Copyright is owned by the Author of the thesis. Permission is given for a copy to be downloaded by an individual for the purpose of research and private study only. The thesis may not be reproduced elsewhere without the permission of the Author.

GEOMETRICAL INTERPRETATIONS OF BÄCKLUND TRANSFORMATIONS AND CERTAIN TYPES OF PARTIAL DIFFERENTIAL EQUATIONS

A THESIS PRESENTED IN PARTIAL FULFILMENT
OF THE REQUIREMENTS FOR
THE DEGREE OF MASTER OF SCIENCE IN
MATHEMATICS AT
MASSEY UNIVERSITY

Anton Raviraj Selvaratnam
2000

Abstract

Gauss' *Theorema Egregium* contains a partial differential equation relating the Gaussian curvature K to components of the metric tensor and its derivatives. Well-known partial differential equations such as the Schrödinger equation and the sine-Gordon equation correspond to this PDE for special choices of K and special coördinate systems. The sine-Gordon equation, for example, can be derived via Gauss' equation for $K = -1$ using the Tchebychef net as a coördinate system.

In this thesis we consider a special class of Bäcklund Transformations which correspond to coördinate transformations on surfaces having a specified Gaussian curvature. These transformations lead to Gauss' PDE in different forms and provide a method for solving certain classes of non-linear second order partial differential equations.

In addition, we develop a more systematic way to obtain a coordinate system for a more general class of PDE, such that this PDE corresponds to the Gauss equation.

Acknowledgements

I am greatly indebted to my supervisors, Dr. Bruce van-Brunt, Assoc. Prof. Dean Halford and Dr. Marijcke Vlieg-Hulstman who have helped me to appreciate the beauty of this subject, and also for their excellent guidance, enthusiasm and help during the course of this work.

I acknowledge my debt to Dr. Mark Johnston and Philip Etheridge for all their help and advice on computing and especially helping me with my LaTeX.

Finally, I would like to thank my fellow graduate students and all the other staff members at the Mathematics Department, my family members, well wishers and friends for their continuous support and encouragement.

Contents

List of Figures

Chapter 1

Introduction

1.1 General

The dynamics of interfaces, surfaces, fronts are an important ingredients in numerous nonlinear phenomena arising in classical and quantum physics, and in some cases the dynamics can be modelled by nonlinear partial differential equations (PDEs) that describe the evolution of surfaces in time. As a result of this relationship, the study of the connection between certain types of surfaces and nonlinear PDEs has been one of the classical problems of differential geometry. Curvature, for example, plays an important part in a number of problems of physics and mathematics associated with manifolds.

Often, one has to solve nonlinear PDEs in order to explain the physical phenomena, but solution techniques for nonlinear PDEs are fairly specialized and rare. One of these techniques, a coordinate transformation method, loosely speaking, known as the *Bäcklund Transformation method*, is of interest in this text. It is known [7] that a Bäcklund transformation may be regarded, in geometrical language, as a transformation of a surface S into a new surface $\bar{S}$, where S is a solution of a given PDE, but where the transformed surface $\bar{S}$ may either be a solution of the original PDE or of some other differential equation. Bäcklund transformations, in essence, preserve invariant properties between two differential equations and their solutions, and they relate these equations to one another through a representation of surfaces with the same curvature in some known coordinate systems. They can thus be useful for finding a solution to a given differential equation by relating it to another differential equation with a known solution. In recent times, interest in these transformations have persisted due to their connection with the sine-Gordon equation and its associated soliton theory.

1.2 A Brief Description

The first chapter contains the general introduction and a review of the literature pertaining to the work in this thesis, followed by some definitions and fundamental equations which will be used in the following chapters. In section 1.3 we review some basic definitions which arise in differential geometry. In subsection 1.3.2, the Gauss equation, which plays a central rôle in our discussions, is presented. We then illustrate how some well known PDEs such as the Schrödinger equation, the sine-Gordon equation, the Liouville equation and the Monge-Ampère equation can be generated from the Gauss equation by the appropriate choice of coordinates. In section 1.4 we show how the covariant transformation equations can be used to determine the Bäcklund transformations between two coordinate systems, where each coordinate system represents a specific PDE.

Chapter 2 consists of two major sections. In section 2.1 we look mainly at the solution techniques and Bäcklund transformations developed for various classes of second order quasi-linear partial differential equations [26]. In subsection 2.1.1 we first show how a certain class of second order quasi-linear PDEs of the hyperbolic type can be solved. As an example, a family of solutions for the sine-Gordon equation is derived. The Cauchy problem is then discussed and the sine-Gordon equation is used as an illustration. Further, we establish that the solution obtained for the Cauchy problem of the sine-Gordon equation corresponds to a Beltrami surface. Our approach in deriving solutions through Bäcklund transformations is further illustrated through an example, where a soliton solution of the sine-Gordon equation is used to derive a solution to the Schrödinger equation. Subsections 2.1.2 and 2.1.3 deal with some classes of second order quasi-linear PDEs of the parabolic type and the elliptic type, respectively. Illustrative examples are given wherever appropriate.

In section 2.2, we show how the same technique used in section 2.1 can be implemented to solve a fully non-linear second order PDE, the Monge-Ampère equation, and further discuss the solution to the Cauchy problem for this equation. Finally, we discuss some relationships among the sine-Gordon, the Monge-Ampère and the Schrödinger equations, which Bäcklund transformations elucidate and discuss briefly how a more general class of Monge-Ampère equation can be solved using Bäcklund transformations.

The topics in Chapter 3 pertain to a systematic way of obtaining a coordinate system corresponding to a more general class of PDEs which can be interpreted as the Gauss equation. This complements the material in Chapter 2, where we established some useful solution techniques via Bäcklund transformations for some classes of PDEs. It is noted that in generalising the technique to include a non-constant Gaussian curvature function, we extend significantly to class of PDEs for which this solution method is available.

Section 3.1 provides a brief introduction to the remainder of Chapter 3. Section 3.2 deals with the preliminaries required for the sections to follow. We also provide with a brief review of the literature pertaining to the material in Chapter 3 in this section.

In section 3.3 a complete characterisation is given for the class of differential equations of type

$$u_t = F\left(K(x,t), u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^k u}{\partial x^k}\right).$$

Illustrative examples such as the generalised Burgers equation and the generalised KdV equation are provided to show how we can, in principle, determine the coordinate systems for these types of equations.

Section 3.4 consists the complete characterisation for the class of differential equations of type

$$u_{xt} = F\left(K(x,t), u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^k u}{\partial x^k}\right).$$

Once again, we provide illustrative examples to show how we can determine the coordinate systems for these types of equations. The generalised sine-Gordon equation and the generalised sinh-Gordon equation are used as examples.

In Chapter 4, we conclude the thesis by summing up particular results and proposing certain matters which need further investigation.

1.3 Some Geometrical Aspects

In this section, we review some basic definitions which arise in differential geometry [7, 34, 35]. Let S be a surface in E^3, Euclidean 3-space, and let Γ be a curve on S. If (u, v) denote curvilinear coordinates on S, then the curve Γ can be described by an implicit relationship of the form

$$\phi(u, v) = 0.$$

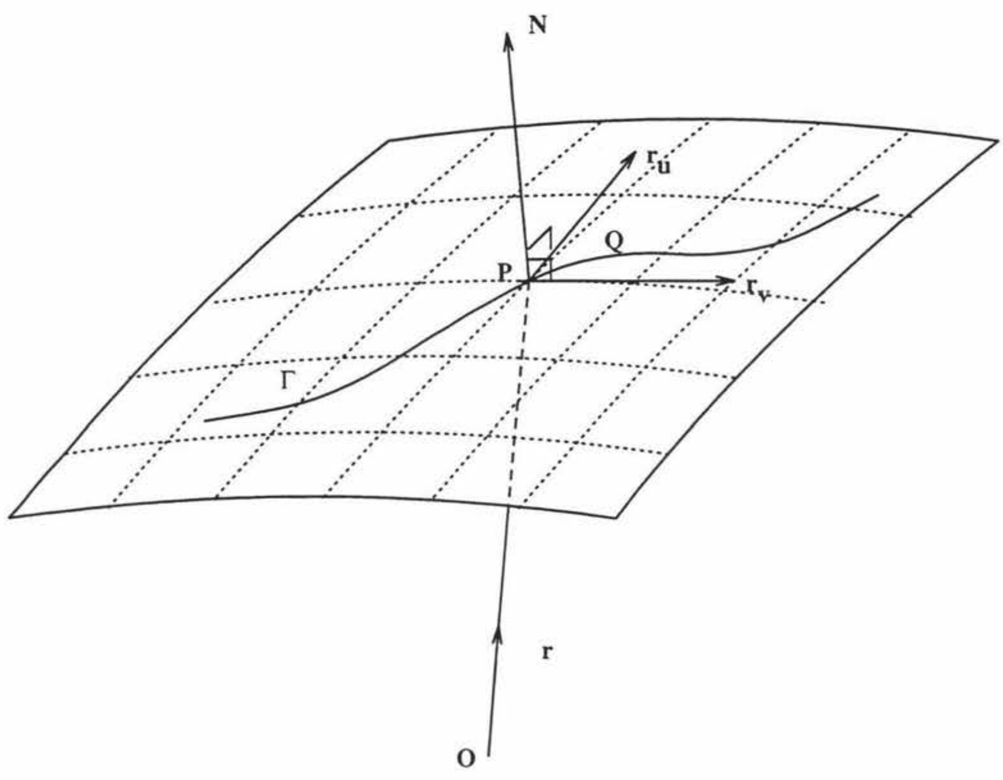

Figure 1.1: The surface S and the curve Γ

The curve Γ defined above can also be given in parametric form:

$$u = u(t), \qquad v = v(t). \tag{1.1}$$

Let $\mathbf{r}$ be the position vector of a point P on the curve. Then the vector $d\mathbf{r}/dt = \dot{\mathbf{r}}$, given by

$$\dot{\mathbf{r}} = \mathbf{r}_u \dot{u} + \mathbf{r}_v \dot{v}, \tag{1.2}$$

is tangent to the curve and therefore to the surface (cf. Fig.1.1). Here the subscripts u and v denote partial differentiation with respect to u and v respectively. Equation (1.2) can also be written (in a form independent of the choice of parameter) as,

$$d\mathbf{r} = \mathbf{r}_u \, du + \mathbf{r}_v \, dv. \tag{1.3}$$

If Q is in a neighbourhood of P on the curve, then the distance ds, between P and Q on the curve can be expressed as

$$I = ds^2 = d\mathbf{r}.d\mathbf{r} = E\,du^2 + 2F\,du\,dv + Gdv^2, \tag{1.4}$$

where

$$E = \mathbf{r}_u.\mathbf{r}_u \,,\ F = \mathbf{r}_u.\mathbf{r}_v \,,\ G = \mathbf{r}_v.\mathbf{r}_v. \tag{1.5}$$

The quadratic form in equation (1.4) is called the *first fundamental form* for the surface S.

The functions E, F and G depend on u and v and are called the *components of the metric tensor* or the *components of the first fundamental form.*

The quantity

$$|\mathbf{r}_u \wedge \mathbf{r}_v| = H = \sqrt{EG - F^2}\,, \tag{1.6}$$

corresponds to the *differential area element.* The angle θ between the coordinate curves is

$$\cos\theta = \frac{\mathbf{r}_u.\mathbf{r}_v}{|\mathbf{r}_u|\,|\mathbf{r}_v|} = \frac{F}{\sqrt{EG}}. \tag{1.7}$$

If $\mathbf{t}$ is the unit tangent vector at P to the curve Γ on the surface S and $\mathbf{N}$ is the unit surface normal, then the curvature vector of Γ at P, $\mathbf{k}$, can be decomposed as

$$d\mathbf{t}/ds = \mathbf{k} = \mathbf{k}_n + \mathbf{k}_g\,,$$

where $\mathbf{k}_n$ is parallel to $\mathbf{N}$ and orthogonal to $\mathbf{k}_g$ (see Fig. 1.2).

The vector $\mathbf{k}_g$ is called the *tangential curvature vector* or *geodesic curvature vector* and

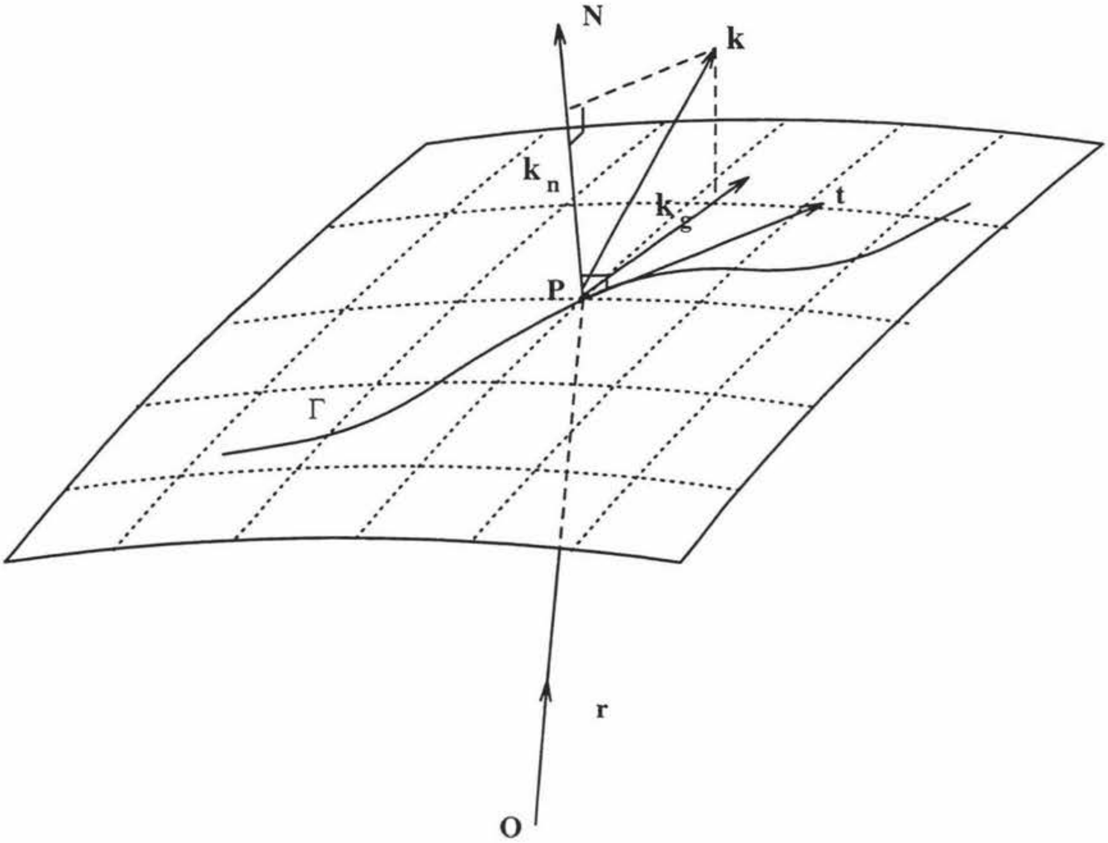

Figure 1.2: The *normal* and *tangential* curvature vectors

the vector $\mathbf{k}_n$ is called the *normal curvature vector*. The latter can be expressed by

$$\mathbf{k}_n = \kappa_n \mathbf{N},$$

where κ_n is known as the *normal curvature*. The normal curvature is given by

$$\kappa_n = \frac{e\,du^2 + 2f\,du\,dv + g\,dv^2}{E\,du^2 + 2F\,du\,dv + G dv^2} \tag{1.8}$$

where, in terms of vector triple products,

$$e = \frac{(\mathbf{r}_{uu}, \mathbf{r}_u, \mathbf{r}_v)}{H}, \quad f = \frac{(\mathbf{r}_{uv}, \mathbf{r}_u, \mathbf{r}_v)}{H}, \quad g = \frac{(\mathbf{r}_{vv}, \mathbf{r}_u, \mathbf{r}_v)}{H}. \tag{1.9}$$

The numerator of equation (1.8), written as

$$II = -\,d\mathbf{r}.d\mathbf{N} = e\,du^2 + 2f\,du\,dv + g\,dv^2 \tag{1.10}$$

is defined as the *second fundamental form.* The functions e, f and g are known as the *components of the second fundamental form.*

1.3.1 Gaussian and Mean Curvatures

The normal curvature given in equation (1.8), when considered in the direction $\lambda = du/dv$ is

$$\kappa_n = \frac{e + 2f\,\lambda + g\,\lambda^2}{E + 2F\,\lambda + G\,\lambda^2} = \kappa_n(\lambda)\,. \tag{1.11}$$

Extrema for κ_n w.r.t λ are characterized by

$$d\kappa_n/d\lambda = 0\,,$$

and this condition implies

$$\kappa_n = \frac{II}{I} = \frac{f + g\lambda}{F + G\lambda} = \frac{e + f\lambda}{E + F\lambda}\,.$$

The above equation indicates that

$$(Fg - Gf)\,\lambda^2 + (Eg - Ge)\,\lambda + (Ef - Fe) = 0,$$

which determines two directions dv/du, in which κ_n obtains an extreme value, unless II vanishes or unless II and I are proportional. One value must be maximum, the other a minimum. These directions are called the *directions of principal curvature* or *curvature directions* and the corresponding values for κ_n denoted by κ_1 and κ_2 are defined as the *principal curvatures.*

The quantities

$$\mathcal{H} = \frac{1}{2}\,(\kappa_1 + \kappa_2) = \frac{Eg - 2fF + eG}{2\,(EG - F^2)} \tag{1.12}$$

and

$$K = \kappa_1 \kappa_2 = \frac{eg - f^2}{EG - F^2} \tag{1.13}$$

are invariants, and are called respectively the *mean curvature* and the *Gaussian curvature* of the surface.

1.3.2 The Gauss Equation and some well-known PDEs

A key result in classical differential geometry is Gauss' *Theorema Egregium* [34], which asserts that the Gaussian curvature depends purely on the components of the first fundamental form. Specifically, we have the *Gauss Equation*:

$$K(u,v) = \frac{1}{2H}\left(\left(\frac{F}{HE}E_v - \frac{1}{H}G_u\right)_u + \left(\frac{2}{H}F_u - \frac{1}{H}E_v - \frac{F}{HE}E_u\right)_v\right). \tag{1.14}$$

This equation will play a central rôle in our discussion. Many nonlinear and, some linear PDEs of interest, correspond to the Gauss equation on a surface of prescribed curvature parametrized in an appropriate coordinate system. In certain coordinate systems the Gauss equation takes a particularly simple form. Well known partial differential equations such as the Schrödinger equation, the sine-Gordon equation, the Liouville equation and the Monge-Ampère equation are the classical examples[4, 18]. We illustrate below how these PDEs can be generated from the Gauss equation by the appropriate choice of coordinates.

1.3.2.1 The Schrödinger Equation

Our first example is the Schrödinger equation,

$$\psi_{uu} + K(u,v)\,\psi = 0,$$

which, as will be seen, corresponds to the Gauss equation for surfaces of Gaussian curvature $K(u,v)$ in geodesic polar coordinates.

In a neighbourhood of every point on a smooth surface, a geodesic polar coordinate system exists[34]; hence, we can always construct such a local coordinate system for the surface with Gaussian curvature $K(u,v)$. For a geodesic polar coordinate system $E=1$ and $F=0$, equation (1.4) reduces to

$$ds^2 = du^2 + Gdv^2,$$

and equation (1.14) becomes,

$$K(u,v) = -G^{-1/2}(G^{1/2})_{uu}.$$

Using $H=\sqrt{G}$ we have,

$$H_{uu} + K(u,v)H = 0. \tag{1.15}$$

The solution to Schrödinger's equation (1.15) thus corresponds to the differential area element for a surface of curvature $K(u,v)$ in the geodesic coordinates.

1.3.2.2 The sine-Gordon Equation

When $E=G=1$, the coordinate system forms a *Tchebychef Net* [6, 34], which exists for sufficiently smooth surfaces[34], and equation (1.4) becomes,

$$ds^2 = du^2 + 2Fdudv + dv^2.$$

If θ is the angle through which the coordinate vector $\mathbf{r}_u$ must be turned to bring it into coincidence with $\mathbf{r}_v$ then we have,

$$F = \cos\theta$$

(from equation (1.7)). Now equation (1.14) takes the form

$$K = \frac{1}{\sqrt{1-F^2}}\left(\frac{1}{H}F_u\right)_v$$

i.e.

$$\theta_{uv} = -K(u,v)\sin\theta. \tag{1.16}$$

This is a second order hyperbolic PDE for the function θ, with $u=$ constant and

$v =$ constant as the characteristics. For $K(u,v) = -1$, we get the familiar *sine-Gordon Equation*,

$$\theta_{uv} = \sin\theta\,. \tag{1.17}$$

1.3.2.3 The Liouville Equation

Let $E = G = 0$ so that, the coordinate curves are the minimal lines. We note that this makes the surface representation complex. Equation (1.14) becomes,

$$(\ln F)_{uv} + KF = 0$$

i.e.

$$\Phi_{uv} + K\,e^{\Phi} = 0, \tag{1.18}$$

where

$$F = e^{\Phi}.$$

For $K =$ constant, equation (1.18) corresponds to the *Liouville Equation.*

1.3.2.4 The Monge-Ampère Equation

Consider a surface described by

$$\mathbf{r} = (u, v, Z(u,v))\ .$$

Then the components E, F and G of the first fundamental form, for graphical coordinates will be given by

$$E = 1 + Z_u^2\,, \quad F = Z_u\,Z_v\,, \quad G = 1 + Z_v^2\,,$$

and thus equation (1.4) becomes

$$ds^2 = (1 + Z_u^2)du^2 + 2Z_uZ_vdudv + (1 + Z_v^2)dv^2.$$

The Gauss equation (1.14) reduces to

$$K(u,v) = \frac{Z_{uu}\,Z_{vv} \;-\; Z_{uv}^2}{(1 + Z_u^2 + Z_v^2)^2}\,, \tag{1.19}$$

which also can be written as

$$Z_{uu}\,Z_{vv} \;-\; Z_{uv}^2 \;-\; K(u,v)\,\left(1 + Z_u^2 + Z_v^2\right)^2 = 0$$

which is an equation of the *Monge-Ampère* type.

Certain partial equations can thus be interpreted as statements of Gauss' Theorem on a surface of curvature K in an appropriate coordinate system. This observation motivates a strategy for solving these equations based on Bäcklund transformations which correspond to curvilinear coordinate transformations on the surface defined intrinsically by K.

1.4 Gauss Equation and Bäcklund Transformations

Given a PDE, the idea here is to first find a coordinate system such that the PDE corresponds to the Gauss equation for a surface of known Gaussian curvature. Then we seek another PDE that can be solved, and determine a coordinate system such that this PDE corresponds to the Gauss equation for the same Gaussian curvature. Using the covariant transformation equations for the two determined coordinate systems yields a system of non-linear PDEs. Solutions to this system define the Bäcklund transformations between the two coordinate systems, thus enabling us to obtain solutions to the given PDE by transforming the known solution of the other PDE.

In order to further describe this method, let us consider two partial differential equations $\mathcal{D}(\phi) = 0$ and $\mathcal{E}(\chi) = 0$ which are of the same order. Assume that the PDE $\mathcal{D}(\phi) = 0$ is the given equation to be solved and the other is a PDE with a known solution.

Further, we assume that these two PDEs can be identified as the Gauss equation with the same K, and that the corresponding components of their first fundamental forms are E, F, G and $\hat{E}, \hat{F}, \hat{G}$ respectively. Let the respective coordinates be (u, v) and (x, y)

(see Fig. 1.3).

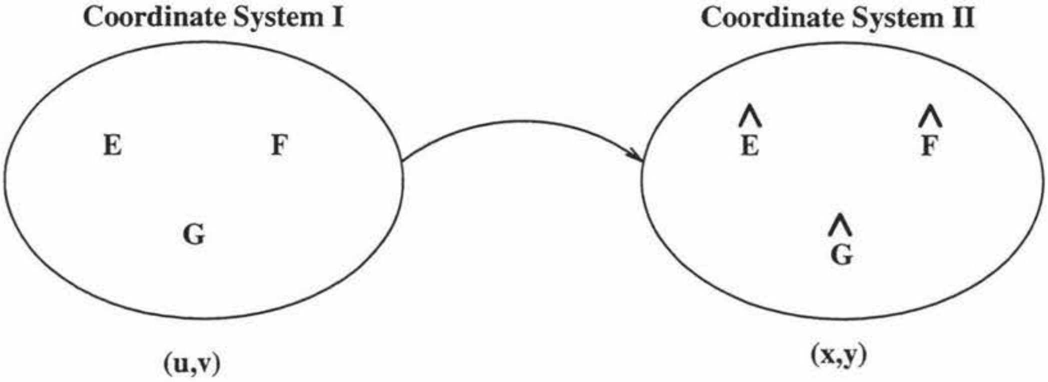

Figure 1.3: Coordinate transformation from coordinate system I to the coordinate system II.

From the tensor formula,

$$\hat{g}_{ij} = g_{lm} \frac{\partial X^l}{\partial \hat{X}_i} \cdot \frac{\partial X^m}{\partial \hat{X}_j}, \tag{1.20}$$

for coordinate transformations, where $g_{11} = E$, $g_{12} = g_{21} = F$, $g_{22} = G$, $\hat{g}_{11} = \hat{E}$, $\hat{g}_{12} = \hat{g}_{21} = \hat{F}$, $\hat{g}_{22} = \hat{G}$ and then by using the specific values for $\hat{g}_{ij}$'s and g_{lm}'s we obtain the system

$$E\, u_x^2 + 2F\, u_x\, v_x + G\, v_x^2 = \hat{E} \tag{1.21}$$

$$E\, u_x\, u_y + F\,(u_x\, v_y + v_x\, u_y) + G\, v_x\, v_y = \hat{F} \tag{1.22}$$

and

$$E\, u_y^2\,, + 2F\, u_y\, v_y + G\, v_y^2 = \hat{G}. \tag{1.23}$$

We need to solve this system of non-linear PDEs to determine the required Bäcklund transformations.

When applying the method described above in solving a PDE, we are aware of the fact that we may have difficulties, first in identifying the given PDE as the Gauss equation; i.e., to determine the corresponding coordinate system, and then in solving the system of PDEs which determines the Bäcklund transformations. The latter could be relatively harder than the original problem. Further, it should be noted that, imposing different initial conditions on this system of PDEs yields different Bäcklund transformations. This shows that all the solutions to the given PDE cannot be obtained by using one set of Bäcklund transformations, and thus we only end up with certain classes of solutions. This certainly is a weakness in our method, especially when we are looking for all possible solutions.

Chapter 2

Some second order PDEs and Gauss' Equation

In this chapter we develop the technique outlined in Chapter 1 and apply it to specific types of PDEs. In the first section we show how a certain class of second order quasi-linear PDEs of the hyperbolic type can be solved. As an example, we obtain a family of solutions to the sine-Gordon equation. The Cauchy problem is then discussed and the sine-Gordon equation is used as an illustration. Also, we analyse the possibilities of tackling some classes of second order quasi-linear PDEs of the parabolic type and the elliptic type.

In section 2.2, we show how the same technique can be implemented to solve a fully non-linear second order PDE, the Monge-Ampère equation and then discuss the solution to the Cauchy problem for this equation. There are interesting relationships among the sine-Gordon equation, the Monge-Ampère equation and the Schrödinger equation which the Bäcklund transformations expose.

2.1 Solving a class of second order quasi-linear PDEs

Consider a surface with local coordinates u and v, and suppose the coefficients of the first fundamental form E, F and G are of the form

$$E = E(\phi),\, F = F(\phi),\, \text{and}\, G = G(\phi)\,, \tag{2.1}$$

where ϕ is some function of u and v and

$$H^2 = EG - F^2 > 0. \tag{2.2}$$

Under this assumption, the Gauss equation (1.15) becomes

$$\begin{aligned}
&\tfrac{1}{4}\,\{4F_\phi\,H^2\,\phi_{uv} - 2G_\phi\,H^2\,\phi_{uu} - 2E_\phi\,H^2\,\phi_{vv} \\
+ &\quad (G_\phi E_\phi G + E(G_\phi)^2 - 2G_\phi F_\phi F)\,(\phi_u)^2 - 2\,H^2\,G_{\phi\phi}(\phi_u)^2 \\
+ &\quad (G_\phi E_\phi E + G(E_\phi)^2 - 2E_\phi F_\phi F)\,(\phi_v)^2 - 2\,H^2\,E_{\phi\phi}(\phi_v)^2 \\
- &\quad 2\,(F_\phi E_\phi G + F_\phi G_\phi E - 2F(F_\phi)^2 + 4\,H^2)\;F_{\phi\phi}\phi_u\phi_v \\
- &\quad 4K(EG - F^2)^2\} \qquad = 0\,.
\end{aligned} \tag{2.3}$$

This is a PDE of the form

$$G_\phi\phi_{uu} \;-\; 2F_\phi\phi_{uv} \;+\; E_\phi\phi_{vv} \;+\; 2K\,H^2 + \Theta(u, v, \phi, \phi_u, \phi_v) \;=\; 0\,, \tag{2.4}$$

where

$$\Theta(u, v, \phi, \phi_u, \phi_v) = \Theta_1(u, v, \phi)(\phi_u)^2 + \Theta_2(u, v, \phi)(\phi_v)^2 + \Theta_3(u, v, \phi)\phi_u\phi_v\,, \tag{2.5}$$

and

$$\Theta_1(u, v, \phi) = -\,\frac{\{G_\phi E_\phi G + E(G_\phi)^2 - 2G_\phi F_\phi F - 2\,H^2\,G_{\phi\phi}\}}{2\,H^2}\,, \tag{2.6}$$

$$\Theta_2(u, v, \phi) = -\,\frac{\{G_\phi E_\phi E + G(E_\phi)^2 - 2E_\phi F_\phi F - 2\,H^2\,E_{\phi\phi}\}}{2\,H^2}\,, \tag{2.7}$$

$$\Theta_3(u, v, \phi) = \frac{\{F_\phi E_\phi G + F_\phi G_\phi E - 2F(F_\phi)^2 - 2\,H^2\,F_{\phi\phi}\}}{H^2}\,. \tag{2.8}$$

Equation (2.4) is of the form

$$A\phi_{uu} + 2B\phi_{uv} + C\phi_{vv} + D = 0\,, \tag{2.9}$$

and is thus a second order quasi-linear PDE, with coefficients A, B and C depending on u, v and ϕ, and D depending exclusively on u, v, ϕ, ϕ_u and ϕ_v.

If $B^2 - AC$ does not change sign, equation (2.9) can be classified into one of three types: hyperbolic, parabolic, and the elliptic, corresponding to $B^2 - AC > 0$, $B^2 - AC = 0$, or $B^2 - AC < 0$ respectively. Equation (2.4) is thus:

(i) hyperbolic iff $(F_\phi)^2 - E_\phi G_\phi > 0$;
(ii) parabolic iff $(F_\phi)^2 - E_\phi G_\phi = 0$; and
(iii) elliptic iff $(F_\phi)^2 - E_\phi G_\phi < 0$.

We shall discuss below the necessary and sufficient conditions required for equation (2.4) to be of one of these forms, and then investigate the possibilities of obtaining solutions through Bäcklund Transformations for each of these types.

2.1.1 Equations of Hyperbolic Type

Consider a second order quasi-linear hyperbolic PDE of the type

$$\phi_{uv} = f(\phi, \phi_u, \phi_v). \tag{2.10}$$

Equation (2.4) will be of this form if $F_\phi \neq 0$ and either

(a) $E_\phi = G_\phi = 0$ or
(b) $G_\phi \phi_{uu} + E_\phi \phi_{vv} = 0$.

We shall discuss the above two cases separately, and find the conditions which should be imposed on equation (2.10).

2.1.1.1 Case(a): $E_\phi = G_\phi = 0$

If $E_\phi = G_\phi = 0$ then E and G are constants.

By an appropriate scaling of the coordinates u and v we can make E and G equal and without loss of generality take $E = 1$ and $G = 1$. Thus we use a Tchebychef net to represent the given PDE (2.10).

So, for these choices of E and G equation (2.4) reduces to

$$-2F_\phi \phi_{uv} + 2K(1 - F^2) + \frac{[-2F(F_\phi)^2 - 2(1 - F^2)F_{\phi\phi}]}{(1 - F^2)} \phi_u \phi_v = 0\,,$$

i.e.

$$\phi_{uv} = \frac{KH^2}{F_\phi} - \frac{1}{F_\phi} \left\{ F_{\phi\phi} + \frac{F(F_\phi)^2}{H^2} \right\} \phi_u \phi_v\,, \tag{2.11}$$

(since $F_\phi \neq 0$). Equation (2.11) is of the form

$$\phi_{uv} = M(\phi) + A(\phi)\phi_u \phi_v\,, \tag{2.12}$$

where

$$M(\phi) = \frac{K(1 - F^2)}{F_\phi} \tag{2.13}$$

and

$$A(\phi) = \frac{-1}{F_\phi} \left\{ F_{\phi\phi} + \frac{F(F_\phi)^2}{(1 - F^2)} \right\}. \tag{2.14}$$

These equations indicate that, if a second order quasi-linear hyperbolic PDE is of the form (2.12), then it can be identified as Gauss' equation, where the (u, v) coordinate system corresponds to a Tchebychef net ($E = 1$, $G = 1$ and $F = F(\phi)$), on a surface of curvature

$$K = \frac{M(\phi)\, F_\phi}{(1 - F^2)}\,,$$

where the function F can be determined from (2.14).

Let Σ (see Fig. 2.1) denote the surface described by the position vector function $\mathbf{r}(u, v)$. Then

$$\begin{aligned} \mathbf{r}_u \cdot \mathbf{r}_u &= E = 1, \\ \mathbf{r}_v \cdot \mathbf{r}_v &= G = 1, \text{ and} \\ \mathbf{r}_u \cdot \mathbf{r}_v &= F. \end{aligned} \tag{2.15}$$

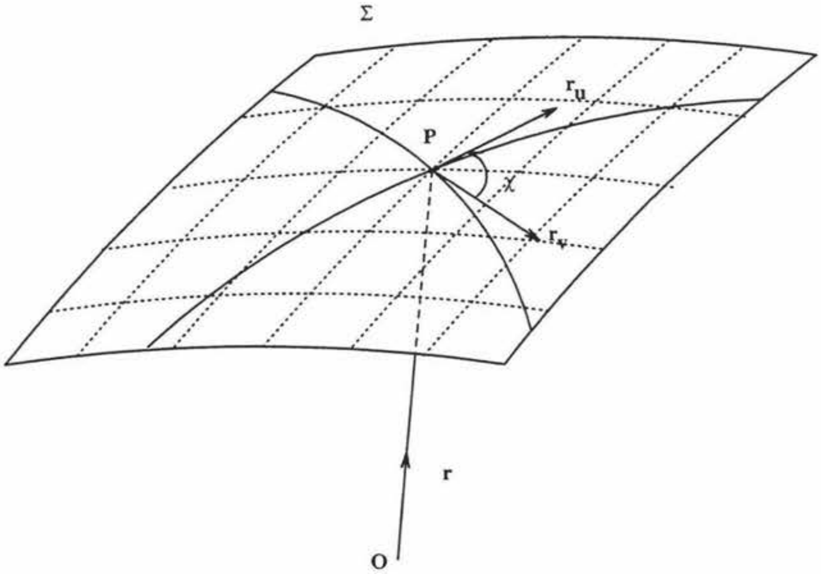

Figure 2.1: Surface Σ described by the vector $\mathbf{r}(u,v)$.

But

$$\frac{\mathbf{r}_u \cdot \mathbf{r}_v}{\|\mathbf{r}_u\| \, \|\mathbf{r}_v\|} = \cos\chi,$$

where $\chi(\phi)$ is the angle between the coordinate lines on the surface; thus,

$$F = \cos\chi(\phi)\,, \tag{2.16}$$

since $\|\mathbf{r}_u\| = \|\mathbf{r}_v\| = 1$. Consequently,

$$\begin{aligned} F_\phi &= -\sin\chi(\phi)\,\chi'(\phi), \\ F_{\phi\phi} &= -\sin\chi(\phi)\,\chi''(\phi) - \cos\chi(\phi)\,(\chi'(\phi))^2\,, \end{aligned} \tag{2.17}$$

and substituting these expressions into (2.14) we have

$$A(\phi) = -\frac{\chi''}{\chi'}\,. \tag{2.18}$$

Therefore,

$$\ln\chi' = -\int A(\phi)\,d\phi + \ln|c|\,,$$

where c is a constant of integration, and so

$$\chi'(\phi) = c \exp\left(-\int A(\phi)\, d\,\phi\right).$$

Thus,

$$\chi(\phi) = c \int \exp\left(-\int_0^{\phi} A(\zeta)\, d\,\zeta\right) d\,\phi + c_1\,, \tag{2.19}$$

where c_1 is some constant of integration. In terms of $\chi(\phi), the$ Gaussian curvature is

$$K = -\frac{M(\phi)}{\sin\chi(\phi)}\,\chi'(\phi)\,.$$

We can thus identify the PDE given by equation (2.12) with a surface defined intrinsically by the quantities

$$E = 1,\ F = \cos\chi(\phi),\ G = 1 \text{ and } K = -\frac{M(\phi)}{\sin\chi(\phi)}\,\chi'(\phi)\,. \tag{2.20}$$

Note that if $A(\phi) = 0$ then we have

$$\chi(\phi) = \lambda\phi + \lambda_1\,,$$

and so,

$$F = \cos(\lambda\phi + \lambda_1), \tag{2.21}$$

where λ and λ_1 are arbitrary constants. From (2.19) we have

$$\chi_u = \lambda \exp\left(-\int A(\phi)\, d\,\phi\right)\,\phi_u\,,$$

$$\chi_v = \lambda \exp\left(-\int A(\phi)\, d\,\phi\right)\,\phi_v\,,$$

and from equation (2.12)

$$\chi_{uv} = \lambda \exp\left(-\int A(\phi)\, d\,\phi\right)\{\phi_{uv} - A(\phi)\,\phi_u\,\phi_v\} = \lambda \exp\left(-\int A(\phi)\, d\,\phi\right) M(\phi)\,.$$

Hence solving equation (2.19) for ϕ, we have a PDE of the form

$$\chi_{uv} = M_1(\chi)\,. \tag{2.22}$$

We can thus reduce a PDE of the form of (2.12) to one of the form (2.22) by using the transformation defined by (2.19). Hence it is sufficient to limit the investigation to the case when $A(\phi) = 0$.

If $A(\phi) = 0$, equation (2.12) reduces to

$$\phi_{uv} = M(\phi), \tag{2.23}$$

and from (2.20) and (2.21),

$$E = 1,\ F = \cos(\lambda\,\phi + \lambda_1),\ G = 1,\ ,\ K = -\frac{\lambda\,M(\phi)}{\sin(\lambda\,\phi + \lambda_1)}\,. \tag{2.24}$$

The constants λ and λ_1 correspond respectively to the magnification and the shift of the angle between the characteristics. Since λ_1 is merely the reference point from which the angles are measured, we can choose $\lambda_1 = 0$. Using the transformation

$$\chi = \lambda\phi,$$

which involves the magnification factor λ, we may transform (2.23) to a PDE of the form

$$\chi_{uv} = M_1(\chi).$$

Thus without loss of generality we may choose $\lambda = 1$. Choosing $\lambda = 1$ and $\lambda_1 = 0$ we identify a PDE of the form (2.23) as Gauss' equation on a surface Σ with

$$E = 1,\ F = \cos\phi,\ G = 1,\ K = -\frac{M(\phi)}{\sin\phi}\,. \tag{2.25}$$

We shall now derive a transformation on Σ from a geodesic coordinate system (x, y) to the (u, v) coordinate system (Tchebychef net).

Let $\hat{E}$, $\hat{F}$ and $\hat{G}$ be the coefficients of the first fundamental form in the geodesic coordinate system. In geodesic coordinates $\hat{E} = 1$ and $\hat{F} = 0$. Moreover, Gauss' equation becomes the Schrödinger equation

$$\hat{H}_{xx} + K(x,y)\,\hat{H} = 0, \tag{2.26}$$

where $\hat{H}^2 = \hat{E}\hat{G} - \hat{F}^2 = \hat{G}$.

If we can determine a coordinate transformation between geodesic coordinates and Tchebychef net coordinates, and if we can solve this Schrödinger equation, then we can find a solution to (2.23).

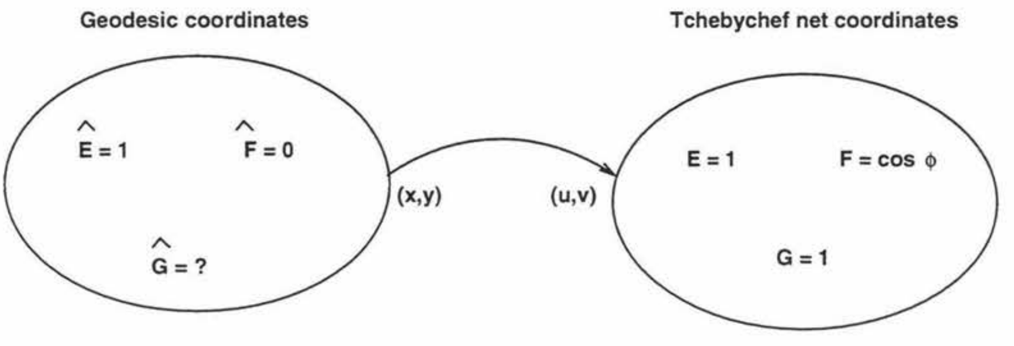

Figure 2.2: Coordinate transformation from geodesic coordinate system to the Tchebychef net coordinate system.

The usual tensor properties (where $g_{11} = E$, $g_{12} = g_{21} = F$, $g_{22} = G$, $\hat{g}_{11} = \hat{E}$, $\hat{g}_{12} = \hat{g}_{21} = \hat{F}$, $\hat{g}_{22} = \hat{G}$) yield the following relations between the coordinates:

$$x_u^2 + \hat{H}^2\,y_u^2 = 1\,, \tag{2.27}$$

$$x_u\,x_v + \hat{H}^2\,y_u\,y_v = F = \cos\phi\,, \tag{2.28}$$

$$x_v^2 + \hat{H}^2\,y_v^2 = 1. \tag{2.29}$$

The general solution to the above system of PDEs provides the required coordinate transformations between the two coordinate systems. Solving this system could be formidable. Thus we look at one specific solution which we can be determined by assuming,

$$x_u = x_v = \left\{1 - \hat{H}^2(x,y)\right\}^{1/2} \quad \text{and} \quad y_u^2 = y_v^2 = 1 \tag{2.30}$$

provided $y_u \neq y_v$. From the last equation we define a transformation implicitly by

$$x = \int \left\{1 - \hat{H}^2(x,y)\right\}^{1/2} du + f(v) \tag{2.31}$$

and

$$y = u - v, \tag{2.32}$$

where $f(v)$ satisfies,

$$f'(v) + \frac{\partial}{\partial v} \left\{ \int \left(1 - \hat{H}^2(x,y)\right)^{1/2} du \right\} = \left\{1 - \hat{H}^2(x,y)\right\}^{1/2}. \tag{2.33}$$

Clearly this transformation is non-singular since the Jacobian is non- zero. i.e.

$$x_u y_v - x_v y_u = -2\sqrt{(1 - \hat{H}^2)} \neq 0.$$

We also note that $\hat{H} < 1$ unless $K = 0$, i.e., $M(\phi) = 0$.

We are aware of the fact that, by choosing particular solutions as described above for the system of PDEs (2.27)- (2.29), we restrict ourselves into obtaining only special classes of solutions of the PDE (2.23) and not its general solution.

Equations (2.28) and (2.30) imply that

$$F = \cos\phi = (1 - 2\hat{H}^2), \tag{2.34}$$

hence

$$\hat{H} = \sin\frac{\phi}{2}. \tag{2.35}$$

The derivatives of $\hat{H}$ with respect to x are thus

$$\hat{H}_x = \frac{1}{2}\cos\frac{\phi}{2}\,\phi_x$$

and

$$\hat{H}_{xx} = \frac{1}{2}\cos\frac{\phi}{2}\,\phi_{xx} - \frac{1}{4}\sin\frac{\phi}{2}\,(\phi_x)^2.$$

Substituting $\hat{H}$ and $\hat{H}_{xx}$ into the Schrödinger equation (2.26), and after some simplification we obtain

$$\phi_{xx} - \frac{1}{2}\tan\frac{\phi}{2}(\phi_x)^2 = M(\phi)\sec^2\frac{\phi}{2}. \tag{2.36}$$

Equation (2.36) is a second order PDE which does not have the y derivative terms and can be treated as a second order ODE. The two arbitrary 'constants' of integration will be arbitrary functions of y. The ODE

$$\frac{d^2\phi}{dx^2} - \frac{1}{2}\tan\frac{\phi}{2}\left(\frac{d\phi}{dx}\right)^2 = M(\phi)\sec^2\frac{\phi}{2} \tag{2.37}$$

can be solved by the use of the substitution

$$\frac{d\phi}{dx} = p\,. \tag{2.38}$$

We have that

$$\frac{d^2\phi}{dx^2} = \frac{d}{dx}\left(\frac{d\phi}{dx}\right) = \frac{d}{d\phi}(p).\frac{d\phi}{dx} = p\frac{dp}{d\phi},$$

and (2.37) thus reduces to

$$\frac{dp}{d\phi} - \frac{1}{2}\tan\frac{\phi}{2}p = M(\phi)\sec^2\frac{\phi}{2}p^{-1}. \tag{2.39}$$

Let

$$z = p^2 \tag{2.40}$$

then equation (2.39) reduces to the linear first order ODE

$$\frac{dz}{d\phi} - \tan\frac{\phi}{2}z = 2M(\phi)\sec^2\frac{\phi}{2}. \tag{2.41}$$

To solve this, we find the integrating factor

$$\exp\left(\int -\tan\frac{\phi}{2}\,d\phi\right) = \cos^2\frac{\phi}{2}, \tag{2.42}$$

and reduce equation (2.41) to

$$\frac{d}{d\phi}\left\{z\cos^2\frac{\phi}{2}\right\} = 2\,M(\phi). \tag{2.43}$$

Therefore,

$$z = 2\sec^2\frac{\phi}{2}\left\{\int M(\phi)d\phi + c_1(y)\right\}. \tag{2.44}$$

Now the combination of (2.38), (2.40) and (2.44) yields,

$$p = \frac{d\phi}{dx} = 2\sec\frac{\phi}{2}\left\{\int M(\phi)\,d\phi + c_1(y)\right\}^{1/2},$$

and so we have

$$x = \frac{1}{2}\int\frac{\cos\frac{\phi}{2}}{\left\{\int M(\phi)\,d\phi + c_1(y)\right\}^{1/2}}\,d\phi + c_2(y) \tag{2.45}$$

where $c_1(y)$ and $c_2(y)$ are arbitrary functions of y.

Now (2.45) defines a relationship

$$\mu(x, y, \phi) = 0 \tag{2.46}$$

between x, y and ϕ and by assuming that one set of values x_0, y_0, ϕ_0 can be found to satisfy (2.46) and that, near (x_0, y_0, ϕ_0), μ and its first partial derivatives are continuous and $\frac{\partial\mu}{\partial\phi} \neq 0$, the *implicit function theorem*[22] states that in a region of the xy plane containing (x_0, y_0), there is precisely one differentiable function

$$\phi = \alpha(x, y) \tag{2.47}$$

which reduces (2.46) to an identity and is such that $\phi_0 = \alpha(x_0, y_0)$.

Under these assumptions, we now have from (2.35) and (2.47) that,

$$\hat{H} = \sin\frac{1}{2}(\alpha(x, y)) \tag{2.48}$$

and so

$$x_u = x_v = \cos\frac{1}{2}(\alpha(x, y)). \tag{2.49}$$

Noting that $\alpha(x, y)$ contains the two arbitrary functions $c_1(y)$ and $c_2(y)$, and that $y = u - v$ we have

$$\frac{\partial x}{\partial u} = \frac{\partial x}{\partial v} = , \cos\frac{1}{2}(\alpha(x, u - v)). \tag{2.50}$$

From the above expression we can, in principle, determine x in terms of u and v, say,

$$x = \beta(u,v).$$

Hence, there is a coordinate transformation

$$\left.\begin{aligned} x &= \beta(u,v) \\ y &= u-v \end{aligned}\right\}, \tag{2.51}$$

and since

$$\frac{\partial(x,y)}{\partial(u,v)} \neq 0$$

this transformation is invertible:

$$\left.\begin{aligned} u &= \gamma(x,y) \\ v &= \delta(x,y) \end{aligned}\right\}. \tag{2.52}$$

Hence the general solution to the given PDE (2.12) is

$$\begin{aligned} \phi &= \alpha(x,y) \\ &= \alpha(\beta(u,v), u-v) \end{aligned} \tag{2.53}$$

where u and v are given by (2.52).

The above method shows that at least formally, a solution can be obtained; however, the method involves some inversions, which may prove formidable. We should also note that the functions c_1 and c_2 depend on y (or $u-v$). In the next section we consider as an example, the *sine-Gordon equation*, since this is of the form (2.23), and investigate the possibilities of solving initial value problems.

2.1.1.2 Example: sine-Gordon equation

The sine-Gordon equation

$$\phi_{uv} = \sin\phi, \tag{2.54}$$

is of the form (2.23) with $M(\phi) = \sin\phi$ and $K = -1$; hence, (2.45) becomes

$$x = \frac{1}{2}\int \frac{\cos\frac{\phi}{2}}{\{-\cos\phi + c_1(y)\}^{1/2}}\, d\phi + c_2(y) \tag{2.55}$$

where $c_1(y)$ and $c_2(y)$ are arbitrary functions of y.

Using some trigometrical identities (2.55) can be expressed as

$$x = \frac{1}{2}\int \left\{ \frac{\cos\left(\frac{\phi}{2}\right)}{\sqrt{\left(\frac{c_1(y)-1}{2}\right) + \sin^2\left(\frac{\phi}{2}\right)}} \right\} d\phi + c_2(y). \tag{2.56}$$

The substitution

$$t = \sin\left(\frac{\phi}{2}\right)$$

puts (2.56) into the form

$$x = \int \frac{1}{\sqrt{\left(\sqrt{\frac{c_1(y)-1}{2}}\right)^2 + t^2}}\, dt + c_2(y), \qquad c_1(y) \neq 1 \tag{2.57}$$

so

$$x = \ln\left| \frac{t + \sqrt{\left(\sqrt{\frac{c_1(y)-1}{2}}\right)^2 + t^2}}{\sqrt{\frac{c_1(y)-1}{2}}} \right| + c_2(y). \tag{2.58}$$

Substituting for t, and solving for $\phi(x,y)$ we obtain

$$\phi(x,y) = 2\sin^{-1}\left\{ \sqrt{\frac{c_1(y) - 1}{2}}\, \sinh\{x - c_2(y)\} \right\} \tag{2.59}$$

or

$$\sin\frac{\phi}{2} = \left\{ \sqrt{\frac{c_1(y) - 1}{2}}\, \sinh\{x - c_2(y)\} \right\}.$$

Hence from (2.35)

$$\hat{H} = \left\{ \sqrt{\frac{c_1(y) - 1}{2}}\, \sinh\{x - c_2(y)\} \right\}. \tag{2.60}$$

The expression for $\hat{H}$ obtained above can be easily verified to be a solution of the Schrödinger equation,

$$\hat{H}_{xx} - \hat{H} = 0.$$

Now from equations (2.30) and (2.60) we have

$$x_u = x_v = \left\{ 1 - \left(\frac{c_1(y) - 1}{2} \right) \sinh^2 (x - c_2(y)) \right\}^{1/2}, \tag{2.61}$$

thus,

$$\int \frac{1}{\left\{ 1 - \left(\frac{c_1(y)-1}{2} \right) \sinh^2 (x - c_2(y)) \right\}^{1/2}} \, dx = u + \rho(v)$$

and

$$\int \frac{1}{\left\{ 1 - \left(\frac{c_1(y)-1}{2} \right) \sinh^2 (x - c_2(y)) \right\}^{1/2}} \, dx = v + \sigma(u)$$

where $\rho(v)$ and $\sigma(u)$ are arbitrary functions of v and u respectively.

If we consider the first equation of the above two, and make a substitution

$$\sin\theta = \sqrt{\frac{c_1(y) - 1}{2}} \sinh(x - c_2(y)) \tag{2.62}$$

we get

$$u + \rho(v) = m \int \frac{1}{\left\{ 1 + m^2 \sin^2 \theta \right\}^{1/2}} \, d\theta \tag{2.63}$$

where

$$m = \left\{ \frac{c_1(y) - 1}{2} \right\}^{-1/2}. \tag{2.64}$$

Therefore

$$u + \rho(v) = \frac{k}{2} F \, (\alpha \, , k) = \frac{k}{2} \mathrm{sn}^{-1} (\sin\alpha \, , k) \, , \tag{2.65}$$

where

$$k = m(1 + m^2)^{-1/2}$$

and F is an *elliptic function of the first kind*[12, 21]; this implies that $\rho(v)$ depends on the arbitrary functions c_1 and c_2 and is such that

$$\rho'(v) = \frac{\partial}{\partial v}\left\{\frac{k}{2}\ \mathrm{sn}^{-1}(\sin\alpha\,,\, k)\right\}$$

and moreover

$$\alpha = \sin^{-1}\left\{\frac{1}{k}\tanh\left(x - c_2(y)\right)\right\}, \tag{2.66}$$

$$k = \sqrt{\frac{2}{c_1(y)+1}}\,. \tag{2.67}$$

From equation (2.65) we have

$$\mathrm{sn}\left(\frac{2}{k}\left(u\,+\,\rho(v)\right)\right) \,=\, \sin\alpha$$

and so by using equation (2.66) we get

$$x \,=\, \tanh^{-1}\left\{k\,\mathrm{sn}\left(\frac{2}{k}(u\,+\,\rho(v))\,,\,k\right)\right\} + c_2(y) \tag{2.68}$$

where $y \,=\, u \,-\, v$.

Hence, from (2.59), (2.67) and (2.68) (after some algebraic manipulations) we obtain the following family of solutions for the *sine-Gordon equation*:

$$\phi(u,v) = 2\sin^{-1}\left\{\frac{\mathrm{sn}\left(\frac{2}{k}(u+\rho(v))\,,\,k\right)}{\left\{1+\frac{k^2}{(1-k^2)}\,\mathrm{cn}^2\left(\frac{2}{k}(u+\rho(v))\,,\,k\right)\right\}^{1/2}}\right\} \tag{2.69}$$

where k is an arbitrary function of $u - v$, and $\rho(v)$ is an arbitrary function of v.

2.1.1.3 Initial Value Problems

In this section we consider the *Cauchy Problem* [17] for equations of the form (2.12).

The Cauchy Problem:

The *Cauchy Problem* for equation (2.12) consists of solving this PDE for given initial data along a non-characteristic curve. Let Λ be a smooth curve with no self-intersections, defined in the plane in parametric form by $u = U(t)$ and $v = V(t)$.

Let

$$\phi_u = p \quad \text{and} \quad \phi_v = q$$

and suppose that on Λ the Cauchy Data are

$$\phi = \Phi\,(t), \quad p = P\,(t)\,, \quad \text{and} \quad q = Q\,(t)\,. \tag{2.70}$$

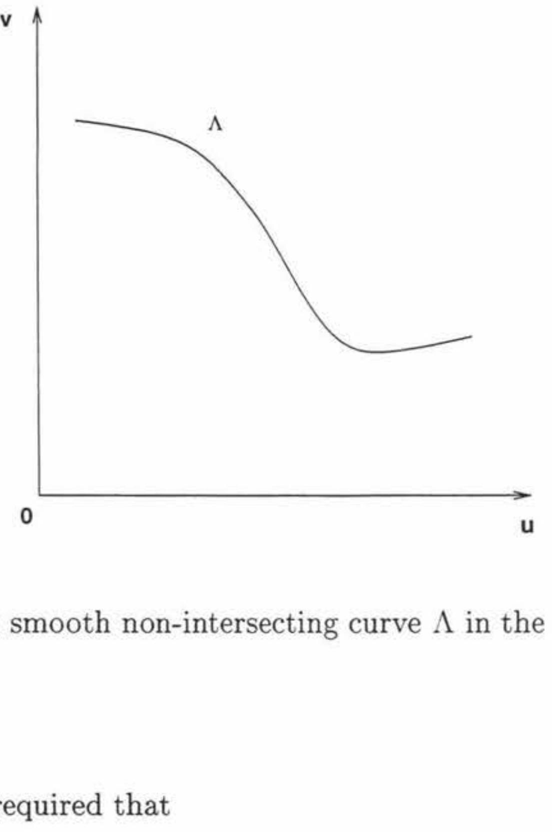

Figure 2.3: A smooth non-intersecting curve Λ in the $u - v$ plane.

For compatibility it is required that

$$\dot{\Phi}\,(t) = P\,(t)\,\dot{U}\,(t) + Q\,(t)\,\dot{V}\,(t)\,.$$

Then the following system of equations is satisfied along Λ:

$$\begin{aligned} \phi_{uv} &= f\,(\,\Phi\,(t)\,,\,P\,(t)\,,Q\,(t)\,) \\ \dot{U}\,(t)\,\phi_{uu} + \dot{V}\,(t)\,\phi_{uv} &= \dot{P}\,(t) \\ \dot{U}\,(t)\,\phi_{uv} + \dot{V}\,(t)\,\phi_{vv} &= \dot{Q}\,(t)\,. \end{aligned}$$

Since the determinant of the coefficients of the above system is

$$\begin{vmatrix} 0 & 1 & 0 \\ \dot{U} & \dot{V} & 0 \\ 0 & \dot{U} & \dot{V} \end{vmatrix} = -\dot{U}\,\dot{V},$$

we note that the Cauchy data are non-characteristic if

$$\dot{U} \neq 0 \quad \text{and} \quad \dot{V} \neq 0\,.$$

We use the sine-Gordon equation as an example to illustrate the above initial value problem.

Example: sine-Gordon Equation

To solve the Cauchy problem for the sine-Gordon equation (2.54), we shall take ϕ as in (2.59) (using (x, y) geodesic coordinates) rather than as in (2.69) which uses the Tchebychef net (u, v). The calculations for ϕ_u and ϕ_v are easier this way.

Recall from equation (2.59), that

$$\phi = 2\,\sin^{-1}\left\{\sqrt{\frac{c_1(y)\,-\,1}{2}}\,\sinh\{x\,-\,c_2(y)\}\right\};$$

therefore,

$$\phi_x = \frac{2}{\left\{1-\left(\frac{c_1(y)-1}{2}\right)\sinh^2(x-c_2(y))\right\}^{1/2}}\sqrt{\frac{c_1(y)-1}{2}}\,\cosh(x-c_2(y)), \tag{2.71}$$

and

$$\begin{aligned}\phi_y \;&=\; \frac{2\left\{\sqrt{\frac{c_1(y)-1}{2}}\,\cosh(x-c_2(y))(-\,c_2'(y))\,+\sinh(x-c_2(y))\,\frac{1}{2\sqrt{\frac{c_1(y)-1}{2}}}\,\frac{1}{2}\,c_1'(y)\right\}}{\left\{\,1\,-\,\left(\frac{c_1(y)-1}{2}\right)\,\sinh^2\left(x-c_2(y)\right)\,\right\}^{1/2}} \\ &=\; \frac{\left\{-\,4\left(\frac{c_1(y)-1}{2}\right)\,\cosh(x\,-\,c_2(y))\,c_2'(y)+\,\sinh(x\,-\,c_2(y))\,c_1'(y)\right\}}{2\sqrt{\frac{c_1(y)-1}{2}}\,\left\{\,1\,-\,\left(\frac{c_1(y)-1}{2}\right)\,\sinh^2\left(\,x\,-\,c_2(y)\,\right)\,\right\}^{1/2}}.\end{aligned} \tag{2.72}$$

Recall from (2.51) and (2.61) that

$$x_u = x_v = \left\{ 1 - \left(\frac{c_1(y) - 1}{2} \right) \sinh^2 (x - c_2(y)) \right\}^{1/2},$$

and

$$y_u = 1, \quad y_v = -1.$$

Hence

$$\begin{aligned} \phi_u &= \phi_x x_u + \phi_y y_u \\ &= 2\sqrt{\tfrac{c_1(y)-1}{2}} \cosh(x - c_2(y)) + \phi_y \end{aligned}$$

and

$$\begin{aligned} \phi_v &= \phi_x x_v + \phi_y y_v \\ &= 2\sqrt{\tfrac{c_1(y)-1}{2}} \cosh(x - c_2(y)) - \phi_y \end{aligned}$$

where ϕ_y is as in (2.72).

Adding the expressions for ϕ_u and ϕ_v yields

$$\phi_u + \phi_v = 4\sqrt{\frac{c_1(y) - 1}{2}} \cosh(x - c_2(y))$$

and subtracting them gives

$$\phi_u - \phi_v = \frac{\left\{ -4\left(\frac{c_1(y)-1}{2}\right) \cosh(x - c_2(y))\, c_2'(y) + \sinh(x - c_2(y))\, c_1'(y) \right\}}{\sqrt{\frac{c_1(y)-1}{2}} \left\{ 1 - \left(\frac{c_1(y)-1}{2}\right) \sinh^2(x - c_2(y)) \right\}^{1/2}}.$$

Suppose that the initial conditions in the (u, v) coordinate system(Tchebychef net) $\phi_u = P(t)$ and $\phi_v = Q(t)$ are transformed to $P_1(t)$ and $Q_1(t)$ in the (x, y) (geodesic) coordinate system. Then from the last two expressions we have by substitution that

$$P_1 + Q_1 = 4\sqrt{\frac{c_1(y_0) - 1}{2}} \cosh(x_0 - c_2(y_0)), \tag{2.73}$$

and

$$P_1 - Q_1 = \frac{\left\{-4\left(\frac{c_1(y_0)-1}{2}\right)\cosh(x_0 - c_2(y_0))\, c_2'(y_0) + \sinh(x_0 - c_2(y_0))\, c_1'(y_0)\right\}}{\sqrt{\frac{c_1(y_0)-1}{2}}\left\{1 - \left(\frac{c_1(y_0)-1}{2}\right)\sinh^2\left(x_0 - c_2(y_0)\right)\right\}^{1/2}} \tag{2.74}$$

where $x_0(t)$ and $y_0(t)$ correspond to the initial curve in the (x, y) geodesic coordinate system.

From equation (2.73),

$$\cosh(x_0 - c_2(y_0)) = \frac{P_1 + Q_1}{4\sqrt{\frac{c_1(y_0)-1}{2}}}, \tag{2.75}$$

and using the identity $\cosh^2\theta - \sinh^2\theta = 1$ this becomes

$$\sinh(x_0 - c_2(y_0)) = \frac{\left\{\left(\frac{P_1+Q_1}{4}\right)^2 - \left(\frac{c_1(y_0)-1}{2}\right)\right\}^{1/2}}{\sqrt{\frac{c_1(y_0)-1}{2}}}. \tag{2.76}$$

Equation (2.75) implies

$$c_2(y_0) = x_0 - \cosh^{-1}\left\{\frac{P_1 + Q_1}{4\sqrt{\frac{c_1(y_0)-1}{2}}}\right\}. \tag{2.77}$$

Equation (2.76) indicates that

$$1 - \left(\frac{c_1(y_0) - 1}{2}\right)\sinh^2(x_0 - c_2(y_0)) = \left(\frac{c_1(y_0) + 1}{2}\right) - \left(\frac{P_1 + Q_1}{4}\right)^2 \tag{2.78}$$

and differentiating equation (2.77) with respect to y_0 implies

$$c_2'(y_0) = \frac{P_1 + Q_1}{16\left(\frac{c_1(y_0)-1}{2}\right)\left\{\left(\frac{P_1+Q_1}{4}\right)^2 - \left(\frac{c_1(y_0)-1}{2}\right)\right\}^{1/2}}\, c_1'(y_0) \tag{2.79}$$

Next we substitute the expressions given in equations (2.75), (2.76), (2.77) (2.78) and (2.79) into equation (2.74), and after some simplifications we get,

$$
\begin{aligned}
c_1'(y_0) &= (Q_1 - P_1)\left\{\left[\left(\frac{c_1(y_0)+1}{2}\right) - \left(\frac{P_1+Q_1}{4}\right)^2\right]\left[\left(\frac{P_1+Q_1}{4}\right)^2 - \left(\frac{c_1(y_0)-1}{2}\right)\right]\right\}^{1/2} \\
&= (Q_1 - P_1)\left\{\left[\frac{1}{2} - \left(\left(\frac{P_1+Q_1}{4}\right)^2 - \frac{c_1(y_0)}{2}\right)\right]\left[\frac{1}{2} + \left(\left(\frac{P_1+Q_1}{4}\right)^2 - \frac{c_1(y_0)}{2}\right)\right]\right\}^{1/2} \\
&= (Q_1 - P_1)\left\{\frac{1}{4} - \left(\left(\frac{P_1+Q_1}{4}\right)^2 - \frac{c_1(y_0)}{2}\right)^2\right\}^{1/2} \\
&= -\frac{(P_1 - Q_1)}{2}\left\{1 - \left(\left(\frac{P_1+Q_1}{2\sqrt{2}}\right)^2 - c_1(y_0)\right)^2\right\}^{1/2} \\
&= A\left\{1 - (B - c_1(y_0))^2\right\}^{1/2}, \qquad (2.80)
\end{aligned}
$$

where

$$
A = -\frac{P_1 - Q_1}{2} \quad \text{and} \quad B = \left(\frac{P_1+Q_1}{2\sqrt{2}}\right)^2. \qquad (2.81)
$$

Equation (2.80) is a first order ODE which can be solved for $c_1(y_0)$, and so $c_2(y_0)$ can be determined from equation (2.73). Hence, the Cauchy Problem for the sine-Gordon equation can be reduced to solving a nonlinear first order ODE and inversions.

For example, suppose Λ is denoted in the Tchebychef net coordinate system by

$$
u + v = c_0 \quad \text{(constant)}, \qquad (2.82)
$$

and on Λ,

$$
\phi = \Phi_0, \quad p = P_0 \quad \text{and} \quad q = Q_0 \qquad (2.83)
$$

where Φ_0, P_0 and Q_0 are *constants*. It can be shown using equation (2.53) that the initial conditions given in equation (2.83), when transformed to the geodesic (x, y) coordinates, have the same form. i.e.

$$
\phi = \Phi_1, \quad p = P_1 \quad \text{and} \quad q = Q_1 \qquad (2.84)
$$

where Φ_1, P_1 and Q_1 are *constants.* Then A and B (given in equation (2.81)) are constants and so, solving for $c_1(y_0)$ using equation (2.80) and making a substitution

$$B \,-\, c_1(y_0) \,=\, \sin\theta \tag{2.85}$$

we obtain

$$c_1'(y_0) \,=\, -\,\cos\theta\,.\,\theta'\,.$$

Substituting the last two expressions in equation (2.80) gives

$$\theta' \,=\, A,$$

i.e.

$$\theta \,=\, A\,y_0 \,+\, C; \tag{2.86}$$

thus

$$c_1(y_0) \,=\, B \,-\, \sin(\,A\,y_0 \,+\, C\,).$$

To determine C, we recall that $\phi \,=\, \Phi_1$ on Λ. Using the expressions for $c_1(y_0), c_2(y_0)$ and equation (2.59), gives

$$C \,=\, \sin^{-1}\left\{2\sin^2\left(\frac{\Phi_1}{2}\right) - 1\right\} \,-\, A\,y_0\,. \tag{2.87}$$

Hence, in general,

$$c_1(y) \,=\, B \,-\, \sin(\,A\,y \,+\, C\,)\,, \tag{2.88}$$

and from equations (2.77) and (2.87), we find that

$$c_2(y) \,=\, x_0 \,-\, \cosh^{-1}\sqrt{\frac{B}{(B-1) - \sin(A\,y \,+\, C)}}\,. \tag{2.89}$$

A solution to the Cauchy Problem for the sine-Gordon equation in geodesic coordinates is given by equation (2.59), where $c_1(y)$ and $c_2(y)$ are defined by equations (2.88) and (2.89) respectively. To get a solution in terms of the original coordinates we can use equations (2.32) and (2.45) to express x and y in terms of u and v.

Derivation of the Beltrami Surface

Though it is perhaps not obvious, this solution corresponds to a *Beltrami surface* [28] (see Fig.2.4). Recall that the Beltrami surface is a surface of revolution described by

$$\mathbf{r} = \left(\sin X \cos Y, \sin X \sin Y, \cos X + \ln \tan \frac{X}{2}\right),$$

where the (X, Y) coordinates correspond to the lines of curvature.

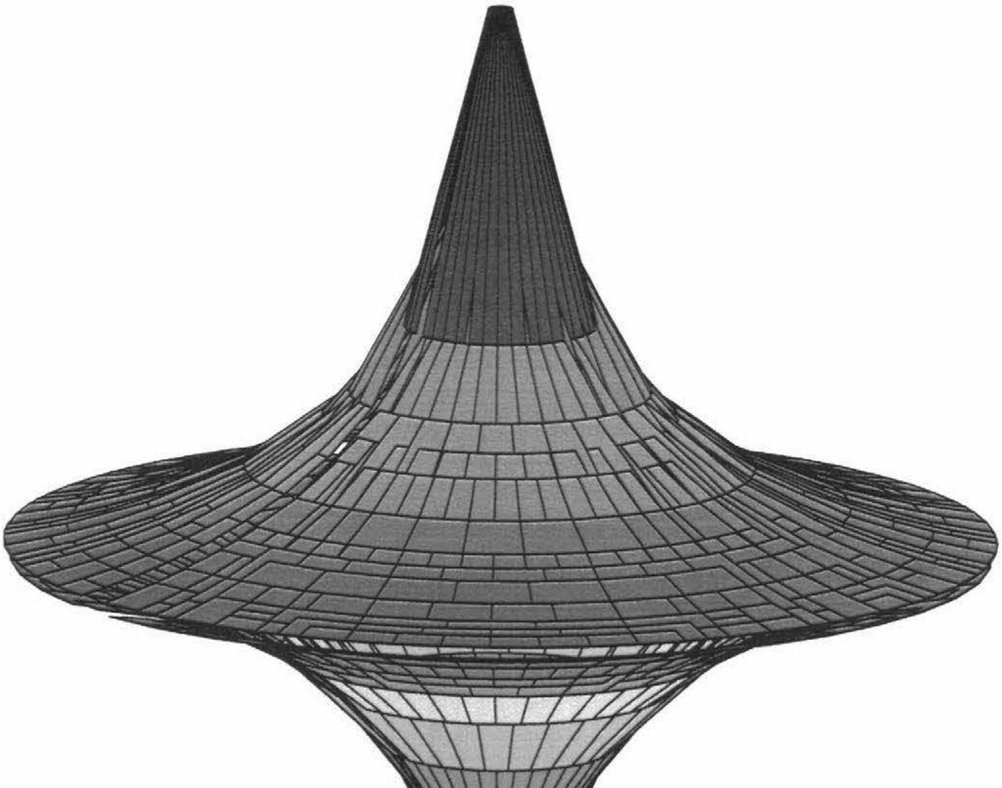

Figure 2.4: *Beltrami surface* in the (X, Y) coordinate system

The coefficients E, F and G of the first fundamental form are

$$E = \cot^2 X, \quad F = 0, \quad G = \sin^2 X,$$

and thus $H = \sqrt{EG - F^2} = \cos X$.

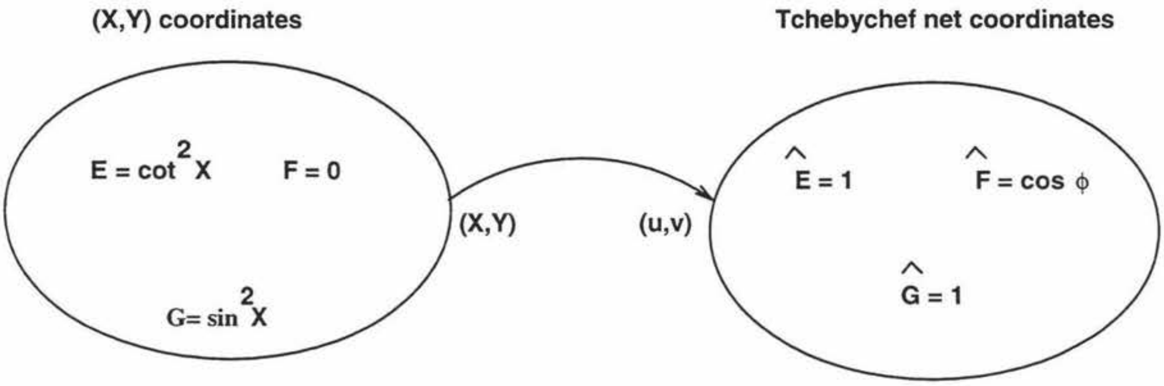

Figure 2.5: Coordinate transformation from (X, Y) coordinate system to the Tchebychef net coordinate system.

A transformation between the (X, Y) coordinate system and the Tchebychef net (u, v) coordinate system is defined by the system

$$X_u^2 \cot^2 X + Y_u^2 \sin^2 X = 1 , \tag{2.90}$$

$$X_u X_v \cot^2 X + Y_u Y_v \sin^2 X = \hat{F} = \cos\phi , \tag{2.91}$$

$$X_v^2 \cot^2 X + Y_v^2 \sin^2 X = 1 . \tag{2.92}$$

One solution to this system can be determined under the assumption that

$$Y_u^2 = Y_v^2 = 1 .$$

Choosing

$$Y = u - v , \tag{2.93}$$

equations (2.90) and (2.92) give

$$X_u = X_v = \sin X.$$

A suitable transformation is defined by (2.93) and

$$u + v = \ln \mid \csc X - \cot X \mid . \tag{2.94}$$

This non-characteristic curve in the (u, v) coordinates (Tchebychef net) can be taken as the initial curve for the Cauchy Problem of the sine-Gordon equation. Also, we have from equations (2.95) and (2.96) that

$$\phi = c_1 , \quad \phi_u = c_2 \quad \text{and} \quad \phi_v = c_3 , \tag{2.98}$$

where c_1, c_2 and c_3 are constants.

The last two equations are the same initial conditions as in equations (2.82) and (2.83). Since the Cauchy data are analytic, the *Cauchy-Kowalewski theorem* [9] guarantees a unique solution. Thus, the solution which we had for the Cauchy Problem of the sine-Gordon Equation corresponds to a *Beltrami surface.*

Further Illustration

If we take a known soliton solution of the sine-Gordon equation, (2.54),

$$\phi = 4\tan^{-1}(\exp(u + v + b)), \tag{2.99}$$

where b is a constant, then from equation (2.35) we obtain

$$\hat{H} = \operatorname{sech}(u + v + b),$$

and thus from equation (2.30), we get

$$x_u = x_v = \tanh(u + v + b).$$

Since $y = u - v$, we have $y_u = 1$ and $y_v = -1$. Now using the identities [11]

$$u_x = \frac{y_v}{(x_u y_v - x_v y_u)}, \quad v_x = \frac{y_u}{(x_v y_u - x_u y_v)},$$

we obtain expressions for u_x and v_x as

$$u_x = v_x = 2\coth(u + v + b).$$

Since

$$\hat{H}_x = \hat{H}_u u_x + \hat{H}_v v_x,$$

we obtain after substitution and some simplifications,

$$\hat{H}_x = \operatorname{sech}(u + v + b).$$

Similarly using

$$\hat{H}_{xx} = (\hat{H}_x)_u u_x + (\hat{H}_x)_v v_x$$

we get

$$\hat{H}_{xx} = \operatorname{sech}(u + v + b),$$

which clearly illustrates that $\hat{H} = \operatorname{sech}(u + v + b)$ is the corresponding solution to the Schrödinger equation (2.26), where $K = -1$.

Further, solving

$$x_u \;=\; x_v \;=\; \tanh(u + v + b)$$

and choosing the arbitrary functions of integration appropriately, yields the coordinate transformation for x as

$$x = \ln\,|\cosh(u + v + b)|\,.$$

The above illstration clearly indicates that we can generate classes of solutions to either of the equations by using our technique with known solution to one of the equations.

2.1.1.4 Case(b): $G_\phi \phi_{uu} + E_\phi \phi_{vv} \;=\; 0.$

In this case having

$$G_\phi \phi_{uu} + E_\phi \phi_{vv} \;=\; 0 \tag{2.100}$$

equation (2.4) becomes

$$\phi_{uv} = M(\phi) + A(\phi)\phi_u^2 + B(\phi)\phi_v^2 + C(\phi)\phi_u\phi_v \tag{2.101}$$

where

$$M(\phi) \;=\; \frac{KH^2}{F_\phi}, \quad A = \frac{\Theta_1}{2F_\phi}, \quad B = \frac{\Theta_2}{2F_\phi}, \quad C = \frac{\Theta_3}{2F_\phi}.$$

Thus we have to solve a system of two partial differential equations (equations (2.100) and (2.101)), where obviously, both E and G are functionally dependent on the unknown function ϕ. This condition makes it more difficult to solve this system and hence to determine the required coordinate system for the interpretation of the Gauss equation. Therefore we refrain from further pursuing this case.

2.1.2 Equations of Parabolic Type

Consider a second order quasi-linear parabolic PDE of the type

$$\phi_{uu} = f(\phi, \phi_u, \phi_v). \tag{2.102}$$

Equation (2.4) will be of this form if $G_\phi \neq 0$ and either

(a) $E_\phi = F_\phi = 0$ or

(b) $E_\phi \phi_{vv} - 2F_\phi \phi_{uv} = 0$.

2.1.2.1 Case(a): $E_\phi = F_\phi = 0$

If $E_\phi = F_\phi = 0$ then equation (2.101) will be of the form

$$\phi_{uu} = N(\phi) + A(\phi)\phi_u^2 \tag{2.103}$$

where

$$N(\phi) = \frac{-2KH^2}{G_\phi} \quad \text{and} \quad A(\phi) = \frac{1}{G_\phi}\left\{\frac{EG_\phi^2}{2H^2} - G_{\phi\phi}\right\}. \tag{2.104}$$

Equation (2.103) is a second order ODE in ϕ, which can be solved using standard methods[13].

Example:

Consider the PDE

$$\phi_{uu} = 1 - \phi_u^2.$$

Then we have

$$N(\phi) = 1 \quad \text{and} \quad A(\phi) = -1.$$

Since $E_\phi = F_\phi = 0$, we may choose

$$E = F = 1.$$

Solving (2.104)) for G and K, we obtain

$$G = 1 + e^{2\phi} \quad \text{and} \quad K = -1.$$

Finally we solve the given problem by using the methods adapted in equations (2.38)-(2.45) and obtain the solution

$$\phi = \ln\left|\frac{e^{(u-c_2)} - c_1 e^{-(u-c_2)}}{2}\right|,$$

where c_1 and c_2 are arbitrary constants.

2.1.2.2 Case(b): $E_\phi \phi_{vv} - 2F_\phi \phi_{uv} = 0$.

In this case, we end up with the system of PDEs

$$E_\phi \phi_{vv} - 2F_\phi \phi_{uv} = 0 \tag{2.105}$$

and

$$\phi_{uu} = M(\phi) + A(\phi)\phi_u^2 + B(\phi)\phi_v^2 + C(\phi)\phi_u\phi_v \tag{2.106}$$

where

$$M(\phi) = \frac{-2KH^2}{G_\phi} \quad \text{and} \quad A(\phi) = \frac{1}{G_\phi}\left\{\frac{EG_\phi^2}{2H^2} - G_{\phi\phi}\right\}.$$

This scenario is similar to that we had in the second case for the hyperbolic PDEs. Due to the same reasons described in that particular case, we are unable to further pursue this case.

2.1.3 Equations of Elliptic Type

Consider a second order quasi-linear elliptic PDE of the type

$$\phi_{uu} + \phi_{vv} = f(\phi, \phi_u, \phi_v). \tag{2.107}$$

Equation (2.4) is of this form if

$$E_\phi = G_\phi \quad \text{and} \quad F_\phi = 0,$$

and in this case equation (2.4) reduces to

$$\phi_{uu} + \phi_{vv} = M(\phi) + A(\phi)(\phi_u^2 + \phi_v^2) \tag{2.108}$$

where

$$M(\phi) = -2KH^2 \quad \text{and} \quad A(\phi) = \frac{(E+G)}{2H^2}.$$

Note that for a real coordinate system to exist $A(\phi) \neq 0$, we define implicitly a new coordinate system by taking

$$E = G = g(\phi) \quad \text{and} \quad F = 1 \tag{2.109}$$

where $g(\phi)$ is given by (using the expression for $A(\phi)$)

$$g(\phi) = \frac{1 \pm \sqrt{1 + 4A(\phi)^2}}{2A(\phi)}.$$

Hence we have our new coordinate system defined by

$$E = G = g(\phi)\,, \quad F = 1\,, \quad \text{and} \quad K = \frac{-M(\phi)}{2\{g(\phi)^2 - 1\}}\,. \tag{2.110}$$

As we have done before in section (2.2.1) we consider the coordinate transformations between the geodesic coordinate system and the coordinate system defined above, by using the transformation equations,

$$x_u^2 + \hat{H}^2 y_u^2 = g(\phi), \tag{2.111}$$

$$x_u\, x_v \,+\, \hat{H}^2\, y_u\, y_v \,=\, F \,=\, 1\,, \tag{2.112}$$

$$x_v^2 \,+\, \hat{H}^2\, y_v^2 \,=\, g(\phi). \tag{2.113}$$

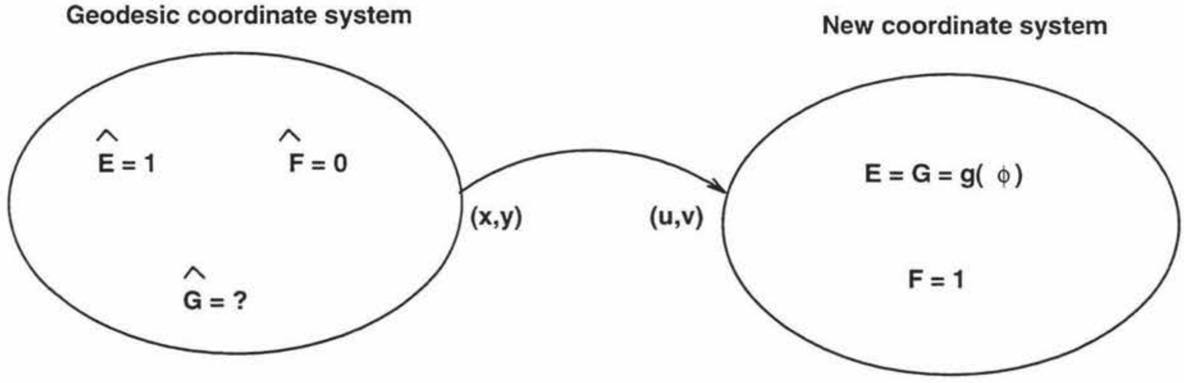

Figure 2.7: Coordinate transformation from geodesic coordinate system to the new coordinate system.

One solution to the above system of equations (2.111)- (2.113), which we shall use in the sequel can be determined by assuming,

$$x_u \,=\, x_v \,=\, \left\{g(\phi) \,-\, \hat{H}^2(x,y)\right\}^{1/2} \quad \text{and} \quad y_u^2 \,=\, y_v^2 \,=\, 1$$

provided $y_u \,\neq\, y_v$. From the last equation we define a transformation implicitly by

$$x \,=\, \int \left\{g(\phi) \,-\, \hat{H}^2(x,y)\right\}^{1/2}\, du \,+\, f_1(v)$$

and

$$y \,=\, u \,-\, v,$$

where $f_1(v)$ satisfies,

$$f_1'(v) \,+\, \frac{\partial}{\partial\, v}\left\{\int \left(g(\phi) \,-\, \hat{H}^2(x,y)\right)^{1/2}\, du\right\} \,=\, \left\{g(\phi) \,-\, \hat{H}^2(x,y)\right\}^{1/2},$$

and $\hat{H}$ becomes

$$\hat{H} \,=\, \frac{1}{\sqrt{2}}\{g(\phi) - 1\}^{1/2}.$$

Thus from equation (2.112) we have

$$x_u = x_v = \frac{1}{\sqrt{2}}\{1 + g(\phi)\}^{1/2}. \tag{2.114}$$

As we have described in section (2.2.1), we find the first and second derivatives of $\hat{H}$ with respect to x and then by substitution we obtain from the Schrödinger equation (2.26), a PDE

$$\phi_{xx} = \frac{\{2(g-1)g'' - (g')^2\}}{2(1-g)g'}(\phi_x)^2 + \frac{2M(\phi)}{(g+1)g'}$$

which can be treated as a second order ODE, as it does not have the y derivative terms. The ODE

$$\frac{d^2\phi}{dx^2} = \alpha(\phi)\left(\frac{d\phi}{dx}\right)^2 + \beta(\phi), \tag{2.115}$$

where

$$\alpha(\phi) = \frac{2(g-1)g'' - g'^2}{2(1-g)g'} \quad \text{and} \quad \beta(\phi) = \frac{2M(\phi)}{(g+1)g'} \tag{2.116}$$

is solved using the same techniques as used in section (2.1.1); thus,

$$x = \frac{1}{2}\int \frac{e^{-\int \alpha(\phi)d\phi}}{\left\{\int 2\beta(\phi)e^{-2\int \alpha(\phi)d\phi}\,d\phi + q_1(y)\right\}^{1/2}}\,d\phi + q_2(y) \tag{2.117}$$

where $q_1(y)$ and $q_2(y)$ are arbitrary functions of y.

Now (2.117) defines a relationship

$$\mu(x, y, \phi) = 0 \tag{2.118}$$

between x, y and ϕ and by assuming that one set of values x_0, y_0, ϕ_0 can be found to satisfy (2.46) and that, near (x_0, y_0, ϕ_0), μ and its first partial derivatives are continuous and $\frac{\partial\mu}{\partial\phi} \neq 0$, the implicit function theorem states that in a region of the xy plane containing (x_0, y_0), there is precisely one differentiable function

$$\phi = \tau(x, y) \tag{2.119}$$

which reduces (2.117) to an identity and is such that $\phi_0 = \tau(x_0, y_0)$.

Under these assumptions, we now obtain

$$\hat{H} = \frac{1}{\sqrt{2}}\{g(\tau(x,y)) - 1\}^{1/2}, \tag{2.120}$$

and so,

$$\frac{\partial x}{\partial u} = \frac{\partial x}{\partial v} = , \frac{1}{\sqrt{2}}\{1 + g(\tau(x, u-v))\}^{1/2} \tag{2.121}$$

where $\phi = \tau(x,y) = \tau(x, u-v)$ (since $y = u - v$).

Since from equation (2.116) we will be able to determine x in terms of u and v, say, $x = \vartheta(u,v)$, we finally obtain the general solution of the PDE (2.106) as

$$\phi = \tau(\vartheta(u,v), u-v).$$

Next we shall consider some examples of the above discussed type of PDE and analyse the possibilities of finding the general solution.

2.1.3.1 Example 1:

First let us consider the equation

$$\phi_{uu} + \phi_{vv} = \frac{4(\phi^2+4)}{\phi^2(\phi^2+8)}\left\{\phi_u^2 + \phi_v^2\right\} \tag{2.122}$$

of the form (2.107), then we have

$$M(\phi) = 0 \quad \text{and} \quad A(\phi) = \frac{4(\phi^2+4)}{\phi^2(\phi^2+8)}.$$

Also we have

$$E = G = g(\phi) = \frac{1}{4}(\phi^2+4), \quad F = 1 \quad \text{and} \quad K = 0$$

which implies

$$\alpha(\phi) = 0 \quad \text{and} \quad \beta(\phi) = 0$$

from equation (2.116). Hence from equation (2.115) we obtain

$$\phi = q_1(y)x + q_2(y)$$

which yields

$$x = \frac{1}{q_1(y)}\{\phi - q_2(y)\}.$$

By substituting the expression for ϕ obtained above into equation (2.114) we get

$$x_u = x_v = \frac{1}{2\sqrt{2}}\left\{(q_1(y)x + q_2(y))^2 + 8\right\}^{1/2}$$

which yields

$$\int \frac{\sqrt{2}}{\{(q_1(y)x + q_2 y)^2 + 8\}^{1/2}}\,dx = u + \rho_0(v), \tag{2.123}$$

and

$$\int \frac{\sqrt{2}}{\{(q_1(y)x + q_2 y)^2 + 8\}^{1/2}}\,dx = v + \sigma_0(u), \tag{2.124}$$

where $\rho_0(v)$ and $\sigma_0(u)$ are arbitrary functions of v and u respectively.

If we consider equation (2.123) and make the substitution

$$t = q_1 x + q_2$$

we obtain

$$u + \rho_0(v) = \frac{4}{q_1}\ln\left|\frac{t + \sqrt{t^2+8}}{2\sqrt{2}}\right|, \tag{2.125}$$

which implies that $\rho_0(v)$ depends on the arbitrary functions q_1 and q_2 and is such that

$$\rho_0'(v) = \frac{\partial}{\partial v}\left\{\frac{4}{q_1}\ln\left|\frac{t + \sqrt{t^2+8}}{2\sqrt{2}}\right|\right\}.$$

Equation (2.125) implies that

$$t = 2\sqrt{2}\sinh\left\{\frac{q_1}{4}(u + \rho_0(v))\right\}$$

and therefore since $y = u - v$,

$$x = \frac{1}{q_1(u-v)}\left\{2\sqrt{2}\sinh\left[\frac{q_1(u-v)}{4}(u+\rho_0(v))\right] - q_2(u-v)\right\}.$$

Hence the general solution for the PDE in equation (2.120) is

$$\phi(u,v) = 2\sqrt{2}\sinh\left\{\frac{q_1(u-v)}{4}(u+\rho_0(v))\right\}.$$

2.1.3.2 Example 2:

The equation

$$\phi_{uu} + \phi_{vv} = \frac{\phi}{16}(\phi^2+8) + \frac{4(\phi^2+4)}{\phi^2(\phi^2+8)}\left\{\phi_u^2 + \phi_v^2\right\} \tag{2.126}$$

is of the form (2.108), where we identify

$$M(\phi) = \frac{\phi}{16}(\phi^2+8) \quad \text{and} \quad A(\phi) = \frac{4(\phi^2+4)}{\phi^2(\phi^2+8)}.$$

Hence we get

$$E = G = g(\phi) = \frac{1}{4}(\phi^2+4), \quad F = 1 \quad \text{and} \quad K = \frac{-1}{2\phi}$$

which implies

$$\alpha(\phi) = 0 \quad \text{and} \quad \beta(\phi) = 1$$

from equation (2.116). Hence from equation (2.115),

$$\phi = \frac{1}{2}\left\{(x - r_2(y))^2 - r_1(y)\right\},$$

where r_1 and r_2 are arbitrary functions of $y = u - v$.

Using the same procedure as in Example 1 we obtain the general solution for the PDE (2.122) as

$$\phi(u,v) = \frac{1}{2}\left\{\left[\frac{2\sqrt{2}}{r_1}\sinh\left(\frac{r_1}{4}(u+\rho_1(v))\right) - \frac{r_2}{r_1}[1-r_1]\right]^2 - r_1\right\}.$$

2.2 Solving a class of second order non-linear PDEs

In this section we consider a class of Monge-Ampère equation [1, 23, 41] and discuss a method for constructing solutions. We first identify the Monge-Ampère equation as the Gauss equation and then reformulate it as the sine-Gordon equation via Bäcklund transformations. The sine-Gordon equation may be solved by the methods discussed in the previous sections, and this will yield the corresponding solution to the Monge-Ampère equation. It will be shown that instead of solving the Monge-Ampère equation, we can solve a transformed equation-a first order non-linear PDE using the solution of the sine-Gordon equation.

2.2.1 A class of Monge-Ampère equation

Consider a surface described by

$$\mathbf{r} = (X, Y, Z(X,Y)).$$

The components of the first fundamental form (1.4) are

$$E = 1 + Z_X^2, \quad F = Z_X Z_Y, \quad G = 1 + Z_Y^2,$$

and the components of the second fundamental form (1.11) are

$$e = \frac{Z_{XX}}{\sqrt{1 + Z_X^2 + Z_Y^2}},$$

$$f = \frac{Z_{XY}}{\sqrt{1 + Z_X^2 + Z_Y^2}},$$

and

$$g = \frac{Z_{YY}}{\sqrt{1 + Z_X^2 + Z_Y^2}}.$$

In terms of this parametrization, the Gaussian and the mean curvatures are

$$K = \frac{Z_{XX} Z_{YY} - Z_{XY}^2}{(1 + Z_X^2 + Z_Y^2)^2} \tag{2.127}$$

and

$$\mathcal{H} = \frac{Z_{XX}(1+Z_Y^2) - 2Z_X Z_Y Z_{XY} + Z_{YY}(1+Z_X^2)}{2\,(1+Z_X^2+Z_Y^2)^{3/2}}\,. \tag{2.128}$$

We note that the Gauss equation reduces to equation (2.127) for graphical coordinates. Further, when substituting K and $\mathcal{H}$ in the linear Weingarten relation [36]

$$K + 2b\mathcal{H} + c = 0 \tag{2.129}$$

where b and c are arbitrary constants, we obtain

$$\frac{Z_{XX}\,Z_{YY} - Z_{XY}^2}{(1+Z_X^2+Z_Y^2)^2} + b\,\frac{Z_{XX}(1+Z_Y^2) - 2Z_X Z_Y Z_{XY} + Z_{YY}(1+Z_X^2)}{2\,(1+Z_X^2+Z_Y^2)^{3/2}} + c = 0\,. \tag{2.130}$$

For the choice of $b = 0$ and $c = 1$ equation (2.129) describes a pseudosphere and equation (2.130) reduces to

$$\frac{Z_{XX}\,Z_{YY} - Z_{XY}^2}{(1+Z_X^2+Z_Y^2)^2} = -1\,. \tag{2.131}$$

2.2.1.1 A special class of Monge-Ampère equation and Bäcklund transformations

In this section we consider a special case of the Monge-Ampère equation, equation (2.131) and discuss some solution techniques by using Bäcklund transformations.

If the surface is described in Tchebychef net coordinates (u, v) the components of the fundamental form are $E = 1$, $F = \cos\phi$ and $G = 1$, where ϕ is the angle between the coordinate curves on the surface (section 2.1.1.2). For a pseudosphere with $K = -1$, Gauss's equation yields the sine-Gordon equation (1.18).

In graphical coordinates, the Gauss equation produces the Monge-Ampére equation (2.131). These two PDEs are evidently connected by a Bäcklund transformation and, once such a transformation is known, the Monge-Ampère equation can be solved if the sine-Gordon equation can be solved, (and vice-versa).

The Bäcklund transformation is essentially a transformation on the solution surface from graphical to Tchebychef net coordinates. Interpreting the tensor formulae given in equation (1.20) as

$$g_{ij} = \hat{g}_{lm} \frac{\partial \hat{X}^l}{\partial X^i} \frac{\partial \hat{X}^m}{\partial X^j}$$

for coordinate transformations we obtain the system of nonlinear first order PDEs

$$(1+p^2)X_u^2 + 2pqX_uY_u + (1+q^2)Y_u^2 = 1\,, \tag{2.132}$$

$$(1+p^2)X_uX_v + pq(X_uY_v + X_vY_u) + (1+q^2)Y_uY_v = \cos\phi\,, \tag{2.133}$$

$$(1+p^2)X_v^2 + 2pqX_vY_v + (1+q^2)Y_v^2 = 1\,, \tag{2.134}$$

where $p = Z_X$ and $q = Z_Y$.

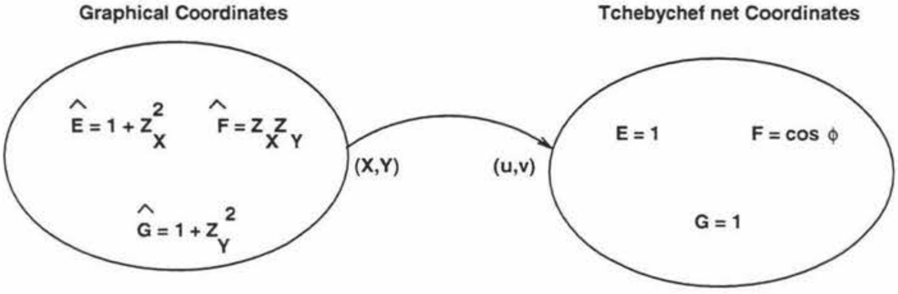

Figure 2.8: Coordinate transformation from graphical coordinate system to the Tchebychef net coordinate system.

Let

$$M = \begin{pmatrix} \frac{q}{\sqrt{p^2+q^2}} & \frac{-p}{\sqrt{p^2+q^2}} \\ \frac{p}{\sqrt{p^2+q^2}} & \frac{q}{\sqrt{p^2+q^2}} \end{pmatrix},$$

and

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} = M \begin{pmatrix} X_i \\ Y_i \end{pmatrix}$$

where $i = u, v$. Then the system (2.132)-(2.134) transforms to the system

$$x_u^2 + (1+p^2+q^2)y_u^2 = 1\,, \tag{2.135}$$

$$x_ux_v + (1+p^2+q^2)y_uy_v = \cos\phi\,, \tag{2.136}$$

$$x_v^2 + (1+p^2+q^2)y_v^2 = 1\,. \tag{2.137}$$

Here (x, y) corresponds to geodesic coordinates since $\bar{E} = 1$, $\bar{F} = 0$ and $\bar{G} = (1+p^2+q^2)$. Also we note that

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = M_1 \begin{pmatrix} x_i \\ y_i \end{pmatrix} \tag{2.138}$$

where

$$M_1 = \begin{pmatrix} \frac{q}{\sqrt{p^2+q^2}} & \frac{p}{\sqrt{p^2+q^2}} \\ \frac{-p}{\sqrt{p^2+q^2}} & \frac{q}{\sqrt{p^2+q^2}} \end{pmatrix} = M^{-1}.$$

We now have a Bäcklund transformation between the (X, Y) and the (x, y) coordinate systems. Next, we shall derive a transformation between the geodesic and the Tchebychef net coordinate systems (see Fig.2.9). We note that solutions to the system of PDEs (2.135)-(2.137) will provide us with such a transformation. Since we have already dealt with a similar system in section 2.2.1, it motivates us to choose the following set of solutions to this system.

Let

$$y = a(u - v) \tag{2.139}$$

where $-1 < a < 1$, $a \neq 0$, and

$$x_u = x_v = \left\{1 - a^2(1 + p^2 + q^2)\right\}^{1/2}. \tag{2.140}$$

Note that the Jacobian

$$\frac{\partial(x, y)}{\partial(u, v)} = x_u y_v - x_v y_u = -2a\left\{1 - a^2(1 + p^2 + q^2)\right\}^{1/2} \neq 0\,;$$

hence the transformation between the (x, y) geodesic and the (u, v) Tchebychef net coordinate systems is non-singular, provided that $a^2(1 + p^2 + q^2) \neq 1$.

Now by substituting (2.139) and (2.140) in (2.136) we get

$$\cos\phi = 1 - 2a^2(1 + p^2 + q^2)$$

or

$$(1 + p^2 + q^2) = \frac{1}{a^2}\sin^2\frac{\phi}{2}\,. \tag{2.141}$$

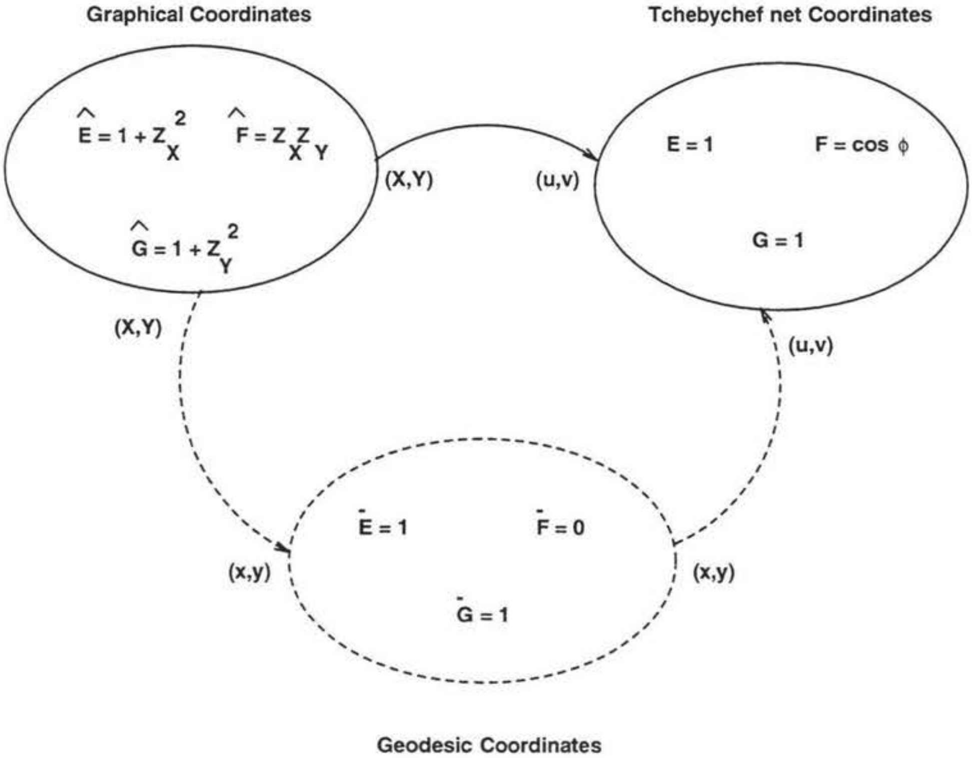

Figure 2.9: Coordinate transformations between graphical, geodesic and Tchebychef net coordinate systems.

Equation (2.141) is a first order non-linear PDE which can be solved for $Z(X, Y)$, a solution to the Monge-Ampère equation, provided that we have the solution $\phi(u, v)$ of the sine-Gordon equation, in terms of X and Y. That is, we have to relate both set of transformations (given in (2.138) and (2.139)-(2.140) respectively), the transformations between graphical-geodesic $((X,Y) - (x,y))$ and the geodesic-Tchebychet net $((x,y)-(u,v))$ and thus determine the corresponding transformation between graphical-Tchebychef net $((X,Y) - (u,v))$ coordinate systems.

We note from (2.140) that

$$x_u = x_v = \cos\frac{\phi}{2}\,. \tag{2.142}$$

If we denote

$$A = \frac{p}{\sqrt{p^2+q^2}} \quad \text{and} \quad B = \frac{q}{\sqrt{p^2+q^2}} \tag{2.143}$$

then we have

$$A^2 + B^2 = 1 \tag{2.144}$$

and from (2.138) we find

$$X_u = Bx_u + Ay_u, \tag{2.145}$$

$$Y_u = -Ax_u + By_u, \tag{2.146}$$

$$X_v = Bx_v + Ay_v, \tag{2.147}$$

and

$$Y_v = -Ax_v + By_u\,. \tag{2.148}$$

By solving (2.145) and (2.146) for A and B we get

$$A = \frac{X_u y_u - Y_u x_u}{x_u^2 + y_u^2} \quad \text{and} \quad B = \frac{X_u x_u + Y_u y_u}{x_u^2 + y_u^2}\,. \tag{2.149}$$

Similarly from (2.147) and (2.148) we obtain

$$A = \frac{X_v y_v - Y_v x_v}{x_v^2 + y_v^2} \quad \text{and} \quad B = \frac{X_v x_v + Y_v y_v}{x_v^2 + y_v^2}\,. \tag{2.150}$$

Substituting (2.149) in (2.144) yields

$$X_u^2 + Y_u^2 = x_u^2 + y_u^2 \tag{2.151}$$

and similarly by substituting (2.150) in (2.144) gives us

$$X_v^2 + Y_v^2 = x_v^2 + y_v^2\,. \tag{2.152}$$

Now by equating the two different expressions for A and B in (2.149) and (2.150) we have

$$\frac{X_u y_u - Y_u x_u}{x_u^2 + y_u^2} = \frac{X_v y_v - Y_v x_v}{x_v^2 + y_v^2} \tag{2.153}$$

and

$$\frac{X_u x_u + Y_u y_u}{x_u^2 + y_u^2} = \frac{X_v x_v + Y_v y_v}{x_v^2 + y_v^2}\,. \tag{2.154}$$

First by substituting

$$x_u = x_v = \cos\frac{\phi}{2}, \quad y_u = a \quad \text{and} \quad y_v = -a$$

in equations (2.151),(2.152), (2.153) and (2.154), and then by comparing them we determine that

$$X_u = X_v = -\cos\frac{\phi}{2}, \quad Y_u = a \quad \text{and} \quad Y_v = -a \tag{2.155}$$

a solution to the system (2.151)-(2.154), where A and B become

$$A = \pm\frac{2a\cos\frac{\phi}{2}}{a^2 + \cos^2\frac{\phi}{2}}\,, \quad \text{and} \quad B = \frac{a^2 - \cos^2\frac{\phi}{2}}{a^2 + \cos^2\frac{\phi}{2}}.$$

We note that particular choices of solutions for coordinate transformations similar to what we have here, will not help us obtain the general solution to the original problem, but only help us find a special class of solutions. From (2.155) it is evident that given $\phi(u, v)$, a solution to the sine-Gordon equation we can determine X and Y in terms of u and v. In other words, we determine the required Bäcklund transformations (say)

$$\left.\begin{array}{rcl} X &=& \alpha\,(u, v) \\ Y &=& \beta\,(u, v) \end{array}\right\} \tag{2.156}$$

and since the Jacobian $J = 2a\cos\left(\frac{\phi}{2}\right) \neq 0$ for solutions of the sine-Gordon equation such that $\cos\left(\frac{\phi}{2}\right) \neq 0$, the transformations in (2.155) are invertible and thus we obtain

$$\left.\begin{array}{rcl} u &=& \gamma\,(X, Y) \\ v &=& \delta\,(X, Y) \end{array}\right\}. \tag{2.157}$$

Hence using (2.157) we can write (2.141) as a first order non-linear PDE

$$\Theta\,(X, Y, Z, p, q) \,=\, p^2 + q^2 + 1 - \frac{1}{a^2}\sin^2\left(\frac{\Phi(X, Y)}{2}\right) \,=\, 0\,, \tag{2.158}$$

where

$$\Phi(X, Y) = \phi(\gamma(X, Y), \delta(X, Y)).$$

Equation (2.158) can, in principle, be solved using *characteristics* [33] which helps us to write (2.158) as a system of five ODEs

$$X'(t) = \Theta_p = 2p, \tag{2.159}$$

$$Y'(t) = \Theta_q = 2q, \tag{2.160}$$

$$Z'(t) = p\Theta_p + q\Theta_q = 2(p^2 + q^2), \tag{2.161}$$

$$p'(t) = -\Theta_X - p\Theta_Z = -\Theta_X, \tag{2.162}$$

and

$$q'(t) = -\Theta_Y - q\Theta_Z = -\Theta_Y, \tag{2.163}$$

where $\Theta_Z = 0$. Solving this system of ODEs will enable us to determine the solution $Z = Z(X, Y)$ for the Monge-Ampére equation (2.131).

Next we shall illustrate how a solution of the sine-Gordon equation can be used to determine a solution to the Monge-Ampère equation in the form of (2.131).

Example: A class of solutions

The function

$$\phi = 4\tan^{-1}\left(\mathrm{e}^{(u+v)}\right) \tag{2.164}$$

is a (soliton) solution of the sine-Gordon equation (2.54). Now

$$\cos\frac{\phi}{2} = \frac{1 - \tan^2\frac{\phi}{4}}{1 + \tan^2\frac{\phi}{4}},$$

and by using (2.164) we obtain

$$\cos\frac{\phi}{2} = -\tanh(u + v)\,; \tag{2.165}$$

hence, equation (2.155) indicates that

$$X_u = X_v = \tanh(u + v) \tag{2.166}$$

and thus

$$X = \ln|\cosh(u+v)| \,. \tag{2.167}$$

Further, we have

$$Y = a(u-v)\,. \tag{2.168}$$

Equation (2.168) implies that

$$(u+v) = \cosh^{-1}\left(\mathrm{e}^{X}\right)\,,$$

and (2.165) implies that

$$\sin\frac{\phi}{2} = \operatorname{sech}(u+v)\,;$$

consequently,

$$\sin^2\frac{\phi}{2} = \mathrm{e}^{-2X}.$$

Equation (2.158) thus reduces to

$$\Theta\left(X,Y,Z,p,q\right) \,=\, p^2+q^2+1-\frac{1}{a^2}\,\mathrm{e}^{-2X} \,=\, 0\,, \tag{2.169}$$

with characteristic equations

$$X'(t) = \Theta_p = 2p, \tag{2.170}$$

$$Y'(t) = \Theta_q = 2q, \tag{2.171}$$

$$Z'(t) = p\Theta_p + q\Theta_q = 2(p^2+q^2), \tag{2.172}$$

$$p'(t) = -\Theta_X = -\Theta_X = -\frac{2}{a^2}\,\mathrm{e}^{-2X} \tag{2.173}$$

and

$$q'(t) = -\Theta_Y = -\Theta_Y = 0. \tag{2.174}$$

Equation (2.174) yields

$$q(t) = \text{constant} = c_1(\text{say}) \tag{2.175}$$

and so, from equation (2.171) we obtain

$$Y(t) = 2c_1t + c_2 \tag{2.176}$$

where c_2 is an arbitrary constant. Now from equations (2.170) and (2.173) we have

$$X''(t) = 2p'(t) = -\frac{4}{a^2}\,\mathrm{e}^{-2X}$$

which is a second order ODE in the form

$$X''(t) + \frac{4}{a^2}\,\mathrm{e}^{-2X} = 0\,,$$

from which we obtain a solution

$$X(t) = \frac{1}{2}\tanh^{-1}\left\{\frac{1-\left[\frac{a\sqrt{c_3}}{2}\mathrm{sech}\left(\sqrt{c_3}(t+c_4)\right)\right]^4}{1+\left[\frac{a\sqrt{c_3}}{2}\mathrm{sech}\left(\sqrt{c_3}(t+c_4)\right)\right]^4}\right\} \tag{2.177}$$

where c_3 and c_4 are arbitrary constants. Equation (2.177) implies

$$X'(t) = \sqrt{c_3}\tanh\left(\sqrt{c_3}(t+c_4)\right),$$

and equation (2.170) implies that

$$p = \frac{\sqrt{c_3}}{2}\tanh\left(\sqrt{c_3}(t+c_4)\right).$$

Hence (2.172) reduces to

$$Z'(t) = 2(p^2+q^2) = \frac{c_3}{2}\tanh^2\left(\sqrt{c_3}(t+c_4)\right) + 2c_1^2,$$

and therefore

$$Z(t) = \frac{\sqrt{c_3}}{4}\left\{\ln\left[\frac{\tanh\left(\sqrt{c_3}(t+c_4)\right)+1}{\tanh\left(\sqrt{c_3}(t+c_4)\right)-1}\right] - 2\tanh\left(\sqrt{c_3}(t+c_4)\right)\right\}+2c_1^2t+c_5 \tag{2.178}$$

where c_5 is a constant.

Hence we have solved the system of five ODEs (2.170)- (2.174), which has generated five arbitrary constants c_1, c_2, c_3, c_4 and c_5, and thus we have determined a solution to the Monge-Ampère equation which corresponds to the soliton solution (2.164) of the sine-Gordon equation.

The elimination of the parameter t and some of the arbitrary constants in equations (2.176), (2.177) and (2.178) yields the general solution [33] of the Monge-Ampere equation which corresponds to the solution of the sine- Gordon equation given in (2.164).

A relationship amoung the Monge-Ampère equation, sine-Gordon equation and the Schrödinger equation

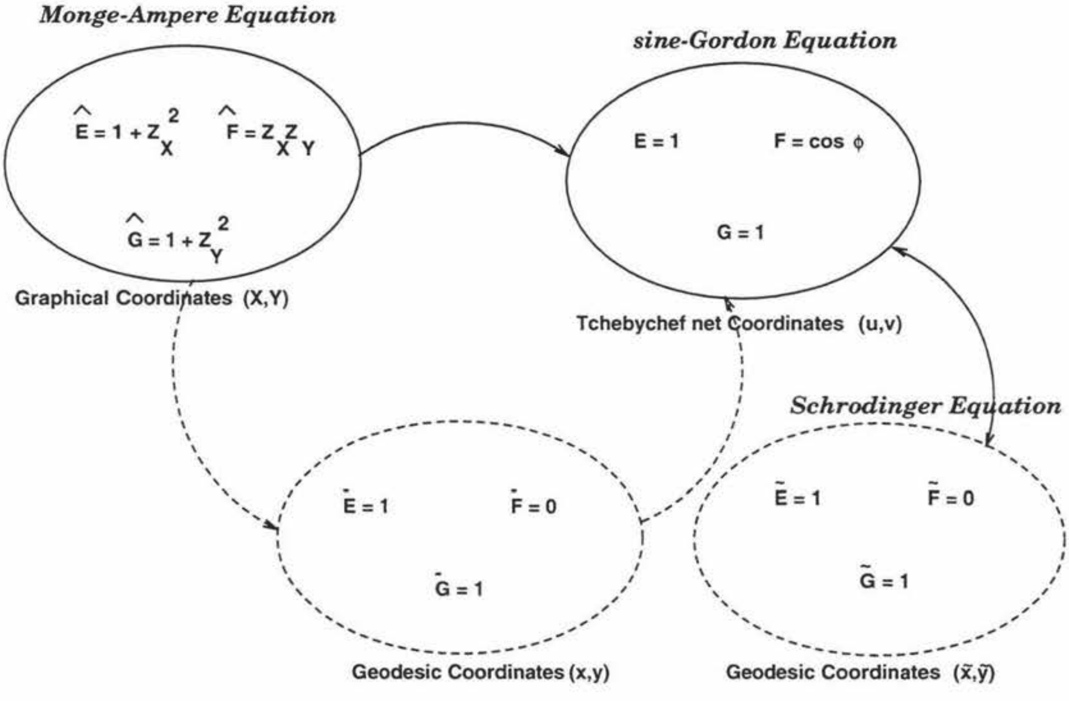

Figure 2.10: Relationships among the sine-Gordon equation, the Monge- Ampère equation & the Schrödinger equation.

The method described in section 2.2.1 requires a solution to the sine- Gordon equation or in other words we have to solve the sine-Gordon equation to find solutions to the Monge-Ampère equation. Solving the sine-Gordon equation using Bäcklund transformations has already been discussed in previous sections where the Bäcklund transformations were defined from the (u, v) Tchebychef net to the geodesic coordinate system, and then we had to solve the Schrödinger equation in order to get solutions to the sine-Gordon equation. Fig.2.10 depicts the situation.

Initial value problems: The Cauchy problem

Suppose we have to find a solution of the Monge-Ampère equation which passes through a curve Γ defined by

$$X = \bar{\alpha}(s)\,, \quad Y = \bar{\beta}(s)\,, \quad Z = \bar{\gamma}(s)\,, \tag{2.179}$$

such that $XY \neq 0$, then, we have to actually use these initial conditions to solve equation (2.158) and find the unique solution to the Monge- Ampère equation.

We solve the system of ODEs (2.159)-(2.163) subject to the initial conditions for X, Y and Z as

$$X_0 = \bar{\alpha}(s)\,, \quad Y_0 = \bar{\beta}(s)\,, \quad Z_0 = \bar{\gamma}(s)\,, \tag{2.180}$$

in the solutions

$$X = X(p_0, q_0, X_0, Y_0, Z_0, t_0, t)\,, \quad \text{etc.} \tag{2.181}$$

The corresponding initial values of p_0, q_0 are determined by the relations

$$\bar{\gamma}'(s) = p_0\,\bar{\alpha}'(s) + q_0\,\bar{\beta}'(s)$$

and

$$\Theta\left\{\bar{\alpha}(s), \bar{\beta}(s), \bar{\gamma}(s), p_0, q_0\right\} = 0\,.$$

If we substitute these values of X_0, Y_0, Z_0, p_0, q_0 and the appropriate value of t_0 in equation (2.162), we obtain X, Y, Z in terms of the two parameters t, s such that

$$X = X_1(t,s)\,, \quad Y = Y_1(t,s)\,, \quad Z = Z_1(t,s)\,. \tag{2.182}$$

Eliminating s, t from these three equations yields the required solution in the form,

$$\Psi(X, Y, Z) = 0\,.$$

2.2.1.2 A more general class of Monge-Ampère equation and Bäcklund transformations

Let us now consider the more general class of Monge-Ampère equation in the form given in equation (2.126). Since to every surface satisfying a linear Weingarten relation we can find a pseudosphere with Gaussian curvature $K = -1$ among its parallel surfaces[38], there will exist a geometrically tractable transformation from the familiar Monge-Ampère equation (2.127) to the generalised Monge-Ampère equation (2.126). Now suppose that we have a solution to the Monge-Ampère equation for the pseudosphere. Then we can use the Bäcklund transformations we have developed in section 2.2.1.1 to get a solution to the sine–Gordon equation (2.54) or vice versa. Once we have the solution to the sine–Gordon equation we can use known Bäcklund transformations[36] to get it into the lines of curvature. But using the concept of parallel surfaces we know that the new fundamental components will be in terms of the lines of curvature for the parallel surface. Also we know the Bäcklund transformations to get this in terms of the characteristics(which corresponds to the Tchebychef net) coordinates. Thus, we have a solution to the generalised sine-Gordon equation, and from this we can determine the corresponding solution of the generalised Monge-Ampère equation. Fig.2.11 given below illustrates this concept.

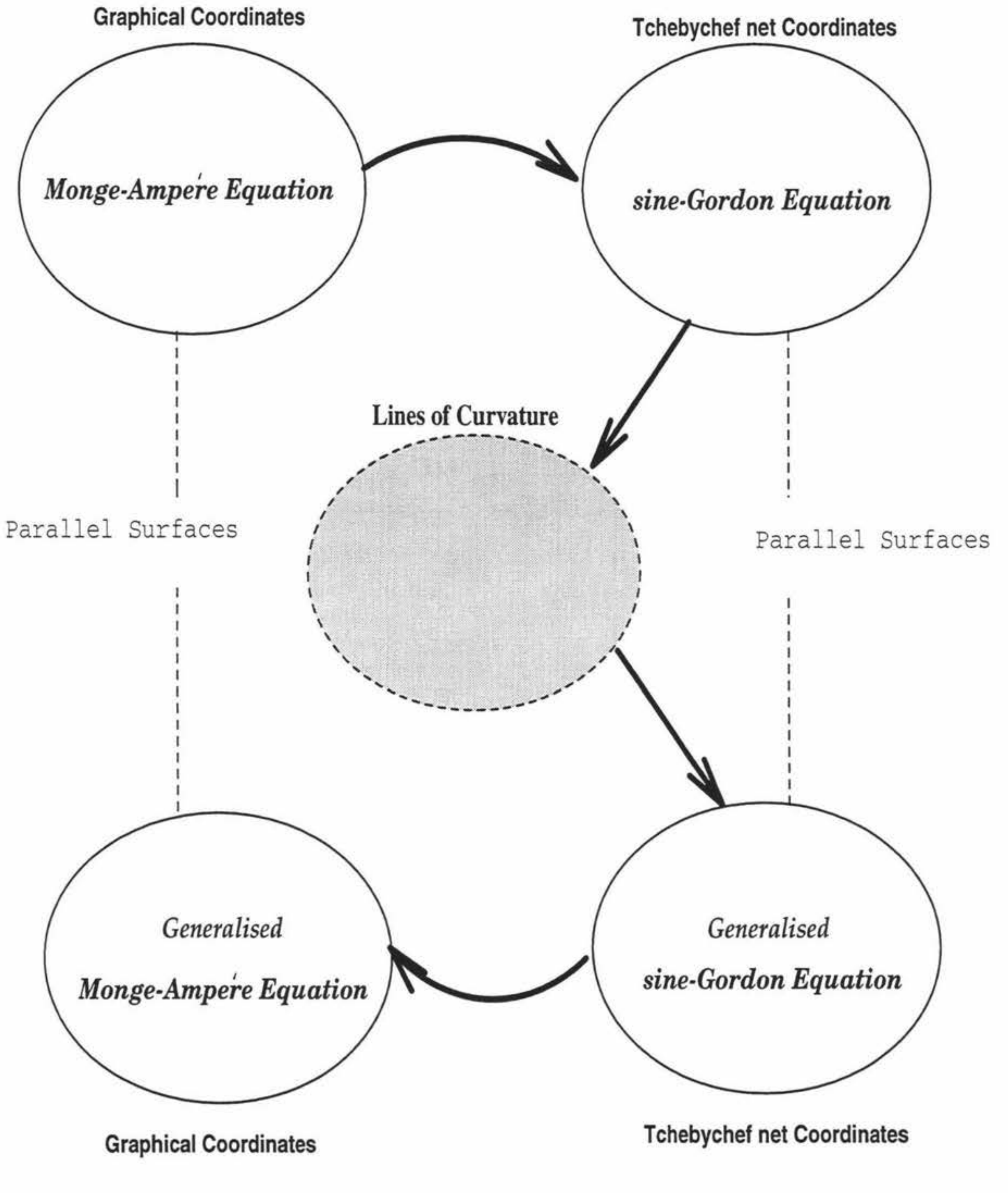

Figure 2.11: Relationships between the sine-Gordon, the generalised sine- Gordon, the Monge-Ampère and the generalised Monge-Ampère equations.

Chapter 3

PDEs, Coordinate Systems & the Gauss Equation

3.1 Introduction

In Chapter 2 we identified some coordinate systems which enable us to interpret the Gauss equation as some well known partial differential equations such as the Schrödinger equation, the sine-Gordon equation, the Liouville equation and the Monge-Ampére equation. We used this interpretation to investigate some of the connections between these equations and solutions techniques using coordinate transformations.

We have yet, however, to address the matter of how a given partial differential equation can be interpreted as the Gauss equation. That is, given a PDE, is there a systematic way to obtain a coordinate system such that the PDE corresponds to the Gauss equation? We address this question here for a more general class of PDEs.

In the next section we describe briefly some of the work carried out by Chern, Tenenblat, Kamran, Jorge and Sasaki[4, 15, 16, 31, 32], who developed a technique to determine coordinate systems in which certain types of evolution equations such as the Korteweg-de Vries(KdV) equation, the modified Korteweg-de Vries(MKdV) equation and the sine-Gordon equation describing pseudospherical surfaces(p.s.s) can be interpreted as the Gauss equation.

In section 3 and 4 we extend this technique to a more general class of PDEs and illustrate through examples how to determine the coordinate systems for these types of equations.

3.2 Preliminaries

In this section, we first summarise the essential concepts of how Sasaki[31, 32] used the language of exterior differential forms[8] to interpret the AKNS formulation of the inverse scattering method(the ISM)[27] for evolution equations. Then we look at Chern's and Tenenblat's[4] formulation of a definition(based upon this interpretation of Sasaki's), for a differential equation to describe a pseudospherical surface.

Given a non-linear PDE, Sasaki showed the basic steps in the AKNS method consisted of:

(a) setting up an appropriate 2x2 linear scattering(eigenvalue) problem in the "space" variable in which the solution of the non-linear equation plays the rôle of the potential;

(b) choosing the"time" dependence of the eigenfunctions in such a way that the eigenvalues remain invariant as the potential evolves according to the non-linear equation;

(c) solving the direct scattering problem at the initial "time" and determining the "time" dependence of the scattering data;

(d) doing the inverse scattering problem at later "times", namely reconstructing the potential from the scattering data.

He then summarises the essence of the first two steps as follows:

Find three one forms ω_1, ω_2 and ω_3 consisting of independent and dependent variables and their derivatives, such that the non-linear equation is given by

$$\Theta = d\Omega - \Omega \wedge \Omega = 0 \tag{3.1}$$

where

$$\Omega = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & -\omega_1 \end{pmatrix}$$

and Tr $\Omega = 0$.

Based on this interpretation, Chern and Tenenblat formulated the following definition for a differential equation to describe a pseudospherical surface as follows:

Definition 3.2.1 *Let $\mathcal{M}$ be a two-dimensional differentiable manifold parametrised by coordinates x, t. A differential equation for a real function $u(x,t)$* describes a pseudo-spherical surface(p.s.s) *if and only if there exist differentiable functions $f_{\alpha\beta}$, $1 \leq \alpha \leq 3$, $1 \leq \beta \leq 2$, depending on u and its derivatives, $f_{21} = \eta$(a parameter), such that the $1-$forms*

$$\omega_\alpha = f_{\alpha 1}\, dx \,+\, f_{\alpha 2}\, dt \tag{3.2}$$

satisfy the structure equations

$$d\omega_1 = \omega_3 \wedge \omega_2 \tag{3.3}$$

$$d\omega_2 = \omega_1 \wedge \omega_3 \tag{3.4}$$

$$d\omega_3 = \omega_1 \wedge \omega_2, \tag{3.5}$$

of a p.s.s. Here ω_1 and ω_2 are the forms which determine the metric on M, and ω_3 is the connection form.

In the above definition Chern and Tenenblat had chosen $f_{21} = \eta$(a constant parameter), so that the problem may be reduced to the inverse scattering problem considered by Ablowitz et al. in [27], where η corresponds to the spectral parameter.

Note that equations (3.3) and (3.4) are the structure equations which determine the connection form ω_3; equation (3.5) corresponds to the Gauss equation. In the pseudospherical case the Gaussian curvature of $\mathcal{M}$ is -1.

The above definition led Chern and Tenenblat to perform a complete classification of the evolution equations of the form

$$u_t = F\left(u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^k u}{\partial x^k}\right)$$

which describe p.s.s. Further they gave a geometrical method for constructing Bäcklund transformations and conservation laws for these equations. The classification and solution by inverse scattering of equations of more general type than the above equation, which describe pseudospherical surfaces were considered in subsequent papers [15, 2, 25], still under the assumption that f_{21} is a constant parameter.

In [15], Jorge and Tenenblat did a complete classification for equations of the type

$$u_{tt} = F\left(u, u_x, u_{xx}, u_t\right)$$

gave similar results to those of Chern and Tenenblat in [4]. Further, Jorge and Tenenblat applied their theory to show that the Liouville equation,

$$u_{tt} + u_{xx} = \delta\, e^{2u},$$

which is associated with minimal surfaces, also describes a pseudospherical surface. It is noted that in the above equation δ is a constant.

In [16], Kamran and Tenenblat generalised the results of Chern and Tenenblat by classifying the evolution equations of the form

$$u_t = F\left(u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^k u}{\partial x^k}\right)$$

which describe pseudospherical surfaces, without making any assumption that f_{21} is a constant parameter or any other *a priori* assumptions. Further, they have proved a local existence theorem to the effect that given any two differential equations describing pseudospherical surfaces (not necessarily evolutionary), such that one of the functions f_{ij} is an invertible function of u only, there exists locally a smooth mapping transforming any generic solution of one equation into a generic solution of the other.

Poznyak and Popov [29] used the work of Sasaki, Chern and Tenenblat to describe a number of problems related to a certain geometrical approach. This geometrical approach is to interpret differential equations and to base them as relations that are generated in some way by special coordinate nets on surfaces with prescribed Gaussian curvature.

The generalization by Poznyak and Popov motivates us to analyse more general classes of differential equations which may correspond to surfaces of variable Gaussian curvature $K(x,t)$. We thus focus first on extending definition 3.2.1, and then on devising a method whereby the functions $f_{\alpha\beta}$ can be determined for the general case.

A straightforward extension of definition 3.2.1 is as follows:

Definition 3.2.2 *Let $\mathcal{M}$ be a two-dimensional differentiable manifold parametrised by coordinates x, t. A differential equation for a real function $u(x,t)$* describes a surface of Gaussian curvature $K(x,t)$ *if and only if there exist differentiable functions $f_{\alpha\beta}$, $1 \leq \alpha \leq 3$, $1 \leq \beta \leq 2$, depending on u and its derivatives, $f_{21} = \eta$ (a parameter), such that the $1-$forms*

$$\omega_1 = f_{11}\, dx \,+\, f_{12}\, dt \tag{3.6}$$

$$\omega_2 = \eta\, dx \,+\, f_{22}\, dt \tag{3.7}$$

$$\omega_3 = f_{31}\, dx \,+\, f_{32}\, dt \tag{3.8}$$

satisfy the structure equations

$$d\omega_1 = \omega_3 \wedge \omega_2 \tag{3.9}$$

$$d\omega_2 = \omega_1 \wedge \omega_3 \tag{3.10}$$

$$d\omega_3 = \,-K(x,t)\, \omega_1 \wedge \omega_2, \tag{3.11}$$

of a surface of Gaussian curvature $K(x,t)$. Here ω_1 and ω_2 are the forms which determine the metric on $\mathcal{M}$, and ω_3 is the connection form.

It follows from this definition that for each non-trivial solution u of the differential equation, one gets a metric defined on $\mathcal{M}$, whose Gaussian curvature is $K(x,t)$. We note that $\mathcal{M}$ is a p.s.s. whenever $K = -1$.

3.3 Characterisation I

In this section, we consider differential equations for $u(x,t)$ of the form

$$u_t = F\left(K(x,t),\, u,\, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^k u}{\partial x^k}\right) \tag{3.12}$$

which correspond to surfaces of variable Gaussian curvature $K(x,t)$ in some coordinate system. It is assumed that the functions $f_{\alpha\beta}$ depend on $u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^k u}{\partial x^k}$, $1 \leq \alpha \leq 3$, $1 \leq \beta \leq 2$, except that $f_{21} = \eta$ is a constant.

We first obtain necessary conditions on the functions $f_{\alpha\beta}$ in Lemma 3.3.1. In particular, it is shown that f_{11} and f_{31} depend only on u. By imposing a generic condition on f_{11} and f_{31} we obtain Theorem 3.3.2. The non- generic cases are given by Theorems 3.3.3, 3.3.4 and 3.3.5. Our proof techniques follow those of Chern et. *al.*

From now on we will use the following notation:

$$z_0 = u, \quad z_1 = \frac{\partial u}{\partial x}, \ldots, z_k = \frac{\partial^k u}{\partial x^k}, \tag{3.13}$$

and

$$z_{0,t} = \frac{\partial u}{\partial t}. \tag{3.14}$$

Lemma 3.3.1 *Let*

$$z_{0,t} = F\left(K(x,t),\, u,\, z_0,\, z_1, \ldots, z_k\right) \tag{3.15}$$

be a differential equation which corresponds to a surface of variable Gaussian curvature $K(x,t)$, with associated 1−forms $\omega_\alpha = f_{\alpha 1}\, dx \,+\, f_{\alpha 2}\, dt$, $1 \leq \alpha \leq 3$, where $f_{21} = \eta$ is a parameter. If $f_{\alpha\beta}$ are functions of $z_0, z_1, z_2, \ldots, z_k$ then

$$f_{11,z_i} = f_{31,z_i} = 0, \qquad 1 \leq i \leq k \tag{3.16}$$

$$f_{12,z_k} = f_{22,z_k} = f_{32,z_k} = 0 \tag{3.17}$$

$$f_{22,z_{k-1}} = 0, \tag{3.18}$$

$$f_{11,z_0}^2 + f_{31,z_0}^2 \neq 0. \tag{3.19}$$

Moreover,

$$-F\,f_{11,z_0} + \sum_{i=0}^{k-1} z_{i+1}\,f_{12,z_i} + (\eta\,f_{32} - f_{22}f_{31}) = 0 \tag{3.20}$$

$$\sum_{i=0}^{k-2} z_{i+1}\,f_{22,z_i} + (f_{12}\,f_{31} - f_{11}f_{32}) = 0 \tag{3.21}$$

$$-F\,f_{31,z_0} + \sum_{i=0}^{k-1} z_{i+1}\,f_{32,z_i} + K\,(f_{11}\,f_{22} - \eta\,f_{12}) = 0 \tag{3.22}$$

Proof: Since

$$dz_i = \frac{\partial z_i}{\partial x}\,dx + \frac{\partial z_i}{\partial t}\,dt,$$

we have for $0 \leq i \leq k-1$

$$dz_i \wedge dt = z_{i+1}\,dx \wedge dt \tag{3.23}$$

and

$$\begin{aligned} dz_0 \wedge dx &= \left(\frac{\partial z_0}{\partial x}\,dx + \frac{\partial z_0}{\partial t}\,dt\right) \wedge dx \\ &= z_{0,t}\,dt \wedge dx \\ &= -z_{0,t}\,dx \wedge dt \\ &= -F\,dx \wedge dt\,, \end{aligned} \tag{3.24}$$

from equation (3.12).

Substituting (3.6), (3.7) and (3.8) in (3.9) we have

$$d(f_{11}dx + f_{12}dt) = (f_{31}dx + f_{32}dt) \wedge (\eta\,dx + f_{22}dt),$$

i.e.

$$\sum_{i=0}^{k} \frac{\partial f_{11}}{\partial z_i}\,dz_i \wedge dx + \sum_{i=0}^{k} \frac{\partial f_{12}}{\partial z_i}\,dz_i \wedge dt = (f_{22}f_{31} - \eta\,f_{32})dx \wedge dt.$$

The above equation can be written as

$$\sum_{i=0}^{k} f_{11,z_i}\, dz_i \wedge dx + \sum_{i=0}^{k} f_{12,z_i}\, dz_i \wedge dt + (\eta\, f_{32} - f_{22} f_{31}) dx \wedge dt = 0,$$

which implies that

$$\begin{aligned} f_{11,z_0}\, dz_0 \,\wedge\, dx \,+\, \sum_{i=1}^{k} f_{11,z_i}\, dz_i \,\wedge\, dx \,+ f_{12,z_k}\, dz_k \,\wedge\, dt & \\ + \sum_{i=0}^{k-1} f_{12,z_i}\, dz_i \,\wedge\, dt \,+\, (\eta\, f_{32} - f_{22} f_{31}) dx \,\wedge\, dt &= 0\,. \end{aligned} \tag{3.25}$$

Substituting (3.23) and (3.24) in (3.25) we obtain

$$\begin{aligned} -f_{11,z_0}\, F\, dx \,\wedge\, dt \,+\, \sum_{i=1}^{k} f_{11,z_i}\, dz_i \,, \wedge\, dx \,+\, f_{12,z_k}\, dz_k \,\wedge\, dt & \\ + \sum_{i=0}^{k-1} f_{12,z_i}\, z_{i+1}\, dx \,\wedge\, dt \,+\, (\eta\, f_{32} - f_{22} f_{31}) dx \,\wedge\, dt &= 0\,, \end{aligned} \tag{3.26}$$

and this reduces to

$$\begin{aligned} f_{12,z_k}\, dz_k \,\wedge\, dt \,+\, \sum_{i=1}^{k} f_{11,z_i}\, dz_i \,\wedge\, dx & \\ + \left(-F f_{11,z_0} \,+\, \sum_{i=0}^{k-1} z_{i+1} f_{12,z_i} \,+\, (\eta\, f_{32} - f_{22} f_{31}) \right) dx \,\wedge\, dt &= 0. \end{aligned} \tag{3.27}$$

Equating the coefficients on both sides of equation (3.26) yields

$$f_{11,z_i} = 0\,, \qquad 1 \leq i \leq k\,, \tag{3.28}$$

$$-F\, f_{11,z_0} \,+\, \sum_{i=0}^{k-1} z_{i+1}\, f_{12,z_i} + (\eta\, f_{32} - f_{22} f_{31}) \,=\, 0\,, \tag{3.29}$$

and

$$f_{12,z_k} = 0\,. \tag{3.30}$$

Now substituting (3.7), (3.8) and (3.6) in (3.10) we obtain

$$d(\eta\, dx + f_{22}dt) = (f_{11}dx + f_{12}dt) \wedge (f_{31}dx + f_{32}dt).$$

After some manipulations and substitutions this last equation reduces to

$$f_{22,z_k}\, dz_k \wedge dt + \left(\sum_{i=0}^{k-1} z_{i+1} f_{22,z_i} + (-f_{11}f_{32} + f_{12}f_{31})\right) dx \wedge dt = 0. \tag{3.31}$$

By equating the coefficients on both sides of (3.31), we obtain

$$f_{22,z_k} = 0\,, \tag{3.32}$$

and

$$\sum_{i=0}^{k-1} z_{i+1}\, f_{22,z_i} + (-f_{11}f_{32} + f_{12}f_{31}) = 0\,. \tag{3.33}$$

Finally, substituting (3.6), (3.7) and (3.8) in (3.11) we obtain

$$d(f_{31}dx + f_{32}dt) = -K(x,t)\,(f_{11}dx + f_{12}dt) \wedge (\eta\, dx + f_{22}dt),$$

which, after substitution and some manipulations reduces to

$$\begin{aligned} &\sum_{i=1}^{k} f_{31,z_i}\, dz_i \wedge dx + f_{32,z_k}\, dz_k \wedge dt \\ + &\left(-F f_{31,z_i} + \sum_{i=0}^{k-1} z_{i+1} f_{32,z_i} + K(x,t)\,(f_{11}f_{22} - \eta\, f_{12})\right) dx \wedge dt \;=\; 0. \end{aligned} \tag{3.34}$$

Equating the coefficients on both sides yields

$$f_{31,z_k} = 0\,, \qquad f_{31,z_i} = 0\,, \quad , \;\; 1 \leq i \leq k\,, \tag{3.35}$$

$$-F\, f_{31,z_0} + \sum_{i=0}^{k-1} z_{i+1}\, f_{32,z_i} + K(x,t)(f_{11}f_{22} - \eta\, f_{12}) = 0\,. \tag{3.36}$$

By taking the partial derivative w.r.t z_k of both sides of equation (3.33) we obtain

$$\frac{\partial}{\partial z_k}\left(\sum_{i=0}^{k-1} z_{i+1}\, f_{22,z_i} + (-f_{11}f_{32} + f_{12}f_{31})\right) = 0,$$

which yields

$$f_{22,z_{k-1}} = 0.$$

From (3.33) and (3.36) it is obvious that if $f_{11,z_0} = 0$ and $f_{31,z_0} = 0$ simultaneously, then (3.12) cannot be the necessary and sufficient condition for the w_α to satisfy the structure equations of a surface with Gaussian curvature $K(x,t)$.

Thus we have

$$f_{11,z_0}^2 + f_{31,z_0}^2 \neq 0.$$

Hence the lemma. □

Next we introduce the notation

$$L = \begin{vmatrix} f_{11} & f_{31} \\ f_{11,z_0} & f_{31,z_0} \end{vmatrix}, \qquad H = \begin{vmatrix} -K\, f_{11} & f_{31} \\ f_{31,z_0} & f_{11,z_0} \end{vmatrix}, \tag{3.37}$$

$$P = \begin{vmatrix} f_{11,z_0} & f_{31,z_0} \\ f_{11,z_0z_0} & f_{31,z_0z_0} \end{vmatrix}, \qquad M = f_{31,z_0}^2 + K f_{11,z_0}^2, \tag{3.38}$$

and

$$B = \sum_{i=0}^{k-2} z_{i+1}\, f_{22,z_i}\,. \tag{3.39}$$

For $L \neq 0$ we define A^j recursively as follows:

$$A^{k-1} = 0,$$

and for $0 \le j \le k-2$,

$$A^j = -\sum_{i=0}^{k-1} z_{i+1}\, A^{j+1}_{z_i} + \frac{1}{L}\,(z_1 L_{z_0} + \eta\, H)\, A^{j+1} + \frac{1}{L}\,(-z_1 P + \eta\, M)\, B_{z_{j+1}} + f_{22,z_{j+1}} H\,. \tag{3.40}$$

We consider the generic case when $HL \neq 0$ in Theorem 3.3.2. The non- generic cases will be discussed in Theorems 3.3.3 - 3.3.5.

Theorem 3.3.2 *Let $f_{\alpha\beta}$, $1 \leq \alpha \leq 3$, $1 \leq \beta \leq 2$, be differentiable functions of $z_0, z_1, \ldots, z_k$ such that (3.16)-(3.19) holds, except that $f_{21} = \eta$, a non-zero constant. Suppose $HL \neq 0$. Then*

$$z_{0,t} = F\left(K(x,t), z_0, z_1 \ldots, z_k\right)$$

corresponds to a surface of variable Gaussian curvature $K(x,t)$, with associated $1-$forms $\omega_\alpha = f_{\alpha 1}\, dx + f_{\alpha 2}\, dt$ if and only if

$$F = \frac{1}{L}\sum_{i=0}^{k-1} z_{i+1} B_{z_i} + \frac{1}{HL}\left(-z_1\frac{L}{\eta} + f_{31}^2 + K f_{11}^2\right)\sum_{i=0}^{k-2} z_{i+1}A^i + \frac{B}{HL}(z_1 M + \eta L) + z_1\frac{f_{22}}{\eta} \tag{3.41}$$

and

$$f_{12} = \frac{f_{11}f_{22}}{\eta} + \frac{1}{H}\left(\frac{-f_{11}}{\eta}\sum_{i=0}^{k-2} z_{i+1}A^i + f_{31,z_0}B\right), \tag{3.42}$$

$$f_{32} = \frac{f_{31}f_{22}}{\eta} - \frac{1}{H}\left(\frac{f_{31}}{\eta}\sum_{i=0}^{k-2} z_{i+1}A^i + K f_{11,z_0}B\right), \tag{3.43}$$

where f_{11}, f_{31}, f_{22} satisfy the following differential equation:

For $0 \leq j \leq k-1$,

$$\frac{L}{\eta}f_{22,z_j} - \frac{L}{\eta}\sum_{i=0}^{k-2}\left(z_{i+1}\frac{A^i}{H}\right)_{z_j} + A^j + \frac{M}{H}B_{z_j} + \frac{B}{H^2}(LP + M^2)\delta_{j0} = 0, \tag{3.44}$$

where $\delta_{j0} = 0$ if $j \neq 0$ and $\delta_{00} = 1$.

Proof: ($\Longrightarrow$) Suppose $z_{0,t} = F\left(K(x,t),\, u,\, z_0,\, z_1, \ldots, z_k\right)$ describes a surface with Gaussian curvature $K(x,t)$.

Then from Lemma 3.3.1 it follows that (3.20)- (3.22) is satisfied. Noting that by hypothesis $f_{11,z_0} \neq 0$ and $f_{31,z_0} \neq 0$ simultaneously, (3.20)$\times$ $f_{31,z_0}$$-(3.20)\times$ f_{11,z_0} simplifies to

$$\sum_{i=0}^{k-1} z_{i+1} \left(f_{12,z_i} f_{31,z_0} - f_{32,z_i} f_{11,z_0}\right) + \eta \left(f_{32} f_{31,z_0} + K f_{12} f_{11,z_0}\right) + f_{22} H = 0, \tag{3.45}$$

where H is defined in (3.37). Using (3.39), equation (3.21) can be written as

$$B - f_{11} f_{32} + f_{12} f_{31} = 0. \tag{3.46}$$

Considering (3.20)$\times$ $f_{31}$$-(3.22)\times$ f_{11} we obtain

$$FL + \sum_{i=0}^{k-1} z_{i+1} \left(f_{12,z_i} f_{31} - f_{32,z_i} f_{11}\right) + \eta \left(f_{32} f_{31} + K f_{12} f_{11}\right) - f_{22} \left(f_{31}^2 + K f_{11}^2\right) = 0, \tag{3.47}$$

where L is defined in (3.37).

Taking the partial derivative w.r.t z_k of both sides of (3.46) we get

$$f_{12,z_{k-1}} f_{31,z_0} - f_{32,z_{k-1}} f_{11,z_0} = 0, \tag{3.48}$$

and taking the partial derivative w.r.t z_{k-1} of both sides of (3.46) we obtain

$$f_{12,z_{k-1}} f_{31} - f_{32,z_{k-1}} f_{11} = -B_{z_{k-1}}, \tag{3.49}$$

where we have used the results

$$f_{11,z_i} = f_{31,z_i} = 0\,, \qquad 1 \leq i \leq k \tag{3.50}$$

of Lemma 3.2.1.

Solving (3.48) and (3.49) for $f_{12,z_{k-1}}$ and $f_{32,z_{k-1}}$ we obtain

$$f_{12,z_{k-1}} = \frac{f_{11,z_0}}{L} B_{z_{k-1}}\,, \qquad f_{32,z_{k-1}} = \frac{f_{31,z_0}}{L} B_{z_{k-1}}. \tag{3.51}$$

Now, taking the z_{k-1} derivative of (3.46) we obtain

$$f_{31,z_0}f_{12,z_{k-2}} - f_{11,z_0}f_{32,z_{k-2}} = -\eta\left(f_{31,z_0}f_{32,z_{k-1}} + Kf_{11,z_0}f_{12,z_{k-1}}\right) - f_{22,z_{k-1}}H\,.$$

Substituting the expressions given in equation (3.51) in the above yields

$$f_{31,z_0}f_{12,z_{k-2}} - f_{11,z_0}f_{32,z_{k-2}} = -\frac{\eta}{L}MB_{z_{k-1}} - f_{22,z_{k-1}}H\,, \tag{3.52}$$

where M is defined in (3.38).

Taking the partial derivative w.r.t z_{k-2} of both sides of (3.46) one gets

$$f_{31}f_{12,z_{k-2}} - f_{11}f_{32,z_{k-2}} = -B_{z_{k-2}}\,. \tag{3.53}$$

Then solving (3.49) and (3.52) for $f_{12,z_{k-2}}$ and $f_{32,z_{k-2}}$ produces

$$f_{11,z_{k-2}} = -\frac{1}{L}\left(f_{11}A^{k-2} - f_{11,z_0}B_{z_{k-2}}\right)\,, \tag{3.54}$$

and

$$f_{32,z_{k-2}} = -\frac{1}{L}\left(f_{31}A^{k-2} - f_{31,z_0}B_{z_{k-2}}\right)\,, \tag{3.55}$$

where A^j is defined in (3.40).

Recursively taking the z_{j+1} derivative of (3.45) and the z_j derivative of (3.46) for $1 \le j \le k-1$,we obtain

$$f_{12,z_j} = -\frac{1}{L}\left(f_{11}A^j - f_{11,z_0}B_{z_j}\right)\,, \tag{3.56}$$

and

$$f_{32,z_j} = -\frac{1}{L}\left(f_{31}A^j - f_{31,z_0}B_{z_j}\right)\,. \tag{3.57}$$

Now by taking the z_1 derivative of (3.45) we get

$$f_{12,z_0}f_{31,z_0} - f_{32,z_0}f_{11,z_0} + A^0 = 0. \tag{3.58}$$

Writing (3.45) in the form of

$$\sum_{i=0}^{k-2} z_{i+1}\left(f_{12,z_i}f_{31,z_0} - f_{32,z_i}f_{11,z_0}\right) + z_k\left(f_{12,z_{k-1}}f_{31,z_0} - f_{32,z_{k-1}}f_{11,z_0}\right)$$
$$+\eta\left(f_{32}f_{31,z_0} + Kf_{12}f_{11,z_0}\right) + f_{22}\,H = 0,$$

and by substituting (3.48) yields

$$\sum_{i=0}^{k-2} z_{i+1}\left(f_{12,z_i}f_{31,z_0} - f_{32,z_i}f_{11,z_0}\right) + \eta\left(f_{32}f_{31,z_0} + Kf_{12}f_{11,z_0}\right) + f_{22}H = 0.$$

The above can be expressed as

$$f_{32}f_{31,z_0} + Kf_{12}f_{11,z_0} - \frac{1}{\eta}\sum_{i=0}^{k-2} z_{i+1}A^i + \frac{f_{22}}{\eta}H = 0. \tag{3.59}$$

Solving (3.46) and (3.59) for f_{11} and f_{32} yields

$$f_{12} = \frac{f_{11}f_{22}}{\eta} + \frac{1}{H}\left(\frac{-f_{11}}{\eta}\sum_{i=0}^{k-2} z_{i+1}A^i + f_{31,z_0}B\right), \tag{3.60}$$

and

$$f_{32} = \frac{f_{31}f_{22}}{\eta} - \frac{1}{H}\left(\frac{f_{31}}{\eta}\sum_{i=0}^{k-2} z_{i+1}A^i + Kf_{11,z_0}B\right). \tag{3.61}$$

Using equations (3.47), (3.56), (3.57), (3.60) and (3.61), after some manipulations we obtain

$$F = \frac{1}{L}\sum_{i=0}^{k-1} z_{i+1}\,B_{z_i} + \frac{1}{HL}\left(-z_1\frac{L}{\eta} + f_{31}^2 + Kf_{11}^2\right)\sum_{i=0}^{k-2} z_{i+1}A^i + \frac{B}{HL}(z_1M + \eta\,L) + z_1\frac{f_{22}}{\eta}. \tag{3.62}$$

In order to get the differential equation given in (3.44), we first differentiate both sides of equation (3.60) w.r.t z_j and obtain

$$f_{12,z_j} = \frac{1}{\eta}f_{11}f_{22,z_j} + \frac{1}{H}\frac{\partial}{\partial z_j}\left(-\frac{f_{11}}{\eta}\sum_{i=0}^{k-2} z_{i+1}A^i + f_{31,z_0}B\right) - \frac{1}{H^2}\frac{\partial H}{\partial z_j},$$

which simplifies to

$$f_{12,z_j} = \frac{f_{11}}{\eta} f_{22,z_j} - \frac{f_{11}}{\eta H}\left(\sum_{i=0}^{k-2} z_{i+1}A^i\right)_{z_j} + \frac{f_{31,z_0}}{H} B_{z_j}\,. \tag{3.63}$$

Similarly, by taking the z_j derivative of (3.61), after some simplifications we obtain

$$f_{32,z_j} = \frac{f_{31}}{\eta} f_{22,z_j} - \frac{f_{31}}{\eta H}\left(\sum_{i=0}^{k-2} z_{i+1}A^i\right)_{z_j} + \frac{K f_{31,z_0}}{H} B_{z_j}\,. \tag{3.64}$$

Considering (3.64)$\times f_{11,z_0}-$(3.63)$\times f_{31,z_0}$ and using equations (3.56), (3.57), (3.63) and (3.64) we obtain

$$\begin{aligned}
&\tfrac{f_{31,z_0}}{L}\left(f_{11}A^j - f_{11,z_0}B_{z_j}\right) - \tfrac{f_{11,z_0}}{L}\left(f_{31}A^j - f_{31,z_0}B_{z_j}\right)\\
=\;& -f_{31,z_0}\left(\tfrac{f_{11}}{\eta} f_{22,z_j} - \tfrac{f_{11}}{\eta H}\left(\textstyle\sum_{i=0}^{k-2} z_{i+1}A^i\right)_{z_j} + \tfrac{f_{31,z_0}}{H}B_{z_j}\right)\\
+\;& f_{11,z_0}\left(\tfrac{f_{31}}{\eta} f_{22,z_j} - \tfrac{f_{31}}{\eta H}\left(\textstyle\sum_{i=0}^{k-2} z_{i+1}A^i\right)_{z_j} + \tfrac{K f_{11,z_0}}{H}B_{z_j}\right),
\end{aligned} \tag{3.65}$$

i.e.

$$\begin{aligned}
\left(f_{11}f_{31,z_0} - f_{31}f_{11,z_0}\right)\frac{A^j}{L} &= \frac{\left(f_{31}f_{11,z_0} - f_{11}f_{31,z_0}\right)}{\eta} f_{22,z_j}\\
&+ \frac{\left(f_{11}f_{31,z_0} - f_{31}f_{11,z_0}\right)}{\eta H}\left(\sum_{i=0}^{k-2} z_{i+1}A^i\right)_{z_j}\\
&- \frac{1}{H}\left(f_{31,z_0}^2 + K f_{11,z_0}^2\right)B_{z_j},
\end{aligned} \tag{3.66}$$

which reduces to (3.44). The necessary part of the theorem is thus proved.

($\Longleftarrow$) Conversely, assume that F, f_{12} and f_{32} are given by the expressions in equations (3.41)–(3.44).

Suppose f_{11}, f_{31}, f_{22} satisfy the expressions given in (3.16)–(3.18), then the 1−forms ω_α satisfy the structure equations of the surface of Gaussian curvature $K(x,t)$, provided that F satisfy the equations (3.20) and (3.22). i.e. F satisfies

$$-F\,f_{11,z_0} + \sum_{i=0}^{k-1} z_{i+1}\,f_{12,z_i} + (\eta\,f_{32} - f_{22}f_{31}) = 0 \tag{3.67}$$

and

$$-F\,f_{31,z_0} + \sum_{i=0}^{k-1} z_{i+1}\,f_{32,z_i} + K(x,t)\,(f_{11}\,f_{22} - \eta\,f_{12}) = 0. \tag{3.68}$$

Since we have

$$f_{11,z_i} = f_{31,z_i} = 0\,, \qquad 1 \le i \le k$$

and $f_{22,z_{k-1}} = 0$, we can write (3.62) as

$$\begin{aligned} &\sum_{i=0}^{k} f_{11,z_i} dz_i \wedge dx + f_{12,z_k} dz_k \wedge dt \\ +&\left(-F f_{11,z_0} + \sum_{i=0}^{k-1} z_{i+1} f_{12,z_i} + (\eta\,f_{32} - f_{22}f_{31})\right) dx \wedge dt = 0. \end{aligned} \tag{3.69}$$

But from the structure equation we have relation (3.25), which could be written as

$$\begin{aligned} f_{11,z_0}\,dz_0 \wedge dx &+ \sum_{i=1}^{k} f_{11,z_i}\,dz_i \wedge dx + f_{12,z_k}\,dz_k \wedge dt \\ &+ \left(\sum_{i=0}^{k-1} f_{12,z_i} + (\eta\,f_{32} - f_{22}f_{31})\right) dx \wedge dt = 0\,. \end{aligned} \tag{3.70}$$

Subtracting equation (3.69) from equation (3.70) yields

$$f_{11,z_0}\,dz_0 \wedge dx + f_{11,z_0} F\,dx \wedge dx = 0,$$

i.e.

$$(dz_0 - F dt) \wedge dx = 0,$$

which implies

$$z_{0,t} = F.$$

Thus the sufficient part is proved and hence the theorem.□

In the non-generic case either

Case(i): $L = 0$ or

Case(ii): $L \neq 0$ and $H = 0$.

For Case(i), we have that

$$f_{11} f_{31,z_0} - f_{31} f_{11,z_0} = 0$$

which, for a non-trivial solution, can be classified into the following sub-cases:
Sub-case(a): $f_{11} = 0$ or $f_{31} = 0$

and

Sub-case(b): $\frac{f'_{11}}{f_{11}} = \frac{f'_{31}}{f_{31}}$ which implies

$$f_{11} = \lambda f_{31}$$

where λ is independent of z_0.

Theorems 3.3.3 and 3.3.4 deal with sub-cases (a) and (b) respectively.

From Case(ii), we have that $H = 0$ and thus

$$K f_{11} f_{11,z_0} + f_{31} f_{31,z_0} = 0$$

which implies

$$f_{31}^2 + K f_{11}^2 = c$$

where $c \neq 0$, and c does not depend on z_0.

This case is discussed in Theorem 3.3.5.

Theorem 3.3.3 *Let $f_{\alpha\beta}$, $1 \leq \alpha \leq 3$, $1 \leq \beta \leq 2$, be differentiable functions of $z_0, z_1, \ldots, z_k$ such that (3.16)-(3.19) holds, except that $f_{21} = \eta$, a non-zero constant. Suppose $f_{11} = 0$ and $f_{31} \neq 0$ or $f_{11} = 0$ and $f_{31} \neq 0$. Then*

$$z_{0,t} = F\left(K(x,t), z_0, z_1, \ldots, z_k\right)$$

corresponds to a surface of variable Gaussian curvature $K(x,t)$, with associated $1-$forms $\omega_\alpha = f_{\alpha 1}\,dx + f_{\alpha 2}\,dt$ if and only if

$$f_{22,z_{k-2}} = 0$$

and

$$F = \frac{1}{\eta\, f_{31,z_0}} \sum_{i=0}^{k-1} z_{i+1} \left(\left(\sum_{j=0}^{k-2} z_{j+1} \left(\frac{B}{f_{31}} \right)_{z_j} \right)_{z_i} + (f_{22} f_{31})_{z_i} \right) + \frac{\eta\, K B}{f_{31} f_{31,z_0}}, \tag{3.71}$$

$$f_{12} = -\frac{B}{f_{31}}, \tag{3.72}$$

$$f_{32} = \frac{1}{\eta} \left(\sum_{i=0}^{k-2} z_{i+1} \left(\frac{B}{f_{31}} \right)_{z_i} + f_{22} f_{31} \right) \tag{3.73}$$

if $f_{31} \neq 0$ or

$$F = \frac{1}{\eta\, f_{11,z_0}} \sum_{i=0}^{k-1} z_{i+1} \left(\left(-\sum_{j=0}^{k-2} z_{j+1} \left(\frac{B}{f_{31}} \right)_{z_j} \right)_{z_i} + (f_{22} f_{11})_{z_i} \right) - \frac{\eta\, K B}{f_{11} f_{11,z_0}}, \tag{3.74}$$

$$f_{12} = \frac{1}{\eta} \left(-\sum_{i=0}^{k-2} z_{i+1} \left(\frac{B}{f_{11}} \right)_{z_i} + f_{22} f_{11} \right) ,, \tag{3.75}$$

$$f_{32} = \frac{B}{f_{11}} \tag{3.76}$$

if $f_{11} \neq 0$.

Proof: ($\Longrightarrow$) Suppose $z_{0,t} = F\left(K(x,t),\, u,\, z_0,\, z_1, \ldots, z_k\right)$ describes a surface with Gaussian curvature $K(x,t)$. Then from Lemma 3.3.1 it follows that (3.20)–(3.22) is satisfied. Substituting $f_{11} = 0$ (by hypothesis) in (3.20)–(3.22) we obtain

$$\sum_{i=0}^{k-1} z_{i+1}\, f_{12,z_i} + \eta\, f_{32} - f_{22} f_{31} \;=\; 0, \tag{3.77}$$

$$B \,+\, f_{12}\, f_{31} \;=\; 0, \tag{3.78}$$

and

$$-F\, f_{31,z_0} + \sum_{i=0}^{k-1} z_{i+1}\, f_{32,z_i} - K\,\eta\, f_{12} = 0. \tag{3.79}$$

Taking the partial derivative of (3.77) on both sides w.r.t z_k yields

$$f_{12,z_{k-1}} = 0,$$

and by taking the partial derivative of (3.78) on both sides w.r.t z_{k-1} we obtain

$$B_{z_{k-1}} = 0.$$

Hence we have

$$f_{12,z_{k-1}} = B_{z_{k-1}} = 0. \tag{3.80}$$

From equation (3.78) we get

$$f_{12} = -\frac{B}{f_{31}}, \tag{3.81}$$

and from equation (3.77), we obtain

$$f_{32} = \frac{1}{\eta}\left(\sum_{i=0}^{k-2} z_{i+1}\left(\frac{B}{f_{31}}\right)_{z_i} + f_{22} f_{31}\right). \tag{3.82}$$

Substituting (3.81) and (3.82) in (3.79) yields

$$F f_{31,z_0} = \sum_{i=0}^{k-1} z_{i+1}\left(\frac{1}{\eta}\left(\sum_{j=0}^{k-2} z_{j+1}\left(\frac{B}{f_{31}}\right)_{z_j} + f_{22} f_{31}\right)\right)_{z_i} + \frac{\eta K B}{f_{31}},$$

which reduces to

$$F f_{31,z_0} = \sum_{i=0}^{k-1} z_{i+1}\left(\frac{1}{\eta}\left(\sum_{j=0}^{k-2} z_{j+1}\left(\frac{B}{f_{31}}\right)_{z_j}\right)_{z_i} + \frac{1}{\eta}\left(f_{22} f_{31}\right)_{z_i}\right) + \frac{\eta K B}{f_{31}},$$

from which we obtain the required expression for F given in equation (3.71).

The proof of the expressions given in equations (3.74)– (3.76) can be obtained in a similar fashion by substituting $f_{31} = 0$ (and $f_{11} \neq 0$) in equations (3.20)–(3.22).

($\Longleftarrow$) The converse is a straightforward computation similar to that in the converse part of Theorem 3.3.2. $\square$

Theorem 3.3.4 *Let $f_{\alpha\beta}$, $1 \le \alpha \le 3$, $1 \le \beta \le 2$, be differentiable functions of $z_0, z_1, \ldots, z_k$ such that (3.16)–(3.19) holds, and $f_{21} = \eta$, a non-zero parameter. Suppose $f_{31} = \lambda f_{11} \neq 0$, where λ does not depend on z_0. Then*

$$z_{0,t} = F(K(x,t), z_0, z_1, \ldots, z_k)$$

corresponds to a surface of variable Gaussian curvature $K(x,t)$, with associated 1–forms $\omega_\alpha = f_{\alpha 1}\, dx + f_{\alpha 2}\, dt$ if and only if

(a) f_{22} does not depend on z_i, $0 \le i \le k$, $f_{32} = \lambda f_{12}$, and

$$F = \frac{1}{f_{11,z_0}} \left(\sum_{i=0}^{k} z_{i+1} f_{12,z_i} + \lambda(\eta f_{12} - f_{11} f_{22}) \right) \tag{3.83}$$

whenever $\lambda^2 + K = 0$; or

(b) $f_{22,z_{k-2}} = 0$, and

$$F = \frac{1}{(\lambda^2 + K) f_{11,z_0}} \left(\sum_{i=0}^{k-1} \frac{z_{i+1}}{\eta} \left(\left(- \sum_{j=0}^{k-2} z_{j+1} \left(\frac{B}{f_{11}} \right)_{z_j} \right)_{z_i} + (\lambda^2 + K) \left(f_{22} f_{11} \right)_{z_i} \right) + \frac{\eta K B}{f_{11}} \right), \tag{3.84}$$

$$f_{12} = \frac{1}{(\lambda^2 + K)} \left(\frac{1}{\eta} \sum_{i=0}^{k-2} z_{i+1} \left(\frac{B}{f_{11}} \right)_{z_i} - \frac{\lambda B}{f_{11}} \right) + \frac{f_{11} f_{22}}{\eta}, \tag{3.85}$$

$$f_{32} = \frac{1}{(\lambda^2 + K)} \left(\frac{\lambda}{\eta} \sum_{i=0}^{k-2} z_{i+1} \left(\frac{B}{f_{11}} \right)_{z_i} + \frac{K B}{f_{11}} \right) + \frac{\lambda}{\eta} f_{11} f_{22} \tag{3.86}$$

whenever $\lambda^2 + K \neq 0$.

Proof: ($\Longrightarrow$) Suppose $z_{0,t} = F\,(K(x,t),\, u,\, z_0,\, z_1, \ldots, z_k)$ describes a surface with Gaussian curvature $K(x,t)$. Then from Lemma 3.3.1 it follows that (3.20)–(3.22) is satisfied.

Substituting $f_{31} = \lambda f_{11}$ in (3.20)–(3.22), we obtain

$$-F\, f_{11,z_0} + \sum_{i=0}^{k-1} z_{i+1}\, f_{12,z_i} + (\eta\, f_{32} - \lambda\, f_{22} f_{11}) = 0\,, \tag{3.87}$$

$$B - f_{11}(f_{32} - \lambda\, f_{12}) = 0\,, \tag{3.88}$$

and

$$-\lambda F\, f_{11,z_0} + \sum_{i=0}^{k-1} z_{i+1}\, f_{32,z_i} + K(f_{11}\, f_{22} - \eta\, f_{12}) = 0 \tag{3.89}$$

respectively.

(a) Suppose $\lambda^2 + K = 0$.

Considering $-1\times$(3.88)$+\lambda\times$(3.90), we obtain

$$\sum_{i=0}^{k-1} z_{i+1}(\lambda\, f_{12,z_i} - f_{32,z_i}) + \lambda(\eta\, f_{32} - \lambda\, f_{22} f_{11}) - K(f_{11} f_{22} - \eta\, f_{12}) = 0,$$

which can be written as

$$\sum_{i=0}^{k-1} z_{i+1}(\lambda\, f_{12,z_i} - f_{32,z_i}) + \eta(\lambda\, f_{32} + K f_{12}) - (\lambda^2 + K) f_{11} f_{22} = 0. \tag{3.90}$$

Substituting $\lambda^2 + K = 0$ in (3.90) we obtain

$$\sum_{i=0}^{k-1} z_{i+1}(\lambda\, f_{12,z_i} - f_{32,z_i}) - \eta\,\lambda(\lambda\, f_{12} - f_{32}) = 0. \tag{3.91}$$

Taking the partial derivative of (3.91) on both sides w.r.t z_k we obtain

$$\lambda\, f_{12,z_{k-1}} - f_{32,z_{k-1}} - \eta\,\lambda(\lambda\, f_{12,z_k} - f_{32,z_k}) = 0$$

which reduces to

$$\lambda\, f_{12,z_{k-1}} - f_{32,z_{k-1}} = 0 \tag{3.92}$$

upon substituting $f_{12,z_k} = f_{32,z_k} = 0$ by Lemma 3.3.1.

The z_{k-1} derivative of (3.92) yields

$$\lambda f_{12,z_{k-2}} - f_{32,z_{k-2}} - \eta\lambda(\lambda f_{12,z_{k-1}} - f_{32,z_{k-1}}) = 0,$$

which reduces to

$$\lambda f_{12,z_{k-2}} - f_{32,z_{k-2}} = 0 \tag{3.93}$$

upon using (3.92). Continuing this process, taking successive derivatives of (3.91) with respect to $z_{k-2}, z_{k-3}, \ldots, z_2, z_1$, we finally obtain

$$\lambda f_{12} - f_{32} = 0. \tag{3.94}$$

Substituting (3.94) into (3.88), we obtain

$$B = 0,$$

i.e.

$$\sum_{i=0}^{k-2} z_{i+1} f_{22,z_i} = 0,$$

which implies

$$f_{22,z_i} = 0, , \qquad 0 \leq i \leq k-2 .$$

But from Lemma 3.3.1 we already have that

$$f_{22,z_k} = f_{22,z_{k-1}} = 0.$$

Hence we have

$$f_{22,z_i} = 0, , \qquad 0 \leq i \leq k ,$$

i.e. f_{22} does not depend on z_i, for $0 \leq i \leq k$.
Now by substituting (3.94) in (3.87), we obtain

$$F f_{11,z_0} = \sum_{i=0}^{k-1} z_{i+1} f_{12,z_i} + (\eta\lambda f_{12} - \lambda f_{22} f_{11})$$

which yields the required result (3.83).

(b) Suppose $\lambda^2 + K \neq 0$.

Considering $K\times$(3.87)$+\lambda\times$(3.89), we have

$$-(\lambda^2 + K)F f_{11,z_0} + \sum_{i=0}^{k-1} z_{i+1}(K f_{12,z_i} + \lambda f_{32,z_i}) + K\,\eta(f_{32} - \lambda f_{11}) = 0. \tag{3.95}$$

Taking the z_k derivative of (3.90) once again yields

$$\lambda f_{12,z_{k-1}} - f_{32,z_{k-1}} = 0 \tag{3.96}$$

upon substituting $f_{12,z_k} = f_{32,z_k} = 0$ by Lemma 3.3.1.

Taking the z_{k-1} derivative of (3.88) yields

$$f_{22,z_{k-2}} + f_{11}(\lambda f_{12,z_{k-1}} - f_{32,z_{k-1}}) + (\lambda f_{12} - f_{32})f_{11,z_{k-1}} = 0.$$

Substituting (3.96) and $f_{11,z_{k-1}} = 0$ (from Lemma 3.3.1), we obtain

$$f_{22,z_{k-2}} = 0. \tag{3.97}$$

From (3.88), we now obtain

$$\lambda f_{12} - f_{32} = -\frac{B}{f_{11}} \tag{3.98}$$

which yields

$$\sum_{i=0}^{k-2} z_{i+1}(\lambda f_{12,z_i} - f_{32,z_i}) = -\sum_{i=0}^{k-2} z_{i+1}\left(\frac{B}{f_{11}}\right)_{z_i}.$$

But, by using (3.96), we have that

$$\sum_{i=0}^{k-1} z_{i+1}(\lambda f_{12,z_i} - f_{32,z_i}) = \sum_{i=0}^{k-2} z_{i+1}(\lambda f_{12,z_i} - f_{32,z_i}).$$

Hence

$$\sum_{i=0}^{k-1} z_{i+1}(\lambda f_{12,z_i} - f_{32,z_i}) = -\sum_{i=0}^{k-2} z_{i+1}\left(\frac{B}{f_{11}}\right)_{z_i}. \tag{3.99}$$

Substituting (3.99) in (3.90), we now obtain

$$\eta(\lambda f_{32} + K f_{12}) = \sum_{i=0}^{k-2} z_{i+1} \left(\frac{B}{f_{11}}\right)_{z_i} + (\lambda^2 + K) f_{11} f_{22}. \tag{3.100}$$

Solving (3.98) and (3.100) simultaneously for f_{11} and f_{32}, we obtain

$$f_{12} = \frac{1}{(\lambda^2 + K)} \left(\frac{1}{\eta} \sum_{i=0}^{k-2} z_{i+1} \left(\frac{B}{f_{11}}\right)_{z_i} - \frac{\lambda B}{f_{11}}\right) + \frac{f_{11} f_{22}}{\eta},$$

and

$$f_{32} = \frac{1}{(\lambda^2 + K)} \left(\frac{\lambda}{\eta} \sum_{i=0}^{k-2} z_{i+1} \left(\frac{B}{f_{11}}\right)_{z_i} + \frac{KB}{f_{11}}\right) + \frac{\lambda}{\eta} f_{11} f_{22},$$

respectively, which are the required results (3.85) and (3.86).

Finally, by substituting (3.98) and (3.100) in (3.95), we obtain the expression for F given by equation (3.84).

Hence the necessary part.

($\Longleftarrow$) The converse part is a straightforward computation similar to that in the converse part of Theorem 3.3.2. $\square$

For the next Theorem we need to introduce the following notation:

$$E^{k-1} = 0$$

$$E^j = -\sum_{i=0}^{k-1} z_{i+1} E^{j+1}_{z_i} + \left(-z_1 \frac{L}{c} + \eta\right) B_{z_{j+1}}\,, \qquad ,\ 0 \le j \le k-2 \tag{3.101}$$

Theorem 3.3.5 *Let $f_{\alpha\beta}$, $1 \le \alpha \le 3$, $1 \le \beta \le 2$, be differentiable functions of $z_0, z_1, \ldots, z_k$ such that (3.16)-(3.19) holds, except that $f_{21} = \eta$, a non-zero constant. Suppose that $f_{31}^2 + K f_{11}^2 = c$, where $c \neq 0$ and c does not depend on z_i, and that $L \neq 0$. Then*

$$z_{0,t} = F(K(x,t), z_0, z_1, \ldots, z_k)$$

corresponds to a surface of variable Gaussian curvature $K(x,t)$, with associated 1−forms

$\omega_\alpha = f_{\alpha 1}\,dx + f_{\alpha 2}\,dt$ *if and only if*

$$F = \frac{1}{L}\left(\sum_{i=0}^{k-1} z_{i+1}B_{z_i} - z_1(f_{12,z_0}f_{31} - f_{32,z_0}f_{11}) - \eta\,(f_{31}f_{32} + Kf_{11}f_{12}) + cf_{22}\right) \quad (3.102)$$

where f_{12} and f_{32} are functions of f_{11}, f_{31}, f_{22} which satisfy, for $1 \le j \le k-1$,

$$f_{12,z_j} = -\frac{1}{c}\left(f_{31}B_{z_j} + K\,\eta\,f_{11}B_{z_{j+1}}\right), \quad (3.103)$$

$$f_{32,z_j} = \frac{K}{c}\left(f_{11}B_{z_j} - \eta\,f_{31}B_{z_{j+1}}\right), \quad (3.104)$$

$$Kf_{12,z_0}f_{11} + f_{32,z_0}f_{31} + K\,\eta\,B_{z_1} = 0, \quad (3.105)$$

$$-f_{11}f_{32} + f_{12}f_{31} + B = 0, \quad (3.106)$$

and f_{11}, f_{32}, f_{22} satisfy the differential equation,

$$-\sum_{i=0}^{k-1} z_{i+1}B_{z_{i+1}} + B = 0. \quad (3.107)$$

Proof: ($\Longrightarrow$) Suppose $z_{0,t} = F\,(K(x,t),\, u,\, z_0,\, z_1, \ldots, z_k)$ describes a surface with Gaussian curvature $K(x,t)$. Then from Lemma 3.3.1 it follows that (3.20)–(3.22) is satisfied. Also by hypothesis we have $H = 0$. i.e.

$$f_{31}f_{31,z_0} + Kf_{11}f_{11,z_0} = 0. \quad (3.108)$$

Considering $Kf_{11}\times$(3.20)+$f_{31}\times$(3.22) and substituting (3.108) we obtain

$$\sum_{i=0}^{k-1} z_{i+1}(Kf_{11}f_{12,z_i} + f_{31}f_{32,z_i}) + K\eta\,(f_{11}f_{32} - f_{11}f_{31}) = 0,$$

i.e.

$$\sum_{i=0}^{k-1} z_{i+1}(Kf_{11}f_{12,z_i} + f_{31}f_{32,z_i}) + K\eta\,B = 0, \quad (3.109)$$

where B is given in (3.39).

Considering $f_{31}\times(3.20)-f_{11}\times(3.22)$, we obtain

$$\begin{aligned} F(-f_{31}f_{11,z_0}+f_{11}f_{31,z_0})+\sum_{i=0}^{k-1} z_{i+1}(f_{31}f_{12,z_i}-f_{11}f_{32,z_i}) & \\ +\eta\, f_{31}f_{32}-f_{31}^2 f_{22}-Kf_{11}(-\eta\, f_{12}+f_{22}f_{11}) &= 0\,. \end{aligned} \tag{3.110}$$

By substituting $f_{31}^2+Kf_{11}^2=c$ and L from (3.37), equation (3.110) reduces to

$$LF+\sum_{i=0}^{k-1} z_{i+1}(f_{31}f_{12,z_i}-f_{11}f_{32,z_i})+\eta\,(f_{31}f_{32}+Kf_{11}f_{12})-f_{22}c=0. \tag{3.111}$$

Taking the partial derivative of (3.109) on both sides w.r.t z_{j+1} we obtain

$$Kf_{11}f_{12,z_j}+f_{31}f_{32,z_j}+K\eta\, B_{z_{j+1}}=0, \tag{3.112}$$

and taking the partial derivative of (3.88) on both sides w.r.t z_j for $j=k-1,k-2,\ldots,2,1$, we get

$$B_{z_j}-f_{11}f_{32,z_j}-f_{32}f_{11,z_j}+f_{12}f_{31,z_j}+f_{31}f_{12,z_j}=0,$$

which reduces to

$$B_{z_j}=f_{11}f_{32,z_j}-f_{31}f_{12,z_j}, \tag{3.113}$$

since $f_{11,z_i}=f_{31,z_i}=0$, for $1\le i\le k$ from Lemma 3.3.1.

Solving equations (3.112) and (3.113) simultaneously for f_{12,z_j} and f_{32,z_j} we obtain

$$f_{12,z_j}=-\frac{1}{c}\left(f_{31}B_{z_j}+K\,\eta\, f_{11}B_{z_{j+1}}\right), \tag{3.114}$$

and

$$f_{32,z_j}=\frac{K}{c}\left(f_{11}B_{z_j}-\eta\, f_{31}B_{z_{j+1}}\right), \tag{3.115}$$

respectively, for $1\le j\le k-1$.

Taking the z_1 derivative of (3.109), we obtain

$$K f_{11} f_{12,z_0} + f_{31} f_{32,z_0} + K\eta\, B_{z_1} = 0$$

which can be written as

$$K f_{11} f_{12,z_0} + f_{31} f_{32,z_0} = -K\eta\, B_{z_1}. \tag{3.116}$$

Considering (3.109), we have

$$\begin{aligned} K\eta\, B &= -\sum_{i=0}^{k-1} z_{i+1}(K f_{11} f_{12,z_i} + f_{31} f_{32,z_i}) \\ &= -(K f_{11} f_{12,z_0} + f_{31} f_{32,z_0}) z_1 - \sum_{i=1}^{k-1} z_{i+1}(K f_{11} f_{12,z_i} + f_{31} f_{32,z_i}) \\ &= z_1 K\eta\, B_{z_1} - \sum_{i=1}^{k-1} z_{i+1}(K f_{11} f_{12,z_i} + f_{31} f_{32,z_i}). \end{aligned} \tag{3.117}$$

The last step follows from the substitution of (3.116). Substituting (3.114) and (3.115) in (3.117), after some simplifications we obtain

$$-\sum_{i=0}^{k-1} z_{i+1} B_{z_{i+1}} + B = 0,$$

a differential equation satisfied by the functions f_{11}, f_{31} and f_{22}.

Now from (3.111), we have

$$LF = -\sum_{i=0}^{k-1} z_{i+1}(f_{31} f_{12,z_i} - f_{11} f_{32,z_i}) - \eta\,(f_{31} f_{32} + K f_{11} f_{12}) + f_{22} c. \tag{3.118}$$

But, from (3.88), we obtain for $1 \le i \le k-1$,

$$B_{z_i} = -(f_{31} f_{12,z_i} - f_{11} f_{32,z_i}). \tag{3.119}$$

Hence, substituting (3.119) in (3.118) yields the required expression for F given in equation (3.102), and thus the necessary part.

($\Longleftarrow$) The converse part is a straightforward computation similar to that in the converse part of Theorem 3.3.2. □

3.3.1 Illustrative Examples

In this section we apply the above results to some examples to determine the components of the first fundamental form. Recall that the components of the first fundamental form E, F, G can be determined in terms of $f_{\alpha\beta}$, $1 \le \alpha \le 3$, $1 \le \beta \le 2$, *viz.*

$$E = f_{11}^2 + f_{21}^2 , \tag{3.120}$$

$$F = f_{11}f_{12} + f_{21}f_{22} , \tag{3.121}$$

$$G = f_{12}^2 + f_{22}^2 . \tag{3.122}$$

Example 1: Generalised Burgers Equation

Using the notations given in (3.13) and (3.14), a generalisation of the Burgers equation is

$$u_t = (1 + K(x,t) + u)u_x + u_{xx},$$

which can be written as

$$z_{0,t} = (1 + K(x,t) + z_0)z_1 + z_2.$$

With $k = 2$, Theorem 3.3.2 implies

$$f_{11} = \frac{z_0}{2}, \qquad f_{12} = \frac{z_0^2}{4} + \frac{z_1}{2} ,$$

$$f_{21} = \eta, \qquad f_{22} = \frac{\eta}{2} z_0 \, , ,$$

and

$$f_{31} = -\eta, \qquad f_{32} = -\frac{\eta}{2} z_0 \, .$$

Hence the components of the first fundamental form E, F, G are given by

$$E = \left(\frac{u^2}{4} + \eta^2\right),$$

$$F = \frac{\eta^2 u}{2} + \frac{u}{4}\left(\frac{u^2}{2} + u_x\right),$$

and

$$G = \left(\frac{u^2}{4} + \frac{u_x}{2}\right)^2 + \frac{\eta^2}{4}u^2.$$

Example 2: Generalised KdV Equation

A generalised KdV equation

$$u_t = (1 + K(x,t) + 6u)u_x + u_{xxx}$$

can be written as

$$z_{0,t} = (1 + K + 6z_0)z_1 + z_3.$$

With $k = 3$, Theorem 3.3.2 implies

$$f_{11} = 1 - z_0, \qquad f_{12} = -z_2 + \eta\, z_1 - 2z_0^2 - \eta^2\, z_0 + 2z_0 + \eta^2,$$

$$f_{21} = \eta, \qquad f_{22} = -2z_1 + 2\eta\, z_0 + \eta^3,$$

and

$$f_{31} = -1 - z_0, \qquad f_{32} = -z_2 + \eta\, z_1 - 2z_0^2 - \eta^2\, z_0 - 2z_0 - \eta^2.$$

Hence the components of the first fundamental form E, F, G are given by

$$E = (1-u)^2 + \eta^2,$$

$$F = (1-u)(-u_{xx} + \eta\, u_x - \eta^2\, u - 2u^2 + \eta^2 + 2u) + \eta\,(\eta 3 + 2\eta\, u - 2u_x),$$

and

$$G = (-u_{xx} + \eta\, u_x - \eta^2\, u - 2u^2 + \eta^2 + 2u)^2 + (\eta 3 + 2\eta\, u - 2u_x)^2.$$

Example 3: Generalised MKdV Equation

A generalisation of the MKdV equation is

$$u_t = \left(1 + K(x,t) + \frac{3}{2}u^2\right) u_x + u_{xxx},$$

which may be written as

$$z_{0,t} = \left(1 + K + \frac{3}{2}z_0^2\right) z_1 + z_3.$$

With $k = 3$, Theorem 3.3.3 implies

$$f_{11} = 0, \qquad f_{12} = -\eta\, z_1\,,$$

$$f_{21} = \eta, \qquad f_{22} = \frac{1}{2}\eta\, z_0^2 + \eta^3\,,$$

and

$$f_{31} = z_0 \qquad f_{32} = -z_2 + \frac{1}{2}z_0^3 + \eta^2\, z_0\,.$$

Hence the components of the first fundamental form E, F, G are given by

$$E = \eta^2\,,$$

$$F = \eta\, \left(\frac{\eta\, u^2}{2} + \eta^3\right)\,,$$

and

$$G = \eta^2\, u_x^2 + \left(\frac{\eta\, u^2}{2} + \eta^3\right)^2\,.$$

3.4 Characterisation II

In this section, we characterise equations of the type

$$z_{1,t} = F(K(x,t), z_0, z_1, \ldots, z_k) \tag{3.123}$$

which correspond to a surface of variable Gaussian curvature $K(x,t)$.

Theorem 3.4.1 *Let* $f_{\alpha\beta}$, $1 \leq \alpha \leq 3$, $1 \leq \beta \leq 2$, *be differentiable functions of* $z_0, z_1, \ldots\ldots\ldots, z_k$ *except that* $f_{11} = 0, f_{32} = 0$, *and* $f_{21} = \eta$, *a non-zero parameter. Then*

$$z_{1,t} = F(K(x,t), z_0, z_1, \ldots, z_k)$$

corresponds to a surface of variable Gaussian curvature $K(x,t)$, *with associated* $1-$*forms* $\omega_\alpha = f_{\alpha 1}\, dx + f_{\alpha 2}\, dt$ *if and only if*

$$F = -\frac{K\eta}{a}\,(b \sin\, az_0 + d \cos\, az_0) \tag{3.124}$$

and

$$f_{12} = b \sin\, az_0 + d \cos\, az_0, \tag{3.125}$$

$$f_{22} = b \cos\, az_0 - d \sin\, az_0, \tag{3.126}$$

$$f_{31} = az_1, \tag{3.127}$$

where $a \neq 0$ *and* b, d *do not depend on* z_i, $0 \leq i \leq k$.

Proof: Since

$$dz_i = \frac{\partial z_i}{\partial x}\, dx + \frac{\partial z_i}{\partial t}\, dt,$$

we have, for $0 \leq i \leq k-1$,

$$dz_i \wedge dt = z_{i+1}\, dx \wedge dt \tag{3.128}$$

and

$$\begin{aligned} dz_1 \wedge dx &= \left(\frac{\partial z_1}{\partial x}\, dx + \frac{\partial z_1}{\partial t}\, dt\right) \wedge dx \\ &= z_{1,t}\, dt \wedge dx \end{aligned}$$

$$= \; -z_{1,t}\, dx \wedge dt$$

$$= \; -F\, dx \wedge dt\,, \tag{3.129}$$

from equation (3.123).

Substituting (3.6), (3.7) and (3.8) in (3.9), we have

$$d(f_{11}dx + f_{12}dt) = (f_{31}dx + f_{32}dt) \wedge (\eta\, dx + f_{22}dt)$$

which reduces to

$$d(f_{12}dt) = (f_{31}dx) \wedge (\eta\, dx + f_{22}dt)$$

since $f_{11} = f_{32} = 0$ and $f_{21} = \eta$. i.e.

$$\sum_{i=0}^{k} \frac{\partial f_{12}}{\partial z_i}\, dz_i \wedge dt - (f_{31}f_{22})dx \wedge dt = 0.$$

The above equation can be written as

$$f_{12,z_k}\, dz_k \wedge dt + \sum_{i=0}^{k-1} f_{12,z_i}\, dz_i \wedge dt - f_{22}f_{31}\, dx \wedge dt = 0.$$

Substituting (3.128) and (3.129) in the above equation yields

$$f_{12,z_k}\, dz_k \wedge dt + \left(\sum_{i=0}^{k-1} z_{i+1} f_{12,z_i} - f_{22}f_{31} \right) dx \wedge dt = 0. \tag{3.130}$$

Equating the coefficients on both sides of equation (3.130) yields

$$f_{12,z_k} = 0\,, \tag{3.131}$$

and

$$\sum_{i=0}^{k-1} z_{i+1}\, f_{12,z_i} - f_{22}f_{31} = 0\,. \tag{3.132}$$

Now substituting (3.6), (3.7) and (3.8) in (3.10), we obtain

$$d(\eta\, dx + f_{22}dt) = (f_{11}dx + f_{12}dt) \wedge (f_{31}dx + f_{32}dt).$$

After some manipulations and substitutions this equation reduces to

$$f_{22,z_k}\, dz_k \wedge dt + \left(\sum_{i=0}^{k-1} z_{i+1} f_{22,z_i} + f_{12}f_{31}\right) dx \wedge dt = 0. \tag{3.133}$$

By equating the coefficients on both sides of (3.133), we obtain

$$f_{22,z_k} = 0\,, \tag{3.134}$$

and

$$\sum_{i=0}^{k-1} z_{i+1}\, f_{22,z_i} + f_{12}f_{31} = 0\,. \tag{3.135}$$

Finally, substituting (3.6), (3.7) and (3.8) in (3.11), we obtain

$$d(f_{31}dx + f_{32}dt) = -K(x,t)\,(f_{11}dx + f_{12}dt) \wedge (\eta\, dx + f_{22}dt),$$

which, after substitution and some manipulations reduces to

$$\sum_{i=2}^{k} f_{31,z_i}\, dz_i \wedge dx - (K\eta\, f_{12} + F f_{31,z_1}) dx \wedge dt = 0. \tag{3.136}$$

Equating the coefficients on both sides yields

$$f_{31,z_i} = 0\,, \qquad i \neq 1\,, \tag{3.137}$$

$$F\, f_{31,z_1} + K\eta\, f_{12} = 0\,. \tag{3.138}$$

Taking the partial derivative of (3.132) on both sides w.r.t z_k yields

$$f_{12,z_{k-1}} - f_{31}f_{22,z_k} - f_{22}f_{31,z_k} = 0$$

which reduces to

$$f_{12,z_{k-1}} = 0$$

since $f_{22,z_k} = f_{31,z_k} = 0$.

Taking the partial derivative of (3.135) on both sides w.r.t z_k we obtain

$$f_{22,z_{k-1}} + f_{31}f_{12,z_k} + f_{12}f_{31,z_k} = 0$$

which reduces to

$$f_{22,z_{k-1}} = 0,$$

since $f_{31,z_k} = f_{12,z_k} = 0$.

The partial derivative on both sides of (3.132) w.r.t z_{k-1} yields

$$f_{12,z_{k-2}} - f_{31}f_{22,z_{k-1}} - f_{22}f_{31,z_{k-1}} = 0$$

which reduces to

$$f_{12,z_{k-2}} = 0,$$

since $f_{22,z_{k-1}} = f_{31,z_{k-1}} = 0$.

Taking the partial derivative of (3.135) on both sides w.r.t z_{k-1} we obtain

$$f_{22,z_{k-2}} + f_{31}f_{12,z_{k-1}} + f_{12}f_{31,z_{k-1}} = 0$$

which reduces to

$$f_{22,z_{k-2}} = 0,$$

since $f_{31,z_{k-1}} = f_{12,z_{k-1}} = 0$.

Continuing this process of taking the derivatives of (3.132) and (3.135) with respect to $z_{k-2}, z_{k-3}, \ldots, z_2$, we finally end up with

$$f_{12,z_i} = f_{22,z_i} = 0\,, \qquad 1 \leq i \; leq\, k. \tag{3.139}$$

Now, taking the double derivative of (3.132) with respect to z_1, we obtain

$$f_{31,z_1,z_1} = 0$$

which implies

$$f_{31} = az_1 + e\,, \tag{3.140}$$

where a and e are independent of z_i for $0 \leq i \leq k$.
By substituting (3.139) and (3.140) in (3.132) one obtains

$$z_1 f_{12,z_0} - (az_1 + e) f_{22} = 0$$

which can be written as

$$(f_{12,z_0} - af_{22})z_1 - ef_{22} = 0\,. \tag{3.141}$$

Similarly by substituting (3.139) and (3.140) in (3.135) one obtains

$$z_1 f_{22,z_0} + (az_1 + e) f_{12} = 0$$

which can be written as

$$(f_{22,z_0} + af_{12})z_1 + ef_{12} = 0\,. \tag{3.142}$$

Taking the partial derivatives on both sides of (3.141) and (3.142) w.r.t z_1, and then by using (3.139) we obatin

$$f_{12,z_0} - af_{22} = 0\,, \tag{3.143}$$

and

$$f_{22,z_0} + af_{12} = 0 \tag{3.144}$$

respectively and thus

$$e = 0\,. \tag{3.145}$$

Taking the partial derivative of (3.143) on both sides w.r.t z_0 we obtain

$$f_{12,z_0z_0} - af_{22,z_0} = 0,$$

which, when substituting (3.144) yields

$$f_{12,z_0z_0} + a^2 f_{12} = 0\,, \tag{3.146}$$

a differential equation for f_{12}. Solving (3.146) we have

$$f_{12} = b \sin az_0 + d \cos az_0 \,,$$

where b, d do not depend on z_i, $0 \leq i \leq k$. Similarly by taking the partial derivative of (3.144) on both sides w.r.t z_0 and then by substituting (3.143), we get a differential equation for f_{22} which yields

$$f_{22} = b \cos az_0 - d \sin az_0 \,.$$

We note that if $a = 0$ then $\omega_3 = 0$, contradicting the fact that ω_3 is the connection form. Hence $a \neq 0$. Finally, substituting the expressions for f_{12} and f_{31,z_1} in equation (3.138) yields

$$F = -\frac{K\eta}{a}\,(b \sin az_0 + d \cos az_0)\,,$$

which is the required result, and hence the necessary part follows.

($\Longleftarrow$) The converse part is a straightforward computation similar to that in the converse part of Theorem 3.3.2. □

Using arguments similar to the above theorem, we can prove the following results.

Theorem 3.4.2 *Let $f_{\alpha\beta}$, $1 \leq \alpha \leq 3$, $1 \leq \beta \leq 2$, be differentiable functions of $z_0, z_1, \ldots, z_k$ except that $f_{12} = 0, f_{13} = 0$, and $f_{21} = \eta$, a non-zero parameter. Then*

$$z_{1,t} = F(K(x,t), z_0, z_1, \ldots, z_k)$$

corresponds to a surface of variable Gaussian curvature $K(x,t)$, with associated $1-$forms $\omega_\alpha = f_{\alpha 1}\, dx + f_{\alpha 2}\, dt$ if and only if

$$F = -\frac{K\eta}{a}\,(b \cosh az_0 + d \sinh az_0) \tag{3.147}$$

and

$$f_{11} = az_1, \tag{3.148}$$

$$f_{22} = b \sinh az_0 + d \cosh az_0, \tag{3.149}$$

$$f_{32} = b \cosh az_0 + d \sinh az_0, \tag{3.150}$$

where $a \neq 0$ and b, d do not depend on z_i, $0 \leq i \leq k$.

3.4.1 Illustrative Examples

In this section we shall look at examples which illustrate the above two theorems by providing us with the required coordinate systems.

Example 1: Generalised sine-Gordon Equation

Using the notations given in (3.13) and (3.14), the generalised sine-Gordon equation

$$u_{xt} = -K(x,t) \sin u$$

can be written as

$$z_{1,t} = -K(x,t) \sin z_0.$$

Using Theorem 3.4.1 with the choices of

$$a = 0, \quad b = \frac{1}{\eta}, \quad \text{and} \quad c = 0,$$

we obtain

$$f_{11} = 0, \qquad f_{12} = \frac{1}{\eta} \sin z_0,$$

$$f_{21} = \eta, \qquad f_{22} = \frac{1}{\eta} \cos z_0,$$

and

$$f_{31} = -z_1, \qquad f_{32} = 0.$$

Hence the components of the first fundamental form E, F, G are given by

$$E = \eta^2,$$

$$F = \cos z_0 = \cos u\,,$$

and

$$G = \frac{1}{\eta^2}\,,$$

respectively.

Example 2: Generalised sinh-Gordon Equation

Using the notations given in (3.13) and (3.14), the generalised sine-Gordon equation

$$u_{xt} = -K(x,t)\sinh u$$

can be written as

$$z_{1,t} = -K(x,t)\sinh z_0.$$

Using Theorem 3.4.2 with the choices of

$$a = 1, \quad b = 0\,, \quad \text{and} \quad , c = \frac{1}{\eta},,$$

we obtain

$$f_{11} = z_1, \qquad f_{12} = 0\,,$$

$$f_{21} = \eta, \qquad f_{22} = \frac{1}{\eta}\cosh z_0\,,$$

and

$$f_{31} = 0\,, \qquad f_{32} = \frac{1}{\eta}\sinh z_0\,.$$

Hence the components of the first fundamental form E, F, G are given by

$$E = \eta^2 + z_1^2 = \eta^2 + u_x^2\,,$$

$$F = \cosh z_0 = \cosh u\,,$$

and

$$G = \frac{1}{\eta^2} \cosh^2 u$$

respectively.

Chapter 4

Conclusions

In this thesis, we exploited classical differential geometry, to find Bäcklund transformations and hence solve certain classes of non-linear partial differential equations. The observation that certain partial differential equations can be interpreted as a statement of Gauss' theorem in an appropriate coordinate system is fruitful and produces some useful strategies for solving PDEs based on Bäcklund transformations.

In Chapter 1 we outlined some basic concepts from differential geometry, especially the Gauss equation which plays a central rôle. In this chapter we outline a strategy for solving a given a PDE. Essentially, if we can determine a coordinate system such that the PDE corresponds to the Gauss equation for a surface of known Gaussian curvature, and if another (simpler) PDE can be found that also corresponds to the Gauss equation for the same Gaussian curvature, then the covariant transformation equations can be used to determine (in principle) the Bäcklund transformations between the two coordinate systems. If we can solve the latter PDE then we can obtain solutions to the original PDE by transforming the known solution of the other PDE.

The above strategy has some stumbling blocks. Firstly, given a PDE, we need to determine a coordinate system such that it corresponds to the Gauss equation. Secondly, we need to identify the Gaussian curvature of the surface and find a simpler PDE to solve. Thirdly, we then need to solve the system of PDEs which arise from the covariant transformation equations. Fourthly, we need to consider potential restrictions on the initial data for the system.

In Chapter 2 we focused on a simple, specific case and used the sine-Gordon equation to illustrate the ideas. Chapter 2 thus served to introduce our techniques for determining a family of solutions ((2.69)) to the sine-Gordon equation. Even though we were successful in deriving a family of solutions, we had to ackowledge the fact that our method involved some inversions, which proved tedious and in some instances, formidable. Moreover, the family of solutions depended crucially on the solution to the transformation equations, which without initial data is not unique.

To further illustrate the techniques we solved a simple Cauchy problem for the sine-Gordon equation. The solution to this problem corresponds to a Beltrami surface (though perhaps not obvious) and provides a simple geometrical example. We also investigated a soliton solution ((2.99)) to the sine-Gordon equation and derived a solution to the Schrödinger equation through Bäcklund transformations.

An investigation for case(b) in sections 2.1.1.4 and 2.1.2.2 for the classes of second order quasi-linear PDEs of the hyperbolic type and the parabolic type, was not fruitful because the systems of PDEs were *prima facie* harder to solve than the original problem. We did not investigate these cases further, but it would be worthwhile to find a solution technique for these cases.

The motivation to solve a class of second order quasi-linear PDEs of the elliptic type considered in section 2.1.3 was provided to the effect that a transformation some what similar to the one used in the case for the sine-Gordon equation might work.

Near the end of Chapter 2 we investigated a class of fully nonlinear PDEs, in particular the Monge-Ampère equation, which can be interpreted as the Gauss equation if the surface is parametrised in graphical coordinates. More complicated non-linear PDEs also could be attacked using our techniques, and the initial value problems can be reduced to problems involving the solution of a first order ODE. The key, however is to find a suitable geometrical interpretation as a statement of Gauss' theorem.

In section 2.2.1.2 we briefly visited the concepts relating to a solution technique for a more general class of Monge-Ampère equation based on parallel surfaces. This is one other area which signals further work.

In Chapter 3, we considered the crucial problem of determining a coordinate system such that the given PDE corresponds to the Gauss equation. The discussion in Chapter 3 was devoted to a reasonably detailed investigation of this question. In this chapter we extended the work of Chern et.*al.* to surfaces of non-constant Gaussian curvature. This enables us to apply our techniques to a more general class of second order PDEs. The material in Chapter 3 focused primarily on finding a systematic way to determine a coordinate system for a given PDE.

The motivation for the complete classification investigated in section 3.3, for differential equations of the form

$$u_t = F\left(K(x,t),\, u,\, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^k u}{\partial x^k}\right)$$

was not only due to the fact that we were trying to answer the above query, but also that we were trying to extend our solution techniques to third order PDEs such as the generalised KdV equation

$$u_t = (1 + K(x,t) + 6u)u_x + u_{xxx}.$$

The characterisation in section 3.4 focused on the PDEs of the form

$$u_{xt} = F\left(K(x,t),\, u,\, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^k u}{\partial x^k}\right),$$

which includes the generalised sine-Gordon equation and the generalised sinh-Gordon equation.

Although we did not investigate specific examples of some of the generalised classes of PDEs discussed in Chapter 3, we note here that the methods detailed in Chapter 2 could potentially shed some light on solution families for these more complicated PDEs. The crux of the problem is to solve the transformation equations which may be more formidable than the original equation. Nonetheless if specific solutions to the transformation equations can be found, then at least some families of solutions can be identified. There is certainly scope for further investigation here.

Bibliography

[1] T. Aubin. *Nonlinear Analysis on Manifolds.* Springer-Verlag, NY, Inc., 1982.

[2] M.J. Ablowitz R. Beals and K.Tenenblat. On the Solution of the Generalized Wave and Generalized sine-Gordon Equations. *Studies in Applied Mathematics*, 74:177–203, 1986.

[3] J.A. Cavalcante and K. Tenenblat. Conservation laws for nonlinear evolution equations. *Journal of Mathematical Physics*, 24(4):1044–1049, apr 1988.

[4] S.S. Chern and K.Tenenblat. Pseudospherical Surfaces and Evolution Equations. *Studies in Applied Mathematics*, 74:55–83, 1986.

[5] T. Chou. A generalization of Bäcklund theorem. *Chinese Science Bulletin*, 40(22):1854–1860, nov 1995.

[6] M. Crampin and D.J. Saunders. The sine-Gordon Equation, Tchebyshev nets, and Harmonic Maps. *Reports of Mathematical Physics*, 23(3):327–340, 1986.

[7] L.P. Eisenhart. *A Treatise on the Differential Geometry of Curves and Surfaces.* Dover, 1909 (reprint 1960).

[8] H. Flanders. *Differential Forms.* Academic Press, New York, 1963.

[9] P.R. Garabedian. *Partial Differential Equations.* John Wiley & Sons, INC, 1964.

[10] J.P. Gazeau and P.Winternitz. Symmetries of variable coefficient Korteweg-de Vries equations. *Journal of Mathematical Physics*, 33(12):4087–4102, dec 1992.

[11] R.P. Gillespie. *Partial Differentiation.* Oliver and Boyd, Edinburgh, 1954.

[12] I.S. Gradshteyn and I.M. Ryzhik. *Table of Integrals, Series and Products.* Academic Press, 1980.

[13] E.L. Ince. *Ordinary Differential Equations.* Longmans, Green and Co., London, 1956.

[14] M. Johnston. Geometry and the sine-Gordon Equation. Technical Report AMR 95/16, University of New South Wales, Australia, 1995.

[15] L.P. Jorge and K. Tenenblat. Linear Problems Associated with Evolution Equations of the Type $u_{tt} = f(u, u_x, u_{xx}, u_t)$. *Studies in Applied Mathematics*, 77:103–117, 1987.

[16] N. Kamran and K. Teneblat. On differential equations describing pseudo-spherical surfaces. *Journal of Differential Equations*, 115:75–98, 1995.

[17] J. Kevorkian. *Partial Differential Equations, Analytical Solution Techniques.* Chapman & Hall, Inc., London, 1993.

[18] B.G. Konopelchenko. Solitons of Curvature. *Acta Applicandae Mathematicae*, 39:379–387, 1995.

[19] B.G. Konopelchenko. Soliton curvature of surfaces and spaces. *Journal of Mathematical Physics*, 38(1):434–457, 1997.

[20] B.G. Konopelchenko and I.A. Taimanov. Constant mean curvature surfaces via an integrable dynamical system. *Journal of Physics A: Math. Gen.*, 29:1261–1265, 1996.

[21] D.F. Lawden. *Elliptic Functions and Applications.* Springer-Verlag, 1989.

[22] L. Marden. *Calculus of Several Variables.* George Allen & Unwin Ltd, 1971.

[23] C. Miranda. *Partial Differential Equations of Elliptic Type.* Springer-Verlag, NY, 1970.

[24] R.M. Miura. The Korteweg-de Vries equation: A Survey of Results. *SIAM REVIEW*, 18(3), jul 1976.

[25] R.Beals M.Rabelo and K.Tenenblat. Bäcklund Transformations and Inverse Scattering Solutions for Some Pseudospherical Surface equations. *Studies in Applied Mathematics*, 81:125–151, 1989.

[26] A.R. Selvaratnam B.van-Brunt M.Vlieg-Hulstman and W.D. Halford. On the Solution of a class of second order quasi-linear PDEs and the Gauss equation. *Journal of Australian Mathematical Society*, to appear.

[27] M.J. Ablowitz D.J. Kaup A.C. Newell and H. Segur. Soliton Equations. *Physics Review Letters*, 31:125, 1973.

[28] A.V. Pogorelov. *Differential Geometry.* P.Noordhoff N.V - Groningen - The Netherlands, 1989.

[29] E.G. Poznyak and A.G. Popov. Non-Euclidean Geometry: The Gauss Formula and An Interpretation of Partial Differntial Equations. *Journal of Mathematical Sciences*, 78(3):241–252, feb 1996.

[30] C. Rogers and W.F. Shadwick. *Bäcklund Transformations and Their Applications.* Academic Press, 1982.

[31] R. Sasaki. Geometrization of Soliton Equations. *Physics Letters*, 71A(5,6):390–392, may 1979.

[32] R. Sasaki. Soliton Equations and Pseudospherical Surfaces. *Nuclear Physics B*, 154:343–357, 1979.

[33] I.N. Sneddon. *Elements of Partial Differential Equations.* McGraw-Hill Book Company, Inc. NY, 1957.

[34] J.J. Stoker. *Differential Geometry.* John Wiley & Sons, Inc., 1989.

[35] D.J. Struik. *Lectures on Classical Differential Geometry.* Addison-Wesley, 1961.

[36] B. van Brunt and K. Grant. Hyperbolic Weingarten surfaces. *Math. Proc. Camb. Phil. Soc.*, 116:489–504, 1994.

[37] J. Weiss. The Painlevê property for partial differential equations. ii: Bäcklund transformation, lax pairs, and the schwarzian derivative. *Journal of Mathematical Physics*, 24(6):1405–1413, jun 1983.

[38] T.J. Willmore. *An Introduction to Differential Geometry.* Oxford University Press, 1959.

[39] P. Winternitz and J.P.Gazeau. Allowed transformations and symmetry classes of variable coefficient Korteweg-de Vries equations. *Physics Letters A*, 167:246–250, 1992.

[40] H. Wu. On Bäcklund Transofrmations for Nonlinear Partial Differential Equations. *Journal of Mathematical Analysis and Applications*, 192:151–179, 1995.

[41] D. Zwillinger. *Handbook of Differential Equations.* Academic Press, Inc., 1992.

Errata

Page	Line	Reads as	Corrected to
1	1	…are an important…	….are important…..
3	11	…to class ….	….to classes …..
	14	…provide with a …	….provide a ….
	22	…consists the ..	…consists a …
7	7	$\frac{du}{dv}$	$\frac{dv}{du}$
11	6	…partial equations …	partial differential equations
15	3	..Gauss equation (1.15) …	Gauss equation (1.14) …
24	4	p = ….= 2 sec….	p = …. = √2 sec…..
	6	$x = \frac{1}{2}$…..	$x = \frac{1}{\sqrt{2}}$ …..
26	2	$x = \frac{1}{2}$…..	$x = \frac{1}{\sqrt{2}}$ …..
38	last line	$u_x = v_x = 2$ coth…..	$u_x = v_x =$ _ coth…..
40	13	….equation (2.101)…	…equation (2.4) ….
41	14	$M(\phi) =$ ….	$M(\phi) =$ …., $A(\phi) = \frac{\Theta_1}{G_\phi}$, $B(\phi) = \frac{\Theta_2}{G_\phi}$ and $C(\phi) = \frac{\Theta_3}{G_\phi}$
42	17	…in section (2.2.1) …	….in section (2.1.1) ….
44	1	Thus from equation (2.112)	Thus we have …..
	3	…in section (2.2.1) …	…in section (2.1.1)….
	18	…satisfy (2.46) …	….satisfy (2.118) …
	last line	..reduces (2.117) …	….reduces (2.118)….
45	7	…PDE (2.106)…	…PDE (2.107) ….
47	3	…equation (2.120) …	…equation (2.122) …
	18	(2.122) as	(2.126) as
48	15	…form (1.11) ….	….form (1.10) ….

Page	Line	Reads as	Corrected to
49	17	…(section 2.1.1.2).	…(section 2.1.1.1).
	18	…equation (1.18).	…equation (1.17).
51	10	…in section 2.2.1,…	…in section 2.1.1,
56	5	Equation (2.168) …	Equation (2.167) …
68	13	from equation (3.12).	from equations (3.12) and (3.14).
69	12	…equation (3.26) yields	…equation (3.27) yields
70	15	$f_{31,}z_k = 0$ ….	$f_{32,}z_k = 0$ ….
74	1	…derivative of (3.46) …	…derivative of (3.45)…
	9	$f_{11,}z_{k-2} =$ …….	$F_{12,}z_{k-2} =$ ……
77	7	…write (3.62) as	…write (3.26) as
	14	…+ $f_{11,}z_0 F\ dx \wedge dx = 0$,	…+ $f_{11,}z_0 F\ dx \wedge dt = 0$,
78	21	…or $f_{11} = 0$ and $f_{31} \neq 0$.	…or $f_{11} \neq 0$ and $f_{31} = 0$.
82	8	$-1 \times (3.88) + \lambda \times (3.90)$, ..	$-1 \times (3.89) + \lambda \times (3.87)$, ..