first is the *externally connected* ($\boldsymbol{EC}$) relation. It describes cases in which two spatial regions *only* touch their boundary parts. The second relation represents the configurations in which there are no interior-interior intersections between the entities. However, their common part is limited to the boundary parts (i.e. there is an intersection between the boundary of one region and the interior part of the other region) This is referred to as the *interior contact* relation ($\boldsymbol{IC}$). The third relation, *overpass* ($\boldsymbol{OV}$), describes the case where there is an interior-interior intersection. The formal definitions for these relations are:

Definition 4.1.14. $(\boldsymbol{IMB}(x,y) \leftrightarrow \boldsymbol{O}(x,y) \wedge \neg\boldsymbol{P}(x,y) \wedge \neg\boldsymbol{P}(y,x))$

Definition 4.1.15. $\boldsymbol{EC}(x,y) \leftrightarrow \boldsymbol{IMB}(x,y) \wedge \forall z\ (\boldsymbol{P}(z,x) \wedge \boldsymbol{P}(z,y) \rightarrow \boldsymbol{B}(z,x) \wedge \boldsymbol{B}(z,y))$

Definition 4.1.16. $\boldsymbol{IC}(x,y) \leftrightarrow \boldsymbol{IMB}(x,y) \wedge \neg\exists p\ (\boldsymbol{I}(p,x) \wedge \boldsymbol{I}(p,y)) \wedge \neg(\forall z\ (\boldsymbol{P}(z,x) \wedge \boldsymbol{P}(z,y) \rightarrow (\boldsymbol{B}(z,x) \wedge \boldsymbol{B}(z,y))))$

Definition 4.1.17. $\boldsymbol{OV}(x,y) \leftrightarrow \boldsymbol{IMB}(x,y) \wedge \exists z\ (\boldsymbol{I}(z,x) \wedge \boldsymbol{I}(z,y))$

Note that Definition 4.1.17 does not exclude the possibility that some boundary of x may intersect with either interior or boundary of y (or vice versa). The imbricate relation ($\boldsymbol{IMB}$) and all of its refinements (i.e. $\boldsymbol{OV}, \boldsymbol{EC}$, and $\boldsymbol{IC}$) are provably irreflexive and symmetric. More examples of the proposed relations of this theory are shown in Subsection 4.3.2.

4.1.5 Relational Lattice

In the previous Subsections, we presented our formalism in first-order logic to represent a multidimensional space. In this Subsection, we reveal the conceptual structure supported by the theory. The structure is in the form of a lattice in which every node corresponds to a relation. It shows the specialization among the introduced relations in our theory. The structure, known as a *relational lattice*, also expresses which subset of the introduced relations has the JEPD property (or is the base relation). As we mentioned throughout this thesis, it is essential for constraint-based reasoning.

In the relational lattice, the most general relation is interpreted as a tautology ($\top$) and is the top node, while the most specific concept is interpreted as a contradiction ($\bot$) and is the bottom node. The remaining nodes of the lattice link the defined mereotopological relations according to two rules: *specialization* (rule I) and *subsumption* (rule II).

I) Specialization: Where there is an edge between two nodes (or relations) in a lattice, some source relation $\boldsymbol{S}$ (lower in the lattice) implies the target relation $\boldsymbol{T}$ (further up in the lattice):

$$\boldsymbol{S}(x,y) \rightarrow \boldsymbol{T}(x,y)$$

For example, $\boldsymbol{P}(x,y) \rightarrow \boldsymbol{O}(x,y)$.

When one relation points to (i.e. specializes) more than a single relation (e.g. $\boldsymbol{EQ}$ specializes $\boldsymbol{P}$ and $\boldsymbol{P}^{-1}$), then the specialized relation implies all of the relations it points to. For example, $\boldsymbol{EQ}(x,y) \rightarrow \boldsymbol{P}(x,y) \wedge \boldsymbol{P}^{-1}(x,y)$.

II) Subsumption: Where two (or more) relations $\boldsymbol{S}_1$ to $\boldsymbol{S}_n$ specialize a single relation $\boldsymbol{T}$ (e.g. $\boldsymbol{PP}$ and $\boldsymbol{EQ}$ specialize $\boldsymbol{P}$), then the disjunction of the specialized relations is equivalent to the target relation:

$$\boldsymbol{T}(x,y) \leftrightarrow \boldsymbol{S}_1(x,y) \vee \cdots \vee \boldsymbol{S}_n(x,y)$$

For instance, $\boldsymbol{P}(x,y) \leftrightarrow \boldsymbol{PP}(x,y) \vee \boldsymbol{EQ}(x,y)$.

According to these rules, the weakest relationships are discrete $\boldsymbol{D}$ and overlap $\boldsymbol{O}$ relations which are directly connected to the top node. This means that every pair of spatial regions are either discrete ($\boldsymbol{D}$) from each other or overlapping ($\boldsymbol{O}$) in this theory. The overlap configuration is then refined by other defined relations, including part-of ($\boldsymbol{P}$), part-of converse ($\boldsymbol{P}^{-1}$), equality ($\boldsymbol{EQ}$), imbricate ($\boldsymbol{IMB}$), interior contact (**IC**), overpass (**OV**), externally connected ($\boldsymbol{EC}$), proper part ($\boldsymbol{PP}$), tangential proper part ($\boldsymbol{TP}$**P**), interior part ($\boldsymbol{I}$), and their converses $\boldsymbol{PP}^{-1}$, $\mathbf{TPP}^{-1}$, $\boldsymbol{I}^{-1}$. Applying these rules is shown as a lattice of relations in Figure 4.6.

To confirm the JEPD properties of the subset of introduced relations, we check the jointly exhaustive (*JE* or (rule III)) and pairwise disjointness (*PD* or (rule IV)) characteristics. The former property checks the satisfaction of at least one of the defined relations ($\boldsymbol{R}_i$) in a domain for every pair of regions. The latter checks that no spatial configuration satisfies more than one $\boldsymbol{R}_i$ at a time.

III) $\boldsymbol{R}_1(x,y) \vee \boldsymbol{R}_2(x,y) \vee ... \vee \boldsymbol{R}_n(x,y)$ (jointly exhaustive),

IV) $\boldsymbol{R}_i(x,y) \rightarrow \neg\boldsymbol{R}_j(x,y)$ (pairwise disjoint).

in which $\boldsymbol{R}_i \in \{\boldsymbol{D}, \boldsymbol{O}, \boldsymbol{IMB}, \boldsymbol{P}, \boldsymbol{P}^{-1}, \boldsymbol{PP}, \boldsymbol{PP}^{-1}, \boldsymbol{TPP}, \boldsymbol{EQ}, \boldsymbol{I}, \boldsymbol{TPP}^{-1}, \boldsymbol{I}^{-1}, \boldsymbol{OV}, \boldsymbol{IC}\}$.

By checking the rules III and IV over the set of introduced mereotopological relations, we obtain a subset of relations, including $\boldsymbol{D}$, $\boldsymbol{TPP}$, $\boldsymbol{EQ}$, $\boldsymbol{I}$, $\boldsymbol{TPP}^{-1}$, $\boldsymbol{I}^{-1}$, $\boldsymbol{OV}$, $\boldsymbol{IC}$, and $\boldsymbol{EC}$, provably the set of JEPD relations in this theory. They form the most specific relations in the lattice, directly connected to the bottom node and shown by light color boxes in Figure 4.6.

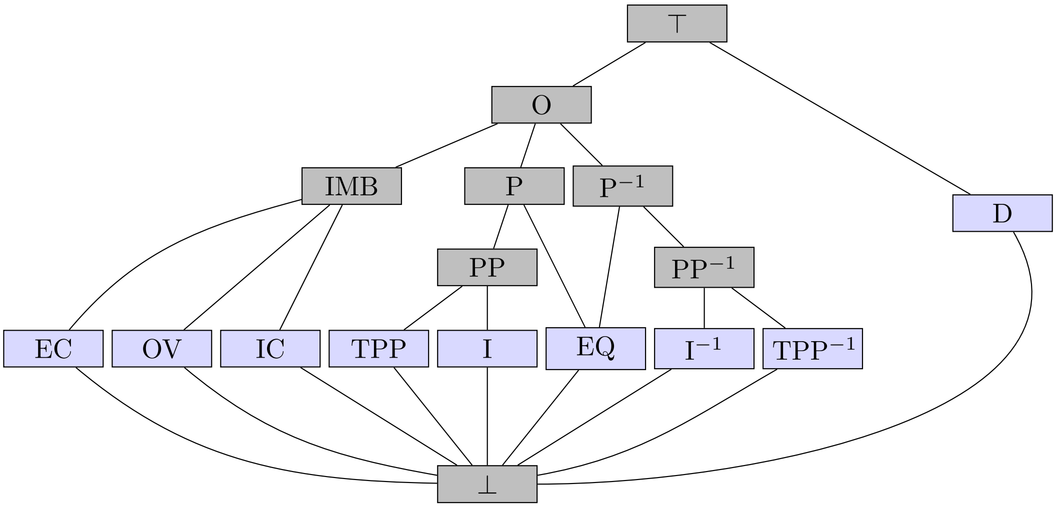


Figure 4.6: The lattice of the relations. The light-colored boxes represent the JEPD set of relations.

4.1.6 Theory Extension

In some existing theories that have attempted to model mereotopological relationships in multidimensional space, a relation in which the regions have *only* an interior-interior intersection has been introduced. Our initial proposed theory also had specialization of the overpass relation ($\boldsymbol{OV}$) described in this configuration. Its name was crosses relation ($\boldsymbol{X}$) with the following definition:

Definition 4.1.18. $\boldsymbol{X}(x, y) \leftrightarrow \boldsymbol{IMB}(x, y) \wedge \forall z\ (\boldsymbol{P}(z, x) \wedge \boldsymbol{P}(z, y) \rightarrow \boldsymbol{I}(z, x) \wedge \boldsymbol{I}(z, y))$.

However, this relation was not recognized as necessary during the cognitive validity experiments. Human participants grouped the given models of objects of various dimensions according to the nine JEPD relations (i.e. $\{\boldsymbol{D}, \boldsymbol{TPP}, \boldsymbol{EQ}, \boldsymbol{I}, \boldsymbol{TPP}^{-1}, \boldsymbol{I}^{-1}, \boldsymbol{OV}, \boldsymbol{IC}$, and $\boldsymbol{EC}\}$) without considering the crosses relation ($\boldsymbol{X}$) as a separate group of models (we will talk about the cognitive adequacy of the proposed formalism in Chapter 5).

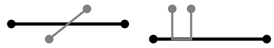
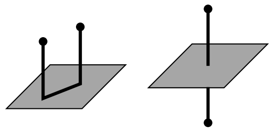
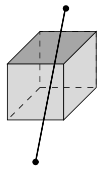
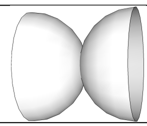
In addition, the limited number of spatial configurations described by this relationship (see Table 4.1), leading us to remove this relationship from our final theory and modify the overpass relation ($\boldsymbol{OV}$) to cover the crosses situations.

4.2 Relationships to other Mereotopological Theories

In this Section, we relate our theory to other mereotopological theories with similar expressiveness (i.e. can express the same spatial configurations). In particular, we formally compare the relations that can be defined in the equidimensional and multidimensional versions of the region connection calculi, i.e. RCC-8 (Randell et al., 1992b) and RCC*′-9. RCC*′-9 is our modified version of RCC*-9 (Clementini and Cohn, 2014), in which we resolved an inconsistency in the original theory (see Appendix Appendix B) (Izadi et al., 2019).

RCC-8 only accepts closed (or bounded) equidimensional regions and defines all of the mereotopological relations based on a single *connection* primitive. A set of eight relations with JEPD properties were

Table 4.1: Crosses relations

Set	Diagrams representing "Crosses" relation
Line-Line	
Line-Polygon	
Line-Polyhedron	
Polygon-Polygon	
Polygon-Polyhedron	Not possible
Polyhedron-Polyhedron	Not possible

introduced based on that primitive. Since entities of co-dimension greater than zero (like a point or line on a plane) do not have an interior, these objects cannot have a non-tangential proper part which is a fundamental assumption of RCC-8. RCC*′-9 proposed tackling this problem and representing spatial relations among regions of various dimensions (i.e. from zero to two) on a plane. A set of nine relations with JEPD properties were introduced based on a single primitive *connection relation*, $\boldsymbol{C}(x, y)$. However, as we said in Chapter 2, the defined relations in RCC*′-9 could not represent relations among objects embodied in higher-dimensional space (e.g. points, lines and polygons in three-dimensional space). We addressed this issue in our proposed theory using two primitives: *part-of* and *boundary part*. Furthermore, we defined a set of nine relations with JEPD properties based on the primitives.[6] A general comparison of these theories is provided in Table 4.2.

Now, we study every JEPD relation of RCC-8 (Randell et al., 1992b), RCC*′-9 (Izadi et al., 2019), and our proposed theory to determine whether there are one-to-one mappings among them. The formal comparison of the definitions of these relations shows that there is a one-to-one mapping between four RCC-8's, RCC*′-9's and our relations (see Table 4.3). Their names are tangential proper part ($\boldsymbol{TPP}$), non-tangential proper part ($\boldsymbol{NTPP}$), and their converses (in the RCC family) map to tangential proper part ($\boldsymbol{TPP}$), interior part ($\boldsymbol{I}$) and their converses (in our theory), respectively. However, our relations are

[6]We also provide formal proof of the correctness of our composition table, which was not available for RCC*-9. We will represent it in Chapter 6

Table 4.2: General comparison among RCC-8, RCC*-9 and our theory.

	RCC-8	RCC*-9	Our theory
Number of primitives	1	1	2
Primitives	Connection	Connection	Part-of and Boundary-part
Accepted regions	Equidimensional	Multidimensional	Multidimensional
Co-dimensional value	0	≤ 2	≤ 3
Number of JEPD relations	8	9	9

Table 4.3: One-to-one mappings among mereotopological relations in RCC-8, RCC*′-9, and our theory.

RCC-8	RCC*-9	Our theory
Tangential Proper Part (TPP)	Tangential Proper Part (TPP)	Tangential Proper Part (TPP)
Non-tangential Proper Part (NTPP)	Non-tangential Proper Part (NTPP)	Interior Part (I)
Tangential Proper Part Converse (TPP^{-1}))	Tangential Proper Part Converse (TPP^{-1}))	Tangential Proper Part Converse (TPP^{-1}))
Non-tangential Proper Part Converse ($NTPP^{-1}$)	Non-tangential Proper Part Converse ($NTPP^{-1}$))	Interior Part Converse (I^{-1}))

satisfiable for bounded regions and entities with implicit boundaries, which is not the case for the previous theories.

Two of the relations in RCC-8, RCC*′-9, and our theory are equivalent (see Definition 3.2.13): discrete/disconnect ($\boldsymbol{D}/\boldsymbol{DC}$) and equality ($\boldsymbol{EQ}$) (Table 4.4).

The externally connected relation ($\boldsymbol{EC}$) represents almost the same spatial configurations in RCC-8 and our theory (ignoring the participated regions' dimensionality). In both theories, a pair of entities must have only a common part of their boundaries, but restricting the common part to the boundary of *at least one* of the regions is the configuration represented by $\boldsymbol{EC}$ in the RCC*′-9. Our formalism represents this arrangement via the interior contact relation ($\boldsymbol{IC}$).

The partially overlap relation ($\boldsymbol{PO}$) in RCC-8 and RCC*′-9 requires a common interior part according to its axioms and definitions. Due to RCC-8's assumptions, the participated regions in this relation have a common boundary part. It is not a valid constraint in RCC*′-9, which means that a pair of entities may or may not have an intersection on their boundaries. The $\boldsymbol{PO}_{RCC*'-9}$ maps directly to our overpass relation ($\boldsymbol{OV}$), which is the generalised form of the $\boldsymbol{PO}_{RCC-8}$ (see Table 4.5, Table 4.6).

RCC*′-9 has one more relation, i.e. cross ($\boldsymbol{CR}$), than RCC-8, in which the participating entities only have an interior part in common. This configuration is generalized by the $\boldsymbol{OV}$ relation in our theory (see Table 4.6).

According to these explanations, we can say that four relations in our theory map directly to RCC-8, and RCC*′-9 's relations, while the other two have equivalent definitions. Three of our relations (i.e. $\boldsymbol{EC}$, $\boldsymbol{IC}$, and $\boldsymbol{OV}$) generalize other relations in the RCC-8 and RCC*′-9.

4.3 Evaluation

We will evaluate the proposed multidimensional theory from three perspectives. This section deals with logical verification and diagrammatic (intuitive) illustrations of the theory. Then in Chapter 5, we will also validate it from the cognitive point of view.

Table 4.4: Equivalence mereotopological relations among RCC-8, RCC*-9, and our theory.

RCC-8	RCC*-9	Our theory
Disconnect ($\boldsymbol{DC}$)	Disconnect ($\boldsymbol{DC}$)	Discrete ($\boldsymbol{D}$)
Equality ($\boldsymbol{EQ}$)	Equality ($\boldsymbol{EQ}$)	Equality ($\boldsymbol{EQ}$)

Table 4.5: Mapping between RCC-8 and our theory.

RCC-8		Our theory
Externally Connected ($\boldsymbol{EC}$)	$\leftrightarrow$	Externally Connected ($\boldsymbol{EC}$)
Partially Overlap ($\boldsymbol{PO}$)	$\subseteq$	Overpass ($\boldsymbol{OV}$)

Table 4.6: Mapping between RCC*′-9 and our theory.

RCC*′-9		Our theory
Externally Connected ($\boldsymbol{EC}$)	$\leftrightarrow$	Interior Contact ($\boldsymbol{IC}$)
Partially Overlap ($\boldsymbol{PO}$)	$\subseteq$	Overpass ($\boldsymbol{OV}$)
Cross ($\boldsymbol{CR}$)	$\subseteq$	Overpass ($\boldsymbol{OV}$)

4.3.1 Logical Verification

Logical verification checks the correctness of the theory's specifications, including axioms, definitions and theorems. In particular, it shows a model for the axiomatization of the theory in first-order logic. The verification process includes *consistency proofs* and *proving the critical properties* of the given theory.

Consistency checking confirms that the theory (T) does not entail any contradiction, that is, no formula (ϕ) exists such that the axioms and definitions logically entail both ϕ and $\neg\phi$. It typically requires the generation of some finite models. We used automated finite model finders, including Mace4 (McCune, 2006) and Paradox (Claessen and Sörensson, 2003), to prove the consistency of our theory. However, the consistency checking approach is somewhat limited in its usefulness as it constructs only a single, and often the simplest, model. In such a model, many feature classes and spatial relations are empty or universal in every possible interpretation. For example, one simple model contains only a set of detached points without any linear or areal features. The generated model also may not use most of the relations such as $\boldsymbol{TPP}$ or $\boldsymbol{EC}$, while some other relations such as $\boldsymbol{D}$ and $\boldsymbol{P}$ relate features only to themselves. Consequently, these relations could not be instantiated by all the feature classes of the domain. A set of more complex, *non-trivial models* (this approach was first used in (Hahmann, 2013)) can be generated using additional logical constraints. These restrictions ascertain that all relations can be instantiated positively (i.e. a relation should be able to hold for some pair of regions) and negatively (i.e. a relation should not hold for all pairs of regions), by pairs of distinct objects from every feature class. The creation of non-trivial models is forced by existential axioms of the form $\exists x\ \mathbf{Q}(x)$[7] and $\exists x \exists y\ [\mathbf{R}(x,y) \wedge x \neq y]$[8]. The models can be found on GitHub[9]. Appendix C provides a list of GitHub's theorems and models.

We used automated theorem provers, including Prover9 (McCune, 2006) (as our default prover) and Vampire (Riazanov and Voronkov, 2002) (Vampire was occasionally used if prover9 and model finders could not return an answer.), to prove the characteristics of the theory stated via **Theorem**s.

We presented our theory in Common Logic (ISO24707) (Commission, 2007) to facilitate the verification process as a standard language to express an ontology. It is a family of logics extending first-order logic. Common logic using syntactic forms called *dialects* are then mapped to the Common Logic's *abstract syntax*. We used the Common Logic Interchange Format (CLIF) dialect to represent our theory as it facilitates the exchange of knowledge via Macleod. Macleod easily reads this format and translates it into

[7] $\boldsymbol{Q}$ denotes an arbitrary unary predicate, thus instantiating all classes.
[8] R denotes an arbitrary binary relation from the formalism.
[9] `https://github.com/azadehi/PhD-Thesis.git`

other formats (i.e. TPTP and LADER) used by theorem provers and finite model finders (i.e. Vampire, MACE4, Prover9, and Paradox). In this dialect **Definition 4.1.17**, for example, translates into:

(forall(x y)(iff (OV x y)(and (IMB x y) (exists (z) (and (I z x)(I z y))))))

Since there is no reasoner to work directly with the CLIF format, the language must be translated to feed the automated provers and finite model finders. Prover9 and Mace4 accept LADR syntax, while Vampire and Paradox use TPTP format. These reasoners, along with the translation tools, are embedded in a toolkit known as Macleod[10]. This toolkit has previously been used to prove the consistency of RCC-8 and some multidimensional theories like the INCH calculus and CODI/CODIB (Hahmann, 2013, 2020). The toolkit enables the theory to be submitted and tests it against the theorem provers and finite model finders simultaneously. It makes our work easier by deploying all the theorem provers and model finders simultaneously. It also gives us a chance to check the consistency of a given theory.

It is important to note that the formalism we presented in Section 4.1 applies a series of ontological development tasks. Throughout this procedure, we used theorem proving and consistency checking techniques to enhance the theory, modify axioms, definitions, and theorems, and better understand the theory. Specific properties were often not proven during this process, and sometimes we had a counterexample to the properties. It forced us to re-examine and update some of the axioms and definitions and then check the consistency of the formalism every time. Most of the time, we had to amend a too general property and could not be proven. Otherwise, our intended properties are not provable.

4.3.2 Diagrammatic Illustration

We have formally defined a set of spatial relations in our theory and provided proofs for their properties. However, we have not examined the range of different types of geometric configurations that these relations can represent. In this Section, we test several extreme cases to see whether those extreme cases are grouped by the formal definitions of our relations in a way that seems intuitive. We then discuss the groupings that result from applying our relations to this wide range of geometric configurations.

In our formalism, a spatial relation is a logical statement representing a spatial configuration between a pair of objects while referring to their common part. The first and foremost spatial feature of any pair of objects is whether they are connected or not. These properties are reflected in the definitions of the overlap ($\boldsymbol{O}$) and discrete ($\boldsymbol{D}$) relations, respectively (see **Definition 4.1.2** and **Definition 4.1.3**). Some of the configurations of the discrete relation are shown in Table 4.7.

In contrast to the clarity in the definition of discrete spatial arrangements, the overlapping configuration describes a different range of connections between a pair of objects. The intuitive meaning of this relation says that x overlaps y iff there is a common part (z) between them (see **Definition 4.1.2**). The further refinements of this relation are based on the type of the common part, i.e. whether the common element is part of one of the objects or forming all of it. The former arrangement was defined via the *imbricate* relation (see **Definition 4.1.14**), and seen in all the diagrams labelled with $\boldsymbol{EC}$, $\boldsymbol{OV}$ and $\boldsymbol{IC}$ in Table 4.12, Table 4.8, and Table 4.9, respectively. Meanwhile, the latter configuration represented the part-of relation

[10] `https://github.com/thahmann/macleod.git`

($\boldsymbol{P}$) or its converse being seen in all the diagrams labelled with $\boldsymbol{I}$ (which is similar to its converse $\boldsymbol{I}^{-1}$), $\boldsymbol{TPP}$ (which is similar to its converse $\boldsymbol{TPP}^{-1}$) and $\boldsymbol{EQ}$ in Table 4.13, Table 4.10, and Table 4.11, respectively.

The further specializations of the imbrication arrangement (i.e. $\boldsymbol{EC}$, $\boldsymbol{OV}$ and $\boldsymbol{IC}$) are also based on the properties of the common part of two entities, whether it placed on the boundary parts, interior parts, or mixture of these parts of in both entities. By definition, every non-zero-dimensional entity has an interior part and might also have a boundary. When a pair of objects meet each other *only* at their outermost parts (i.e. boundaries), we say that they are externally connected ($\boldsymbol{EC}$) as defined in **Definition 4.1.15**. Some of the layouts illustrating this relation are shown in Table 4.12.

In contrast to the externally connected relationship, two imbricated objects might have a common interior part (with or without the intersections at their boundaries). This arrangement is defined by the overpass relation (see **Definition 4.1.17**) and covers all the cases shown in Table 4.8.

If a pair of imbricated objects, x and y, have only interior-boundary intersection (i.e. $int(x) \cap bdy(y)$), they represent an interior contact configuration defined in **Definition 4.1.16** and are shown in Table 4.9.

When an object such as x is completely covered by (i.e. part-of) another object like y, y may either extend beyond x, or coincide with it if they have the same size. The former configuration is represented by the proper part relation ($\boldsymbol{PP}$), defined in **Definition 4.1.11**, while the latter represents the equality arrangement (EQ) defined in **Definition 4.1.1** and shown in Table 4.11.

A pair of objects in a proper part configuration is refined based on the existence or non-existence of internal tangency. In this case, the existence of the internal tangency leads to the configuration defined in **Definition 4.1.9** and known as tangential proper part relation ($\boldsymbol{TPP}$), while lack of it is defined by the interior part relation, $\boldsymbol{I}$ (see **Definition 4.1.11**). Their geometrical configurations are shown in Table 4.10 and Table 4.13, respectively.

4.4 Summary

In this Chapter, we have introduced a multidimensional mereotopological theory that works independently of the dimension of the involved spatial entities. It is based on mereological and topological primitives. A set of mereotopological relations is defined based on these primitives. The relations are represented in a hierarchy. On its coarser level, we have a description of discrete and overlap spatial configurations. The latter is then refined to define more specific spatial arrangements. Eventually, a small set of the most specialized relations with the property of jointly exhaustive and pairwise disjoint (JEPD) is introduced. Note that the JEPD relations (excluding the *interior contact* relation) represent the same spatial arrangements by the eight mereotopological relations defined in the spatial calculus RCC-8 (Randell et al., 1992b) and 9-IM (Egenhofer, 1989, 1991; Egenhofer and Franzosa, 1991; Egenhofer and Herring, 1991) for bounded regions in the unidimensional space.

Our theory is specified using a subset of Common Logic (ISO 24707) (Commission, 2007) that has standard first-order semantics to assist in verifying the theory. Since no reasoners accept input in Common Logic syntax, we translated it into the available reasoners' accepted formats. We applied the consistency proofs method to verify the theory. Consistency is a property that is expected from any theory to enable inference. Consistency was proved by constructing a model of the theory. Also, we checked the existence of a model of a theory such that every relation is satisfied. We also proved the properties of the theory via automated theorem provers.

In addition to the logical verification of the proposed theory, we explored the introduced relations visually. It was a valuable tool for discovering the sufficiency of the JEPD relations, even in extreme cases. A set of diagrams are provided for every JEPD relation in which we had spatial objects of various dimensions. These provide an intuitive presentation of the relations.

With its various definitions, this theory could be extended by defining the crosses relation in which the pair of participating entities only have the interior-interior intersection. Its introduction would require us to modify the other defined relations to ensure the JEPD property. If the crosses relation, along with the other relations, satisfy the property of JEPD, we would have a more expressive theory than the one presented. However, the crosses relation was not shown to be cognitively meaningful in our experiments (see the following Chapter), which led to its removal from our proposed theory.

The proposed theory in this Chapter is the foundation for future Chapters. We will investigate the cognitive plausibility of this theory in Chapter 5, using the set of base relations as stimulus material for empirical experiments. We will then, in Chapter 6, show how to infer further knowledge from this theory in a flood event scenario.

Table 4.7: Discrete relation

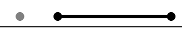
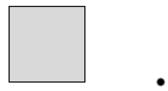
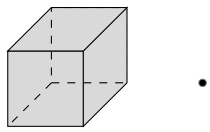
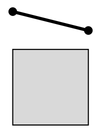
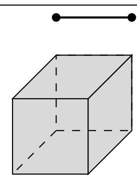
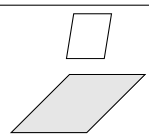
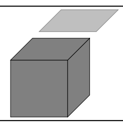
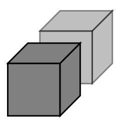
Set	**Diagrams representing “Discrete” relation**
Point-Point	
Point-Line	
Point-Polygon	
Point-Polyhedron	
Line-Line	
Line-Polygon	
Line-Polyhedron	
Polygon-Polygon	
Polygon-Polyhedron	
Polyhedron -Polyhedron	

Table 4.8: Overpass relations

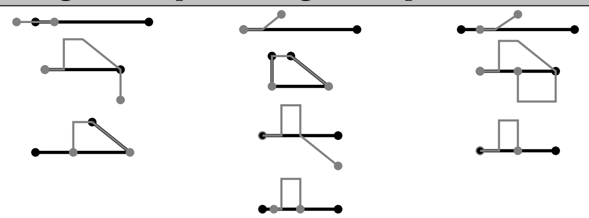
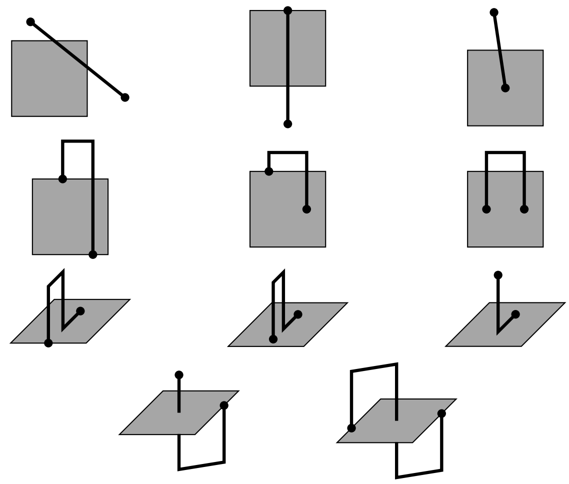
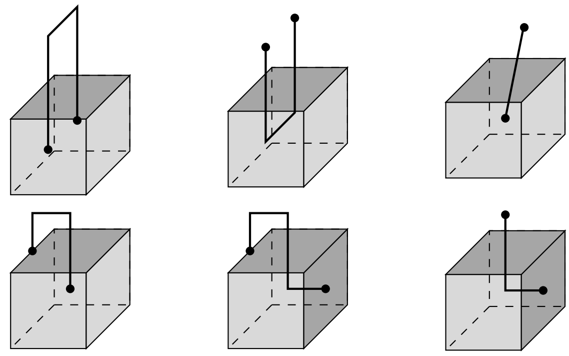
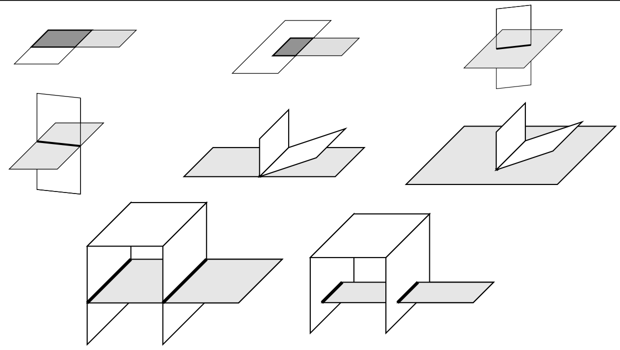
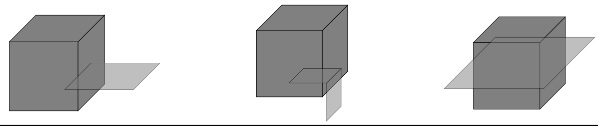
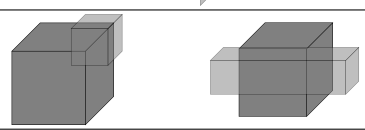
Set	Diagrams representing "Overpass" relation
Line-Line	
Line-Polygon	
Line-Polyhedron	
Polygon-Polygon	
Polygon-Polyhedron	
Polyhedron-Polyhedron	

Table 4.9: Interior contact relation

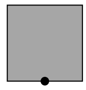
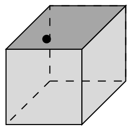
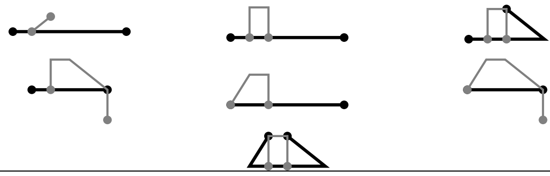
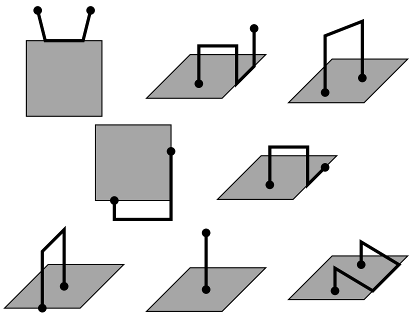
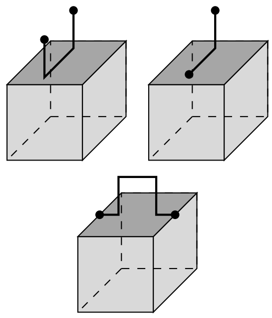
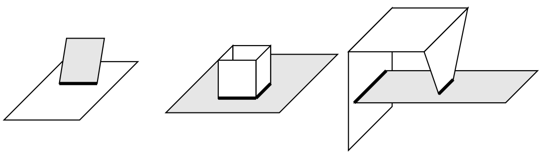
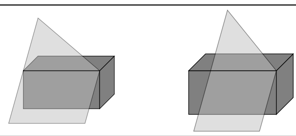
Set	**Diagrams representing "Interior contact" relation**
Point-Line	
Point-Polygon	
Point-Polyhedron	
Line-Line	
Line-Polygon	
Line-Polyhedron	
Polygon-Polygon	
Polygon-Polyhedron	
Polyhedron-Polyhedron	Not possible

Table 4.10: Tangential proper part relation

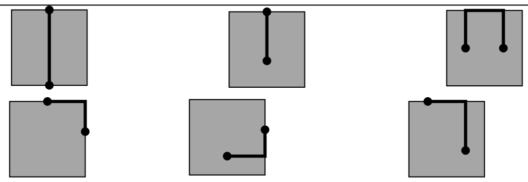
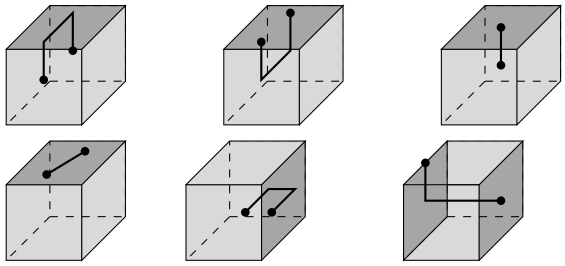
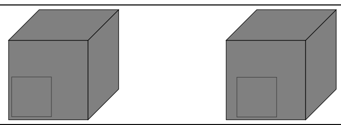
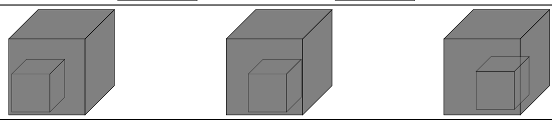
Set	Diagrams representing "Tangential proper part" relation
Line-Line	
Line-Polygon	
Line-Polyhedron	
Polygon-Polygon	
Polygon-Polyhedron	
Polyhedron-Polyhedron	

Table 4.11: Equality relation

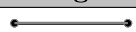
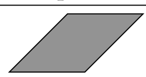
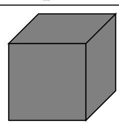
Set	Diagrams representing "Equality" relation
Line-Line	
Line-Polygon	Not possible
Line-Polyhedron	Not possible
Polygon-Polygon	
Polygon-Polyhedron	Not possible
Polyhedron-Polyhedron	

Table 4.12: Externally connected relation

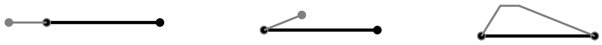
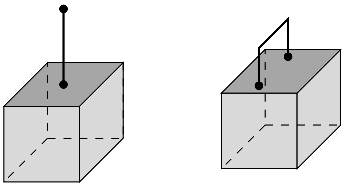
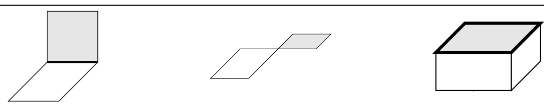
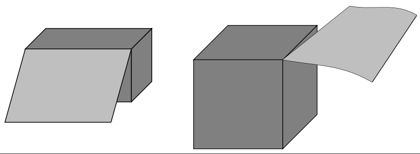
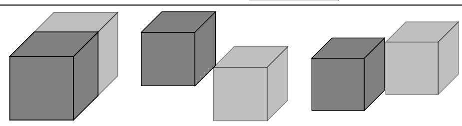
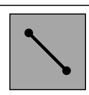
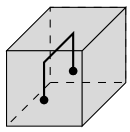
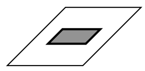
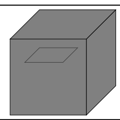
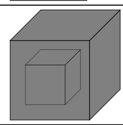
Set	Diagrams representing "Externally Connected" relation
Line-Line	
Line-Polygon	
Line-Polyhedron	
Polygon-Polygon	
Polygon-Polyhedron	
Polyhedron-Polyhedron	

Table 4.13: Interior relation

Set	Diagrams representing "Interior" relation
Line-Line	
Line-Polygon	
Line-Polyhedron	
Polygon-Polygon	
Polygon-Polyhedron	
Polyhedron-Polyhedron	

Chapter 5

Cognitive Validity of Our Mereotopological Relations in a Multidimensional Space

Having presented our formalism in Chapter 4, we now investigate our set of introduced spatial relations' cognitive adequacy (the term clarified in this Chapter). Studying whether a formal approach can be a model of human perception has always been of interest to researchers, especially in the field of spatial knowledge representation (Egenhofer and Mark, 1995a; Knauff et al., 1995, 1997; Renz et al., 2000).

The cognitive properties of famous theories such as Interval calculus (Allen, 1983), Region Connection Calculus (RCC) (Randell et al., 1992a) and the 9-Intersection Model (9-IM) (Egenhofer and Herring, 1989) have been studied in the literature. However, this previous research has mostly evaluated the spatial relations between a pair of one-dimensional (like Interval Calculus), two-dimensional (like RCC), or one and two-dimensional (like 9-IM) objects. For instance, their materials showed a pair of lines on an axis, polygons on a plane, or a line and a polygon on a plane. However, human cognition can easily model objects using different geometries from various dimensions. Many findings in cognition already support this notion (see (Lynch, 1960) for more detail). Thus, we defined the spatial relations among the geometries regardless of their dimensions (or the dimension of their embedded space) in our theory; we must consider whether it makes sense to presume that the relations in our theory reflect spatial representations of (mereo)topological concepts in the human mind.

The contribution of the current Chapter is a step in this direction by justifying the correspondence between the set of (mereo)topological relations describing the spatial relationships among objects of various dimensions embedded in a three-dimensional space presented in the previous Chapter and cognitively relevant concepts.

In this Chapter, we report our empirical investigation in which the cognitive validity, known as "cognitive adequacy" in the QSR community, of the formalism proposed in Section 4.1 will be discussed. The formalism introduced a number of spatial relations. Similar to other investigations (Egenhofer and Mark, 1995a; Knauff et al., 1995, 1997; Renz et al., 2000), we will carry out a grouping task followed by describing the generated groups to give us an insight into the subjects' perceptions.

The remainder of this Chapter is structured as follows. Firstly, we will explain the notion of cognitive adequacy and discuss how it has been evaluated over the existing formal theories in Section 5.1. Then we will explain our empirical investigation in Section 5.2, followed by the relevant findings in Section 5.3. Finally, we will discuss the results from both psychological and computational points of view in Section 5.4.

5.1 Cognitive Adequacy of Mereotopological Theories

In this Section, we will look at the traditional claim of closeness between spatial theories (more specifically mereotopological theories) and a common-sense understanding of spatial relations. The recognition of this aspect of human spatial cognition plays an essential role in designing intuitive spatial information systems. The question about this claim is what people mostly distinguish spatial relations. Among various studied aspects of space, the significant role of invariant characteristics of the spatial configurations under certain transformations (i.e. (mereo)topology) in human cognition has been confirmed by many empirical investigations (Knauff et al., 1995; Egenhofer and Mark, 1995a; Knauff et al., 1997; Renz et al., 2000). In the experiments, the given theory's cognitive validity, also known as “cognitive adequacy”, is evaluated.

The concept of cognitive adequacy has been theorised as a value on a continuum from strong to weak (Strube, 1992). While the strong (idealized) form of cognitive adequacy represents an accurate model of human cognition, the weaker form confirms its ergonomics and user-friendliness. According to the literature, two types of cognitive adequacy have been identified: conceptual and inferential (Knauff et al., 1995). A theory is conceptually adequate if its representation symbolizes a model of human conceptual knowledge. However, if the reasoning system of the calculus is similar to the reasoning structure of people, it is considered an inferentially adequate system. A conceptually and inferentially adequate system results in an intuitive system. This thesis focuses on conceptual adequacy, as our concentration is on the representation rather than the reasoning aspects of our proposed multidimensional formalism presented in Chapter 4.

As we said earlier, mereotopological relations are proposed in different calculi (Allen, 1983; Egenhofer and Franzosa, 1991; Egenhofer and Herring, 1991; Randell and Cohn, 1989; Randell et al., 1992b; Cohn et al., 1997; Gotts, 1996; Galton, 1996; Smith, 1996; Hahmann and Grüninger, 2011; Hahmann, 2018), but only the cognitive adequacy of some of them, including the Interval calculus, 9-IM, and RCC, has been confirmed in (Knauff et al., 1995; Knauff, 1997), (Egenhofer and Mark, 1994) and (Knauff et al., 1997; Renz et al., 2000) respectively. The main objective among them was to indicate the aspects of spatial relations that are important to people. However, these cognitive adequacy investigations have been customized to suit the limitations of every theory. The constraints are mainly rooted in the theories' assumptions about the accepted entities in a domain and the codimension value. For instance, the given objects in the Interval calculus (Knauff et al., 1995; Knauff, 1997) are two line segments, as one-dimensional spatial entities representing on an axis, while they are two polygons, as two-dimensional spatial entities, on a plane in the RCC (Knauff et al., 1997; Renz et al., 2000).

Human mental perception about the relationships between two line segments has been investigated in (Knauff, 1997), in which thirteen possible situations between two line segments (as represented by the Interval calculus (Allen, 1983)) were used as stimulus material. In this calculus, there is an order among the mereotopological relations between a pair of line segments in which two adjacent situations can be transformed from one to another only by one movement. The empirical experiment had two parts: recall and recognise. Participants were asked to draw the next spatial relation according to the learned order after receiving a cue stimulus during the recall section. In the recognition phase, they had to pick a requested relation while having instances of other relations as distractors. The results were evaluated based on the number of incorrect answers and the time taken to complete the tasks. The findings show that participants consistently recognised the mereotopological distinctions between the line segments. However, they could not remember the order of them easily. This experiment also showed the power of mereotopological relations irrespective of their dependence on language.

The cognitive adequacy of RCC-8 (Knauff et al., 1997; Renz et al., 2000) has been checked at two levels of complexity. In the simpler version (Knauff et al., 1997), the participants were asked to group a set of figures (shown on the computer screen) of various spatial arrangements of two different coloured circles based on RCC-8 relations. After that, respondents were asked to to describe every group. Following research (Renz et al., 2000) included other object shapes to discover more about people's perception of space. The experimental procedure was similar in both studies. In both experiments, the groups identified by respondents were analysed to understand their similarities. The outcomes reassert the prominent role of the mereotopological information defined in RCC-8, confirming the cognitive plausibility of the set of six topological relations (excluding the converse forms of tangential and non-tangential proper parts). Moreover, the researchers found some explanations about other sorts of connectivity in the participants' verbalisation, referring to the dimension of the common part and the number of common points between a pair of polygons. The researchers claimed that these aspects can be used to refine the topological relations in a future extension of formalism.

The 9-IM relations come from the emptiness or non-emptiness of the intersections between different parts of the entities. The cognitive adequacy of 9-IM among objects of different dimensions was checked via two experiments (Egenhofer and Mark, 1994), both using drawing cards showing nineteen acceptable spatial configurations between a line and polygon. Subjects were told that the line and polygon are the representation of a road and a park respectively. In the first experiment, the subjects were asked to group the cards in such a way that all the spatial relations in a group were describable via “the same phrase or sentence” and then select a card from each group as its representative. Dependent measures were a repetition of the number of generated groups, the selected prototypes and the researchers analysed the similarity of the generated categories among the subjects. In the second experiment, a collection of drawing cards and two sentences were given to the subjects and asked them to rate the appropriateness of the sentences for every card. The results of the two experiments not only confirmed the importance of mereotopological relations in human perception but also validated a large number of the 9-IM relations as a foundation for characterising space.

The above-mentioned cognitive adequacy checking experiments are summarized in Table 5.1 in terms of their stimulus material, participants, and methods. As can be seen, grouping the given items and providing descriptions for the generated groups were commonly used in these experiments, and we adopted this technique to check the cognitive adequacy of our proposed multidimensional mereotopological theory. However, we differ in the stimulus materials and the participants.

The difference in materials is due to differences in the fundamental mereotopological theory that we intend to assess. In contrast to the Interval calculus and RCC, we evaluate the spatial relations among entities of various dimensions. Thus, our stimuli comprise items of different dimensions, including the three-dimensional entities that have not been investigated previously. Also, we use a set of *three-dimensional models* to represent the spatial configurations in our space, while previous works have been limited to two-dimensional drawings. This enrichment gives our participants a chance to have a tangible experience of the spatial entities and to view the configurations from different angles and perspectives, which is essential in three-dimensional spatial representation.

Furthermore, participants in our experiments are not limited to students because we believe that (mereo)topological theories and, subsequently, the systems developed based on them should be usable by the general public. Therefore, people with different educational backgrounds participated in our experiments. Our experiments will be detailed in the following Sections.

5.2 Materials and Method

In this Section, we will talk about the materials being used in our experiments to evaluate the cognitive adequacy of our proposed multidimensional mereotopological theory. We will also explain the process of our experiments.

Table 5.1: Material and methods in the cognitive adequacy investigations

Formalism	Material	Participants	Method
Interval calculus	Images of various configurations of two lines on cards	Students	- Solving three–term series problems - Qualifying the given descriptions
RCC-8	Images of various configurations of two polygons on screen	Students	- Grouping images - Writing descriptions for generated groups (in German)
9-IM	Images of various configurations of a line and a polygon on cards	Students and non-students	- Grouping cards - Writing descriptions for generated groups (in various languages)

5.2.1 Materials

The materials used in the experiment consisted of a set of 163 models made by a 3D-printer machine from Polylactide (PLA) material[1]. The models were designed using a computer-aided design (CAD) technique. Some of our models are shown in Figure 5.1 - Figure 5.6, and the complete set of them is shown in Appendix D. We used 3D models instead of the 2D images (which were used in the previous studies (Egenhofer and Franzosa, 1991; Egenhofer and Herring, 1991; Knauff et al., 1995, 1997; Renz et al., 2000)) to enable observers to move a model in their hands and investigate the type of connection between two modelled objects in more detail. This opportunity would not be available with 2D representations or 3D-visualizations.

In the 163 models, we included items representing a range of different types of (mereo)topological relations between two objects. Specifically, we ensured that all of the JEPD relations in our theory were represented with multiple models showing different combinations and configurations of one, two and three-dimensional objects for each spatial relation (i.e. two or three models for every relation in each set). In every model, the mereotopological relation between a pair of objects, A and B, is constructed based on the existence of an intersection between an interior, and(or) boundary parts of A and an interior, and(or) boundary parts of B in a model. For instance, all the samples of the externally connected relation (**EC**), regardless of the shape of the objects, only have boundary-boundary intersections. Note that zero-dimensional objects were excluded from this experiment because any representation of zero-dimensional entities that people can see makes it difficult to distinguish them from two- or three-dimensional objects.

We created the wavy thin shaped objects to represent one-dimensional entities (i.e. wavy worm-like artifacts) and circular-shaped objects to represent the two and three-dimensional entities (i.e. a disk and a ball respectively). They were chosen because some spatial configurations are only representable via wavelike structures. Also, we wanted to keep the shape of every n-dimensional object consistent among all the sets to avoid the risk of subjects grouping the models by shape (and similarly aimed for consistency of other aspects of the models). The only models in which the general shape of the curvy lines, wavy disks and balls differed were a few cases in which changes were required due to differences in size.

As we said, making the design of the models as consistent as possible was one of our concerns, so we could be sure that participants were grouping the items by considering their (mereo)topological relations, not colour,or texture. As a result, the two participating objects in every model are the same colour as each other, except in the configurations that we have a pair of one or two-dimensional objects. In these cases, while one of the objects has a non-cream colour (i.e. one of the lines in the Line-Line model is red, and one of the surfaces in the surface-surface one is blue), the other one is cream in these models. This exception is used because of the difficulty in differentiating the objects from each other in some of the models containing objects of the same dimension, which is straightforward if one of the objects is in another dimension.

As has been shown in (Renz et al., 2000), people do not usually distinguish between a relation and its converse form (e.g. between the tangential proper part ($\boldsymbol{TPP}$) relation and its converse $\boldsymbol{TPP}^{-1}$ in

[1]This material was selected because it is better suited to models with fine details.

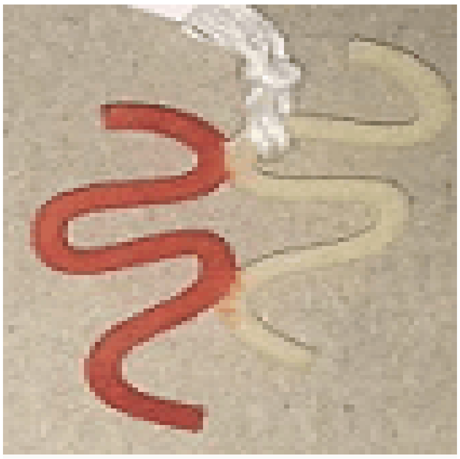


Figure 5.1: Model in Line-Line set

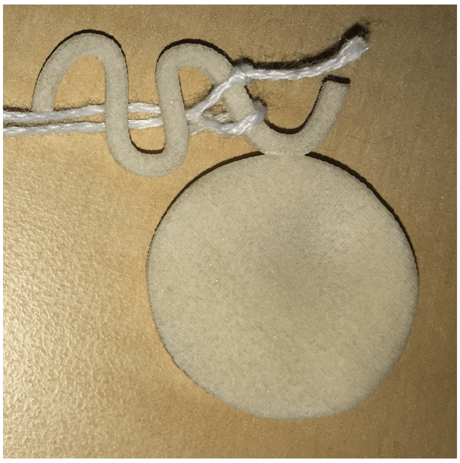


Figure 5.2: Model in Line-Surface set

Figure 5.3: Model in Line-Volume set

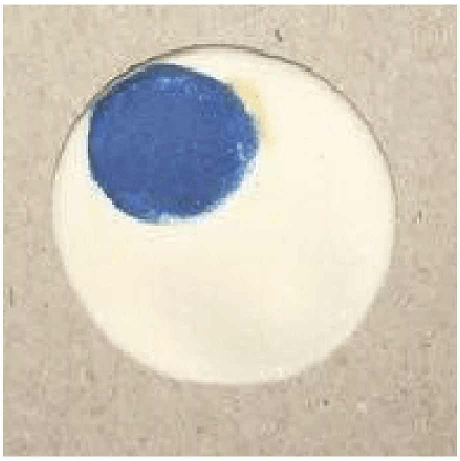


Figure 5.4: Model in Surface-Surface set

Figure 5.5: Model in Surface-Volume set

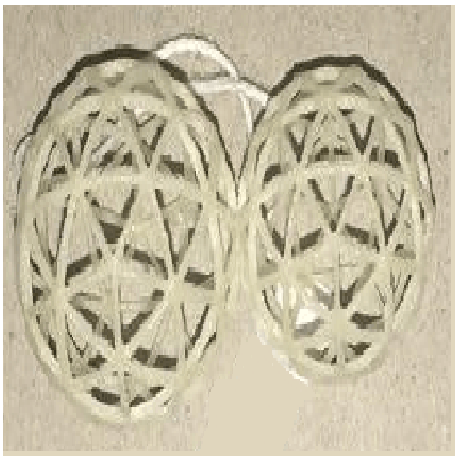


Figure 5.6: Model in Volume-Volume set

Table 5.2: Summary of the number of items given to a participant per set.

Set of items	Number of items
Line-Line	31
Line-Surface	33
Line-Volume	27
Surface-Surface	29
Surface-Volume	23
Volume-Volume	20

RCC-8). Although this indeterminacy is theoretically important, it does not create a paradox in showing the cognitive adequacy of the formalism. For instance, if subjects properly categorise the instances of the $\boldsymbol{TPP}$ relation, this result can be generalized to its converse form as well. Thus, we do not consider our experiment's converse forms of relations. In other words, our investigation only includes the instances of the eight relations (i.e. {discrete ($\boldsymbol{D}$), externally connected ($\boldsymbol{EC}$), interior contact ($\boldsymbol{IC}$), cross ($\boldsymbol{X}$), interior part ($\boldsymbol{I}$), overpass ($\boldsymbol{OV}$), tangential proper part ($\boldsymbol{TPP}$) and equal ($\boldsymbol{EQ}$)}). Although the cross-relation ($\boldsymbol{X}$) is not part of our final set of JEPD relations (shown in Figure 4.6), it was initially included in our empirical experiment. Twelve of 163 models represented the cross-relation. However, they are now classified as the overpass relation. The result of the experiment led us to remove the cross-relation which led us to remove the cross-relation from our final version of the JEPD relations. (we will discuss this in Section 5.3).

All the items had identification numbers which were randomly selected and attached as labels. These numbers were mapped to the named topological relations in the analysis step.

A summary of the models is shown in Table 5.2. We had 31 items showing different configurations of two lines, for instance. We presented objects in sets by their dimension. For example, we only have pairs of one-dimensional entities in the Line-Line set of items.

5.2.2 Participants

We recruited participants by publishing advertisements on social media (e.g. Facebook or local websites like Neighbourly[2]) and posting paper advertisements on noticeboards around the campus and community noticeboards in the neighbourhood (e.g. the public library). We also handed out flyers on campus to encourage participation. We ran tests with 32 people, of whom five participated in the pilot study and the rest in the main experiment.

The pilot study participants were all females between 20 to 29 years old, having a bachelor's degree in various fields of science. We do not think that the fact that all our pilot study participants were female would have affected our results since no specific gender differences were found in our entire study's results, in which we had participants of both genders. In the full study, we tested 27 participants[3] (17 female and 10 male). The sample size of 27 is due to challenges with recruitment. However, we consider that 27 participants are enough to get useful results in comparison with 27, 20 and 19 participants in (Egenhofer

[2] `www.neighbourly.co.nz`

[3] Since some of the participants could not finish the experiment within the allocated time, the number of participants per set of models differed.

and Mark, 1994), (Knauff et al., 1997) and (Renz et al., 2000) studies respectively. The participants' age and education level are shown Figure 5.7. Furthermore, we had an equal number of native and non-native English speakers participants. This equality is in line with the Sapir-Whorf hypothesis (Whorf, 1940) that language influences thought and could suggest differences among native and non-native speakers.

All subjects provided written informed consent before participating in the empirical investigation. The university human ethics committee[4] approved the experimental design and materials. Subjects could only participate if they were not computer scientists and were not familiar with artificial intelligence qualitative spatial representation and reasoning. These limitations assured us that non-experts evaluated the perceptual nature of our proposed theory. The participants received NZ$20 shopping vouchers for their participation.

5.2.3 Method

Our method follows the tradition and spirit of (Knauff et al., 1995; Egenhofer and Mark, 1994; Knauff et al., 1997; Renz et al., 2000). Since, like this previous work, we are interested in the common-sense interpretations of spatial relations, we followed a *category construction* method (Lakoff, 2008). The categorisation is a well-understood method for assessing conceptual knowledge since categorisation is a basic human cognitive ability (Lakoff, 2008; Harnad, 2017). In this method, participants are asked to group objects into categories that share similar characteristics.

The key feature of this method is that subjects do not receive a predetermined number of categories and are free to make any number of groups appropriate for the stimuli given. In this experiment, participants were given groups of objects by set. It simplifies grouping by helping the participants identify common patterns among the models in each set.

Before running the main experiment, we did a pilot study with five of our participants, giving them the following instruction:

> "*Here are a set of different models of two objects. Please classify them into several groups based on their* <u>*similarities*</u>*. After you complete your grouping, you will be asked to write a description of each group.*"

There is a resemblance between the wording of the above instruction and the ones used in (Egenhofer and Franzosa, 1991; Egenhofer and Herring, 1991). Also, like (Knauff et al., 1997; Renz et al., 2000), we asked

[4]The Massey University ethics approval number is 4000020635 and is entitled as "Cognitive properties of topological spatial relations in a multidimensional space".

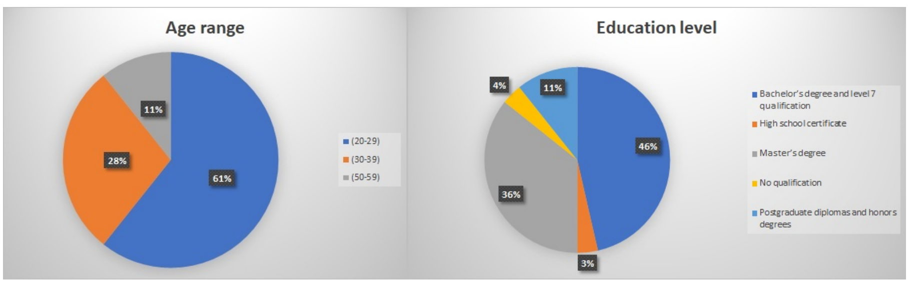


Figure 5.7: Frequency of age groups and educational level among the participants

participants to describe their thinking. However, almost all of the participants in our pilot sessions had difficulty understanding the *similarity* among the given models. They interpreted similarity based on the objects' size, distance, or texture. The misunderstanding resulted in our changing the wording in the main experiment to:

> "*Here are a set of different models of two objects. Please classify them into several groups based on how they are connected. Please do not consider other factors like the size, distance, texture of the objects. After you complete your grouping, you will be asked to write a description of each group.*"

The same instruction was used for all the sets of models.

The experiment was held in a meeting room at our department in which the sets of models were arranged on top of a meeting desk and given randomly to the participants, who completed the experiment one at a time. The subject sat in the closest chair to the set he/she was grouping and changed his/her position during the experiment according to the location of the set. There was no limitation on the number of generated groups.

The participants worked set by set, grouping the models and then describing before moving on to the next set. In the description task, participants were asked to describe and write the properties of the groups on a computer in English. It gave them a chance to describe their thinking processes and eliminated the potential for random grouping. Also, we did this to acquire additional information from the descriptions. The participants could modify their grouping while writing their descriptions.

The experiment (including grouping and description tasks) was sat for an hour, and every subject grouped and described as many sets as he/she could in the time. The experiment only continued after the hour if a participant was willing to do (in limited cases). The experimenter was in the room during the experiment and recorded the members of every group.

Although the majority of the participants could finish the tasks in all the sets within the given time, some took longer, and so we collected a different number of responses per set. There were 26 responses for the Line-Volume and Line-Surface sets of objects, and 23 for the Line-Line set, while there were 27 for the rest of sets of models.

5.3 Results

Results were extracted using two types of analysis. First, we extracted information by analysing the composition of the groups created by respondents. After that, we analysed the descriptions that were provided for each group. We will explain both of these analysis methods and findings in the following Sections.

5.3.1 Grouping task analysis

Subjects needed from 55 to 105 minutes with an average of 65 minutes to group the models and then describe their groupings. By collecting the grouping answers, we analysed the frequency of the number of

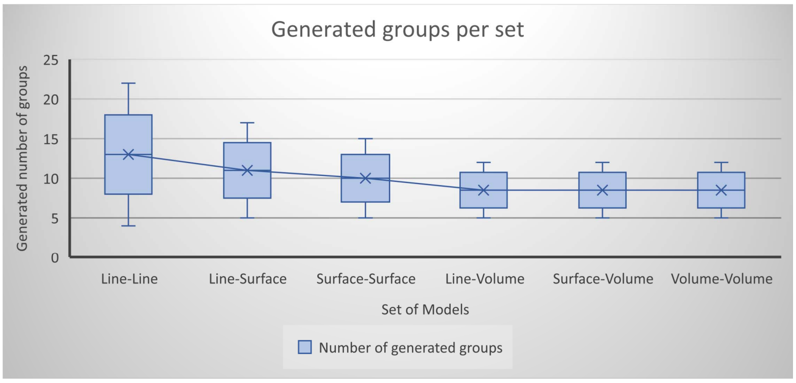


Figure 5.8: Box-plot showing the minimum, first quartile, median, third quartile, and a maximum number of groups defined by the subjects in every set.

generated groups, which gave us an overview of the responses.

To gain insight into the variation in the participants' groupings, we extracted the number of generated groups in every set (see Figure 5.8), which varied widely (for instance, a minimum of 4 and the maximum of 24 groups are generated in the Line-Line set of models). Also, the maximum number of produced groups is seen in this set, which indicates people's difficulty to describe these models in comparison to the other sets.

In addition to the number of generated groups in every set, their content analysis is also informative. Their categorisation helped us to explore the similarity among the participants' replies. This is the goal of the following Section.

5.3.1.1 Clustering

Using a similar method to (Knauff et al., 1997; Renz et al., 2000), we generated a similarity matrix to express each participant's responses for the grouping task. We encode similarity as a binary value: 1 in any cell expresses that a participant placed the pair of models denoted by the row and column in the same group, 0 if the two models were not in the same group. We then generated an overall similarity matrix by averaging over all participants' similarity matrices for every grouping task and then applied clustering algorithms to uncover the underlying concepts shown by the groupings across all participants. After that, we examined the degree to which those clusters reflected our JEPD relations.

We use agglomerative hierarchical clustering with the average linkage method, which is particularly useful since we want to examine the clusters at different levels of granularity. The hierarchical clustering approach either accepts distances between data points, or raw data in the form of a matrix as an input. Since our data is the overall similarity matrix (and not a raw data matrix) for every set, we need to extract the distance matrix from it. We calculate the maximum similarity for all pairs in each set. The maximum similarity is 1 because every object is identical to itself. We then subtract from this the similarity value of the pair presented in every cell of the similarity matrix (i.e. $distance = 1 - pairwise\ similarity$)[5].

[5] Note that this formula computes dissimilarity, which is not always the same as distance, since its returned value may or may not be a metric (a distance function is by definition a metric), however, the satisfaction of the metric axioms (precisely the triangle inequality) over the extracted distance values has been checked, and so the presented formula returns a distance value.

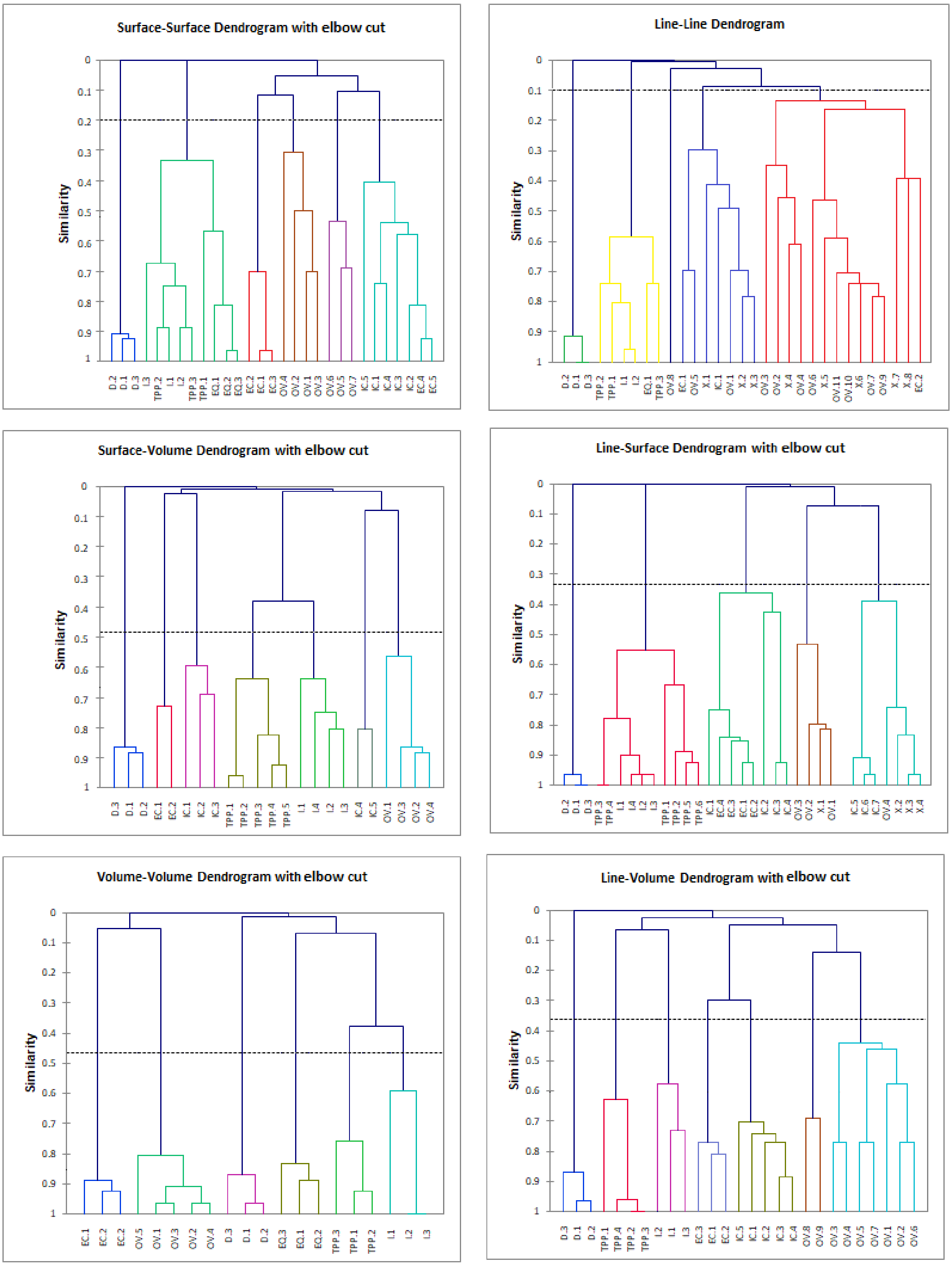


Figure 5.9: Dendrograms with elbow cuts.

Figure 5.9 depicts the dendrograms resulting from the cluster analysis on all the sets of models. On the horizontal axis, the models of every set are represented as leaves of the tree. The ordering along this axis differs among the dendrograms. The names of the models shown on the horizontal axis are derived from the mapping of the models' identification numbers to the JEPD relations introduced in Chapter 4 (note that there may be multiple models for a given relation. Appendix D shows all models for every relation.). The vertical axis shows the level of similarity between two models ranging from 1 to 0. Two models are similar if their similarity level is closer to 1. For instance, in the Volume-Volume set, all the samples of the externally connected relation (i.e. **EC**.1, **EC**.2, and **EC**.3) are very similar to each other (i.e. their similarity value is around 0.9), while the dissimilarity value is high between the collection of these items and samples of the overpass relation (i.e. **OV**.1, **OV**.2, and **OV**.3) in this set (i.e. the similarity value is less than 0.1).

Furthermore, a heatmap applying the same clustering algorithm on every set is presented in Figure 5.10. The heatmap rearranges the similarity matrix's rows and columns to comparable rows and comparable columns. Green tones indicate a high similarity value, while lower values are depicted with red tones. For example, a high similarity between an item and itself (i.e. diagonal cells) can be seen in all the heatmaps, while a low similarity value, for instance, occurs between an item showing "*an object is a tangential proper part of another*" (**TPP**) and the one representing "*an object overpasses the other one*" (**OV**) in the Surface-Surface set.

If our theory is cognitively adequate, the most similar models will appear in the same clusters, but as is noticeable from the graphs, this is not clearly visible in the hierarchical clustering. Therefore, we calculate appropriate truncation levels to study further the match between the relations in our theory and the clusters from the human subjects experiments.

In agglomerative hierarchical clustering, cluster membership is determined by specifying a level of truncation of the dendrogram's branches, and it controls the number of obtained partitions. Over thirty different indices (Milligan and Cooper, 1985; Rousseeuw, 1987; Tibshirani et al., 2001) have been introduced to automatically determine the optimal number of clusters in a data set. Among them, a set of three methods, including elbow (Thorndike, 1953), average silhouette (Rousseeuw, 1987) and gap statistic (Tibshirani et al., 2001), are popular, though they apply a range of different computational techniques to suggest the optimal number of clusters

We used the R programming language to compute thirty indices to decide the optimal number of clusters. According to the majority rule, we selected the optimal value from the indices.

The optimal suggested value of the Elbow method was confirmed by the majority of the other automatic techniques in our calculations. Applying this method to all sets of models shows that the optimal number of clusters is seven in the Line-Volume and Surface-Volume sets, while it is six in the Volume-Volume and Surface-Surface sets. This value drops to five groups in the Line-Surface set[6].

To identify the specified number of clusters for each set, we moved the horizontal line from the top of each dendrogram down until it breaks the numbered branches defined by the optimal number of clusters

[6]As can be seen in Figure 5.9 and Figure 5.10, clustering over the set of Line-Line models behaves differently from the other sets. So, we have analysed them separately (see **finding pattern in Line-Line set** paragraph.

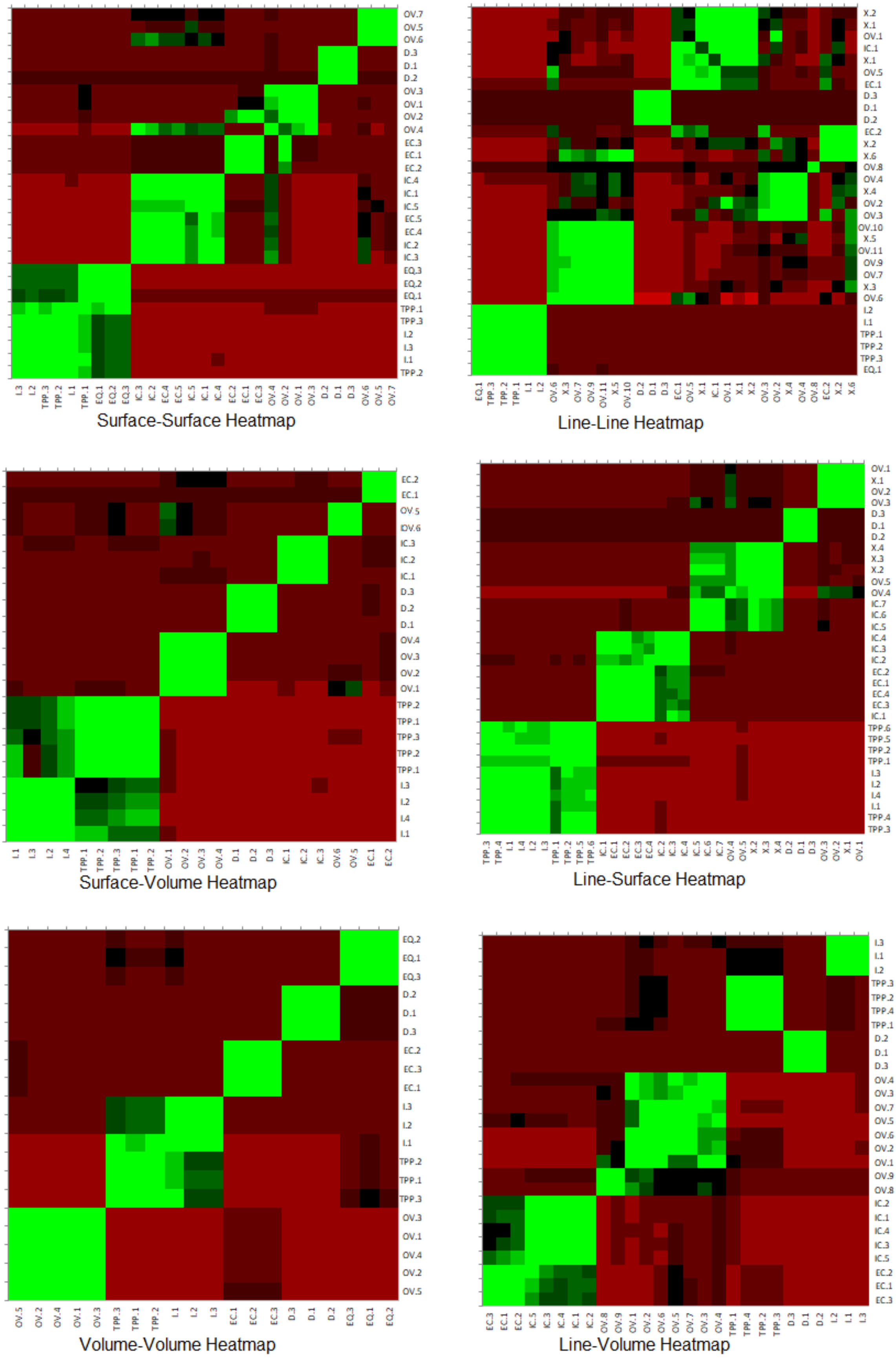


Figure 5.10: Heatmaps of all sets.

Table 5.3: The members of clusters per set generated based on the Elbow cuts

General relation	**No overlap**				**Overlap**			
	Discrete		**Part of**			**Imbricate**		
Set	**Cluster1**	**Cluster2**	**Cluster3**	**Cluster4**	**Cluster5**	**Cluster6**	**Cluster7**	**Cluster8**
Line-Surface	D1	I1			EC1	OV1	OV4	
	D2	I2			EC2	OV2	X2	
	D3	I3			EC3	OV3	X3	
		I4			EC4	X1	X4	
		TPP1			IC1		IC5	
		TPP2			IC2		IC6	
		TPP3			IC3		IC7	
		TPP4			IC4			
		TPP5						
		TPP6						
Line-Volume	D1	I1	TPP1		EC1	IC1	OV1	OV8
	D2	I2	TPP2		EC2	IC2	OV2	OV9
	D3	I3	TPP3		EC3	IC3	OV3	
			TPP4			IC4	OV4	
						IC5	OV5	
							OV6	
							OV7	
Surface-Surface	D1	I1			EC1	EC4	OV1	OV5
	D2	I2			EC2	EC5	OV2	OV6
	D3	I3			EC3	IC1	OV3	OV7
		TPP1				IC2	OV4	
		TPP2				IC3		
		TPP3				IC4		
		EQ1				IC5		
		EQ2						
		EQ3						
Surface-Volume	D1	I1	TPP1		EC1	IC1	OV1	OV5
	D2	I2	TPP2		EC2	IC2	OV2	OV6
	D3	I3	TPP3			IC3	OV3	
		I4	TPP4				OV4	
			TPP5					
Volume-Volume	D1	I1	TPP1	EQ1	EC1	OV1		
	D2	I2	TPP2	EQ2	EC2	OV2		
	D3	I3	TPP3	EQ3	EC3	OV3		
						OV4		
						OV5		

(see Figure 5.9). For instance, the six clusters of the Volume-Volume models are distinguished when the similarity value is around 0.45. Table 5.3 shows the clusters and their members in every set according to the defined cuts on the dendrograms.

Having considered the individual clusters as defined by the Elbow cut-off, we also consider the higher-level groupings, in which clusters are combined to show a higher level of granularity.

As can be seen in almost all of the dendrograms in Figure 5.9 (except for the Line-Line set), the models for the externally connected ($\boldsymbol{EC}$), internal contact ($\boldsymbol{IC}$) and overpass ($\boldsymbol{OV}$) relations are the most similar (closest convergence to the truncation line in every set) and form the granular cluster of the imbricate ($\boldsymbol{IMB}$) relation. This integrated cluster then links to the clusters that contain samples of part-of relation (i.e. interior par ($\boldsymbol{I}$), tangential proper part ($\boldsymbol{TPP}$) and equal ($\boldsymbol{EQ}$)), which then covers all the instances of the set after merging with the samples of discrete ($\boldsymbol{D}$) models. This finding is compatible with our lattice of relations (see Figure 4.6) and shows the semantic relationships among the introduced spatial relations in our formalism.

Finding patterns in the Line-Line set: As we said, the clustering results for the Line-Line set, in which every model consists of two curvy lines, were different from the other sets, so we carried out further analysis to reveal the implicit concepts in the grouping of these models. However, further analysis in this set does not mean that our analysis approach is completely different here in comparison to the other sets.

By applying the elbow method on the clustering outputs of the Line-Line set shown in Figure 5.9, five clusters were identified as an optimal number. The five clusters would be generated by placing the cut-off around the similarity level of 0.1 on the Line-Line set dendrogram in this figure. Although it is possible to

Figure 5.11: Sample of two lines on the same plane(a), on non-parallel planes(b).

see a comprehensible pattern in two of the generated clusters (i.e. clusters containing discrete ($\mathbf{D}_n$) and part-of ($\mathbf{EQ}_n$, $\mathbf{I}_n$, and $\mathbf{TPP}_n$) models), further understandable trends within the other clusters are not clearly visible.

The groups' descriptions provided a clear picture of grouped models within these clusters. The descriptions revealed that the grouping of the items was firstly based on whether or not two lines were connected. If they were connected, the participants considered the position of an imaginary plane containing a line with respect to the other line's plane. It seems that the grouping criteria were based on a participant's answers to the following question when two lines have an intersection:

Are the lines placed on the same plane in this item?

Every "yes" reply to this query leads to a specific categorisation. For instance, the model shown in Figure Figure 5.11(a) has a positive reply, while the question gets a negative response for the configuration shown in Figure 5.11(b). Note that every "no" answer reveals that the lines are placed in different planes.

By considering these answers in a Boolean form, i.e. "yes"(1) and "no"(0), for every item and appending them as extra features to the Line-Line's overall similarity matrix (generated previously), we produced a new data set. This dataset has a range of similarity values from zero to one between every pair of items (like the other sets) as well as yes (1) or no (0) answers. After applying the same clustering algorithm, i.e. agglomerative hierarchical clustering with the average linkage method, the resulting dendrogram almost confirms our hypothesis in this set of models (see Figure 5.12).

The optimal number of clusters determined via the elbow method is four in the new dendrogram. The dendrogram thus says that a line might be discrete from, part-of, or imbricated another line. The imbricated lines have been further categorised by their planar information (i.e. either on the same or different planes). The members of these clusters are shown in Table 5.4.

After determining the clusters and their members, we evaluated the merging process among all clusters in the Line-Line set. By looking at the branches above the truncation line in the set's dendrogram (Figure 5.12), we see that the clusters with models of the same and different planes join in the closest node. Their closeness reflects that the lines are overlapping but not part of each other in these models. Then they combined with the cluster of $\boldsymbol{TPP}$, $\boldsymbol{I}$, and $\boldsymbol{EQ}$ models (or in other words, the group of "part-of" models) in which a line is (partially/fully) covered by another. Finally, all these clusters merged with the cluster containing discrete models to form all the models.

To put it in a nutshell, what stands out in the clustering of the Line-Line set of models is consistent

Table 5.4: Summary of the Line-Line set clusters.

General relation	No overlap		Overlap	
	Discrete	Part of	Imbricate	
Set	Cluster1	Cluster2	Cluster3	Cluster4
			Same planes	Different planes
Line-Line	D1	TPP1	EC1	EC2
	D2	TPP2	OV1	OV6
	D3	TPP3	OV2	OV7
		I1	OV3	OV9
		I2	OV4	OV10
		I3	OV5	OV11
		EQ	OV8	X4
			IC1	X5
			X1	X8
			X2	
			X3	
			X6	
			X7	

recognition of the discrete relation. Moreover, the samples of part-of and imbricate relations are strongly distinguished from the rest of the relations. However, the refinements of these relations (as shown in the lattice of relations) are not straightforward. Participants considered some extra information to refine these relations.

To assess the clustering performance over all sets, the results of the hierarchical clustering should be measured by computing the cophenetic correlation coefficient for the dendrograms. The clustering output (a dendrogram) is an appropriate summary of the input data if the cophenetic correlation between the input and output is high (R Core Team, 2013). Intuitively, cophenetic correlation measures how accurately a dendrogram maintains the distances/dissimilarities in the input data (the distance matrix). The closer

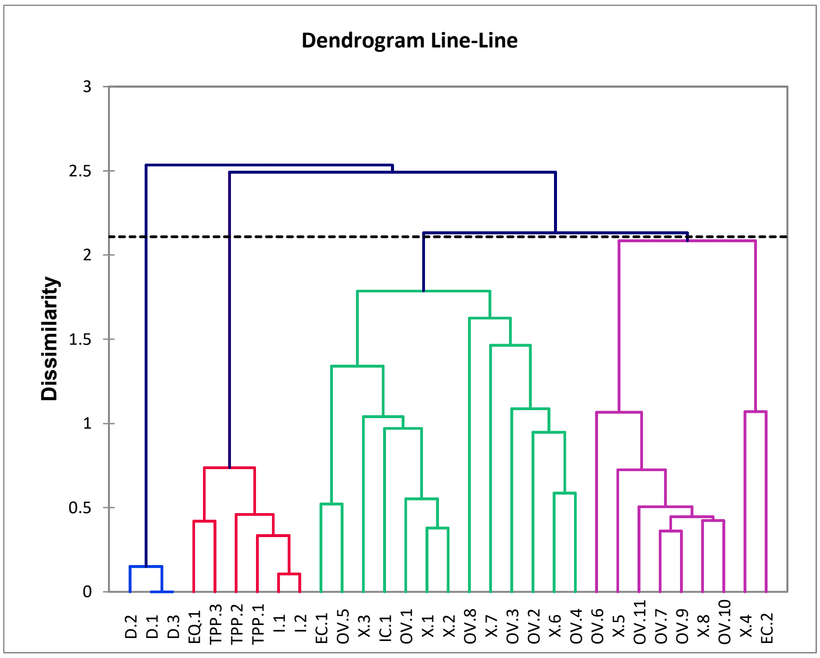


Figure 5.12: Dendrogram of the Line-Line items after adding the co-planarity information.

the clustering output is to the matrix, the closer the value is to 1, which indicates a high-quality clustering procedure.

To obtain the cophenetic correlation, we compute the cophenetic distance between two data points,es become members of the same cluster during the clustering process. In other words, it is the height of the dendrogram where the two branches that include the two data points merge into a single branch (i.e. dendrogrammatic distance).

Table 5.6 shows the cophenetic correlation values for all the generated dendrograms for the given sets. As can be seen, these values are close to 1, confirming the quality of the clustering process.

5.3.1.2 Discussion

According to the clustering results shown in Table 5.3 and Table 5.4 and distilled in Table 5.5, it seems that the main grouping factor is connectivity/disconnectivity between the objects in a model since samples of the *discrete* relation were consistently recognised in all sets of models by participants. This relation thus shows a high level of cognitive plausibility.

When the objects are connected (i.e. representatives of the *overlap* relation in our theory), the most important categorisation factor was whether an object is completely inside another one (i.e. Does the model show the *part-of* configuration?). For every positive reply, a member was added to the group containing models of the *part-of* relation, while for every negative answer, a model was added to the group representing the *imbricate* relation. However, these findings were not confirmed by the results in the Line-Line set of models since the grouping factors were different in this set of items in comparison to the other sets (see finding patterns in Line-Line set in subsubsection 5.3.1.1 for more detail). A possible explanation for this might be the difference value between an object (here a line (or a one-dimensional entity)) and its embedded space (here the three-dimensional world) which is known as the *co-dimensional* value. The higher the co-dimension value, the more factors (e.g. placing a pair of objects on an imaginary plane in the Line-Line set) are considered in the grouping. These groups confirm the cognitive adequacy of the most general set of relations, including discrete, part-of, and imbricate in our formalism.

The members of the *part-of* group were divided into three classes showing *interior part*, *tangential proper part* and *equality* spatial arrangements. We can see the role of co-dimensional value in these groupings. When it is zero for one or both objects in a model (like in Line-Volume, Surface-Volume, and Volume-Volume sets), participants considered properties like existence/non-existence of connection in the boundaries of two objects to be more specific in describing the connection status. However, they ignored these properties in cases like Line-Line, Line-Surface, Surface-Surface sets where the co-dimensional value is non-zero.

Similarly, we can see the impact of the co-dimensional value on the refinements of the *imbricate* relation. Moreover, we will see (in the following Section) that participants used some directional information when describing samples of this relation in the Line-Line, Line-Volume, Surface-Surface, and Surface-Volume sets.

Most importantly, the given models in our empirical investigations were frequently grouped by the type

Table 5.5: A summary of the clusters' analysis.

Relation	Are all the samples of the relation kept together?	Are samples of other relations also in group?	Conclusion/Decision
D	Consistently grouped together in all sets.	Separate from all other relations in all sets.	Excellent support for D as a separate and distinct relation. Although the two items are positioned in non-zero angle with respect to each other in some of the items, they have constantly grouped with zero-angle located items. It shows that directional information has the second priority in comparison to the connection in humans' perception.
EQ	Consistently grouped together in all sets. It is only valid in the sets that two participated objects are having the same dimensions. i.e. surface-surface & volume-volume here. They are consistently grouped together. Also, they were recognised as a separate group in the volume-volume set.	They are grouped with other instances of the part-of relation, i.e. I and TPP, in the surface-surface set.	Strong support for EQ as a separate relation. Also, it seems that the recognition of the two equal balls was easier than two concatenated disks from the human subjects' perspective. It might be a good support for the idea of better spatial understanding when there is an equi-dimensionality between the objects and their embedding space.
EC	They grouped together in 4 out of 5 sets, i.e. line-surface, line-volume, surface-volume and volume-volume. Moreover, they have represented a separate standalone group of the items in 3 of those sets, i.e. line-volume, surface-volume and volume-volume.	Grouped with some samples of ICs in two sets (i.e. line-surface and surface-surface sets).	Good support for EC as a separate relation. All ECs represent the spatial configurations only have boundary-boundary intersection (or in another word no interior-interior intersection). The combined ICs also do not have any interior-interior intersections while they have other types of connections. So the first common property between them is lack of interior-interior intersection. Another common property between them is the directional information. Either the line or surface positioned in non-zero angle with respect to the (other) surface in the named sets. As a result, we can rank the directional information as a refinement of the connection type.
OV	They grouped together and generated a standalone group in 2 of 5 sets, i.e. volume-volume and surface-volume sets. However, they have generated two groups of (only) OVs in another two sets, i.e. line-volume and surface-surface sets.	Only one instantiation of OVs is combined with Xs and ICs in the line-surface set.	Good support for OV as a separate relation but could be subdivided. The main reason for the subdivision of OVs is that one of the objects is either entered and passed through the other one or not (in both Line-Volume and surface-surface sets), while they have the interior-interior intersection. This property affects the participants' perception in the grouping task. Moreover, since the X relation represents only the interior-interior intersection cases, the combination of OVs with Xs in the line-surface set shows that this type of intersection is more important than other kinds of intersections. The common property among all the combined ICs, Xs and OVs are directional information.
TPP	Consistently grouped together in all sets. Its samples generate an individual group in 3 of 5 sets, i.e. line-volume, surface-volume and volume-volume.	Combined with Is in the line-surface, and with Is and EQs in the surface-surface set.	Strong support for TPP as a separate relation. The combination of TPP with Is and EQs is strong support for the claim that the EQ, TPP and I items are different kinds of the part-of spatial configuration. Moreover, their representation as a standalone group in the set of having volume as (at least) one of the participated objects is another confirmation of better spatial understanding when there is an equi-dimensionality between (at least) one of the objects and their embedding space.
I	Consistently grouped together in all sets. Its samples generate an individual group in 3 of 5 sets, i.e. line-volume, surface-volume and volume-volume.	Combined with TPPs in the line-surface and TPPs and EQs in the surface-surface set.	Strong support for I as a separate relation. Furthermore, the combination of Is with TPPs and EQs is secure support for the claim that the EQ, TPP and I items are different kinds of the part-of spatial configuration.
X	It is only valid between two lines, or a line and a surface. In other words, it sounds only when the dimensional sum of the participated objects is less or equal to the dimension of their embedding space. By excluding the line-line set as an exceptional set of items, we can see that almost all (except one) of the Xs are grouped together in the line-surface set.	One X is grouped with OVs in the line-surface set, while most of the members of this group are ICs.	A weak support for X relation as a distinct relation. We have seen some Xs and ICs are combined in the line-surface set. While the line passes through the surface in Xs, its boundary meets the interior of the surface in ICs. The line is perpendicular on the surface in all these ICs and Xs. It seems that the directional information plays the main role after being connected (regardless of the type of connection) in these cases.
IC	Almost consistently grouped in all sets (i.e. 3 out of 4 sets), except in the line-surface set subdividing into two groups. They made a standalone group among 2 of the 3 sets. It is not a valid relation between two volumes.	Grouped with ECs in the surface-surface and line-surface sets. We can see some samples of Xs in the latter set as well.	Good support for IC relation. Both ECs and ICs define the configurations without any interior-interior intersection. As explained before the common property among them is the directional information. On the other hand, the grouping of ICs and Xs with similar directional information is another strong evidence for the role of orientation. This information was important in the grouping of the referred ICs, ECs and Xs since the objects were somehow perpendicular in these items.

Table 5.6: Cophenetic Correlation values

Set of items	Cophenetic Correlation
Line-Line	0.961
Line-Surface	0.993
Line-Volume	0.983
Surface-Surface	0.972
Surface-Volume	0.989
Volume-Volume	0.993

of connection between the participating objects. It shows that the topological relationships, as stipulated in Chapter 4, seem to form a meaningful level of cognitive adequacy. However, there is one exception in which the participants hardly differentiate the *crosses* cases from *overpass* or *interior contact* relations. Thus, a generalisation of the overpass relation representing the crosses configurations is replaced. In the following Section, we will do further analysis on the description.

5.3.2 Description Analysis

The grouping task was followed by eliciting a verbal description for every generated group in our experiment. We then analysed the descriptions to identify the characteristics that connected objects in every group produced by an individual and looked for common themes across participants.

We applied computerised text analysis, specifically thematic analysis (TA) introduced by Braun and Clarke (Braun and Clarke, 2006). It is a common method for analysing qualitative data (including texts) to extract the patterns of meaning (or themes). A theme reveals an important feature of the descriptions in regard to our research questions, and the "variants of TA appear to share some level of theoretical flexibility but can vary enormously in terms of both the fundamental philosophy and the theme development procedures" (Braun et al., 2019, p. 1). Here we used reflexive thematic analysis (Braun and Clarke, 2019).

Reflexive thematic analysis is a widely-cited approach and is a modified terminology of TA. It acknowledges that researchers can say that the themes 'emerge' from the data as an underlying theory and confirm their involvement in creating the themes. Among various types of reflexive thematic analysis, we used the inductive approach in which the descriptions determine the themes without any preconceived background. The steps of the reflexive thematic analysis are as follows:

I) Familiarization with the descriptions,

II) Assign preliminary codes to the descriptions in order to describe the content,

III) Search for patterns or themes in codes across the different descriptions,

IV) Review themes,

V) Define and name themes.

The detail of each phase is provided in the following paragraphs. Please note that this was not conducted in a *linear* fashion, but rather *iteratively*. Although this research method is qualitative, we provide basic quantitative information to aid understanding of the data.

5.3.2.1 Familiarization with Descriptions

In this step, we explored the data deeply by repeated reading. Multiple re-reading has shown promising results in extracting any patterns or themes. Our data consists of twenty-seven reports in this experiment. Every report has 1400-1500 words on average. We gained some prior thoughts and knowledge of data during the data collection process. For instance, the wide variations in grouping among participants indicated that participation of linear objects in a model increased the difficulty of decision making for the human subjects. Nevertheless, we needed to immerse ourselves in the depth of the description content. Immersion was achieved through repeated reading and searching for the themes.

The entire data set was read three times before coding, which shaped our possible themes. We became familiar with all aspects of the data during reading, such as the word choices to describe the connection of the entities and how they had been supported with other types of spatial information, e.g. direction, size, etc. During this phase, some notes and ideas for coding were generated and then used in the formal coding process.

5.3.2.2 Coding

After gaining some knowledge of the descriptions, we produced an initial list of codes. Codes highlighted important segments or elements of the descriptions relevant to our research questions. They identified particular features of descriptions, i.e. the connection of entities, the connection type, and other supporting information (see Table 5.7, for examples). Different coloured text was used to indicate the presence of the respective code in Table 5.7, e.g. "equal in size" mapped to the "object size" code (see Appendix F for guidelines in mapping between codes and segments of the descriptions). We applied every code to every description for each set. Several codes might encode some segments.

Another important consideration in our coding process was the level at which "meaning" was extracted and coded. Some of the codes were identified from the "outer layer" of the descriptions, known as semantic codes. They consider the explicit meaning, for instance, in "Two lines are attached to each other's end". The verb of this sentence states the connection status. On the other hand, some codes, known as latent, were looking for an implicit meaning. For instance, as humans, we perceived the connection style in the following sentence: "Parts of the curve are on the outside of the ball but not the remaining". However, the connection has not been clearly expressed in this sentence. Our coding system covered both.

Coding had been done with the help of word processing software MAXQDA 2018, in which we coded by naming and tagging segments of texts in description.

The codes were generalised by the themes as the broader concepts in the following phase. The interpretive analysis of the descriptions occured via the themes.

5.3.2.3 Search for Themes in Codes

After collecting information and generating a list of codes, we analysed and sorted the codes to extract broader themes. The repeatedly extracted patterns in the descriptions are shown in the thematic mind-map

Table 5.7: Group description with codes applied

Group Description	Coded for
This group is made up 4 items. Each item comprises of a circle and a wavy object is tightly stacked to each other (in two points) in a vertical direction. The wavy objects are stacked from both ends* to the inner sides of the circles seem as if they divide the circle into two parts, though they do not. All objects are equal in size as of circles, unlike some the wavy objects which are larger than others.	Shape of the entities Connection Degree of freedom in the connection Number of connection points Direction in connection Connection in the line's boundary Connection in the surface's interior Object size

*Every segment of a description can be coded several times, here "both end" statement is coded both as "Connection in the line's end point" and "Small overlapping area".

(see Figure 5.13). The map shows the relationships among codes and their higher-level categorisations, e.g. among the main overarching themes, a *position of connection* covers two sub-themes, i.e. connection *on the boundaries* of both objects and one of the objects. As can be seen, some initial codes form sub-themes (unfilled boxes in Figure 5.13), which are then subsumed by some general themes (solid filled boxes in Figure 5.13). We did not discard any codes at this stage.

We ended this phase with a collection of themes, as shown in the solid boxes in Figure 5.13. According to this set of themes, the importance of the *connection* in these descriptions were confirmed.

5.3.2.4 Review Themes

In this step, we reviewed the extracted themes, sub-themes, and codes from the previous step and checked whether they should be retained, or not play an effective role in our analysis.

Through our reviewing process, it became evident that themes like *a proportion of overlapping area*, *the position of connection*, and *type of connection* have enough data to introduce a super theme describing the *type of connection* between two objects.

On the other hand, the *shape* theme must be omitted since the participants referred to the shape of the objects just for naming, and it remained unchanged while describing groups in a set or even sets. For instance, a participant wrote:

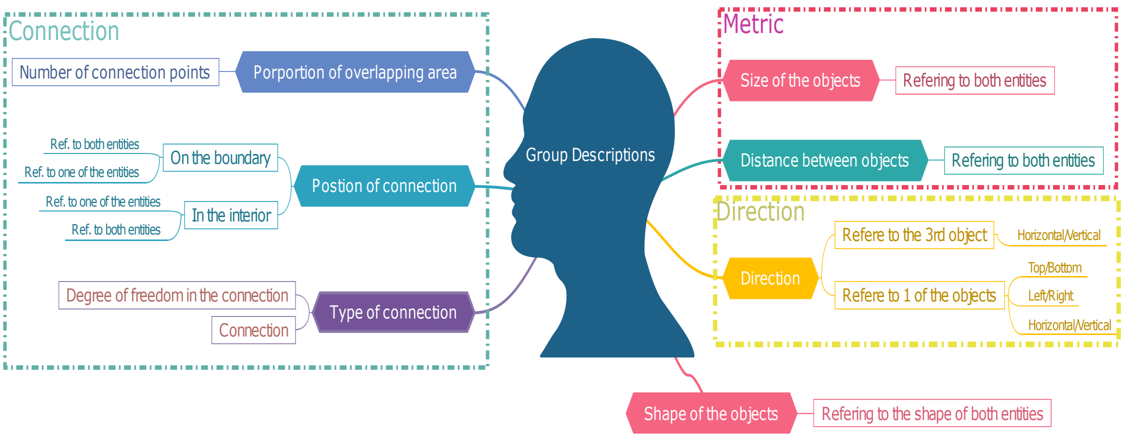


Figure 5.13: Thematic map

> Subject12(Line-Volume set)
>
> ...No interaction between the *plate* (i.e. two-dimensional object) and the *tail* (i.e. one-dimensional object)...

while described a group of models in the Line-Surface set. The participant used these names not only to describe all the objects (having the same dimensionality) in this set, but also these names had been used in the descriptions of other groups in the remaining sets (like in the Line-Line and Surface-Surface sets). In other words, these words were descriptors and not reasons for grouping.

Furthermore, the other two themes, *distance* and *size*, should be removed, since they are also used as descriptors. The following example demonstrate this idea:

> Subject8(Surface-Surface set)
>
> ...A circle is inside another one in all models of this group. However, in some of them a blue circle has a **smaller** white circle inside while its vice versa in others...

5.3.2.5 Define and Name Themes

In this phase, we identified the most vital parts of the themes found in the previous step and explained the aspects of the descriptions being captured by them. The super themes were extracted via the sub-themes. These super themes had been labelled as *connection* and *direction*. The themes'names were ordered according to their best reflect the descriptions' perspective. The *connection* concept and its various interpretations were mentioned repeatedly in the descriptions. Thus, it was straightforward to use this criterion for categorisation. After that, the *directional* information was found helpful to describe the connection by several participants. We will describe them in the following paragraphs.

Of course, other aspects of the descriptions extended across these themes. They provided elucidation of the participants' conceptualisations, showing that their perception was never made up of a single aspect but of a collection of linked aspects.

Theme 1: Connection

The connection theme was defined via recognition and nomination of the *connection* between a pair of entities as the prominent factor in the categorisation process. The theme encapsulated each of the participants' struggle to describe the *connection* of two objects. It covers a linkage type, the position of the connection, and the size of the overlapping area.

Participants described the *connection* using different verbs like:

> Subject17(Line-Line set)
>
> Wavy objects are **attached** together...
>
> ...but the wavy objects are **not placed** in the...

Sometimes, these verbs refer to the type of connection:

Subject27(Line-Volume set)
The line **passes** through the ball...

During description analysis, we realized that one-third of the participants used similar ideas to verbalize the *connection* of two objects. They identified and named different parts of the entities while describing the *connection*. Either implicitly:

Subject4(Volume-Volume set)
...the disks **are touching** ...

or explicitly:

Subject21(Line-Line set)
Lines **meet** at their endpoints...

They also referred to the position of the *connection* (i.e. the interior connection or boundary connection):

Subject3(Line-Surface set)
... have the squiggly line **inside** of the circle...

Subject20(Line-Volume set)
Line is on the **edge** of ball with...

In addition, they often highlighted the *size* of the common part(s) between two entities. For instance, by characterising the common part as points:

Subject20(Line-Volume set)
Line meets ball in **one point**...

It showed a deeper level of understanding. There were some pieces of evidence that the *connection* of objects might be supported by the type of the common part, the number of common parts or even both. In the majority of the descriptions, the *connection* were supported by the next theme, i.e. *direction*. The key difference between these themes is that the former is purely related to the properties of the common part between two objects, while the latter is about the common part's order.

Theme 2: Direction

The detection of *connection* was often facilitated by *directional* information. *Directional* information played an important role in descriptions; for example, some participants considered the higher-dimensional entity in a model as a primary object and used it to describe the *direction*.

Subject26(Surface-Volume set)
...it was meant to be placed **above** the ball...

Sometimes, the directional information was refined by axial information expressed via adverbs like: top, bottom, vertical, left, right, and horizontal:

Subject3(Line-Line set)
...running flat on **top** of ...

Subject21(Line-Volume set)
...the noodle **vertically** sticking outwards and only contacting the ball...

Participants also incorporated the *angle* in terms of degree or their names (e.g. right angle) and viewing it as a solution to provide an accurate description of the *connection*.

Subject14(Line-Line set)
...but the **angle** is less than **90 degrees**...

These examples are pieces of evident to show the incorporation of *directional* information in humans' conceptualisation.

Many other aspects were highlighted in the descriptions (e.g. *shape*), but more focus was given to the *size* and *distance* between objects, leading to the next theme that reveals *metric* information. However, the frequency of this theme was incredibly less than the connection and direction themes.

Although flexible thematic analysis is a qualitative method, we support it with an indication of the quantities of descriptions in each theme to gain more insight from the descriptions in the next Section.

5.3.3 Quantitative Examination of Thematic Analysis

We also analysed the outputs of the thematic analysis quantitatively. We quantified the extracted themes (i.e. *connection* and *direction*) and their combinations, as they had been recognised as the main reasons for grouping. We exploited the codes to quantify their status in the descriptions. We counted all the codes representing every theme and their combinations. Then, we computed their percentage with respect to all thematic codes.

Figure 5.14[7] shows an overview of the quantitative analysis. Around two-third of the participants used *connection* information for grouping purposes. However, about one-quarter of them supported this information with *directional* information in their categorisations. Also, almost the same proportion of the participants considered just *directional* information for their grouping. Moreover, this figure says that all descriptions contains either *connection* or *directional* information or even both .

5.4 General Discussion

This study sat out to assess the cognitive adequacy of our proposed theory, presented in Chapter 4. The most prominent finding to emerge from the quantitative examination of the thematic analysis is the importance of connection (or (mereo)topological) relations in the human assessment of spatial arrangements,

[7]Total number of the themes and their combination is 2706.

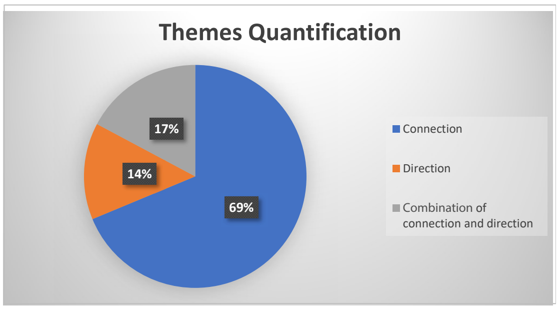


Figure 5.14: Themes and their combination quantification.

which has already been found in our clustering analysis. It is consistent with the studies reported in (Knauff et al., 1997; Renz et al., 2000).

Moreover, some descriptions support the claim of refining (mereo)topological information by other information, such as the ratio of overlapping sectors. This assertion is already found in (Renz et al., 2000) that said the RCC-8's externally connected and tangential proper part relations could be refined by referring to the overlapping area. This finding is re-expressing what we found in clustering analysis, where various refinements of the part-of and overlap relations are introduced.

Interestingly, participants described spatial configurations via the connection between the spatial entities and then supported it with directional information. Meanwhile, they used shape, relative size and distance information as descriptors. This finding revisits the importance of directional information, which was already shown by our clustering results and found in the literature (Moratz et al., 2003; Clementini and Bellizzi, 2019).

We found good correspondence between the proposed multidimensional theory and the (mereo)topological relationships conceptualised by humans. The set of relations with a finer level of granularity (i.e. JEPD relations) has also been confirmed.

5.5 Summary

In this Chapter, the cognitive adequacy of our multidimensional mereotopological theory involving objects of various dimensions has been investigated. The experiment investigated the conceptual distinctions defined in the theory. Our general approach (i.e. categorising given models and describing the groups) was similar to other relevant research in the field. However, we performed further analysis on the provided descriptions as we sought to extract features that might explain the reasons for distinctions between relations. The result confirmed that the mereotopological distinctions introduced by the formalism are cognitively adequate.

The collected results highlight some important findings on the process of human spatial understanding and reveal the importance of evaluating the plausibility of any formal spatial theory.

Chapter 6

Reasoning with the Multidimensional Mereotopological Theory

In this Chapter, we will develop a reasoning strategy about the spatial relations represented in our proposed multidimensional mereotopological formalism. Moreover, we will show two case studies that can be formally represented via introduced relations in the formalism. We will also discuss about the accomplishments of the reasoning problem in spatial analysis via the case study.

In the previous Chapters, we introduced our multidimensional mereotopological formalism in the language of first-order logic (in Chapter 4) and evaluated the cognitive validity of the spatial relations in it (in Chapter 5). Introducing a set of multidimensional mereotopological relations that is both *expressive* and *cognitively adequate* by itself is interesting. However, a spatial system's ability to infer new spatial knowledge from the existing one makes it intelligent.

Other multidimensional mereotopological theories have not underestimated the importance of reasoning. Most of the current work (discussed in Chapter 2) relies on inference rules for deducing new information. Although their method is powerful, it is not efficient from the computational point of view and makes them impossible to implement on machines.

The most efficient qualitative reasoning method is the *constraint-based* technique (Renz et al., 2000) which is our reasoning mechanism. To use this method for reasoning purposes, it is necessary to exploit the composition of binary relations having the JEPD property. For instance, if we have R_1 relation between "resident" and "building A", and R_2 relation between "building A" and "water" in a flood event scenario. Then the composition of R_1 and R_2 limits the possible relations between "resident" and 'water". The resulting composition of any pair of relations is pre-computed and stored in a table known as a *composition table* to improve the speed of the reasoning mechanism. However, the table's entries underlying proofs and satisfiable models are difficult to generate in some cases (Randell et al., 1992a).

The importance of the existence of an efficient reasoning procedure for a theory is demonstrable in spatial analysis. Through spatial analysis, one can interact with GIS to answer questions and support decisions. For instance, without an efficient reasoning mechanism queries like "*Are all appliances in the basement of the building in water?* could not be answered as there is no finite-sized counterexample for it. The inference issue transforms to a *query answering* problem in our case study in which the spatial system performs reasoning to return all answers for a given query. For instance, we expect that a spatial system can reply to "Are any residents of building A surrounded by water?" and help the emergency management in a disaster such as a flood event disaster.

This Chapter is organised as follows: we will explain *constraint-based reasoning* in Section 6.1. Then, we will show the result of enforcing the constraint-based reasoning on a flood even in Section 6.2.

6.1 Qualitative Spatial Reasoning Using the Constraint-based Technique

As already said, our proposed binary mereotopological relations can be considered as a set of binary constraints. Reasoning over the spatial objects represented by these constraints can be formulated as a binary *constraint satisfaction problem* (CSP) in which the state of spatial objects must satisfy a number of mentioned constraints.

A binary CSP consists of a set of variables $\mathscr{V} = \{x_1, x_2, ...x_m\}$ which represent the objects in the domain $\mathscr{D}$, and a set of constraints (θ) consisting of binary relations ($R_i \subseteq D^n \times D^n$) over a pair of variables (x_i, x_j). The two sets (i.e. $\mathscr{V}$ and θ) together form a constraint network graph (Dechter, 1992). Nodes of the graph represent variables (x_i), and every edge is a binary relation (R_{ij}) between a pair of variables (i and j are objects). We can then apply graph analysis techniques to find a solution for the binary CSP. A solution returns instances for the variables of $\mathscr{V}$ from the domain ($\mathscr{D}$) values such that all constraints are satisfied (or `True`). If a CSP has a solution, it is consistent (Renz and Ligozat, 2005).

Generally, assessing the consistency of a (binary) CSP over an infinite domain (like space) is undecidable. Thus, an alternative solution is determining the CSP's inconsistency. The algorithm known as the *path-consistency* algorithm (Mackworth, 1977) is exploited. A binary CSP is path-consistent if and only if there exists an instantiation of a third variable for a pair of instantiated variables to form a network of satisfied constraints over the three variables. A CSP where the path-consistency cannot be enforced is not consistent, but a CSP is not necessarily consistent where path-consistency can be enforced (Renz, 2002). An effective approach to achieve path-consistency on a binary CSP is successively enforcing the *composition* operation until a fixed point is attained.

The structure in which the composition operation is applicable must be a relation algebra. Thus, the CSP should be formulated as a relation algebra for consistency checking purposes. A relation algebra has a set of binary relations ($\mathscr{R}$) which is equipped with basic operations including `union`[1] ($\cup$), `intersec-`

[1] It is also known as `sum` or `disjunction`.

tion[2] ($\cap$), complement ($\tilde{\cdot}$), converse ($\breve{\cdot}$) and composition ($\circ$). Union, intersection and complement are defined in the standard way, while the converse and composition operations are defined as follows respectively:

$$\breve{R} := \{(i,j)|(j,i) \in R\}$$

$$R \circ S := \{(i,j)|\exists k : (i,k) \in R \wedge (k,j) \in S\}$$

A relation algebra also has the universal relation ($*$), which is a valid relationship between any pair of the domain's objects, empty relation ($\emptyset$), which does not hold between any pair of the domain's objects, and the identity relation (Id), which is a valid relation between an entity and itself.

Furthermore, relation algebra structure assumes at least one relation for every pair of spatial objects in the set of constraints (θ). If there is no such relation between the objects, the universal relation will be considered, i.e. $R_{i,j} = *$ between the objects i and j. Also, the existence of a relation between a pair of objects R_{ij} in θ, requires the presence of its converse, $\breve{R_{ij}}$ (or R_{ji}), in θ as well.

If the relation algebra is based on the set of binary JEPD relations, the algebraic structure can facilitate the process of finding a solution for the CSP over the infinite domain. The reason is to classify an infinite domain into finite disjoint relational classes by the JEPD relations and exploit such categorized space for reasoning purposes (Dubba et al., 2012, p. 116).

By definition, the composition operation must be calculated for the JEPD relations since the non-JEPD ones are only the disjunction of the JEPD ones. However, there is no guarantee for the generation of the composition of the JEPD relations. Thus, approximating the true resultant relations of the operation is required. The approximated relation is the strongest relation, which contains the possible output of the composition operation for a pair of relations. An operation known as *weak composition* returns this relation and is used to check the consistency of the binary CSP instead of the composition operation in path-consistency. It is the case for many well-known calculi like RCC-8 (Randell et al., 1992b). The weak composition operation is defined as:

$$R_{ij} := R_{ij} \cap (R_{ik} \circ R_{kj})$$

where R_{kj} , and R_{ik} are members of $\mathscr{R}$. The output of the weak composition operation (R_{ij}) is either a member of $\mathscr{R}$ or one of the particular relations (i.e. universal relation, empty relation or identity relation). If the operation's output is an empty relation, then the CSP is inconsistent; otherwise, it is algebraic-closed or a-closed[3]. The algorithm that enforces the weak-composition operation to check the consistency of the CSP is known as the "*algebraic-closure* algorithm." Figure 6.1 shows the steps of this algorithm (proposed in (van Beek, 1992)), which we will also use over the set of spatial relations introduced in our formalism. The resulting CSP, achieved by applying the weak composition operation, is equivalent to the original CSP (Renz and Nebel, 2007) since it has the same set of solutions.

[2]It is also known as product or conjunction.
[3]A phrase "algebraic-closure" is more suitable here. However, it is more common to use "path-consistent" in the literature which is not, as we discussed the reason in the text, exactly applicable here.

Algorithm: ALGEBRAIC-CLOSURE
Input: A set θ of binary constraints over the variables x_1, x_2,...x_n of θ.
Output: `fail`, if θ is inconsistent; algebraic-closure consistent set equivalent to θ, otherwise.

I $Q \leftarrow \{(i,j,k),(k,i,j) | i < j, k \neq i, k \neq j\}$; ($i$ shows the i-th variable of θ. Similarly for j and k)

II while $Q \neq \emptyset$ do;

III select and delete a path (i,k,j) from Q;

IV if $REVISE(i,k,j)$ then:

V if $R_{ij} = \emptyset$ then return `fail`:

VI else $Q \leftarrow Q \cup \{(i,j,k),(k,i,j) | k \neq i, k \neq j\}$;

Function: $REVISE(i,k,j)$
Input: three variables i, k and j
Output: `true`, if R_{ij} is revised; `false` otherwise.
Side effects: R_{ij} and R_{ji} revised using the operations $\cap$ and $\circ$ over the constraints involving i, k and j.

I old R $:= R_{ij}$;

II $R_{ij} := R_{ij} \cap (R_{ik} \circ R_{kj})$;

III if (old $R := R_{ij}$) then return $false$;

IV $R_{ji} := \breve{R_{ij}}$;

V return `true`.

Figure 6.1: Algebraic-closure algorithm (van Beek, 1992)

6.1.1 Constructing (Unrestricted) Composition Tables

As we discussed in the previous section, the composition of a pair of mereotopological relations is essential in constraint-based reasoning to extract the strongest relation between any pair of objects. Intuitively, the composition operation is a chaining process in which the spatial relation between two objects is extracted through their relation to a third common entity.

In a particular theory (T) which supports a set of binary JEPD relationships (R_i), the composition operation is defined as a conjunction of a pair of relations expressing pairwise configurations of three entities, a, b and c (i.e. $R_1(a,b) \wedge R_2(b,c)$), which is mapped to a disjunction of all possible binary relations between a and c in the theory (shown by $R_3(a,c)$). The mapping for every pair of relations corresponds to a theorem and is stored in a cell of a matrix, called a *composition table*, for every ordered pair of relations $R_1(a,b)$ and $R_2(b,c)$. These relations (i.e. $R_1(a,b)$ and $R_2(b,c)$) show a respective row and column of the matrix. Since our theory supports nine binary JEPD relationships, eighty-one (i.e. 9×9) entries will appear in the composition table.

Constructing the composition table has two characteristics to be considered. The first is to prove that $R_1(a,b) \wedge R_2(b,c) \rightarrow R_3(a,c)$ (Eschenbach, 2001). $R_3(a,c)$ might be one of the JEPD relations or a disjunction of some of them, i.e. $R_{3_1}(a,c) \vee R_{3_2}(a,c) \vee ... \vee R_{3_n}(a,c)$. This says that there is at least one relation between a and c in the theory. The second feature looks for a model for every possible binary relationship between a and c ($R_{3_i}(a,c)$), which participates in the disjunction of the relations (Randell et al., 1992a). The second feature reveals the possible relation(s) between a and c. Sometimes there is no binary relation to be omitted from all the possible ones in $R_3(a,c)$; in this case, $R_3(a,c)$ is true for every defined relation in theory (i.e. $R_3(a,c)$ is the *universal* relation).

We typically cannot directly prove the given statement (i.e. $R_1(a,b) \wedge R_2(b,c) \rightarrow R_3$) for two reasons. First of all, trying to prove the disjunction statement is difficult (Grüninger et al., 2011), and second, we should also look at many different potential combinations to prove the disjunction statement. So, instead of proving the disjunction statement, we checked the provability of the following theorem:

Theorem 6.1.1. $R_1(a,b) \wedge R_2(b,c) \rightarrow \neg R_3(a,c)$

in which R_is are mapped to JEPD relations. For instance, the table entry as $I(a,b)$ and $EC(b,c)$ is $D(a,c)$ corresponds to an instance of the theorem:
$I(a,b) \wedge EC(b,c) \rightarrow \neg R_i(a,c)$ where $R_i \in \{\mathbf{EC}, \mathbf{IC}, \mathbf{OV}, \mathbf{EQ}, \mathbf{TPP}, \mathbf{I}, \mathbf{TPP}^{-1}, \mathbf{I}^{-1}\}$.
We are essentially eliminating options one at a time. At the same time, we try to find a model for the possible (or not eliminated) relation $R_3(a,c)$.

Table 6.1 shows the results of computing the compositions from Theorem 6.1.1 for every pair of JEPD relations in our theory. We used automated theorem provers and finite model finders to generate the compositions.

By considering the properties of resolution techniques in automated theorem proving (see Chapter 3) along with the composition table generation, it is clear that most, if not all of our theory's axioms (that contribute to the axioms and definitions expressing the properties of multidimensional mereotopological relations) can be used to generate the table.

6.1.2 Constructing (Restricted) Composition Tables

As you remember from Chapter 3, our spatial taxonomy consists of two types of regions in terms of having or not having (referable) boundaries (i.e. manifolds with boundaries (or bounded regions) and closed manifolds with boundaries(or unbounded regions)). In Subsection 6.1.1, we only generated a composition table without explicitly expressing any information about the existence of boundaries for the regions in Theorem 6.1.1. This is a piece of valuable information specifically when addressing:(a) cases in which *ALL* regions are unbounded, and (b) cases in which *ALL* regions are bounded. A suitable statement adds these restrictions to Theorem 6.1.1. For example, if an object a has a boundary, it is expressed via $\exists x(\mathbf{B}(x,a))$, whereas its boundaryless property is stated by $\neg\exists y(\mathbf{B}(y,a))$. The options where only one object is bounded (and the other two are not) or only two objects are bounded (and the other one is not) do not make much sense, since permutation (of bounded or unbounded) in the triple Theorem 6.1.1 leads to different results that are not important for a user in practice. For instance, the case where regions a and c are roads (or line segments as samples of bounded regions) and b is a square (or a ring as sample of unbounded regions) differs than b and c are roads, and a is a square. The mentioned restrictions were added to the premises in Theorem 6.1.1. Thus, we have different variants of Theorem 6.1.1 as follows:

- All the entities are boundaryless:
  $(\neg\exists x\mathbf{B}(x,a) \wedge \neg\exists y\mathbf{B}(y,b) \wedge \neg\exists z\mathbf{B}(z,c) \wedge R_1(a,b) \wedge R_2(b,c)) \rightarrow \neg R_{3_i}(a,c)))$

Table 6.1: Composition table for the 9 basic relations (without any restrictions on the given entities).

	D	EC	TPP	TPP^{-1}	OV	IC	EQ	I	I^{-1}
D	{TPP^{-1}, TPP, EC, I^{-1}, EQ, I, OV, D, IC}	{OV, TPP, I, D, IC, EC}	{EC, I, TPP, D, IC, OV}	{D}	{I, OV, D, EC, TPP, IC}	{TPP, I, EC, OV, D, IC}	{D}	{D, TPP, IC, EC, OV, I}	{D}
EC	{TPP^{-1}, OV, I^{-1}, IC, D, EC}	{TPP^{-1}, TPP, D, IC, EC, OV, EQ}	{I, OV, EC, TPP, IC}	{D, EC, TPP^{-1}, IC}	{I, IC, D, EC, TPP, OV}	{ OV, TPP, TPP^{-1}, I, D, IC, EC}	{EC}	{IC, TPP, OV, I}	{D}
TPP	{D}	{IC, TPP, EC, D}	{TPP, I}	{I, D, EQ, TPP, TPP^{-1}, I^{-1}, OV, EC, IC}	{TPP, D, I, OV, EC, IC}	{EC, D, I, TPP, OV, IC}	{TPP}	{ I}	{ I^{-1}, D, OV, EC, TPP^{-1}, IC}
TPP^{-1}	{EC, OV, I^{-1}, TPP^{-1}, D, IC}	{IC, I^{-1}, TPP^{-1}, EC, OV}	{OV, IC, TPP, EQ, TPP^{-1}, EC}	{TPP^{-1}, I^{-1}}	{IC, TPP^{-1}, OV, I^{-1}}	{EC, IC, OV, I^{-1}, TPP^{-1}}	{TPP^{-1}}	{IC, TPP^{-1}, I, OV, TPP}	{TPP^{-1}, I^{-1}}
OV	{EC, OV, D, TPP^{-1}, I^{-1}, IC}	{IC, TPP^{-1}, EC, D, OV, I^{-1}}	{TPP, I, OV, IC}	{D, I^{-1}, OV, TPP^{-1}, EC, IC}	{TPP^{-1}, D, I^{-1}, I, EC, OV, TPP, IC, EQ}	{TPP, IC, EC, I^{-1}, OV, TPP^{-1}, I, D}	{OV}	{OV, TPP, I, IC}	{OV, D, I^{-1}, EC, TPP^{-1}, IC}
IC	{OV, I^{-1}, EC, TPP^{-1}, D, IC}	{D, TPP^{-1}, OV, IC, I^{-1}, EC, TPP}	{EC, TPP, OV, I, IC}	{EC, I^{-1}, OV, D, TPP^{-1}, IC}	{EC, IC, TPP^{-1}, OV, TPP, D, I}	{OV, EC, IC, I, I^{-1}, EQ, D, TPP, TPP^{-1}}	{ IC}	{ I, IC, TPP, OV}	{TPP^{-1}, D, EC, OV, IC}
EQ	{D}	{EC}	{TPP}	{TPP^{-1}}	{OV}	{ IC}	{I^{-1}, I, EQ}	{II, I, EQ}	{I^{-1}, I, EQ}
I	{D}	{D}	{I, TPP}	{D, TPP, I, EC, OV, IC}	{I, D, OV, EC, TPP, IC}	{D, TPP, EC, IC}	{EQ, I, I^{-1}}	{ I^{-1}, I, EQ}	{EC, I, TPP^{-1}, TPP, I^{-1}, OV, EQ, D, IC}
I^{-1}	{OV, TPP^{-1}, EC, IC, D, I^{-1}}	{I^{-1}, OV, IC, TPP^{-1}}	{IC, TPP, OV, I^{-1}, TPP^{-1}}	{ I^{-1}}	{OV, TPP^{-1}, I^{-1}}	{ TPP^{-1}, IC, I^{-1}, OV}	{EQ, I, I^{-1}}	{ EQ, OV, TPP^{-1}, TPP, I^{-1}, I}	{EQ, I^{-1}, I}

- All of the entities have boundaries:
  $(\exists x\mathbf{B}(x,a) \wedge \exists y\mathbf{B}(y,b) \wedge \exists z\mathbf{B}(z,c) \wedge R_1(a,b) \wedge R_2(b,c)) \rightarrow \neg R_{3_i}(a,c)))$

Each of these cases led to the construction of a new composition table. Appendix E shows the results of computing the compositions for the variants of Theorem 6.1.1. The production of these tables, even with the help of automated tools, is a tedious task in which the process of triangle checking frequently applies for every statement. We checked 648 theorems by Prover9, Vampire, Mace4 and Paradox. Some of them required a day to finish.

The composition tables facilitate reasoning about how different multidimensional regions are related to other regions. The asserted relations (or the data given) among spatial entities are maintained in a network that reveals their pairwise relationships. Adding new relations to the network would expand, and the constraint-based reasoning can infer what relations (i.e. edges) can hold between two not explicitly related regions by looking at what "triangles" are allowed in the network.

In the next section, we will develop constraint-based reasoning over a sample data-set using the composition tables.

6.2 A Use Case for Constraint-based Reasoning

In this case study, we use a flood event occurrence in a residential zone as a spatial scenario to demonstrate the practical application of reasoning with our formalism. Flooding is a widespread natural disaster causing significant damage to buildings, human lives, and the ecosystem. It led to about 39.26% of natural disasters worldwide and caused damage worth USD 397.3 billion between 2000 and 2014 (Guha-Sapir et al., 2015). These statistics indicate that flooding can significantly negatively impact human life, property, crops, and livestock. To reduce the impacts, responding to emergencies in real-time is crucial and could be achieved via a geographical information system in which emergency management experts can express their required information. The system should precisely return the requested information relaying on inferences over the given knowledge. Here, the reasoning mechanism would determine the possible mereotopological relations between a pair of entities.

6.2.1 Flood Event Scenario

Flooding is the excess or surplus of water not being accommodated by its natural bed (i.e. river, water channel, etc.). The overflow might affect a residential area, a piece of land using for houses, apartments, nursing homes, schools, childcare facilities or prisons. Although a residential zone is usually comprises several buildings, we have only considered a single house as a sample here (shown in Figure 6.2). In this scene, the house, its residents, and all around it are depicted surrounded by water. Each house consists of several rooms containing various appliances and furniture. All the rooms are filled with different amounts of water.

In order to plan rescue operations, an emergency team requires some data which surveyors collected.

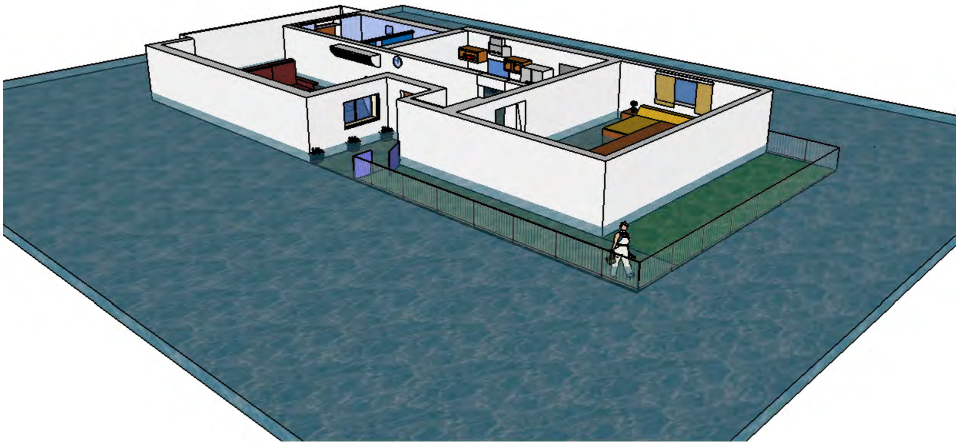


Figure 6.2: Flood event environment

After spatial data collection, the data is stored in a geospatial database using a vector data model[4] (consult (Worboys and Duckham, 2004) for other spatial data models) in which spatial objects were qualified by giving a unique name to each of them (e.g. socket1 and power cable2). Furthermore, the spatial objects are represented by abstract entities of various dimensions such as sockets (as 0-dimensional regions), power cables (as 1-dimensional regions), walls (as 2-dimensional regions) and floodwater (as 3-dimensional regions) in this modelling. Our formalism allows this particular scene representation in different ways according to the level of detail required. Some of the spatial objects and their corresponding dimensions are listed in Table 6.2[5]. Also, the vector model keeps some (mereo)topological information of the collected data such as arc-node topology, polygon-arc topology and etc (see (Worboys and Duckham, 2004) for more

[4]The vector data model is selected because it represents geographic features similar to maps.

[5]Here we considered the residential area a 2D object which is formed by a collection of buildings footprints. According to this assumption, every building (including the house) is part of the residential area if its footprint is part of the area. This assumption rolls out a counter-intuitive idea of placing the house as a three-dimensional object *in* a residential area as a two-dimensional object.

Table 6.2: Spatial objects' dimensions in the dataset.

Spatial Object	**0-dimensional**	**1-dimensional**	**2-dimensional**	**3-dimensional**
Cable's endpoints	✓			
Plug	✓			
Socket	✓			
Power Cable		✓		
Power Cord		✓		
Drainage and sewage systems		✓		
Fuse board			✓	
House Path			✓	
Street			✓	
Carpet			✓	
Balcony			✓	
Walls			✓	
Flood surface (water level)			✓	
Building's footprint			✓	
Residential area			✓	
Flood zone			✓	
County			✓	
floodwater				✓
House’s room				✓
House appliances				✓
House				✓

information). For example, a (mereo)topological table might have information about the house's outer walls defined as boundary of a green area. The (mereo)topological information was manually mapped into our proposed relations for future inference purposes. The assigned relation is one of the following ones: discrete ($\boldsymbol{D}$), externally connected ($\boldsymbol{EC}$), overpass ($\boldsymbol{OV}$), interior contact ($\boldsymbol{IC}$), interior part ($\boldsymbol{I}$), tangential proper part ($\boldsymbol{TPP}$), equal ($\boldsymbol{EQ}$), has an interior part ($\boldsymbol{I}^{-1}$), and has a tangential proper part ($\boldsymbol{TPP}^{-1}$) or their disjunction.

6.2.2 Reasoning Problem for the Flood Event Scenario

As we said, a spatial information system can help the rescue teams by providing their required information. It formulates their geographical queries and feeds them to an inference mechanism that finds answers. Some possible queries that an emergency management personnel might be interested in be as follows:

(A) Are there any residential areas **in** the flood zone?

(B) Has water fully/partially **run into** the house?

(C) Has water **entered** the kitchen?

(D) Is the property **accessible**?

(E) Is the property's basement **covered with** water?

(F) Are the property drainage and sewage pipes **inundate by** flood?

(G) Is a power fuse board **above** the water level?

(H) Are any power cables cut?

- (a) If yes, is the cable **submerged in** water?
    - i. If yes, is there any part of the cable **connected to** the fuse board?
    - ii. If no, are any endpoints of the cable **near** the water level?

(I) Are any power cables **in** the water?

(J) Are all the sockets **out** of the water?

(K) Are the properties' appliances **standing in** water?

- (a) If yes, are any wet appliances' power plugs **inserted in** sockets there?

The focus of every query is on a specific spatial relation (shown in bold letters in the above list) between a pair of spatial objects. Table 6.3 shows the pair of objects and their relationships in every query. As can be seen, these queries cover almost all of the combinations of relations between regions of various dimensions.

Like the spatial database translation, the queries also must be translated into the introduced mereotopological relations in our proposed theory for the same purpose, i.e. reasoning. The mapping from natural language queries (listed in Subsection 6.2.1) into our mereotopological relations has been done by considering various resources. At first, the meanings of the spatial terms were looked up in the Merriam-Webster dictionary (Merriam-Webster editors, 2020). After that, we reviewed literature in which the mapping between the previously introduced mereotopological relations (in other formalisms) and people's use of natural language relations has been evaluated (Rashid et al., 1998; Schwering, 2007; Ferreira and Delazari, 2019). By considering these resources, and our knowledge and experience, the equivalent relation is extracted for every natural language spatial term from our multidimensional formalism. This mapping is shown in Table 6.4.

The queried spatial relations and their meanings from the dictionary are listed in the first and second columns of Table 6.4, respectively. Regardless of the words' functionality (e.g. preposition), we can see an overlap between some of their meanings of the words. For instance, *run into* and *enter* express very similar meaning, i.e. go or flow into something (see Table 6.4). However, checking their equivalence in the literature might help us provide a stronger foundation for the mapping. Reviewing the literature on the queried spatial terms is summarized and presented in the third column of Table 6.4. Although some of the terms have not been investigated in the literature, the majority of them have been studied, and the results of these studies are adopted here. These interpretations are mostly limited to the spatial entities representing lines and polygons on a plane, but they are extendable to the higher dimensional entities and spaces.

The meaning of the first queried term, "in" (i.e. located in), is compatible with its interpretation in the literature, i.e. contained in (Schwering, 2007). Both expressions focus on full coverage of one of the objects by another one. So, this spatial configuration maps into our proposed interior relation (**I**). When an object "run into" the other object, they are in contact. However, some part of the former object is inside the latter, and the rest of it is outside. The "enter" expression also has the same interpretation as "run into". By following (Schwering, 2007; Rashid et al., 1998), the represented spatial arrangements of these phrases are mapped into the overpass relation (**OV**) in our formalism.

The "connected to" spatial term has more than one interpretation in the literature. The common property among all these interpretations is overlapping areas between one of the objects' boundaries and any part of the other (Rashid et al., 1998). However, there must be some uncommon parts between the objects. It is equivalent to the imbricate relation (***IMB***) in our formalism, and its interpretation might be any of this relation's refinements (i.e. externally connected ***EC***, overpass ***OV***, or interior contact ***IC*** relations). By considering the relevant query (named (H) in the list of queries) in which a power cable must pass the boundary of the fuse board to join the inner branch board, the "connected to" term mostly

Table 6.3: Objects and their spatial relationships in the given queries.

Query	Object1 and its dimension	Spatial relation	Object2 and its dimension
Are there any residential areas in the flood zone?	residential area(2D)	in	flood zone(2D)
Has water fully/partially run into the house?	floodwater(3D)	run into	house(3D)
Has water entered the kitchen?	floodwater(3D)	entered	kitchen(3D)
Is the property accessible?	house path(2D)	accessible	house(3D)
Is the property's basement covered with water?	basement(3D)	covered with	floodwater(3D)
Are the property drainage and sewage pipes inundated by flood?	drainage and sewage pipes(1D)	inundated by	floodwater(3D)
Is a power fuse board above the water level?	fuse board(2D)	above	water level(2D)
Are any power cables cut?			
If yes, is the cable submerged in water?	power cable(1D)	submerged in	floodwater(3D)
If yes, is there any part of the cable connected to the fuse board?	power cable(1D)	connected to	fuse board(2D)
If no, are any endpoints of the cable near the water level?	endpoints(0D)	near	water level(2D)
Are any power cables in the water?	power cables(1D)	in	floodwater(3D)
Are all the sockets out of water?	sockets(0D)	out	floodwater(3D)
Are the properties' appliances standing in water?	house appliances(3D)	standing in	floodwater(3D)
If yes, are any wet appliances' power plugs inserted in sockets there?	power plugs(0D)	inserted in	sockets(0D)

Table 6.4: Equivalence of spatial terms from Merriam Webster dictionary and literature in natural language processing.

Queried spatial relations	Meaning in dictionary	Equivalent from literature	Literature
in	Located in; surrounded by(Preposition)	An object is completely contained in the interior of another object	(Schwering, 2007)
run into	To flow into(Verb)	Going from an object's interior to its outside	(Rashid et al., 1998)
enter	To come or go into(Verb)	Going from an object's interior to its exterior	(Schwering, 2007)
accessible	Capable of being reached(Adjective)	Not provided	
covered with	To lay or spread something over(Verb)	A thing which lies on, over, or around something.	Not provided
inundated by	Fill or cover completely, usually with water(Verb)	Not provided	
above	At a higher position than(Preposition)	Not provided	
submerged in	Covered with water(Verb)	Not provided	
connected to	Be or become joined or united or linked(Verb)	1. An object goes from outside of another object to its boundary 2. Going from an object's interior to its outside 3. The interior of an object is in contact with the other ones' boundary	(Rashid et al., 1998)
near	Not far distant in time, space, degree or circumstances(Preposition)	An object is completely outside of another object	(Ferreira and Delazari, 2019)
out	Outside or external(Adjective)	An object is completely outside of another object	(Rashid et al., 1998)
standing in	cover, substitute(Verb)	Not provided	
inserted in	Place onto or put into something(Verb)	Not provided	

can be interpreted via the overpass relation $\boldsymbol{OV}$ here.

Both "near" and "out" show a pair of (mereotopologically) disjoint objects (Ferreira and Delazari, 2019; Rashid et al., 1998). Although there might be some differences in their (implicit) metric information, we are not interested in this information in our formalism. Thus, these terms have been represented by our discrete relation ($\boldsymbol{D}$).

In addition to the investigated expressions in the literature, there is a set of "non investigated" spatial query terms here including "accessible", "covered with", "inundated by", "above", "submerged in", "standing in", and "inserted in". These terms have been interpreted based on their meanings and our understanding. The "accessibility" of a spatial entity refers to its reachability. For instance, a house is reachable if there exists a (usable) path to it. In other words, we are interested in the connectivity (or, more specifically, external connectivity ($\boldsymbol{EC}$)) between the house and its path.

An entity is "covered with" another entity if the other one entirely surrounds it. For example, the disaster manager would like to know whether the basement is full of water in the relevant query (named (E) in the list of queries). So, the spatial configuration between the basement and the floodwater can be represented by the "part of" relation ($\boldsymbol{P}$). Moreover, "inundated by" could represent any overlapping spatial configurations between a pair of objects. It is an equivalent of the imbricate relation ($\boldsymbol{IMB}$) in our theory. "Submerged in" is specifically about elevation but allowing any kind of parthood ($\boldsymbol{P}$).

Also, the similarity among the mereotopological meanings of "near","out", and "above" expressions motivated us to consider the same relation, i.e. the discrete relation, for "above" by ignoring its implicit directional information[6]. The "stand in" statement has various synonyms, like "cover" and "substitute". So, the relevant queried relations (named (K) in the list of queries) can be equivalent to tangential proper part ($\boldsymbol{TPP}$) like the "covered with" expression. However, the "stand in the water" phrase usually depicts an arrangement in which an object is over-passing another one ($\boldsymbol{OV}$). Finally, the "inserted in" term describes a pair of entities in which one of them is closely fitted inside the other (that are power cable plugs and sockets in the (K-a) query). This interpretation is expressed by the equality relation ($\boldsymbol{EQ}$).

The summary of mappings between the natural-language spatial relations used in the queries and the

[6]The directional information is out of the scope of this work.

(a) Queried spatial relations and their equivalent topological relations.

Queried spatial relation	Mapped relation
in	I
run into	OV
entered	OV
accessible	EC
covered with	P
inundated by	IMB
above	D
submerged in	P
connected to	OV
near	D
out	D
standing in	OV
inserted in	EQ

(b) Objects and their mapped spatial relationships in the given queries.

Object1 and its dimension	Spatial relation	Object2 and its dimension
residential area(2D)	I	flood zone(2D)
floodwater(3D)	OV	house(3D)
floodwater(3D)	OV	kitchen(3D)
house path(2D)	EC	house(3D)
basement(3D)	P	floodwater(3D)
Drainage and sewage pipes(1D)	IMB	floodwater(3D)
fuse board(2D)	D	water level(2D)
power cable(1D)	P	floodwater(3D)
power cable(1D)	OV	fuse board(2D)
endpoints(0D)	D	water level(2D)
power cable(1D)	I	floodwater(3D)
sockets(0D)	D	floodwater(3D)
house appliances(3D)	OV	floodwater(3D)
power plugs(0D)	EQ	sockets(0D)

Table 6.5: Mapping between queries and topological relations.

mereotopological relations in our formalism is shown in Table 6.5a. According to these mappings, the given queries can be expressed via the language of our theory. For instance, the query "*Are there any residential areas* ***in*** *the flood zone?*" is translated to "**I** (residential area, flood zone)". The translations of the given queries are shown in Table 6.5b. They will feed into the inference system, i.e. algebraic-closure algorithm, later.

6.2.3 Reasoning with the Multidimensional Formalism

As mentioned, one advantage of using a formalism that supports a set of JEPD relationships is the possibility of using constraint-based reasoning. This reasoning method can compute answers to answer-seeking queries in intelligent systems. In this section, we will talk about the process of using the reasoning technique on the collected data from the flood event scenario to find answers for the given queries listed in Subsection 6.2.2.

The process by which the queries are answered is shown in Figure 6.3. First, the collected qualitative spatial data (discussed in Subsection 6.2.1) form a constraint network (explained in Section 6.1). Then, each qualitative question is reformulated (see Subsection 6.2.2) and added to the network. After that, the algebraic-closure algorithm (explained in Figure 6.1) propagates the given constraints to find the feasible answer(s) for a given query. The algorithm returns a modified network of constraints representing possible relations between a pair of queried spatial entities.

The returned set might have a single member or a number of members that reveal a consistent scenario. If the set *only* has the queried relation as its member, it is interpreted as a precise "Yes" answer to the given query. In the case of returning a multi-member set, the sought answer *might* be "Yes" if the queried relation is a member of the set. If none of the above conditions is met, the answer to the asked-query is "No".

To apply the reasoning algorithm, we used the publicly available `SparQ` reasoning toolbox developed at the University of Bremen (Wallgrün et al., 2007) to enforce the algebraic-closure algorithm. We supplied the toolbox with our formalism by introducing it via its arity, list of JEPD relations and their corresponding converse relations, identity relation, and the (unrestricted) composition table. More details about the

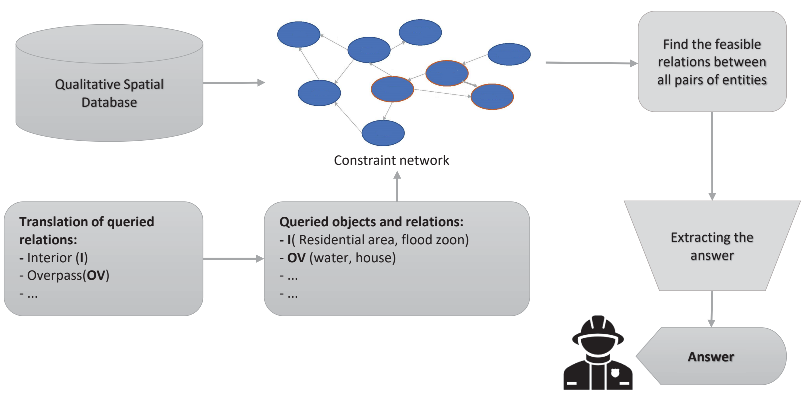


Figure 6.3: Process of finding an answer for a given query.

toolbox are provided in (Dylla et al., 2017).

The qualitative spatial information (i.e. the collected data along with every query) is translated to a `SparQ` command as per the following sample:

```
sparq constraint-reasoning multi-dim algebraic-closure
((Kitchen TPP House)(Kitchen OV Water)(Fridge I Kitchen)(Fridge EC Stove)
(PowerCore5 EC Stove)(Plug5 D Stove)(plug5 TPP PowerCore5)(Plug5 EQ Socket5)
(ResidentalArea I FloodSurface)(FloodSurface TPP Water)...)
```

where every inner parenthesis shows a pair of objects and their relationships using infix notation. The result of this reasoning command is a refined network.

The modified constraint network has the same representation as the one given in the toolbox. However, we have depicted the network (for the first query as an example) to provide a better understanding of the answer-seeking process for a reader (see Figure 6.4). As you remember, the first query was "Are there any residential areas in the flood zone?" which is then mapped to the following statement **I**(`Residential area`, `Flood zone`). The query's network has a definite "Yes" reply to the first query since the returned set only contains the interior relation (**I**).

It is also possible that the response to a query may be "possibly yes". For example, for the second question (i.e. "Has water fully/partially run into the house?"), the algorithm returned several relations, i.e. $\{\boldsymbol{IC},\ \boldsymbol{I}^{-1},\ \boldsymbol{OV},\ \boldsymbol{TPP}^{-1}\}$, indicating that any of these relations could be valid between the queried entities. Thus, the overpass relation ($\boldsymbol{OV}$) is likely a true relation between "floodwater" and "the house". This is the approach for interpreting all the *possible* cases in Table 6.6.

Some of the queries were mapped to the disjunction of relations such as $\boldsymbol{P}$ and $\boldsymbol{IMB}$ shown in Table 6.6. In these cases, the system's response was checked more cases compared to the queries being mapped to a specific JEPD relation. For instance, the response set for the $\boldsymbol{IMB}$ query includes checking all the specializations of this relation (i.e. $\boldsymbol{IC}$, $\boldsymbol{EC}$ and $\boldsymbol{OV}$).

On the other hand, the non-existence of the queried spatial relations in the output set of feasible relations for the seventh query (i.e."Is the cable submerged in water?") shows a precise "No" answer to

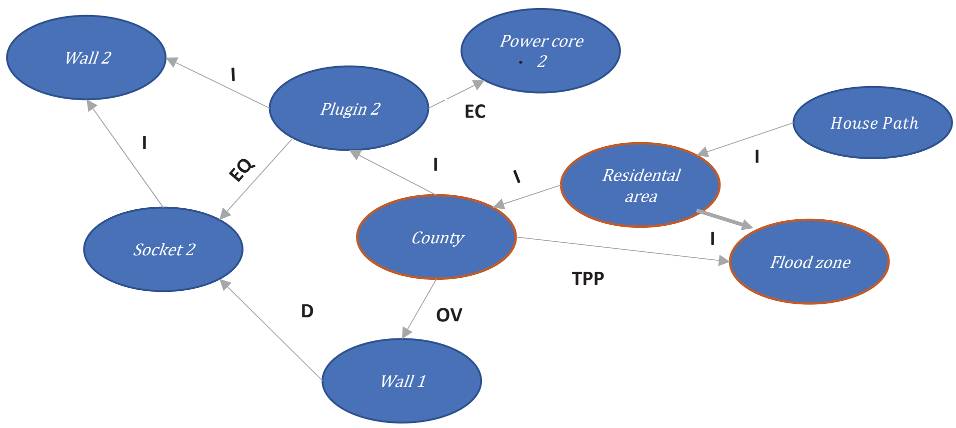


Figure 6.4: Partially depiction of the modified constraint network for the first query.

Table 6.6: Returned answers to the listed queries.

Object1 and its dimension	Spatial relation	Object2 and its dimension	Answer to the query
residential area(2D)	I	flood zone(2D)	Yes
floodwater(3D)	OV	house(3D)	Possibly Yes {IC, I^{-1}, OV, TPP^{-1}}
floodwater(3D)	OV	kitchen(3D)	Possibly Yes {IC, I^{-1}, OV, TPP^{-1}}
house path(2D)	EC	house(3D)	Yes
basement(3D)	P	floodwater(3D)	Possibly Yes {I, TPP}
drainage and sewage pipes(1D)	IMB	floodwater(3D)	Yes
fuse board(2D)	D	water level(2D)	No
power cable(1D)	P	floodwater(3D)	Possibly Yes {P, IMB}
power cable(1D)	OV	fuse board(2D)	Possibly Yes {OV, EC}
endpoints(0D)	D	water level(2D)	Possibly Yes {D, I}
power cable(1D)	I	floodwater(3D)	Yes
sockets(0D)	D	floodwater(3D)	Yes
appliances(3D)	OV	floodwater(3D)	Possibly Yes {IC, I, OV, TPP}
power plugs(0D)	EQ	sockets(0D)	Yes

it. The same reply would be interpreted if the output is an empty set of relations (i.e. found inconsistency in the network). This process of comparing the existence or non-existence of the queried relation in the modified networks of constraints were done manually for the rest of the queries listed in Table 6.5[7]. As the reasoning mechanism could find answers for various types of given queries, it is a reliable method. The result of this comparison is summarized in Table 6.6.

6.2.4 Second Use Case for Reasoning Problem

Map reading is another application of our formalism. Maps provide an abstract representation of space, and their depictions can be used for navigation. Map reading is an important skill for humans and intelligent agents. Maps are designed for various purposes such as highway maps, country maps, hiking maps etc. For example, a map of Big Morongo Canyon Preserve Trail System (Figure 6.5[8]) shows the various connected trails (all linear features) with the trail markers (point features). It also shows areal features including benches, decks, buildings and marsh areas with some trails passing by or through them. Our goal here is to find trail markers of every trail and the marshes to which the trail passes through.

As with the flood event scenario, we first collect and store mereotopological relations among the trails, marshes, buildings etc. Note that most of the trails consist of several parts connected via trail markers. We enumerated every part of a tail based on the order of its markers. For instance, a part of the Marsh trail located between marker 1 and marker 2 is MarshTrail1, while the part between marker 2 and marker 4 is MarshTrail2. Since marshes do not have any specific names on the map, we name them by their nearest trail with the "Marsh" word as a prefix. For example, a marsh near the MarshTrail2 is MarshMarshTrail2.

[7]It is feasible to check the existence or non-existence of the queried relation by machines for a large set of queries.
[8]`https://upload.wikimedia.org/wikipedia/commons/4/4d/TrailMap2.jpg`

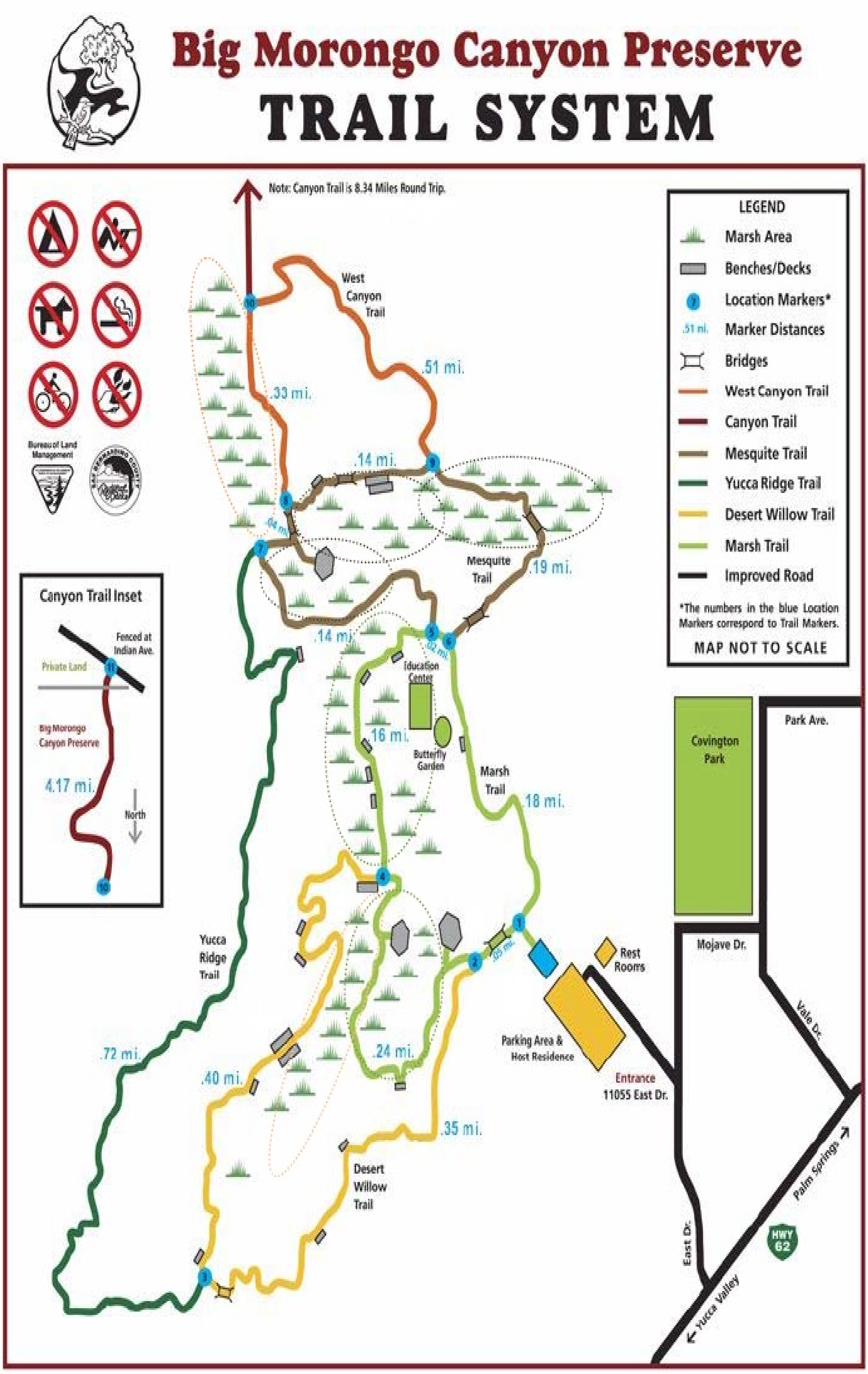


Figure 6.5: Schematic trail map of Big Morongo Canyon Preserve

Sample of the dataset is as bellow:

```
sparq constraint-reasoning multi-dim algebraic-closure
((MarshWestCanyonTrail1 IC WestCanyonTrail1)(WestCanyonTrail1 TPP
Marker8)(WestCanyonTrail1 EC MesquiteTrail3)(MarshTrail1 D WestCanyonTrail1)(MarshTrail2
OV MarshMarshTrail2) ...)
```

Note that we do not store all the spatial information represented on the map, though it is small. We want to simulate large maps with many entities for which it is not practical to collect and store all the quantitative data.

As the next step, we map the queried relation to our defined mereotopological relation(s). Our goal mereotopological relation is “passing through”. The meaning of “passing through”, according to the Meriam Webster dictionary, is making something move through something else, or infiltrate. This meaning is close to the definition of the overpass relation (**OV**) in our formalism. Now, the goal would be paraphrased as `(Trail OV Marsh)` in which Trail is one of the trails (or trails part) and Marsh would be one of the marshes near the relevant trail parts. We will iterate our query over trails and marshes and feed every iterated query along with the dataset into a constraint-based reasoner. For instance, the first queried relation is `(MarshTrail1 OV MarshMarshTrail2)` the reasoning algorithm generates a constraints network like Figure 6.4 over the trail map dataset and returns a singleton set containing only a disconnect relation **D**. The same approach would be repeated over all the trails and marshes. The result of the reasoning algorithm for every pair of trail and mash is shown in Table 6.7. According to this table, MarshTrail2 is possibly passing through (i.e. **OV**) Marsh MarshTrail2, but MarshTrail3 and MesquiteTrail4 would pass through (i.e. **OV**) Marsh MarshTrail3 and Marsh MesquiteTrail4, respectively.

To retrieve all the trail markers of the overpassed trails, we need to refine the returned set of spatial entities (i.e. MarshTrail2, MarshTrail3, MesquiteTrail4, MarshMarshTrail2, MarshMarshTrail3, MarshMesquite-Trail4) to the linear ones. Thus, we logically define a linear feature:

Definition 6.2.1. $\forall x\ \boldsymbol{LinearFeature}(x) \longleftrightarrow \exists y_1 \exists y_2(\boldsymbol{bdy}(x) = y_1 \wedge \boldsymbol{bdy}(x) = y_2 \wedge \boldsymbol{D}(y_1, y_2))\wedge$

Table 6.7: Result of applying constraint-based reasoning over the queried trails and marshes

Trail\Marsh	Marsh West Canyon Trail1	Marsh Mesquite Trail1	Marsh Mesquite Trail2	Marsh Mesquite Trail3	Marsh Mesquite Trail4	Marsh Desert Willow Trail2	Marsh Marsh Trail2	Marsh Marsh Trail3
WestCanyonTrail1	IC	D	IC	D	D	D	D	D
WestCanyonTrail2	D	D	D	D	D	D	D	D
MesquiteTrail1	D	IC	D	D	D	D	D	IC
MesquiteTrail2	IC	IC	D	IC	D	D	D	D
MesquiteTrail3	D	D	D	IC	D	D	D	D
MesquiteTrail4	D	D	D	D	OV	D	D	D
YuccaRidgeTrail	D	D	D	D	D	D	D	D
DesertWillowTrail1	D	D	D	D	D	D	D	D
DesertWillowTrail2	D	D	D	D	D	IC	D	D
MarshTrail1	D	D	D	D	D	D	D	D
MarshTrail2	D	D	D	D	D	IC	IC, OV	D
MarshTrail3	D	D	D	D	D	D	D	OV
MarshTrail4	D	D	D	D	D	D	D	D

$\forall y_1 \forall y_2 \forall y_3 (\boldsymbol{bdy}(x) = y_1 \wedge \boldsymbol{bdy}(x) = y_2 \wedge \boldsymbol{bdy}(x) = y_3 \rightarrow y_1 = y_2 \wedge y_2 = y_3 \wedge y_1 = y_3)$

And then check this attribute over the returned set of entities. The following features would satisfy the linearity property MarshTrail2, MarshTrail3, MesquiteTrail4 and their boundaries (returned by our ***bdy*** function (ref. Definition 4.1.13) would be the queried markers.

6.3 Summary

Although much of the work in multidimensional mereotopological representation and reasoning has concentrated on spatial representation, a reasoning mechanism over the represented qualitative data-set is also important. In this Chapter, we looked at a reasoning mechanism for query-answering purposes that were supported by our proposed theory.

Our case study was illustrated via a geographical data-set containing geometries of various dimensions and their pairwise multidimensional mereotopological relations representing a flood event scenario. The given queries were answered by means of the constraint-based reasoning technique (i.e. algebraic closure algorithm), which specified whether a queried relation is consistent with, inconsistent with or a consequence of the data-set. The query-answering performance was evaluated in terms of whether any answer was found for a given query and how informative the returned answer was. Interestingly, answers were returned for all the given queries. The answers' correctness was then visually checked in three-dimensional modelling of the scenario. The results demonstrated the practicality of the reasoning method. The success of this reasoning method makes our proposed formalism transferable to other uses.

Chapter 7

Summary and Outlook

In this thesis, we mentioned that spatial reasoning is an important task in artificial intelligence (AI), enabling machines to mimic human common-sense. One of the primary factors contributing to this interest is that spatial reasoning provides qualitative spatial information similar to humans spatial representation of the world. A particular form of the qualitative spatial relationships (i.e. mereotopological relations), in which spatial relations are preserved under translation, rotation or scaling, is broadly used for representing spatial information in AI. These relations have been promoted as a fundamental connector between human spatial perception and intelligent agents (Randell and Cohn, 1989; Varzi, 1994; Asher and Vieu, 1995; Eschenbach and Heydrich, 1995; Borgo and Masolo, 2010; Knauff et al., 1995, 1997; Klippel et al., 2013). The traditional mereotopological theories only represent the relationships between equidimensional spatial entities like a pair of line segments or polygons. However, some applications, such as map reading, are required to describe spatial relations among objects of various dimensions, like a road (as a line segment) and a parcel (as a polygon), which are not supported by the traditional theories. This necessity led to the introduction of mereotopological theories (including Smith's work (Smith, 1996; Smith et al., 2000), Take-dim (Galton, 1996), the INCH calculus (Gotts, 1996), CODI family (Hahmann, 2013, 2020) and RCC*-9 (Clementini and Cohn, 2014)) accepting entities of various dimensions in recent studies. Nevertheless, the different levels of expressivity in the language of these theories can not support an efficient reasoning strategy. Indeed, these theories do not balance the trade-off between representation expressivity and the computational cost of reasoning. Theories using the language of higher-order logic (HOL) are highly expressive, but their reasoning mechanism is highly computationally expensive. Although theories proposed in the language of first-order logic (FOL) are not expressive as the HOL ones, their reasoning mechanism is still computationally expensive.

The current study aimed to fill this gap by proposing a multidimensional mereotopological formalism in which representation expressivity and computational reasoning cost are balanced. Our theory developed a cognitively plausible formalism to represent mereotopological relations among spatial objects of dimension zero to three and constructed the composition table(s) necessary for constraint-based reasoning.

In this Chapter, we will give a general overview of our main contributions and summarize the important points of every Chapter in Section 7.1. Then we will answer the research questions raised in the Introduction in Section 7.2. After pointing out the limitations of our proposed theory in Section 7.3, we will refer to its possible extension paths in Section 7.4. Section 7.5 will provide a synthesis of the key points of the research.

7.1 Summary of Contributions

To design a multidimensional mereotopological theory for use with efficient reasoning, we made the following contributions:

I *A comprehensive review of the existing multidimensional mereotopological theories*: A detailed account of the existing multidimensional mereotopological theories were provided in Chapter 2. Their properties were compared from the representation and reasoning points of view. We considered their representative language, foundation primitive(s), supported dimension of the entities, mereotopological and topological properties, and reasoning mechanisms. We also talked about the importance of representing the boundary of spatial objects in human perception and referred to the difficulty of its representation in mereotopological theories. Based on this comparison, we understood the role of evaluated aspects of the logical theories and how to use them to meet the applications' requirements. For instance, we realized the importance of FOL both in representing spatial knowledge and supporting efficient reasoning. To the best of our knowledge, such a comparison has not been made before. The comparison was the basis of our proposed theory in this work and enabled us to design it according to the existing research gap. Moreover, it can provide a foundation for expanding the existing theories or even the establishment of a new multidimensional mereotopological theory.

II *Multidimensional mereotopological formalism*: We specified the ontological properties of the spatial entities of various dimensions (independent of their dimensions) and their mereotopological relations in Chapter 4. The theory has been represented by the use of first-order logic in which the entities include point, linear, areal, and volume features. The proposed mereotopological relations between the entities were axiomatized based on two spatial primitives: *part-of* and *boundary part* relations. The former holds when a common part exists between a pair of entities. Meanwhile, the latter represents the boundary of an entity. The proper axiomatization of the primitives provided a basis for defining a set of further mereotopological relations including discrete ($\boldsymbol{D}$), overlap ($\boldsymbol{O}$), imbricate ($\boldsymbol{IMB}$), externally connected ($\boldsymbol{EC}$), interior contact ($\boldsymbol{IC}$), overpass ($\boldsymbol{OV}$), the converse of the part-of, equality ($\boldsymbol{EQ}$), proper part ($\boldsymbol{PP}$), interior part ($\boldsymbol{I}$), tangential proper part ($\boldsymbol{TPP}$) and their converses (i.e. $\boldsymbol{PP}^{-1}$, $\boldsymbol{I}^{-1}$, and $\boldsymbol{TPP}^{-1}$). A relational lattice represented the inheritance structure among the introduced relations. However, a subset of these relations, including $\boldsymbol{D}$, $\boldsymbol{EC}$, $\boldsymbol{IC}$, $\boldsymbol{OV}$, $\boldsymbol{EQ}$, $\boldsymbol{TPP}$, $\boldsymbol{I}$ and their converse (i.e. $\boldsymbol{I}^{-1}$, and $\boldsymbol{TPP}^{-1}$), can partition the whole domain. These relationships form a set of nine base (or JEPD) relations in our proposed formalism. The properties and consistency of the theory were then verified with the help of automatic theorem provers and finite model finders. Prior to this study, there was no multidimensional mereotopological theory in which a set of JEPD relations was introduced, and thus no theory supported an efficient reasoning strategy (i.e. constraint-based reasoning) for mereotopological relations among multidimensional objects.

III *Evaluating the cognitive adequacy of the multidimensional mereotopological formalism*: In Chapter 5, we validated the set of introduced mereotopological relations to see whether they are consistent with human cognition. We designed and made a set of models showing different spatial configurations between a pair of objects representing various dimensions, i.e. Line-Line, Line-Surface, Line-Volume, Surface-Surface, Surface-Volume, and Volume-Volume sets. Then we asked participants (of different genders, age groups, educational backgrounds, and languages) to group the given models based on the similarity of the connection between the objects in them and provide descriptions for their grouping. The collected data from the grouping task was then analyzed both quantitatively and qualitatively. In the quantitative analysis, we applied hierarchical clustering on the overall similarity matrix constructed from the generated groups in every set of models. The clustering outputs were shown in the form of dendrograms in which the optimal number of groups were extracted. We used reflexive thematic analysis for qualitative analysis of the descriptions of groups of similar models and identified common themes that respondents used to form groups. The results of both methods demonstrated that our formalism is a reliable model of people's spatial conceptualization in a multidimensional space, although the agreement among participants in our study was much more convincing for some relations than for other relations. As an example of the less convincing relations, we can name the cross-relation a member of our initial set of mereotopological relations. This relation could not be conceptualized in the empirical investigations. Thus, it triggered some changes in our formalism, which led to the relation removal in the current theory. Before this work, evidence for the cognitive plausibility of any multidimensional mereotopological theories was not available. The availability of the mereotopological representation of multidimensional space close to human cognition facilitates human interaction with spatial information systems.

IV *Reasoning over the multidimensional mereotopological formalism*: We developed an efficient reasoning strategy, i.e. constraint-based reasoning (in Chapter 6). In our study, the reasoning problem is translated to a constraint satisfaction problem (CSP). Since the language for a CSP representation is based on constraints, it allows a tractable algorithm for reasoning, which is not available for reasoning over existing FOL multidimensional theories. Here, the problem space was represented by the finite mereotopological relations (or constraints) over variables (instantiated by the spatial objects) and solved by the algebraic closure algorithm. The algorithm searched for solutions by looking up the composition tables. The composition table was generated by applying the composition operator over a pair of JEPD relations over a common object. The operator demonstrated a disjunction of all the possible introduced mereotopological relations holding between a pair of given objects over the common one. We also generated separate composition tables based on the boundary properties of the objects. The inference mechanism was then tested on a case study, which was a hypothetical flood event occurring in a residential area. In this case study, spatial data describing the event was translated into our proposed mereotopological relations between the spatial objects. Also, the rescue team's questions were mapped to the introduced mereotopological relations to simulate the querying procedure. All qualitative data was fed into the constraint-based reasoning algorithm to facilitate query answering procedure (i.e. automated reasoning). This scenario was a simple example of the potential real-world applications of our formalism. By adopting our proposed formalism, finding answers for the given queries over spatial entities of various dimensions would be answerable with less computational complexity as a well-known efficient algorithm has generated them. In contrast, the accuracy of the answers is still practical. Future studies about the computational properties of reasoning over our formalism (either formal or empirical evaluations)) that have not been studied here would identify its further characteristic.

7.2 Answering the Research Hypothesis

Based on the research carried out and the findings obtained in this thesis, we can answer the research questions as follows:

I *Introducing a set of dimension-independent mereotopological relationships, which is cognitively plausible and supports constraint-based reasoning to describe the relative locations of spatial entities of various dimensions (i.e. from zero to three dimensions), is possible in a three-dimensional space.*

The set of mereotopological relationships to describe the relative locations of spatial entities of various dimensions (i.e. from zero to three dimensions) is discrete (***D***), overlap (***O***), imbricate (***IMB***), externally connected (***EC***), interior contact (***IC***), overpass (***OV***), equality (***EQ***), the part-of (***P***), proper part (***PP***), interior part (***I***), tangential proper part (***TPP***) and their converses.

II *We can identify a subset of jointly exhaustive and pairwise disjoint (JEPD) multidimensional relations that generate an algebra for constraint-based reasoning.*

Nine of the introduced relations including discrete (***D***), externally connected (***EC***), interior contact (***IC***), overpass (***OV***), equality (***EQ***), interior part (***I***), tangential proper part (***TPP***) and their converses have the JEPD properties.

III *Our proposed set of multidimensional mereotopological relations has a subset of nine relations (excluding crosses relation) which closely aligns with people's conceptual understanding of spatial relationships among objects of various dimensions (i.e. cognitively adequate).*

We can confirm that empirical evidence provides adequate grounds for the assuming that our proposed multidimensional mereotopological relations reflect the human perception of spatial relationships between objects of different dimensions.

IV *Constraint-based reasoning over a set of multidimensional mereotopological relations is possible in a dimension-independent way.*

Our introduced mereotopological relations allow efficient reasoning, i.e. constraint-based reasoning. We also generated the compositions of the relations and stored them in a composition table.

7.3 Limitations

Although we intended to do comprehensive research based on the available resources, this study has some limitations, as listed below, which can lead to meaningful future research directions:

- In our formalism, we mostly focused on binary mereotopological relations expressing the spatial position of a pair of simple entities (i.e. one-piece and without any holes). There are, of course, some interest in expanding the domain of discourse by including multi-piece entities and entities with holes. Their consideration allows the introduction of much finer-grained relations than the existing mereotopological relations. The set of axioms and definitions can introduce these relations. Because of the complexity in representing a multidimensional space, multi-pieces entities entities with holes were out of the scope of this study. Future work needs to look at whether the axioms and definitions are extendable to other types of entities.
- Our second limitation is closely related to the previous one. We have not introduced any Boolean operators such as sum, product, differentiate, and complement in our formalism since the output of these operations might be a multi-piece entity that was not accepted in our domain. The operators have been defined in some of the existing mereotopological theories(Hahmann, 2020; Galton, 1996). However, their outputs are not always definable (i.e. would be an empty set) in all of these theories (i.e. an empty set (or ***null*** entity) has not been defined in them). In contrast, we have defined the null entity making the operators always definable (or total).
- A series of topological functions are required to assign precise semantics to the user's queries. These functions reflect the complexity and variety of spatial entities and their relations being not expressible via mereotopological relations. For instance, properties of a region consisting of several disjointed parts can be described in terms of *separation number*, which returns the maximum number of mutually disconnected parts of a region. This concept is also helpful in describing an intersection of a pair of regions that consists of several segments. Knowing some information about the dimensionality of the intersection part(s) is another valuable example of the topological functions. However, these concepts are not definable in our mereotopological relations due to a lack of supporting axiomatization.
- In our empirical investigation, we had difficulties accessing people to participate in our study. Increasing the number of participants might provide more substantial evidence for the claim of cognitive adequacy of our proposed formalism. Another interesting challenge is how participants group models when they consist of multiple pieces or have holes. It would show us whether participants continue to categorize the given models according to our proposed mereotopological relations. Answering this question will also assess the adequacy of the regions' level of detail necessary to define spatial relationships.
- Our main concern in the formalism was expressing the static characteristics of space instead of establishing a calculus to describe how the spatial configurations of regions change over time. However, some domains require dynamic reasoning in which possible transitions among the relations are

considered. Further study is necessary to extract the continuous transitions among the introduced mereotopological relations (Aameri and Gruninger, 2013; Davis, 2001).

7.4 Future Perspectives

In addition to the further development of our theory prompted by the limitations already described, future work would benefit from extending the theory's representation and reasoning facets.

Many disciplines, such as GIS, require an enhanced representation of spatial objects, including objects with uncertain boundaries, such as urban areas, clouds, noise, areas of specific vegetation, etc. The concern would be that our current proposed theory can be used or expanded to model these types of entities. This issue might be tackled either by extending the axioms and definitions to represent this type of entities, or preassuming a minimum and maximum extents for the vague boundaries (Smith and Mark, 1998; Smith et al., 2000) and then extracting the possible mereotopological relations between these types of entities. Both of these approaches would improve the representation aspects of the formalism.

Increasing the expressivity of the theory and introducing a set of finer-grained relations would pose challenges for the construction of a composition table. Of course, one can employ any first-order theorem prover; however, the enriched representation makes this approach impractical for many computational problems. Thus, a researcher may need to look for alternative approaches like encoding the theory in propositional logic, or extracting a tractable subset of the introduced relations.

Since mereotopological relations play an important role in the semantics of natural language expressions describing the relative location of spatial objects, one possible direction for our formalism could be to work on its semantics. This would be achieved by constructing a taxonomy of entities or expanding the set of relations that express the semantic structures required in natural language. Additional axioms, specifically existential ones, for the theory would be required. The interpretation of the existential quantifier axiom requires (at least) an entity in the model to substitute the quantifier's bounded variable. This interpretation is compatible with the semantics of many (pro)nouns in English sentences in which (at least) one entity in the domain of discourse must be selected. So, one way to treat the (pro)nouns is to translate them into existentially quantified variables ranging over the entities in the model.

7.5 Conclusion

Throughout this thesis, we have shown that abstracting out useful invariant properties of spatial objects of various dimensions in human perception is a fundamental aspect of understanding the nature of common-sense spatial knowledge. One of these properties is mereotopological relationships between spatial objects, which are preserved even independent of their dimension. We proposed an ontological system in which these relationships were represented. Because of the introduction of a subset of relations with the JEPD characteristics in the formalism, our proposed system develops a composition table that supports an efficient reasoning mechanism. Also, the proposed theory was validated to show its cognitive adequacy , which makes it a user-friendly system. Thus, we can say that we developed an extensive theory to express multidimensional space.

To capture more common-sense aspects of space, the formalism needs to be extended by either addressing the limitations or expanding its representation and reasoning aspects. The usage context of the theory might be considered as an extension criterion; for instance, enhancing the expressive power of the theory by axiomatizing the shape of the spatial objects or improving its computational efficiency.

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Appendix A

Multidimensional Mereotopological Theories And Their Comparison

A.1 Multidimensional Meretopological Theories

The mereotopological theories introduced in Subsection 2.1.1 are all restricted to entities that have the same dimensions and that are located in an embedding space of the same dimension. Consequently, they do not accommodate regions of different dimensions such as boundaries of regions as dependent entities or other independent lower-dimensional entities. The insistence on dimensional homogeneity by these theories is the result of certain paradoxes arising when allowing entities of different dimensions to co-exist. Assume as an example a land parcel (as a two-dimensional entity) on a map and its boundary (as a one-dimensional entity). If they are both assumed as entities of a domain, what is the spatial relationship between them? Is the peel part of the apple or not? If yes, what is the difference between the whole apple and the apple without its peel? If the peel is not part of the apple, is it part of the air? These questions arise when abstract lower-dimensional entities are permitted.

On the other hand, people often describe the relationships between spatial entities in a way that treats them as being of different dimensions, and in particular, spatial databases and geographic information systems require that entities be conceptualised as geometry types of different dimensions. For instance, people say "a road (as a one-dimensional entity) goes into a park (as a two-dimensional entity)" or "a road (as a one-dimensional entity) borders a park (as a two-dimensional entity)". A multidimensional mereotopological framework should be able to provide formal descriptions of such situations. Thus, a few researchers have tried to increase the expressivity of the mereotopological theories to describe the spatial arrangements of entities of various dimensions.

While the majority of the equidimensional theories were proposed in the language of *first-order logic*

(FOL) (in which the variables and quantifiers only range over individuals), more expressive theories sometimes abstract properties of the entities by their type in a multidimensional space. This has resulted in the introduction of theories using *higher-order logic* (HOL) (Smith, 1996; Smith et al., 2000; Galton, 1996)in addition to those using FOL(Gotts, 1996; Hahmann, 2013; Clementini and Cohn, 2014; Hahmann, 2020). In HOL, the wide range of variables and quantifiers over individuals, relations and functions can be formalized which leads to the expression of infinite sets of properties and allows infinite fusion.

This section reviews the axiomatisations of the few proposed multidimensional mereotopological theories in the literature both in FOL and HOL. We will survey and compare them from different points of view and we summarize them in Table Table A.1.

A.1.1 Multidimensional Meretopological Theories in First-Order Logic

In this section, we review the three main multidimensional theories using the language of first-order logic (FOL) to represent their domain of discourse: *The INCH calculus* (Gotts, 1996), *Containment-dimension family* (CODI) (Hahmann, 2013) (Hahmann, 2018) and *Region connection calculus*-9* (RCC*-9) (Clementini and Cohn, 2014). They can be divided into two types based on their inferential approach. INCH and CODI are axiomatic systems in which further knowledge is extractable by using *theorem proving*, while RCC*-9 exploits an alternative approach (i.e. *constraint-based reasoning*) which gives it the advantage of being able to use *composition tables* to infer further knowledge from the given information.

Nevertheless, the common property of all the proposed FOL theories is in their assumptions (and consequently their methods) regarding representation of entities. Lower-dimensional entities are represented independently in these theories. For instance, they accept a line segment as an individual entity without requiring it to be a boundary of some higher-dimensional entity.

The INCH calculus (Gotts, 1996): This was a response to the inability of the RCC (Randell et al., 1992a)[1] to accept lower-dimensional regions. It was developed based on a single binary primitive "*x includes a chunk of y*", denoted as $\mathbf{INCH}(x, y)$, where a chunk of y refers to an equi or lower-dimensional part of it being represented by x. Almost the same properties are also expressed by the CODI's primitive, **Cont**(x,y), which implies full containment. The **INCH** relation is neither a purely mereological relation, as it is a notion of overlap rather than a refinement of parthood (which is common in the mereologoical theories), nor a topological relation, though it requires a connection between the entities. The connection alone is not sufficient for **INCH** to hold; for example; a pair of two-dimensional regions sharing only a part of their boundaries do not satisfy the **INCH** relation.

Moreover, the condition for each entity is that it consists of a (closed) set of points of uniform dimension (i.e. regular closed regions). No further constraints are imposed on the entities, so they may, for example, consist of multiple disconnected pieces. Furthermore, for mathematical simplicity, the theory introduced a unique *null* entity via the predicate **ZEX**.

The INCH calculus is axiomatised through a set of axioms and definitions specified in Figure A.1. The

[1] The most well known qualitative spatial representation and reasoning method in the field of artificial intelligence.

IN-D1. $\mathbf{ZEX}(x) \leftrightarrow \neg\mathbf{INCH}(x,x)$ (x has zero region)

IN-D2. $\mathbf{GED}(x,y) \leftrightarrow \mathbf{ZEX}(y) \vee \exists z[\mathbf{INCH}(x,z) \wedge \mathbf{INCH}(z,y)]$

(The dimensionality of x is at least y)

IN-D3. $\mathbf{ED}(x,y) \leftrightarrow \mathbf{GED}(x,y) \wedge \mathbf{GED}(y,x)$ (x and y have the same dimensionality)

IN-D4. $\mathbf{GD}(x,y) \leftrightarrow \mathbf{GED}(x,y) \wedge \neg\mathbf{GED}(y,x)$ (The dimensionality of x is greater than y)

IN-D5. $\mathbf{CS}(x,y) \leftrightarrow \forall z[\mathbf{INCH}(x,z) \rightarrow \mathbf{INCH}(y,z)]$ (x is a constituent of y)

IN-D6. $\mathbf{OV}(x,y) \leftrightarrow \mathbf{INCH}(x,y) \wedge \mathbf{INCH}(y,x)$ (x overlaps y)

IN-D7. $\mathbf{CO}(x,y) \leftrightarrow \exists z[\neg\mathbf{ZEX}(z) \wedge \mathbf{CS}(z,x) \wedge \mathbf{CS}(z,y)]$ (x is connected to y)

IN-D8. $\mathbf{CH}(x,y) \leftrightarrow \mathbf{INCH}(x,y) \wedge \forall z[\mathbf{OV}(x,z) \rightarrow \mathbf{OV}(y,z)]$ (x is a chunk of y)

IN-D9. $\mathbf{EL}(x,y) \leftrightarrow \mathbf{CS}(x,y) \wedge \neg\mathbf{INCH}(x,y)$ (x is an element (lower-dimensional part) of y)

IN-D10. $\mathbf{ATOM} \leftrightarrow \forall y[\mathbf{CH}(y,x) \rightarrow y = x]$ (Atom)

IN-D11. $\mathbf{PT}(x) \leftrightarrow \forall y[\mathbf{CS}(y,x) \rightarrow \mathbf{ZEX}(y) \vee y = x]$ (point or zero-dimensional entity)

IN-D12. $\mathbf{MDSL}(x,y) \leftrightarrow \mathbf{EL}(x,y) \wedge \forall v \forall w[\mathbf{CH}(v,y) \wedge \mathbf{EL}(x,v) \wedge \mathbf{CH}(w,y) \wedge \mathbf{EL}(x,w) \rightarrow \mathbf{OV}(v,w)]$

(superficial element x has one less dimension region y it bounds)

IN-D13. $\mathbf{SEL}(x,y) \leftrightarrow \exists z[\mathbf{MDSL}(z,y) \wedge \mathbf{CS}(x,z)]$ (superficial element)

IN-D14. $\mathbf{BD}(x,y) \leftrightarrow \forall z[\mathbf{CH}(z,x) \leftrightarrow \mathbf{MDSEL}(z,y)]$ (x is the boundary of y)

IN-A1. $x = y \leftrightarrow \forall z[\mathbf{INCH}(x,z) \leftrightarrow \mathbf{INCH}(y,z)]$

IN-A2. $x = y \leftrightarrow \forall z[\mathbf{INCH}(z,x) \leftrightarrow \mathbf{INCH}(z,y)]$

IN-A3. $\mathbf{INCH}(x,y) \rightarrow \mathbf{INCH}(x,x)$

IN-A4. $\mathbf{GED}(x,y) \vee \mathbf{GED}(y,x)$

IN-A5. $\mathbf{GED}(x,y) \wedge \mathbf{GED}(y,z) \rightarrow \mathbf{GED}(x,z)]$

IN-A6. $[\mathbf{INCH}(x,y) \wedge \mathbf{INCH}(y,z) \wedge \mathbf{INCH}(z,x)] \rightarrow \mathbf{INCH}(y,x)$

IN-A7. $\mathbf{INCH}(x,y) \rightarrow \exists z[\mathbf{CS}(z,x) \wedge \mathbf{OV}(y,z)]$

IN-A8. $\mathbf{CH}(x,y) \rightarrow \mathbf{CS}(x,y)$

IN-A9. $\mathbf{ED}(x,y) \rightarrow \exists z \forall w[\mathbf{INCH}(z,w) \leftrightarrow \mathbf{INCH}(x,w) \vee \mathbf{INCH}(y,w)]]$

IN-A10. $\mathbf{ED}(x,y) \rightarrow \exists z \forall w[\mathbf{INCH}(z,w) \leftrightarrow \exists v[\mathbf{INCH}(v,w) \wedge \mathbf{CH}(v,x) \wedge \neg\mathbf{OV}(v,y)]]$

IN-A11. $\mathbf{INCH}(x,x) \rightarrow \mathbf{INCH}(Ue,x)$ (The universal (Ue) entity includes (a chunk of) all entities)

IN-A12. $\neg\mathbf{ZEX}(x) \wedge \mathbf{PT}(x) \rightarrow \exists y[\mathbf{CH}(y,x) \wedge \neg\exists z[\mathbf{SEL}(z,x) \wedge \mathbf{INCH}(y,z)]]$ (Non-atomistic axiom)

IN-A13. $[\neg\mathbf{ZEX}(x) \rightarrow \exists y[\mathbf{CH}(y,x) \wedge \mathbf{ATOM}(y)]$ (Atomistic axiom)

Figure A.1: Defined relations (top) and axioms (bottom) in the **INCH** calculus (Gotts, 1996).

defined relations include dimensional comparison of regions (such as **GED**, **ED**, **GD** and etc.), specialized notions of parthood (e.g. **CS**, **CH**, and **EL**), and an overlap relations (well-known in equidimensional mereotopological theories).

To combine a pair of regions and represent a new one, the Boolean operators, i.e. `sum` and `difference`, were defined. However, they are only defined over the equidimensional regions. Further operators, such as `product` and `complement`, were defined in terms of `sum` and `difference`. Two axioms (axioms IN-A9 and IN-A10 in Figure A.1) together with **ZEX** guarantee that the operators are well-defined (i.e. always results in a unique entity) for all pairs of equidimensional regions.

Containment-Dimension Family (CODI and CODIB) (Hahmann, 2013, 2018): This family of theories is inspired by the INCH calculus, with the difference that various connection relations have been defined in this family, which improves its expressivity in comparison to the INCH calculus. For instance, the externally connected status of a pair of regions is definable.

The family consists of two theories: Containment-dimension (CODI), and Containment-dimension boundary (CODIB). The latter is an extension of the former by adding a concept of boundary containment. Both members of the family accept simple regions and their collection (known as a complex region), which are regular closed regions of various possible dimensions. The only restriction on a complex region is that its elemental regions can only overlap on their boundaries. For example, finite sets of line segments are (only) connected in their boundaries to form a complex linear feature. Also, the same predicate as the INCH calculus is used to represent the *null* entity with a further restriction on it as a unique minimal dimensional entity of the domain.

CODI is based on two primitive relations: (1) a mereological primitive "containment", **Cont**(x,y) which says that "x is a part of y" where x's dimension is less than or equal to y's dimension and (2) a dimension relation, $x \leq_{\text{dim}} y$, which compares the dimensionality of two spatial regions and is interpreted as "the dimensionality of x is at most equal to the dimensionality of y". The theory is defined by a set of axioms and definitions shown in Figure A.2. The defined relations include relations for comparing dimensions ($<_{\text{dim}}, =_{\text{dim}}$ and $\prec_{\text{dim}}$), representing the minimal and maximal dimensions of the entities in the domain (**MaxDim**, and **MinDim**) repectively, equidimensional variants of the containment (**P** and **PP**), indivisible (**Min**) and the biggest element (**Max**) within a dimension, and more specifically three specialized forms of the connection of two entities:

- *Partial overlap* (**PO**) holds between equidimensional regions and represents the strongest form of the contact relation. The dimensionality of the common part is same as the both participants. This relation is a reflexive and symmetric relation for non-zero regions.
- *Incidence* (**Inc**) holds when the dimensionality of the common part is equal to exactly one of the participants. It is an irreflexive and symmetric relation.
- *Superficial contact* (**SC**) is the weakest form of contact and holds when the shared region is of a lower-dimension than both participants. It is an irreflexive and symmetric relation.

Further refinement of the **Cont** relation leads to the introduction of the second member of the family, CODIB. The refinement checks whether a common part of two spatial regions is a boundary part of any of them. The new primitive relation, "boundary-containment" (**BCont**(x, y)), basically specializes the shared concept between the containment (**Cont**) and incident (**Inc**) relations. Figure A.3 shows the added axioms in CODIB.

Moreover, CODI's axiomatisation is extended in (Hahmann, 2018) to lift the limitation of the Boolean operators (i.e. `sum`, `product`, `complement` and `difference`) of being only valid over the equidimensional entities in the initial version of the formalism (Hahmann, 2013) to regions of any dimensions.

Region Connection Calculus*-9 (RCC*-9) (Clementini and Cohn, 2014): This theory, like the INCH calculus, was introduced to extend the RCC family to accept a range of regions of various dimensions. However, it benefits from a more efficient reasoning mechanism, i.e. constraint-based reasoning,

CO-D1.	$x <_{\mathrm{dim}} y \leftrightarrow x \leq_{\mathrm{dim}} y \wedge y \not\leq_{\mathrm{dim}} x$	(x's dimension is less than y's dimension)
CO-D2.	$x =_{\mathrm{dim}} y \leftrightarrow x \leq_{\mathrm{dim}} y \wedge y \leq_{\mathrm{dim}} x$	(x and y have the same dimension)
CO-D3.	$\mathbf{MaxDim}(x) \leftrightarrow \forall y[y \leq_{\mathrm{dim}} x]$	(Maximal-dimensional entities)
CO-D4.	$\mathbf{MinDim}(x) \leftrightarrow \neg\mathbf{ZEX}(x) \wedge \forall y[\neg\mathbf{ZEX}(y) \rightarrow y \geq_{\mathrm{dim}} x]$	(Minimal nonzero dimensional ent.)
CO-D5.	$x \prec_{dim} y \leftrightarrow x \leq_{\mathrm{dim}} y \wedge \forall z[z \leq_{\mathrm{dim}} y \vee y \leq_{\mathrm{dim}} z]$	(Next highest dimension)
CO-D6.	$\mathbf{C}_{\mathrm{CODI}}(x,y) \leftrightarrow \exists z[\mathbf{Cont}(z,x) \wedge \mathbf{Cont}(z,y)]$	(Contact)
CO-D7.	$\mathbf{P}_{\mathrm{CODI}}(x,y) \leftrightarrow \mathbf{Cont}(x,y) \wedge x =_{\mathrm{dim}} y$	(equidimensional parthood)
CO-D8.	$\mathbf{PP}_{\mathrm{CODI}}(x,y) \leftrightarrow \mathbf{P}_{\mathrm{CODI}}(x,y) \wedge x \neq y$	(equidimensional proper parthood)
CO-D9.	$\mathbf{Max}(x) \leftrightarrow \neg\mathbf{ZEX}(x) \wedge \forall y[\neg\mathbf{PP}(x,y)]$	(Maximal in dimension)
CO-D10.	$\mathbf{Min}(x) \leftrightarrow \neg\mathbf{ZEX}(x) \wedge \forall y[\neg\mathbf{PP}(y,x)]$	(Minimal in dimension)
CO-D11.	$\mathbf{PO}_{\mathrm{CODI}}(x,y) \leftrightarrow \exists z[\mathbf{P}_{\mathrm{CODI}}(z,x) \wedge \mathbf{P}_{\mathrm{CODI}}(z,y)]$	(Overlap in a Part)
CO-D12.	$\mathbf{Inc}(x,y) \leftrightarrow \exists z[(\mathbf{Cont}(z,x) \wedge \mathbf{P}_{\mathrm{CODI}}(z,y) \wedge z <_{\mathrm{dim}} x) \vee (\mathbf{P}_{\mathrm{CODI}}(z,x) \wedge \mathbf{Cont}(z,y) \wedge z <_{\mathrm{dim}} y)]$	(Incidence)
CO-D13.	$\mathbf{SC}(x,y) \leftrightarrow \exists z[\mathbf{Cont}(z,x) \wedge \mathbf{Cont}(z,x)] \wedge \forall z[\mathbf{Cont}(z,x) \wedge \mathbf{Cont}(z,x) \rightarrow z <_{\mathrm{dim}} x \wedge z <_{\mathrm{dim}} y]$	(Superficial Contact)
CO-D14.	$\mathbf{Con}(x) \leftrightarrow \forall y[\mathbf{PP}(y,x) \rightarrow \mathbf{C}_{\mathrm{CODI}}(y, x-y)$	(Self-connectedness)
CO-A1.	$x \leq_{\mathrm{dim}} y$	($\leq_{\mathrm{dim}}$ reflexive)
CO-A2.	$x \leq_{\mathrm{dim}} y \wedge y \leq_{\mathrm{dim}} z \rightarrow x \leq_{\mathrm{dim}} z$	($\leq_{\mathrm{dim}}$ transitive)
CO-A3.	$\mathbf{ZEX}(x) \wedge \mathbf{ZEX}(y) \rightarrow x = y$	(unique zero region)
CO-A4.	$\mathbf{ZEX}(x) \rightarrow x \leq_{\mathrm{dim}} y$	(zero region has lowest dimension)
CO-A5.	$\exists x[\mathbf{MinDim}(x)]$	(a region of lowest dimension exists)
CO-A6.	$\neg\mathbf{ZEX}(x) \leftrightarrow \mathbf{Cont}(x,x)$	(**Cont** reflexive and definition **ZEX**)
CO-A7.	$\mathbf{Cont}(x,y) \wedge \mathbf{Cont}(y,x) \rightarrow x = y$	(**Cont** anti-symmetric)
CO-A8.	$\mathbf{Cont}(x,y) \wedge \mathbf{Cont}(y,z) \rightarrow \mathbf{Cont}(x,z)$	(**Cont** transitive)
CO-A9.	$\mathbf{ZEX}(x) \rightarrow \forall y[\neg\mathbf{Cont}(z,x) \wedge \neg\mathbf{Cont}(y,x)]$	(null region never in **Cont** relation)
CO-A10.	$\mathbf{Cont}(x,y) \rightarrow x \leq_{\mathrm{dim}} y$	(a contained entity is of the same or lesser dimension)

Figure A.2: Defined relations (top) and axioms (bottom) in the **CODI** (Hahmann, 2013, 2018).

BC-A1.	$\mathbf{BCont}(x,y) \rightarrow \mathbf{Cont}(x,y) \wedge \mathbf{Inc}(x,y)$	(boundary-containment as special kind of containment and incidence)
BC-A2.	$\mathbf{BCont}(x,y) \wedge \mathbf{Cont}(z,x) \rightarrow \mathbf{BCont}(z,y)$	(**BCont** transitive wrt. **Cont**)
BC-A3.	$\mathbf{SC}(x,y) \wedge \mathbf{Min}(x) \wedge \mathbf{Cont}(z,x) \wedge \mathbf{P}(x,v) \wedge \mathbf{Cont}(y,v) \rightarrow \mathbf{BCont}(z,x)$	
BC-A4.	$\mathbf{SC}(x,y) \wedge \mathbf{P}(x,v) \wedge \mathbf{P}(y,v) \wedge \mathbf{Cont}(z,x) \wedge \mathbf{Cont}(z,y) \wedge \prec_{dim} v \rightarrow \neg\mathbf{BCont}(x,z)$	
BC-A5.	$\mathbf{BCont}(x,y) \wedge \mathbf{P}(y,z) \wedge \forall v,w[\mathbf{P}(v,z) \wedge \neg\mathbf{PO}(v,y) \wedge \mathbf{P}(w,x) \rightarrow \neg\mathbf{Cont}(w,v)] \rightarrow \neg\mathbf{BCont}(x,z)$	

Figure A.3: Axioms of the **CODIB** (Hahmann, 2013) (Hahmann, 2018).

than the INCH calculus (and similarly than the CODI family).

The theory accepts regular closed regions. Although the dimensionality of the regions is always less than or equal to their embedding space, they can have various dimensions from zero to two. A region might be made of single or multi-pieces of equidimensional regions.

Following the RCC family, the *connection* relation is kept as the only topological primitive, and a set of nine JEPD relations are defined based on it (see Figure A.4). The definitions of the introduced relations, including non-tangential proper part (**NTPP**), tangential proper part (**TPP**), overlap (**O**), partially overlap (**PO**), and externally connected (**EC**) relations, are redefined (in comparison to RCC-8, the most well-known member of the family). The refinement is due to the introduction of the second primitive, the boundary relation (**B**(x,y)). The new primitive relation is axiomatised to represent a boundary part (x) of the referred region (y). The boundary relation, along with the connection primitive lead to the introduction of a new spatial relation to describe a cross configuration (**CR**). The inheritance properties among the introduced relations are represented in the form of a relational lattice.

Compared to RCC-8, RCC*-9 does not introduce any Boolean operators over regions. Thus, their introduction is a possible extension to this theory. In this case, the formalism would also require a predicate to guarantee the definability of the operators in the domain of discourse.

RCC-D1.	$\mathbf{DC}(x,y) \leftrightarrow \neg\mathbf{C}(x,y)$	(x and y are disconnected)
RCC-D2.	$\mathbf{P}(x,y) \leftrightarrow \forall z[\mathbf{C}(z,x) \rightarrow \mathbf{C}(z,y)]$	(x is part of y)
RCC-D3.	$\mathbf{PP}(x,y) \leftrightarrow \mathbf{P}(x,y) \wedge \neg\mathbf{P}(y,x)$	(x is a proper part of y)
RCC-D4.	$\mathbf{EQ}(x,y) \leftrightarrow \mathbf{P}(x,y) \wedge \mathbf{P}(y,x)$	(x equals to y)
RCC-D5.	$\mathbf{NTPP}(x,y) \leftrightarrow \mathbf{PP}(x,y) \wedge \forall z[\mathbf{B}(z,y) \rightarrow \mathbf{DC}(x,z)]$	(x is a non-tangential proper part of y)
RCC-D6.	$\mathbf{TPP} \leftrightarrow \mathbf{PP}(x,y) \wedge \neg\mathbf{NTPP}(x,y)$	(x is a tangential proper part of y)
RCC-D7.	$\mathbf{O}(x,y) \leftrightarrow \exists z[\mathbf{NTPP}(z,x) \wedge \mathbf{NTPP}(z,y)] \wedge \exists t[\mathbf{TPP}(t,x) \wedge \mathbf{TPP}(t,y)]$	(x overlaps y)
RCC-D8.	$\mathbf{PO}(x,y) \leftrightarrow \mathbf{O}(x,y) \wedge \neg\mathbf{P}(x,y) \wedge \neg\mathbf{P}(y,x)$	(x partially overlaps y)
RCC-D9.	$\mathbf{EC}(x,y) \leftrightarrow \mathbf{C}(x,y) \wedge \neg\mathbf{O}(x,y) \wedge \forall z[[\mathbf{P}(z,x) \wedge \mathbf{P}(z,y)] \rightarrow \mathbf{TPP}(z,x) \vee \mathbf{TPP}(z,y)]$	(x is externally connected to y)
RCC-D10.	$\mathbf{CR}(x,y) \leftrightarrow \mathbf{C}(x,y) \wedge \neg\mathbf{O}(x,y) \wedge \mathbf{EC}(x,y)$	(x crosses y)
RCC-D11.	$\mathbf{Pi}(x,y) \leftrightarrow \mathbf{P}(y,x)$	(x has a part y)
RCC-D12.	$\mathbf{PPi}(x,y) \leftrightarrow \mathbf{PP}(y,x)$	(x has a proper part y)
RCC-D13.	$\mathbf{TPPi}(x,y) \leftrightarrow \mathbf{TPP}(y,x)$	(x has a tangential proper part y)
RCC-D14.	$\mathbf{NTPPi}(x,y) \leftrightarrow \mathbf{NTPP}(y,x)$	(x has a non-tangential proper part y)
RCC-A1.	$\mathbf{C}(x,x)$	(**C** reflexive)
RCC-A2.	$\mathbf{C}(x,y) \rightarrow \mathbf{C}(y,x)$	(**C** symmetric)
RCC-A3.	$\mathbf{B}(x,y) \rightarrow \mathbf{PP}(x,y)$	(x is the boundary of y)

Figure A.4: Defined relations (top) and axioms (bottom) in the RCC*-9 (Clementini and Cohn, 2014).

TD-D1.	$\mathbf{O}(x, y) \leftrightarrow \exists z\ \mathbf{P}(z, x) \wedge \mathbf{P}(z, y)$	(x overlaps y)
TD-D2.	$\mathbf{PT}(x) \wedge \mathbf{PT}(y) \rightarrow \mathbf{ED}(z, y)$	(x is a point)
TD-D3.	$\mathbf{ED}(x, y) \leftrightarrow \exists z(\mathbf{P}(x, z) \wedge \mathbf{P}(z, y))$	(Equal dimensionality of x and y)
TD-D4.	$\mathbf{D}_0(x) \leftrightarrow \exists z[\mathbf{PT}(z) \wedge \mathbf{ED}(z, x)]$	(Zero-dimensional entity)
TD-D5.	$\mathbf{LD}(x, y) \leftrightarrow \exists v \exists w[\mathbf{ED}(x, v) \wedge \mathbf{B}(v, w) \wedge \mathbf{ED}(w, y)]$	(x of lower dimension than y)
TD-D6.	$\mathbf{IN}(x, y) \leftrightarrow \mathbf{P}(x, y) \vee \forall z[\mathbf{P}(z, x) \rightarrow \neg \mathbf{B}(z, y) \wedge \exists w \exists v[\mathbf{P}(v, z) \wedge \mathbf{P}(w, y) \wedge \mathbf{B}(v, w)]]$	(x is within y)
TD-A1.	$\mathbf{P}(x, x)$	($\mathbf{P}$ reflexive)
TD-A2.	$\mathbf{P}(x, y) \wedge \mathbf{P}(y, z) \rightarrow x = y$	($\mathbf{P}$ anti-symmetric)
TD-A3.	$\mathbf{P}(x, y) \wedge \mathbf{P}(y, z) \rightarrow \mathbf{P}(x, z)$	($\mathbf{P}$ transitive)
TD-A4.	$\mathbf{O}(x, y) \rightarrow \exists z\ [\mathbf{P}(x, z) \wedge \mathbf{P}(y, z)]$	(existence of upper bound for every pair of overlapped entities)
TD-A5.	$[\mathbf{P}(z, y) \rightarrow \mathbf{O}(z, x)] \rightarrow \mathbf{P}(y, x)$	(every none part of x is disjoint from it)
TD-A6.	$\exists x \Phi(x) \wedge \exists x \forall y\ (\Phi(y) \rightarrow \mathbf{P}(y, x)) \rightarrow \exists x \forall y\ (\mathbf{O}(y, x) \leftrightarrow \exists z\ (\Phi(z) \wedge \mathbf{O}(y, z))$	(existence of unique fusion of the Φ regions)
TD-A7.	$\mathbf{PT}(x) \wedge \mathbf{PT}(y) \rightarrow \mathbf{ED}(z, y)$	(All points are equidimensional)
TD-A8.	$\mathbf{D}_0(x) \rightarrow \exists y\ \mathbf{PT}(y) \wedge \mathbf{P}(y, x)$	(Every zero-dimensional region has a point part)
TD-A9.	$\mathbf{B}(x, y) \rightarrow \neg \mathbf{ED}(x, y)$	(a region can not bound a region equidimensional it)
TD-A10.	$\mathbf{B}(x, y) \wedge \mathbf{B}(y, z) \rightarrow \mathbf{B}(x, z)$	($\mathbf{B}$ transitivity)
TD-A11.	$\mathbf{P}(x, y) \wedge \mathbf{B}(y, z) \rightarrow \mathbf{B}(x, z)$	(any part of a region bounding z also bounds z)
TD-A12.	$\mathbf{B}(x, z) \wedge \mathbf{ED}(z, y) \rightarrow \exists u \exists v[\mathbf{ED}(x, u) \wedge \mathbf{B}(u, v) \wedge \mathbf{P}(v, y)]$	
TD-A13.	$\mathbf{B}(y, x) \rightarrow \neg \mathbf{D}_0(x)$	(no boundary for zero-dimensional entity)
TD-A14.	$\neg \mathbf{D}_0(x) \rightarrow \exists y \exists z(\mathbf{B}(y, z) \wedge \mathbf{P}(z, x))$	(a none zero dimensional region has a bounded part)
TD-A15.	$\mathbf{B}(y, x) \rightarrow \exists z(\mathbf{PT}(z) \wedge \mathbf{B}(z, x))$	(every bounded region is bounded by a point)
TD-A16.	$\mathbf{B}(x, y) \rightarrow \exists z \forall w(\mathbf{B}(w, y) \leftrightarrow \mathbf{IN}(w, z)$	(existence of unique boundary for every bounded region y (named as ∂y))
TD-A17.	$\mathbf{B}(u, x) \wedge \mathbf{B}(v, y) \rightarrow [\mathbf{ED}(x, y) \leftrightarrow \mathbf{ED}(\partial x, \partial y))$	(equidimensionality of two bounded regions with equidimensional boundaries)

Figure A.5: Axioms and defined relations in the Take-Dim (Galton, 1996).

A.1.2 Multidimensional Meretopological Theories in Higher-Order Logic

In this section, we review the multidimensional mereotopological theories proposed in the language of higher-order logical (HOL), i.e. Galton and Smith's theories. The main reason for using HOL is the ability to use infinity fusion for the *sum*. Moreover, these theories' approach to representing the lower-dimensional entities and Boolean operators differs significantly. The lower-dimensional regions arise only as boundaries of the higher-dimensional ones in these theories. For example, a line segment is only definable as a boundary of a surface which itself is a boundary of a solid in a three-dimensional space. Also, the Boolean operators are defined by using the definite description operator which prevents them from always being definable in the domain.

Take Dimension Seriously (Take-dim) (Galton, 1996): Galton presents his theory, subsequently referred to as *Take-dim*, in second-order logic in which separate mereological and topological structures are combined. The mereological aspect of the theory is based on the *parthood* relation, which is only valid over equidimensional regions. This constraint does not allow the summation of regions of different dimensions which consequently classifies them with respect to their dimensions. The dimensionality of each class of regions is ascertained by the *equidimensional* relation (**ED**).

Take-dim also uses a *bounds* relation ($\mathbf{B}(x,y)$) as a topological primitive. This relation denotes that "x bounds y", and any part of x also bounds y. This primitive, unlike the first one, only relates a pair of non-equidimensional regions. Every entity of dimension m ($m \neq 0$) is bounded by some lower-dimensional entitie(s) of dimension n ($n < m$), which is part of y (i.e. regions are regular closed). If the bounded region is from the next lower-dimension class of regions (with respect to the referred one) and maximal (i.e. the whole boundary), it represents *the* boundary of the referred region. The boundary plays the role of interface between the host region and its complement.

This pair of primitives are supported by a set of axioms (see Figure A.5) and used to define a set of spatial relations such as a lower-dimension relation (**LD**). This relation, in turn, helps to define the next lower-dimension (**NLD**) which is crucial in describing the boundary of a region.

Moreover, the Boolean operators: `sum`, `product` and `difference` are defined as significant tools to describe more regions. However, they are valid between a pair of equidimensional regions and only definable while their resultant region is still in the same dimension.

Smith's Theory of Parts and Boundaries (Smith, 1996; Smith et al., 2000): This theory, like Take-dim, uses separate primitives to represent mereological and topological aspects of the spatial domain. The mereological characteristics are shown by a reflexive, anti-symmetric and transitive *parthood* relation, and then supplemented by the topological primitive relation, *interior parthood*, $\mathbf{IP}(x,y)$. The second primitive denotes that x is a part of y but does not include any of the boundary points of y. These two primitives are combined to define a *boundary part* relation ($\mathbf{B}(x,y)$) in which boundary elements of any region are represented. The set of further relations are defined between the entities based on these relations and are shown in Figure A.6.

SM-D1.	$\mathbf{O}(x,y) \leftrightarrow \exists z\ (\mathbf{P}(z,x) \wedge \mathbf{P}(z,y))$	(x overlaps y)
SM-D2.	$\mathbf{D}(x,y) \leftrightarrow \neg\mathbf{O}(x,y)$	(x is discrete from y)
SM-D3.	$\mathbf{Pt}(x,y) \leftrightarrow \forall y\ (\mathbf{P}(y,x) \rightarrow y = x$	(x is a point)
SM-D4.	$\sigma x(\Phi(x)) \leftrightarrow \iota y(\forall w\ (\mathbf{O}(w,y) \leftrightarrow \exists v(\Phi(v) \wedge \mathbf{O}(w,v))$	(sum of Φers)
SM-D5.	$\mathbf{X}(x,y) \leftrightarrow \neg\mathbf{P}(x,y) \wedge \neg\mathbf{D}(x,y)$	(x crosses y)
SM-D6.	$\mathbf{St}(x,y) \leftrightarrow \forall z(\mathbf{IP}(x,z) \rightarrow \mathbf{X}(z,y))$	(x straddles y)
SM-D7.	$\mathbf{B}(x,y) \leftrightarrow \forall z(\mathbf{P}(z,x) \rightarrow \mathbf{St}(z,y)$	(x is boundary of y)
SM-D8.	$\mathbf{T}(x,y) \leftrightarrow \exists z(\mathbf{P}(z,x) \wedge \mathbf{B}(z,y))$	(x is tangent of y)
SM-D9.	$cl(x) \leftrightarrow x \cup \sigma y(\mathbf{B}(y,x)$	(topological closure)
SM-D10.	$bdy(x) \leftrightarrow \sigma y(\mathbf{B}(y,x)$	(topological boundary)
SM-D11.	$int(x) \leftrightarrow \sigma y(\mathbf{IP}(y,x)$	(topological interior)
SM-D12.	$\mathbf{Bd}(x) \leftrightarrow \exists y\mathbf{B}(x,y)$	(x is a boundary)
SM-D13.	$\mathbf{S}(x,y) \leftrightarrow \mathbf{D}(cl(x),y) \wedge \mathbf{D}(x,cl(y))$	(x and y are separate)
SM-D14.	$\mathbf{Cn}(x) \leftrightarrow \neg\exists y\exists z(\mathbf{S}(y,z) \wedge x = y \cup z)$	(x is a connected entity)
SM-D15.	$\mathbf{IB}(x,y) \leftrightarrow \mathbf{IP}(x,y) \wedge \mathbf{B}(x,x)$	(x is interior boundary of y)
SM-D16.	$\mathbf{A}(x,y) \leftrightarrow \mathbf{Pt}(x) \wedge \forall z(\mathbf{IP}(x,z) \wedge x \neq z \rightarrow \forall t(\mathbf{P}(t,z) \wedge \mathbf{D}(t,x) \wedge \mathbf{O}(t,y)$	(x is a neighbourhood of y)
SM-D17.	$\mathbf{IPt}(x,y) \leftrightarrow \mathbf{Pt}(x) \wedge \mathbf{IP}(x,y)$	(x is interior point of y)
SM-D18.	$\mathbf{BPt}(x,y) \leftrightarrow \mathbf{Pt}(x) \wedge \mathbf{B}(x,y)$	(x is boundary point of y)
SM-D19.	$cm(x) \leftrightarrow \sigma y(\mathbf{P}(x,y) \wedge \mathbf{Cn}(y))$	(x is maximally connected entity)
SM-A1.	$\mathbf{P}(x,x)$	($\mathbf{P}$ reflexive)
SM-A2.	$\mathbf{P}(x,y) \wedge \mathbf{P}(y,z) \rightarrow x = y$	($\mathbf{P}$ anti-symmetric)
SM-A3.	$\mathbf{P}(x,y) \wedge \mathbf{P}(y,z) \rightarrow \mathbf{P}(x,z)$	($\mathbf{P}$ transitive)
SM-A4.	$\mathbf{P}(x,y) \leftrightarrow \forall z\ (\mathbf{O}(z,x) \rightarrow \mathbf{O}(z,y)$	($\mathbf{P}$ is a special sort of $\mathbf{O}$)
SM-A5.	$\exists x\Phi(x) \rightarrow \exists y\forall w(\mathbf{O}(w,y) \leftrightarrow \exists v(\Phi(v) \wedge \mathbf{O}(w,v)))$	(existence of unique fusion of the Φ regions)
SM-A6.	$\mathbf{IP}(x,y) \rightarrow \mathbf{P}(x,y)$	($\mathbf{IP}$ is a special sort of $\mathbf{P}$)
SM-A7.	$\mathbf{IP}(x,y) \wedge \mathbf{P}(y,z) \rightarrow \mathbf{IP}(x,z)$	(left monotonicity wrt $\mathbf{IP}$)
SM-A8.	$\mathbf{P}(x,y) \wedge \mathbf{IP}(y,z) \rightarrow \mathbf{IP}(x,z)$	(right monotonicity wrt $\mathbf{IP}$)
SM-A9.	$\mathbf{IP}(x,y) \wedge \mathbf{IP}(x,z) \rightarrow \mathbf{IP}(x,(y \cap z)$	(condition on finite intersections)
SM-A10.	$(\exists x\ \Phi(x) \wedge \forall x\ (\Phi(x) \rightarrow \mathbf{IP}(x,y)) \rightarrow \mathbf{IP}(\sigma x(\Phi(x)),y))$	(condition on arbitrary unions)
SM-A11.	$\exists y\ \mathbf{IP}(x,y)$	(existence of an entity contains all other entities of the domain)
SM-A12.	$\mathbf{IP}(x,y) \rightarrow \mathbf{IP}(x,\sigma t(\mathbf{IP}(t,y)))$	
SM-A13.	$\exists x\mathbf{B}(x,y)$	(all entities have boundary)
SM-A14.	$\mathbf{Bd}(x) \rightarrow \exists z\exists t(\mathbf{B}(x,z) \wedge \mathbf{P}(x,z) \wedge \mathbf{IP}(t,z)$	(Dependency of boundaries)
SM-A15.	$\mathbf{Pt}(x) \rightarrow \forall y(\mathbf{B}(y,x) \leftrightarrow x = y)$	

Figure A.6: Defined relations (top) and axioms (bottom) in Smith's theory (Smith, 1996; Smith et al., 2000).

In a spatial domain, these relations only support three-dimensional entities which are bounded by lower-dimensional entities. Their boundaries might be either elements of them or just bound them from outside without being their elements. However, boundaries are always considered dependent entities whose existence relies on their higher-dimension entity.

The formalization also includes an infinitary operator – the so-called fusion operator – as well as definite description operator (i.e. iota ι). The former allows the arbitrary summation to arbitrary sets of entities, including infinite sets. The latter lets the theory describe unique entities that satisfy a specific property. These operators are used to define topological operators, like `closure`, `interior` and `boundary`, and Boolean operators including `sum`, `product`, `difference` and `complement`. However, their definablity has not been guaranteed[2].

A.2 Comparison of Multidimensional Theories

In this section, we will compare the introduced multidimensional mereotopological theories in more detail. This comparative analysis will highlight their differences and commonalities of each theory and enable the reader to make a more informed choice for which theory to adopt as a multidimensional mereotopology for a specific domain or application.

We first review and compare the fundamental choices of each theory in Subsection A.2.1, then consider and describe their mereological and topological characteristics (in Subsection A.2.2 and Subsection A.2.3 respectively)– including the treatment of boundaries – before examining the theories' capabilities for automated reasoning in Subsection A.2.4. A summary of the comparison is shown in Table A.1.

A.2.1 Foundation

This section covers some of the fundamental ontological choices that all mereotopological and more specifically multidimensional mereotopologies have to address. Among them, we have choices of language; type of interaction between mereological and topological aspects of the theory with respect to its primitive relations; dimensionality; and type of accepted spatial regions.

Representation Language: FOL vs. HOL

An ontology language is a formal language used to encode the relationships between a set of concepts within a domain. There are a variety of logical languages available to represent ontological concepts. Languages currently used for specifying multidimensional mereotopological ontologies are of two kinds: *first-order* logic (FOL), which is used in the INCH calculus, CODI (and CODIB) and RCC*-9, and *higher-order* logic (HOL), which is employed for Take-dim and Smith's theory.

Higher-order logic is more expressive than first-order logic (Väänänen, 2019) in that it incorporates "for all properties" into the syntax, while first-order logic can only say "for all elements". Thus, the property-forming operators such as decomposition operators (i.e. `sum`, `product`, `difference` and `complement`)

[2]The operators are not *total* due to the absence of the *null* entity in the domain of discourse (see Chapter 4 for further discussion)

and topological operators (i.e. *interior*, *closure* and *boundary*) can be immediately formulated in HOL. The output of every operator is an entity consisting of all those things satisfying the required condition in the operator. For instance, `product`(x, y) satisfies the condition in which there exists a region (z) collecting all those things that are part of both x and y. However, the FOL theories must consider some limitations on their operators in order to be definable. For instance, the Boolean operators are formulated by reference to equidimensional regions in the INCH calculus, while CODI improves on this and operates on the highest dimensional elements of the regions (Hahmann, 2018).

Although the HOL theories can represent the formalism in a natural way, making inferences and automated reasoning is more challenging in it than in FOL. This is because of the complexity of the process of solving equations between properties (i.e. unification) in HOL. So, as long as a theory is intended to be implemented on intelligent machines (i.e. supporting automated reasoning), first-order logic is an efficient choice.

Basis of the Theory: Topological vs. Mereological

Casati and Varzi (Casati and Varzi, 1999) classified mereotopological theories into three categories according to the interaction between mereological and topological components. Although their categorizations were limited to the equidimensional mereotopological theories, it can be generalized to classify the multi-dimensional mereotopological theories as well.

Extending mereology by topology is one of the introduced classes in (Casati and Varzi, 1999) in which a reflexive, anti-symmetric and transitive parthood primitive relation from the mereological side of the theory is supported by a topological primitive. This is the approach with which Take-dim and Smith's theory are formulated. However, the parthood relation only holds between equidimensional entities in Take-dim, while there is no restriction on the dimensionality of the participating entities in this primitive in Smith's theory. After axiomatizing the parthood relation, these theories introduce another primitive to describe the topological aspect of the domain. Take-dim uses the *bounds* (**B**) relation as a topological primitive, while the *interior parthood* (**IP**) relation is used for this purpose in Smith's theory. Both of the concepts expressed by these primitives are fundamental notions of mathematical topological space. Direct monotonicity with respect to the parthood relation in Smith's theory, and indirect monotonicity (i.e. via the equidimensional relation (`TD-D3`)) in Take-dim connect the mereological aspect of the theory to its topological facet.

Another category of approaches to combining mereology and topology uses the topological component as a basis of a theory and then adds mereology to it. This is the approach followed by all the traditional theories (i.e. Whiteheadian theories), like RCC-8. Since RCC*-9 extends the structure of RCC-8 for a multidimensional space, it follows this approach. RCC*-9 is based on a connection relation (**C**) as a single reflexive and symmetric topological primitive and then defines the mereological relation, parthood, in terms of it.

The third way of representing a mereotopological theory involves viewing topology as a sort of mereology. In the above mentioned mereotopological theories, an *overlap* relation has been introduced either based on the parthood relation or the connection relation. The satisfaction of this relation requires that the common part of the entities has the same dimensionality of at least one of the participating entities. For instance, a common part of a line segment and a polygon is a line segment. This restriction is lifted in the INCH calculus and CODI, and a range of various overlap configurations is permitted regardless of the dimensionality of the common part. This is achieved by basing the theories on a single spatial primitive, **INCH** in the INCH calculus and **Cont** in CODI, which require fulfilment of parthood and connectivity at the same time. Interested readers can consult (Casati and Varzi, 1999) for the pros and cons of these three categories of mereotopological theories.

Supported Dimensionality

As mentioned earlier, the spatial models presented in the equidimensional theories were based on a simplification and limitation of the space occupied by the equidimensional entities. Although this assumption does not prevent them (for example, RCC-8) from representing the spatial relationships between two-dimensional entities as well as the relationships between three-dimensional entities, it is not possible to model the relationships between entities of different dimensions in these theories.

Because of the importance of the boundaries of the spatial entities both in the mathematical topological space and in the ontology of space, Smith defined his spatial relations over the domain of three-dimensional entities (known as *realia* in philosophy), and considering boundaries for them either as fiat or bona fide entities.

Galton (Galton, 1996) asserts that we should not consider any form of spatial entity more fundamental than others. In this regard, he proposed Take-dim in which entities of any dimension (especially points, lines and surfaces) were adapted and separated from the rest of the world by their boundaries. This is the approach also followed in the INCH calculus, CODI and RCC*-9, although CODI extends its domain by considering three-dimensional entities and four-dimensional spatio-temporal regions into it.

Accepted Entities

Spatial regions are *regular* subsets of a topological space in all the introduced theories. This means that each of them has a uniform dimension. Regular regions (regardless of their dimensions) might always have a boundary (or be a regular closed set[3]), only have it as an optional part (or be a regular open set), or not necessarily either of them (i.e. neither an open or a closed set[4]). The INCH calculus, CODI and RCC*-9 all accept (regular) closed regions, whereas the regions may not be open or closed in Take-dim and Smith's theory.

Moreover, boundaries might be defined as *thin* (bodiless) or *thick* (bulky) components of the space. Take-dim, the INCH calculus and RCC*-9 consider boundaries as thin regions of the domain. On the other hand, boundaries are deemed to be of the same dimension as the bounded regions in Smith's theory. CODI's extensions, however, cover both thin and thick boundaries.

[3]Let X be a topological space. A subset A of X is called a regular closed set if A is equal to the closure of the interior of itself. It is called a regular open set if A is equal to the interior of the closure of itself.

[4]A subset A of a topological space is equal to the interior of the closure and the closure of the interior of itself.

A.2.2 Mereological Properties

So far, we have talked about the least restrictive mereological theory (Casati and Varzi, 1999), known as *ground mereology* (i.e. a theory develops based on the parthood as a reflexive, antisymmetric and transitive relation). However, this core basis can be made stronger, by adding principles, to meet certain criteria in the relevant theory. These principles guarantee the existence of some mereological properties in presence of some others. The current and the following sections give a brief review of these principles in the introduced multidimensional formalisms.

In this section, we express the principles which develop the mereological aspect of a theory by taking us from a whole to its parts. These can be found in all mereotopological theories either as an axiom or theorem regardless of their choice of primitives (i.e. mereological base, topological base or the mixture theory (already discussed in Subsection A.2.1)).

Supplementation Principle

One possible assumption to extend the ground mereological theory follows the intuition that a whole can never be divided into only one proper part. According to this idea, there exists more than one proper part for every divisible region.

This principle is known as the *supplementation* principle, which has two versions. Its first form, the *weak supplementation principle* (WSP), says that every proper part must be 'supplemented' by another, disparate part. The stronger expression of the principle, i.e. *strong supplementation principle* (SSP) says that if an entity, x, fails to have another entity, y, as its parts, then there must be something that parts of y and does not overlap x.

Among the introduced multidimensional mereotopological theories, only Smith developed his theory based on the WSP. According to this work, a three-dimensional spatial entity (note that the theory only models *realia* (i.e. real-world objects from spatial points of view) consists of more than one three-dimensional element. This version of the supplementation principle is compatible with the assumption of the fundamentality of the realia in this theory.

Since none of the spatial entities is more fundamental than the other in Take-dim, the INCH calculus and CODI, it is necessary for each entity to have parts with its own dimension to meet the intuition of the supplementation principle. Thus, these theories have considered the stronger version of the principle.

Unfortunately, RCC*-9 does not provide a complete enough axiomatization to cover this principle in either of its versions.

Unrestricted Fusion

One common way of extending a ground mereology is by adding a principle which guarantees the existence of a non-empty object which satisfies a specific condition. The principle is known as *infinitary closure condition* and is expressed via a fusion axiom. Expressing this principle requires to referring to the classes of the entities satisfying a specified condition which is not straight forward in the language of first-order logic (since the variables and quantifiers only range over individuals in this language). Thus, this principle has been seen in the theories using the language of higher-order logic (i.e. Take-dim and Smith's theory).

Take-dim uses a weak form of the fusion axiom (i.e. `TD-A6`) which limits the arbitrary fusion to finite classes of regions of the same dimensionality. However, the strong form of it is used in Smith's theory (i.e. `SM-D4`) which its existence (if definable) is guaranteed by `SM-A5`.

Mereological Operators

Another suggestion for expanding the theory of ground merereology is based on the need for the domain to be closed under mereological operator (including `sum`, `product`, `difference` and `complement`).

The `sum` operator defines the existence of a whole that *exactly* consists of the given entities, while the `product` operator describes the common entity as the intersection of a given pair of entities. While the `difference` operator returns the difference (or reminder) between an entity and its part, the `complement` operator gives back the remainder of the universe excluding a given entity. By the existence of the universal entity in the domain, the `sum` operator is always definable. However, the `product`, `difference` and `complement` are not always definable. For instance, the `product` of a pair of discrete regions is empty.

The formalisms were represented in the higher-order logical language (i.e. Take-dim and Smith's theory) use the fusion axiom to define the mereological operators, which gives them great power. For instance, the fusion of a cube and a line segment would be an entity consisting of both entities. However, Take-dim disallows arbitrary fusions (i.e. weakening the fusion axiom) and limits the operators to operate only on the equidimensional entities.

The mereological operators are defined in some of the first-order theories. The INCH calculus defines `sum` and `difference` operations via a pair of axioms (i.e. `IN-A9` and `IN-A10`) over the equidimensional entities and their uniqueness is guaranteed by another axiom. They then define the `product` operation in terms of `difference` while the `complement` can be defined as a special case of `difference`. CODI's axiomatization has been extend in (Hahmann, 2018) to define all the operations. However, the operations are not limited to the equidimensional entities here. RCC*-9, however, does not provide any definitions for these operations.

Atomism

The final assumption to strengthen a mereological theory is about whether there is any indivisible entity without any proper parts (i.e. *atom*) in the domain. All the introduced multidimensional mereotopological theories defined the atom except the RCC*-9 but under different assumptions.

Some mereotopological theories refuse to accept atoms in their domain. If a theory is axiomatized to be *atomless*, it assumes a proper part for all the entities which is enough to prove an infinite number of proper parts for all of them. Thus, these theories only can have models with infinitely many parts which leads to the continuity of the domain of discourse. An obvious consequence of developing an atomless theory is difficulty in referring to categories of entities of different dimensions. Also, the zero-extended entities (i.e. points) are only definable via nesting the higher dimensional entities in this type of theories. Thus, the multidimensional mereotopological theories either accept only bulky boundaries (i.e. being of the same dimensionality as the bounded entity) in their atomless version like CODI, or restrict the dimensionality of the accepted entities to three and their two-dimensional boundaries such as Smith's theory. In the presence of SSP in atomless theories, the interior of a closed region is its proper part. However, there is no entity to be a proper part of the region and overlap its interior. Therefore, the difference between the region closure and interior (i.e. the region's boundaries) could not overlap with any other entities. RCC*-9 might also be of this type if it adopts all RCC-8's assumptions.

In contrast to the atomless theories, some mereotopological theories (i.e. the INCH calculus and CODI) have a mutually disjoint version in which they assume everything is eventually made of atoms, i.e. *atomistic* mereological theories. They expect an end to the divisibility of all entities, and thus their domain of discourse is discrete. In other words, the atomistic theories assume the sum (join) of a collection of atoms to form every non-zero entity of the domain. This assumption classifies the entities based on their dimensions

Some other mereotopological theories (i.e. the INCH calculus, Take-dim) adopt mixed space in which atoms and non-atomic entities are acceptable, i.e. *atom-tolerant* mereological theories. They are a generalized form of atomless and atomistic theories. The atom-tolerant theories accept the zero extend entities (i.e. points) and consider them as a different ontological species compared to one to three-dimension entities.

A.2.3 Topological Properties

In this section, we will talk about topological aspects of the introduced multidimensional mereotopological theories. The principles discussed so far can extend any ground mereological theory. However, combining mereology with topology allows the self-connectedness and various topological operators to be defined in a theory.

Self-connectedness

The differences between solid wholes (like different parcels) and scattered wholes (like islands of a country) are expressible via the concept of *self-connectedness*. A region is self-connected if any two parts that make up the whole of the region are connected to each other.

In Smith's theory, the self-connectedness is defined in `SM-D4`, and is then extended to represent the maximal connected entity containing the self-connected entity (`SM-D19`). CODI also defines this concept in `CO-D14` and then makes it stronger by limiting it to interior self-connectedness. Although Take-dim and the INCH calculus have the potential to define the concept of self-connectedness (since they have required axioms and definitions), this property has not been defined in those works. One important advantage of the existence of a predicate to define self-connectedness (or the potential for its existence) is that it can support the definition of the universal entity as the maximal connected entity in the domain of discourse. Therefore, every smaller entity than the universal entity is connected to its complement.

Topological Operators

The topological operators return the topological notions of *interior*, *closure* and *boundary* of a given region. The topological operators are mainly defined to make the region-based theories comparable with their point-based counterparts. The *interior* of a point-set is defined as the union of all open sets contained in it. The *closure* of a point-set is defined as the intersection of all closed sets containing it. The *boundary* of a point-set is defined as the difference between the point-set's closure and interior.

Depending on whether the proposed theory intends to provide a standard topological space or not, it has a different approach to defining the topological operators. Smith's goal was to represent a counterpart space for the mathematical topological space. So, all these operators are explicitly defined by the use of the fusion axiom (see `SM-D11`, `SM-D10` and `SM-D9`).

The other multidimensional mereotopological theories do not aim to represent the standard topological space. Thus, the topological operators have not been defined in them. However, the importance of the boundary of the entities motivated researchers (including Smith) to use predicates to describe this element, but not as a function. The significance of the boundary gives an incentive to Take-dim and CODIB to count on it as primitives.

A.2.4 Reasoning

What we have discussed in the previous sections is an account of mereotopological representation of spatial information in a multidimensional space. Although knowledge representation might be enough for some applications, an intelligent system is of limited practical use without a reasoning mechanism. Spatial reasoning is concerned with the cognitive, formal and computational aspects of making logical inferences about the spatial environment (Worboys and Duckham, 2004). Cognitive scientists and psychologists are interested in the cognitive aspects of inference. However, logicians and computer scientists examine the formal, and the computational aspects of reasoning respectively. The focus of the current section is mostly closer to the logicians and computer scientist's point of view than the other researchers.

The general concept of spatial reasoning involves expressing the semantic interrelation of the spatial relationships discussed in an axiomatic theory. Axiomatic descriptions express ontological concepts concerning the domain of spatial relations and propose a taxonomic system of them. The axiomatic system, on the one hand, is used to identify the relations and, on the other hand, supports the interaction of the expected meaning. In this system, ontological verification is treated as a reasoning problem (i.e. consistency checking) but the other kind of reasoning (querying, etc.) is also supported. The system can show that whether a statement (the conjecture) is a logical consequence of the given definitions and axioms. The theories' inference mechanisms are different based on the logical languages they use. The multidimensional mereotopological theories proposed in FOL (i.e. the INCH calculus and CODI) use the *resolution* technique (i.e. rule of inference leading to a refutation theorem-proving) with the help of automated theorem provers while the ones proposed in HOL (i.e. Take-dim and Smith's theory) exploit natural deduction manually.

An alternative though more efficient approach to axiomatic systems is *constraint-based reasoning*. It is applicable in theories that support a set of jointly exhaustive and pairwise disjoint (JEPD) binary relationships. The core of this method is the *composition* operator which is applied over a pair of JEPD relations showing the spatial configurations of three entities in a chain[5]. The operation returns the strongest relationship between the two entities of the chain from the powerset of the JEPD relations, i.e. the set of all possible disjunctions of the JEPD relations. The results of applying the composition operation on all the pair of JEPD relations are stored in a matrix known as a *composition table*. The table is used as a tool to propagate its constraints over the domain and derive the relationship (or a disjunction of the possible relationships) between a pair of queried entities. The introduction of nine JEPD relations in RCC*-9 led to the construction of the relevant composition table in this formalism (Clementini and Cohn, 2014).

Since the composition technique reduces the domain of a decision variable to all of the relations that are stated over this variable and makes a recommendation (i.e. makes the reasoning problem decidable), theories supporting this technique are widely implemened in intelligent systems. However, as we said, it ignores (unary) relations (such as the ones used to classify spatial entities) and functions of the theory in its computation which are considered in axiomatic systems.

A.3 Summary

A summary of the detailed comparison of the multidimensional mereotopological theories is accounted in Chapter 2. However, the take-home message is equidimensional mereotopological theories have not been able to represent entities of different dimensions. On the other hand, the existing multidimensional mereotopological theories have not yet been tailored to efficient reasoning methods such as constraint-based reasoning. The possibility of having an efficient reasoning system is closely related to the representation method. This means that improvement in the expressivity of the spatial representation comes at the cost of losing reasoning capabilities, and vice versa. In this case, it would be desirable to present a multidimensional mereotopological theory that sits somewhere in the middle, in which there is an acceptable level of expressivity in the spatial representation as well as efficient inference. First-order logic, due to the possibility of quantifying variables, can meet these needs in terms of representing qualitative spatial relationships. Although the inference problems are generally undecidable in first-order logic, they can be solved by considering some conditions. One of them (which we will consider in our theory) is to represent space via binary relations. By using a set of binary relations, a relational network (graph) can be formed over the spatial entities. This network facilitates the search for extracting a new relation(s) between a given pair of entities. The facilitation is also closely related to the introduction of spatial relations with the JEPD property. As a result of this property, the space is categorized such that all possible spatial arrangements between every pair of entities belong to only one of these categories. In this case, the inference of the implicit relationship between the two entities is reduced to the search (in a graph) for at least one third common entity that is related to both given ones. By considering these findings, we proposed our theory

[5]Three entities x, y and z and a pair of binary relations R_1 & R_2 selected from the theory such that $R_1(x, y)$ and $R_2(y, z)$, the composition $R_3(x, z)$ represents a disjunction of all the possible binary relations holding between x and z in theory.

in Chapter 4 and demonstrate its reasoning aspect in Chapter 6.

Table A.1: Comparison of different multidimensional mereotopological theories.

	Properties \ Multidimensional theories	**The INCH calculus**	**CODI /CODIB**	**RCC*-9**	**Take-dim**	**Smith's theory**
Foundation	Representation language	FOL	FOL	FOL	HOL	HOL
	Basis	Mereotopology	Mereotopology	Topology	Mereology	Mereology
	Supported Dimensionality	0D–2D regions	0D–3D regions	0D–2D regions	0D–2D regions	3D regions
	Regions	Closed	Closed	Closed	Open/closed	Open/closed
Mereological properties	Supplementation principle	SSP	SSP	Not available	SSP	WSP
	Unrestricted Fusion	N.A	N.A	N.A	Applicable	Applicable
	Mereological operators	Defined for regions of equal dimension	Defined for all regions	Not defined	Defined for regions of equal dimension	Defined for three-dimensional regions (if exists)
	Atomisim	Atomistic or Atom-tolerant	Atomistic or Atomless	Atomless	Atom-tolerant	Atomless
Topological properties	Self-connectedness	Potential to define	Defined	Not defined	Potential to define	Defined
	Topological operators	Not defined	Not defined	Not defined	Not defined	*interior*, *closure*, *boundary*
	Boundary predicate	Defined	Defined	Defined	Defined	Defined
Reasoning		FOL	FOL	Constraint-based	HOL	HOL
		Theorem proving	Theorem proving		Theorem proving	Theorem proving

Appendix B

A Modification Of RCC*-9

DRC 16

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STATEMENT OF CONTRIBUTION
DOCTORATE WITH PUBLICATIONS/MANUSCRIPTS

We, the candidate and the candidate's Primary Supervisor, certify that all co-authors have consented to their work being included in the thesis and they have accepted the candidate's contribution as indicated below in the *Statement of Originality*.

Name of candidate:	Azadeh Izadi
Name/title of Primary Supervisor:	Senior Lecturer Dr. Kristin Stock
Name of Research Output and full reference:	
A modification of RCC*-9. GeoComputation, 18-21 September, Queenston, New Zealand, 2019. doi:10.17608/k6.auckland.9869117.v3	
In which Chapter is the Manuscript /Published work:	Appendix
Please indicate:	
• The percentage of the manuscript/Published Work that was contributed by the candidate:	85%
and	
• Describe the contribution that the candidate has made to the Manuscript/Published Work:	
The candidate identified a logical inconsistency in a previously defined theory, developed a solution to resolve it and wrote the first draft of the paper. The other authors provided advice, input and editing and verified the solution.	
For manuscripts intended for publication please indicate target journal:	
The paper has been presented and published in conference proceedings.	
Candidate's Signature:	Azadeh Izadi — Azadeh Izadi 2021.07.27 19:47:31 -06'00'
Date:	27/07/2021
Primary Supervisor's Signature:	Kristin Stock — Digitally signed by Kristin Stock Date: 2021.07.28 14:48:33 +12'00'
Date:	28/7/21

(This form should appear at the end of each thesis chapter/section/appendix submitted as a manuscript/ publication or collected as an appendix at the end of the thesis)

A modification of RCC*-9

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Abstract

RCC*-9 is a recently developed member of the region connection calculus family, and was introduced to represent topological relations in multidimensional space. In this paper, we discuss and address an inconsistency found in the RCC*-9 formalism, and propose a modified version of RCC*-9 which we call RCC*′-9. Furthermore, we prove the jointly exclusive and pairwise disjoint property of relations and theorems in the lattice of relations in the modified theory. Finally, we confirm the consistency of RCC*′-9 using a finite model finder.

Keywords: Multidimensional topological space, Region Connection Calculus, Qualitative spatial reasoning, Topological relationships.

1 Introduction

The formalization of topological relations has been an important research topic in the Geographical Information Systems (GIS) and Qualitative Spatial Reasoning (QSR) literature. These relations enable certain kinds of spatial querying, analysis and reasoning in GIS, such as whether two features are connected and if so what type of connection exists between them (e.g. whether one feature is proper part of the other, or whether two features touch or overlap each other). Some examples of topological relations include: a polygon representing a national park overlaps with several adjacent polygons representing different countries; census blocks lie entirely within a census tract; parcels, have edges that touch each other.

There are two main approaches to modeling topological relations between spatial entities in the literature. The first of these follows an intersection strategy, and includes the 9-intersection model (9-IM) (Egenhofer and Herring, 1991) its dimensionally extended version (DE-9IM) (Clementini et al., 1994), and the calculus-based method (CBM) (Clementini et al., 1993). Their spatial features consist of points, line segments and areas, thereby describing spatial objects in terms of their dimension which is similar to geometric data standards, such as the Open GeoSpatial Consortium (OGC). 9-IM (and DE-9IM) considers each object's interior (^{o}), boundary (δ) and exterior (^{e}) parts and then represents the spatial relation between pairs of objects using a matrix in which rows

*

represent the aspects (interior, boundary and exterior) of one object and columns represent the aspects of the other object, cell values being 0 or 1, indicating whether or not the object aspects intersect (dimension of the common part is also considered in DE-9IM). Because it considers every possible combination of these three aspects of an object, 9-IMs[1] introduce a large number of relations in a multidimensional space, and are thus not very user friendly. Also, it can only extract further knowledge over areal features. CBM reduces the number of relations by excluding the exterior aspect (i.e. $F1^o$-$F2^e$, $\delta F1$-$F2^e$, $F1^e$-$F2^e$ intersections are omitted)[2]. Instead, it describes the other intersections (interior and boundary) in terms of the participant objects (i.e., whether the common part is equal to one of the participants). Although it introduces a set of practical relations for human use, it does not support qualitative reasoning.

The second main approach adopts axiomatic systems such as (Clarke, 1981; Clark, 1985; Randell and Cohn, 1992) to represent topological relations. These systems are not only capable of introducing a set of relations for end users, but also supporting automated reasoning, i.e., inferring the existence of the relations to additional, unnamed entities, over all of the features in the spatial domain either by constructing a composition table based on the set of finite *jointly exclusive and pairwise disjoint* (JEPD) relations or directly through automated reasoners such as as first-order theorem provers or finite model finders. However, with few exceptions (Gotts work (Gotts, 1996), Galton's work (Galton, 1996) and CODI (Hahmann and Grüninger, 2011; Hahmann, 2018)) they only accept equidimensional spatial entities (e.g., only areal features) in their domain. For instance, equidimensional axiomatic systems are not able to describe a topological relationship between a road and park, if these objects are represented as a one-dimensional and two-dimensional spatial features, respectively, in a geographical dataset.

The development of a comprehensive theory of topological relations to overcome the respective shortcomings of the intersection approach (a large number of relations without a practical reasoning strategy) and axiomatic systems (inability to handle objects of different dimensions in the spatial domain) has long been an open problem in GISience (Galton, 2004). Much existing work (Gotts, 1996; Galton, 1996; Hahmann and Grüninger, 2011; Hahmann, 2018) on extending the axiomatic approach to a truly multidimensional theory has focused on developing a first-order logical axiomatization that affords reasoning with theorem provers. This work studies an alternative approach, the RCC*-9 (Clementini and Cohn, 2014), which more closely follows the early work in qualitative spatial reasoning by aiming to identify a set of JEPD relations to support composition based reasoning with the help of a composition table, which allows simply looking up the results of combinations of spatial relations. A second notable difference is that unlike (Gotts, 1996; Galton, 1996; Hahmann, 2013), (Clementini and Cohn, 2014) does not include a predicate (or predicates) for comparing the dimension of the participant entities. (Hahmann, 2018) includes a primitive relation of "lesser or equal dimension" and (Gotts, 1996; Galton, 1996) define similar relations in their formalism. RCC*-9 (Clementini and Cohn, 2014) aims to define a multidimensional theory without any dimensional comparison tool. However, a closer analysis of the relations in RCC*-9 reveals that they do not represent the expected spatial configuration, which means that the relations are not JEPD and therefore do not satisfy the lattice of the relations. In this paper, we modify the treatment of topological relations among multidimensional features introduced in RCC*-9 to address this problem.

This paper is structured as follows. In Section 2, we explain RCC-8 and RCC*-9 as homogeneous

[1] Although the earlier version of 9-IM (Egenhofer and Franzosa, 1991) is not for multidimensional cases, it follows the same strategy. However, it uses rules to eliminate unnecessary combinations.

[2] $F1$ and $F2$ are two sample spatial features.

and heterogeneous dimensional theories in the region connection calculus framework, and distinguish their attributes and capabilities. In Section 3, we focus on RCC*-9's formalism to explain its weaknesses and propose a solution. Theorems of JEPD properties and a lattice of relations are also proved over the solution. Finally, we check the consistency of the whole formalism in Section 4. Possibilities for extending this work are discussed in Section 5.

2 Region Connection Calculi (RCC)

RCC is a family of qualitative representations of topological relations between regions, and is used for constraint-based qualitative spatial reasoning. It has been extended in various directions, and in the following sections, we review one of the most well-known members of this family, RCC-8, and a more recent multidimensional version, RCC*-9, which is the focus of the current work.

2.1 RCC-8

RCC-8 (Randell et al., 1992) proposed a point-free topological space. Entities of the domain are known as regions in this theory. The regions are (non-empty) chunks of space occupied by physical objects. There is no dimensional difference between the regions and the universal embedding space (zero co-dimension). Moreover, every region only consists of equi-dimensional parts (regular subsets of the space). Also, there is no requirement for the regions to be internally connected (multi-piece regions are permitted).

RCC-8 is based on a single primitive binary relation: **C**(x,y), read as 'x connects with y', which is reflexive and symmetric. This relation holds when there is an overlap between the closures[3] of x and y, that is, when $\mathrm{cl}(x) \cap \mathrm{cl}(y) \neq \varnothing$.

Based on the **C** relation, the additional topological relations shown in Figure 1 are defined. Among them, a set of eight relations (numbered relations in Figure 1) form jointly exhaustive and pairwise disjoint (JEPD) set of relations, called *base relations*. It means that each pair of spatial regions of the considered domain is in exactly one of the eight JEPD relations. Since the spatial domain is indefinite, reasoning techniques mostly rely on verified composition of two base relations. So, an 8 by 8 composition table of base relations has been constructed for RCC-8.

Furthermore, a set of Boolean operations, including `sum`, `product`, `difference`, and `complement` are defined in the logical axiomatization of RCC. The domain is closed under these operations, as guaranteed by the introduction of the *NULL* entity, being defined as the `product` of the discrete regions in the domain. Likewise, all the regions are connected to a specific region known as the *universal* region which is an upper bound of the domain.

2.2 RCC*-9

RCC*-9 (Clementini and Cohn, 2014) extends RCC-8 by admitting the coexistence of regions of heterogeneous dimensions[4] Indeed, the regions are not lumps of space filled by physical objects as

[3]In point-set topology, the closure of a subset S of points in a topological space consists of all points in S together with all limit points of S.

[4]The reason for the asterisk in the name of RCC*-9 does not only indicate a change in the number of relations in comparison to RCC-8, there is also an additional spatial primitive that the new calculus is able to deal with.

1. $\mathbf{DC}(x,y) \equiv_{def} \neg\mathbf{C}(x,y)$ (x disconnected from y)
   $\mathbf{P}(x,y) \equiv_{def} \forall z[\mathbf{C}(z,x) \rightarrow \mathbf{C}(z,y)]$ (x is part of y)
   $\mathbf{PP}(x,y) \equiv_{def} \mathbf{P}(x,y) \wedge \neg\mathbf{P}(y,x)$ (x is proper part of y)
2. $\mathbf{EQ}(x,y) \equiv_{def} \mathbf{P}(x,y) \wedge \mathbf{P}(y,x)$ (x equals to y)
   $\mathbf{O}(x,y) \equiv_{def} \exists z[\mathbf{P}(z,x) \wedge \mathbf{P}(z,y)]$ (x overlaps y)
   $\mathbf{DR}(x,y) \equiv_{def} \neg\mathbf{O}(x,y)$ (x is discrete from y)
3. $\mathbf{NTPP} \equiv_{def} \mathbf{PP}(x,y) \wedge \neg\exists z[\mathbf{EC}(z,x) \wedge \mathbf{EC}(z,y)]$ (x is non-tangential proper part of y)
4. $\mathbf{TPP} \equiv_{def} \mathbf{PP}(x,y) \wedge \exists z[\mathbf{EC}(z,x) \wedge \mathbf{EC}(z,y)]$ (x is tangential proper part of y)
5. $\mathbf{PO}(x,y) \equiv_{def} \mathbf{O}(x,y) \wedge \neg\mathbf{P}(x,y) \wedge \neg\mathbf{P}(y,x)$ (x partially overlaps y)
6. $\mathbf{EC}(x,y) \equiv_{def} \mathbf{C}(x,y) \wedge \neg\mathbf{O}(x,y)]$ (x externally connected to y)
   $\mathbf{Pi}(x,y) \equiv_{def} \mathbf{P}(y,x)$ (y has part x)
   $\mathbf{PPi}(x,y) \equiv_{def} \mathbf{PP}(y,x)$ (y has proper part x)
7. $\mathbf{TPPi}(x,y) \equiv_{def} \mathbf{TPP}(y,x)$ (y has tangential proper part x)
8. $\mathbf{NTPPi}(x,y) \equiv_{def} \mathbf{NTPP}(y,x)$ (y has non-tangential proper part x)

Figure 1: Defined relations in the RCC-8 from Randell and Cohn (1992)

they are in RCC-8. They are represented based on the terminology of the features in the OGC (OGC, 2010), and can be points, linear or areal features. It is assumed that linear features are topologically closed (i.e. bounded by two, possibly coincident, endpoints), and that areal features are regularly closed (i.e. bounded by a single or multiple linear regions). To follow the OGC standard, objects with holes or multiple parts must also be supported.

The theory not only has the **C** relation as a primitive, but also utilizes a second primitive relation, **B**(x,y), read as 'x is boundary of y', such that x must be a proper part of y (i.e. ∀x∀y **B**(x,y) → **PP**(x,y) is an axiom). So, the boundary of an areal feature is its limiting closed curve, and the set of endpoints are considered the boundary of a linear feature. A point (or set of points) does not have any boundary.

Based on these primitive relations, the set of spatial relations shown in Figure 3 are defined. The definitions of **DC**, **P**, **PP**, and **EQ** are preserved from the RCC-8. However, the introduction of the new primitive relation causes some alteration in the definitions of other relations in RCC-8 such as **DR**, **NTPP**, **TPP**, **NTPPi**, **TPPi**, **O**, **PO**, and **EC**. Moreover, the boundary relation facilitates the introduction of a new spatial relation, **CR**(x,y), read as 'x crosses y', which can only hold between two linear features. The nine numbered relations in Figure 3 have again the JEPD property. Clementini and Cohn (2014) provide a composition table for RCC*-9 (see Figure 4 on p.14), and RCC*-9's composition table has an extra row and column relative to the RCC-8 composition table, corresponding to the **CR** relation. Also, whenever the composition of the two relations returns an overlap relation (or its special case, **PO**) in RCC-8's table, there is a possibility of seeing the **CR** relation between the participants in the entry table as well.

In short, RCC*-9 differs mostly from RCC-8 by accepting entities of different dimensions. It is a boundary-tolerant theory, in contrast to RCC-8, and is based on two primitives, **C** & **B**. The introduction of a new base relation, **CR** in RCC*-9 increases its expressiveness. Since RCC*-9 is based on the OGC's definitions of features (OGC, 2010), it may be considered more applicable in the geographic domain than RCC-8. A summary of the differences is shown in Table 1.

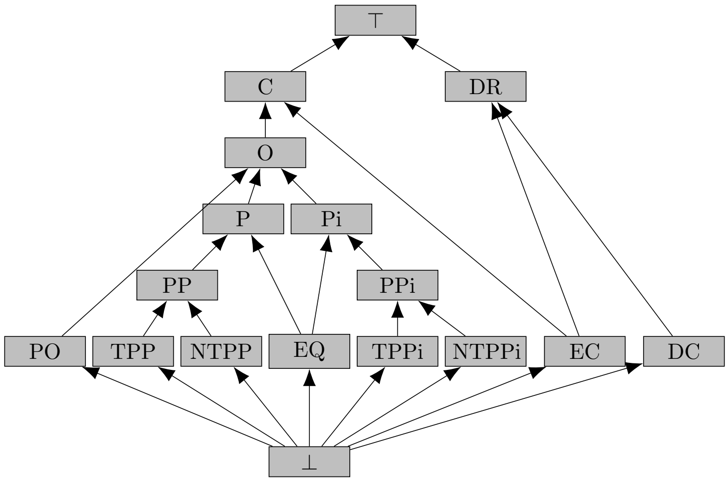

Figure 2: The lattice of RCC-8 topological relations from Randell and Cohn (1992). $\top$ denotes the universal relation that applies to any pair of regions (i.e., true) and $\perp$ denotes the empty relation that never holds (i.e., is always false). Arrows denote a specialization, e.g., the arrow from **EC** to **DR** shows that **EC** specializes **DR**, i.e. if $\mathbf{EC}(x, y)$ is true for arbitrary x and y then $\mathbf{DR}(x, y)$ must also be true. The lattice consists of jointly exhaustive relation. For example, if $\mathbf{DR}(x, y)$ holds for arbitrary x asn y, then either $\mathbf{EC}(x, y)$ or $\mathbf{DC}(x, y)$ is implied. The relations are pairwise disjoint, for example, $\mathbf{EC}(x, y)$ and $\mathbf{DC}(x, y)$ cannot be true at the same time for a pair x and y.

Table 1: Comparison of RCC-8 and RCC*-9

Theory / Properties	RCC-8	RCC*-9
Entities	Homogeneous regions Regular closed	Heterogeneous regions Regular closed
Boundary	Without	With
Primitive relation(s)	1	2
Number of base relations	8	9
Boolean operators	Yes	No
Composition table	Available(8*8)	Available(9*9)
Application	All physical processes	Geographical domain

3 RCC*-9 under surveillance

When we study RCC*-9 in more detail, its lattice of relations (Figure 4) gives rise to some logical statements that must be true in the logical axiomatization of RCC*-9, in order for the lattice to be correct. These statements can be divided into two types: *specialization* and *subsumption*. Specialization is captured by rule I, while rule II captures subsumption.

I) Where there is an edge between two relations in a lattice, some source relation **S** (lower in the lattice) implies the target relation **T** (further up in the lattice):

$$\mathbf{S}(x, y) \rightarrow \mathbf{T}(x, y)$$

For example, $\mathbf{CR}(x, y) \rightarrow \mathbf{C}(x, y)$.
When one relation points to (i.e. specializes) more than a single relation (e.g., **EC** specializes **C** and **DR**), then the specialized relation implies all of the relations it points to. For example, $\mathbf{EC}(x, y) \rightarrow \mathbf{DR}(x, y) \wedge \mathbf{C}(x, y)$.

1. $\mathbf{DC}(x,y) \equiv_{def} \neg\mathbf{C}(x,y)$ (x disconnected from y)
   $\mathbf{P}(x,y) \equiv_{def} \forall z[\mathbf{C}(z,x) \to \mathbf{C}(z,y)]$ (x is part of y)
   $\mathbf{PP}(x,y) \equiv_{def} \mathbf{P}(x,y) \wedge \neg\mathbf{P}(y,x)$ (x is proper part of y)
2. $\mathbf{EQ}(x,y) \equiv_{def} \mathbf{P}(x,y) \wedge \mathbf{P}(y,x)$ (x equals to y)
3. $\mathbf{NTPP}(x,y) \equiv_{def} \mathbf{PP}(x,y) \wedge \forall z[\mathbf{B}(z,y) \to \mathbf{DC}(x,z)]$ (x is non-tangential proper part of y)
4. $\mathbf{TPP}(x,y) \equiv_{def} \mathbf{PP}(x,y) \wedge \neg\mathbf{NTPP}(x,y)$ (x is tangential proper part of y)
   $\mathbf{O}(x,y) \equiv_{def} \exists z[\mathbf{NTPP}(z,x) \wedge \mathbf{NTPP}(z,y)] \wedge \exists t[\mathbf{TPP}(t,x) \wedge \mathbf{TPP}(t,y)]$ (x overlaps y)
5. $\mathbf{PO}(x,y) \equiv_{def} \mathbf{O}(x,y) \wedge \neg\mathbf{P}(x,y) \wedge \neg\mathbf{P}(y,x)$ (x partially overlaps y)
6. $\mathbf{EC}(x,y) \equiv_{def} \mathbf{C}(x,y) \wedge \neg\mathbf{O}(x,y) \wedge \forall z[[\mathbf{P}(z,x) \wedge \mathbf{P}(z,y)] \to \mathbf{TPP}(z,x) \vee \mathbf{TPP}(z,y)]$($x$ externally connected to y)
   $\mathbf{DR}(x,y) \equiv_{def} \mathbf{EC}(x,x) \vee \mathbf{DC}(x,y)$ (x is discrete from y)
7. $\mathbf{CR}(x,y) \equiv_{def} \mathbf{C}(x,y) \wedge \neg\mathbf{O}(x,y) \wedge \neg\mathbf{EC}(x,y)$ (x crosses y)
   $\mathbf{Pi}(x,y) \equiv_{def} \mathbf{P}(y,x)$ (y has part x)
   $\mathbf{PPi}(x,y) \equiv_{def} \mathbf{PP}(y,x)$ (y has proper part x)
8. $\mathbf{TPPi}(x,y) \equiv_{def} \mathbf{TPP}(y,x)$ (y has tangential proper part x)
9. $\mathbf{NTPPi}(x,y) \equiv_{def} \mathbf{NTPP}(y,x)$ (y has non-tangential proper part x)

Figure 3: Defined relations in the RCC*-9 from Clementini and Cohn (2014).

II) Where two (or more) relations $\mathbf{S}_1$ to $\mathbf{S}_n$ specialize a single relation $\mathbf{T}$ (e.g. **CR**, **O** and **EC** all specialize **C**), then the disjunction of the specialized relations is equivalent to the target relation:

$$\mathbf{T}(x,y) \leftrightarrow \mathbf{S}_1(x,y) \vee \cdots \vee \mathbf{S}_n(x,y)$$

Also, the nine base relations, $\mathbf{R}_i$, of RCC*-9 ($\mathbf{R}_i\ i : 1, 2, ..., 9$) must satisfy the following two properties as well:

III) $\neg\mathbf{R}_{i1}(x,y) \vee \neg\mathbf{R}_{i2}(x,y)$ (pairwise disjoint),

IV) $\mathbf{R}_1(x,y) \vee \mathbf{R}_2(x,y) \vee\ ... \vee \mathbf{R}_9(x,y)$ (jointly exhaustive).

To verify the lattice of RCC*-9, we applied and checked these rules (I - IV) over the relations. However, we identified some problems. According to the above mentioned properties in the lattice of RCC*-9, the overlap relation is a generalized form of the **PO**, **P** and $\mathbf{P}^{-1}$ relations, so according

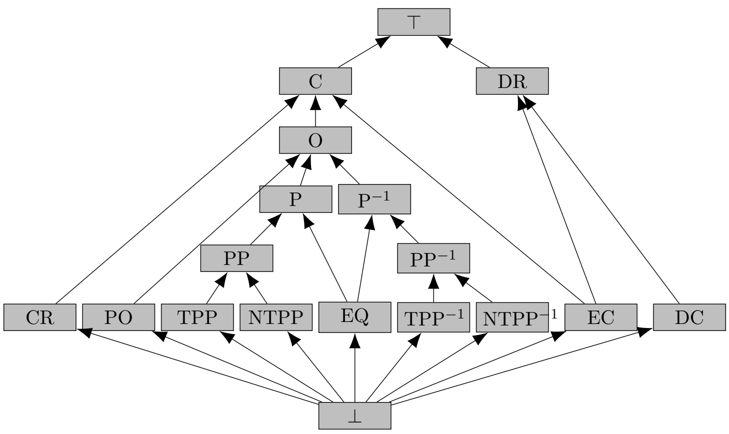

Figure 4: The lattice of the RCC*-9's spatial relations from Clementini and Cohn (2014).

to rule II, we have:

$$\mathbf{O}(x,y) \leftrightarrow \mathbf{PO}(x,y) \vee \mathbf{P}(x,y) \vee \mathbf{P}^{-1}(x,y) \quad (A)$$

This predicate means that not only must the overlap relation (**O**(x,y)) imply its specialized relations, the specialized relations must also imply the overlap relations. In other words, the overlap relation must cover all of thee subsumed relations directly (**P**,**Pi** and **PO**) and indirectly (**PP**,$\mathbf{PP}^{-1}$,**NTPP**, $\mathbf{NTPP}^{-1}$, **TPP** and $\mathbf{TPP}^{-1}$).

On the other hand, the definition of the **O**(x,y) from Figure 3 says:

$$\exists z[\mathbf{NTPP}(z,x) \wedge \mathbf{NTPP}(z,y)] \wedge \exists t[\mathbf{TPP}(t,x) \wedge \mathbf{TPP}(t,y)]$$

By considering (A), we expect that **O**(x,y) is entailed by **P**(x,y):

$$\mathbf{P}(x,y) \rightarrow \mathbf{O}(x,y) \quad (B)$$

Since we fail in showing that the union of a set of all axioms and defined relations (Γ), and the negation of (B) is not satisfiable, there must be a model for it. The finite model finder, Mace4 (McCune, 2006), searches for finite models of it. For a given size two domain ({**0**,**1**}), all instances of the union over this domain are generated (see Table 2). As you can see, while there is a model for **P**(x,y) (i.e.,"1"s in P: table), there is not any model for **O**(x,y) (i.e., "0"s in O: table). Alternatively, the model is a counter-model for $\Gamma \cup (\mathbf{P}(x,y) \rightarrow \mathbf{O}(x,y))$, and so **P**(x,y) does not entail **O**(x,y).

Table 2: Model provided by Mace4 for $\Gamma \cup \neg(\mathbf{P}(x,y) \rightarrow \mathbf{O}(x,y))$.

B:	**0**	**1**
0	0	0
1	0	0

C:	**0**	**1**
0	1	1
1	1	1

DC:	**0**	**1**
0	0	0
1	0	0

NTPP:	**0**	**1**
0	1	1
1	1	1

O:	**0**	**1**
0	0	0
1	0	0

P:	**0**	**1**
0	1	1
1	1	1

PP:	**0**	**1**
0	1	1
1	1	1

TPP:	**0**	**1**
0	0	0
1	0	0

The source of this unexpected behavior seems to be the definition of **O**(x,y). Its truth depends on the existence of an object (t) that is the tangential proper part of x and y at the same time. Such a t must be found in 'all' the specialized relations of **O**. However, if x is a non-tangential proper part of y, there is no such t (see Figure 5). So, RCC*-9 has defined a different notion of overlap than RCC-8. RCC*-9's overlap relation actually captures the partially overlap relation (see Figure

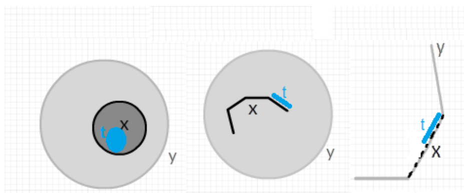

Figure 5: NTPP relation in RCC*-9

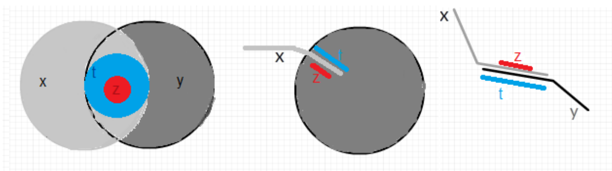

Figure 6: PO relation in RCC*-9

6) rather than overlap. In order to reflect this, we rename **O**(x,y) to partially overlap relation with prime, $\mathbf{PO'}$ (all the relations in our new theory use $'$ [prime] to distinguish them from the original versions):

$$\mathbf{PO'}(x,y) \equiv_{def} \exists z[\mathbf{NTPP}(z,x) \wedge \mathbf{NTPP}(z,y)] \wedge \exists t[\mathbf{TPP}(t,x) \wedge \mathbf{TPP}(t,y)]$$

and we redefine the overlap relation, $\mathbf{O'}(x,y)$, as follows:

$$\mathbf{O'}(x,y) \equiv_{def} \mathbf{PO'}(x,y) \vee \mathbf{P}(x,y) \vee \mathbf{P}^{-1}(x,y)$$

Thus **O**(x,y) is replaced by $\mathbf{O'}$(x,y) in all the relations and theorems, so the definitions of $\mathbf{EC'}$(x,y) and $\mathbf{CR'}$(x,y) relations are modified consequently:

$$\mathbf{EC'}(x,y) \equiv_{def} \mathbf{C}(x,y) \wedge \neg\mathbf{O'}(x,y) \wedge \forall z[[\mathbf{P}(z,x) \wedge \mathbf{P}(z,y)] \rightarrow \mathbf{TPP}(z,x) \vee \mathbf{TPP}(z,y)]$$

$$\mathbf{CR'}(x,y) \equiv_{def} \mathbf{C}(x,y) \wedge \neg\mathbf{O'}(x,y) \wedge \neg\mathbf{EC'}(x,y)$$

We name this modified version of the theory RCC*′-9. The next step is to check the rules of the lattice over this modified set of relations. To do it, we check rules I and II on all of the relations. These rules construct a set of theorems that are listed and proved in Appendix A.

Moreover, the problem in the definition of overlap that has previously been mentioned has the result that overlap does not satisfy rule III over its subset relations. For instance, $\neg\mathbf{PO}(x,y) \vee \neg\mathbf{TPP}(x,y)$ is not provable, since partially overlap and proper part relations (by their definitions) do not represent completely distinguished spatial arrangements. Since the relations must have the JEPD property in order to support reasoning, we must also confirm that the modified set of relations are also JEPD. We achieve this by proving all the theorems shown in Appendix B, which are generated by applying rules III and IV.

Further clarification of the theory is also necessary to provide a clear description of the topological domain. To achieve this goal, more theorems are needed to put more restrictions on the RCC′*-9 relations, and these are contained in Appendix C (all references beginning with Ext.T in Table 3 and the following paragraphs refer to Appendix C). Specifically, the axioms of the theory imply some properties for relations as can be seen in Table 3. Here, the identity of two features is a special case of their equality (Ext.T.9), consequently two non-identical entities are not equal (Ext.T.10). Also, we conclude that a boundary part of a feature is its tangential proper part as well (Ext.T.29).

Table 3: Properties of the relations in the RCC*′-9 and their relevant theorems in Appendix C

Relation	**Properies**
DC(x,y)	Irreflexive (Ext.T.1), Symmetric (Ext.T.2)
P(x,y)	Reflexive (Ext.T.3), Anti-symmetric (Ext.T.4), Transitive (Ext.T.5).
EQ(x,y)	Reflexive (Ext.T.6), Symmetric (Ext.T.7), Transitive (Ext.T.8)
PP(x,y)	Irreflexive (Ext.T.11), Asymmetric (Ext.T.12), Transitive (Ext.T.13)
$\mathbf{O'}$(x,y)	Reflexive (Ext.T.14), Symmetric (Ext.T.15)
DR(x,y)	Irreflexive (Ext.T.16), Symmetric (Ext.T.17)
$\mathbf{PO'}$(x,y)	Irreflexive (Ext.T.18), Symmetric (Ext.T.19)
$\mathbf{EC'}$(x,y)	Irreflexive (Ext.T.20), Symmetric (Ext.T.21)
TPP(x,y)	Irreflexive (Ext.T.22), Asymmetric (Ext.T.23)
NTPP(x,y)	Irreflexive (Ext.T.24), Asymmetric (Ext.T.25), Transitive (Ext.T.26)
$\mathbf{CR'}$(x,y)	Irreflexive (Ext.T.27), Symmetric (Ext.T.28)

Theorems (Ext.T.30) and (Ext.T.31) show the relationship between the RCC*′-9 and the classical mereotopological calculus (Leonard and Goodman, 1940). With the absence of the cross relation in the set of supported relations, RCC*′-9 collapses to RCC-8. Finally, the theorem (Ext.T.32) says that a non-tangential proper part of a region is part of its interior, so the connection of a feature with any interior part of another feature implies either overlap or cross. Any part of the non-tangential proper part of a region is its non tangential proper part (Ext.T.33). If an entity is both part of a second entity and connected to a third entity, the two other (second and third) entities are connected (Ext.T.34).

4 Logical verification

To check the correctness of RCC*′-9, we exploit consistency checking, which is a standard technique in first order logic. It confirms that the formalism does not entail any contradiction after instantiating all the axioms and a set of provable theorems ($< F >$) over the domain. Mathematically, there must be no formula (ϕ) such that ϕ and $\neg\phi$ are a member of $< F >$ simultaneously.

This technique involves generating some finite models via a finite model finder. This technique is implemented in the Macleod suit of tools [5] that was previoulsy used to check the consistency of RCC-8 and some other theories with the help of the finite model finder Mace4. We used the same approach to prove the consistency of RCC*′-9.

5 Conclusion and further work

Since spatial features may be point, linear or areal features in GIS, having a qualitative theory that can support querying over multidimensional data is crucial. RCC*-9 aims to meet this goal. However, we demonstrate that the **O** relation in RCC*-9 does not capture the intended spatial configuration. The main contribution of this paper is the introduction of RCC*′-9 as a modification of RCC*-9 that resolves the identified problem. We prove the JEPD properties of the relations and theorems relevant to the lattice of relations in RCC*′-9 and we evaluate the consistency of the theory by finding finite models via Mace4.

Further research is needed to check whether the composition table of RCC*′-9 remains unchanged and also provide a formal proof of its correctness. In addition, verifying the theory with some sample data sets would be a further useful verification technique to further ensure the consistency and appropriateness of the model in real world scenarios.

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⋄	DC	EC	PO	TPP	NTPP	TPPi	NTPPi	EQ	CR
DC	*	DR, PO, PP, CR	DR, PO, PP, CR	DR, PO, PP, CR	DR, PO, PP, CR	DC	DC	DC	DR, PO, PP, CR
EC	DR, PO, PPi, CR	DR, PO,TPP, TPPi, EQ, CR	DR, PO, PP, CR	EC, PO, PP, CR	PO, PP, CR	DR	DC	EC	DR, PO, PP, CR
PO	DR, PO, PPi, CR	DR, PO, PPi, CR	*	PO, PP, CR	PO,PP, CR	DR, PO, PPi, CR	DR, PO, PPi, CR	PO	DR, PO, PP, PPi, CR
TPP	DC	DR	DR, PO, PP, CR	PP	NTPP	DR, PO, TPP, TPPi, EQ, CR	DR, PO PPi, CR	TPP	DR, PO, PP, CR
NTPP	DC	DC	DR, PO, PP, CR	NTPP	NTPP	DR, PO, PP, CR	*	NTPP	DR, PO, PP, CR
TPPi	DR, PO, PPi, CR	EC,PO PPi, CR	PO, PPi CR	PO, EQ, TPP, TPPi	PO, PP, CR	PPi	NTPPi	TPPi	PO, PPi, CR
NTPPi	DR, PO, PPi, CR	PO, PPi, CR		PO, PPi, CR	PO, PPi, CR	O, CR NTPPi	NTPPi	NTPPi	PO, PPi, CR
EQ	DC	EC	PO	TPP	NTPP	TPPi	NTPPi	EQ	CR
CR	DR, PO, PPi, CR	DR, PO, PPi, CR	DR, PO, PPi,PP, CR	PO, PP, CR	PO, PP, CR	DR, PO, PPi, CR	DR, PO, PPi, CR	CR	*

Table 4: Composition table of relations in RCC*-9

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Appendices

A Theorems of lattice of relations in RCC*′9

A diagrammatic representation of the lattice is contained in (Figure.4) [6]. Here we provide a set of proved theorems. All these theorems are provable by using the refutation technique on the CNFs of the formulas.

$(T.1)$ $\forall x \forall y C(x, y) \vee DR(x, y)$
$(T.2)$ $\forall x \forall y C(x, y) \leftrightarrow CR'(x, y) \vee O'(x, y) \vee EC'(x, y)$
$(T.3)$ $\forall x \forall y DR(xy) \leftrightarrow EC'(x, y) \vee DC(x, y))$
$(T.4)$ $\forall x \forall y DC(x, y) \rightarrow DR(x, y)$
$(T.5)$ $\forall x \forall y EC'(x, y) \rightarrow C(x, y)$
$(T.6)$ $\forall x \forall y EC'(x, y) \rightarrow DR(x, y)$
$(T.7)$ $\forall x \forall y EC'(x, y) \rightarrow C(x, y) \wedge DR(x, y)$
$(T.8)$ $\forall x \forall y O'(x, y) \rightarrow C(x, y)$
$(T.9)$ $\forall x \forall y O'(x, y) \leftrightarrow P(x, y) \vee Pi(x, y) \vee PO'(x, y)$
$(T.10)$ $\forall x \forall y Pi(x, y) \rightarrow O'(x, y)$
$(T.11)$ $\forall x \forall y PPi(x, y) \rightarrow Pi(x, y)$
$(T.12)$ $\forall x \forall y PPi(x, y) \leftrightarrow TPPi(x, y) \vee NTPPi(x, y)$
$(T.13)$ $\forall x \forall y TPPi(x, y) \rightarrow PPi(x, y)$
$(T.14)$ $\forall x \forall y NTPPi(x, y) \rightarrow PPi(x, y)$
$(T.15)$ $\forall x \forall y EQ(x, y) \rightarrow Pi(x, y)$
$(T.16)$ $\forall x \forall y EQ(x, y) \rightarrow P(x, y)$
$(T.17)$ $\forall x \forall y P(x, y) \rightarrow O'(x, y)$
$(T.18)$ $\forall x \forall y PP(x, y) \rightarrow P(x, y)$
$(T.19)$ $\forall x \forall y PP(x, y) \leftrightarrow TPP(x, y) \vee NTPP(x, y)$
$(T.20)$ $\forall x \forall y TPP(x, y) \rightarrow PP(x, y)$
$(T.21)$ $\forall x \forall y NTPP(x, y) \rightarrow PP(x, y)$
$(T.22)$ $\forall x \forall y PO'(x, y) \rightarrow O'(x, y)$
$(T.23)$ $\forall x \forall y CR'(x, y) \rightarrow C(x, y)$

B Theorems of JEPD property of the RCC*′9

Below are assembled together a set of theorems that define the JEPD property of the RCC*′-9's relations. All these theorems are provable by using the refutation technique on the CNFs of the formulas.

$(T.24)$ $\forall x \forall y \neg CR'(x, y) \vee \neg DC(x, y)$
$(T.25)$ $\forall x \forall y \neg CR'(x, y) \vee \neg EC'(x, y)$
$(T.26)$ $\forall x \forall y \neg CR'(x, y) \vee \neg NTPPi(x, y)$
$(T.27)$ $\forall x \forall y \neg CR(x, y) \vee \neg TPPi(x, y)$
$(T.28)$ $\forall x \forall y \neg CR'(x, y) \vee \neg EQ(x, y)$
$(T.29)$ $\forall x \forall y \neg CR'(x, y) \vee \neg NTPP(x, y)$
$(T.30)$ $\forall x \forall y \neg CR(x, y) \vee \neg TPP(x, y)$

[6]These theorems are for RCC*′-9

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(T.31) $\forall x \forall y \neg CR(x,y) \vee \neg PO(x,y)$
(T.32) $\forall x \forall y \neg PO(x,y) \vee \neg DC(x,y)$
(T.33) $\forall x \forall y \neg PO(x,y) \vee \neg EC(x,y)$
(T.34) $\forall x \forall y \neg PO(x,y) \vee \neg NTPPi(x,y)$
(T.35) $\forall x \forall y \neg PO(x,y) \vee \neg TPPi(x,y)$
(T.36) $\forall x \forall y \neg PO(x,y) \vee \neg EQ(x,y)$
(T.37) $\forall x \forall y \neg PO(x,y) \vee \neg NTPP(x,y)$
(T.38) $\forall x \forall y \neg PO(x,y) \vee \neg TPP(x,y)$
(T.39) $\forall x \forall y \neg TPP(x,y) \vee \neg DC(x,y)$
(T.40) $\forall x \forall y \neg TPP(x,y) \vee \neg EC(x,y)$
(T.41) $\forall x \forall y \neg TPP(x,y) \vee \neg NTPPi(x,y)$
(T.42) $\forall x \forall y \neg TPP(x,y) \vee \neg TPPi(x,y)$
(T.43) $\forall x \forall y \neg TPP(x,y) \vee \neg EQ(x,y)$
(T.44) $\forall x \forall y \neg TPP(x,y) \vee \neg NTPP(x,y)$
(T.45) $\forall x \forall y \neg NTPP(x,y) \vee \neg DC(x,y)$
(T.46) $\forall x \forall y \neg NTPP(x,y) \vee \neg EC(x,y)$
(T.47) $\forall x \forall y \neg NTPP(x,y) \vee \neg NTPPi(x,y)$
(T.48) $\forall x \forall y \neg NTPP(x,y) \vee \neg TPPi(x,y)$
(T.49) $\forall x \forall y \neg NTPP(x,y) \vee \neg EQ(x,y)$
(T.50) $\forall x \forall y \neg EQ(x,y) \vee \neg TPPi(x,y)$
(T.51) $\forall x \forall y \neg EQ(x,y) \vee \neg NTPPi(x,y)$
(T.52) $\forall x \forall y \neg EQ(x,y) \vee \neg EC(x,y)$
(T.53) $\forall x \forall y \neg EQ(x,y) \vee \neg DC(x,y)$
(T.54) $\forall x \forall y \neg TPPi(x,y) \vee \neg NTPPi(x,y)$
(T.55) $\forall x \forall y \neg TPPi(x,y) \vee \neg EC(x,y)$
(T.56) $\forall x \forall y \neg TPPi(x,y) \vee \neg DC(x,y)$
(T.57) $\forall x \forall y \neg EC'(x,y) \vee \neg DC(x,y)$
(T.58) $\forall x \forall y \neg EC'(x,y) \vee \neg NTPPi(x,y)$
(T.59) $\forall x \forall y CR'(x,y) \vee PO'(x,y) \vee NTPP(x,y) \vee TPP(x,y) \vee EQ(x,y) \vee TPP^{-1}(x,y) \vee NTPP^{-1}(x,y) \vee EC'(x,y) \vee DC(x,y)$

C Other necessary theorems

Below are the set of theorems that are necessary to support the defined spatial relations in RCC*′-9.
(Ext.T.1) $\forall x \neg DC(x,x)$ (*From reflexivity of C and definition of DC*)
(Ext.T.2) $\forall x \forall y DC(x,y) \rightarrow DC(y,x)$ (*From symmetry of C and definition of DC*)
(Ext.T.3) $\forall x P(x,x)$ (*From definition of P*)
(Ext.T.4) $\forall x \forall y P(x,y) \wedge P(y,x) \rightarrow EQ(x,y)$ (*From definition of EQ*)
(Ext.T.5) $\forall x \forall y \forall z P(x,y) \wedge P(y,z) \rightarrow P(x,z)$ (*From definition of P*)
(Ext.T.6) $\forall x EQ(x,x)$ (*From definitions of P EQ*)
(Ext.T.7) $\forall x \forall y EQ(x,y) \rightarrow EQ(y,x)$ (*From definition of EQ*)
(Ext.T.8) $\forall x \forall y \forall z EQ(x,y) \wedge EQ(y,z) \rightarrow EQ(x,z)$ (*From definition of EQ and Ext.T.5*)
(Ext.T.9) $\forall x \forall y (x = y) \rightarrow EQ(x,y)$
(Ext.T.10) $\forall x \forall y (x \neq y) \rightarrow \neg EQ(x,y)$
(Ext.T.11) $\forall x \neg PP(x,x)$ (*From definition of PP*)
(Ext.T.12) $\forall x \forall y PP(x,y) \rightarrow \neg PP(y,x)$ (*From definition of PP*)

$(Ext.T.13)$	$\forall x \forall y \forall z PP(x,y) \wedge PP(y,z) \rightarrow PP(x,z)$	*(From definition of PP and Ext.T.5)*
$(Ext.T.14)$	$\forall x O'(x,x)$	*(From definitions of P, O)*
$(Ext.T.15)$	$\forall x \forall y O'(x,y) \rightarrow O'(y,x)$	*(From definition of O)*
$(Ext.T.16)$	$\forall x \neg DR(x,x)$	*(From definitions of DC, O, DR)*
$(Ext.T.17)$	$\forall x \forall y DR(x,y) \rightarrow DR(y,x)$	*(From definitions of O, DR)*
$(Ext.T.18)$	$\forall x \neg PO'(x,x)$	*(From definitions of DC PO)*
$(Ext.T.19)$	$\forall x \forall y PO'(x,y) \rightarrow PO'(y,x)$	*(From definitions of PO, O)*
$(Ext.T.20)$	$\forall x \neg EC'(x,x)$	*(From definitions of DC, O, EC)*
$(Ext.T.21)$	$\forall x \forall y EC'(x,y) \rightarrow EC'(y,x)$	*(From symmetry C and definitions of O, EC)*
$(Ext.T.22)$	$\forall x \neg TPP(x,x)$	*(From definitions of DC TPP)*
$(Ext.T.23)$	$\forall x \forall y TPP(x,y) \rightarrow \neg TPP(y,x)$	*(From definitions of TPP and Ext.T.22)*
$(Ext.T.24)$	$\forall x \neg NTPP(x,x)$	*(From symmetry C and definition of NTPP)*
$(Ext.T.25)$	$\forall x \forall y NTPP(x,y) \rightarrow \neg NTPP(y,x)$	*(From definitions of NTPP Ext.T.24)*
$(Ext.T.26)$	$\forall x \forall y \forall z NTPP(x,y) \wedge NTPP(y,z) \rightarrow NTPP(x,z)$	*(From definitions of NTPP, EQ and Ext.T.5)*
$(Ext.T.27)$	$\forall x \neg CR'(x,x)$	*(From definitions of DC, O, CR)*
$(Ext.T.28)$	$\forall x \forall y CR'(x,y) \rightarrow CR'(y,x)$	*(From definition of CR)*
$(Ext.T.29)$	$\forall x \forall y B(x,y) \rightarrow TPP(x,y)$	*(From definition of TPP and boundary axiom)*
$(Ext.T.30)$	$\forall x \forall y \neg EC(x,y) \wedge \neg CR'(x,y) \leftrightarrow (C(x,y) \leftrightarrow O'(x,y))$	*(From all axioms and definitions of DC, O, EC, CR)*
$(Ext.T.31)$	$\forall x \forall y [\neg \exists z EC'(z,x) \wedge \neg CR'(z,x) \rightarrow [P(x,y) \leftrightarrow \forall u [O'(u,x) \rightarrow O'(u,y)]]]$	*(From definitions of CR, EC, and Ext.T.5)*
$(Ext.T.32)$	$\forall x \forall y \forall z NTPP(x,y) \wedge C(z,x) \rightarrow O'(z,y) \vee CR'(z,y)$	*(From definitions of O, NTPP, CR, and Ext.T.5)*
$(Ext.T.33)$	$\forall x \forall y \forall z P(x,y) \wedge NTPP(y,z) \rightarrow NTPP(x,z)$	*(From definitions of P, NTPP, and Ext.T.5)*
$(Ext.T.34)$	$\forall x \forall y C(x,y) \leftrightarrow \exists z P(z,y) \wedge C(z,x)$	*(From definitions of P, and Ext.T.5)*

Appendix C

Github's Folders

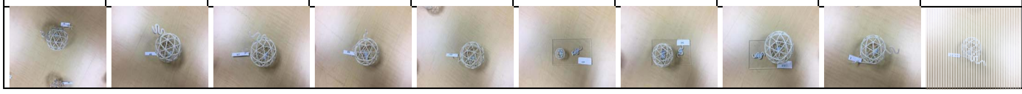
Github's directory is accessible via `https://github.com/azadehi/PhD-Thesis.git`. It consists of several folders containing 1256 theorems as follow:

- Axioms and definitions folders consisting of CLIF files of all the formalism's axioms and definitions and their models.
- Theorems folder have all the provable properties of the formalism such as irreflexivity of the proper parthood relation etc.
- JEPD folder contains proofs of jointly exclusive and pairwise disjoint property for a subset of mereotopological relations introduced in the formalism.
- Lattice folder contains proofs relevant to the hierarchical structure of introduced mereotopological relations.
- Composition folder contains proofs and models of the eneries of composition table.

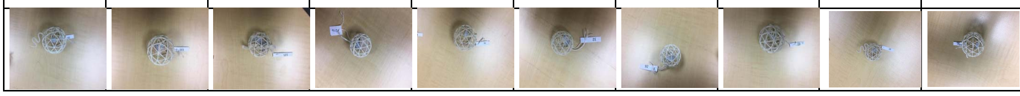
Appendix D

Stimulus Materials For The Empirical Investigations

EC3	EC2	IC1	OV7	I1	D3	D2	D1	OV8	IC2	Relation	Line-Volume
										Model	

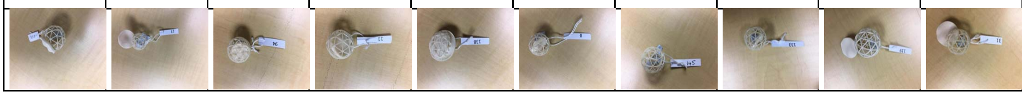


EC1	TPP1	OV1	I3	OV5	TPP3	TPP4	I3	IC3	OV6	Relation	Line-Volume
										Model	

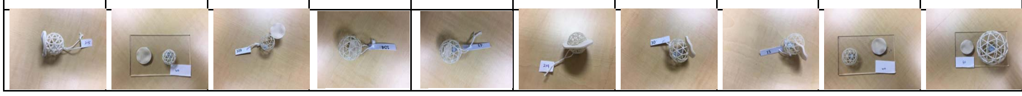


IC4	OV2	IC5	I2	OV4	OV3	OV9	Relation	Line-Volume
							Model	

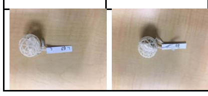
IC1	OV3	TPP6	TPP1	OV5	TPP2	TPP4	I2	EC2	OV1	Relation	Surface-Volume
										Model	



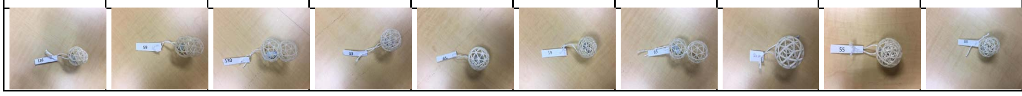
IC2	D1	EC1	I4	TPP3	OV6	IC3	OV4	D3	D2	Relation	Surface-Volume
										Model	



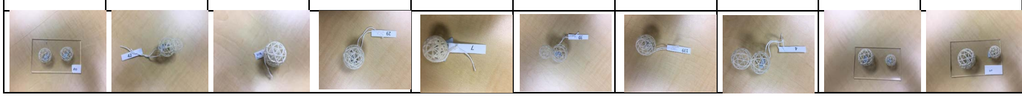
I1	OV2	Relation	Surface-Volume
		Model	



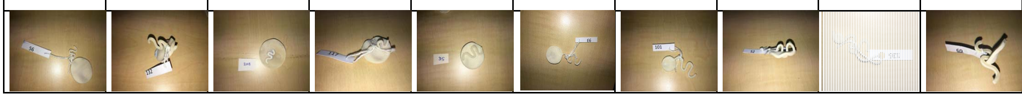
OV3	OV4	OV1	EQ3	EQ1	TPP2	EC3	I2	TPP3	OV5	Relation	Volume-Volume
										Model	



D1	OV2	I3	I1	EQ2	EC1	TPP1	EC2	D3	D2	Relation	Volume-Volume
										Model	

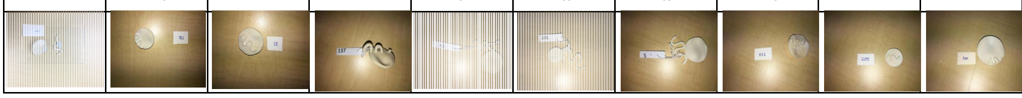


EC2	X4	I3	OV2	TPP2	EC4	EC3	X3	IC2	OV1	Relation	Line-Surface
										Model	



D3	D2	IC7	IC6	X2	I2	IC1	OV3	IC5	Relation	Line-Surface
									Model	

D1	TPP5	I1	X1	IC4	EC3	IC3	TPP3	I4	TPP1	Relation	Line-Surface
										Model	



Appendix E

Composition Tables

Table E.1: Composition table for a triple of bounded regions

	D	EC	TPP	TPP^{-1}	OV	IC	EQ	I	I^{-1}
D	{I^{-1}, I, IC, TPP^{-1}, TPP, EC, EQ, OV, D}	{OV, TPP, I, D, IC, EC}	{EC, I, TPP, D, IC, OV}	{D}	{I, OV, IC, D, EC, TPP}	{OV, TPP, D, I, IC, EC}	{D}	{D, I TPP, IC, EC, OV}	{D}
EC	{TPP^{-1}, OV, I^{-1}, IC, D, EC}	{TPP^{-1}, TPP, D, IC, EC, OV, EQ}	{I, OV, EC, TPP, IC}	{D, EC, TPP^{-1}, IC}	{I, IC, D, EC, TPP, OV}	{OV, TPP, TPP^{-1}, I, D, IC, EC}	{EC}	{IC, TPP, OV, I}	{D}
TPP	{D}	{IC, TPP, EC, D}	{TPP, I}	{I, I^{-1} D, EQ, TPP, IC, TPP^{-1}, OV, EC}	{IC, TPP, D, I, OV, EC}	{EC, D, IC, I, TPP, OV}	{TPP}	{I}	{I^{-1}, IC, D, OV, EC, TPP^{-1}}
TPP^{-1}	{EC, OV, I^{-1}, TPP^{-1}, D, IC}	{IC, I^{-1} I, TPP^{-1}, EC, OV}	{OV, IC, TPP, EQ, TPP^{-1}, EC}	{TPP^{-1}, I^{-1}}	{IC, TPPI, OV, I^{-1}}	{IC, OV, EC, I^{-1}, TPP^{-1}}	{TPP^{-1}}	{TPP^{-1}, I, IC, OV, TPP}	{TPP^{-1}, I^{-1}}
OV	{EC, OV, D, IC, TPPI, I^{-1}}	{IC, TPP^{-1}, EC, D, OV, I^{-1}}	{TPP, I, OV, IC}	{D, I^{-1}, OV, TPP^{-1}, IC, EC}	{TPPI, D, I^{-1}, I, EC, OV, TPP, IC, EQ}	{TPP, IC, I^{-1}, OV, TPP^{-1}, D, EC}	{OV}	{OV, TPP, I}	{OV, D, I^{-1}, EC, IC, TPP^{-1}}
IC	{OV, I^{-1}, D, IC, EC, TPP^{-1}}	{D, TPPI, OV, IC, I^{-1}, EC, TPP}	{TPP, OV, I, IC, EC}	{EC, I^{-1}, OV, D, IC, TPP^{-1}}	{OV EC, TPP, IC, TPP^{-1}, D, I}	{I, I^{-1}, IC, OV, EQ, D, EC, TPP, TPP^{-1}}	{IC}	{I, IC, TPP, OV}	{TPP^{-1}, IC, D, EC}
EQ	{D}	{EC}	{TPP}	{TPP^{-1}}	{OV}	{IC}	{EQ}	{I}	{I^{-1}}
I	{D}	{D}	{I, TPP}	{I IC, D, TPP, EC, OV}	{I, D, OV, EC, IC, TPP}	{IC D, TPP, EC}	{I}	{I}	{I, TPPI, TPP, I^{-1}, D, EC, OV, EQ, IC}
I^{-1}	{D, I^{-1} OV, TPP^{-1}, EC, IC}	{I^{-1}, OV, IC, TPP^{-1}}	{TPP, I^{-1} IC, OV, TPP^{-1}}	{I^{-1}}	{OV, TPP^{-1}, I^{-1}}	{IC TPP^{-1}, I^{-1}, OV}	{I^{-1}}	{TPP^{-1}, TPP, I^{-1}, I, EQ, OV}	{I^{-1}}

Table E.2: Composition table for a triple of unbounded regions

	D	EC	TPP	TPP^{-1}	OV	IC	EQ	I	I^{-1}
D	$\{I^{-1}, EQ, I, OV, D\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{I, OV, D\}$	$\emptyset$	$\{D\}$	$\{D, OV, I\}$	$\{D\}$
EC	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
TPP	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
TPP^{-1}	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
OV	$\{OV, D, I^{-1}\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{D, I^{-1}, I, OV, EQ\}$	$\emptyset$	$\{OV\}$	$\{OV, I\}$	$\{OV, D\}$
IC	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
EQ	$\{D\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{OV\}$	$\emptyset$	$\{I^{-1}, I, EQ\}$	$\{I^{-1}, I, EQ\}$	$\{I^{-1}, I, EQ\}$
I	$\{D\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{I, D, OV\}$	$\emptyset$	$\{EQ, I, I^{-1}\}$	$\{I^{-1}, I, EQ\}$	$\{I, I^{-1}, OV, EQ, D\}$
I^{-1}	$\{OV, D, I^{-1}\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{OV, I^{-1}\}$	$\emptyset$	$\{EQ, I, I^{-1}\}$	$\{EQ, OV, I^{-1}, I\}$	$\{EQ, I^{-1}, I\}$

Table E.3: Composition table for two bounded and one unbounded regions

	D	EC	TPP	TPP^{-1}	OV	IC	EQ	I	I^{-1}
D	$\{IC, TPP^{-1}, I^{-1}, I, OV, D\}$	$\emptyset$	$\emptyset$	$\{D\}$	$\{I, OV, IC, D\}$	$\{OV, D, I, IC\}$	$\emptyset$	$\{D, IC, OV, I\}$	$\{D\}$
EC	$\{TPP^{-1}, OV, I^{-1}, IC, D\}$	$\emptyset$	$\emptyset$	$\{D, TPP^{-1}, IC\}$	$\{I, IC, D, OV\}$	$\{OV, TPP^{-1}, I, D, IC\}$	$\emptyset$	$\{IC, OV, I\}$	$\{D\}$
TPP	$\{D\}$	$\emptyset$	$\emptyset$	$\{I, D, IC, TPP^{-1}, I^{-1}, OV\}$	$\{IC, D, I, OV\}$	$\{D, IC, I, OV\}$	$\emptyset$	$\{I\}$	$\{IC, I^{-1}, D, OV, TPP^{-1}\}$
TPP^{-1}	$\{OV, I^{-1}, TPP^{-1}, D, IC\}$	$\emptyset$	$\emptyset$	$\{TPP^{-1}, I^{-1}\}$	$\{IC, TPP^{-1}, OV, I^{-1}\}$	$\{IC, OV, I^{-1}, TPP^{-1}\}$	$\emptyset$	$\{IC, TPP^{-1}, I, OV\}$	$\{TPP^{-1}, I^{-1}\}$
OV	$\{OV, D, IC, TPP^{-1}, I^{-1}\}$	$\emptyset$	$\emptyset$	$\{D, I^{-1}, OV, TPP^{-1}, IC\}$	$\{TPP^{-1}, D, I^{-1}, I, OV, IC\}$	$\{IC, TPP^{-1}, D, I^{-1}, OV\}$	$\emptyset$	$\{OV, I\}$	$\{OV, D, I^{-1}, IC, TPP^{-1}\}$
IC	$\{OV, I^{-1}, D, IC, TPP^{-1}\}$	$\emptyset$	$\emptyset$	$\{I^{-1}, OV, D, IC, TPP^{-1}\}$	$\{IC, TPP^{-1}, D, I, OV\}$	$\{IC, OV, I, I^{-1}, D, TPP^{-1}\}$	$\emptyset$	$\{I, IC, OV\}$	$\{TPP^{-1}, IC, D, OV\}$
EQ	$\{D\}$	$\emptyset$	$\emptyset$	$\{TPP^{-1}\}$	$\{OV\}$	$\{IC\}$	$\emptyset$	$\{I\}$	$\{I^{-1}\}$
I	$\{D\}$	$\emptyset$	$\emptyset$	$\{IC, D, I, OV\}$	$\{I, D, OV, IC\}$	$\{D\}$	$\emptyset$	$\{I\}$	$\{I, TPP^{-1}, I^{-1}, OV, D, IC\}$
I^{-1}	$\{OV, TPP^{-1}, IC, D, I^{-1}\}$	$\emptyset$	$\emptyset$	$\{I^{-1}\}$	$\{OV, TPP^{-1}, I^{-1}\}$	$\{TPP^{-1}, I^{-1}, OV\}$	$\emptyset$	$\{OV, TPP^{-1}, I^{-1}, I\}$	$\{I^{-1}\}$

Table E.4: Composition table for a one bounded and two unbounded regions

	D	EC	TPP	TPP^{-1}	OV	IC	EQ	I	I^{-1}
D	{IC, TPP^{-1}, I^{-1}, I, OV, D}	∅	∅	∅	{I, OV, IC, D}	∅	{D}	{D, IC, OV, I}	{D}
EC	∅	∅	∅	∅	∅	∅	∅	∅	∅
TPP	∅	∅	∅	∅	∅	∅	∅	∅	∅
TPP^{-1}	{OV, I^{-1}, TPP^{-1}, D, IC}	∅	∅	∅	{IC, TPP^{-1}, OV,				
I^{-1} }	∅	{TPP^{-1} }	{IC, TPP^{-1}, I, OV}	{TPP^{-1}, I^{-1} }					
OV	{OV, D, IC, TPP^{-1}, I^{-1} }	∅	∅	∅	{TPP^{-1}, D, I^{-1}, I, OV, IC}	∅	{OV}	{OV, I}	{OV, D, I^{-1}, TPP^{-1}, IC}
IC	{OV, I^{-1}, D, TPP^{-1}, IC}	∅	∅	∅	{IC, TPP^{-1}, D, I, OV}	∅	{IC}	{I, IC, OV}	{TPP^{-1}, IC, D}
EQ	∅	∅	∅	∅	∅	∅	∅	∅	∅
I	{D}	∅	∅	∅	{I, D, OV, IC}	∅	{I}	{ I}	{I, TPP^{-1}, I^{-1}, OV, D, IC}
I^{-1}	{OV, TPP^{-1}, IC, D, I^{-1} }	∅	∅	∅	{OV, TPP^{-1}, I^{-1} }	∅	{I^{-1} }	{ OV, TPP^{-1}, I^{-1}, I}	{I^{-1} }

Appendix F

Mapping Between The Themes/codes And Selected Passages Of The Provided Description

The following table shows the mapping between themes/code and selected passages of the provided description used in thematic analysis (Chapter 5).

Theme/Code	Selected passages of the description
Connection	Inside, Meet, Glue, Attach, Stick, Touch, Place, Through, Contact, Separate, Within, Connect, Stack, Intersect, Sit on, Contain, Tangent, Enclosed, Pass
Degree of freedom in connection	Tightly, Loose, (not) Closely, Fix, Dense, Glue, Attach, Stick, Flexible
Portion of overlapping area	A small part, (number of) Points, Half of, Part of, Partly, Fully, Completely (inside or outside)
Connection in boundary	Edge, Tail, End, Outer, Tip, Out side
Connection in interior	Inside, Inner, Surface (of disk)
Direction in connection	Vertically, Horizontally, Above, Below, Up, Top, Side, Bottom, Opposite, Upright, Perpendicular, Flat, (X) degree angle, Sit on
Shape of the objects	Wavy object, Line, Worm like object, Disk, Circle, Ball
Distance between objects	Meet, Far, Exactly, Close, Touch, Tangent
Size of the objects	Small, Big, Equally, Large, Size, Diameter (of a disk)

Glossary

9-IM 9- Intersection model.

CAD computer-aided design.

CLIF Common Logic Interchange Format.

CODI Containment Dimension.

CODIB Containment Dimension Boundary.

CSP constraint satisfaction problem.

DE-9IM Dimensional Extension 9- Intersection model.

FOL First order logic.

GIS Geographic Information Systems.

GML Geography Markup Language.

HOL Higher order logic.

INCH Includes a chunck of.

JEPD Jointly Exhaustive and Pairwise Disjoint.

NLP Natural Language Processing.

OBO Open Biomedical Ontology.

OGC Open Geospatial Consortium.

PLA Polylactide.

QSR Qualitative Spatial Reasoning.

QSRR Qualitative Spatial Representation and Reasoning.

RCC Region Connection Calculus.

SSP Strong Supplementation Principle.

TA thematic analysis.

WSP Weak Supplementation Principle.