Advanced Second Order Functional Differential Equations

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To **Hyoung-Kuen**

Abstract

Hall and Wake [1989] showed that an advanced first order equation arising in a cell growth model has a Dirichlet series solution. If the effects of dispersion are included, the cell growth model leads to a second order equation. We show that this equation also has a Dirichlet series solution, which is unique and positive and that it has one maximum. We then investigate the general second order equation with constant coefficients, and show that these equations also have Dirichlet series solutions and that certain qualitative properties such as uniqueness and positivity are preserved for a range of coefficients. Although the solution to the equation arising in a cell growth model with dispersion is a probability density function of the cell size, $y(0) \neq 0$. There are however parameter choices such that $y(0) = 0$ and this motivates our study of the eigenvalue problem. Our final chapter concerns general equations with variable coefficients. We can express a first order equation as a Fredholm integral equation of the second kind and the existence of a solution thus follows using results for Fredholm equations. In addition, we study some classes of second order equations, and show that certain equations have a series solution involving Bessel or Airy functions.

Acknowledgements

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Contents

Chapter 1

Introduction

Functional differential equations of the form

$$y'(x) = by(x) + cy(\alpha x), \tag{1.1}$$

where $x \geq 0, \alpha > 0, \alpha \neq 1$ and $b, c \in \mathbb{C}$, arise in numerous applications. For example, if $\alpha > 1$, equation (1.1) describes the absorption of the light in the milky way (cf. V.A. Ambartsumian [1944]) and steady size distributions in cell populations (cf. A.J. Hall [1991]). The case when $0 < \alpha < 1$ also arises in many applications; e.g. the dynamics of an overhead current collection system for an electric locomotive (cf. J.R. Ockendon and A.B. Tayler [1971]), a special partition problem in additive number theory (cf. K. Mahler [1940]) and nonlinear dynamical systems (cf. G. Derfel [1990]).

There are basic differences between the retarded case $0 < \alpha < 1$ and the advanced case $\alpha > 1$. For example, the initial value problem for the retarded functional differential equations is well-posed, but the same problem for the advanced functional differential equations is ill-posed. In other words, while the initial value problem with a retarded equation has a unique solution, the initial value problem with an advanced equation has an infinite number of infinitely differentiable solutions. This makes it possible to prescribe additional conditions for advanced equations. Another fundamental difference between the equations is that there is an entire solution to the retarded problem, but there is no solution to the advanced problem which is holomorphic at the origin. Whereas the retarded problem has power series solutions, the advanced problem can have (among other solution forms) Dirichlet series solutions. The asymptotics, as $x \to \infty$, of the retarded and the advanced problem are thus very different.

Although this thesis is concerned primarily with advanced second order equations, we review solution methods for first order functional differential equations in this chapter since they can, to some degree, be extended to second order functional differential equations.

In fact, like the first order case, we will show in this thesis that advanced second order functional differential equations with constant coefficients have a Dirichlet series solution under certain conditions. The equation arising in a cell growth model is investigated in Chapter 2 as an application of second order functional differential equations. In Chapter 3 we consider the more general equation with constant coefficients. Since the initial problem for an advanced equation is ill-posed, it is possible to get the eigenfunctions which have the initial value $y(0) = 0$. We investigate this eigenvalue problem and derive expressions for the eigenvalues and eigenfunctions in Chapter 4. The focus in these chapters is on deriving the solutions using a Dirichlet series along with qualitative properties of the solutions such as uniqueness. Finally, attention is turned to equations with variable coefficients. Here we extend some of the results to some special classes of equations. We assume throughout this thesis that x is real, non-negative variable unless otherwise specified.

In this chapter, we review some general results concerning functional differential equations (mainly first order) proved by T. Kato and J.B. McLeod [1971], L. Fox, D.F. Mayers, J.R. Ockendon and A.B. Tayler [1971], J. Carr and J. Dyson [1974] and A. Iserles [1991] among others. We summarize their results concerning the existence and uniqueness of solutions, the form of solutions, and the asymptotic behaviour as $x \to \infty$. We will also compare and contrast advanced functional differential equations with retarded functional differential equations.

1.1 Retarded Functional Differential Equations

Equation (1.1) with $0 < \alpha < 1$ along with the initial condition $y(0) = 1$, for succinctness, will be referred to as *Problem 1.1*.

T. Kato and J.B. McLeod [1971] proved that Problem 1.1 has one and only one solution for $x > 0$ and that the solution can be expressed as a power series, i.e.

$$y(x) = \sum_{n=0}^{\infty} \frac{\prod_{m=1}^{n}(b + c\alpha^{m-1})}{n!} x^n, \tag{1.2}$$

where we use the convention $\prod_{m=1}^{0}(b + c\alpha^{m-1}) = 1$. For the case when $b = 0$, the solution to Problem 1.1 is

$$y(x) = \sum_{n=0}^{\infty} \frac{\alpha^{\frac{1}{2}(n-1)n}}{n!}(cx)^n. \tag{1.3}$$

Here, if $c \in R$ and $c < 0$, then the solution (1.3) has an infinite number of zeros and if $c > 0$, then the solution has no zeros for $x \geq 0$ so that $y(x) > 0$ since $y(0) = 1 > 0$ (cf A. Iserles [1992]).

1.1.1 Asymptotic Behaviour of Solutions

We consider in this section the asymptotic behaviour of the solutions to Problem 1.1 as $x \to \infty$. The asymptotic behaviour of the solutions depends heavily on the sign of b. T. Kato and J.B. McLeod [1971] started the investigation of the asymptotic behaviour by obtaining the possible order of polynomials decaying or growing as x goes to infinity.

Suppose that y grows/decays like x^k as $x \to \infty$, then y' is negligible compared with y so that

$$0 = bx^k + c(\alpha x)^k,$$

and thus

$$k = \frac{log(-b/c)}{Log\alpha}, \tag{1.4}$$

where Log indicates the principal value of the complex logarithm and log is any value. Now k is not uniquely determined, but if k_0 is any particular solution to equation (1.4), then the complete family

$$k_m = k_0 + 2m\pi i/Log\alpha, \qquad m \in Z,$$

can be determined and thus the asymptotic of the solution is

$$\begin{aligned} & x^{k_0} exp\{2m\pi i \frac{Logx}{Log\alpha}\} \\ = \; & x^{k_0} g(Logx), \end{aligned}$$

where g is periodic in $Logx$ of period $|Log\alpha|$. This observation is the starting point for the study of the asymptotic behaviour of the solutions. The periodic function $g(Logx)$ plays an important role in determining the asymptotic form of the solutions. Let $\kappa = Re(k_0) = \frac{Log|b/c|}{Log\alpha}$, then $\kappa < 0$ if $|c| < |b|$ and $\kappa > 0$ if $|c| > |b|$. T. Kato and J.B. McLeod (*op. cit.*) classified the asymptotic behaviour of the solutions for the case that $b \in R$ and $c \in \mathbb{C}$ according to the signs of b. For the case when $b < 0$, they showed that every solution is $O(x^\kappa)$ and no solutions are $o(x^\kappa)$ as $x \to \infty$. Figure 1.1.1 illustrates the solution $y(x)$ to Problem 1.1 and compares the solution with $x^{1/2}y(x)$ and $xy(x)$ when $b = -1$, $c = 1/2$ and $\alpha = 1/2$. In this case, the solution $y(x)$ is $O(x^{-1})$ since $\kappa = -1$, but for $h < 1$, $y(x)$ is $o(x^{-h})$ as $x \to \infty$. They also showed that given any infinitely differentiable function $g(Logx)$ which is

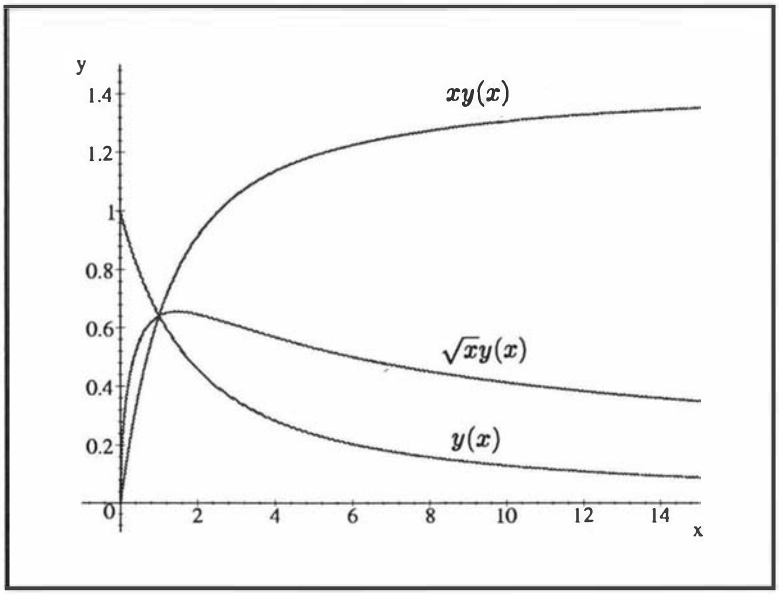

Figure 1.1.1: The solution y to the equation $y'(x) = -y(x) + \frac{1}{2}y(\frac{1}{2}x)$, $x^{1/2}y$ and xy.

periodic of period $|Log\alpha|$ such that, for some positive K and for all $n = 0, 1, 2, \ldots$, if $|g^{(n)}(Logx)| \leq K^{n+1}\alpha^{-\frac{n^2}{2}}$, then there is one solution which has the asymptotic form

$$x^{k_0}(g(Logx) + o(1)), \tag{1.5}$$

and any solution which has this asymptotic behaviour is of the form

$$y(x) = x^{k_0}\{g(Logx) + \sum_{n=1}^{\infty} \frac{x^{-n}g_n(Logx)}{b^n \prod_{m=1}^{n}(1-\alpha^{-m})}\},$$

where the functions g_n are determined by the differential equation
$g_n' = -(k_0 - n)g_n + g_{n+1}$ and $g_0 = g$.

If $b > 0$, Kato and McLeod proved that every solution is $O(e^{bx})$ (possibly $o(e^{bx})$) as $x \to \infty$, and if $|c| < b$, a particular solution is given by

$$y_L(x) = Le^{bx}\{1 + \sum_{n=1}^{\infty} \frac{(-c)^n e^{-b(1-\alpha^n)x}}{b^n \prod_{m=1}^{n}(1-\alpha^m)}\}, \tag{1.6}$$

where L is some constant which depends on the initial value. Note that $y_L(x)$ is a rearrangement of the series (1.2).

They also showed that any other solution $\hat{y}_L$ satisfies
the condition $y_L - \hat{y}_L = O(x^\kappa)$ and that no solution can be $o(x^\kappa)$ as $x \to \infty$. In addition, given any function $g(Logx)$ of the type described for the case $b < 0$, there is one and only one solution which has the asymptotic form (1.5) and every solution has the form

$$y(x) = y_L(x) + \hat{y}(x),$$

where $\hat{y}$ is a solution having the asymptotic form (1.5).

Lastly, they considered the case that $b = 0$. Let $a = Log\alpha$, and

$$\phi(Logx) = x^k (Logx)^h exp(-\frac{1}{2}a^{-1}(Logx - LogLogx)^2), \tag{1.7}$$

where $k = \frac{1}{2} - a^{-1} - a^{-1}Log(-ac)$ and $h = -1 + a^{-1}Log(-ac)$, then every solution is $O(\phi(Logx))$ and no solution is $o(\phi(Logx))$ as $x \to \infty$. Moreover, given any function $g(x)$ of the form

$$g(x) = \sum_{n=-\infty}^{\infty} \gamma_n exp(2n\pi i \frac{x}{|a|}),$$

where $\gamma_n = O\{exp(-\frac{\pi^2|n|}{|a|} - \frac{(log|n|)^2}{2|a|} + Clog|n|)\}$ for some constant C, every solution has the asymptotic form

$$\phi(Logx)\{g(Logx - LogLogx) + o(1)\}.$$

G.R. Morris, A. Feldstein and E.W. Bowen [1972] proved that the solution $y(x)$ to the equation

$$y'(x) = -y(\alpha x),$$

oscillates unboundedly as $x \to \infty$. Figures 1.1.2, 1.1.3 and 1.1.4 illustrate the solution $y(x)$ to Problem 1.1 when $\alpha = 1/2$, $b = 0$ and $c = -1$.

The above observation about the asymptotic behaviour of the solutions when $0 < \alpha < 1$ indicates that equation (1.1) has a solution approaching zero only if

$$b < 0 \quad \text{and} \quad |c| < |b|, \tag{1.8}$$

and they decay no faster than x^κ as $x \to \infty$.

A. Feldstein, A. Iserles and D. Levin [1991] studied the more general equation

$$y'(x) = b(x)y(x) + c(x)y(\theta(x)), \tag{1.9}$$

where $b(x), c(x) \in \mathbb{C}$ and θ is a given differentiable function which satisfies $\theta(0) = 0$ and $0 \leq \theta(x) \leq x$ for $x > 0$. They established the conditions

$$Re(b(x)) < 0, \qquad ||c||_\infty < -\sup_{x \geq 0} Re(b(x)) \inf_{x \geq 0} \sqrt{\theta'(x)}, \tag{1.10}$$

for uniformly bounded solutions. Note that the conditions (1.8) can be derived from the conditions (1.10) by setting $\theta(x) = \alpha x$, $b(x) = b$ and $c(x) = c$.

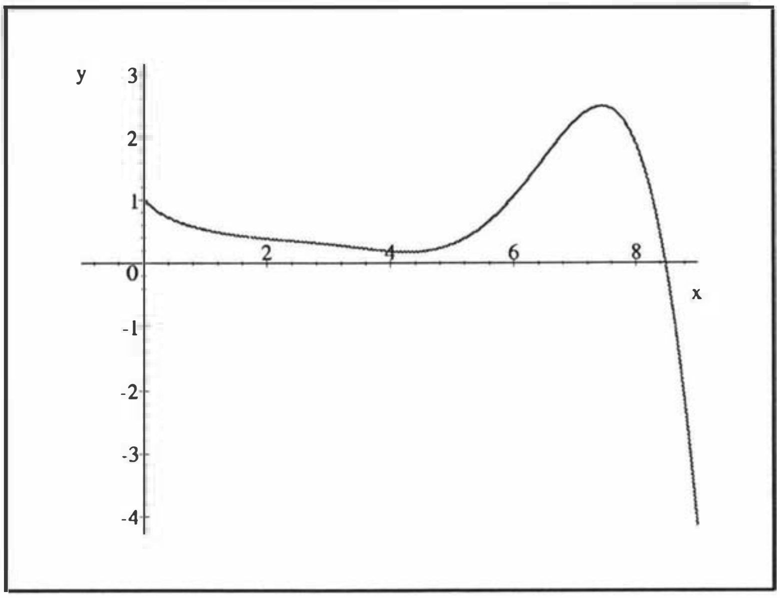

Figure 1.1.2: The solution y to the equation $y'(x) = -y(\frac{1}{2}x)$ in $x \in [0, 10]$.

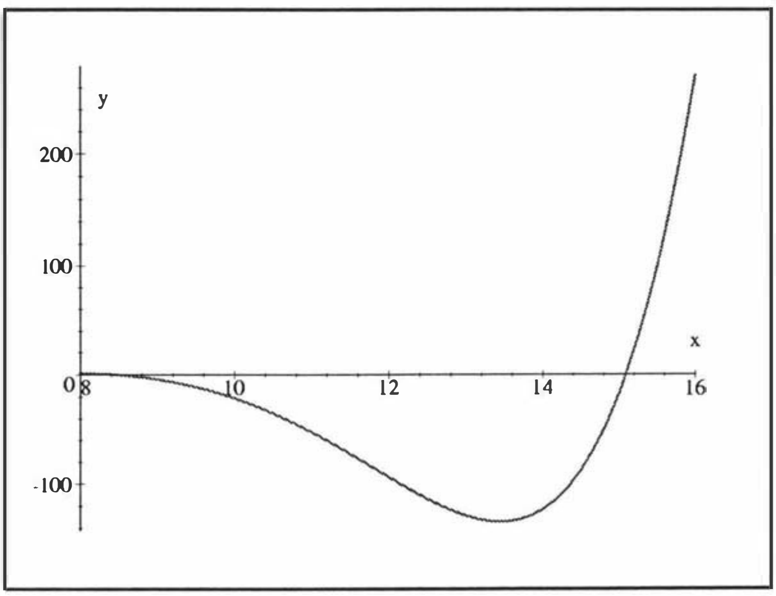

Figure 1.1.3: The solution y to the equation $y'(x) = -y(\frac{1}{2}x)$ in $x \in [8, 16]$.

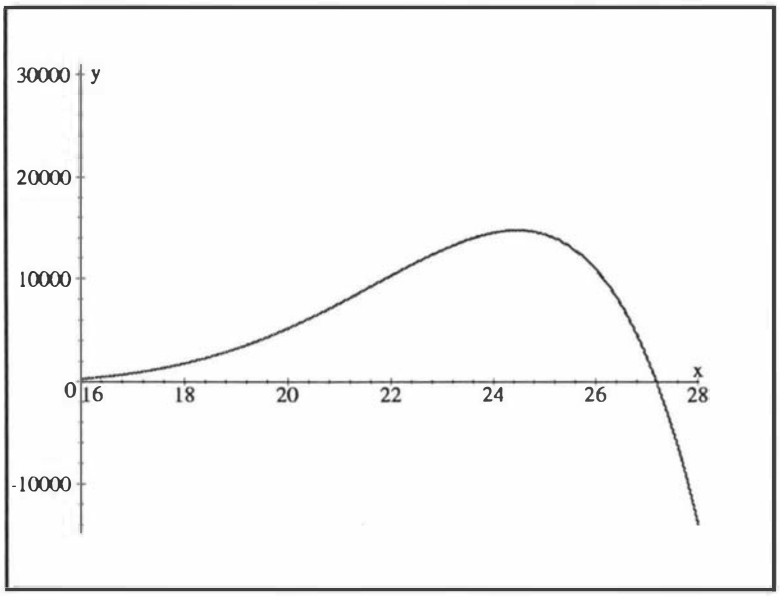

Figure 1.1.4: The solution y to the equation $y'(x) = -y(\frac{1}{2}x)$ in $x \in [16, 28]$.

1.1.2 Systems of Functional Differential Equations

J. Carr and J. Dyson [1974] generalized the asymptotic results about equations with scalar coefficients to equations with matrix coefficients. Their study was motivated by a current collection problem in which the applied force is linked to the displacement through a system of equations. We discuss briefly the asymptotic behaviour of the solutions to the equation

$$y'(x) = By(x) + Cy(\alpha x), \tag{1.11}$$

where B and C are $d \times d$ matrices such that $B^{-1}C$ is diagonalisable. If $y(x) \sim x^k V$ for a constant vector V, then we can determine k and V from the equation

$$(B + \alpha^k C)V = 0. \tag{1.12}$$

Denote the eigenvalues of diagonalisable matrix $B^{-1}C$ by $-\alpha^{-k_1}, -\alpha^{-k_2}, \ldots, -\alpha^{-k_n}$, where, for $h_i = Re(k_i)$, $h_1 \geq h_2 \geq \ldots \geq h_n$. Then the value h_1 plays the same role as that of κ in the previous section. The asymptotic behaviour of the solutions to equation (1.11) depends on the signs of the real parts of the eigenvalues of B like the scalar case depending on the signs of b and the results about the asymptotic behaviour are very similar to those classified by T. Kato and J.B. McLeod (*op.cit.*). For example, if the real parts of the eigenvalues of B are all negative, then every solution to equation (1.11) is $O(x^{h_1})$ and if there exists an eigenvalue b_1 of diagonalisable matrix B such that $Re(b_1) > 0$ and it is the maximum among the real values of the eigenvalues $b_1, b_2, \ldots, b_n$, then $y(x) = O(e^{b_1 x})$.

A. Iserles [1992] studied the generalized pantograph equation

$$y'(x) = By(x) + Cy(\alpha x) + Dy'(\alpha x), \quad y(0) = y_0, \tag{1.13}$$

where B, C, D are $d \times d$ complex matrices and y_0 is a column vector in $\mathbb{C}^d$. Note that equation (1.11) is the pure delay form of equation (1.13). An ordered pair $\{P, Q\}$ is α-canonical if, for $\sigma(P) = \{\lambda_1, \lambda_2, \dots, \lambda_d\}$ and $\sigma(Q) = \{\mu_1, \mu_2, \dots, \mu_d\}$, where $\sigma(\cdot)$ is a set of eigenvalues, it is true that $\mu_k \neq \alpha^l \lambda_j$ for all $k, j \in \{1, 2, \dots, d\}$ and $l = 1, 2, \dots$. Iserles showed that the initial problem is well posed if and only if the pair $\{D, \alpha^{-1}I\}$ is α-canonical, and in this case the solution can be expressed by the power series

$$y(x) = \left(\sum_{n=0}^{\infty} \frac{1}{n!} \prod_{m=1}^{n} \frac{B + \alpha^{m-1}C}{(I - \alpha^{m-1}D)} x^n \right) y_0. \tag{1.14}$$

It is clear that when $D = 0$, the above initial problem is always well posed since $\{0, \alpha^{-1}I\}$ is α-canonical and in this case if the coefficients of equation (1.13) are scalar and $y(0) = 1$, then the solution (1.14) is the same as the solution (1.2).

Iserles examined the asymptotic behaviour of the solution to equation (1.13) using a Dirichlet series of the form

$$Y(x) = \sum_{n=0}^{\infty} E_n e^{\alpha^n x B} V, \tag{1.15}$$

where E_n and V are $d \times d$ matrices, which are independent of x such that $detV \neq 0$. This Dirichlet series representation is valid only if

$$\{B, B\} \quad \text{is } \alpha\text{-canonical}, \tag{1.16}$$

and the spectral radius $\rho(\cdot)$ satisfies the inequality

$$\rho(-CB^{-1}) < 1. \tag{1.17}$$

He established $\lim_{x \to \infty} Y(x) = 0$ only if conditions (1.16) and (1.17) are satisfied along with the condition

$$Re(\sigma(B)) < 0.$$

Note that these conditions include the conditions $b < 0$ and $|c| < |b|$ for the scalar case. Moreover, like the periodic solution to equation (1.1) when $b = 0$, $Y(x)$ is almost periodic when $\max Re(\sigma(B)) = 0$ for a diagonalisable matrix B, along with the conditions (1.16), (1.17) and $detB \neq 0$.

These results were extended by Iserles to the higher-order pantograph equations of the form

$$Ay(x) = By(\alpha x),$$

where

$$A = \sum_{k=0}^{n} a_k \frac{d^k}{dx^k}, \qquad B = \sum_{k=0}^{n} b_k \frac{d^k}{dx^k}.$$

1.2 Advanced Functional Differential Equations

A feature of the advanced functional differential equations is that the initial condition $y(0) = y_0$ does not in contrast with the retarded case, lead to a unique solution. Indeed, T. Kato and J.B. McLeod [1971] proved that an infinite number of non-trivial solutions exists for the initial condition $y(0) = 1$. We thus can consider the problem of solving equation (1.1) with $\alpha > 1$ subject to the general initial condition $y(0) = k$, where k is possibly 0. This initial value problem also differs from the retarded case in that there are no solutions which are holomorphic at the origin. Specifically, if it is assumed that a solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ exists, then we find that

$$\frac{a_n}{a_{n-1}} = \frac{b + c\alpha^n}{n},$$

and since $\alpha > 1$, $\frac{a_n}{a_{n-1}} \to \infty$ as $n \to \infty$ so that the series is divergent for all $x \neq 0$. We refer to equation (1.1) when $\alpha > 1$ along with the condition $y(0) = k$ as *Problem 1.2.*

We start this section with introducing the first order advanced equation arising in a cell growth model studied by A.J. Hall and G.C. Wake [1989] and A.J. Hall [1991].

1.2.1 The Equation Arising in the Cell Growth Model without Dispersion

Hall and Wake (*op. cit.*) derived a cell growth model which leads to the equation

$$y'(x) = -by(x) + b\alpha y(\alpha x), \tag{1.18}$$

where $b > 0$ and $\alpha > 1$. In this model x is the size of cells so that $x \geq 0$ and the function $y(x)$ represents a probability density function. Hall and Wake assumed that each parent cell divides evenly to produce exactly α daughters all of the same size and birth/death and heredity processes are stochastic, but the growth rates are deterministic. This is called a deterministic growth model. In contrast, a model with stochastic cell growth rates leads to the advanced second order functional differential equation with a dispersion coefficient. The qualitative properties of the solution to the second order equation are similar to those to the first order equation except the initial value at $x = 0$, i.e. for the first order equation, $y(0) = 0$, while for the second order equation, $y(0) \neq 0$; the second order equation will be studied in Chapter 2.

The problem Hall and Wake considered was solving equation (1.18) subject to the "zero-flux" boundary condition

$$y(0) = 0, \qquad y(\infty) = 0, \tag{1.19}$$

and the normalizing condition

$$\int_0^{\infty} y(t)\,dt = 1. \tag{1.20}$$

They showed that there is a unique solution to equation (1.18) satisfying the conditions (1.19) and (1.20) and found it by Laplace transforms. Taking the Laplace transform of each side of equation (1.18), noting that $y(0) = 0$, leads to

$$p\bar{y}(p) = b\bar{y}(\frac{p}{\alpha}) - b\bar{y}(p),$$

where $\bar{y}(p)$ denotes the Laplace transformation of $y(x)$, and we thus have

$$\bar{y}(p) = \frac{1}{1+\frac{p}{b}}\bar{y}(\frac{p}{\alpha}), \tag{1.21}$$

with

$$\bar{y}(0) = 1.$$

Equation (1.21) implies

$$\bar{y}(\frac{p}{\alpha}) = \frac{1}{1+\frac{p}{b\alpha}}\bar{y}(\frac{p}{\alpha^2}),$$

and so repeating the arguments $n+1$ times gives

$$\bar{y}(p) = \frac{1}{\prod_{m=0}^{n}(1+\frac{p}{b\alpha^m})}\bar{y}(\frac{p}{\alpha^{n+1}}).$$

Now, $\lim_{n\to\infty}\bar{y}(\frac{p}{\alpha^{n+1}}) = \bar{y}(0) = 1$ and so we get

$$\bar{y}(p) = \frac{1}{\prod_{m=0}^{\infty}(1+\frac{p}{b\alpha^m})}. \tag{1.22}$$

Equation (1.22) can be inverted for $y(x)$ to derive the Dirichlet series solution

$$y(x) = \frac{b}{K}\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{\prod_{m=1}^{n}(\alpha^m - 1)}e^{-\alpha^n bx}, \tag{1.23}$$

where $K = \prod_{n=1}^{\infty}(1-\alpha^{-n})$. Here, we use the convention $\prod_{m=1}^{0}(\alpha^m - 1) = 1$. Note that $y(0) = 0$ since

$$\sum_{n=0}^{\infty} \frac{(-\alpha)^n}{\prod_{m=1}^{n}(\alpha^m - 1)} = \prod_{n=0}^{\infty}(1 - \alpha\alpha^{-(n+1)}) = 0,$$

where we use the Euler identity (cf. G.E. Andrews [1976] p.19). We discuss the Euler identity in detail in Chapter 4. In fact, the equation

$$y'(x) = -by(x) + \lambda y(\alpha x)$$

has a solution satisfying the conditions (1.19) and (1.20) only if $\lambda = b\alpha$. This result will be presented in Chapter 4. Hall and Wake proved that the solution $y(x)$ is unique and positive. We will show briefly that the solution has one maximum and determine some bounds on the maximum critical point. Suppose that there exists a minimum critical point x_1, then equation (1.18) yields $\alpha y(\alpha x_1) = y(x_1)$ and so $y(\alpha x_1) < y(x_1)$ since $\alpha > 1$ and y is positive. This implies that there exists a maximum critical point $x_1 < X_1 < \alpha x_1$ such that $\alpha y(\alpha X_1) = y(X_1)$. Therefore,

$$\alpha y(\alpha X_1) = y(X_1) > y(x_1) = \alpha y(\alpha x_1),$$

and hence $y(\alpha X_1) > y(\alpha x_1)$. This inequality indicates the existence of another minimum critical point $x_2 > x_1$ satisfying $y(\alpha x_2) < y(x_2)$. This process can be repeated to obtain the sequences $\{x_n\}$ and $\{X_n\}$ such that $x_n \to \infty$, $X_n \to \infty$ and $X_n > x_n$, i.e., the solution is oscillating from the first minimum critical point. However, the Dirichlet series solution implies that $y(x) \sim \frac{b}{K}e^{-bx}$ as $x \to \infty$ so that the solution is decreasing for large x. Thus, there is no minimum critical point and exactly one maximum critical point since $y(0) = 0$.

Let us obtain the bounds on the maximum critical point X_m. Integrating equation (1.18) from 0 to X_m yields

$$y(X_m) = b\int_{X_m}^{\alpha X_m} y(x)\,dx < b(\alpha - 1)X_m y(X_m), \tag{1.24}$$

and

$$y(X_m) > b(\alpha - 1)X_m y(\alpha X_m).$$

Since y achives a maximum at $x = X_m$, equation (1.18) implies that $y(X_m) = \alpha y(\alpha X_m)$ so that

$$y(X_m) > \frac{b(\alpha - 1)}{\alpha} X_m y(X_m). \tag{1.25}$$

Combining the inequalities (1.24) and (1.25) leads to the bounds

$$\frac{1}{b(\alpha - 1)} < X_m < \frac{\alpha}{b(\alpha - 1)}.$$

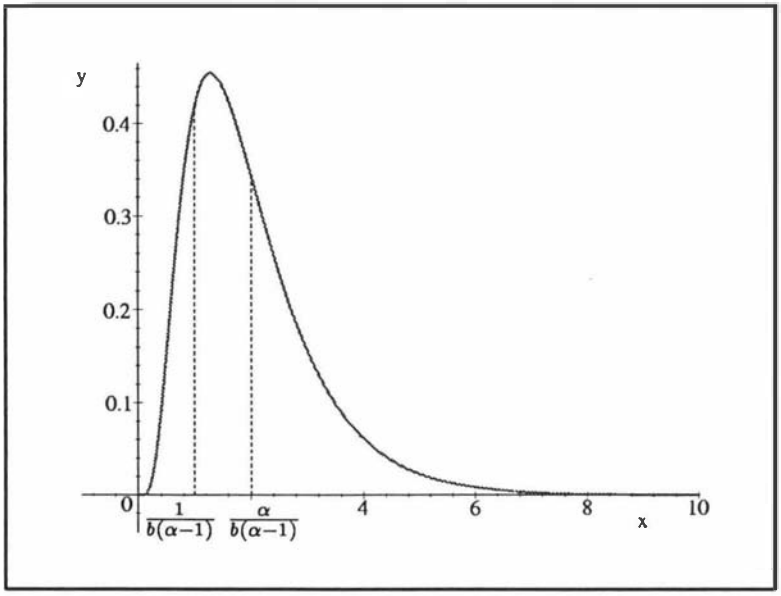

Figure 1.1.5: The solution y to the equation $y'(x) = -y(x) + y(2x)$.

Figure 1.1.5 illustrates the solution which has one maximum for the case when $\alpha = 2$ and $b = 1$.

"Faster decaying" solutions to the advanced equation are typically represented by a Dirichlet series. Indeed, we can derive the solution (1.23) by substituting a Dirichlet series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n e^{-\alpha^n r x},$$

into equation (1.18). Here, the coefficients a_n and the number r can be determined by equating terms of like exponential coefficients, i.e. the sequence $\{e^{-\alpha^n r x}\}$ is regarded as a basis for the solution. The process leads to the equation $r = b$, and the recurrence equation

$$\frac{a_n}{a_{n-1}} = \frac{-\alpha}{\alpha^n - 1}.$$

This approach is more direct than the Laplace transformation method (though essentially equivalent) and we will use it to solve advanced second order equations in later chapters. Generally, we can expect Dirichlet series solutions, and some conditions for the existence of a Dirichlet series solution to advanced equations are discussed in the next subsection.

1.2.2 Dirichlet Series Solutions

P. Frederickson [1971] showed that Problem 1.2 has a solution which is continuous in the closed half plane $Re(bx) \leq 0$, analytic in its interior, and is also in $L_p[0,\infty)$ for $1 \leq p \leq \infty$ if $Re(b) < 0$. He proved these results using a Dirichlet series of the form

$$\phi(x,\beta) = \sum_{n}^{\infty} a_n e^{\beta\alpha^n x}, \tag{1.26}$$

involving two variables x and β. If the above series is substituted into equation (1.1), then

$$(\alpha^n\beta - b)a_n = ca_{n-1}. \tag{1.27}$$

Suppose $\beta = b\alpha^{-l}$, $b \neq 0$ for some integer l, then the recurrence relation (1.27) becomes

$$a_n = \frac{c/b}{\alpha^{n-l}-1}a_{n-1}, \quad n > l. \tag{1.28}$$

This relation indicates that equation (1.1) has a Dirichlet series solution in $L_p[0,\infty)$ for $1 \leq p \leq \infty$ provided $Re(b) < 0$. Note that if $Re(b) = 0$, then the solutions are almost periodic on $(-\infty,\infty)$. Now if $|b|/|c| < \mu < 1$, the recurrence relation (1.28) leads to $|a_{n-1}| < \mu|a_n|$ for all large negative n, and so $|a_n| < K\alpha^n$ for some positive constant K and all negative n. Therefore, if $|b| < |c|$ and $\beta \neq b\alpha^{-N}$ for a positive integer N, then we have a one parameter family of Dirichlet series solutions $\phi(x,\beta)$ in $L_p[0,\infty)$ where $\max(1, \frac{log\alpha}{log|c/b|}) < p \leq \infty$ if $Re(\beta) < 0$. When $|b| < |c|$, Bowen suggested a solution of the form

$$y(x) = C\sum_{n=0}^{\infty}\prod_{m=1}^{n}(\gamma\alpha^m - 1)^{-1}(\frac{c}{b})^n e^{b\gamma\alpha^n x} + C'\sum_{n=1}^{\infty}\prod_{m=0}^{n-1}(\gamma\alpha^{-m} - 1)(\frac{b}{c})^n e^{b\gamma\alpha^{-n}x},$$

where C, C' are real constants, $\gamma \neq \alpha^{-N}$ for any positive integer N and $Re(b\gamma) < 0$. (cf. L. Fox, D.F. Mayers, J.R. Ockendon and A.B. Tayler [1971]).

Note that if $|b| \geq |c|$ and $Re(b) < 0$, then there is the only one solution represented by a Dirichlet series and it has the recurrence relation (1.28) (cf. P. Frederickson [1971]).

An example for the case when $|b| = |c|$ and $b < 0$ appears in a paper on probability theory by Ferguson [1971]. He showed there is a unique solution to the equation and the boundary conditions

$$y'(x) = y(\alpha x) - y(x), \qquad y(0) = 0, \quad y(\infty) = 1,$$

such that the solution can be expressed as

$$y(x) = 1 - h \sum_{n=0}^{\infty} c_n e^{-\alpha^n x},$$

where $c_0 = 1$, $c_n = \prod_{m=1}^{n} (1 - \alpha^m)^{-1}$ and $h = (\sum_{n=0}^{\infty} c_n)^{-1}$.

For the case $b = 0$, G.R. Morris, A. Feldstein and E.W. Bowen [1972] examined a Dirichlet series solution of the form

$$y(x) = \sum_{n=-\infty}^{\infty} a_n e^{-r_n x},$$

where $x \in R$. Substituting the above series into the equation

$$y'(x) = y(\alpha x), \tag{1.29}$$

gives

$$r_{n+1} = \frac{1}{\alpha} r_n, \quad \text{and} \quad a_{n+1} = -r_n a_n.$$

From these relations, there exists a solution

$$y(x) = \sum_{n=-\infty}^{\infty} (-1)^n r_0^n \alpha^{-\frac{1}{2}n(n-1)} e^{-r_0 \alpha^{-n} x},$$

provided $Re(r_0 x) \geq 0$, i.e.

$$\begin{aligned} Re(r_0) &> 0 \quad \text{if } x \geq 0, \\ Re(r_0) &< 0 \quad \text{if } x \leq 0. \end{aligned}$$

This result implies that all solutions valid for $x \geq 0$ can be joined continuously to the solutions valid for $x \leq 0$. The value at $x = 0$, however, is not uniquely determined.

Yet another interesting representation of the solutions to Problem 1.2 when $b = 0$, has been suggested by Bowen (cf. L. Fox, D. Mayers, J. Ockendon and A. Tayler [1971]) as follows:

$$y_\beta(x) = C_\beta \sum_{n=-\infty}^{\infty} (-c)^{-n} \alpha^{-\frac{1}{2}(n+\beta)^2} e^{-\alpha^{-n-\beta-\frac{1}{2}} x},$$

where $\beta \in R$ and C_β is some constant.

P.O. Frederickson [1971] examined the equation

$$y'(x) = F(y(2x)), \tag{1.30}$$

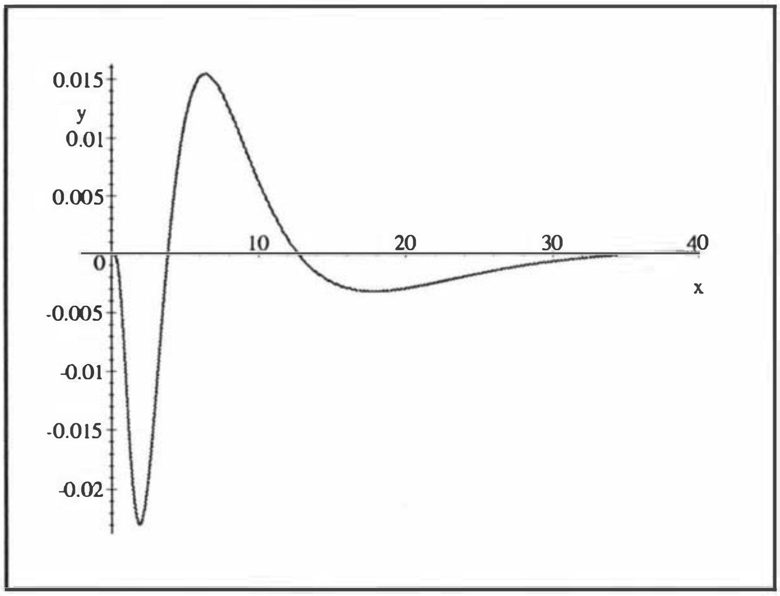

Figure 1.1.6: The solution y to the equation $y'(x) = -y(2x)$.

where $F(x) > 0$ for $x > 0$, $F(-x) = -F(x)$ and $F(x)$ is continuous, and proved there is a solution $y(x)$ with the property that $|y(x)|$ is periodic on R^+. If we set $F(x) = x$, then equation (1.30) is the special case of equation (1.29). Figure 1.1.6 depicts a solution $y(x)$ to Problem 1.2 when $\alpha = 2$, $b = 0$ and $c = -1$.

The theory underlying the advanced equation (1.1) can be extended to the equation of the form

$$y'(x) = by(x) + \sum_{n=0}^{\infty} c_n y(\alpha_n x),$$

containing an infinite number of functional terms.
Like the previous case, the above equation has a Dirichlet series solution analytic in a half plane if $Re(b) < 0$ (cf. P. Frederickson [1971]).

1.2.3 Equations with variable coefficients

A generalization of equation (1.1) consists of variable coefficients instead of constant coefficients. In the cell growth model devised by A.J. Hall and G.C. Wake [1990] and A.J. Hall [1991], the coefficients correspond to growth and birth rates which are not generically constant. This generalization has not been studied as intensely as the constant coefficients case and we first present some results concerning this case based on the work of Hall and Wake (*op. cit.*) and then the more general case will be considered. Hall and Wake studied the equation with variable coefficients of a polynomial form. Specifically, let $k > 0$ and consider the boundary value problem

$$Z'(x) = x^{k-1}(-bZ(x) + b\alpha^k Z(\alpha x)), \tag{1.31}$$

satisfying

$$Z(0) = 0, \quad Z(\infty) = 0,$$

and

$$\int_0^\infty x^{k-2} Z(x)\, dx = \frac{1}{b(\alpha - 1)}. \tag{1.32}$$

In fact, $Z(x) = x^2 y(x)$, where $y(x)$ is a probability density function, but here we concentrate on the solution $Z(x)$ to the boundary value problem.

A solution to this equation can be obtained using substitutions $z = \frac{x^k}{k}$ and $Y(z) = Z(x)$, and equation (1.31) can be converted into the equation

$$Y'(z) = -bY(z) + b\alpha^k Y(\alpha^k z), \tag{1.33}$$

which is essentially the same as the previous equation with constant coefficients. Hence, the solution to equation (1.31) is given by

$$Y(z) = C \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{kn}}{\prod_{m=1}^{n} (\alpha^{km} - 1)} e^{-b\alpha^{kn} z},$$

where C is determined by the condition (1.32), so that

$$Z(x) = C \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{kn}}{\prod_{m=1}^{n} (\alpha^{km} - 1)} e^{-b\frac{\alpha^{kn} x^k}{k}}. \tag{1.34}$$

Clearly,

$$Z(0) = C \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{kn}}{\prod_{m=1}^{n} (\alpha^{km} - 1)} = \prod_{n=0}^{\infty} (1 - \alpha^{-kn}) = 0,$$

where we use the Euler identity. It can be shown the solution is unique, positive and has only one maximum using the same method as that used in the proof of the constant case. The bounds on the maximum critical point are obtained by integrating equation (1.31) from 0 to X_m so that

$$\frac{1}{\sqrt[k]{b(\alpha - 1)}} < X_m < \frac{\alpha}{\sqrt[k]{b(\alpha - 1)}}.$$

Figure 1.1.7 depicts solutions to equation (1.31) corresponding to $k = 1/2$, $k = 1$ and $k = 2$ when $\alpha = 2$ and $b = 1$.

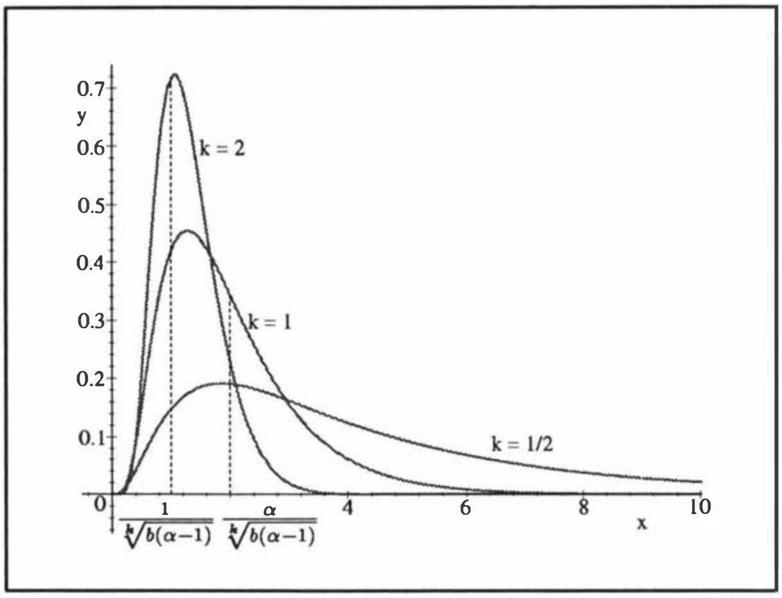

Figure 1.1.7: The solution y to the equation $Z'(x) = x^{k-1}(-Z(x) + 2^k Z(2x))$ when $k = 1/2$, $k = 1$ and $k = 2$ so that $b(\alpha - 1) = 1$.

Let us consider the more general equation

$$y'(x) = -p(x)y(x) + \lambda q(x)y(\alpha x), \tag{1.35}$$

where λ is a constant and $x \geq 0$, satisfying the condition

$$y(\infty) = 0.$$

Multiplying each side of equation (1.35) by $e^{\int_0^x p(\eta)d\eta}$ leads to

$$\frac{d}{dx}(y(x)e^{\int_0^x p(\eta)d\eta}) = e^{\int_0^x p(\eta)d\eta}\lambda q(x)y(\alpha x),$$

and integrating the above equation from 0 to x implies that

$$y(x)e^{\int_0^x p(\eta)d\eta} = y(0) + \frac{\lambda}{\alpha}\int_0^{\alpha x} e^{\int_0^{s/\alpha} p(\eta)d\eta} q(s/\alpha)y(s)ds.$$

Therefore,

$$y(x) = y(0)e^{-P(x)} + \frac{\lambda}{\alpha}\int_0^{\alpha x} e^{-P(x)+P(s/\alpha)} q(s/\alpha)y(s)ds, \tag{1.36}$$

where $P(x) = \int_0^x p(\eta)d\eta$. Let $f(x) = y(0)e^{-P(x)}$ and $Ky(x) = \int_0^\infty K(x,s)y(s)\,ds$, where

$$K(x,s) = \begin{cases} \frac{1}{\alpha}e^{-P(x)+P(s/\alpha)}q(s/\alpha) & \text{if } 0 < s < \alpha x, \\ 0 & \text{if } s > \alpha x. \end{cases}$$

Then

$$y(x) = f(x) + \lambda K y(x), \tag{1.37}$$

and this is a Fredholm equation of the second kind. It is known that if K is a bounded operator with the property

$$\|Ky_1 - Ky_2\| \le M\|y_1 - y_2\|, \quad M < \infty,$$

then equation (1.37) has a unique solution $y \in L_1[0,\infty)$ for all $f \in L_1[0,\infty)$ and sufficiently small $|\lambda|$. In fact, the exact range of λ is obtained by the contraction mapping theorem.

The next theorem shows some conditions for the functions $p(x)$ and $q(x)$ which the above method can be applied.

Theorem 1.2.1 *Let $p(x)$ be a nondecreasing positive function on $[0,\infty)$ and $|q(x)| \le p(x)$ for all $x \in [0,\infty)$. Then for any λ such that $|\lambda| < \alpha$, there is precisely one solution in $L_1[0,\infty)$ of equation (1.35) satisfying the condition $y(\infty) = 0$.*

Proof: Let

$$Ty = y(0)e^{-P(x)} + \frac{\lambda}{\alpha}\int_0^{\alpha x} e^{-P(x)+P(s/\alpha)} q(s/\alpha) y(s)ds.$$

Then for $y \in L_1[0,\infty)$,

$$\|Ty\| \le C + \frac{|\lambda|}{\alpha}\int_0^{\infty}\int_0^{\alpha x} e^{-P(x)+P(s/\alpha)} q(s/\alpha)|y(s)|dsdx,$$

where $C = |y(0)|\int_0^\infty e^{-P(x)}dx$. Here, $p(x)$ is positive so that $P(x) = \int_0^x p(\eta)d\eta$ is a positive and strictly increasing function and therefore $e^{-P(x)} \in L_1[0,\infty)$; thus, $C < \infty$. Now, the order of integration can be changed and $|q(s/\alpha)| \le p(s/\alpha) \le p(x)$ for all $s \in [0, \alpha x)$ since $p(x)$ is nondecreasing so that

$$\begin{aligned}
\|Ty\| &\le C + \frac{|\lambda|}{\alpha}\int_0^\infty \left(\int_{s/\alpha}^\infty e^{-P(x)+P(s/\alpha)} p(x)dx\right)|y(s)|ds \\
&= C + \frac{|\lambda|}{\alpha}\int_0^\infty |y(s)|ds \\
&= C + \frac{|\lambda|}{\alpha}\|y\|.
\end{aligned}$$

This implies that T maps $L_1[0,\infty)$ into $L_1[0,\infty)$. Now for $y_1, y_2 \in L_1[0,\infty)$,

$$\begin{aligned}
\|Ty_1 - Ty_2\| &= \frac{|\lambda|}{\alpha}\int_0^\infty \left|\int_0^{\alpha x} e^{-P(x)+P(s/\alpha)} q(s/\alpha)(y_1(s) - y_2(s))ds\right| dx \\
&\le \frac{|\lambda|}{\alpha}\int_0^\infty \left(\int_{s/\alpha}^\infty e^{-P(x)+P(s/\alpha)} p(x)dx\right)|y_1(s) - y_2(s)|ds \\
&= \frac{|\lambda|}{\alpha}\|y_1 - y_2\|.
\end{aligned}$$

Therefore, we get the result from the contraction mapping theorem.■

Corollary 1.2.2 *Let $p(x)$ be defined as in Theorem 1.2.1. Then for $|\lambda| < \alpha$, the equation*

$$y'(x) + p(x)(y(x) - \lambda y(\alpha x)) = 0,$$

satisfying the condition $y(\infty) = 0$, has precisely one solution in $L_1[0, \infty)$.

Corollary 1.2.3 *Suppose $p(x)$ is defined as in Theorem 1.2.1 and for some constant L, $p(\alpha x) \leq Lp(x)$. Then for $|\lambda| < \alpha/L$, the equation*

$$y'(x) + p(x)y(x) - \lambda p(\alpha x)y(\alpha x) = 0,$$

satisfying the condition $y(\infty) = 0$, has precisely one solution in $L_1[0, \infty)$.

We note that the equation in Corollary 1.2.3 is motivated by the cell growth model where $p(x)$ is a polynomial with positive constants, i.e.

$$p(x) = \sum_{m=0}^{n} a_m x^m,$$

where $a_m \geq 0$ for $m = 0, 1, \ldots, n$. Evidently, $p(x)$ is a nondecreasing and positive function, and $p(\alpha x) \leq \alpha^n p(x)$ for all $x \in [0, \infty)$.

Now, for $f \in L_1[0, \infty)$, the solution can be expressed as a Neumann series such that

$$y = f + \lambda K f + \lambda^2 K^2 f + \ldots + \lambda^n K^n f + \ldots,$$

where

$$K^n f(x) = \int_0^\infty K_n(x, y) f(y)\, dy,$$

and

$$\begin{aligned} K_n(x, y) &= \int_0^\infty K(x, z) K_{n-1}(z, y)\, dz, \qquad n = 2, 3, \ldots, \\ K_1(x, y) &= K(x, y). \end{aligned}$$

As an example, we consider the equation

$$y'(x) = -xy(x) + \lambda x y(\alpha x), \tag{1.38}$$

where $\alpha > 1$. The solution $y(x)$ can be expressed as

$$y(x) = f(x) + \lambda \int_0^\infty K(x, s) y(s)\, ds,$$

where $f(x) = y(0)e^{-\frac{1}{2}x^2}$ and

$$K(x,s) = \begin{cases} \frac{s}{\alpha^2}e^{-\frac{1}{2}x^2+\frac{1}{2\alpha^2}s^2} & \text{if } 0 < s < \alpha x, \\ 0 & \text{if } s > \alpha x. \end{cases}$$

Since $f(x) \in L_1[0,\infty]$, we calculate terms of the Neumann series to obtain a basis of the solution. The second term of the Neumann series is

$$\begin{aligned} \lambda K f(x) &= \lambda \int_0^\infty K(x,s) f(s)\, ds \\ &= \lambda \int_0^{\alpha x} \frac{s}{\alpha^2} e^{-\frac{1}{2}x^2+\frac{1}{2\alpha^2}s^2} y(0) e^{-\frac{1}{2}s^2} ds \\ &= y(0)\frac{\lambda}{\alpha^2} e^{-\frac{1}{2}x^2} \int_0^{\alpha x} s e^{-\frac{1}{2}(1-\frac{1}{\alpha^2})s^2}\, ds \\ &= y(0)\frac{\lambda}{1-\alpha^2}[e^{-\frac{1}{2}\alpha^2 x^2} - e^{-\frac{1}{2}x^2}]. \end{aligned}$$

Now, for $y < \alpha^2 x$,

$$\begin{aligned} K_2(x,y) &= \int_0^\infty K(x,s)K(s,y)\, ds \\ &= \frac{y}{\alpha^4} e^{-\frac{1}{2}x^2+\frac{1}{2\alpha^2}y^2} \int_{y/\alpha}^{\alpha x} s e^{-\frac{1}{2}(1-\frac{1}{\alpha^2})s^2}\, ds \\ &= \frac{y}{\alpha^2(1-\alpha^2)} e^{-\frac{1}{2}x^2+\frac{1}{2\alpha^2}y^2} [e^{-\frac{1}{2}(\alpha^2-1)x^2} - e^{-\frac{1}{2}\frac{\alpha^2-1}{\alpha^4}y^2}] \\ &= \frac{y}{\alpha^2(1-\alpha^2)} [e^{-\frac{1}{2}\alpha^2 x^2+\frac{1}{2\alpha^2}y^2} - e^{-\frac{1}{2}x^2+\frac{1}{2\alpha^4}y^2}], \end{aligned}$$

and for $y > \alpha^2 x$, $K_2(x,y) = 0$. So,

$$\begin{aligned} \lambda^2 K^2 f(x) &= \lambda^2 \int_0^\infty K_2(x,s) f(s) ds \\ &= y(0)\frac{\lambda^2}{\alpha^2(1-\alpha^2)} e^{-\frac{1}{2}\alpha^2 x^2} \int_0^{\alpha^2 x} s e^{-\frac{1}{2}(1-\frac{1}{\alpha^2})s^2} ds \\ &\quad -y(0)\frac{\lambda^2}{\alpha^2(1-\alpha^2)} e^{-\frac{1}{2}x^2} \int_0^{\alpha^2 x} s e^{-\frac{1}{2}(1-\frac{1}{\alpha^4})s^2} ds \\ &= y(0)\frac{\lambda^2}{(1-\alpha^2)^2}[e^{-\frac{1}{2}\alpha^4 x^2} - e^{-\frac{1}{2}\alpha^2 x^2}] \\ &\quad -y(0)\frac{\alpha^2\lambda^2}{(1-\alpha^4)(1-\alpha^2)}[e^{-\frac{1}{2}\alpha^4 x^2} - e^{-\frac{1}{2}x^2}] \\ &= y(0)\frac{\lambda^2}{1-\alpha^2}[-\frac{\alpha^2}{1-\alpha^4}e^{-\frac{1}{2}x^2} \\ &\quad -\frac{1}{1-\alpha^2}e^{-\frac{1}{2}\alpha^2 x^2} + (1-\frac{\alpha^2}{1-\alpha^4})e^{-\frac{1}{2}\alpha^4 x^2}]. \end{aligned}$$

A basis for the solution to equation (1.38) is $e^{-\frac{1}{2}x^2}, e^{-\frac{1}{2}\alpha^2x^2}, e^{-\frac{1}{2}\alpha^4x^2}, \ldots$ and so we have a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n e^{-\frac{1}{2}\alpha^{2n}x^2}.$$

Substituting the series into equation (1.38) leads to the recurrence relation

$$(\alpha^{2n} - 1)a_n = -\lambda a_{n-1},$$

and so

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{2m} - 1)} e^{-\frac{1}{2}\alpha^{2n}x^2}, \tag{1.39}$$

where a_0 can be determined by the condition $\int_0^\infty y(x)\,dx = 1$ so that

$$a_0 = \frac{\sqrt{2}}{\Gamma(1/2)} \left(\sum_{n=0}^{\infty} \frac{(-\lambda/\alpha)^n}{\prod_{m=1}^{n}(\alpha^{2m} - 1)} \right)^{-1}.$$

This solution is not in the form of the Neumann series since a_0 depends on λ. In order to determine the value λ for $y(0) = 0$, we use the Euler identity. Since

$$\begin{aligned} y(0) &= a_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{2m} - 1)} \\ &= a_0 \prod_{n=0}^{\infty} (1 - \frac{\lambda}{\alpha^{2(n+1)}}) \\ &= \frac{\sqrt{2}}{\Gamma(1/2)} W^{-1}(\lambda/\alpha) W(\lambda), \end{aligned}$$

we have $y(0) = 0$ when $\lambda = \alpha^{2n}$ for $n = 1, 2, \ldots$. We note that if $\lambda = \alpha^2$, then the solution (1.39) is the same as the solution (1.34) when $b = 1$ and $k = 2$ provided that the same integral condition is prescribed.

On the other hand, the Neumann series solution implies that $y = y(x, \lambda)$ is holomorphic in λ sufficiently small, and so we can express the solution in the form

$$y(x) = \sum_{n=0}^{\infty} \lambda^n y_n(x).$$

Substituting the series into equation (1.35) leads to the equations

$$y_0'(x) + p(x)y_0(x) = 0,$$

and for $n \geq 1$,

$$y_{n+1}'(x) + p(x)y_{n+1}(x) = q(x)y_n(\alpha x).$$

As an example, we consider equation (1.38). Since

$$y_0(x) = a_0 e^{-\int_0^x \eta d\eta} = a_0 e^{-\frac{1}{2}x^2},$$

we have $y_1'(x) + xy_1(x) = a_0 x e^{-\frac{1}{2}\alpha^2 x^2}$ so that

$$y_1(x) = \frac{a_0}{1-\alpha^2} e^{-\frac{1}{2}\alpha^2 x^2}.$$

Now $y_2'(x) + xy_2(x) = \frac{a_0}{1-\alpha^2} x e^{-\frac{1}{2}\alpha^4 x^2}$ and so we have

$$y_2(x) = \frac{a_0}{(1-\alpha^4)(1-\alpha^2)} e^{-\frac{1}{2}\alpha^4 x^2}.$$

Repeating this process produces the solution (1.39) to equation (1.38). We use this method in Chapter 5 to seek a series solution to special classes of second order equations with variable coefficients.

1.2.4 Asymptotic Behaviour of Solutions

Like the retarded equations, the asymptotic behaviour of advanced equations depends crucially on the signs of b. Let κ and k_0 be the same as defined in the case of retarded equations. We first consider the case $b < 0$. It is generally known that Problem 1.2 has a solution decaying like Le^{bx} as $x \to \infty$ and the solution is of the form

$$y_L(x) = Le^{bx}\{1 + \sum_{n=1}^{\infty} \frac{c^n e^{-b(1-\alpha^n)x}}{b^n \prod_{m=1}^{n}(\alpha^m - 1)}\},$$

where L is determined by an initial condition. T. Kato and J.B. McLeod [1971] showed that a constant multiple of y_L is the only solution satisfying $y = o(x^\kappa)$ as $x \to \infty$ and stated the existence of a slower decay solution than y_L. If g is periodic of period $Log\alpha$ and Hölder-continuous with exponent θ, $0 < \theta \leq 1$, then they showed that there is a solution such that

$$y(x) = x^{k_0} g(Logx) + O(x^{\kappa-\theta}) \text{ as } x \to \infty, \tag{1.40}$$

and this is unique up to addition of a constant multiple of y_L. They also proved that if $g(Logx)$ has an q^{th} derivative that is Hölder-continuous with exponent θ, then $y(x)$ has the asymptotic form

$$y(x) = x^{k_0}\{g(Logx) + \sum_{n=1}^{q} \frac{x^{-n} g_n(Logx)}{b^n \prod_{m=1}^{n}(1-\alpha^{-m})} + O(x^{-q-\theta})\}, \tag{1.41}$$

where $g'_n = -(k_0 - n)g_n + g_{n+1}$ and $g_0 = g$.

For the case $b > 0$, Kato and McLeod proved that there is no solution satisfying $o(x^\kappa)$ as $x \to \infty$, but there is a solution (1.40) and an asymtotic form (1.41) provided that $g(Logx)$ satisfies the same conditions as those in the previous case that $b < 0$.

Lastly, for the case when $b = 0$, they showed that for $\phi(Logx)$ given by (1.7), there is no solution $o(\phi(Logx))$ as $x \to \infty$. However, if g is periodic of period $Log\alpha$ and g' is Hölder-continuous with exponent θ, $0 < \theta < 1$, there is the only one solution of the form

$$y(x) = \phi(Logx)\{g(Logx - LogLogx) + O[(Logx)^{-\theta}]\}.$$

We now conclude that equation (1.1) when $\alpha > 1$ can have a solution decaying to zero as x goes to infinity only if

$$b < 0.$$

1.2.5 Systems of Functional Differential Equations

L. Fox, *et al.* [1971] investigated the asymptotic behaviour of solutions to the equations with matrix coefficients when $\alpha > 0$. They considered the asymptotic behaviour of solutions according to whether the coefficients are non-singular or singular. For the case $0 < \alpha < 1$, we reviewed the asymptotic behaviour studied by J. Carr and J. Dyson [1974] in the previous subsection, and the results examined in two papers are essentially the same if the coefficients are non-singular. Here, we will introduce the results obtained by Fox, *et al.* (*op.cit.*) for the case that $\alpha > 1$.

Consider equation (1.11) when $\alpha > 1$. For non-singular B and C, L. Fox, *et al.* showed that there are two possibilities for an asymptotic decay form. If β_s denotes an eigenvalue of B and b_s is the corresponding eigenvector,

$$y \sim b_s e^{\beta_s x},$$

provided $Re\beta_s < 0$. For $(B + \alpha^v C)a_s = 0$,

$$y \sim a_s x^v,$$

so that $v = \frac{log(-r_s)}{Log\alpha}$, where r_s is an eigenvalue of BC^{-1}, and y decays if $|r_s| < 1$. Equation (1.11) can have an non oscillating solution only if r_s is real and negative.

They obtained other forms when B or C is singular.

If C has rank $R_C < d$ and it is diagonalisable, then they proved that a linear transformation can be found such that the system becomes

$$\frac{d\bar{y}}{dx} = \bar{B}\bar{y}(x) + (\gamma_i\delta_{ij})\bar{y}(\alpha x),$$

where $\gamma_i = 0$ for $i = R_C + 1, \dots, d$. Let

$$\bar{y} = \begin{bmatrix} \bar{y}_0 \\ \bar{y}_1 \end{bmatrix}, \quad \bar{C}_0 = diag(\gamma_1, \gamma_2, \dots, \gamma_{R_C}), \quad \bar{B} = \begin{bmatrix} \bar{B}_0 & \bar{B}_2 \\ \bar{B}_3 & \bar{B}_1 \end{bmatrix},$$

where $\bar{y}_0$ contains the first R_C components of $\bar{y}$ and $\bar{y}_1$ contains the last $d - R_C$ components of $\bar{y}$. The matrix $\bar{B}_1$ is the reduced matrix obtained from $\bar{B}$ by removing its first R_C rows and columns. Suppose $\bar{B}_k$ are non-singular for $k = 0, 1, 2, 3$. Let $Y_1 \gg Y_2$ denote the relation that $Y_1(x) \geq Y_2(x)$ as $x \to \infty$. Then for the case that $|\bar{y}_1| \gg |\bar{y}_0|$, we have

$$\begin{aligned} \frac{d}{dx}\bar{y}_0(x) &= \bar{B}_2\bar{y}_1(x) + \bar{C}_0\bar{y}_0(\alpha x), \\ \frac{d}{dx}\bar{y}_1(x) &= \bar{B}_1\bar{y}_1(x), \end{aligned}$$

so that

$$|\bar{y}_0(x)| \sim e^{\bar{\beta}_s x/\alpha}, \quad |\bar{y}_1(x)| \sim e^{\bar{\beta}_s x},$$

where $\bar{\beta}_s$ is an eigenvalue of $\bar{B}_1$, provided that $Re(\bar{\beta}_s) > 0$. However, when $|\bar{y}_1| \ll |\bar{y}_0|$, there are no asymptotic forms of the solution.

For the case that B has rank $R_B < d$ and it is diagonalisable, they transformed the system to

$$\frac{d\hat{y}}{dx} = (\hat{\beta}_i\delta_{ij})\hat{y}(x) + \hat{C}\hat{y}(\alpha x),$$

where $\hat{\beta}_i = 0$ for $i = R_B + 1, \dots, d$. Using a notation similar to those used in the previous case, i.e.

$$\hat{y} = \begin{bmatrix} \hat{y}_0 \\ \hat{y}_1 \end{bmatrix}, \quad \hat{B}_0 = diag(\hat{\beta}_1, \hat{\beta}_2, \dots, \hat{\beta}_{R_B}), \quad \hat{C} = \begin{bmatrix} \hat{C}_0 & \hat{C}_2 \\ \hat{C}_3 & \hat{C}_1 \end{bmatrix},$$

they showed that there are the following asymptotic forms if the $\hat{C}_k$ are nonsingular for $k = 0, 1, 2, \dots$: If $|\hat{y}_1| \gg |\hat{y}_0|$, then

$$\begin{aligned} \frac{d}{dx}\hat{y}_0(x) &= \hat{B}_0\hat{y}_0(x) + \hat{C}_2\hat{y}_1(\alpha x), \\ \frac{d}{dx}\hat{y}_1(x) &= \hat{C}_1\hat{y}_1(\alpha x), \end{aligned}$$

and if $|\hat{y}_1| \ll |\hat{y}_0|$, then

$$\begin{aligned} \frac{d}{dx}\hat{y}_0(x) &= \hat{B}_0\hat{y}_0(x) + \hat{C}_0\hat{y}_0(\alpha x), \\ \frac{d}{dx}\hat{y}_1(x) &= \hat{C}_3\hat{y}_0(\alpha x). \end{aligned}$$

From these forms, we get the asymptotic forms

$$|\hat{y}_0| \sim exp\{-\frac{1}{2}\frac{(Log\alpha x)^2}{Log\alpha}+\dots\},$$
$$|\hat{y}_1| \sim exp\{-\frac{1}{2}\frac{(Log x)^2}{Log\alpha}+\dots\},$$

or

$$|\hat{y}_0| \sim e^{\hat{\beta}_s x}, \quad |\hat{y}_1| \sim e^{\alpha\hat{\beta}_s x},$$

provided that $Re(\hat{\beta}_s) < 0$.

They also obtained further forms using a perturbation method, which may occur for the case when any of $\hat{C}_k$, $\bar{B}_k$ are singular or B, C are not diagonalisable matrices.

1.3 Equations with Advanced and Retarded Terms

G.A. Derfel and S.A. Molchanov [1990] and G.A. Derfel [1990] studied the equation

$$y''(x) = y(x/\alpha) + y(\alpha x) + \lambda y(x), \tag{1.42}$$

where $\alpha > 1$, having bounded solutions. In order to get the bounded solutions to equation (1.42), they converted the problem into the Schrödinger difference equation

$$a_{n+1} + a_{n-1} + [\lambda - \alpha^{2n}w^2]a_n = 0,$$

where $w \in [1, \alpha]$, by substituting the series

$$y(x) = \sum_{n=-\infty}^{\infty} a_n e^{-w\alpha^n x},$$

into equation (1.42). With this observation, Derfel and Molchanov (*op.cit.*) obtained some results about bounded solutions to the more general equation

$$y''(x) = \sum_{n=0}^{1} c_n y(\alpha_n x) + \lambda y(x), \tag{1.43}$$

where $\alpha_n = q^{r_n}$ for $q > 1$ and $r_n \neq 0$ are rational numbers. They proved that there exists $K > 0$ such that if $\lambda < -K$, equation (1.43) has a non-trivial almost periodic solution and any bounded solutions are periodic. They also proved that if $\lambda > K$, then equation (1.43) has no solution bounded on the whole axis.

We note that for the case when $\alpha_n > 1$, $n = 1, 2$, we can find a sum of two Dirichlet series solutions (cf. Appendix B) and the solution does not exist on the whole axis if $\lambda > 0$, but there exists a bounded oscillating solution if $\lambda < 0$ (cf. Chapter 3).

Chapter 2

The Equation Arising in the Cell Growth Model with Dispersion

We consider in this chapter the following equation which is derived from the cell growth model with dispersion:

$$dy''(x) - y'(x) - by(x) + b\alpha y(\alpha x) = 0, \tag{2.1}$$

where $b > 0$, $d > 0$ and $\alpha > 1$, satisfying the boundary conditions

$$dy'(0) - y(0) = 0, \quad y(\infty) = 0, \tag{2.2}$$

and the normalizing condition

$$\int_0^\infty y(x)\, dx = 1. \tag{2.3}$$

For simplicity, we refer to equation (2.1) along with conditions (2.2) and (2.3) as *Problem 2*. Note that the condition $dy'(0) - y(0) = 0$ could be replaced by $y'(\infty) = 0$, as can be seen by integrating equation (2.1) from 0 to ∞ and conditions $y(\infty) = 0$ and $\int_0^\infty y(x)\, dx = 1$.

We investigate here the uniqueness and positivity of solutions to Problem 2 and construct a solution in the form of a Dirichlet series. Some qualitative properties of the solutions and the limiting cases when b, $d \to 0$ or ∞ are also examined. Most of this material can be found in G.C Wake, S. Cooper, H.K. Kim and B. van-Brunt [1998]. The author's contribution to that paper is Section 2.4 of this thesis. First, however, we will discuss the background of Problem 2.

2.1 The Cell Growth Model

Consider a growing cohort of cells. (The growth could possibly be symplectic as in plant root cells, but it is not necessary for there to be any spatial structure, and thus this theory can apply to situations involving leaf cells or plant-plankton cells.) These cells are undergoing growth and regular constant cell-fission, and so the number density of cells over a size variable will disperse over size in accordance with the modified Fokker-Planck equation

$$n_t(x,t) = -(gn)_x + (Dn)_{xx} + a\alpha^2 n(\alpha x, t) - an(x,t), \tag{2.4}$$

where $x > 0$ is a dimensional size variable, $t > 0$ is time and $n(x,t)$ is a cell number density function, i.e. the number of cells of size x at time t (cf. Gardiner [1983]). Here g, D (= variance/2), a, α are respectively the growth rate (length per time) dispersion coefficient (length^2 per time), frequency of splitting (per time), and the (constant) number of cells obtained from a fission-event (cells of size αx split into cells of size x and so $\alpha > 1$).

We consider the case g, D, and a are all positive and constant. The equation (2.4) is supplemented by the boundary condition (a zero flux condition)

$$Dn_x(0,t) - gn(0,t) = 0, \tag{2.5}$$

and an arbitrary initial condition

$$n(x,0) = n_0(x). \tag{2.6}$$

Clearly, we also need the finiteness condition

$$\lim_{x\to\infty} n(x,t) = 0, \tag{2.7}$$

and we assume the condition

$$\lim_{x\to\infty} n_x(x,t) = 0, \tag{2.8}$$

which is reasonable physically: the density of cells of a given size must tend to zero as the size tends to infinity.

A solution of the steady size distribution (SSD) form to equation (2.4) corresponds to a separable solution: let

$$n(x,t) = y(x)N(t). \tag{2.9}$$

(In other contexts (Heijmans [1985]) it can be shown that solutions of the form (2.9) are attracting for the set of all solutions to (2.4), with arbitrary initial conditions, for large time t.) Now the function $y(x)$ (under a suitable normalization) corresponds to a probability density function and therefore the condition (2.3) must be satisfied. Substituting equation (2.9) into (2.4) gives the relations

$$\frac{N'(t)}{N(t)} = \frac{-gy'(x)}{y(x)} + \frac{Dy''(x)}{y(x)} + a\alpha^2 \frac{y(\alpha x)}{y(x)} - a = \lambda, \tag{2.10}$$

where λ is the separation constant. These relations yield

$$N(t) = N_0 e^{\lambda t},$$

where N_0 is some constant, and the functional differential equation

$$Dy''(x) - gy'(x) - (a+\lambda)y(x) + a\alpha^2 y(\alpha x) = 0. \tag{2.11}$$

The boundary condition (2.5) translates to

$$Dy'(0) - gy(0) = 0. \tag{2.12}$$

Integrating equation (2.11) from zero to infinity and using relations (2.3), (2.8) and (2.12) we see that

$$\lambda = a(\alpha - 1);$$

consequently,

$$N(t) \sim e^{a(\alpha-1)t},$$

which indicates exponential growth since $\alpha > 1$. Let $d = D/g$, $b = a\alpha/g$. Then equation (2.11) simplifies to the functional differential equation (2.1) and, the zero flux condition (2.5) and finiteness condition (2.7) imply the conditions (2.2).

2.2 Positivity and Uniqueness of Solutions

In this section we will show that the solution to Problem 2 is positive and, given the numbers b, d, and α, that it is unique.

An integration of the functional differential equation (2.1) from x to ∞ using the condition (2.2) yields the integro-differential equation

$$-dy'(x) + y(x) - b\int_x^\infty y(\xi)d\xi + b\int_{\alpha x}^\infty y(\xi)d\xi = 0, \tag{2.13}$$

and using the transformation

$$\delta(x) = \int_x^\infty y(\xi)d\xi,$$

this becomes

$$d\delta''(x) - \delta'(x) - b(\delta(x) - \delta(\alpha x)) = 0. \tag{2.14}$$

The condition (2.3) implies

$$\delta(0) = 1, \tag{2.15}$$

and

$$\delta(\infty) = 0. \tag{2.16}$$

We will refer to equation (2.14) along with conditions (2.15) and (2.16) as *Problem 2.1*. We first show that any solution to Problem 2 must be positive.
Since $\delta'(x) = -y(x)$, it suffices to show that any solution to Problem 2.1 has the property that $\delta' < 0$. We begin with a lemma which shows that δ cannot take a constant value in any interval of the form $[\hat{x}, \alpha\hat{x}]$ where $\hat{x} > 0$.

Lemma 2.2.1 *There does not exist a solution to Problem 2.1 continuous on* $[0, \infty)$ *for any* $\alpha > 1$ *which is constant on any interval of the form* $[\hat{x}, \alpha\hat{x}]$*, where* $\hat{x} > 0$.

Proof: Suppose $\delta(x) = \Lambda$ for $x \in [\hat{x}, \alpha\hat{x}]$, where $\hat{x} > 0$, and Λ is a constant; then, $\delta''(x) = \delta'(x) = 0$ in this interval, and equation (2.14) thus indicates that $\delta(x) = \Lambda$ for all $x \in [\alpha\hat{x}, \alpha^2\hat{x}]$. This argument can be repeated using the interval $[\alpha\hat{x}, \alpha^2\hat{x}]$ to show that $\delta(x) = \Lambda$ for all $x \in [\alpha^2\hat{x}, \alpha^3\hat{x}]$, and by induction it is clear that $\delta(x) = \Lambda$ for all $x \in [\alpha^k\hat{x}, \alpha^{k+1}\hat{x}]$, where k is any natural number. Since $\alpha > 1$ this means that $\delta(x) = \Lambda$ for all $x \in [\hat{x}, \infty)$, and the condition (2.16) thus implies that $\Lambda = 0$.

On the other hand, if $\delta(x) = 0$ for all $x \in [\hat{x}, \alpha\hat{x}]$, then

$$d\delta''(x) - \delta'(x) - b\delta(x) = 0, \tag{2.17}$$

for all $x \in [\hat{x}/\alpha, \hat{x}]$. Now δ is twice differentiable in $(0, \infty)$ and thus δ' is continuous; therefore, $\delta(\hat{x}) = \delta'(\hat{x}) = 0$. The ordinary differential equation (2.17) has a unique solution satisfying these initial conditions at $x = \hat{x}$, which by inspection is $\delta(x) = 0$; thus, $\delta(x) = 0$ in the interval $[\hat{x}/\alpha, \hat{x}]$. By induction it can be shown that $\delta(x) = 0$ in any interval of the form $[\hat{x}/\alpha^{k+1}, \hat{x}/\alpha^k]$, where k is any natural number. Therefore, $\lim_{x\to 0^+} \delta(x) = 0$ contradicting the condition (2.15) and the continuity of the solution at $x = 0$. ■

Lemma 2.2.2 *Suppose that a solution* δ *exists to Problem 2.1. If* $\delta'(0) < 0$*, then* $\delta'(x) < 0$ *for all* $x \in (0, \infty)$.

Proof: Suppose there is some point $x^* \in (0,\infty)$ at which $\delta'(x) \geq 0$, then the continuity of $\delta'(x)$ implies that δ' must have a zero somewhere in $(0, x^*]$. Let x_1 denote the smallest value of $x > 0$ for which $\delta'(x) = 0$. The point x_1 cannot correspond to a local maximum for the function δ and therefore $\delta''(x_1) \geq 0$. Equation (2.14) implies that $\delta(x_1) \geq \delta(\alpha x_1)$. If $\delta''(x_1) = 0$, then Lemma 2.2.1 precludes the possibility that $\delta(x) = const.$ on the interval $[x_1, \alpha x_1]$ and thus there must be some point $\hat{x} \in (x_1, \alpha x_1)$ at which $\delta'(\hat{x}) > 0$. If $\delta''(x_1) > 0$, then there are points close to x_1 for which $\delta'(x) > 0$. In either case there must be some point $\tilde{x}_1$ in the interval $(x_1, \alpha x_1)$ such that $\delta'(\tilde{x}_1) > 0$. The change in sign of the derivative in $[x_1, \alpha x_1]$ indicates the presence of a local minimum for the function δ in this interval. Let γ_1 denote the value of $x \in (0, \tilde{x}_1)$ where the minimum occurs. At γ_1, $\delta''(\gamma_1) \geq 0$, and equation (2.14) implies that $\delta(\gamma_1) \geq \delta(\alpha\gamma_1)$; therefore, $\delta(\tilde{x}_1) \geq \delta(\gamma_1) \geq \delta(\alpha\gamma_1)$ and thus there must be some point $\Gamma_1 \in (\gamma_1, \alpha\gamma_1)$ at which δ achieves a local maximum. Now $\delta''(\Gamma_1) \leq 0$ and equation (2.14) indicates that $\delta(\Gamma_1) \leq \delta(\alpha\Gamma_1)$, but $\delta(\alpha\gamma_1) \leq \delta(\Gamma_1)$ so that there must be a local minimum at some point γ_2 for δ in the interval $(\Gamma_1, \alpha\Gamma_1)$. Equation (2.14) yields the inequality $\delta(\gamma_2) \geq \delta(\alpha\gamma_2)$, and since $\alpha\Gamma_1 \in (\gamma_2, \alpha\gamma_2)$ and $\delta(\alpha\Gamma_1) > \delta(\gamma_2)$, there must be a local maximum at some point $\Gamma_2 \in (\gamma_2, \alpha\gamma_2)$. In this manner a sequence of local minima $\{\gamma_k\}$ and a sequence of local maxima $\{\Gamma_k\}$ can be constructed such that

$$\begin{aligned} \Gamma_k &\in (\gamma_k, \alpha\gamma_k), \\ \gamma_{k+1} &\in (\Gamma_k, \alpha\Gamma_k), \\ \delta(\Gamma_k) &\leq \delta(\Gamma_{k+1}), \\ \delta(\gamma_k) &\geq \delta(\gamma_{k+1}). \end{aligned}$$

Let $\mu = \delta(\Gamma_1) - \delta(\gamma_1)$. Then $\mu > 0$ and $\delta(\Gamma_k) - \delta(\gamma_k) \geq \mu$ for all natural numbers k. The continuity of δ precludes the existence on any finite points of accumulation for either sequence and consequently $\Gamma_k \to \infty$ as $k \to \infty$ and $\gamma_k \to \infty$ as $k \to \infty$. By hypothesis $\delta(x) \to 0$ as $x \to \infty$, but even if $\lim_{x\to\infty}(\delta(\Gamma_k) - \delta(\gamma_k))$ existed it could not be zero and consequently even if $\lim_{x\to\infty} \delta(x)$ exists it cannot be zero. We conclude thus that $\delta' < 0$ for all $x \in [0,\infty)$. ■

Note that in the above proof the occurrence of a zero for δ' in $(0,\infty)$ led to a never ending oscillation for the function δ. If it were assumed that $\delta'(0) > 0$ then a similar construction would show that either $\delta'(x) > 0$ for all $x > 0$ or the solution oscillates between local maxima and local minima always at least a finite distance $\mu > 0$ apart. If the solution oscillates then condition (2.16) cannot be satisfied. The only possibility is thus that $\delta'(x) > 0$ for all $x > 0$. But, $\delta(0) = 1$ and so if $\delta'(x) > 0$, then we still cannot satisfiy condition (2.16). This argument establishes the following result:

Corollary 2.2.3 *There are no solutions to Problem 2.1 with* $\delta'(0) > 0$.

The above corollary means that there are no solutions to Problem 2 if $y(0) < 0$. It remains to consider the important case when $y(0) = 0$, i.e. $\delta'(0) = 0$. The next result shows that there is no solution if $y(0) = 0$.

Lemma 2.2.4 *There are no solutions to Problem 2.1 with* $\delta'(0) = 0$.

Proof: We show first that if any such solution exists it must satisfy $\delta'(x) < 0$ for all $x > 0$. This result is then used to show that no such solution can exist. Suppose there exists a solution to Problem 2.1 and that $\delta'(0) = 0$. If $\delta'(\tilde{x}) > 0$ for any $\tilde{x} > 0$, then the proof of Corollary 2.2.3 applied to the interval $(\tilde{x}, \infty)$ instead of $(0, \infty)$ can be used to show that δ will not satisfy the prescribed conditions. Thus, $\delta'(x) \leq 0$ for all $x > 0$. Lemma 2.2.1 implies that there is no number $\xi > 0$ such that $\delta'(x) = 0$ for all $x \in (0, \xi]$ and since δ' cannot be positive, we conclude that there exists points arbitrarily close to 0 such that δ' is negative. The proof of Lemma 2.2.2 can be applied to any interval of the form $[\epsilon, \infty)$, where $\epsilon > 0$ can be arbitrarily small; thus, $\delta'(x) < 0$ for all $x > 0$.

Equation (2.14) can be recast into the form

$$d(e^{-\frac{x}{d}}\delta'(x))' - b(e^{-\frac{x}{d}}(\delta(x) - \delta(\alpha x))) = 0,$$

and because $\delta'(0) = 0$,

$$de^{-\frac{x}{d}}\delta'(x) = \int_0^x be^{-\frac{\xi}{d}}(\delta(\xi) - \delta(\alpha\xi))d\xi.$$

Now $\delta(x) - \delta(\alpha x) > 0$ for all $x > 0$ because $\delta(x)$ is strictly decreasing. But $b > 0$ and $d > 0$, and the above equation indicates that $\delta'(x) > 0$. ■

In terms of $y(x)$ we can summarize the above results in the following theorem:

Theorem 2.2.5 *If a solution $y(x)$ to Problem 2 exists , then $y(x) > 0$ for all $x \geq 0$.*

The analysis underlying Theorem 2.2.5 can also be used to establish the following result:

Theorem 2.2.6 *There exists at most one solution to Problem 2.*

Proof: Suppose that two distinct solutions $y_1(x)$, $y_2(x)$ exist and let $z(x) = y_1(x) - y_2(x)$. Then z satisfies the equation

$$dz''(x) - z'(x) - b(z(x) - \alpha z(\alpha x)) = 0, \tag{2.18}$$

and the conditions

$$\begin{aligned} dz'(0) - z(0) &= 0, \\ \int_0^\infty z(x)\,dx &= 0, \\ z(\infty) = 0. \end{aligned}$$

As with the original equation, this differential equation can be converted into the equation

$$d\Delta''(x) - \Delta'(x) - b(\Delta(x) - \Delta(\alpha x)) = 0, \tag{2.19}$$

where

$$\Delta(x) = \int_x^\infty z(x)\,dx. \tag{2.20}$$

The conditions which Δ must satisfy are

$$\begin{aligned} \Delta(0) &= 0, \\ \Delta(\infty) &= 0. \end{aligned}$$

This problem is essentially the same as the original one involving δ except that $\Delta(x) = 0$ is a valid solution. The analysis developed for δ can be applied to Δ *mutatis mutandis.* Since $\Delta(0) = 0$ and $\Delta(\infty) = 0$, there must exist at least one point $\tilde{x}$ such that $\Delta'(\tilde{x}) = 0$. Now Δ' is not identically zero since the solutions are assumed distinct. We can thus use the analysis in the proofs of Lemma 2.2.2 and Corollary 2.2.3 to show that the function $\Delta(x)$ oscillates away from zero as $x \to \infty$ and therefore $\Delta(\infty) \neq 0$. ∎

2.3 The Dirichlet Series Solution

A solution to Problem 2 can be obtained by use of Laplace transforms; indeed, this method was used by Hall and Wake (*op. cit.*) to solve the first order case. However, as we have seen in Chapter 1, functional differential equations such as (2.1) with $d = 0$ admit solutions in the form of a Dirichlet series (cf. Kato and McLeod [1971], Fredrickson[1971]) and motivated by this observation, we seek a solution to Problem 2 of the form

$$y(x) = \sum_{n=0}^{\infty} a_n e^{-\alpha^n r x}, \tag{2.21}$$

where the coefficients a_n and the parameter r are to be determined. Formally, we have

$$\begin{aligned} y'(x) &= -r\sum_{n=0}^{\infty} a_n\alpha^n e^{-\alpha^n rx}, \\ y''(x) &= r^2\sum_{n=0}^{\infty} a_n\alpha^{2n} e^{-\alpha^n rx}, \\ y(\alpha x) &= \sum_{n=1}^{\infty} a_{n-1} e^{-\alpha^n rx}. \end{aligned}$$

Substituting these into equation (2.1) leads to the expression

$$\begin{aligned} & dr^2\sum_{n=0}^{\infty} a_n\alpha^{2n} e^{-\alpha^n rx} + r\sum_{n=0}^{\infty} a_n\alpha^n e^{-\alpha^n rx} \\ - & \; b\sum_{n=0}^{\infty} a_n e^{-\alpha^n rx} + b\alpha\sum_{n=1}^{\infty} a_{n-1} e^{-\alpha^n rx} = 0. \end{aligned}$$

Equating coefficients of e^{-rx} yields the indicial equation

$$dr^2 + r - b = 0, \tag{2.22}$$

while equating coefficients of $e^{-\alpha^n rx}$ for $n \geq 1$, gives the recurrence relation

$$(d\alpha^{2n}r^2 + \alpha^n r - b)a_n = -b\alpha a_{n-1}. \tag{2.23}$$

Since the series (2.21) will in general diverge for positive x if $r < 0$, we choose r to be the positive root of (2.22). The recurrence relation (2.23) implies that

$$a_n = \frac{(-b\alpha)^n}{\prod_{m=1}^{n}(d\alpha^{2m}r^2 + \alpha^m r - b)} a_0, \tag{2.24}$$

so that a_n can be written as

$$\begin{aligned} a_n &= \frac{(-b\alpha)^n}{\prod_{m=1}^{n}(d\alpha^{2m}r^2 + \alpha^m r - dr^2 - r)} a_0 \\ &= \frac{(-b\alpha)^n}{\prod_{m=1}^{n} r(\alpha^m - 1)(d\alpha^m r + dr + 1)} a_0 \\ &= \frac{q^{n^2}}{\prod_{m=1}^{n}(1 - q^m)(1 + (1 + \frac{1}{dr})q^m)} \left(\frac{-b}{dr^2}\right)^n a_0, \end{aligned} \tag{2.25}$$

where $q = \alpha^{-1}$. Define

$$(p; q)_n = \prod_{j=0}^{n-1}(1 - pq^j), \qquad \text{for } n = 1, 2, 3, \cdots,$$

and let $(p;q)_0 = 1$ (cf. G.E. Andrews [1976] p.17). Using this notation, equation (2.25) is given by

$$a_n = \frac{q^{n^2}}{(q;q)_n(-(1+\frac{1}{dr})q;q)_n}\left(\frac{-b}{dr^2}\right)^n a_0. \tag{2.26}$$

If we let

$$l = -(1+1/dr), \tag{2.27}$$

then equation (2.22) indicates that

$$\frac{b}{dr^2} = \frac{dr^2+r}{dr^2} = 1 + \frac{1}{dr} = -l,$$

and hence (2.26) simplifies to

$$a_n = \frac{a_0 l^n q^{n^2}}{(q;q)_n(lq;q)_n}.$$

Therefore, $y(x)$ is given by

$$\begin{aligned} y(x) &= a_0 \sum_{n=0}^{\infty} \frac{(-b\alpha)^n}{\prod_{m=1}^{n}(d\alpha^{2m}r^2 + \alpha^m r - b)} e^{-\alpha^n rx} \\ &= a_0 \sum_{n=0}^{\infty} \frac{l^n q^{n^2} e^{-q^{-n}rx}}{(q;q)_n(lq;q)_n}, \end{aligned} \tag{2.28}$$

and a_0 can be determined by the condition (2.3) so that

$$\begin{aligned} a_0 &= r\left(\sum_{n=0}^{\infty} \frac{(-b)^n}{\prod_{m=1}^{n}(d\alpha^{2m}r^2 + \alpha^m r - b)}\right)^{-1} \\ &= r\left(\sum_{n=0}^{\infty} \frac{l^n q^{n^2+n}}{(q;q)_n(lq;q)_n}\right)^{-1}. \end{aligned} \tag{2.29}$$

Since $d > 0$ and $\alpha > 1$ we see that

$$0 < q < 1 \text{ and } l < -1.$$

We can check directly that the solution given by (2.28) and (2.29) satisfies the boundary condition (2.2). Integrating both sides of equation (2.1) from 0 to ∞ gives

$$dy'(\infty) - dy'(0) - y(\infty) + y(0) - b\int_0^{\infty} y(x)dx + b\alpha\int_0^{\infty} y(\alpha x)dx = 0.$$

Therefore any solution of equation (2.1) which satisfies

$$y'(\infty) = y(\infty) = 0, \quad \int_0^\infty y(x)dx = \text{finite} \tag{2.30}$$

will automatically satisfy the boundary condition $dy'(0) - y(0) = 0$. Since the solution given by equation (2.28) clearly satisfies the conditions (2.30), it will therefore also satisfy the boundary condition. The solution defined by equation (2.28) is uniformly convergent in any compact interval of $[0, \infty)$ and thus represents the solution to Problem 2.

2.4 Qualitative Properties and the Limiting Cases

In this section we study some qualitative properties of the solution to Problem 2. It was shown in section 2.2 that the solution is unique and positive for $x \in [0, \infty)$. In this section we will show that the solution has precisely one local maximum, and obtain bounds for the location of the maximum. The influence of the dispersion coefficient on the shape of the graph is investigated using some simple, rough bounds. It is also shown that the limiting case as $d \to 0^+$ corresponds to the non-dispersive case solution obtained by Hall and Wake (*op. cit.*). Moreover, the solutions when b approaches 0^+ or ∞ are also examined.

2.4.1 Shape of the Solution

Note that the boundary condition (2.2) and Theorem 2.2.5 indicate that $y'(0) > 0$ and consequently y must have at least one local maximum in $(0, \infty)$. We show that y has precisely one local maximum, but first we need to establish the following lemma:

Lemma 2.4.1 *There exists a $Z \in R$ such that the solution $y(x)$ to Problem 2 satisfying $y'(x) \leq 0$ and $y''(x) \geq 0$ for all $x \geq Z$.*

Proof: Now $y'(x) = -\sum_{n=0}^{\infty} a_n \alpha^n r e^{-\alpha^n r x}$ and the ratio of successive terms are

$$\begin{aligned} R_n(x) &= |\frac{a_n}{a_{n-1}}| \alpha e^{-\alpha^{n-1} r x(\alpha - 1)} \\ &= |\frac{-b\alpha}{d\alpha^{2n} r^2 + \alpha^n r - b}| \alpha e^{-H(x)}. \end{aligned} \tag{2.31}$$

Since $H(x) \to \infty$ as $x \to \infty$, there exists some Z such that

$$R_n(x) < 1,$$

for all $x \geq Z$ and n. The series is alternating and so this implies that $y'(x) \geq 0$ if $a_0 < 0$ and $y'(x) \leq 0$ if $a_0 > 0$. However, the positivity of the solution precludes the possibility that $y'(x) \geq 0$ and thus $y'(x) \leq 0$. The second assertion that $y''(x) \geq 0$ for all x sufficiently large can be proved using essentially the same arguments. ■

Theorem 2.4.2 *Suppose y is the solution to Problem 2, then y has precisely one local maximum.*

Proof: Suppose y has two local maxima. Then it must also have a local minimum. Let x_1 denote a point at which y has a local minimum. Now $y'(x_1) = 0$ and $y''(x_1) \geq 0$, so that equation (2.1) implies that $y(x_1) \geq \alpha y(\alpha x_1) > y(\alpha x_1)$. Since $y(x_1) > y(\alpha x_1)$ and $y(x_1+\epsilon) > y(x_1)$ for ϵ small, there must be a point $X_1 \in (x_1, \alpha x_1)$ at which y achieves a local maximum. Now $y'(X_1) = 0$ and $y''(X_1) \leq 0$ and thus

$$\alpha y(\alpha X_1) \geq y(X_1) > y(x_1) \geq \alpha y(\alpha x_1),$$

i.e. $y(\alpha X_1) > y(\alpha x_1)$. Since $\alpha X_1 > \alpha x_1$ this means that there must be a local minimum in the interval $(X_1, \alpha X_1)$ at say $x = x_2$. We can now repeat the arguments used for the minimum at x_1 to assert the existence of a local maximum at some point $X_2 \in (x_2, \alpha x_2)$ and thus assert the existence on a local minimum in $(X_2, \alpha X_2)$ etc. Thus if a local minimum exists, then a sequence of local maxima $\{X_n\}$ and a sequence of local minima $\{x_n\}$ exist such that $X_n \to \infty$ and $x_n \to \infty$ as $n \to \infty$. But Lemma 2.4.1 precludes the possibility of local extrema arbitrarily far away from the origin. Therefore, y has precisely one local maximum. ■

From the above theorem along with positivity of the solution, we now know roughly the shape of the graph of the solution. For more precise knowledge of the shape of the graph of the solution, we need to examine the signs of $y''(x)$.

Theorem 2.4.3 *The solution $y(x)$ has at most one inflexion point in $(0, X_m)$ and this assertion is also true in (X_m, ∞), where X_m is the maximum critical point.*

Proof: We first examine the signs of $y''(x)$ in the interval $(0, X_m)$. At $x = 0$, equation (2.1) yields

$$dy''(0) - y'(0) - b(1-\alpha)y(0) = 0,$$

and so

$$dy''(0) = y'(0) + b(1-\alpha)y(0) = (1/d + b(1-\alpha))y(0),$$

since $dy'(0) - y(0) = 0$. This implies that if $db(\alpha - 1) < 1$, then $y''(0) > 0$ so that there exists at least one inflexion point v satisfying $y'''(v) \leq 0$ in $(0, X_m)$ since $y''(X_m) \leq 0$. Suppose that there is an inflexion point $v_1 \in (0, X_m)$ such that $y''(v_1) = 0$ and $y'''(v_1) \geq 0$. A differentiation of equation (2.1) leads to the equation

$$dy'''(x) - y''(x) - by'(x) + b\alpha^2 y'(\alpha x) = 0, \tag{2.32}$$

and this equation implies that

$$y'(v_1) \geq \alpha^2 y'(\alpha v_1). \tag{2.33}$$

So, whether $y'(\alpha v_1) > 0$ or $y'(\alpha v_1) < 0$, we have the inequality

$$y'(v_1) \geq y'(\alpha v_1).$$

This asserts the existence of a point $v_2 \in (v_1, \alpha v_1)$ such that $y''(v_2) = 0$, $y'''(v_2) \leq 0$ and

$$y'(v_2) > y'(v_1). \tag{2.34}$$

Here, $y'(v_2) > 0$ since $y'(v_1) \geq 0$ so that $v_2 \in (0, X_m)$. Equation (2.32) implies

$$y'(v_2) \leq \alpha^2 y'(\alpha v_2), \tag{2.35}$$

and combining inequalities (2.33),(2.34) and (2.35) yields the inequalities

$$\alpha^2 y'(\alpha v_2) \geq y'(v_2) > y'(v_1) \geq \alpha^2 y'(\alpha v_1),$$

so that $y'(\alpha v_2) > y'(\alpha v_1)$. This indicates the existence of a point $v_3 \in (\alpha v_1, \alpha v_2)$ such that $y''(v_3) = 0$, $y'''(v_3) \geq 0$ and $y'(v_3) < y'(v_2)$. Inequality (2.35) follows that $y'(\alpha v_2) > 0$ and so $v_3 \in (0, X_m)$. Repeating these arguments constructs a sequence $\{v_n\}$ in $(0, X_m)$ such that $y''(v_n) = 0$. However the continuity of y' precludes the possibility that y' is oscillatory in a limited interval; thus, there is at most one inflexion point v satisfying $y'''(v) \leq 0$ in $(0, X_m)$.

Let us examine $y''(x)$ in the interval (X_m, ∞). Lemma 2.4.1 indicates that there exists an inflexion point $w \in (X_m, \infty)$ satisfying $y'''(w) \geq 0$. Suppose there is an inflexion point $w_1 \in (X_m, \infty)$ such that $y''(w_1) = 0$ and $y'''(w_1) \leq 0$. Then equation (2.32), noting that $y'(\alpha w_1) \leq 0$, implies

$$y'(w_1) \leq \alpha^2 y'(\alpha w_1), \tag{2.36}$$

and so $y'(w_1) \leq y'(\alpha w_1)$. Hence there exists
a point $w_2 \in (w_1, \alpha w_1)$ such that $y''(w_2) = 0$, $y'''(w_2) \geq 0$ and

$$y'(w_2) < y'(w_1), \tag{2.37}$$

and therefore,

$$y'(w_2) \geq \alpha^2 y'(\alpha w_2). \tag{2.38}$$

Combining inequalities (2.36), (2.37) and (2.38) yields

$$\alpha^2 y'(\alpha w_2) \leq y'(w_2) < y'(w_1) \leq \alpha^2 y'(\alpha w_1),$$

and thus $y'(\alpha w_2) < y'(\alpha w_1)$. This asserts the existence of another point $w_3 \in (\alpha w_1, \alpha w_2)$ satisfying $y''(w_3) = 0$, $y'''(w_3) \leq 0$ and $y'(w_3) > y'(w_2)$. Repeating this process produces a sequence $\{w_n\}$ such that $w_n \to \infty$ and $y''(w_n) = 0$. This means that there is $\hat{x}_n \in (w_n - \epsilon, w_n + \epsilon)$ for $\epsilon > 0$ and all $n = 2, 3, \ldots$ such that $y''(\hat{x}_n) < 0$, contradicting the result of Lemma 2.4.1. Consequently, there exists exactly one inflexion point w satisfying $y'''(x) \geq 0$ in (X_m, ∞).■

2.4.2 Bounds on the Maximum Critical Point

Some rough bounds on where the maximum occurs can be readily derived from the integro-differential equation (2.13). Let X_m denote the value of x at which the maximum is achieved. Then, $y'(X_m) = 0$ and thus

$$y(X_m) = b\int_{X_m}^{\alpha X_m} y(\xi)d\xi < b(\alpha - 1)X_m y(X_m), \tag{2.39}$$

and

$$y(X_m) > b(\alpha - 1)X_m y(\alpha X_m).$$

Since y achieves a maximum at X_m, equation (2.1) implies that $y(X_m) \leq \alpha y(\alpha X_m)$ and consequently

$$y(X_m) > \frac{b}{\alpha}(\alpha - 1)X_m y(X_m). \tag{2.40}$$

Inequalities (2.39) and (2.40) thus provide the bounds

$$\frac{1}{b(\alpha - 1)} < X_m < \frac{\alpha}{b(\alpha - 1)}. \tag{2.41}$$

Note that these bounds are independent of the dispersion coefficient d. A graph of the solution is given in Figure 2.2.1.

We can get some rough upper bounds on y in terms of the parameters b and d by use of (2.13). Evidently,

$$y(X_m) = b\int_{X_m}^{\alpha X_m} y(\xi)d\xi,$$

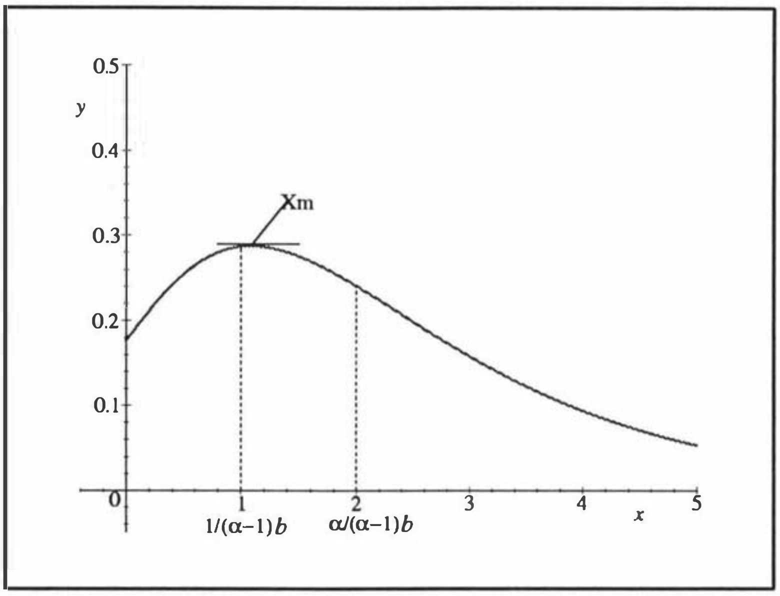

Figure 2.2.1: The bounds on X_m of the solution y when $\alpha = 2$, $b = 1$, $d = 1$.

and since $y > 0$, the normalizing condition (2.3) implies that for all $x \geq 0$,

$$y(x) < b, \tag{2.42}$$

(an upper bound independent of d). A slightly more refined bound can be obtained by integrating equation (2.13) again. This yields the equation

$$dy(x) = dy(0) + \int_0^x y(\xi)d\xi - b\int_0^x \int_\xi^{\alpha\xi} y(\eta)d\eta d\xi,$$

which in turn implies that

$$y(x) < y(0) + \frac{1}{d}. \tag{2.43}$$

In particular, we have that

$$y(0) < y(X_m) < y(0) + \frac{1}{d}, \tag{2.44}$$

so that if the dispersion coefficent d is large, the graph will "flatten out" between $x = 0$ and $x = X_m$ and then decrease steadily to 0 as $x \to \infty$. We see this case in detail in the next subsection.

2.4.3 The Limiting Cases

The uniform convergence of each series of (2.21) w.r.t. b, d or α will be established before we consider the limiting cases.

Uniform Convergence of the Series w.r.t. b or d

The solution y can be regarded as a function of x and the parameters b and d; let $y = y(x, b, d)$. Then

$$y(x,b,d) \leq \sum_{n=0}^{\infty} |a_n e^{-\alpha^n r x}| \leq \sum_{n=0}^{\infty} |a_n|,$$

and the coefficients of the series defined by (2.24) satisfy

$$\begin{aligned} |a_n| &\leq \frac{(b\alpha)^n}{\prod_{m=1}^{n} |d\alpha^{2m} r^2 + \alpha^m r - b|} |a_0| \\ &= \frac{(b\alpha)^n}{\prod_{m=1}^{n} (\alpha^m - 1)|b(\alpha^m + 1) - r\alpha^m|} |a_0| \\ &= C_n(b,d)|a_0|, \end{aligned}$$

where we have used the identity $dr^2 = b - r$. Now, $r = \frac{-1+\sqrt{1+4bd}}{2d}$ from (2.22) so that

$$\begin{aligned} b(\alpha^m + 1) - r\alpha^m &= \frac{2bd(\alpha^m + 1) - (-1 + \sqrt{1+4bd})\alpha^m}{2d} \\ &= \frac{(2bd + 1 - \sqrt{1+4bd})\alpha^m + 2bd}{2d} \\ &\geq \frac{2bd}{2d} \\ &= b, \end{aligned}$$

since $2bd + 1 - \sqrt{1+4bd} \geq 0$. Therefore,

$$C_n(b,d) \leq \frac{(b\alpha)^n}{b^n \prod_{m=1}^{n} (\alpha^m - 1)} = \frac{\alpha^n}{\prod_{m=1}^{n} (\alpha^m - 1)},$$

and so

$$y(x,b,d)/|a_0| \leq \sum_{n=0}^{\infty} \frac{\alpha^n}{\prod_{m=1}^{n} (\alpha^m - 1)}.$$

Since the above series is convergent and independent of b and d, $y(x,b,d)/|a_0|$ is uniformly convergent w.r.t. b and d.

Let $a_0 = rS^{-1}(b,d)$, where

$$S(b,d) = \sum_{n=0}^{\infty} \frac{(-b)^n}{\prod_{m=1}^{n} (d\alpha^{2m} r^2 + \alpha^m r - b)},$$

then

$$
\begin{aligned}
|S(b,d)| &\le \sum_{n=0}^{\infty} \frac{(b\alpha)^n}{\prod_{m=1}^{n} |d\alpha^{2m}r^2 + \alpha^m r - b|} \\
&= \sum_{n=0}^{\infty} C_n(b,d)
\end{aligned}
$$

so that $S(b,d)$ is uniformly convergent.

The Limiting Case as $d \to 0^+$ or $d \to \infty$

If $d \to 0^+$, then the indicial equation (2.22) implies that $r \to b$, and equation (2.27) implies that $l \to -\infty$. Since

$$
\lim_{l \to -\infty} \frac{l^n}{(lq;q)_n} = \frac{(-1)^n}{q^{(n^2+n)/2}},
$$

our solution (2.28) becomes, in the limiting case $d \to 0^+$,

$$
\lim_{d\to 0^+} y(x,b,d) = \lim_{d \to 0^+} a_0 \left(\sum_{n=0}^{\infty} \frac{(-1)^n q^{(n^2-n)/2}}{(q;q)_n} e^{-q^{-n}rx} \right), \tag{2.45}
$$

where

$$
\lim_{d\to 0^+} a_0 = \frac{b}{\sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2}(-1)^n}{(q;q)_n}} = \frac{b}{(q;q)_\infty}. \tag{2.46}
$$

The last equality follows from Gaspar and Rahman [1990, eq. (1.3.16)] with z replaced by $-q$. Equations (2.45) and (2.46) are the same solution as obtained in Hall and Wake (*op. cit.*). Figure 2.2.2 illustrates solutions for small values of d and the solution obtained by Hall and Wake.

For the case $d \to \infty$, the indicial equation (2.22) implies that $r \to 0$ and $dr^2 \to b$ so that $l \to -1$ from (2.27). Therefore, in the limiting case $d \to \infty$,

$$
y(x,b,d) \to a_0 \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q^2;q^2)_n},
$$

and

$$
a_0 = r \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n}}{(q^2;q^2)_n} \to 0.
$$

This result and the earlier observation about the bounds of $y(X_m)$ indicate that $y(x)$ flattens out from $x = 0$ to $x = X_m$ as $d \to \infty$ and the solution approaches the trivial one. (cf. Figure 2.2.3)

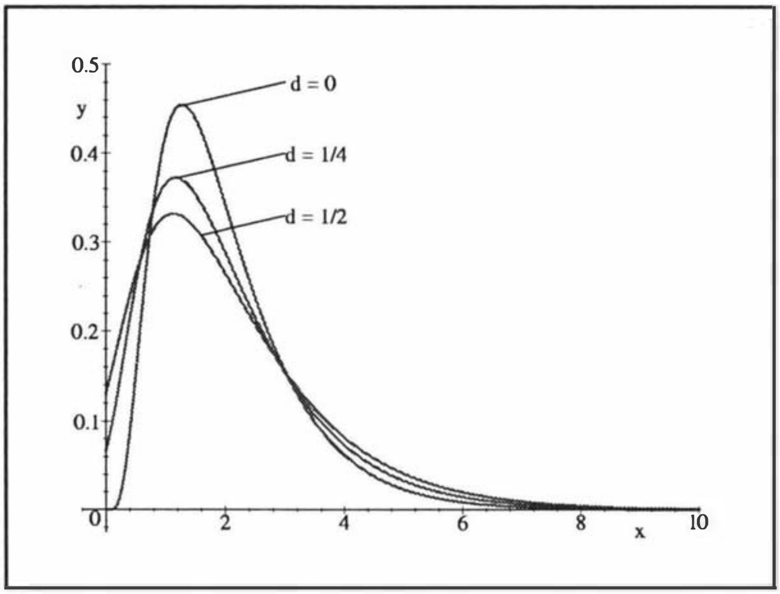

Figure 2.2.2: y for small values of d when $b = 1$ and $\alpha = 2$.

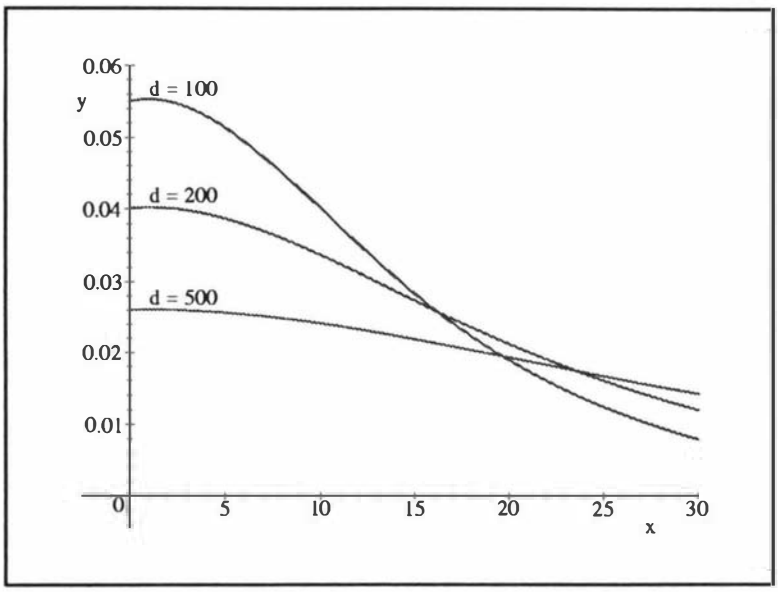

Figure 2.2.3: y for large values of d when $b = 1$ and $\alpha = 2$.

The Limiting Case as $b \to 0^+$ or $b \to \infty$

The bound $y(x) < b$ given by (2.42) implies that $y(x) \to 0$ as $b \to 0^+$. Moreover, the bounds on X_m given by (2.41) imply that $X_m \to \infty$ as $b \to 0^+$. Since $b = \alpha a/g$, where a is the frequency of splitting and g is the growth rate, $b \to 0^+$ means that cells split less frequently or cells grow faster. In either case, the cell size which occupies the biggest proportion in the number of cells is getting bigger.

As $b \to \infty$, $r \to \infty$ so that $l \to -1$ and therefore, the coefficients a_n for $n \geq 1$ approach the same values as those as $d \to \infty$. This implies that $y(x)/a_0 \to 0$ for $x > 0$ since $e^{-\alpha^n rx} \to 0$. Now,

$$a_0 = r\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n}}{(q^2;q^2)_n} \to \infty,$$

and r goes to ∞ slower than e^{-rx} goes to 0 so that $y(x) \to 0$ for $x > 0$. In addition, The bounds on X_m given by (2.41) imply that $X_m \to 0$ and this means that the size which occupies the biggest proportion in the number of cells is getting smaller when cells split more frequently or cells grow slower.

The Limiting Case as $\alpha \to 1^+$ or $\alpha \to \infty$

Suppose the series $y/a_0 = y(\alpha, x)/a_0(\alpha)$ is uniformly convergent w.r.t. α. As $\alpha \to 1^+$, the coefficients of the Dirichlet series solution approach infinity and so the solution does not exist. For the maximum critical point, the bounds (2.41) indicate that the maximum critical point is getting bigger when α is getting closer to 1. For the case when $\alpha \to \infty$, $a_0(\alpha) \to r$ so that $y(\alpha, x) \to re^{rx}$. Figures 2.2.4 and 2.2.5 illustrate the solution $y(x)$ for $\alpha \to 1^+$ and $\alpha \to \infty$ respectively according to $b = 1$ and $d = 1$.

We have shown in this chapter that the solution $y(x)$ is positive for all $x \geq 0$. Since $y(x)$ is a probability density function, the fact that the solution $y(x)$ is not negative is consistent with the model. An interesting feature of this solution is the role $y(0)$ plays in the solution. The initial value $y(0)$ cannot be specified independently of the other conditions and is determined uniquely by the problem. Moreover, any combination of $b > 0$ and $d > 0$ (non-zero finite values) will not produce $y(0) = 0$. This stimulates questions about conditions necessary to make $y(0) = 0$ if we introduce general constant coefficients. We first study the general equations with constant coefficients in Chapter 3 and then return to this question in Chapter 4.

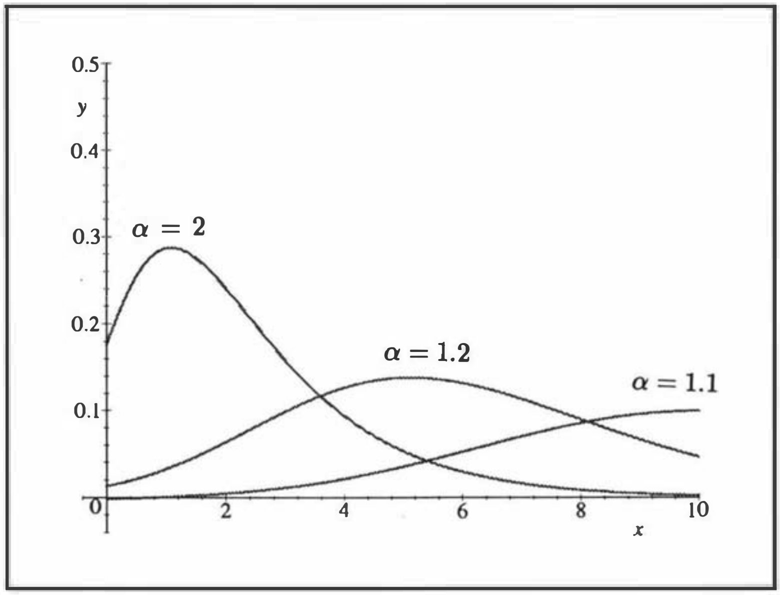

Figure 2.2.4: y when $\alpha = 2$, $\alpha = 1.2$ and $\alpha = 1.1$ according to $b = 1$ and $d = 1$.

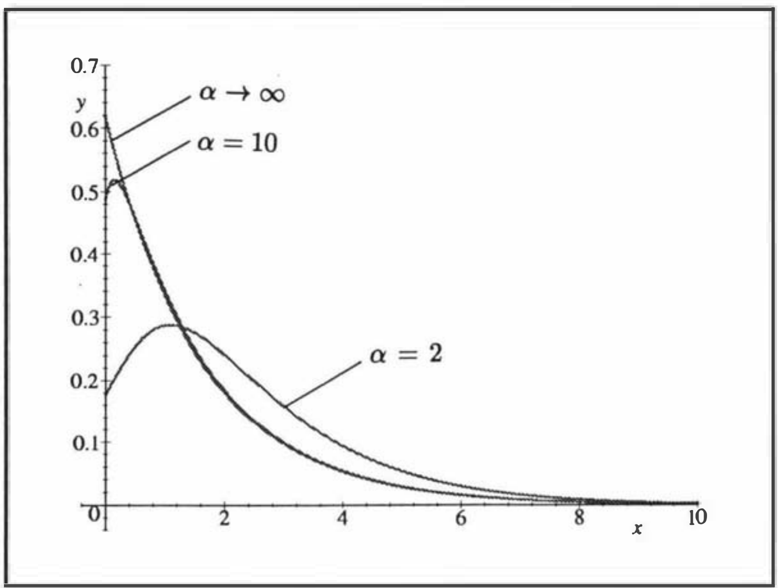

Figure 2.2.5: y when $\alpha = 2$, $\alpha = 10$ and $\alpha \to \infty$ according to $b = 1$ and $d = 1$.

Chapter 3

The General Equations with Constant Coefficients

We investigate in this chapter solutions to a class of second order functional differential equations which are a generalization of the one arising in the cell growth model in Chapter 2. Specifically, we shall consider the equation

$$y''(x) + ay'(x) + by(x) + cy(\alpha x) = 0, \tag{3.1}$$

with the boundary conditions

$$y'(0) + ay(0) = b + c/\alpha, \quad y(\infty) = 0, \tag{3.2}$$

and the normalizing condition

$$\int_0^\infty y(x)\, dx = 1, \tag{3.3}$$

where $a, b, c \neq 0$ are real constants and $\alpha > 1$. Equation (3.1) along with conditions (3.2) and (3.3) will be referred to as *Problem 3*.

3.1 Existence of Solutions

Motivated by Chapter 1, we seek a solution of equation (3.1) which can be represented by a Dirichlet series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n e^{-\alpha^n r x}, \tag{3.4}$$

provided there exists a solution r to the indicial equation

$$r^2 - ar + b = 0, \tag{3.5}$$

such that $Re(r) \geq 0$ $(r \neq 0)$. The indicial equation and the recurrence relation

$$(\alpha^{2n}r^2 - a\alpha^n r + b)a_n = -ca_{n-1}, \tag{3.6}$$

can be obtained by substituting the Dirichlet series (3.4) into equation (3.1) and equating coefficients of like exponential terms. A solution to Problem 3 is thus given by

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-c)^n}{\prod_{m=1}^{n}(\alpha^{2m}r^2 - a\alpha^m r + b)} e^{-\alpha^n r x},$$

where a_0 can be determined by the condition (3.3),

$$a_0 = r\left(\sum_{n=0}^{\infty} \frac{(-c/\alpha)^n}{\prod_{m=1}^{n}(\alpha^{2m}r^2 - a\alpha^m r + b)}\right)^{-1}.$$

The above procedure bears a formed resemblance to the classical Frobenius method for solving linear second order ordinary equations, and like the Frobenius method, the roots to the indicial equation (3.5) indicate the nature of the available Dirichlet series solutions. Convergence of the Dirichlet series requires $Re(r) \geq 0$ $(r \neq 0)$, so that depending on the indicial equation, there are three cases:

(i) no solutions of the form (3.4) exist (no roots to (3.5) such that $Re(r) \geq 0$).
(ii) one solution of the form (3.4) exists (one root to (3.5) such that $Re(r) \geq 0$).
(iii) two solutions of the form (3.4) exist (two roots to (3.5) such that $Re(r) \geq 0$).

The equation studied in the previous chapter illustrates case (ii). We examine briefly case (iii) when two positive roots exist to the indicial equation.

Suppose that there are two distinct positive real roots r_1, r_2 $(r_1 > r_2)$ to the indicial equation (3.5). Then two convergent Dirichlet series solutions $y_1(x) = \sum_{n=0}^{\infty} a_n e^{-\alpha^n r_1 x}$ and $y_2(x) = \sum_{n=0}^{\infty} b_n e^{-\alpha^n r_2 x}$ corresponding to r_1 and r_2, respectively can be constructed. However, $y_2(x)$ has to be treated carefully when $r_1 = \alpha^n r_2$ for some $n \in N$ since

$$\alpha^{2n}r_2^2 - a\alpha^n r_2 + b = (\alpha^n r_2 - r_2)(\alpha^n r_2 - r_1),$$

and so b_n/b_{n-1} has a singularity. This case is roughly analogous to the case where the roots differ by an integer in the Frobenius method. Note that in this case, the recurrence relation w.r.t. r_1 has no singularity since

$$\begin{aligned} \alpha^{2n}r_1^2 - a\alpha^n r_1 + b &= (\alpha^n r_1 - r_1)(\alpha^n r_1 - r_2) \\ &= (\alpha^n r_1 - r_1)(\alpha^{2n} r_2 - r_2) \\ &= r_1 r_2(\alpha^n - 1)(\alpha^{2n} - 1) \\ &> 0, \end{aligned}$$

so that y_1 is a solution to equation (3.1). Suppose that $r_1 = \alpha^k r_2$ for some $k \in N$. We consider the recurrence relation

$$(\alpha^{2k}r_2^2 - a\alpha^k r_2 + b)b_k = -cb_{k-1},$$

and it gives $b_{k-1} = 0$ because $\alpha^{2k}r_2^2 - a\alpha^k r_2 + b = 0$ and so the recurrence relation

$$(\alpha^{2(k-1)}r_2^2 - a\alpha^{k-1}r_2 + b)b_{k-1} = -cb_{k-2},$$

also gives that $b_{k-2} = 0$. The repeated process implies that $b_n = 0$ for $0 \leq n \leq k-1$. Therefore,

$$\begin{aligned} y_2(x) &= \sum_{n=k}^{\infty} b_n e^{-\alpha^n r_2 x} \\ &= \sum_{n=0}^{\infty} b_{n+k} e^{-\alpha^{n+k} r_2 x} \\ &= \sum_{n=0}^{\infty} b_{n+k} e^{-\alpha^n r_1 x}, \end{aligned}$$

and comparing the coefficients b_{n+k} and a_n in $y_1(x)$, we get $b_{n+k} = Aa_n$, where $A = \frac{b_k}{a_0}$ so that $y_1(x) = y_2(x)$ from the condition (3.3). Let us examine the linear independence of solutions when $r_1 \neq r_2$ and $r_1 \neq \alpha^n r_2$. For large x,

$$y_1(x) \sim a_0 e^{-r_1 x} \quad \text{and} \quad y_2(x) \sim b_0 e^{-r_2 x},$$

$$y_1'(x) \sim -a_0 r_1 e^{-r_1 x} \quad \text{and} \quad y_2'(x) \sim -b_0 r_2 e^{-r_2 x}$$

and therefore the Wronskian of y_1 and y_2 becomes

$$y_1(x)y_2'(x) - y_2(x)y_1'(x) = a_0 b_0 (r_1 - r_2) e^{-(r_1+r_2)x} \neq 0$$

so that $y_1(x)y_2'(x) - y_2(x)y_1'(x)$ is not identically zero through all x; thus, y_1 and y_2 are linearly independent. This implies that there is an infinite number of solutions $y(x) = y_1(x) + y_2(x)$ since the first coefficients a_0 and b_0 can be any real value combinations to satisfy the condition (3.3).

We will not pursue the analogy of this procedure with the Frobenius method any further in this thesis. The case when roots of the indicial equation are complex is investigated in the last section.

3.2 Uniqueness of Solutions

In this section, we show that for a certain range of coefficients a, b and c, the solution to Problem 3 is unique.

Theorem 3.2.1 *Given constants a, b, c and α such that $b < 0$, $|c| \leq \alpha|b|$ and $\alpha > 1$, the solution to Problem 3 is unique.*

Proof: Suppose that two distinct solutions $y_1(x), y_2(x)$ to Problem 3 exist and let $z(x) = y_1(x) - y_2(x)$. Then it suffices to show that $z(x) = 0$ for all $x \geq 0$. The function $z(x)$ satisfies the equation

$$z''(x) + az'(x) + bz(x) + cz(\alpha x) = 0, \tag{3.7}$$

along with the conditions

$$z(0) = 0, \quad z(\infty) = 0, \tag{3.8}$$

and

$$\int_0^\infty z(x)\,dx = 0. \tag{3.9}$$

Let

$$\sigma(x) = \int_x^\infty z(t)\,dt, \tag{3.10}$$

then equation (3.7) can be converted into the differential equation

$$\sigma''(x) + a\sigma'(x) + b\sigma(x) + \frac{c}{\alpha}\sigma(\alpha x) = 0, \tag{3.11}$$

where σ satisfies the conditions

$$\sigma(0) = 0, \quad \sigma(\infty) = 0. \tag{3.12}$$

For a non-trivial solution, the conditions (3.12) indicate that $\sigma(x)$ must have a positive maximum or a negative minimum. Suppose that there is a maximum critical point X_1 at which σ achieves a positive value. Then $b\sigma(X_1) + \frac{c}{\alpha}\sigma(\alpha X_1) \geq 0$, and so

$$\sigma(X_1) \leq -\frac{c}{\alpha b}\sigma(\alpha X_1). \tag{3.13}$$

Note that the sign of $\sigma(\alpha X_1)$ depends on the sign of the coefficient c. Let $c > 0$. Then $\sigma(X_1) \le \sigma(\alpha X_1)$ since $0 < -\frac{c}{\alpha b} \le 1$ and therefore, there is another maximum critical point X_2 satisfying $\sigma(X_1) \le \sigma(X_2)$ since $\sigma(\infty) = 0$. This process can be repeated *ad infinitum* to get a sequence $\{X_n\}$ of points at which σ achives a local maximum, and $0 < \sigma(X_1) \le \sigma(X_n)$ so that $\lim_{n\to\infty} \sigma(X_n) \neq 0$, contradicting $\sigma(\infty) = 0$. Thus, $\sigma(x)$ cannot have a positive maximum. In the same manner, it can be shown that σ cannot have a negative minimum. Therefore, $\sigma(x) = 0$ from the condition (3.12) and so $\int_x^\infty z(t)\,dt = \sigma(x) = 0$ for all $x \ge 0$; thus, $z(x) = 0$ from the continuity of $z(x)$. If $c < 0$, then $0 < \frac{c}{\alpha b} \le 1$ so that $\sigma(X_1) \le -\sigma(\alpha X_1)$. This result and the condition $\sigma(\infty) = 0$ imply that there is a point x_1 at which σ achives a negative minimum such that $\sigma(X_1) \le |\sigma(x_1)|$ and $\sigma(x_1) \ge -\frac{c}{\alpha b}\sigma(\alpha x_1) \ge -\sigma(\alpha x_1)$. This asserts the existence of another maximum critical point X_2 such that $\sigma(X_2) \ge |\sigma(x_1)|$. By repeated applications of this argument, we can construct two sequences $\{X_n\}, \{x_n\}$ so that $\lim_{n\to\infty} X_n = \infty$ and $\lim_{n\to\infty} x_n = \infty$, and $\sigma(X_n) - \sigma(x_n) \ge \sigma(X_1) - \sigma(x_1) > 0$ for all $n \in N$, contradicting $\sigma(\infty) = 0$. Hence we get the result.■

If $b < 0$, the roots of the indicial equation (3.5) are real and of opposite signs. Therefore there can be only one Dirichlet series solution to Problem 3. Theorem 3.2.1 then assures us that if also $\alpha > 1$ and $|c| \le \alpha|b|$, there is no other solution. The last condition can be relaxed, provided that an integrability condition is added. This is the content of the next theorem.

Theorem 3.2.2 *If $|c| < \alpha^n|b|$ and $b < 0$, then there exists a unique n^{th} integrable solution to Problem 3 for some $n \in N$. Furthermore, if $b < 0$, then the infinitely integrable solution $y(x)$ is unique.*

Proof: Existence follows from Section 3.1. To prove uniqueness, we begin with the case $n = 2$. Let $z(x)$ and $\sigma(x)$ be defined as those in the proof of Theorem 3.2.1. Since $y(x)$ is twice integrable, $\sigma(x) = \int_x^\infty y(t)\,dt$ is integrable so that we integrate equation (3.11) from 0 to ∞ to get the equation

$$-\sigma'(0) - a\sigma(0) + (b + c/\alpha^2)\int_0^\infty \sigma(t)\,dt = 0.$$

Now $\sigma'(0) = -z(0) = 0$ and $\sigma(0) = \int_0^\infty z(t)\,dt = 0$ so that $\int_0^\infty \sigma(t)\,dt = 0$ since $b + c/\alpha^2 \neq 0$. Applying the transformation

$$\sigma_2(x) = \int_x^\infty \sigma(t)dt, \tag{3.14}$$

to equation (3.11) produces the equation

$$\sigma_2''(x) + a\sigma_2'(x) + b\sigma_2(x) + \frac{c}{\alpha^2}\sigma_2(\alpha x) = 0, \tag{3.15}$$

and

$$\sigma_2(0) = 0, \quad \sigma_2(\infty) = 0. \tag{3.16}$$

Since $|c| < \alpha^2|b|$, we can show that $\sigma_2(x) = 0$ using the same arguments as those used in the proof of $\sigma(x) = 0$ in Theorem 3.2.1; thus, $z(x) = 0$.

For the n^{th} integrable solutions for $n \geq 3$, using the transformation

$$\sigma_n = \int_x^\infty \sigma_{n-1}(t)\, dt,$$

where $\sigma_1(x) = \sigma(x) = \int_x^\infty z(t)\, dt$, we get the equation

$$\sigma_n''(x) + a\sigma_n'(x) + b\sigma_n(x) + \frac{c}{\alpha^n}\sigma_n(\alpha x) = 0, \tag{3.17}$$

and it can be proven that $y(x)$ is unique if $|c| < \alpha^n|b|$ in the arguments similar to the case $n = 2$. This indicates that if $n \to \infty$, then the coefficient c can be any real value so that we get the last result.■

3.3 Positive Solutions

In this section, we investigate the existence of positive solutions and certain qualitative properties of the positive solutions. The interest in these solutions is motivated from the cell growth model.

3.3.1 Existence of Solutions

Theorem 3.3.1 *If $b < 0$ and $|c| \leq \alpha|b|$, then the solution to Problem 3 is positive for $x \in [0, \infty)$.*

Proof: Using the transformation (3.10), the differential equation (3.1) can be converted into the differential equation (3.11), the solution of which satisfies the condition (3.12). Moreover,

$$\sigma'(x) = -y(x)\,.$$

The proof that $\sigma(x)$ cannot have a positive maximum nor a negative minimum is the same as that of Theorem 3.2.1. Therefore, since $\sigma(0) = 1 > 0$ and $\sigma(\infty) = 0$, we have that $\sigma'(x) \leq 0$. We will show that $\sigma(x)$ is strictly decreasing. Suppose that there is a point $w > 0$ such that $\sigma'(w) = 0$, then $\sigma''(w) = 0$ since σ cannot have local maxima nor local minima and hence $b\sigma(w) + \frac{c}{\alpha}\sigma(\alpha w) = 0$. This equation is obviously not satisfied when $\alpha b \leq c < -\alpha b$. If $c = -\alpha b$, then $\sigma(w) = \sigma(\alpha w)$ and this implies that $\sigma(x)$ is constant C in $[w, \alpha w]$. Since equation (3.11) yields $C = 0$, it can be shown that $\sigma(x)$ cannot be constant in any intervals of the form $[\hat{x}, \alpha\hat{x}]$ for $\hat{x} \in [0, \infty)$, using the same method as that used in the proof of Lemma 2.2.1; thus, $\sigma'(x) < 0$ for all $x > 0$. It now remains to prove that $\sigma'(0) < 0$. Suppose $\sigma'(0) = 0$. Equation (3.11) can be recast into the form

$$(e^{ax}\sigma'(x))' + be^{ax}\sigma(x) + \frac{c}{\alpha}e^{ax}\sigma(\alpha x) = 0, \tag{3.18}$$

and integrating the above equation from 0 to x, noting that $\sigma'(0) = 0$, produces the equation

$$e^{ax}\sigma'(x) = -\int_0^x e^{at}(b\sigma(t) + \frac{c}{\alpha}\sigma(\alpha t))\,dt.$$

Now, $b\sigma(t) + \frac{c}{\alpha}\sigma(\alpha t) < 0$ since $|b| \geq |c|/\alpha$ and $\sigma(t) > \sigma(\alpha t)$ for all $t > 0$, so that $\sigma'(x) > 0$. This contradicts that $\sigma'(x) \leq 0$ and thus, $y(x) > 0$ for all $x \geq 0$.■

3.3.2 Qualitative Properties of Solutions

In this section, we examine the conditions which the solution $y(x)$ to Problem 3 is strictly monotonic decreasing and the bounds on $y(x)$ for a certain range of coefficients.

Shape of Solutions

Theorem 3.3.2 *Suppose $b < 0$.*
If $-\alpha|b| \leq c \leq |b| (c \neq 0)$ or if $a > 0$ and $|b| < c < \alpha|b|$, then all solutions $y(x)$ to Problem 3 are strictly monotonic decreasing for all $x \geq 0$.

Proof: For either case in this theorem, the conditions of Theorem 3.3.1 are satisfied and therefore, the solution is positive. If the solution is not strictly monotonic decreasing, then it must have at least one point v such that $y'(v) = 0$ and $y''(v) \leq 0$ in $[0, \infty)$. We use this fact to prove the result.

Let $-\alpha|b| \leq c < 0$ and a point v be mentioned above. Then equation (3.1) implies that

$$by(v) + cy(\alpha v) \geq 0. \tag{3.19}$$

But, the solution $y(x)$ is positive and $b < 0$ so that the inequality (3.19) cannot be satisfied; thus, y is strictly monotonic decreasing in that case.

For the case that $0 < c \leq |b|$, let $X_1 \in [0, \infty)$ be a point satisfying that $y'(X_1) = 0$ and $y''(X_1) \leq 0$. Then $y(X_1) \leq -\frac{c}{b}y(\alpha X_1)$ from equation (3.1) and therefore, $y(X_1) \leq y(\alpha X_1)$. This indicates that y has a local critical point or it is constant in $[X_1, \alpha X_1]$, and in either case there exists a point X_2 such that $y'(X_2) = 0$, $y''(X_2) \leq 0$ and $y(X_2) \geq y(X_1)$. Repeating the arguments thus constructs a sequence $\{X_n\}$ such that $\lim_{n\to\infty} X_n = \infty$ and $y(X_n) \geq y(X_1) > 0$ for all n. This contradicts the condition (3.2).

Lastly, we consider the case that $a > 0$ and $|b| < c < \alpha|b|$. Suppose that there is a point w satisfying $\sigma''(w) \leq 0$, where σ satisfies equation (3.11), then $\sigma(w) < \frac{-c}{b\alpha}\sigma(\alpha w)$ since $\sigma'(w) < 0$ so that $\sigma(w) < \sigma(\alpha w)$. However,

$$\sigma(w) = \int_w^\infty y(x)\,dx > \int_{\alpha w}^\infty y(x)\,dx = \sigma(\alpha w),$$

and this contradicts the previous result. It follows thus that $\sigma''(x) > 0$ for all $x \geq 0$ and so $y'(x) < 0$ since $\sigma''(x) = -y'(x)$.■

From the above theorems, we see that given $b < 0$ and $|c| < \alpha|b|$, the conditions $a < 0$ and $c > 0$ are necessary in order to have a positive solution with a maximum critical point. These sign combinations of the coefficients are the same as those of the cell growth model in Chapter 2. The following theorems can be extended from Theorem 2.4.2 and 2.4.3 if $b < 0$ and $|c| \leq \alpha|b|$:

Theorem 3.3.3 *The solution $y(x)$ to Problem 3 has at most one maximum critical point.*

Theorem 3.3.4 *If a critical point X_m exists, the solution $y(x)$ has at most one inflexion point in each interval of the form $[0, X_m)$ or (X_m, ∞); otherwise, there exists at most one inflexion point in $[0, \infty)$.*

Bounds for Solutions

Assume that $b < 0$ and $|c| \leq \alpha|b|$.

Case1: $a < 0$ and $c > 0$
Since $c/\alpha > 0$, integrating equation (3.18) from x to ∞ leads to the inequality

$$e^{ax}\sigma'(x) = b\int_x^\infty e^{at}\sigma(t)\,dt + \frac{c}{\alpha}\int_x^\infty e^{at}\sigma(\alpha t)\,dt \geq b\int_x^\infty e^{at}\sigma(t)\,dt. \tag{3.20}$$

Now, $\sigma(x) \leq 1$ so that $e^{ax}\sigma'(x) \geq b\int_x^\infty e^{at}\,dt$ and therefore, $\sigma'(x) \geq -b/a$; thus,

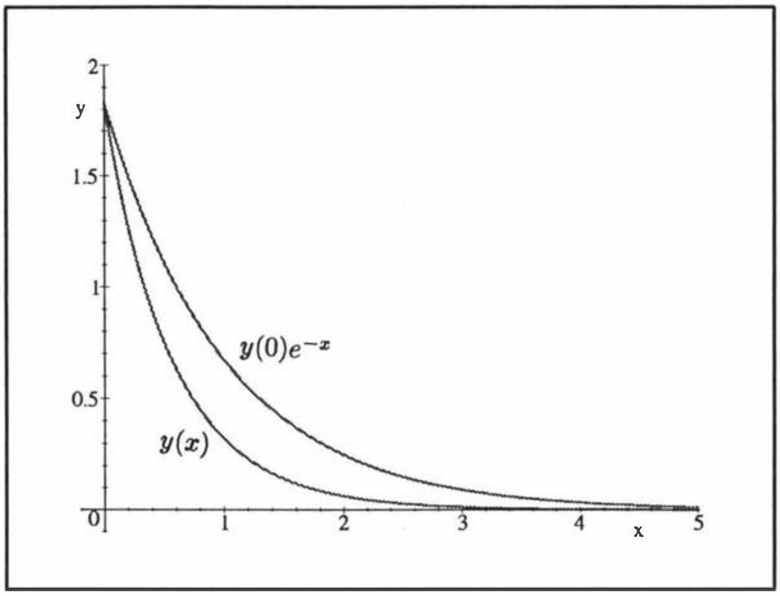

Figure 3.3.1: The bound $y(0)e^{-x}$ for $y(x)$ when $a = 1$, $b = -1$, $c = -1$ and $\alpha = 2$.

$$y(x) \leq b/a,$$

so that if $a \to -\infty$ or $b \to 0^-$, then $y(x) \to 0$.

Case2: $a > 0$
Integrating equation (3.18) from 0 to x produces the equation

$$e^{ax}\sigma'(x) = \sigma'(0) - b\int_0^x e^{at}\sigma(t)\,dt - \frac{c}{\alpha}\int_0^x e^{at}\sigma(\alpha t)\,dt. \tag{3.21}$$

If $c > 0$, then

$$\begin{aligned} e^{ax}\sigma'(x) \geq \sigma'(0) - \frac{c}{\alpha}\int_0^x e^{at}\sigma(\alpha t)\,dt &\geq \sigma'(0) - \frac{c}{\alpha}\int_0^x e^{at}\,dt \\ &= \sigma'(0) - \frac{c}{a\alpha}(e^{ax} - 1), \end{aligned}$$

where we use $\sigma(\alpha x) \leq 1$ for all $x \geq 0$.
The above inequalities thus imply that $\sigma'(x) \geq (\sigma'(0) + \frac{c}{a\alpha})e^{-ax} - \frac{c}{a\alpha}$ so that

$$y(x) \leq (y(0) - \frac{c}{a\alpha})e^{-ax} + \frac{c}{a\alpha}.$$

If $c < 0$, then $e^{ax}\sigma'(x) \geq \sigma'(0)$ from the equality (3.21) and so we have the more elegant bound

$$y(x) \leq y(0)e^{-ax}.$$

Figure 3.3.1 illustrates the bound for $y(x)$ corresponding to the case that $a = 1$, $b = -1$, $c = -1$ and $\alpha = 2$.

3.4 The Limiting Cases and Holomorphicity of Solutions

In this section, Dirichlet series solutions for the limiting cases when a, b or $c \to 0$ or ∞ are considered. Moreover, holomorphicity of the solutions is investigated. Here, we consider only the case when r is real.

Before any limiting cases and holomorphicity of the series are examined, we will first show that the limit of the series can go inside the summation of the series with respect to the parameters a, b and c.

3.4.1 Uniform Convergence of the Series

Let $y = y(x, a, b, c)$, then

$$\begin{aligned} y(x,a,b,c)/a_0 &= \sum_{n=0}^{\infty} \frac{(-c)^n}{\prod_{m=1}^{n}(\alpha^{2m}r^2 - a\alpha^m r + b)} e^{-\alpha^n r x} \\ &\leq \sum_{n=0}^{\infty} \frac{|c|^n}{\prod_{m=1}^{n}|\alpha^{2m}r^2 - a\alpha^m r + b|} \\ &= g(a,b,c), \end{aligned}$$

and since $r^2 = ar - b$,

$$g(a,b,c) = 1 + \sum_{n=1}^{\infty} \frac{|c|^n}{\prod_{m=1}^{n}|(\alpha^m - 1)(ar\alpha^m - b(\alpha^m + 1))|}. \tag{3.22}$$

It is evident that the series for $y(x, a, b, c)/a_0$ is uniformly convergent for c in any bounded domain because the series for $g(a, b, c)$ converges for all finite c.

Let $r_1 = \frac{a+\sqrt{a^2-4b}}{2}$ and $r_2 = \frac{a-\sqrt{a^2-4b}}{2}$, then $r_1 \geq r_2$. If $r = r_1$, then the coefficients a_n becomes

$$a_n = \frac{(-c)^n}{\prod_{m=1}^{n}(\alpha^{2m}r^2 - a\alpha^m r + b)} a_0 = \frac{(-c)^n}{\prod_{m=1}^{n}(\alpha^m r_2 - r_2)(\alpha^m r_2 - r_1)} a_0,$$

so that a singularity occurs when $r_2 = 0$ or $r_1 = \alpha^m r_2$ for some $m \in N$. If $r_1 = \alpha^m r_2$, then $b = \frac{\alpha^m}{(\alpha^m+1)^2}a^2$ since $a = r_1 + r_2 = (\alpha^m + 1)r_2$ and $b = r_1 r_2 = \alpha^m r_2^2$ and this case occurs only when $a > 0$ and $b > 0$. We now get the region of a and b, which doesn't have a singularity. We first except the case when $a \leq 0$ and $b \geq 0$ since a Dirichlet series solution with real $r > 0$ doesn't exist in the region. If $b < 0$, then r_1 can be only a positive root and there is no singularity for all a. If $b \geq 0$ and $a > 0$, then $b \leq a^2/4$ to have a real solution and a singularity occurs at $b = \frac{\alpha^k}{(\alpha^k+1)^2}a^2$, where $k = 1, 2, ...$ for r_2, but not for r_1. From these observations, we consider the uniform convergence of $y(x, a, b, c)/a_0$ w.r.t. a in the following regions:

(i) $b < 0$, $a \in R$ and $r = r_1$
(ii) $b > 0$, $a \geq 2\sqrt{b}$ and $r = r_1$
(iii) $b = 0$, $a > 0$ and $r = r_1$
(iv) $b > 0$, $a \geq 2\sqrt{b}$ and $r = r_2$ except at $a = \sqrt{\frac{(\alpha^k+1)^2}{\alpha^k}b}$, where $k = 1, 2, ...$

After arranging the above regions from the view point of b, we consider the uniform convergence of $y(x, a, b, c)/a_0$ w.r.t. b in the following regions:

(i) $a > 0$, $b \leq a^2/4$ and $r = r_1$
(ii) $a = 0$ and $b < 0$ and $r = r_1$
(iii) $a < 0$, $b < 0$ and $r = r_1$
(iv) $a > 0$, $0 < b \leq a^2/4$ and $r = r_2$ except at $b = \frac{\alpha^k}{(\alpha^k+1)^2}a^2$, where $k = 1, 2, ...$

For $m = 1, 2, \ldots$, let

$$\begin{aligned} g_m(a,b) &= \frac{1}{(\alpha^m - 1)(ar_1\alpha^m - b(\alpha^m + 1))} \\ &= \frac{1}{(\alpha^m - 1)(\frac{1}{2}(a^2 + a\sqrt{a^2 - 4b})\alpha^m - b(\alpha^m + 1))}. \end{aligned} \tag{3.23}$$

Note that $g_m(a, b) > 0$ since

$$\begin{aligned} (\alpha^m - 1)(ar_1\alpha^m - b(\alpha^m + 1)) &= \alpha^{2m}r_1^2 - a\alpha^m r_1 + b \\ &= r_1(\alpha^m - 1)(\alpha^m r_1 - r_2) \\ &> 0, \end{aligned}$$

and consequently, $|g_m(a, b)| = g_m(a, b)$.

Now,

$$a_0 = r\left(\sum_{n=0}^{\infty} \frac{(-c/\alpha)^n}{\prod_{m=1}^{n}(\alpha^{2m}r^2 - a\alpha^m r + b)}\right)^{-1},$$

and

$$\sum_{n=0}^{\infty}\left|\frac{(-c/\alpha)^n}{\prod_{m=1}^{n}(\alpha^{2m}r^2-a\alpha^m r+b)}\right| \leq \sum_{n=0}^{\infty}\frac{|c|^n}{\prod_{m=1}^{n}|\alpha^{2m}r^2-a\alpha^m r+b|} = g(a,b,c),$$

and thus, in order to show the uniform convergence of the series $y(x,a,b,c)/a_0$ and a_0 w.r.t. a or b, we now seek a bound of $g_m(a,b)$ which is independent of a or b, making the series $g(a,b,c)$ be uniformly convergent.

Uniform Convergence of the Series w.r.t. a

Let b be fixed.

Case 1: $b<0$, $a\in R$ and $r=r_1$.
Differentiating $g_m(a,b)$ defined by (3.23) w.r.t. a leads to

$$\frac{\partial g_m(a,b)}{\partial a}=\frac{-\frac{1}{2}(2a+\sqrt{a^2-4b}+a^2/\sqrt{a^2-4b})\alpha^m}{(\alpha^m-1)(\frac{1}{2}(a^2+a\sqrt{a^2-4b})\alpha^m-b(\alpha^m+1))^2}, \tag{3.24}$$

and $\frac{\partial g_m(a,b)}{\partial a}<0$ in the region for $a\in R$. This implies that g_m has a maximum as $a\to-\infty$. To obtain $\lim_{a\to-\infty}g_m(a,b)$, let us first calculate the value $h=\lim_{a\to-\infty}\frac{1}{2}(a^2+a\sqrt{a^2-4b})$. Since

$$\begin{aligned} h &= \lim_{a\to-\infty}\frac{(a^2+a\sqrt{a^2-4b})(a^2-a\sqrt{a^2-4b})}{2(a^2-a\sqrt{a^2-4b})} \\ &= \lim_{a\to-\infty}\frac{a^4-a^2(a^2-4b)}{2(a^2-a\sqrt{a^2-4b})} \\ &= \lim_{a\to-\infty}\frac{2ab}{a-\sqrt{a^2-4b}}, \end{aligned}$$

we apply *l'Hôspital's* rule to get

$$\begin{aligned} h &= \lim_{a\to-\infty}\frac{2b}{1-\frac{a}{\sqrt{a^2-4b}}} \\ &= \frac{2b}{1+1}=b. \end{aligned}$$

Hence

$$\lim_{a\to-\infty}g_m(a,b)=\frac{1}{|b|(\alpha^m-1)},$$

so that

$$g(a,b,c)\leq 1+\sum_{n=1}^{\infty}\frac{|c|^n}{\prod_{m=1}^{n}|b|(\alpha^m-1)},$$

and since this series is convergent, $g(a, b, c)$ is uniformly convergent w.r.t. a; thus, $y(x, a, b, c)/a_0$ and a_0 are uniformly convergent in the region.

Case 2: $b > 0$, $a \geq 2\sqrt{b}$ and $r = r_1$.
From (3.24), $\frac{\partial g_m(a,b)}{\partial a} < 0$ for $a \geq 2\sqrt{b}$ and so in the region, g_m has the maximum $\frac{1}{|b|(\alpha^m-1)^2}$ when $a = 2\sqrt{b}$ so that

$$g(a, b, c) \leq 1 + \sum_{n=1}^{\infty} \frac{|c|^n}{\prod_{m=1}^{n} |b|(\alpha^m - 1)^2}. \tag{3.25}$$

Therefore, $y(x, a, b, c)/a_0$ and a_0 are uniformly convergent in this region.

Case 3: $b = 0$, $a > 0$ and $r = r_1$.
Now, $\frac{\partial g_m(a,b)}{\partial a} < 0$ for $a > 0$ so that g_m has a maximum as $a \to 0^+$ in the region. Since $\lim_{a \to 0^+} g_m(a, b) = \infty$, we use the Weierstrass theory that if X is locally compact, then the series which is convergent on every compact subset in X (i.e. compactly convergent series) is locally uniformly convergent in X. Let

$$C = \{[p, \infty) | p > 0\},$$

then C is locally compact and for any compact set $[p, \infty)$ in C, $g(a, b, c)$ is convergent. Consequently, $g(a, b, c)$ is compactly convergent in C, implying the locally uniform convergence of the function in C; thus, $y(x, a, b, c)/a_0$ and a_0 are locally uniformly convergent w.r.t. a in this region.

Case 4: $b > 0$, $a \geq 2\sqrt{b}$ and $r = r_2$ except at $a = \sqrt{\frac{(\alpha^k+1)^2}{\alpha^k} b}$ for $k = 1, 2, \ldots$.
Since all singularities are isolated in this region, we can divide the region as an infinitely countable number of compact sets. Let

$$D_1 = \left\{ [p, q] | 2\sqrt{b} \leq p < q < \sqrt{\frac{(\alpha + 1)^2}{\alpha} b} \right\},$$

and for $k \geq 1$,

$$D_{k+1} = \left\{ [p, q] | \sqrt{\frac{(\alpha^k + 1)^2}{\alpha^k} b} < p < q < \sqrt{\frac{(\alpha^{k+1} + 1)^2}{\alpha^{k+1}} b} \right\},$$

and $D = \bigcup_{k=1}^{\infty} D_k$. Then the set D is locally compact. For any sum of intervals of the form $[p, q]$ in D, $g(a, b, c)$ is convergent and therefore, $g(a, b, c)$ is locally uniformly convergent in D; thus, $y(x, a, b, c)/a_0$ and a_0 are locally uniformly convergent in D.

Uniform Convergence of the Series w.r.t. b

Let a be fixed.

Case 1: $a > 0$, $b \le a^2/4$ *and* $r = r_1$.
By differentiating $g_m(a,b)$ w.r.t. b , we get

$$\frac{\partial g_m(a,b)}{\partial b} = \frac{a\alpha^m/\sqrt{a^2-4b} + (\alpha^m+1)}{(\alpha^m-1)(\frac{a^2+a\sqrt{a^2-4b}}{2}\alpha^m - b(\alpha^m+1))^2}, \tag{3.26}$$

and $\frac{\partial g_m(a,b)}{\partial b} > 0$ for $b \le a^2/4$. From this observation, we have a maximum of g_m at $b = a^2/4$ and the maximum value is $\frac{4}{a^2(\alpha^m-1)^2}$ in the region. Hence

$$g(a,b,c) \le 1 + \sum_{n=1}^{\infty} \frac{4|c|^n}{\prod_{m=1}^{n} a^2(\alpha^m-1)^2},$$

and thus, $y(x,a,b,c)/a_0$ and a_0 are uniformly convergent w.r.t. b in the region since the above series is convergent.

Case 2: $a = 0$, $b < 0$ *and* $r = r_1$.
Since $\frac{\partial g_m(a,b)}{\partial b} > 0$ for $b < 0$, $g_m(a,b)$ has a maximum as $b \to 0^-$, and $\lim_{b\to 0^-} g_m(a,b) = \infty$. Let

$$E = \{(-\infty, q] | q < 0\},$$

then the set is locally compact. For every compact set $(-\infty, q]$ in E, $g(a,b,c)$ is convergent. Therefore, by the Weierstrass theory, $g(a,b,c)$ is locally uniformly convergent in E; thus, $y(x,a,b,c)/a_0$ and a_0 are locally uniformly convergent w.r.t. b in this region.

Case 3: $a < 0$, $b < 0$ *and* $r = r_1$.
Now, $\frac{\partial g_m(a,b)}{\partial b} > 0$ for $b < 0$ so that g_m achieves a maximum as $b \to 0^-$, and $\lim_{b\to 0^-} g_m(a,b) = \infty$. So, we get the result using the same arguments as those used in the proof of Case 2.

Case 4: $a > 0$, $0 < b \le a^2/4$ *and* $r = r_2$ *except at* $b = \frac{\alpha^k}{(\alpha^k+1)^2}a^2$, *for* $k = 1, 2, \ldots$.
We can prove the locally uniform convergence of $y(x,a,b,c)/a_0$ and a_0 w.r.t. b in this region in the arguments similar to those used in the proof of Case 4 for the locally uniform convergence w.r.t. a by letting

$$F_1 = \{[p,q] | \frac{\alpha}{(\alpha+1)^2}a^2 < p < q \le a^2/4\},$$

and for $k \ge 2$,

$$F_k = \{[p,q] | \frac{\alpha^k}{(\alpha^k+1)^2}a^2 < p < q < \frac{\alpha^{k-1}}{(\alpha^{k-1}+1)^2}a^2\},$$

and $F = \bigcup_{k=1}^{\infty} F_k$.

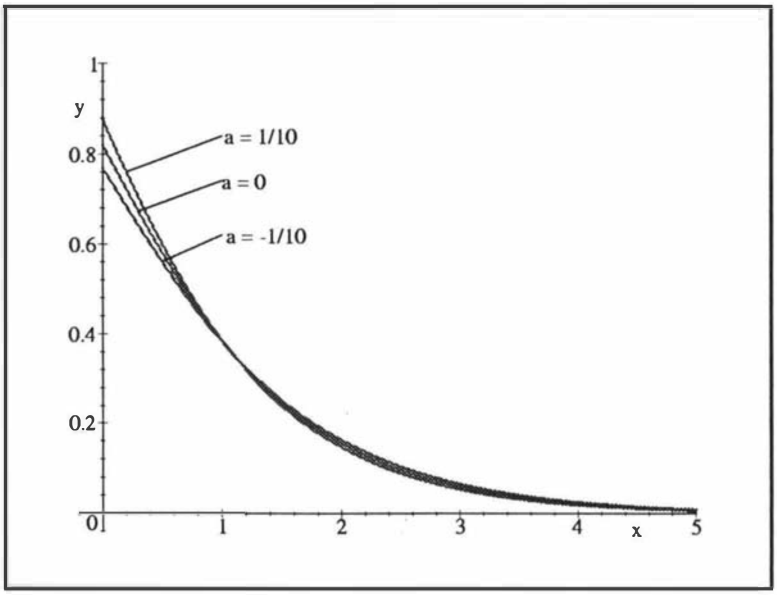

Figure 3.3.2: y for small values of a when $b = -1, c = 1$ and $\alpha = 2$.

3.4.2 The Limiting Cases and Holomorphicity of Solutions

The Limiting Case as $a \to 0$

The limiting case when $a \to 0$ can be considered only when $b < 0$ since $y(x)$ is uniformly convergent at $a = 0$ only when $b < 0$. When $a \to 0$, $r_1 \to \sqrt{-b}$ so that $a_n \to (-c/b)^n / \Pi_{m=1}^n (1 - \alpha^{2m})$; thus,

$$y(x) \to a_0 \sum_0^\infty \frac{(-c/b)^n}{\Pi_{m=1}^n (1 - \alpha^{2m})} e^{-\alpha^n \sqrt{-b} x},$$

and this is the same as the solution to the equation when $a = 0$. This result is the same whether $a \to 0^+$ or $a \to 0^-$. Figure 3.3.2 shows the change of solutions around $a = 0$ when the coefficient a approaches zero from the positive direction or the negative direction. Let us now examine the holomorphicity of the solution at $a = 0$. As we have seen before, if $b < 0$, terms in the series are all holomorphic and therefore, the uniform convergence of the series leads to the holomorphicity of the solution at $a = 0$.

The Limiting Case as $b \to 0$

The case when $a > 0$, $b \leq a^2/4$ and $r = r_1$ is only considered here since $y(x)$ is uniformly convergent at $b = 0$ in this case. When $b \to 0$, we have $r_1 \to a$ so that

$$y(x) \to a \left(\sum_{n=0}^\infty \frac{(-\frac{c}{a^2 \alpha})^n}{\Pi_{m=1}^n \alpha^m (\alpha^m - 1)} \right)^{-1} \sum_{n=0}^\infty \frac{(-c/a^2)^n}{\Pi_{m=1}^n \alpha^m (\alpha^m - 1)} e^{-\alpha^n a x}.$$

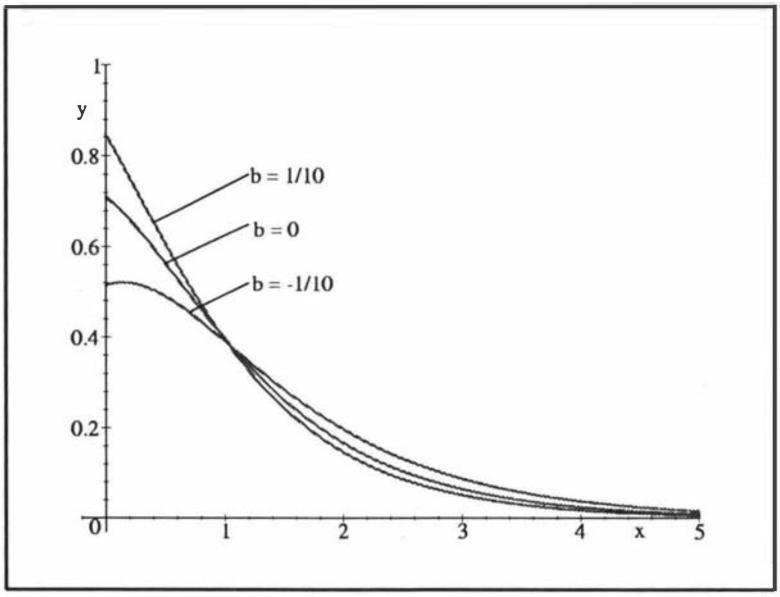

Figure 3.3.3: y for small values of b when $a = 1$, $c = 1$ and $\alpha = 2$.

Figure 3.3.3 shows this result. Moreover, the solution is holomorphic at $b = 0$ in this case since the previous observation indicates that all terms are holomorphic so that the uniform convergence of the series leads to the holomorphicity of the solution at $b = 0$.

The Limiting Case as $c \to 0$

When $c \to 0$, we have $a_n \to 0$ for all $n \geq 1$ since

$$a_n = a_0 \frac{(-c)^n}{\prod_{m=1}^{n}(\alpha^{2m}r^2 - a\alpha^m r + b)},$$

and $a_0 \to r$, so that

$$y(x) \to re^{-rx}. \tag{3.27}$$

Here, the value r is not changed since the coefficient c does not affect it.

It is evident that the solution $y(x)$ is the same as that to the equation

$$u''(x) + au'(x) + bu(x) = 0 \tag{3.28}$$

satisfying the boundary conditions (3.2) and (3.3). Figure 3.3.4 illustrates the shape of the graph of the solution for c small. Moreover, it is evident that every term a_n is holomorphic at $c = 0$ since there are no singular points and consequently the uniform convergence of the series leads to the holomorphicity of the solution at $c = 0$.

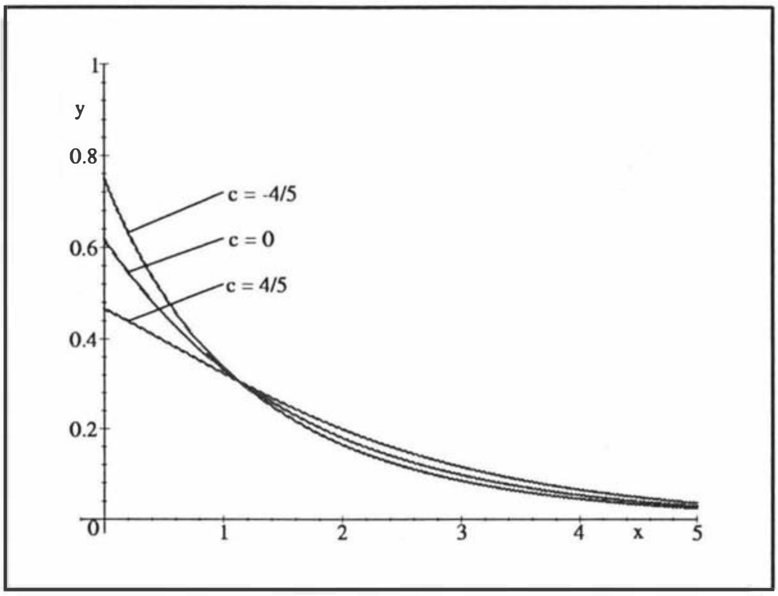

Figure 3.3.4: y for small values of c when $a = -1, b = -1$ and $\alpha = 2$.

The Limiting Case as $a \to \infty$

We consider two cases according to the value of r. If $r = r_1$, then $r \to \infty$ and if $r = r_2$, then $r \to 0$ as $a \to \infty$. Now,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{a_n}{a_0} e^{-\alpha^n r_1 x} &\leq e^{-r_1 x} \sum_{n=0}^{\infty} \left|\frac{a_n}{a_0}\right| \\ &\leq e^{-r_1 x} \sum_{n=0}^{\infty} \frac{|c|^n}{\prod_{m=1}^{n} |\alpha^{2m} r_1^2 - a\alpha^m r_1 + b|} \\ &= e^{-r_1 x} V,\end{aligned}$$

and

$$\begin{aligned}a_0 &\leq r_1 \left| \sum_{n=0}^{\infty} \frac{(-c/\alpha)^n}{\prod_{m=1}^{n} (\alpha^{2m} r_1^2 - a\alpha^m r_1 + b)} \right|^{-1} \\ &= r_1 / W.\end{aligned}$$

As $a \to \infty$, $r_1 \to \infty$ so that $V \to 1$ and $W \to 1$. Consequently, for $x > 0$,

$$\begin{aligned}|y(x)| &\leq \frac{V}{W} r_1 e^{-r_1 x} \\ &\to 0,\end{aligned}$$

and thus, $y(x) \to 0$. At $x = 0$, the value $r_1 x$ can be some constant C or zero or infinity depending on how fast x approches zero according to the value r_1 and so $y(0)$

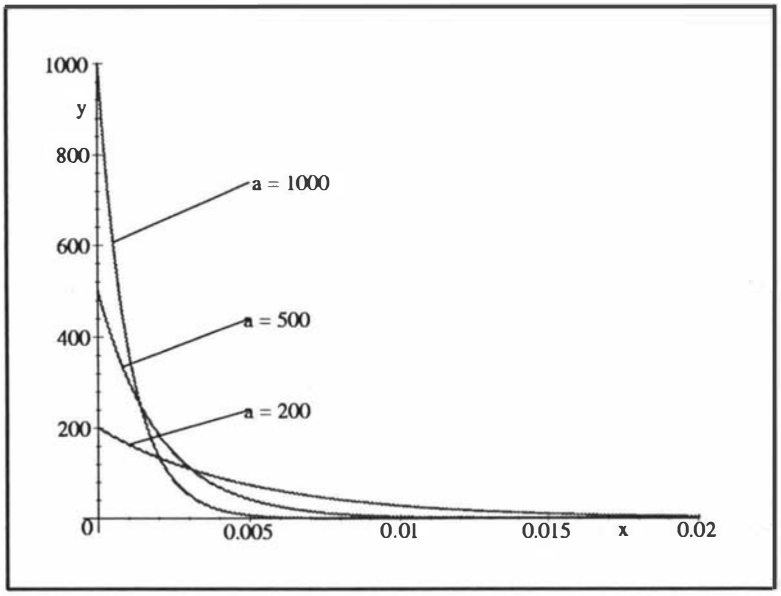

Figure 3.3.5: y for large values of a when $r_1 \to \infty$ according to $b = 1, c = 1$ and $\alpha = 2$.

cannot be determined. Figure 3.3.5 shows that the solution is going to zero for all x except the original point. For the second case, there is a singularity as we observed. However, the singularities are isolated so that we consider only the region which has no singularities. When $r_2 \to 0$, we have $ar_2 \to b$ from the indicial equation (3.5) so that $a_n \to a_0(-c/b)^n/\Pi_{m=1}^{n}(1-\alpha^m)$ and, $a_0 \to 0$. Therefore, $y(x) \to 0$. Figure 3.3.6 depicts the result.

The Limiting Case as $a \to -\infty$

This case is considered only when $b < 0$ and $r = r_1$. In the same manner as the previous case, it can be shown that $y(x) \to 0$ since $r_1 \to 0$ as $a \to -\infty$.

The Limiting Case as $b \to -\infty$

The solution y is uniformly convergent for all a in this case. As $b \to -\infty$, $r_1 \to \infty$ so that we get $y(x) \to 0$ using the same arguments those used in the case when $a \to \infty$ and $r_1 \to \infty$.

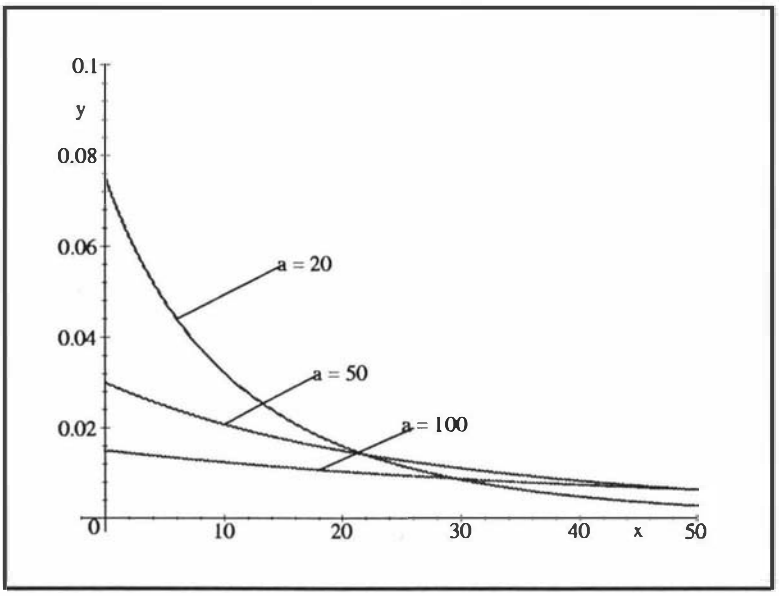

Figure 3.3.6: y for large values of a when $r_2 \to 0$ according to $b = 1, c = 1$ and $\alpha = 2$.

3.5 Oscillating Solutions

If the roots of the indicial equation (3.5) are complex and $Re(r) \geq 0$, then equation (3.1) will have real solutions with Cosine or Sine functions in the terms of the series, and we expect these solutions to be oscillatory . In this section, the two cases when $a = 0$ and $b > 0$ or when $a > 0$ and $b > \frac{a^2}{4}$ which yield complex roots to the indicial equation are investigated since the behaviour of solutions when $a = 0$ is different from that when $a > 0$. Here we seek merely a bounded solution without imposing boundary conditions (3.2) and integrability condition (3.3).

3.5.1 The Case $a = 0$ and $b > 0$

If $a = 0$ and $b > 0$, then equation (3.1) becomes

$$y''(x) + by(x) + cy(\alpha x) = 0. \tag{3.29}$$

The next theorem shows that if a bounded solution exists to equation (3.29), then it is oscillating.

Theorem 3.5.1 *If $-b < c$ and $c \neq 0$, then every bounded non-trivial solution to equation* (3.29) *is oscillating.*

Proof: Suppose y is a bounded function which does not oscillate. Then

$\lim_{x\to\infty} y(x) = C$ for some constant C. So, $\lim_{x\to\infty} y''(x) = -(b+c)C$ from equation (3.29), and since $\lim_{x\to\infty} y''(x) = 0$ and $b+c > 0$, $C = 0$. Suppose a solution to equation (3.29) is not oscillating. Then there exists a point w such that for all $x > w$, $y'(x) \leq 0$ and $y''(x) \geq 0$ if $y(x) \geq 0$ or $y'(x) \geq 0$ and $y''(x) \leq 0$ if $y(x) \leq 0$. Let $y(x) \geq 0$ for all $x > w$, then equation (3.29) implies that $y''(x) < 0$ for all $x > w$ since $by(x) + cy(\alpha x) > 0$. This contradicts that $y''(x) \leq 0$ in that interval and thus there must be a point $w_1 > w$ satisfying $y(w_1) = 0$ and $y(w_1 + \epsilon) < 0$ for $\epsilon > 0$. Suppose that for $x > w_1$, $y(x) \leq 0$, then it can be shown that $y''(x) > 0$ in that interval from equation (3.29). This leads to the existence of another point $w_2 > w_1$ satisfying $y(w_2) = 0$ and $y(w_2 + \epsilon) > 0$ for $\epsilon > 0$. This process can be repeated to get a sequence of zeros $\{w_n\}$, and so we conclude that the solution is oscillatory . ■

A substitution of the Dirichlet series (3.4) into equation (3.29) leads to the recurrence relation

$$(\alpha^{2n} r^2 + b) a_n = -c a_{n-1}, \tag{3.30}$$

and the indicial equation

$$r^2 + b = 0. \tag{3.31}$$

Hence,

$$a_n = \frac{(c/b)^n}{\prod_{m=1}^{n} (\alpha^{2m} - 1)} a_0,$$

and $r = \pm\sqrt{b}i$. In fact, we have the same result whether $r = \sqrt{b}i$ or $r = -\sqrt{b}i$ so that only the case $r = \sqrt{b}i$ is considered here.

Now, the solution $u(x)$ to equation (3.29) is given by

$$\begin{aligned} u(x) &= a_0 \sum_{n=0}^{\infty} \frac{(c/b)^n}{\prod_{m=1}^{n} (\alpha^{2m} - 1)} e^{-\alpha^n \sqrt{b} i x} \\ &= a_0 \sum_{n=0}^{\infty} \frac{(c/b)^n}{\prod_{m=1}^{n} (\alpha^{2m} - 1)} (\cos(\alpha^n \sqrt{b} x) - i \sin(\alpha^n \sqrt{b} x)). \end{aligned}$$

It is known that if the coefficients of a linear equation are real, then real and imaginary part of any complex solutions are again solutions to that equation. We can apply the theory to equation (3.29), and get the two real solutions,

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(c/b)^n}{\prod_{m=1}^{n} (\alpha^{2m} - 1)} \cos(\alpha^n \sqrt{b} x),$$

and

$$y_2(x) = \sum_{n=0}^{\infty} \frac{(c/b)^n}{\prod_{m=1}^{n}(\alpha^{2m}-1)} \sin(\alpha^n \sqrt{b}x).$$

Clearly, $y_1(x)$ is convergent since

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{a_n}{a_0} \cos(\alpha^n r x) &\leq \sum_{n=0}^{\infty} \left|\frac{a_n}{a_0} \cos(\alpha^n r x)\right| \\ &\leq \sum_{n=0}^{\infty} \left|\frac{a_n}{a_0}\right| < \infty, \end{aligned}$$

and so is $y_2(x)$. Let us examine the linear independence of y_1 and y_2. At $x = 0$, $y_2(0) = 0$ and

$$y_1(0) = \sum_{n=0}^{\infty} \frac{(c/b)^n}{\prod_{m=1}^{n}(\alpha^{2m}-1)} = \prod_{n=0}^{\infty} \left(1 + \frac{c/b}{\alpha^{2(n+1)}}\right),$$

where we use the Euler identity (we will study it in detail in Chapter 4). This implies that if $c \neq -b\alpha^{2n}$ for $n \in N$, then $y_1(0) \neq 0$ and so y_1 and y_2 are linearly independent in that case. Now $y_1'(0) = 0$ and

$$y_2'(0) = \sqrt{b} \sum_{n=0}^{\infty} \frac{(\alpha c/b)^n}{\prod_{m=1}^{n}(\alpha^{2m}-1)} = \sqrt{b} \prod_{n=0}^{\infty} \left(1 + \frac{c/b}{\alpha^{2n+1}}\right),$$

so that $y_2'(0) \neq 0$ when $c = -b\alpha^{2n}$ for all $n \in N$. This thus indicates that y_1 and y_2 are linearly independent for any b, c, α. Therefore, a more general real solution of the equation (3.29) is given by

$$\begin{aligned} y(x) &= Py_1(x) + Qy_2(x) \\ &= \sum_{n=0}^{\infty} \frac{(c/b)^n}{\prod_{m=1}^{n}(\alpha^{2m}-1)} (P\cos(\alpha^n\sqrt{b}x) + Q\sin(\alpha^n\sqrt{b}x)), \end{aligned} \tag{3.32}$$

where P, Q are real constants.

The solution $y(x)$ is bounded by $(|P|+|Q|)\sum_{n=0}^{\infty}|a_n/a_0|$, but it is not integrable for $x \geq 0$. Moreover, if $\alpha \in N$, then $y_2(x)$ has zeros at $x = \frac{l\pi}{\sqrt{b}}$ for $l = 1, 2, \ldots$ and at those points, $y_1(x) = (-1)^l P + P\sum_{n=1}^{\infty} \frac{(c/b)^n}{\prod_{m=1}^{n}(\alpha^{2m}-1)}$ if α is even and $y_1(x) = (-1)^l P\sum_{n=1}^{\infty} \frac{(c/b)^n}{\prod_{m=1}^{n}(\alpha^{2m}-1)}$ if α is odd, so that the solution $y(x)$ is oscillatory for $\alpha \in N$. Now given two initial conditions, the solution (3.32) is uniquely determined. Figure 3.3.7 shows the solution $y(x)$ when the initial conditions $y(0) = \sum_{n=0}^{\infty} \frac{1}{\prod_{m=1}^{n}(2^{2m}-1)}$ and $y'(0) = \sum_{n=0}^{\infty} \frac{2^n}{\prod_{m=1}^{n}(2^{2m}-1)}$ are given according to $b = 1, c = 1$ and $\alpha = 2$.

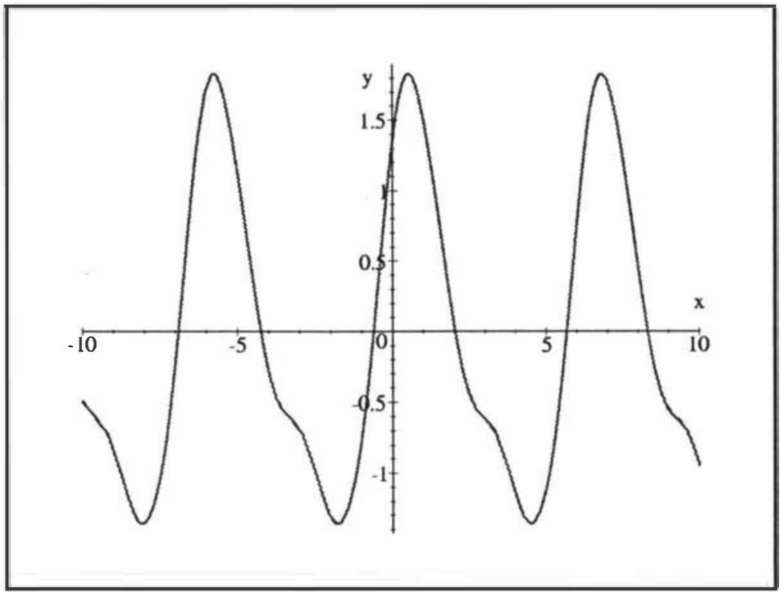

Figure 3.3.7: $y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Pi_{m=1}^{n}(1-2^{2m})}(\cos(2^n x) + \sin(2^n x))$.

3.5.2 The Case $a > 0$ and $b > \frac{a^2}{4}$

In the same manner as getting the recurrence relation and indicial equation from equation (3.1), we have $r = \frac{a \pm i\sqrt{4b-a^2}}{2}$ and the terms a_n satisfying $\sum_{n=0}^{\infty} a_n < \infty$. The terms a_n can be divided into a real part and a complex part and so let $a_n = b_{1n} + ib_{2n}$. Then $\sum_{n=0}^{\infty} b_{1n}$ and $\sum_{n=0}^{\infty} b_{2n}$ exist since $\sum_{n=0}^{\infty} a_n$ exists. As the previous case, the results for $r = \frac{a+i\sqrt{4b-a^2}}{2}$ and $r = \frac{a-i\sqrt{4b-a^2}}{2}$ are the same and so only the case $r = \frac{a+i\sqrt{4b-a^2}}{2}$ is considered here.

Now, the solution $u(x)$ to equation (3.1) when $a > 0$ and $b > 0$ is given by

$$\begin{aligned} u(x) &= \sum_{n=0}^{\infty}(b_{1n} + ib_{2n})e^{-\alpha^n \frac{a+i\sqrt{4b-a^2}}{2}x} \\ &= \sum_{n=0}^{\infty}(b_{1n} + ib_{2n})e^{-\alpha^n \frac{a}{2}x}(\cos(\alpha^n \frac{\sqrt{4b-a^2}}{2}x) - i\sin(\alpha^n \frac{\sqrt{4b-a^2}}{2}x)), \end{aligned}$$

and thus we have the following real solutions:

$$y_1(x) = \sum_{n=0}^{\infty}(b_{1n}\cos(\alpha^n \frac{\sqrt{4b-a^2}}{2}x) + b_{2n}\sin(\alpha^n \frac{\sqrt{4b-a^2}}{2}x))e^{-\alpha^n \frac{a}{2}x},$$

and

$$y_2(x) = \sum_{n=0}^{\infty}(b_{2n}\cos(\alpha^n \frac{\sqrt{4b-a^2}}{2}x) - b_{1n}\sin(\alpha^n \frac{\sqrt{4b-a^2}}{2}x))e^{-\alpha^n \frac{a}{2}x}.$$

We now show the linear independence of these two solutions y_1 and y_2. Let $k = \frac{\sqrt{4b-a^2}}{2}$. For large x,

$$y_1(x) \sim (b_{10}\cos(kx) + b_{20}\sin(kx))e^{-\frac{a}{2}x},$$

and

$$y_2(x) \sim (b_{20}\cos(kx) - b_{10}\sin(kx))e^{-\frac{a}{2}x},$$

so that

$$y_1'(x) \sim -\frac{a}{2}(b_{10}\cos(kx) + b_{20}\sin(kx))e^{-\frac{a}{2}x} - (kb_{10}\sin(kx) - kb_{20}\cos(kx))e^{-\frac{a}{2}x},$$

and

$$y_2'(x) \sim -\frac{a}{2}(b_{20}\cos(kx) - b_{10}\sin(kx))e^{-\frac{a}{2}x} - (kb_{20}\sin(kx) + kb_{10}\cos(kx))e^{-\frac{a}{2}x}.$$

Hence,

$$\begin{aligned} y_1(x)y_2'(x) - y_2(x)y_1'(x) &= -kb_{10}^2(\cos^2(kx) + \sin^2(kx)) - kb_{20}^2(\cos^2(kx) + \sin^2(kx)) \\ &= -k(b_{20}^2 + b_{10}^2)e^{-ax} \\ &\neq 0, \end{aligned}$$

and this implies that the two solutions are linearly independent. Therefore, the real general solution is

$$y(x) = Py_1(x) + Qy_2(x), \tag{3.33}$$

where P and Q are real constants.

Since $|\cos(\alpha^n rx)| \leq 1$, $|\sin(\alpha^n rx)| \leq 1$ and $e^{-\alpha^n ax} \to 0$ for all $n \geq 0$, the solution $y(x) \to 0$ as $x \to \infty$. It is integrable for $x \geq 0$ because

$$|y(x)| \leq (|P| + |Q|)\sum_{n=0}^{\infty}(|b_{1n}| + |b_{2n}|)e^{-\alpha^n \frac{a}{2}x},$$

and the series is in $L_1[0,\infty)$; thus, $y(x)$ is integrable in $[0,\infty)$. Moreover, it is evident that $y(x)$ is infinitely integrable.

We now have two undetermined parameters P and Q and therefore, there is an infinite number of integrable solutions to equation (3.1). Even if the integrability condition (3.3) is imposed, another boundary condition is necessary to be prescribed for the solution $y(x)$ to be determined uniquely. Figure 3.3.8 illustrates the form of a solution $y(x)$.

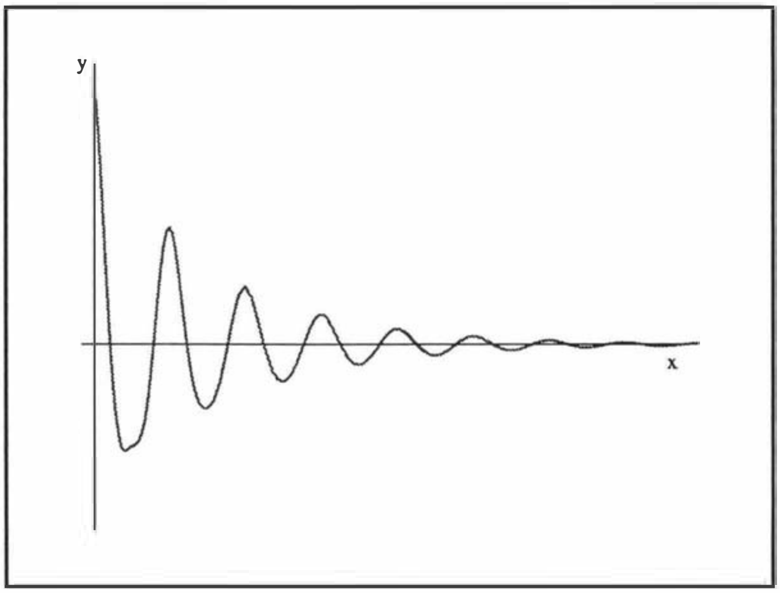

Figure 3.3.8: The expected shape of the graph of the solution $y(x)$ when $a > 0, b > \frac{a^2}{4}$.

Chapter 4

The Eigenvalue Problems

We investigate in this chapter the eigenvalues and eigenfunctions satisfying $y(0) = 0$ to advanced second order functional differential equations $L(y(x)) = \lambda y(\alpha x)$. This is motivated from the equation arising in a cell growth model in Chapter 2, which has non-zero value of the probability density function at $x = 0$. We examine equations with general constant coefficients and seek the values of λ which make $y(0) = 0$. It is known from Chapter 3 that there exists a Dirichlet series solution to advanced functional differential equations if $L(y(x)) = 0$ has a solution such that $y(\infty) = 0$, and we use this solution form to determine the eigenvalues.

4.1 The First Order Problem

We first study the eigenvalue problem for the first order equation

$$y'(x) + by(x) - \lambda y(\alpha x) = 0, \tag{4.1}$$

where $b > 0$ and $\alpha > 1$, satisfying the boundary conditions

$$y(0) = 0, \quad y(\infty) = 0, \tag{4.2}$$

and the normalizing condition

$$\int_0^\infty y(t)\, dt = 1. \tag{4.3}$$

We will refer to equation (4.1) along with conditions (4.2) and (4.3) as *Problem 4.1*. We note that we have one solution satisfying Problem 4.1 for the case $\lambda = b\alpha$ from A.J. Hall and G.C. Wake [1989] and A.J. Hall [1991].

In fact, the value $\lambda = b\alpha$ is the only eigenvalue for equation (4.1) since integrating the equation from 0 to ∞ yields

$$y(0) = (b - \frac{\lambda}{\alpha}) \int_0^\infty y(t)\, dt. \tag{4.4}$$

However, for the second order equations with constant coefficients which will be discussed in this chapter, an integration of the equations from 0 to ∞ doesn't simply produce the eigenvalues λ since there exists an undetermined term $y'(0)$ in the equation after integration. For this reason, we introduce a method of using a Dirichlet series solution to get the eigenvalues, which can be applied to the second order equations.

Substituting a Dirichlet series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n e^{-\alpha^n r x}, \tag{4.5}$$

into equation (4.1) gives the indicial equation $r - b = 0$ and thus the recurrence relation is

$$b(\alpha^n - 1)a_n = -\lambda a_{n-1}.$$

Hence a solution $y(x)$ to equation (4.1) is given by

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-\lambda/b)^n}{\prod_{m=1}^{n}(\alpha^m - 1)} e^{-\alpha^n b x}, \tag{4.6}$$

where we use the convention $\prod_{m=1}^{0}(\alpha^m - 1) = 1$, and the condition (4.3) implies

$$a_0 = b\left(\sum_{n=0}^{\infty} \frac{(\frac{-\lambda}{b\alpha})^n}{\prod_{m=1}^{n}(\alpha^m - 1)}\right)^{-1}.$$

At $x = 0$,

$$y(0) = a_0 \sum_{n=0}^{\infty} \frac{(-\lambda/b)^n}{\prod_{m=1}^{n}(\alpha^m - 1)}.$$

In general $y(0) \neq 0$ so that the initial condition (4.2) is not satisfied. There are, however, certain values for λ, the eigenvalues, which do satisfy this initial condition. Let $q = \frac{1}{\alpha}$, then

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \frac{(-\lambda/b)^n}{\prod_{m=1}^{n}(\alpha^m - 1)} &= 1 + \sum_{n=1}^{\infty} \frac{(-\frac{\lambda q}{b})^n q^{\frac{n(n-1)}{2}}}{\prod_{m=1}^{n}(1 - q^m)} \\ &\equiv F(\lambda). \end{aligned}$$

Now $F(\lambda)$ is a partition function which can be converted into an infinite product using Euler identity. Recall that, for $|t| < 1, |q| < 1$, the *Euler Identity* is

$$1+\sum_{n=1}^{\infty}\frac{t^n q^{\frac{n(n-1)}{2}}}{\prod_{m=1}^{n}(1-q^m)}=\prod_{n=0}^{\infty}(1+tq^n), \tag{4.7}$$

and therefore,

$$F(\lambda)=\prod_{n=0}^{\infty}(1-\frac{\lambda q^{n+1}}{b})=\prod_{n=0}^{\infty}(1-\frac{\lambda}{b\alpha^{n+1}}), \tag{4.8}$$

provided that $\frac{\lambda q}{b}<1$. However, even if $\frac{\lambda q}{b}\geq 1$, we still have the identity (4.8) by analytic continuation since the infinite product converges for all $\lambda \in \mathbb{C}$ and $F(\lambda)$ is an entire function. Note that the denominator of a_0 can also be expressed as an infinite product, and thus

$$a_0=b\left(\prod_{n=0}^{\infty}(1-\frac{\lambda}{b\alpha^{n+2}})\right)^{-1}=bF^{-1}(\lambda/\alpha). \tag{4.9}$$

Now, the zeros of $F(\lambda)$ are given by

$$\lambda=b\alpha^n, \qquad n=1,2,\ldots.$$

However, not all the zeros $\lambda=b\alpha^n$ for $n=1,2,\ldots$ correspond to eigenvalues since equation (4.9) implies that a_0 does not exist when $\lambda=b\alpha^{n+1}$, $n=1,2,\ldots$. Therefore, Problem 4.1 has the only one eigenfunction corresponding to the eigenvalue $\lambda=b\alpha$, which is the solution to the equation arising in the cell growth model shown by Hall and Wake (*op.cit.*). Note that for the values $\lambda=b\alpha^{n+1}$, $n=1,2,\ldots$, the solutions corresponding to those values satisfy the condition $\int_0^\infty y(x)\,dx=0$.

4.2 The Eigenvalue Problem I

In this section, we consider two eigenvalue problems. The first problem corresponds to the equation

$$y''(x)-by(x)+\lambda y(\alpha x)=0, \tag{4.10}$$

where $b>0$ and $\alpha>1$. We will refer to equation (4.10) along with conditions (4.2) and (4.3) as *Problem 4.2.1*. The second problem corresponds to the equation

$$y''(x)-\lambda y(x)+cy(\alpha x)=0, \tag{4.11}$$

where $c > 0$ and $\alpha > 1$. We will refer to equation (4.11) along with the conditions (4.2) and (4.3) as *Problem 4.2.2.*

We will construct solutions to Problem 4.2.1 using a Dirichlet series representation including the parameter λ and investigate some qualitative properties of the solutions. Problem 4.2.2, which is closely related to Problem 4.2.1 will then be considered.

4.2.1 Solutions to Problem 4.2.1

We seek a solution to Problem 4.2.1 of the form (4.5). Substituting the series into equation (4.10) leads to the recurrence relation

$$b(\alpha^{2n} - 1)a_n = -\lambda a_{n-1}, \tag{4.12}$$

and the indicial equation

$$r^2 - b = 0\,.$$

Although there are two solutions to the indicial equation we must choose $r = \sqrt{b}$ for a convergent series. Using the relation (4.12) and $r = \sqrt{b}$, a solution $y(x)$ is thus given by

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-\frac{\lambda}{b})^n}{\prod_{m=1}^{n}(\alpha^{2m} - 1)} e^{-\alpha^n \sqrt{b} x} \quad , \tag{4.13}$$

where a_0 can be determined by the condition (4.3), i.e.

$$a_0 = \sqrt{b} \left(\sum_{n=0}^{\infty} \frac{(-\frac{\lambda}{b\alpha})^n}{\prod_{m=1}^{n}(\alpha^{2m} - 1)} \right)^{-1} .$$

At $x = 0$, we have $y(0) = a_0 G(\lambda)$, where

$$G(\lambda) = \sum_{n=0}^{\infty} \frac{(-\frac{\lambda}{b})^n}{\prod_{m=1}^{n}(\alpha^{2m} - 1)}. \tag{4.14}$$

Let $q = \frac{1}{\alpha^2}$, then

$$
\begin{aligned}
G(\lambda) &= 1+\sum_{n=1}^{\infty}\frac{(-\frac{\lambda}{b})^n}{\prod_{m=1}^{n}(\alpha^{2m}-1)} \\
&= 1+\sum_{n=1}^{\infty}\frac{(-\frac{\lambda q}{b})^n q^{\frac{n(n-1)}{2}}}{\prod_{m=1}^{n}(1-q^m)} \\
&= \prod_{n=0}^{\infty}(1-\frac{\lambda q^{n+1}}{b}) \\
&= \prod_{n=0}^{\infty}(1-\frac{\lambda}{b\alpha^{2(n+1)}}),
\end{aligned}
$$

where the Euler Identity (4.7) has been used, and

$$
a_0 = \sqrt{b}\left(\prod_{n=0}^{\infty}(1-\frac{\lambda}{b\alpha^{2n+3}})\right)^{-1} = \sqrt{b}G^{-1}(\lambda/\alpha). \tag{4.15}
$$

Now, the zeros of $G(\lambda)$ are given by

$$
\lambda = b\alpha^{2n}, \qquad n = 1, 2, ...,
$$

and unlike the first order case, $a_0 \neq \infty$ at the zeros of $G(\lambda)$ and thus all the zeros of $G(\lambda)$ can be used to construct eigenfunctions. The n^{th} eigenfunction $y_n(x)$ is of the form

$$
y_n(x) = a_{0n}\sum_{m=0}^{\infty}\frac{(-\alpha^{2n})^m}{\prod_{k=1}^{m}(\alpha^{2k}-1)}e^{-\alpha^m\sqrt{b}x}, \quad n = 1, 2, ..., \tag{4.16}
$$

where

$$
a_{0n} = \sqrt{b}\left(\prod_{m=0}^{\infty}(1-\frac{\alpha^{2n}}{\alpha^{2m+3}})\right)^{-1} = \sqrt{b}G^{-1}(b\alpha^{2n-1}). \tag{4.17}
$$

Uniqueness of Solutions

We will show that the eigenfunctions are unique, but first we need the following lemma:

Lemma 4.2.1 *Any solution to the functional differential equation*

$$
\tau''(x) - b\tau(x) + b\tau(\alpha x) = 0, \tag{4.18}
$$

where $\alpha > 1, b > 0$, satisfying the condition $\tau(\infty) = 0$, can have neither a positive maximum nor a negative minimum.

Proof: Suppose X_1 is a positive maximum critical point. Because $\tau''(X_1) \leq 0$, we have $\tau(\alpha X_1) \geq \tau(X_1)$ from (4.18). This implies the existence of another maximum critical point $X_2 > X_1$ since $\tau(\infty) = 0$ and clearly, $\tau(X_2) \geq \tau(X_1)$. This process can be repeated to obtain a sequence of maximum critical points $\{X_n\}$ such that $\lim_{n\to\infty} \tau(X_n) \neq 0$, contradicting the condition $\tau(\infty) = 0$. It can be shown that the solutions cannot have negative minima using the same arguments. ■

Theorem 4.2.2 *The only eigenfunction of Problem 4.2.1 in* $L_{2n}[0,\infty)$ *is* y_n *defined in equations* (4.16) *and* (4.17), *whose eigenvalue is* $\lambda = b\alpha^{2n}$.

Proof: We first show this for the case $\lambda = b\alpha^2$. Equation (4.10) for $\lambda = b\alpha^2$ is

$$y''(x) - by(x) + b\alpha^2 y(\alpha x) = 0. \tag{4.19}$$

Suppose that two distinct eigenfunctions $y_{11}(x), y_{12}(x)$ exist and let $z(x) = y_{11}(x) - y_{12}(x)$. Then $z(x)$ satisfies the equation

$$z''(x) - bz(x) + b\alpha^2 z(\alpha x) = 0, \tag{4.20}$$

and the conditions

$$z(0) = 0, \qquad z(\infty) = 0, \tag{4.21}$$

$$\int_0^\infty z(x)\, dx = 0. \tag{4.22}$$

An integration of equation (4.20) using the conditions (4.21) and (4.22) yields the integro-differential equation

$$-z'(x) - b\int_x^\infty z(t)\, dt + b\alpha \int_{\alpha x}^\infty z(t)\, dt = 0. \tag{4.23}$$

Let

$$\sigma(x) = \int_x^\infty z(t)\, dt, \tag{4.24}$$

then $\sigma''(x) = -z'(x)$ and so equation (4.23) can be converted into the equation

$$\sigma''(x) - b\sigma(x) + b\alpha\sigma(\alpha x) = 0. \tag{4.25}$$

Now $y_1(x)$ in $L_2[0,\infty)$ and so $z(x)$ is also in $L_2[0,\infty)$ so that the transformation

$$\tau(x) = \int_x^\infty \sigma(t)dt, \tag{4.26}$$

is well defined. Using this transformation on equation (4.25) yields the equation

$$\tau''(x) - b\tau(x) + b\tau(\alpha x) = 0, \tag{4.27}$$

and clearly,

$$\tau''(x) = -\sigma'(x) = z(x). \tag{4.28}$$

Now, $\tau(\infty) = 0$ so that $\tau(x)$ has neither positive maxima nor negative minima by Lemma 4.2.1, and therefore $\tau(x)$ has to be a decreasing function if $\tau(0) > 0$ or an increasing function if $\tau(0) < 0$. If $\tau(0) = 0$, then $\tau(x)$ must be identically zero in the interval $[0,\infty)$. Suppose that $\tau(x)$ is decreasing, then $\tau(x) \geq \tau(\alpha x)$ so that $\tau''(x) \geq 0$; thus, $z(x) \geq 0$ by (4.28). Similarly, if $\tau(x)$ is increasing, then $z(x) \leq 0$. The condition (4.22) thus implies that $z(x) = 0$ for all x. Therefore the eigenfunction corresponding to the eigenvalue $\lambda = b\alpha^2$ is unique. The next eigenvalue $\lambda = b\alpha^4$, produces the differential equation

$$y''(x) - by(x) + b\alpha^4 y(\alpha x) = 0. \tag{4.29}$$

Let $z(x) = y_{21}(x) - y_{22}(x)$ for two distinct solutions $y_{21}(x), y_{22}(x)$ to equation (4.29), then $z(x)$ satisfies equation (4.29) and the conditions (4.21) and (4.22). Transforming equation (4.29) twice by (4.24) for σ_1 instead of σ and (4.26) for τ_1 instead of τ leads to the equation

$$\tau_1''(x) - b\tau_1(x) + b\alpha^2\tau_1(\alpha x) = 0, \tag{4.30}$$

which is essentially the same as (4.20). Since $z(x)$ in $L_4[0,\infty)$, it is enough to show that the conditions (4.21) and (4.22) are also satisfied to establish that the above equation has the trivial solution. Now, $\tau_1''(0) = -\sigma_1'(0) = z(0)$ and so $\tau_1''(0) = 0$; equation (4.30) thus implies that $\tau_1(0) = 0$. Clearly, $\tau_1(\infty) = 0$, and since $\tau_1'(0) = -\sigma_1(0) = 0$, an integration of equation (4.30) yields $\int_0^\infty \tau_1(t)dt = 0$. Therefore, $\tau_1(x) = 0$ for all x so that $\tau_1''(x) = 0$; thus, $z(x) = 0$. The case $\lambda = b\alpha^6$ can be handled in a similar way, i.e, by mapping solutions back to the $\lambda = b\alpha^4$ case. It can thus be shown by induction that all the eigenfunctions are unique. ■

Qualitative Properties of Solutions

We now examine some qualitative properties of these eigenfunctions defined by series (4.16). The first step in this direction is to show a simple relationship between two consecutive eigenfunctions and thus it makes possible to deduce some properties of the $(n+1)^{th}$ eigenfunction from the information about the n^{th} eigenfunction. Differentiating $y_n(x)$ twice gives

$$y_n''(x) = ba_{0n} \sum_{m=0}^{\infty} \frac{(-\alpha^{2(n+1)})^m}{\prod_{k=1}^{m}(\alpha^{2k}-1)} e^{-\alpha^m \sqrt{bx}},$$

and

$$y_{n+1}(x) = a_{0n+1} \sum_{m=0}^{\infty} \frac{(-\alpha^{2(n+1)})^m}{\prod_{k=1}^{m}(\alpha^{2k}-1)} e^{-\alpha^m \sqrt{bx}}.$$

It is evident that $y_{n+1}(x) = Ay_n''(x)$ for $A = a_{0n+1}/ba_{0n}, n = 1, 2, \ldots$. Here, $A < 0$ since equation (4.17) indicates that the a_{0n} are alternatively positive or negative depending on whether n is even or odd, and the zeros of $y_{n+1}(x)$ correspond to the zeros of $y_n''(x)$. This indicates for example that $y_{n+1}(x) \geq 0$ at a local maximum for $y_n(x)$ and $y_{n+1}(x) \leq 0$ at a local minimum for $y_n(x)$. In order to use the above observation, we need some information about the first eigenfunction, i.e. the solution to Problem 4.2.1 for $\lambda = b\alpha^2$.

Theorem 4.2.3 *The first eigenfunction y_1 of Problem 4.2.1 is positive for all $x > 0$.*

Proof: The result can be eastablished using the same method as that used in the proof of Theorem 4.2.2. Equation (4.19) can be converted into (4.27) using the transformations (4.24) and (4.26). Now, $\tau'(0) = -\sigma(0)$ and $\sigma(0) = 1 > 0$ so that

$$\tau'(0) = -1 < 0. \tag{4.31}$$

Lemma 4.2.1 and the condition (4.31) indicate that $\tau(x)$ is decreasing and $\tau(0) > 0$, and therefore $\tau(x) \geq \tau(\alpha x)$. It can be shown that $\tau(x) \neq \tau(\alpha x)$ using the same arguments as those used in the proof of Lemma 2.2.1 so that $\tau(x) > \tau(\alpha x)$; thus, $y_1(x) = \tau''(x) > 0$ for $x > 0$. ■

The next results, Theorems 4.2.4, 4.2.7 and Lemma 4.2.5, are shown for the first eigenfunction, but they can be extended to all the eigenfunctions without difficulty.

Theorem 4.2.4 *There exists an $\epsilon_1 > 0$ such that $y_1'(x) \geq 0$ and $y_1''(x) \leq 0$ for all $x \in (0, \epsilon_1)$.*

Proof: Integrating both sides of equation (4.10) for $\lambda = b\alpha^2$ from 0 to ∞ yields $y_1'(0) = b(\alpha - 1) > 0$. The continuity of $y_1(x)$ implies that for some $\epsilon_1 > 0$, there is an interval $(0, \epsilon_1)$ in which $y_1(x)$ is increasing from 0 to $\alpha\epsilon_1$; thus, $y_1'(x) \geq 0$ in that interval, and equation (4.10) implies that $y_1''(x) \leq 0$ in that interval since $-by_1(x) + b\alpha^2 y_1(\alpha x) \geq 0$. For the other eigenfunctions, this property follows from the fact that $y_n'(0) = b(\alpha^{2n-1} - 1) > 0$. ■

Lemma 4.2.5 *There exists a $Z_1 > 0$ such that $y_1(x)$ and $y_1'(x)$ are strictly monotone in (Z_1, ∞). In fact, this result is true for all derivatives $y_1^{(k)}(x)$ for all $k \in N$.*

Proof: Using the Dirichlet series representation (4.16) for $y_1(x)$ we have

$$\begin{aligned} -\frac{y_1'(x)}{a_{01}} =& K_0(1 - \frac{\alpha^3}{\alpha^2 - 1} e^{-(\alpha-1)\sqrt{bx}}) \\ &+ K_1(1 - \frac{\alpha^3}{\alpha^6 - 1} e^{-(\alpha-1)\alpha^2\sqrt{bx}}) \\ &+ \ldots, \end{aligned} \tag{4.32}$$

where

$$K_m = \frac{\sqrt{b}\alpha^{6m}}{\prod_{k=1}^{2m}(\alpha^{2k} - 1)} e^{-\alpha^{2m}\sqrt{bx}}, \qquad m = 0, 1, \ldots .$$

Since the exponential function in the first term is decreasing to zero as $x \to \infty$, the first term is positive in (Z_1, ∞) for some point Z_1. Also, the first term is the smallest one among terms in (4.32) and all K_m are positive, so that $\frac{-y_1'(x)}{a_{01}} > 0$ in that interval. Now, a_{01} is positive and therefore, $y_1'(x) < 0$ in (Z_1, ∞) ; thus, $y_1(x)$ is strictly monotone in this interval. A similar construction shows that $\frac{y_1''(x)}{a_{01}} > 0$ so that $y_1''(x) > 0$ and thus $y_1'(x)$ is strictly monotone in (Z_1, ∞). The above result can be extended to the other eigenfunctions by obtaining $\frac{y_n'(x)}{a_{0n}} < 0$ and $\frac{y_n''(x)}{a_{0n}} > 0$. In fact, since the a_{0n} are alternately positive or negative according to n, so are $y_n'(x)$ and $y_n''(x)$. ■

Corollary 4.2.6 *$y_n(x)$ must have finite number of zeros on R.*

Proof: Suppose that there is an infinite number of zeros $\{z_n\}$ of y_n.
Let $\Pi = \{ z \in \mathbb{C} \mid Re(z) > 0 \}$. Then $y_n \in H(\Pi)$, where $H(\Pi)$ is the set of functions holomorphic on Π. There can be no point of accumulation of zeros in the interior of Π, and if there were one at the origin, then $y_n'(0)$ would be either zero or nonexistent, but we already know that $y_n'(0) \neq 0$. This implies that y_n cannot have any finite points of accumulation for zeros of y_n in Π and *a fortiori* there are no finite points of accumulation for zeros on R^+. On the other hand, Lemma 4.2.5 precludes the possibility that y_n has zeros such that $\lim_{n\to\infty} z_n = \infty$. Therefore, we get the result.
■

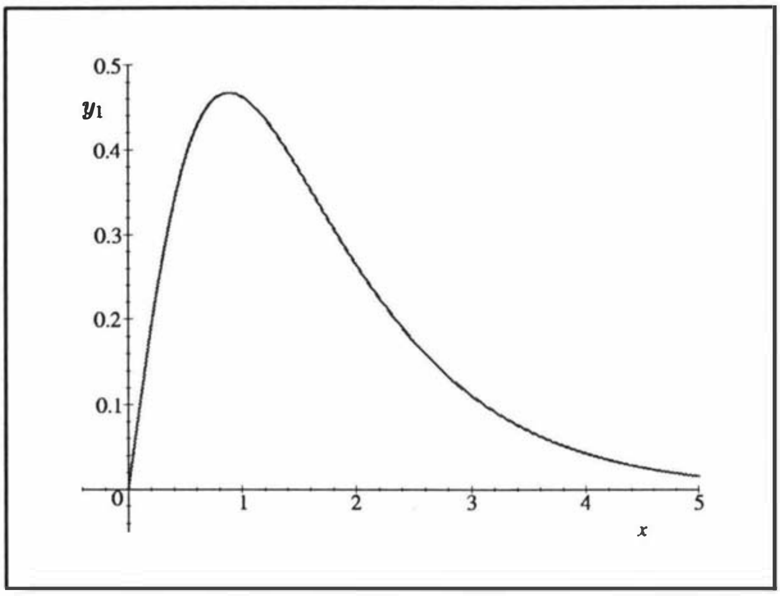

Figure 4.4.1: The first eigenfunction $y_1(x)$ for $\lambda = 4$ to Problem 4.2.1 when $b = 1$ and $\alpha = 2$. Eigenfunctions, here and in later Figures, were calculated from their Dirichlet series using Maple.

Theorem 4.2.7 *The eigenfunction $y_1(x)$ can have neither positive minima nor negative maxima.*

Proof: Suppose $y_1(x)$ has a positive local minimum at $x = x_1 > 0$. Now $y_1'(x_1) = 0$ and $y_1''(x_1) \geq 0$, so that equation (4.19) implies that $y_1(x_1) \geq \alpha^2 y_1(\alpha x_1)$ and so $y_1(x_1) > y_1(\alpha x_1)$. Hence, there must be a point $X_1 \in (x_1, \alpha x_1)$ at which y_1 achieves a positive local maximum. Now $y_1'(X_1) = 0$ and $y_1''(X_1) \leq 0$ and thus,

$$\alpha^2 y_1(\alpha X_1) \geq y_1(X_1) > y_1(x_1) \geq \alpha^2 y_1(\alpha x_1),$$

i.e. $y_1(\alpha X_1) > y_1(\alpha x_1)$. Since $y_1(\alpha X_1) > y_1(\alpha x_1)$, there must be a point which has a local minimum in the interval $(X_1, \alpha X_1)$. Here, the local minimum can be positive or negative. We can now repeat this argument to obtain a sequence of local maxima $\{X_n\}$ and local minima $\{x_n\}$ such that $y_1(X_n) - y_1(x_n) \neq 0$ for all n. This indicates that the eigenfunction is not monotone in any intervals of the form (X, ∞), thus contradicting Lemma 4.2.5. A similar construction shows that the eigenfunction cannot have negative minima.■

Theorems 4.2.3 and 4.2.7 imply the following result:

Corollary 4.2.8 *The first eigenfunction has exactly one maximum.*

The first eigenfunction $y_1(x)$ for $b = 1$ and $\alpha = 2$ is depicted in Figure 4.4.1.

It remains to investigate $y_n''(x)$ in order to obtain $y_{n+1}(x)$. We begin with the following lemma which shows the distribution of zeros of the eigenfunctions:

Lemma 4.2.9 *If $y_n(z_1) = 0$ and $y_n(z_2) = 0$ for $z_1 < z_2$, then $z_2 > \alpha z_1$.*

Proof: By Corollary 4.2.6, the zeros of y_n are isolated. Let z_2 be the next zero beyond z_1, and suppose if possible that $z_2 \le \alpha z_1$. Then y_n is of constant sign in (z_1, z_2). If that sign is negative then $y_n'(z_1) \le 0$ and $y_n'(z_2) \ge 0$. Integrating equation (4.10) for y_n instead of y from z_1 to z_2 and substituting $\lambda = b\alpha^{2n}$ give the integro-differential equation

$$y_n'(z_2) - y_n'(z_1) - b\int_{z_1}^{z_2} y_n(s)\,ds + b\alpha^{2n-1}\int_{\alpha z_1}^{\alpha z_2} y_n(s)\,ds = 0.$$

Now $y_n'(z_2) - y_n'(z_1) \ge 0$ so that $\int_{z_1}^{z_2} y_n(s)\,ds \ge \alpha^{2n-1}\int_{\alpha z_1}^{\alpha z_2} y_n(s)\,ds$ and therefore,

$$\int_{\alpha z_1}^{\alpha z_2} y_n(s)\,ds < 0,$$

since $\int_{z_1}^{z_2} y_n(s)\,ds < 0$. This implies that $y_n(x)$ cannot increase steadily after z_2 and there must be a point $z_3 < \alpha z_2$, which satisfies $y_n(z_3) = 0$. Let z_3 be the smallest zero in $(z_2, \alpha z_2)$. Then $y_n'(z_2) \ge 0$ and $y_n'(z_3) \le 0$, and it can be shown that there exists another zero $z_4 < \alpha z_3$ using the same arguments. Repeating this process produces an infinite number of zeros of $y_n(x)$. This contradicts Corollary 4.2.6 and so we have the result. A similar construction also gives the result if the constant sign of y_n was positive. ■

It is clear from Lemma 4.2.5 that there are finite number of critical points and so we have the following theorem for y_n which can be represented by a Dirichlet series:

Theorem 4.2.10 *Let $\{x_i\}$ be critical points for $i = 1, 2, ..., n$. Then $y_n''(x) < 0$ for $x \in (0, x_1)$, and in any interval of the form (x_i, x_{i+1}) for $i = 1, 2, ..., n-1$ or (x_n, ∞), there is exactly one point s such that $y_n''(s) = 0$.*

Proof: Let us first examine the interval $(0, x_1)$. Suppose $y_n''(\hat{x}) \ge 0$ for $\hat{x} \in (0, x_1)$. Since $y_n''(x_1) \le 0$, there must be a local maximum critical point $s \in (0, x_1)$ of y_n' such that $y_n''(s) = 0$ and $y_n'''(s) \le 0$. Now $y_n''(s) = 0$ so that $y_n(s) = \alpha^{2n} y_n(\alpha s)$ from equation (4.10). Since $y_n'(0) > 0$, $y_n(x) > 0$ for $x \in (0, x_1)$ and therefore, $y_n(\alpha s) > 0$ and $y_n(\alpha s) < y_n(s)$; thus, $\alpha s > x_1$. Hence for the first eigenfunction, $y_1'(\alpha s) \le 0$. For the other eigenfunctions, there is a possibility that the point αs is located in somewhere after more than two zeros of the eigenfunctions. Let the first and the second zero be z_1, z_2 respectively , then $z_2 > \alpha z_1$ from Lemma 4.2.9. This implies that $\alpha s < \alpha z_1$, and since $y_n(\alpha s) > 0$, we have $\alpha s < z_1$; thus, $y_n'(\alpha s) \leq 0$. On the other hand, a differentiation of equation (4.10) yields the equation

$$y_n'''(x) - by_n'(x) + b\alpha^{2n+1} y_n'(\alpha x) = 0. \tag{4.33}$$

Now, $y_n'''(s) \leq 0$ so that $y_n'(s) \leq \alpha^{2n+1} y_n'(\alpha s)$; thus, $y_n'(\alpha s) \geq 0$. The equality $y_n'(\alpha s) = 0$ is obtained when $y_n'(x) = 0$ in $[s, \alpha s]$. Since $y_n'''(x) = 0$ in that interval, equation (4.33) indicates that $y_n'(x) = 0$ in $[\alpha s, \alpha^2 s]$. This process can be repeated to get $y_n'(x) = 0$ for $x \geq s$ using the intervals of the form $[\alpha^k s, \alpha^{k+1} s]$ for $k = 1, 2, \ldots$ successively. So, $y_n'(\alpha s) > 0$, contradicting the previous result that $y_n'(\alpha s) \leq 0$. Thus $y_n''(x) < 0$ for all $x \in (0, x_1)$.

It is clear that there must be at least one inflexion point between any two consecutive critical points as well as between the last critical point and infinity since $y_n''(x)$ is continuous in $(0, \infty)$. Suppose first that there exists an inflexion point s_1 such that $y''(s_1) = 0$ and $y'''(s_1) \leq 0$ between a maximum critical point x_1 and the following minimum critical point x_2. Then the point s_1 is a maximum critical point of y_n'. Since $y_n'''(s_1) \leq 0$, $y_n'(s_1) \leq \alpha^{2n+1} y_n'(\alpha s_1)$ from equation (4.33) and so, whether $y_n'(\alpha s_1) \leq 0$ or $y_n'(\alpha s_1) \geq 0$, we have $y_n'(s_1) \leq y_n'(\alpha s_1)$ since $y_n'(s_1) \leq 0$. This indicates that there must be a point $s_2 \in (s_1, \alpha s_1)$ at which y_n' achieves a local minimum. Note that Lemma 4.2.9 and $y_n'(s_2) < y_n'(s_1) \leq 0$ yield $s_2 \in (x_1, x_2)$. Since $y_n'''(s_2) \geq 0$, $y_n'(s_2) \geq \alpha^{2n+1} y_n'(\alpha s_2)$ so that $y_n'(\alpha s_1) > y_n'(\alpha s_2)$. This follows that another maximum critical point s_3 of y_n' exists in the interval $(s_2, \alpha s_2)$. Here, $y_n'(\alpha s_2) < 0$ and so $s_3 \in (x_1, x_2)$. This argument can be repeated *ad infinitum* to construct an infinite number of critical points $\{s_m\}$ of $y_n'(x)$ such that $s_m \in (x_1, x_2)$, i.e. y_n' is oscillating in a bounded interval, contradicting the continuity of y_n'. Therefore, there is no inflexion point s satisfying $y_n''(s) = 0$ and $y_n'''(s) \leq 0$ between a maximum critical point and the following minimum critical point. It remains to prove that there is no point $\hat{v} \in (x_1, x_2)$ satisfying $y_n''(\hat{v}) = 0$ except one inflexion point. Suppose there is a point v satisfying $y_n''(v) = 0$. Then v is not an inflexion point so that $y_n'''(v) = 0$, and $y_n''(x) \geq 0$ or $y_n''(x) \leq 0$ in $(v - \eta, v + \eta)$ for some $\eta > 0$. Let $y_n''(x) \geq 0$ in that interval. Since y_n'' is not constantly zero in (v, x_2), there exists a minimum critical point $v_1 \geq v$ of y_n'' such that $y_n^{(iv)}(v_1) \geq 0$. Differentiating equation (4.33) leads to the equation

$$y_n^{(iv)}(x) - by_n''(x) + b\alpha^{2n+2} y_n''(\alpha x) = 0, \tag{4.34}$$

and so $0 = y_n''(v_1) \geq \alpha^{2n+2} y_n''(\alpha v_1)$; thus, $y_n''(v_1) \geq y_n''(\alpha v_1)$. This indicates the existence of a maximum critical point $v_2 < \alpha v_1$ of y_n'' such that $y_n''(v_2) < 0$, $y_n'''(v_2) = 0$ and $y_n^{(iv)}(v_2) \leq 0$. Since equation (4.33) yields $y_n'(v_1) = \alpha^{2n+1} y_n'(\alpha v_1)$, we have $y_n'(\alpha v_1) \leq 0$. Therefore, Lemma 4.2.9 leads to $\alpha v_1 < x_1$; thus, $v_2 \in (x_1, x_2)$.

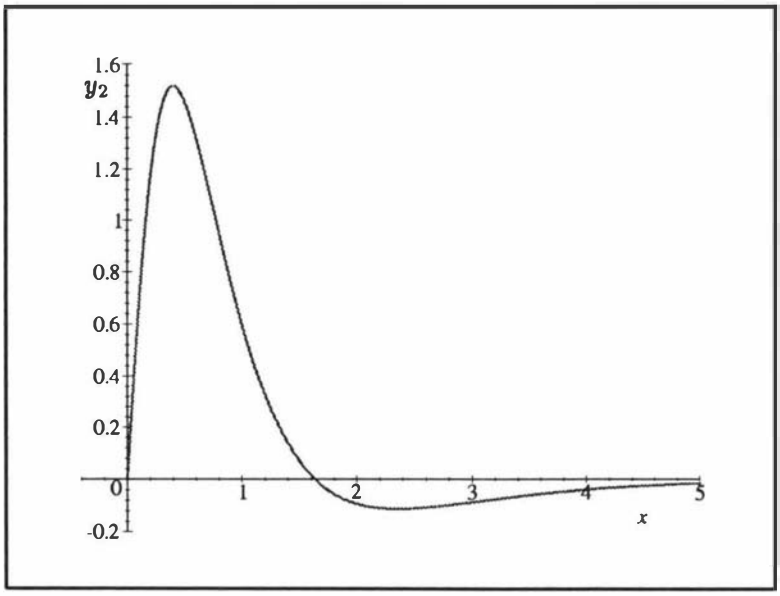

Figure 4.4.2: The second eigenfunction $y_2(x)$ for $\lambda = 16$ to Problem 4.2.1 when $b = 1$ and $\alpha = 2$.

Now $y_n''(v_2) \leq \alpha^{2n+2} y_n''(\alpha v_2)$ so that

$$\alpha^{2n+2} y_n''(\alpha v_2) \geq y_n''(v_2) > y_n''(v_1) \geq \alpha^{2n+2} y_n''(\alpha v_1),$$

and consequently, $y_n''(\alpha v_2) > y_n''(\alpha v_1)$. This implies that there exists another minimum critical point $v_3 > v_2$ of y_n'', and $v_3 \in (x_1, x_2)$ from equation (4.33). This argument can be repeated to get an infinite number of critical points $\{v_n\}$ of y_n'' such that $v_n \in (x_1, x_2)$. This means y_n'' is oscillating in a bounded interval, contradicting the continuity of y_n''. The case when $y_n''(x) \leq 0$ can be proved using a similar argument, and thus there is exactly one point v satisfying $y''(v) = 0$ in (x_1, x_2). For the interval between a minimum critical point and the following maximum critical point or between the last critical point and infinity, we can prove the results using a similar argument. ■

From the first eigenfunction and Theorem 4.2.10, the shape of the graph of the second eigenfunction can be deduced. The function $y_2(x)$ has one positive maximum, one negative minimum and one zero z in $(0, \infty)$. Moreover, there is no inflexion point between 0 and the maximum critical point, but it has one inflexion point between critical points or between the minimum critical point and infinity. The location of inflexion points between critical points can be deduced by considering the sign of $y_2''(z)$. Since $y_2(\alpha z) < 0$, $y_2''(z) > 0$ from equation (4.10). Hence an inflexion point of the eigenfunction lies in between the maximum critical point and the zero z. The second eigenfunction $y_2(x)$ when $b = 1$ and $\alpha = 2$ is depicted in Figure 4.4.2. The other eigenfunctions can be obtained inductively in a similar way, and we have the following corollary:

Corollary 4.2.11 *The eigenfunctions $y_n(x)$ have precisely n zeros including the zero at x=0.*

Moments of $y_1(x)$

The first eigenfunction $y_1(x)$ to Problem 4.2.1 can be treated as a probability density function since it is non-negative with one maximum and satisfies the normalizing condition (4.3) so that we obtain the statistical properties of the graph of the equation by considering the moments of $y_1(x)$. The m^{th} moment of $y_1(x)$ about the origin is

$$\mu_m = \int_0^\infty t^m y_1(t)\,dt.$$

Multiplying equation (4.19) by x^m leads to the equation

$$x^m y_1''(x) - bx^m y_1(x) + b\alpha^2 x^m y_1(\alpha x) = 0,$$

and integrating the above equation from 0 to ∞ produces

$$m(m-1)\int_0^\infty t^{m-2}y_1(t)\,dt - b\int_0^\infty t^m y_1(t)\,dt + b\alpha^{1-m}\int_0^\infty t^m y_1(t)\,dt = 0.$$

Therefore, for $m \geq 2$, we have the recurrence relation

$$m(m-1)\mu_{m-2} = b(1-\alpha^{1-m})\mu_m. \tag{4.35}$$

Clearly, $\mu_0 = 1$ and the mean value μ_1 can be obtained by using the Dirichlet series solution. Now,

$$\mu_1 = \sum_{n=0}^\infty a_n \int_0^\infty te^{-\alpha^n\sqrt{b}t}\,dt,$$

and since

$$\int_0^\infty te^{-\alpha^n\sqrt{b}t}\,dt = \sqrt{b}\alpha^{-n}\int_0^\infty e^{-\alpha^n\sqrt{b}t}\,dt = b^{-1}\alpha^{-2n},$$

$$\begin{aligned} \mu_1 &= a_{01}\sum_{n=0}^\infty \frac{(-1)^n}{\prod_{k=1}^n(\alpha^{2k}-1)} \\ &= b^{-1/2}G(b)/G(b\alpha). \end{aligned} \tag{4.36}$$

The relation (4.35) and $\mu_0 = 1$ indicate that $\mu_2 = \frac{2}{b(1-\alpha^{-1})}$ so that $\mu_4 = \frac{24}{b^2(1-\alpha^{-1})(1-\alpha^{-3})}$, and in general, for $k = 1, 2, \ldots$,

$$\mu_{2k} = \frac{(2k)!}{b^k \prod_{i=1}^{k}(1-\alpha^{-(2i-1)})}.$$

In the same manner, we have $\mu_3 = \frac{6}{b\sqrt{b}(1-\alpha^{-2})}\frac{G(b)}{G(b\alpha)}$ so that $\mu_5 = \frac{120}{b^2\sqrt{b}(1-\alpha^{-2})(1-\alpha^{-4})}\frac{G(b)}{G(b\alpha)}$ and consequently, in general, for $k = 1, 2, \ldots$,

$$\mu_{2k+1} = \frac{(2k+1)!}{b^k\sqrt{b}\prod_{i=1}^{k}(1-\alpha^{-2i})}\frac{G(b)}{G(b\alpha)}.$$

Now, the variance is given by

$$\begin{aligned}\sigma^2 &= \mu_2 - \mu_1^2 \\ &= \frac{2\alpha}{b(\alpha-1)} - \frac{G^2(b)}{bG^2(b\alpha)}.\end{aligned}$$

As an example, when $\alpha = 2$ and $b = 1$, y_1 has the mean value, the variance and the skewness as follows:

$$\begin{aligned}\mu_1 &\approx 1.64, \\ \sigma^2 &\approx 1.31, \\ \sqrt{\beta} = \frac{\mu_3}{\sigma^3} &\approx 8.75.\end{aligned}$$

4.2.2 Solutions to Problem 4.2.2

Attention is now focused on the qualitative properties of the solutions to Problem 4.2.2. The structure of the eigenfunctions arising in Problem 4.2.2 is similar to that arising in Problem 4.2.1. The eigenvalues and eigenfunctions for Problem 4.2.2 can be determined in the same manner as that used in Problem 4.2.1 and we omit the details.

The solution $u(x)$ is given by

$$u(x) = c_0 \sum_{n=0}^{\infty} \frac{(-\frac{c}{\lambda})^n}{\prod_{m=1}^{n}(\alpha^{2m}-1)} e^{-\alpha^n r x},$$

where r is determined from the indicial equation

$$r^2 - \lambda = 0,$$

and c_0 is obtained from the condition (4.3) so that

$$c_0 = \sqrt{\lambda}\left(\sum_{n=0}^{\infty} \frac{(-\frac{c}{\alpha\lambda})^n}{\prod_{m=1}^{n}(\alpha^{2m}-1)}\right)^{-1}.$$

Now,

$$\begin{aligned} u(0) &= c_0(1+\sum_{n=1}^{\infty} \frac{(-\frac{c}{\lambda})^n}{\prod_{m=0}^{n}(\alpha^{2m}-1)}) \\ &\equiv \sqrt{\lambda}\bar{G}^{-1}(\alpha\lambda)\bar{G}(\lambda), \end{aligned} \tag{4.37}$$

and comparing (4.37) with (4.14) leads to the following identity:

$$\bar{G}(\lambda) = \prod_{n=0}^{\infty}(1-\frac{c}{\lambda\alpha^{2(n+1)}}). \tag{4.38}$$

Note that $\bar{G}(\lambda) = G(\frac{1}{\lambda})$ if $b = c$, where G is defined by (4.14). The expression (4.38) shows that the zeros of $\bar{G}(\lambda)$ are given by

$$\lambda = c\alpha^{-2n}, \qquad n = 1, 2, ...,$$

and $c_0 = \sqrt{\lambda}\bar{G}^{-1}(\alpha\lambda) \neq \infty$ at those points so that all the zeros of $\bar{G}(\lambda)$ can be used to construct eigenfunctions. The n^{th} eigenfunction is

$$u_n(x) = c_{0n}\sum_{m=0}^{\infty} \frac{(-\alpha^{2n})^m}{\prod_{k=1}^{m}(\alpha^{2k}-1)} e^{-\alpha^{m-n}\sqrt{cx}}, \tag{4.39}$$

where

$$c_{0n} = \frac{\sqrt{c}}{\alpha^n}\left(\prod_{m=0}^{\infty}(1-\frac{\alpha^{2n}}{\alpha^{2m+3}})\right)^{-1} = \frac{\sqrt{c}}{\alpha^n}\bar{G}^{-1}(c\alpha^{-2n+1}).$$

Comparing (4.39) with the n^{th} eigenfunction $y_n(x)$ arising in Problem 4.2.1 yields the relationship

$$u_n(x) = \frac{c_{0n}}{a_{0n}} y_n(\frac{1}{\alpha^n}\sqrt{\frac{c}{b}}x). \tag{4.40}$$

Note that $c_{0n} = \alpha^{-n}\sqrt{\frac{c}{b}}a_{0n}$; therefore, $\frac{c_{0n}}{a_{0n}} > 0$ for all the value n. The relationship (4.40) implies that the basic features of the eigenfunctions $u_n(x)$ such as the number of critical points, zeros and the change of the sign of the eigenfunctions are essentially the same as those of Problem 4.2.1.

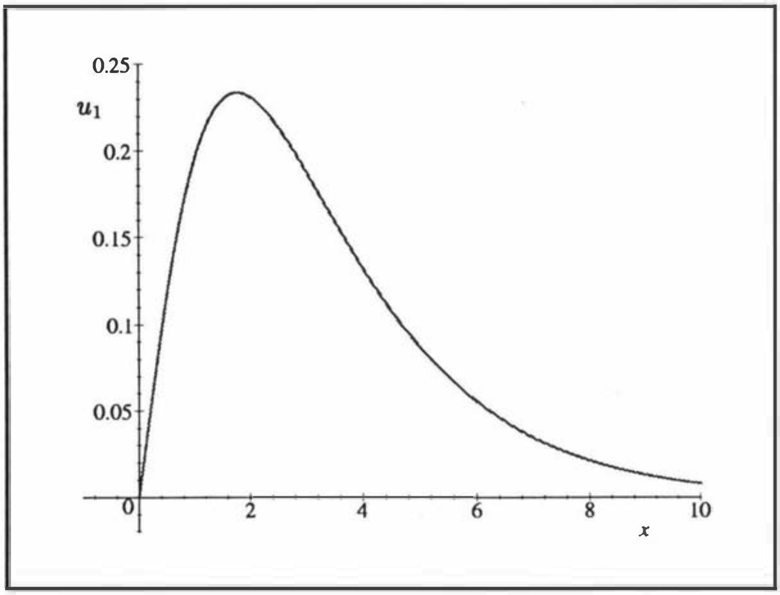

Figure 4.4.3: The first eigenfunction $u_1(x)$ for $\lambda = \frac{1}{4}$ to Problem 4.2.2 when $c = 1$ and $\alpha = 2$.

In contrast with the previous problem, the eigenvalues form a sequence $\{\lambda_n\}$ such that $\lambda_n \to 0$ as $n \to \infty$. The first eigenvalue is $\lambda = c/\alpha^2$, and it can be shown that the corresponding eigenfunction is unique and positive for $x > 0$, and it has exactly one maximum critical point. The number of the zeros for the n^{th} eigenfunction $u_n(x)$ is also n, which is the same as that for the n^{th} eigenfunction $y_n(x)$, but the location of the zeros for the n^{th} eigenfunction $u_n(x)$ is $\sqrt{\frac{b}{c}}\alpha^n$ times far from the origin as that of the n^{th} eigenfunction $y_n(x)$. Figures 4.4.3 and 4.4.4 show the first and second eigenfunctions $u_1(x), u_2(x)$ when $c = 1$ and $\alpha = 2$.

We finish this section with the following corollary :

Corollary 4.2.12 *Let the first zeros of the eigenfunctions arising in Problems 4.2.1 and 4.2.2 be $\{z_{1n}\}$ and $\{z_{2n}\}$ for $n = 1, 2, \ldots,$ respectively. Then* $\lim_{n\to\infty} z_{1n} = 0$ *and* $\lim_{n\to\infty} z_{2n} = 0$.

Proof: There is an inflexion point less than the first zero of the n^{th} eigenfunction and the inflexion point becomes the zero of the $(n+1)^{th}$ eigenfunction, so that the first zero of the $(n+1)^{th}$ eigenfunction is less than that of the n^{th} eigenfunction. Now, $|u_n(x)| \le \frac{\sqrt{c}}{\alpha^n}|\bar{G}^{-1}(c\alpha^{-2n+1})\bar{G}(c\alpha^{-2n})| \to 0$ as $n \to \infty$ and so $u_n(x) \to 0$; thus, $z_{2n}(x) \to 0$ as $n \to \infty$. The relationship (4.40) can be expressed as

$$u_n(\sqrt{\frac{b}{c}}\alpha^n x) = \frac{c_{0n}}{a_{0n}} y_n(x),$$

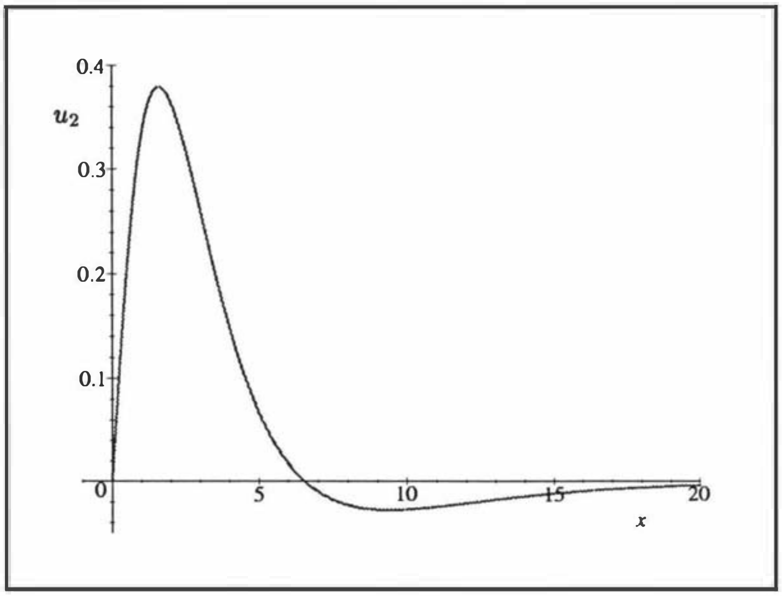

Figure 4.4.4: The second eigenfunction $u_2(x)$ for $\lambda = \frac{1}{16}$ to Problem 4.2.2 when $c = 1$ and $\alpha = 2$.

and $y_n(x) \to 0$ as $n \to \infty$ for $x > 0$ since $u_n(\sqrt{\frac{b}{c}}\alpha^n x) \to 0$ as $n \to \infty$ for $x > 0$; thus, $z_{1n}(x) \to 0$ as $n \to \infty$. ∎

4.3 The Eigenvalue Problem II

Problems 4.2.1 and 4.2.2 did not include first derivative terms. We now consider the more general equation

$$y''(x) - ay'(x) - by(x) + \lambda y(\alpha x) = 0, \tag{4.41}$$

where $a > 0$, $b > 0$ and $\alpha > 1$. We will refer to equation (4.41) along with conditions (4.2) and (4.3) as *Problem 4.3*. In this section we investigate the properties of solutions to Problem 4.3 and generalize the analysis of the previous section. The major complication here is that we cannot determine the eigenvalues explicitly.

Substituting a solution $y(x)$ of the form (4.5) into equation (4.41) yields the indicial equation

$$r^2 + ar - b = 0, \tag{4.42}$$

and the recurrence relation

$$(\alpha^{2n}r^2 + a\alpha^n r - b)a_n = -\lambda a_{n-1}. \tag{4.43}$$

From equations (4.43) and (4.42), $y(x)$ is given by

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{2m}r^2 + a\alpha^m r - b)} e^{-\alpha^n r x},$$

where $r = \frac{-a+\sqrt{a^2+4b}}{2}$, and a_0 can be determined by the condition (4.3), so that

$$a_0 = r\left(\sum_{n=0}^{\infty} \frac{(-\frac{\lambda}{\alpha})^n}{\prod_{m=1}^{n}(\alpha^{2m}r^2 + a\alpha^m r - b)}\right)^{-1}. \tag{4.44}$$

Now,

$$y(0) = a_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{2m}r^2 + a\alpha^m r - b)},$$

and eigenvalues need to be found in order for the solutions to satisfy the condition $y(0) = 0$. Let $y(0) = a_0\hat{G}(\lambda)$, where

$$\hat{G}(\lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{2m}r^2 + a\alpha^m r - b)},$$

then $y'(0) = -ra_0\hat{G}(\alpha\lambda)$ and $y''(0) = r^2a_0\hat{G}(\alpha^2\lambda)$. Substituting these expressions into equation (4.41) yields the advanced equation

$$r^2\hat{G}(\alpha^2\lambda) + ar\hat{G}(\alpha\lambda) + (\lambda - b)\hat{G}(\lambda) = 0. \tag{4.45}$$

If $\lambda \geq b$, the coefficients of the above equation are non-negative and therefore $\hat{G}(\lambda), \hat{G}(\alpha\lambda), \hat{G}(\alpha^2\lambda)$ cannot have the same sign for all $\lambda \geq b$. Hence we have a sequence $\{\lambda_n\}$ of zeros of $\hat{G}(\lambda)$ such that $\lim_{n\to\infty}\lambda_n = \infty$. We now examine $a_0 = r\hat{G}^{-1}(\lambda/\alpha)$ to show all the zeros of $\hat{G}$ correspond to eigenvalues. Suppose $\lambda_n \neq b\alpha^k$ for all $n \in N$ and all $k = 2, 3, \ldots$, and $\hat{G}(\lambda_n/\alpha) = 0$. Substituting $\lambda = \frac{\lambda_n}{\alpha^2}$ into equation (4.45) yields

$$r^2\hat{G}(\lambda_n) + ar\hat{G}(\lambda_n/\alpha) + (\frac{\lambda_n}{\alpha^2} - b)\hat{G}(\lambda_n/\alpha^2) = 0,$$

and this equation implies that $\hat{G}(\lambda_n/\alpha^2) = 0$ since $\hat{G}(\lambda_n) = 0$, $\hat{G}(\lambda_n/\alpha) = 0$ and $\lambda_n \neq b\alpha^2$. Now, by substituting $\lambda = \frac{\lambda_n}{\alpha^3}$ into equation (4.45), we have the advanced equation

$$r^2\hat{G}(\lambda_n/\alpha) + ar\hat{G}(\lambda_n/\alpha^2) + (\frac{\lambda_n}{\alpha^3} - b)\hat{G}(\lambda_n/\alpha^3) = 0,$$

and therefore $\hat{G}(\lambda_n/\alpha^3) = 0$ since $\lambda_n \neq b\alpha^3$. The repeated process implies that

$$\hat{G}(\lambda_n/\alpha^k) = 0,$$

for all $k \in N$ and since $\hat{G}$ is continuous at $\lambda = 0$, $\hat{G}(0) = 0$. On the other hand, substituting $\lambda = 0$ into $\hat{G}(\lambda)$ shows that $\hat{G}(0) = 1$, and we thus conclude that $\hat{G}(\lambda_n/\alpha) \neq 0$ for $n \in N$ and hence $a_0 \neq \infty$ at the zeros of $\hat{G}$. Hence if $\lambda_n \neq b\alpha^k$ for all n and $k = 2, 3, \ldots$, then all the zeros of $\hat{G}$ can thus be used to construct the eigenfunctions y_n for all $n \in N$. In fact, even if $\lambda_n = b\alpha^k$ for some n and k, $\hat{G}(\lambda_n/\alpha) \neq 0$. This will be proven after lemmas.

The result of Lemma 4.2.1 can be extended to the following lemma:

Lemma 4.3.1 *Any solution to the functional differential equation*

$$\tau''(x) - a\tau'(x) - b\tau(x) + k\tau(\alpha x) = 0,$$

where $0 < k \leq b$ *and* $\alpha > 1$, *satisfying the condition* $\tau(\infty) = 0$ *can have neither positive maxima nor negative minima.*

The next lemma provides bounds for the smallest eigenvalue.

Lemma 4.3.2 *Let* λ_1 *denote the smallest eigenvalue. Then* $b\alpha < \lambda_1 < b\alpha^2$.

Proof: Let us first consider lower bound of λ_1. Suppose $\lambda_1 < b\alpha$. Applying the transformation $\sigma(x)$ in (4.24) to equation (4.41) for y_1 instead of y leads to the equation

$$\sigma''(x) - a\sigma'(x) - b\sigma(x) + \frac{\lambda_1}{\alpha}\sigma(\alpha x) = 0,$$

and the conditions $\sigma(0) = 1$ and $\sigma(\infty) = 0$. Lemma 4.3.1 indicates that $\sigma(x)$ can have neither a positive maximum nor a negative minimum since $\lambda_1 < b\alpha$. Therefore, $\sigma'(x) \leq 0$ because $\sigma(0) = 1$ so that $y_1(x) \geq 0$; thus, $y_1'(0) \geq 0$. On the other hand, integrating equation (4.41) for y_1 from 0 to ∞ yields the equation $y_1'(0) = -b + \frac{\lambda_1}{\alpha}$ so that $y_1'(0) < 0$ since $\lambda_1 < b\alpha$. This fact contradicts the earlier result $y_1'(0) \geq 0$. For the case when $\lambda_1 = b\alpha$, it is enough to show that $\hat{G}(b\alpha) > 0$. Since $r_1 + r_2 = -a$ and $r_1 r_2 = -b$ for two roots r_1, r_2 of the indicial equation (4.42),

$$\begin{aligned}\prod_{m=1}^{n}(\alpha^{2m}r^2 + a\alpha^m r - b) &= \prod_{m=1}^{n}(\alpha^m r_1 - r_1)(\alpha^m r_1 - r_2)\\ &= \prod_{m=1}^{n} r_1^2 q^{-2m}(1 - q^m)(1 + \frac{b}{r_1^2}q^m),\end{aligned}$$

where $q = \frac{1}{\alpha}$, and so

$$\hat{G}(\lambda) = \sum_{n=0}^{\infty} \frac{(-\frac{\lambda}{r_1^2})^n q^{n^2+n}}{\prod_{m=1}^{n}(1-q^m)(1+\frac{b}{r_1^2}q^m)}.$$

Let $z = -\frac{b}{r_1^2}$, then

$$\hat{G}(b\alpha) = \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{\prod_{m=1}^{n}(1-q^n)(1-zq^n)},$$

and thus,

$$\hat{G}(b\alpha) = \left(\prod_{n=0}^{\infty}(1-zq^n)\right)^{-1} > 0, \tag{4.46}$$

from the result of well-known hypergeometric series (cf. G. Andrews [1976]). Therefore, $\lambda_1 > b\alpha$. Now, by substituting $\lambda = b$, equation (4.45) can be converted into the following equation:

$$r^2\hat{G}(b\alpha^2) + r\hat{G}(b\alpha) = 0,$$

and this equation implies that there is at least one zero less than $\alpha^2 b$ since

$$\hat{G}(b\alpha^2)\hat{G}(b\alpha) < 0.$$

■

Note that the above lemma indicates that $\hat{G}(b\alpha^2) < 0$. So, if $\hat{G}(b\alpha^3) = 0$, then $\lambda = b\alpha^3$ can be an eigenvalue. Suppose for $k \geq 4$, $\hat{G}(b\alpha^k) = 0$ and $\hat{G}(b\alpha^{k-1}) = 0$. Substituting $\lambda = b\alpha^{k-2}$ into equation (4.45) implies that $\hat{G}(b\alpha^{k-2}) = 0$. If $k = 4$, we have $\hat{G}(b\alpha^2) = 0$, contradicting $\hat{G}(b\alpha^2) < 0$. For the case when $k \geq 5$, we substitute $\lambda = b\alpha^{k-3}$ into equation (4.45) to get $\hat{G}(b\alpha^{k-3}) = 0$. The process will be repeated until we get the equation

$$r^2\hat{G}(b\alpha^4) + ar\hat{G}(b\alpha^3) + b(\alpha^2-1)\hat{G}(b\alpha^2) = 0,$$

and this implies that $\hat{G}(b\alpha^2) = 0$. Thus all the values of λ satisfying $\hat{G}(\lambda) = 0$ can be used to construct eigenfunctions.

Let us consider bounds for the other eigenvlaues. For the first eigenvalue λ_1, there is an eigenvalue λ_2 in $(\alpha\lambda_1, \alpha^2\lambda_1)$ from (4.45) and therefore, $\lambda_2 < b\alpha^4$ since

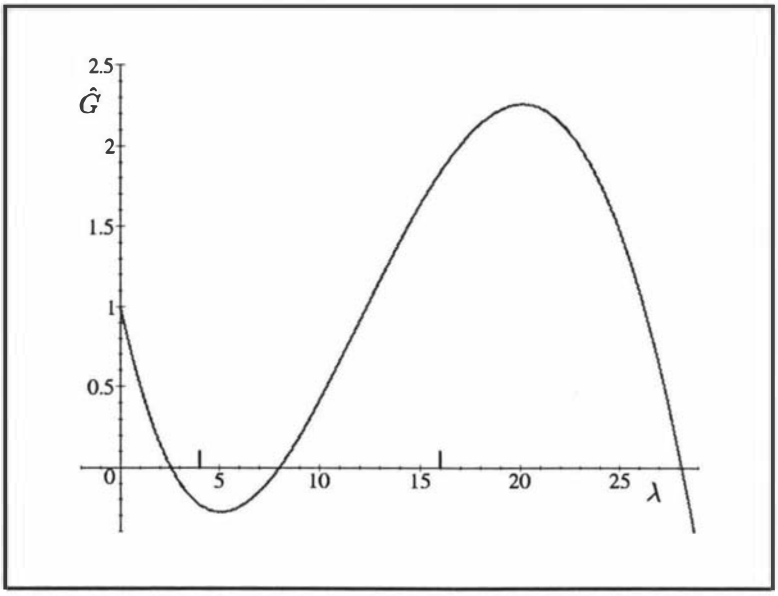

Figure 4.4.5: The bounds $\lambda_1 < 4$, $\lambda_2 < 16$ and $\lambda_3 < 64$ of $\hat{G}(\lambda)$ corresponding to the case that $a = 1$, $b = 1$ and $\alpha = 2$.

$\lambda_1 < b\alpha^2$. In a similar manner, another zero λ_3 such that $\lambda_3 < b\alpha^6$ can be obtained. Repeating this process produces a sequence $\{\lambda_n\}$ such that $\lambda_n < b\alpha^{2n}$. Figure 4.4.5 shows the locations and bounds for zeros of $\hat{G}(\lambda)$.

Now, the n^{th} eigenfunction corresponding to the n^{th} eigenvalue is given by

$$y_n(x) = a_{0n} \sum_{m=0}^{\infty} \frac{(-\lambda_n)^m}{\prod_{k=1}^{m}(\alpha^{2k}r^2 + a\alpha^k r - b)} e^{-\alpha^m r x}, \tag{4.47}$$

where

$$a_{0n} = r \left(\sum_{n=0}^{\infty} \frac{(-\frac{\lambda_n}{\alpha})^m}{\prod_{k=1}^{n}(\alpha^{2k}r^2 + a\alpha^k r - b)} \right)^{-1} = r\hat{G}^{-1}(\lambda_n/\alpha). \tag{4.48}$$

4.3.1 Uniqueness and Qualitative Properties of Solutions

The results of Theorem 4.3.3, Lemmas 4.3.4 and 4.3.5 can be extended to the other eigenfunctions with little difficulty.

Theorem 4.3.3 *The only eigenfunction of Problem 4.3 in $L_2[0,\infty)$ is $y_1(x)$, whose eigenvalue is $\lambda = \lambda_1$.*

Proof: Suppose $z(x) = y_{11}(x) - y_{12}(x)$ for two distinct solutions $y_{11}(x), y_{12}(x)$ to equation (4.41) when $\lambda = \lambda_1$, then we have the equation

$$z''(x) - az'(x) - bz(x) + \lambda_1 z(\alpha x) = 0, \tag{4.49}$$

and the conditions $z(0) = 0, z(\infty) = 0$ and $\int_0^\infty z(x)dx = 0$. Since $y_1(x)$ is at least twice integrable by assumption, we can use the transformations $\sigma(x)$ and $\tau(x)$ defined by (4.24) and (4.26) to equation (4.49) so that

$$\tau''(x) - a\tau'(x) - b\tau(x) + \frac{\lambda_1}{\alpha^2}\tau(\alpha x) = 0, \tag{4.50}$$

and $\tau(0) = 0$, since $\tau'(0) = -\sigma(0) = 0$ and $\tau''(0) = y(0) = 0$. Now, $\frac{\lambda_1}{\alpha^2} < b$ and so $\tau(x) = 0$ from Lemma 4.3.1 and therefore, $\tau''(x) = 0$; thus, $z(x) = 0$ and we get the result. ■

Lemma 4.3.4 *There exists an $\epsilon_1 > 0$ such that $y_1'(x) \geq 0$ and $y_1''(x) \geq 0$ for all $x \in (0, \epsilon_1)$.*

Proof: Integrating equation (4.41) from 0 to ∞ gives $y_1'(0) = \frac{\lambda_1}{\alpha} - b > 0$ and so $y_1''(0) > 0$. The continuity of y_1' and y_1'' thus gives the results. ■

The monotonicity result of Lemma 4.2.5 can be extended to Problem 4.3 using a similar argument.

Lemma 4.3.5 *There exists a $Z_1 > 0$ such that $y_1(x)$ and $y_1'(x)$ are strictly monotone in (Z_1, ∞).*

Theorem 4.3.6 *The first eigenfunction $y_1(x)$ is positive for $x > 0$.*

Proof: Applying the transformations $\sigma(x)$ and $\tau(x)$ defined by (4.24) and (4.26) to the differential equation (4.41) produces the functional differential equation (4.50) and $\tau''(x) = y_1(x)$. Since $\lambda_1 < b\alpha^2$, the result of Lemma 4.3.1 can be applied to equation (4.50). Now, $\tau(0) > 0$ since $\tau'(0) = -1$ and $\tau''(0) = 0$, so that $\tau'(x) \leq 0$ and $\tau(x) \geq 0$. To establish that $y_1(x) > 0$ for $x > 0$, it suffices to show that $\tau''(x) > 0$. Since $y_1(x) \geq 0$ in $x \in (0, \epsilon)$ for some $\epsilon > 0$ by Lemma 4.3.4, $\tau''(x) = y_1(x) \geq 0$ in that interval.

Suppose that $\tau''(\hat{x}) < 0$ for some $\hat{x} \in (\epsilon, \infty)$. Because $\tau''(x) \geq 0$ in $(0, \epsilon)$, there is a maximum critical point s_1 of τ' such that $\tau''(s_1) = 0$ and $\tau'''(s_1) \leq 0$. A differentiation of equation (4.50) gives the following equation:

$$\tau'''(x) - a\tau''(x) - b\tau'(x) + \frac{\lambda_1}{\alpha}\tau'(\alpha x) = 0,$$

and $\frac{\lambda_1}{\alpha}\tau'(\alpha s_1) \geq b\tau'(s_1)$. Since $\tau'(x) \leq 0$ for all x and $\lambda_1 > b\alpha$, $\tau'(\alpha s_1) \geq \tau'(s_1)$. This implies that there must be a minimum critical point s_2 of $\tau'(x)$ in $(s_1, \alpha s_1)$ and thus $\frac{\lambda_1}{\alpha}\tau'(\alpha s_2) \leq b\tau'(s_2)$. Therefore,

$$\frac{\lambda_1}{\alpha}\tau'(\alpha s_2) \leq b\tau'(s_2) < b\tau'(s_1) \leq \frac{\lambda_1}{\alpha}\tau'(\alpha s_1),$$

so that $\tau'(\alpha s_2) < \tau'(\alpha s_1)$. This indicates the existence of another maximum critical point s_3 of $\tau'(x)$ in $(s_2, \alpha s_2)$ and so we have $\tau'(\alpha s_3) \geq \tau'(s_3)$ and $\tau'(s_3) > \tau'(s_2)$ using the same arguments. The repeated process produces an infinite number of critical points of τ' satisfying $\tau''(x) = 0$ so that $y_1(x)$ has an infinite number of zeros, contradicting Lemma 4.3.5 that $y_1(x)$ is monotone in some interval (Z, ∞). Therefore, $\tau''(x) \geq 0$ for all $x > 0$ and so $y_1(x) \geq 0$. It remains to prove that $y_1(x) > 0$. Suppose that there exists a point $\bar{x}$ such that $y_1(\bar{x}) = 0$. Since $y_1(x) \geq 0$, the solution is zero for $x \geq \bar{x}$ or it has a zero minimum. It can be shown that $y_1(x)$ cannot be zero in any intervals of the form $[\hat{x}, \alpha\hat{x}]$ for $\hat{x} \in (0, \infty)$ using the same arguments as those used in the proof Lemma 2.2.1. Let $\bar{x}_1$ denote the point such that $y_1'(\bar{x}_1) = 0$ and $y_1''(\bar{x}_1) \geq 0$. Equation (4.41) indicates that $y_1(\alpha\bar{x}_1) \leq 0$ and so $y_1(\alpha\bar{x}_1) = 0$. Hence there exists another point $\bar{x}_2$ satisfying $y_1'(\bar{x}_2) = 0$ and $y_1''(\bar{x}_2) \geq 0$. Repeating this process constructs an infinite number of zeros $\{\bar{x}_n\}$ of y, contracting Lemma 4.3.5. ■

The next theorem is a straightforward extension of Theorem 4.2.7 and the corollary follows immediately from Theorems 4.3.6 and 4.3.7.

Theorem 4.3.7 *The eigenfunctions $y_n(x)$ to Problem 4.3 can have neither a positive minimum nor a negative maximum.*

Corollary 4.3.8 *The first eigenfunction $y_1(x)$ has exactly one maximum.*

Now, integrating equation (4.41) from 0 to ∞ gives $y_1'(0) = -b + \lambda_1/\alpha$ and since $b\alpha < \lambda_1 < b\alpha^2$, we have $-b + \lambda_1/\alpha > 0$ and thus $y_1''(0) > 0$. With this result and using the same arguments as those used in the proof of Theorem 4.2.10, we have the following theorem, which gives the shape of the graph of the first eigenfunction y_1:

Theorem 4.3.9 *The eigenfunction $y_1(x)$ has only one point s satisfying $y_1''(s) = 0$ in each interval $(0, X_m)$ or (X_m, ∞), where X_m is the maximum critical point.*

Figure 4.4.6 illustrates the first eigenfunction $y_1(x)$ when $a = 1, b = 1$ and $\alpha = 2$.

In Problem 4.2.1, the graphs of the eigenfunctions can be obtained from the first eigenfunction since there is a relationship $y_{n+1}(x) = Ay_n'(x)$ between consecutive eigenfunctions. In the case of Problem 4.3, the eigenvalues are not known explicitly

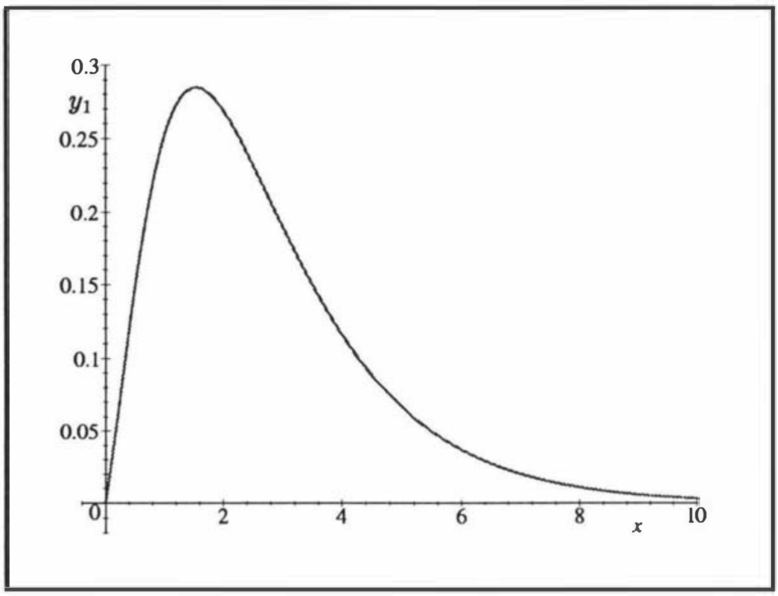

Figure 4.4.6: The first eigenfunction $y_1(x)$ for $\lambda \approx 2.54$ to Problem 4.3 according to $a = 1$, $b = 1$ and $\alpha = 2$.

and there is no obvious relationship between consecutive eigenfunctions. The graphs of the other eigenfunctions for Problem 4.3 are thus not as simple to obtain from the first eigenfunction as those for Problem 4.2.1. However, the numerical evidence suggests that the qualitative features of the n^{th} eigenfunction for Problem 4.3 may be essentially the same as those for Problem 4.2.1.

Chapter 5

The Equations with Variable Coefficients

In this chapter, we investigate linear second order functional differential equations with polynomial variable coefficients. Two kinds of methods are used here to examine the equations. One method is to seek a series type solution like the solutions to the equations with constant coefficients; the other method involves the use of a Green's function to establish the existence of a solution.

5.1 Introduction

We consider the equation

$$L(y(x)) = p(x)y''(x) + q(x)y'(x) + r(x)y(x) = \lambda r(x)y(\alpha x), \tag{5.1}$$

with the boundary condition

$$y(\infty) = 0, \tag{5.2}$$

and the normalizing condition

$$\int_0^\infty y(t)\, dt = 1, \tag{5.3}$$

where $p(x)$, $q(x)$ and $r(x)$ are polynomials and $\alpha > 1$.
We will refer to equation (5.1) along with boundary conditions (5.2) and (5.3) as *Problem 5.*

Motivated from Dirichlet series solutions to the equations with constant coefficients, we look for the following form of solutions to Problem 5:

$$y(x) = \sum_{n=0}^{\infty}(-\lambda)^n y_n(x), \tag{5.4}$$

where $y_0(x)$ is a general solution to the equation

$$L(y_0(x)) = 0, \tag{5.5}$$

satisfying the condition $y_0(\infty) = 0$ (which may not be possible, depending on the coefficients), and

$$L(y_n(x)) = -r(x)y_{n-1}(\alpha x), \tag{5.6}$$

for all $n \geq 1$.

As an example, a Dirichlet series solution to Problem 4.2.1 without the condition $y(0) = 0$ when $b = 1$ can be obtained by using (5.4), (5.5) and (5.6). A solution to the equation

$$L(y_0(x)) = y_0''(x) - y_0(x) = 0,$$

satisfying $y_0(\infty) = 0$ is $y_0(x) = a_0 e^{-x}$, where $a_0 \in R$. Now

$$L(y_1(x)) = y_1''(x) - y_1(x) = a_0 e^{-\alpha x}, \tag{5.7}$$

and substituting $y_1(x) = a_1 e^{-\alpha x}$ into equation (5.7) indicates that $a_1 = \frac{a_0}{\alpha^2 - 1}$ and thus

$$y_1(x) = \frac{a_0}{\alpha^2 - 1} e^{-\alpha x}.$$

We get the third component $y_2(x)$ from the equation

$$L(y_2(x)) = y_2''(x) - y_2(x) = \frac{a_0}{\alpha^2 - 1} e^{-\alpha^2 x},$$

using the arguments similar to those used for obtaining $y_1(x)$, so that $a_2 = \frac{a_0}{(\alpha^4 - 1)(\alpha^2 - 1)}$ and therefore

$$y_2(x) = \frac{a_0}{(\alpha^2 - 1)(\alpha^4 - 1)} e^{-\alpha^2 x}.$$

In general, we have

$$y_n(x) = \frac{a_0}{\prod_{m=1}^{n}(\alpha^{2m} - 1)} e^{-\alpha^n x},$$

and consequently,

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{2m} - 1)} e^{-\alpha^n x}, \tag{5.8}$$

where a_0 can be determined by normalizing the solution so that

$$a_0 = \left(\sum_{n=0}^{\infty} \frac{(-\lambda/\alpha)^n}{\prod_{m=1}^{n}(\alpha^{2m}-1)} \right)^{-1}.$$

Note that $y(x)$ defined by (5.8) is the same as the solution defined by (4.13) when $b = 1$. These calculations suggest a method for solving equations with variable coefficients.

Based on this observation, we first focus on some equations which can be solved using this method. It is known that the following types of equations have a solution $y(x)$ such that $y(\infty) = 0$:

I) $y''(x) - xy(x) = 0, \quad y(x) = Ai(x)$, where $Ai(x)$ is an *Airy function.*

II) $xy''(x) - y'(x) - y(x) = 0, \quad y(x) = xK_2(2\sqrt{x})$, where $K_2(2\sqrt{x})$ is a *modified Bessel function of the second kind.*

III) $xy''(x) - y'(x) - x^3y(x) = 0, \quad y(x) = e^{-\frac{1}{2}x^2}$.

IV) $y''(x) + xy'(x) + y(x) = 0, \quad y(x) = e^{-\frac{1}{2}x^2}$.

V) $(x+k)^2y''(x) + a(x+k)y'(x) + by(x) = 0, \quad y(x) = (x+k)^{rx}$, where $k > 0$ and a, b are real values which produce a negative root r for the equation $r^2 + (a-1)r + b = 0$. Note that if $k = 0$, the solution to the above equation is $y(x) = x^{rx}$ and so it has a singularity at $x = 0$.

In the next sections, we present solutions to the equations

$$\begin{aligned} y''(x) - xy(x) + \lambda xy(\alpha x) &= 0 \\ xy''(x) - y'(x) - y(x) + \lambda y(\alpha x) &= 0 \\ xy''(x) - y'(x) - x^3y(x) + \lambda x^3 y(\alpha x) &= 0, \end{aligned}$$

using the series (5.4). For the equations

$$y''(x) + xy'(x) + y(x) + \lambda y(\alpha x) = 0, \tag{5.9}$$

and

$$(x+k)^2y''(x) + a(x+k)y'(x) + by(x) + \lambda y(\alpha x) = 0, \tag{5.10}$$

where $k > 0$, the term $y_1(x)$ cannot be obtained explicitly from $y_0(x)$ so that the method used for the three previous equations cannot be easily applied here. The existence of a solution for equation (5.10) will be proved by using a Green's function to reform the problem as an integral equation and applying the contraction mapping theorem. However, it is not shown whether or not equation (5.9) has a solution.

5.2 The equation $y''(x) - xy(x) + \lambda xy(\alpha x) = 0$

Substituting $p(x) = 1$, $q(x) = 0$ and $r(x) = -x$ into equation (5.1) leads to the equation

$$y''(x) - xy(x) + \lambda xy(\alpha x) = 0, \tag{5.11}$$

where $\alpha > 1$, and a solution to the equation $L(y(x)) = y''(x) - xy(x) = 0$ satisfying the boundary condition (5.2) is $y(x) = Ai(x)$. The function $Ai(x)$ satisfies

$$Ai(x) \sim \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-\frac{2}{3}x^{\frac{3}{2}}},$$

as $x \to \infty$ (cf. Andrews [1992]) and at $x = 0$, $Ai(x)$ is not defined, but $\lim_{x\to 0^+} Ai(x)$ exists and $Ai(0) = \lim_{x\to 0^+} Ai(x) = \frac{1}{\sqrt[3]{9}}\Gamma(\frac{2}{3})$.

We refer to equation (5.11) along with conditions (5.2) and (5.3) as *Problem 5.1.*

5.2.1 Existence of Solutions

The first component $y_0(x) = a_0 Ai(x)$ for $a_0 \in R$ so that $y_1(x)$ is a solution to the equation

$$y_1''(x) - xy_1(x) = xa_0 Ai(\alpha x), \tag{5.12}$$

and substituting $y_1(x) = a_1 Ai(\alpha x)$ into equation (5.12) produces

$$a_1\alpha^2 Ai''(\alpha x) - xa_1 Ai(\alpha x) = xa_0 Ai(\alpha x). \tag{5.13}$$

Now $Ai''(x) - xAi(x) = 0$ so that

$$Ai''(\alpha x) = \alpha x Ai(\alpha x). \tag{5.14}$$

Therefore, equation (5.13) can be converted into the equation

$$xa_1(\alpha^3 - 1)Ai(\alpha x) = xa_0 Ai(\alpha x),$$

and hence $a_1 = \frac{a_0}{\alpha^3 - 1}$. Consequently,

$$y_1(x) = \frac{a_0}{\alpha^3 - 1} Ai(\alpha x),$$

and so the third component $y_2(x)$ can be obtained from the equation

$$y_2''(x) - xy_2(x) = \frac{a_0 x}{\alpha^3 - 1} Ai(\alpha^2 x). \tag{5.15}$$

Substituting $y_2(x) = a_2 Ai(\alpha^2 x)$ into equation (5.15) and using equation (5.14) imply that $a_2 = \frac{a_0}{(\alpha^3-1)(\alpha^6-1)}$ so that

$$y_2(x) = \frac{a_0}{(\alpha^3-1)(\alpha^6-1)} Ai(\alpha^2 x).$$

In general we have that

$$y_n(x) = \frac{a_0}{\prod_{m=1}^{n}(\alpha^{3m}-1)} Ai(\alpha^n x),$$

and we thus have the formal solution:

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty}(-\lambda)^n a_n Ai(\alpha^n x) \\ &= a_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{3m}-1)} Ai(\alpha^n x), \end{aligned} \tag{5.16}$$

where a_0 can be determined by normalizing $y(x)$ and $y(0) = \sum_{n=0}^{\infty}(-\lambda)^n a_n Ai(0)$.

We now show that the series (5.16) is a solution to Problem 5.1. This entails establishing that series is uniformly convergent in the interval $[0, \infty)$, proving $\lim_{x\to\infty} y(x) = 0$, and showing that the series can be normalized to satisfy condition (5.3).

Let us first recall properties of the function $A_i(x)$. For all $x > 0$,

$$Ai(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}} K_{1/3}(\frac{2}{3}x^{3/2}),$$

(cf. Andrews [1992]). Now, $K_p(x)$ is positive for all $p \in R$ and thus $Ai(x)$ is positive; moreover, $Ai'(x) = -\frac{x}{\sqrt{3}\pi} K_{2/3}(\frac{2}{3}x^{3/2})$ (cf. Luke [1962]), and consequently $Ai'(x) < 0$ so that $Ai(x)$ is decreasing for all $x > 0$.

Since $Ai(x)$ is a decreasing positive function,

$$\sum_{n=0}^{\infty} |a_n| Ai(\alpha^n x) \le \sum_{n=0}^{\infty} |a_n| Ai(0),$$

and so the series(5.16) is uniformly convergent on $x \in [0, \infty)$ because

$$\sum_{n=0}^{\infty} |a_n| = |a_0| \sum_{n=0}^{\infty} \frac{|\lambda|^n}{\prod_{m=1}^{n}(\alpha^{3m}-1)},$$

is convergent. Therefore, the series defining $y(x)$ can be differentiated or integrated term by term. Moreover,

$$y(\infty) = \lim_{x\to\infty} y(x) = \sum_{n=0}^{\infty} a_n \lim_{x\to\infty} A(\alpha^n x) = 0,$$

since $Ai(\alpha^n x) \to 0$ as $x \to \infty$ for all n.

Let us confirm that the series (5.16) satisfies the equation (5.11). Now

$$\begin{aligned} L(y) &= y''(x) - xy(x) \\ &= a_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{3m}-1)} \alpha^{2n} Ai''(\alpha^n x) \\ &\quad - x a_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{3m}-1)} Ai(\alpha^n x), \end{aligned}$$

and since $A''(\alpha^n x) = \alpha^n x Ai(\alpha^n x)$,

$$\begin{aligned} L(y) + \lambda x y(\alpha x) &= x a_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{3m}-1)} (\alpha^{3n}-1) Ai(\alpha^n x) \\ &\quad + x a_0 \lambda \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{3m}-1)} Ai(\alpha^{n+1} x) \\ &= x a_0 \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n-1}(\alpha^{3m}-1)} Ai(\alpha^n x) \\ &\quad - x a_0 \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n-1}(\alpha^{3m}-1)} Ai(\alpha^n x) \\ &= 0. \end{aligned}$$

It is known that

$$\int_0^{\infty} t^q K_p(t)\, dt = 2^{q-1} \Gamma(\frac{q+p+1}{2}) \Gamma(\frac{q-p+1}{2}), \tag{5.17}$$

if $q+p > -1$ and $q-p > -1$ (cf. Luke [1962]). So,

$$\begin{aligned} \int_0^{\infty} Ai(t)\, dt &= \frac{1}{\pi\sqrt{3}} \int_0^{\infty} \sqrt{t} K_{1/3}(2/3 t^{3/2})\, dt \\ &= \frac{1}{\pi\sqrt{3}} \int_0^{\infty} K_{1/3}(t)\, dt \\ &= \frac{1}{2\sqrt{3}\pi} \Gamma(2/3)\Gamma(1/3). \end{aligned}$$

Now

$$\begin{aligned} I_n = \int_0^{\infty} Ai(\alpha^n t)\, dt &= \frac{1}{\alpha^n} \int_0^{\infty} Ai(t)\, dt \\ &= \frac{\Gamma(2/3)\Gamma(1/3)}{2\sqrt{3}\pi\alpha^n}, \end{aligned}$$

so that $I_n < \infty$ for all $n \geq 0$. Hence the uniform convergence along with the integrability of each term in the series implies that $y(x)$ is integrable in $[0, \infty)$. Therefore,

$y(x)$ defined by (5.16) is a solution to Problem 5.1 and a_0 can be determined by normalizing the solution so that

$$a_0 = \frac{2\sqrt{3}\pi}{\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})}\left(\sum_{n=0}^{\infty}\frac{(-\lambda/\alpha)^n}{\prod_{m=1}^{n}(\alpha^{3m}-1)}\right)^{-1}.$$

5.2.2 Uniqueness of Solutions

Theorem 5.2.1 *If $|\lambda| \le \alpha^2$, then there is a unique solution $y(x)$ to Problem 5.1 such that $xy(x)$ is integrable on $[0,\infty)$.*

Proof: Suppose y_1 and y_2 are distinct solutions to Problem 5.1 and let $z(x) = y_1(x) - y_2(x)$. Then z satisfies equation (5.11) and $\int_0^\infty z(t)\,dt = 0$. Note that in equation (5.11), $xy(x)$ is integrable so that $y''(x)$ is integrable on $[0,\infty)$. Integrating equation (5.11) for z instead of y from x to ∞ produces the integro-differential equation

$$-z'(x) - \int_x^\infty tz(t)\,dt + \frac{\lambda}{\alpha^2}\int_{\alpha x}^\infty tz(t)\,dt = 0. \tag{5.18}$$

Using the transformation

$$\xi(x) = \int_x^\infty tz(t)\,dt, \tag{5.19}$$

the above equation can be converted into the equation

$$x\xi''(x) - \xi'(x) - x^2\xi(x) + \frac{\lambda}{\alpha^2}x^2\xi(\alpha x) = 0, \tag{5.20}$$

since $\xi'(x) = -xz(x)$ and $\xi''(x) = -z(x) - xz'(x)$. Furthermore, $\xi(\infty) = 0$ and $\xi(0) = \int_0^\infty tz(t)\,dt$. It can be proven that ξ is increasing or decreasing using the same method as that used in the proof of Lemma 4.2.1. This means that $z(x) \le 0$ or $z(x) \ge 0$ because $\xi'(x) = -xz(x)$ and so the condition $\int_0^\infty z(t)\,dt = 0$ implies that $z(x) = 0$.■

The above theorem indicates that if $|\lambda| \le \alpha^2$, the only solution to Problem 5.1 is given by (5.16).

5.2.3 Qualitative Properties of Solutions

Theorem 5.2.2 *Suppose that $xy(x)$ is integrable on $[0,\infty)$. If $|\lambda| \le \alpha^2$, then the solution to Problem 5.1 is positive for $x \ge 0$.*

Proof: Using the transformation $\kappa(x) = \int_x^\infty ty(t)\,dt$, we have the equation (5.20) for κ instead of ξ and using the same arguments as those used in the proof of Lemma 4.2.1, we can deduce that $\kappa'(x) \le 0$ or $\kappa'(x) \ge 0$. This implies that $y(x) \ge 0$ or $y(x) \le 0$ respectively, since $\kappa'(x) = -xy(x)$. Now $\int_0^\infty y(t)\,dt = 1$ and therefore, $y(x)$ cannot be non-positive for all $x \ge 0$; thus, $y(x) \ge 0$ for all $x \ge 0$. It remains to prove that $y(x) \ne 0$ for any $x \ge 0$. Equation (5.18) for y instead of z gives

$$y'(x) = (-1 + \frac{\lambda}{\alpha^2}) \int_{\alpha x}^{\infty} ty(t)\,dt - \int_x^{\alpha x} ty(t)\,dt, \tag{5.21}$$

and thus $y'(x) \le 0$ since $-1 + \frac{\lambda}{\alpha^2} \le 0$. Therefore y cannot have a maximum so that if there is a point s satisfying $y(s) = 0$, then $y(x) = 0$ for all $x \ge s$. Let $\hat{x}$ be the smallest point satisfying $y(x) = 0$ in $x \in [0, \infty)$. Now, $y(x) = 0$ in $[\hat{x}, \alpha\hat{x}]$ and so equation (5.11) implies

$$y''(x) - xy(x) = 0, \tag{5.22}$$

for all $x \in [\hat{x}/\alpha, \hat{x}]$, and $y(\hat{x}) = 0$. Since y'' exists, y' is continuous so that equation (5.22) satisfies the initial condition $y'(\hat{x}) = 0$. Therefore, the ordinary differential equation (5.22) has the unique solution $y(x) = 0$ in $[\hat{x}/\alpha, \hat{x}]$ and it contradicts the fact that $\hat{x}$ is the smallest zero of y; thus, $y(x) > 0$.■

The following lemma and theorem are satisfied for all λ and $y(x)$ in (5.16):

Lemma 5.2.3 *There exists a $Z \in R$ such that $y'(x) \le 0$ or $y'(x) \ge 0$ for all $x \ge Z$.*

Proof: Since $y(x) = \sum_{n=0}^\infty a_n Ai(\alpha^n x)$,

$$y'(x) = \sum_{n=0}^{\infty} a_n \alpha^n Ai'(\alpha^n x).$$

If $\lambda < 0$, $y'(x) < 0$ since $Ai'(\alpha^n x) < 0$ and $a_n > 0$ for all $n \ge 0$. Let $\lambda > 0$. It is known that $Ai'(x) \sim -\frac{1}{2\sqrt{\pi}} x^{1/4} e^{-\frac{2}{3}x^{3/2}}$ as $x \to \infty$ (cf. Luke [1962]) and so

$$\frac{Ai'(\alpha^n x)}{Ai'(\alpha^{n-1}x)} \sim \alpha^{1/4} e^{-\frac{2}{3}(\alpha^{n-1}(\alpha-1)x)^{3/2}} = \alpha^{1/4} e^{-H(x)}.$$

Since $H(x) \to \infty$ as $x \to \infty$, we can make the value $|\frac{Ai'(\alpha^n x)}{Ai'(\alpha^{n-1}x)}|$ small arbitrarily by increasing x for all n. Thus there exists a $Z \in R$ such that

$$|\frac{a_n}{a_{n-1}}| \alpha \frac{Ai'(\alpha^n x)}{Ai'(\alpha^{n-1}x)} < 1, \tag{5.23}$$

for all n and $x \ge Z$. Now the series is alternating so that $y'(x) \le 0$ if $a_0 > 0$ and $y'(x) \ge 0$ if $a_0 < 0$. ■

Using the result of the above lemma and the same arguments as those used in the proof of Theorem 4.2.7 we can get the following theorem:

Theorem 5.2.4 *The solution $y(x)$ defined by (5.16) can have neither a positive minimum nor a negative maximum for all λ.*

Note that if $|\lambda| < \alpha^2$, then we get $y'(0) < 0$ from the integral equation (5.21). This observation along with Theorem 5.2.4 indicates that $y(x)$ is decreasing in $[0, \infty)$.

5.2.4 The Values λ When $y(0) = 0$

Using arguments similar to those used in Chapter 4, we determine the eigenvalues λ satisfying $y(0) = 0$ and the corresponding eigenfunctions. At $x = 0$, the solution $y(x)$ defined by (5.16) becomes

$$y(0) = a_0 k_1 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{3m} - 1)}, \tag{5.24}$$

where $k_1 = \frac{1}{\sqrt[3]{9}\Gamma(2/3)}$. Let $y(0) = a_0 k_1 V(\lambda)$, then by a method similar to that used in Chapter 4, we get

$$V(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{3m} - 1)} = \prod_{n=0}^{\infty}(1 - \frac{\lambda}{\alpha^{3(n+1)}}).$$

Now, $a_0 = \frac{2\sqrt{3}\pi}{\Gamma(2/3)\Gamma(1/3)} V^{-1}(\lambda/\alpha)$ so that $a_0 \neq \infty$ at the values of $\lambda = \alpha^{3n}$ for $n = 1, 2, \ldots$; thus, the eigenvalues satisfying $y(0) = 0$ are given by $\lambda = \alpha^{3n}$ for all $n = 1, 2, \ldots$.

Let us investigate the eigenfunction y_1 corresponding to the eigenvalue $\lambda = \alpha^3$.

Theorem 5.2.5 *The eigenfunction $y_1(x)$ corresponding to $\lambda = \alpha^3$ is positive for all $x > 0$.*

Proof: Note that the Airy function $Ai(x)$ decays exponentially so that $x^k y(x)$ is integrable on $[0, \infty)$ for any $k \in R$. Multiplying equation (5.11) by x yields

$$xy''(x) - x^2 y(x) + \lambda x^2 y(\alpha x) = 0, \tag{5.25}$$

and by integrating this equation from x to ∞ and substituting $\lambda = \alpha^3$ we get the equation

$$-xy'(x) + y(x) - \int_x^{\infty} t^2 y(t)\, dt + \int_{\alpha x}^{\infty} t^2 y(t)\, dt = 0. \tag{5.26}$$

The transformation

$$\Xi(x) = \int_x^{\infty} t^2 y(t)\, dt,$$

converts the above equation into

$$x\Xi''(x) - 3\Xi'(x) - x^2\Xi(x) + x^2\Xi(\alpha x) = 0, \tag{5.27}$$

since $\Xi'(x) = -x^2y(x)$ and $\Xi''(x) = -2xy(x) - x^2y'(x)$ so that $-xy'(x) = \frac{\Xi''(x)}{x} + 2y(x)$ and $y(x) = \frac{\Xi'(x)}{-x^2}$. Suppose that there is a positive maximum or a negative minimum. Then using the same arguments as those used in the proof of Lemma 4.2.1, we get $\Xi'(x) \leq 0$ or $\Xi'(x) \geq 0$, and the equation $\Xi'(x) = -x^2y(x)$ yields that $y(x) \geq 0$ or $y(x) \leq 0$. However, the condition $\int_0^\infty y(t)\,dt = 1$ precludes the possibility that $y(x) \leq 0$; thus, $y(x) \geq 0$. Suppose there is a point $s > 0$ satisfying $\Xi'(s) = -y(s) = 0$, then $\Xi''(s) = 0$ since Ξ cannot have a positive maximum and so $\Xi(s) = \Xi(\alpha s)$. This implies that $\Xi(x) = C$ for $x \in [s, \alpha s]$, where C is some constant and, equation (5.27) indicates that $\Xi(x) = C$ for $[\alpha s, \alpha^2 s]$. The argument can be repeated *ad infinitum* using the interval $[\alpha s, \alpha^2 s]$ to get the result $\Xi(x) = C$ for all $x \geq s$ and since $y(\infty) = 0$, $C = 0$. Let $\hat{x} > 0$ be the smallest point satisfying $y(x) = 0$ for $x \in [0, \infty)$. Then it can be shown that $y(x) = 0$ in $[\hat{x}/\alpha, \hat{x}]$ using the same manner as that used in the proof of Theorem 5.2.2. This contradicts that $\hat{x}$ is the smallest zero of y and consequently, $y(x) > 0$ for $x > 0$. ■

Note the for $\alpha^2 < |\lambda| < \alpha^3$, it can be proven that a series solution $y(x)$ defined by (5.16) is positive for $x \geq 0$ using the same arguments as those used in the proof of Theorem 5.2.5. If $-\alpha^3 \leq \lambda < \alpha^2$, then equation (5.21) leads to $y'(0) = -1 + \frac{\lambda}{\alpha^2} < 0$ and so, from the result of Theorem 5.2.4, we get y is decreasing for all x. For the case $\lambda = \alpha^2$, equation (5.21) indicates that $y'(x) \leq 0$ for all x. If $\alpha^2 < \lambda \leq \alpha^3$, then $y'(0) > 0$ from (5.21), so that $y(x)$ has at least one maximum critical point. Theorem 5.2.4 implies that the solution cannot have a positive minimum critical point and therefore, it has exactly one maximum critical point.

5.2.5 Bounds on the Maximum Critical Point of y_1

The eigenfunction $y_1(x)$ has exactly one maximum critical point. Let X_m denote the point at which y_1 achieves its maximum. Here we shall consider the bounds for the maximum critical point X_m of $y_1(x)$. Now $y'(X_m) = 0$ and equation (5.26) thus implies that

$$y_1(X_m) = \int_{X_m}^{\alpha X_m} t^2 y_1(t)\,dt.$$

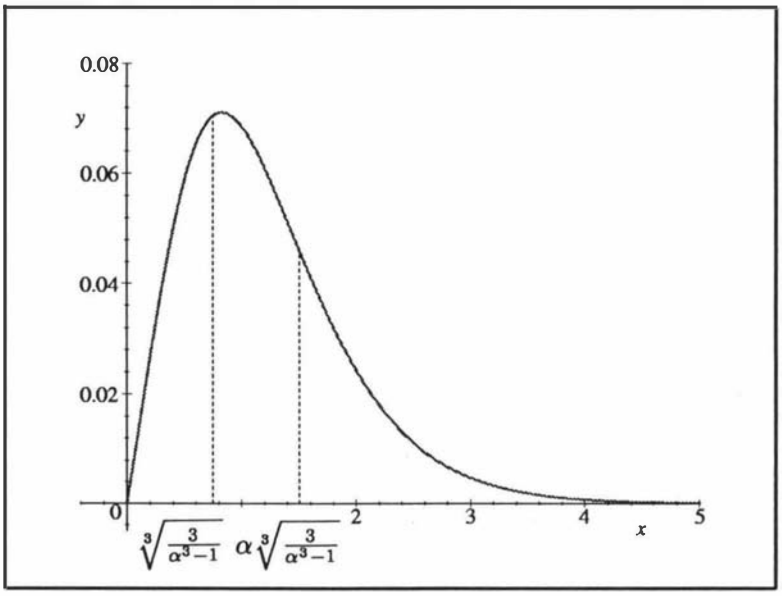

Figure 5.5.1: The bounds on X_m and the solution y of $y''(x) - xy(x) + 8xy(2x) = 0$.

Since $y_1(X_m)$ is the maximum,

$$\begin{aligned} y_1(X_m) &\leq y_1(X_m)\int_{X_m}^{\alpha X_m} t^2\,dt \\ &= \frac{1}{3}y_1(X_m)X_m^3(\alpha^3 - 1), \end{aligned}$$

and hence

$$X_m \geq (\frac{3}{\alpha^3 - 1})^{1/3}. \tag{5.28}$$

On the other hand, $y_1(\alpha X_m)$ is the minimum in $[X_m, \alpha X_m]$ and consequently

$$\begin{aligned} y_1(X_m) &\geq y_1(\alpha X_m)\int_{X_m}^{\alpha X_m} t^2\,dt \\ &= \frac{1}{3}y_1(\alpha X_m)X_m^3(\alpha^3 - 1). \end{aligned}$$

Since $y_1''(X_m) \leq 0$, $y_1(X_m) \leq \alpha^3 y_1(\alpha X_m)$ from equation (5.11) for $\lambda = \alpha^3$ so that

$$y_1(X_m) \geq \frac{1}{3}\alpha^{-3}y_1(X_m)X_m^3(\alpha^3 - 1),$$

and therefore,

$$X_m \leq \alpha(\frac{3}{\alpha^3 - 1})^{1/3}. \tag{5.29}$$

Together inequalities (5.28) and (5.29) indicates that

$$(\frac{3}{\alpha^3-1})^{1/3} \le X_m \le \alpha(\frac{3}{\alpha^3-1})^{1/3}.$$

Figure 5.5.1 illustrates the bounds $(\frac{3}{7})^{1/3} \le X_m \le 2(\frac{3}{7})^{1/3}$ corresponding to the case $\alpha = 2$.

We note that the results in this section can be extended to the equation

$$y''(x) - bxy(x) + \lambda xy(\alpha x) = 0, \tag{5.30}$$

where $b > 0$, provided that we exchange the condition $|\lambda| \le \alpha^3$ for $|\lambda| \le b\alpha^3$ and the condition $\lambda = \alpha^3$ for $\lambda = b\alpha^3$. The solution to the equation $y_0''(x) - bxy_0(x) = 0$ satisfying $y_0(\infty) = 0$ is $y_0(x) = Ai(b^{1/3}x)$ so that a solution to equation (5.30) which satisfies the conditions (5.2) and (5.3) is given by

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-\lambda/b)^n}{\prod_{m=1}^{n}(\alpha^{3m}-1)} Ai(b^{1/3}\alpha^n x),$$

where

$$a_0 = \frac{2\sqrt{3}\sqrt[3]{b}\pi}{\Gamma(2/3)\Gamma(1/3)} \left(\sum_{n=0}^{\infty} \frac{(-\frac{\lambda}{b\alpha})^n}{\prod_{m=1}^{n}(\alpha^{3m}-1)} \right)^{-1}.$$

5.3 The equation $xy''(x) - y'(x) - y(x) + \lambda y(\alpha x) = 0$

A substitution of $p(x) = x$, $q(x) = -1$ and $r(x) = -1$ into equation (5.1) leads to the equation

$$xy''(x) - y'(x) - y(x) + \lambda y(\alpha x) = 0, \tag{5.31}$$

where $\alpha > 1$. We will refer to equation (5.31) along with conditions (5.2) and (5.3) as *Problem 5.2.*

It is known that a solution to the equation $L(y(x)) = xy''(x) - y'(x) - y(x) = 0$ satisfying the boundary condition (5.2) is $y(x) = xK_2(2\sqrt{x})$. The function $K_p(x) \sim \sqrt{\frac{\pi}{2x}}e^{-x}$ for $p \ge 0$ as $x \to \infty$ (cf. Andrews [1992]) so that $xK_2(2\sqrt{x}) \sim \sqrt{\frac{\pi}{4}}x^{3/4}e^{-2\sqrt{x}}$ as $x \to \infty$. At $x = 0$, $xK_2(2\sqrt{x})$ is not defined but the limiting value exists as $x \to 0^+$ and $\lim_{x\to 0^+} xK_2(2\sqrt{x}) = \frac{\Gamma(2)}{2} = \frac{1}{2}$ because $K_p(x) \sim \frac{\Gamma(p)}{2}(\frac{2}{x})^p$ as $x \to 0^+$.

5.3.1 Existence of Solutions

Now $y_0(x) = a_0 x K_2(2\sqrt{x})$ for $a_0 \in R$. Let $K(x) = xK_2(2\sqrt{x})$. Then the function $y_1(x)$ satisfies the equation

$$xy_1''(x) - y_1'(x) - y_1(x) = a_0 K(\alpha x), \tag{5.32}$$

and substituting $y_1(x) = a_1 K(\alpha x)$ into equation (5.32) implies that

$$xa_1\alpha^2 K''(\alpha x) - a_1\alpha K'(\alpha x) - a_1 K(\alpha x) = a_0 K(\alpha x). \tag{5.33}$$

Since $xK''(x) - K'(x) - K(x) = 0$, $\alpha x K''(\alpha x) - K'(\alpha x) = K(\alpha x)$ so that equation (5.33) becomes $a_1(\alpha - 1)K(\alpha x) = a_0 K(\alpha x)$ and therefore, $a_1 = \frac{a_0}{\alpha - 1}$. This implies that

$$y_1(x) = \frac{a_0}{\alpha - 1} K(\alpha x),$$

and so

$$xy_2''(x) - y_2'(x) - y_2(x) = \frac{a_0}{\alpha - 1} K(\alpha^2 x).$$

A substitution of $y_2(x) = a_2 K(\alpha^2 x)$ into the above equation and using the arguments similar to those used for getting $y_1(x)$ give the coefficient $a_2 = \frac{a_0}{(\alpha-1)(\alpha^2-1)}$ and thus,

$$y_2(x) = \frac{a_0}{(\alpha - 1)(\alpha^2 - 1)} K(\alpha^2 x).$$

In general,

$$y_n(x) = \frac{a_0}{\prod_{m=1}^{n}(\alpha^m - 1)} K(\alpha^n x),$$

and therefore, we have the formal solution:

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} (-\lambda)^n a_n K(\alpha^n x) \\ &= \sum_{n=0}^{\infty} (-\lambda)^n a_n \alpha^n x K_2(2\sqrt{\alpha^n x}) \\ &= a_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^m - 1)} \alpha^n x K_2(2\sqrt{\alpha^n x}), \end{aligned} \tag{5.34}$$

where a_0 can be determined by normalizing $y(x)$ and

$$K(0) = \lim_{x \to 0} [\alpha^n x K_2(2\sqrt{\alpha^n x})] = \frac{1}{2}.$$

Consider the convergence of the series (5.34). Now,

$$\begin{aligned} D(x) &= \frac{d}{dx}(\alpha^n x K_2(2\sqrt{\alpha^n x})) \\ &= \alpha^n K_2(2\sqrt{\alpha^n x}) + \alpha^n x K_2'(2\sqrt{\alpha^n x})(\alpha^n x)^{-1/2}\alpha^n \\ &= \alpha^n K_2(2\sqrt{\alpha^n x}) + \alpha^n \sqrt{\alpha^n x} K_2'(2\sqrt{\alpha^n x}), \end{aligned}$$

and since $tK_p'(t) = -pK_p(t) - tK_{p-1}(t)$ for $p \in R$ (cf. Gray [1922]) and $K_p(t) \geq 0$,

$$\begin{aligned} D(x) &= \alpha^n K_2(2\sqrt{\alpha^n x}) + \frac{1}{2}\alpha^n(-2K_2(2\sqrt{\alpha^n x}) - 2\sqrt{\alpha^n x}K_1(2\sqrt{\alpha^n x})) \\ &= -\alpha^n \sqrt{\alpha^n x} K_1(2\sqrt{\alpha^n x}) \qquad (5.35) \\ &\leq 0. \end{aligned}$$

This implies that $\alpha^n x K_2(2\sqrt{\alpha^n x})$ is decreasing for all x so that

$$\begin{aligned} y(x) &\leq \sum_{n=0}^{\infty} |a_n| \alpha^n x K_2(2\sqrt{\alpha^n x}) \\ &\leq \sum_{n=0}^{\infty} |a_n| K(0). \end{aligned}$$

Therefore, the series (5.34) is uniformly convergent and so $y(x)$ is a well defined function which can be differentiated/integrated term by term. Moreover, $y(\infty) = 0$ since $\lim_{x\to\infty}(\alpha^n x K_2(2\sqrt{\alpha^n x})) = 0$ for all n.

Let us confirm that the series (5.34) satisfies equation (5.31). Now

$$\begin{aligned} L(y) &= xy''(x) - y'(x) - y(x) \\ &= xa_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^n - 1)} \alpha^{2n} K''(\alpha^n x) \\ &\quad - a_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^m - 1)} \alpha^n K'(\alpha^n x) \\ &\quad - a_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^m - 1)} K(\alpha^n x), \end{aligned}$$

and since

$$\alpha^n x K''(\alpha^n x) - K'(\alpha^n x) = K(\alpha^n x), \qquad (5.36)$$

$$\begin{aligned} L(y)+\lambda y(\alpha x) &= a_0\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^m-1)}(\alpha^n-1)K(\alpha^n x) \\ &\quad +\lambda a_0\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^m-1)}K(\alpha^{n+1}x) \\ &= a_0\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{\prod_{m=1}^{n-1}(\alpha^m-1)}K(\alpha^n x) \\ &\quad -a_0\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{\prod_{m=1}^{n-1}(\alpha^m-1)}K(\alpha^n x) \\ &= 0. \end{aligned}$$

We now consider the integrability of $\alpha^n x K_2(2\sqrt{\alpha^n x})$ on $[0,\infty)$. Using equation (5.17), we have

$$\begin{aligned} \int_0^\infty \alpha^n x K_2(2\sqrt{\alpha^n x})\,dx &= \frac{1}{8}\alpha^{-n}\int_0^\infty x^3K_2(x)\,dx \\ &= \frac{1}{2}\alpha^{-n}\Gamma(3) \\ &= \alpha^{-n}<\infty, \end{aligned}$$

for all n and so the uniform convergence of the series implies that the series (5.34) is integrable on $[0,\infty)$. Therefore, a solution to Problem 5.2 is given by

$$y(x)=a_0\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^m-1)}\alpha^n x K_2(2\sqrt{\alpha^n x}), \tag{5.37}$$

where

$$a_0=\left(\sum_{n=0}^{\infty}\frac{(-\lambda/\alpha)^n}{\prod_{m=1}^{n}(\alpha^m-1)}\right)^{-1}. \tag{5.38}$$

5.3.2 Uniqueness of Solutions

Theorem 5.3.1 *If $|\lambda|\le\alpha$, then given λ and α, the solution to Problem 5.2 is unique.*

Proof: Suppose that there are distinct solutions y_1 and y_2 to Problem 5.2 and let $z(x)=y_1(x)-y_2(x)$. Then z satisfies the equation

$$xz''(x)-z'(x)-z(x)+\lambda z(\alpha x)=0, \tag{5.39}$$

for all $x>0$, and $\int_0^\infty z(t)\,dt=0$. An integration of equation (5.39) from x to ∞ leads to the integro-differential equation

$$-xz'(x) + 2z(x) - \int_x^\infty z(t)\,dt + \frac{\lambda}{\alpha}\int_{\alpha x}^\infty z(t)\,dt = 0, \tag{5.40}$$

and the transformation

$$\zeta(x) = \int_x^\infty z(t)\,dt \tag{5.41}$$

converts the above equation into the equation

$$x\zeta''(x) - 2\zeta'(x) - \zeta(x) + \frac{\lambda}{\alpha}\zeta(\alpha x) = 0, \tag{5.42}$$

with boundary values $\zeta(\infty) = 0$ and $\zeta(0) = 0$. In the same manner as that used the proof of Lemma 4.2.1, it can be proven that $\zeta'(x) \leq 0$ or $\zeta'(x) \geq 0$ for all $x > 0$ and the condition $\zeta(0) = 0$ implies that $\zeta'(x) = \zeta(x) = 0$. The continuity of $z(x)$ indicates thus that $z(x) = 0$ for all x. ■

5.3.3 Qualitative Properties of Solutions

Theorem 5.3.2 *If $|\lambda| < \alpha$, then for the solution $y(x)$ to problem 5.2, $y(x) > 0$ for $x \geq 0$.*

Proof: If we put

$$\rho(x) = \int_x^\infty y(t)\,dt, \tag{5.43}$$

where $y(x)$ is a solution to Problem 5.2, then ρ satisfies equation (5.42) for ρ instead of ζ and $\rho(0) = 1$. Therefore, in the same manner as that used in the proof of Lemma 4.2.1, we get $\rho'(x) \leq 0$ since $\rho(0) = 1 > 0$ so that $y(x) = -\rho'(x) \geq 0$. Now if $\rho'(s) = 0$ for some $s \geq 0$, then equation (5.42) for ρ instead of ζ indicates that $\rho''(s) > 0$ since

$$-\rho(s) + \frac{\lambda}{\alpha}\rho(\alpha s) < 0.$$

Therefore, $y'(s) = -\rho''(s) < 0$ and this implies that there is an interval $(s, s+\epsilon)$ for $\epsilon > 0$ such that $y(x) < 0$ in that interval, contradicting the result $y(x) \geq 0$; thus, $y(x) > 0$ for $x \geq 0$.■

Lemma 5.3.3 *Suppose $y(x)$ is a solution defined by* (5.37) *to Problem 5.2. Then there exists a $Z \in R$ such that $y'(x) \leq 0$ or $y'(x) \geq 0$ for all $x \geq Z$.*

Proof: For a solution $y(x)$ defined by (5.37),

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty} a_n \frac{d}{dx}(\alpha^n x K_2(2\sqrt{\alpha^n x})) \\ &= -\sum_{n=0}^{\infty} a_n \alpha^n \sqrt{\alpha^n x} K_1(2\sqrt{\alpha^n x}) \end{aligned}$$

(cf. (5.35)). Now if $\lambda < 0$, then $y'(x) < 0$ since $K_1(2\sqrt{\alpha^n x}) > 0$ for all n and $a_n > 0$. Assume that $\lambda > 0$. Since $K_p(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}$ as $x \to \infty$,

$$R(x) = \frac{K_1(2\sqrt{\alpha^n x})}{K_1(2\sqrt{\alpha^{n-1} x})} \sim \alpha^{1/4} e^{-2\sqrt{\alpha^n x} + 2\sqrt{\alpha^{n-1} x}}.$$

The function $R(x)$ can thus be small arbitrarily by increasing x for all n and this implies that there exists a $Z \in R$ such that

$$\begin{aligned} \left|\frac{a_n}{a_{n-1}}\right| \frac{\alpha^n \sqrt{\alpha^n x} K_1(2\sqrt{\alpha^n x})}{\alpha^{n-1}\sqrt{\alpha^{n-1} x} K_1(2\sqrt{\alpha^{n-1} x})} &= \left|\frac{a_n}{a_{n-1}}\right| \alpha\sqrt{\alpha} \frac{K_1(2\sqrt{\alpha^n x})}{K_1(2\sqrt{\alpha^{n-1} x})} \\ &< 1, \end{aligned} \tag{5.44}$$

for all n and $x \geq Z$. Now the series is alternating and so $y'(x) \leq 0$ if $a_0 > 0$ and $y'(x) \geq 0$ if $a_0 < 0$. ■

Using the result of the above lemma and the same arguments as those used in the proof of Theorem 4.2.7 we can get the following theorem:

Theorem 5.3.4 *The solutions $y(x)$ defined by* (5.37) *to Problem 5.2 can have neither a positive minimum nor a negative maximum.*

5.3.4 The Values λ When $y(0) = 0$

We now investigate the eigenvalues λ satisfying $y(0) = 0$ and the corresponding eigenfunctions. Unlike the previous section, we can obtain an eigenvalue directly from equation (5.31). Integrating equation (5.31) from 0 to ∞ leads to the equation

$$\begin{aligned} 2y(0) - (1 - \frac{\lambda}{\alpha}) \int_0^{\infty} y(t)\, dt &= -1 + \frac{\lambda}{\alpha} \\ &= 0, \end{aligned} \tag{5.45}$$

so that there is the only one eigenvalue when $\lambda = \alpha$.

Let us confirm this result from the series solution. At $x = 0$,

$$y(0) = \frac{a_0}{2} \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^m - 1)} = \frac{1}{2} F^{-1}(\lambda/\alpha) F(\lambda),$$

where $F(\lambda)$ is defined in (4.8) when $b = 1$, and so $y(0) = 0$ only when $\lambda = \alpha$ since $a_0 = \infty$ at the points $\lambda = \alpha^n$ for $n = 2, 3, \ldots$. Note that this result is essentially the same as that for the first order equation when $b = 1$ in Chapter 4. The similar results about an eigenvalue between two equations (5.45) and (4.1) can be explained by integrating those equations from 0 to ∞ so that the value λ can be exactly determined by $y(0)$ and $\int_0^\infty y(x)\,dx = 1$. For the case of the second order equation with constant coefficients in Chapter 4 and equation (5.11), integrating those equations from 0 to ∞ gives another unknown value $y'(0)$ which makes the value λ undetermined and so we use the method to obtain eigenvalues using a Dirichlet series solution in these cases.

In the same manner as that used in the proof of Theorem 5.3.2 we have the following theorem:

Theorem 5.3.5 *The eigenfunction y_1 corresponding to $\lambda = \alpha$ is positive for $x > 0$.*

5.3.5 Bounds on the Maximum Critical Point of y_1

Theorems 5.3.4 and 5.3.5 imply that y_1 has exactly one maximum critical point. Let X_m be the maximum critical point. Then $y_1'(X_m) = 0$ so that equation (5.40) for y_1 instead of z with $\lambda = \alpha$ yields

$$y_1(X_m) = \frac{1}{2}\int_{X_m}^{\alpha X_m} y_1(t)\,dt, \tag{5.46}$$

and therefore,

$$y_1(X_m) \le \frac{1}{2} y_1(X_m)(\alpha - 1)X_m,$$

and this implies that

$$X_m \ge \frac{2}{\alpha - 1}. \tag{5.47}$$

Since $y_1(\alpha X_m)$ is the minimum in $[X_m, \alpha X_m]$, inequality (5.46) produces

$$y_1(X_m) \ge \frac{1}{2} y_1(\alpha X_m)(\alpha - 1)X_m,$$

and $\alpha y_1(\alpha X_m) \ge y_1(X_m)$ from (5.31) so that

$$y_1(X_m) \ge \frac{1}{2}\alpha^{-1} y_1(X_m)(\alpha - 1)X_m,$$

and thus

$$X_m \le \frac{2\alpha}{\alpha - 1}. \tag{5.48}$$

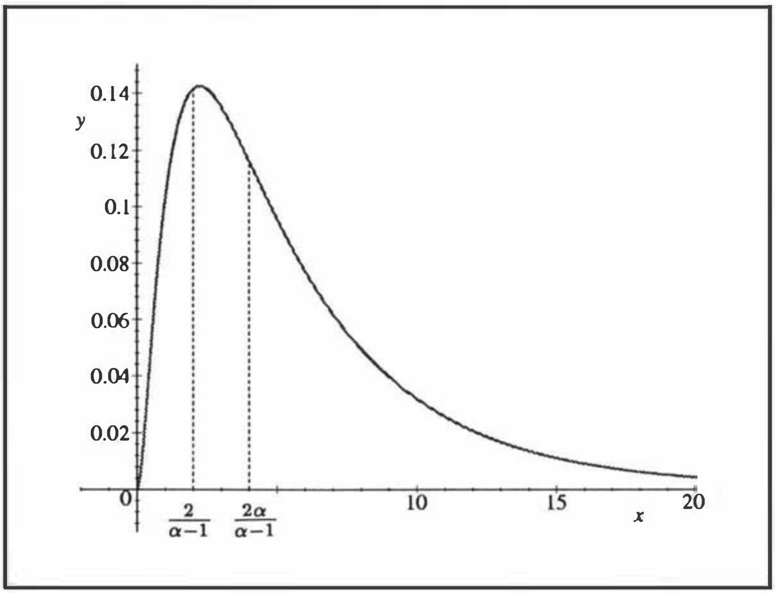

Figure 5.5.2: The bounds on X_m and the solution $y(x)$ of $xy''(x) - y'(x) - y(x) + 2y(2x) = 0$.

Combining inequalities (5.47) and (5.48) gives the bounds

$$\frac{2}{\alpha - 1} \leq X_m \leq \frac{2\alpha}{\alpha - 1}.$$

Figure 5.5.2 shows the bounds $2 \leq X_m \leq 4$ for the case $\alpha = 2$.

The results of this section can be extended to the more general equation

$$xy''(x) - ay'(x) - by(x) + \lambda y(\alpha x) = 0, \tag{5.49}$$

where $a > 0$ and $b > 0$ if the conditions $|\lambda| < \alpha$ and $\lambda = \alpha$ are exchanged for the conditions $|\lambda| < b\alpha$ and $\lambda = b\alpha$ respectively. A solution to the equation $xy_0''(x) - ay_0'(x) - by_0(x) = 0$ satisfying $y_0(\infty) = 0$ is $y_0(x) = x^{(a+1)/2}K_{a+1}(2\sqrt{b}\sqrt{x})$ so that a solution to equation (5.49) which satisfies the conditions (5.2) and (5.3) is given by

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-\lambda/b)^n}{\prod_{m=1}^{n}(\alpha^m - 1)} (\alpha^n x)^{(a+1)/2} K_{a+1}(2\sqrt{b}\sqrt{\alpha^n x}),$$

where a_0 can be obtained by integrating $y(x)$ from 0 to ∞ and using the condition (5.3); thus,

$$a_0 = 2\frac{b^{\frac{a+3}{2}}}{\Gamma(a+2)} \left(\sum_{n=0}^{\infty} \frac{(-\frac{\lambda}{b\alpha})^n}{\prod_{m=1}^{n}(\alpha^m - 1)} \right)^{-1}.$$

5.4 The equation $xy''(x) - y(x) - x^3y(x) + \lambda x^3y(\alpha x) = 0$

Substituting $p(x) = x$, $q(x) = -1$ and $r(x) = -x^3$ into equation (5.1) leads to the equation

$$xy''(x) - y'(x) - x^3y(x) + \lambda x^3y(\alpha x) = 0, \tag{5.50}$$

where $\alpha > 1$ and a solution to the equation $L(y(x)) = xy''(x) - y'(x) - x^3y(x) = 0$ satisfying the boundary condition (5.2) is $y(x) = e^{-\frac{1}{2}x^2}$ so that we seek a Dirichlet series solution to equation (5.50).

We will refer to equation (5.50) along with conditions (5.2) and (5.3) as *Problem 5.3.*

5.4.1 Existence of Solutions

Now $y_0(x) = a_0e^{-\frac{1}{2}x^2}$. Let $E(x) = e^{-\frac{1}{2}x^2}$, then

$$xy_1''(x) - y_1'(x) - x^3y_1(x) = x^3a_0E(\alpha x), \tag{5.51}$$

and substituting $y_1(x) = a_1E(\alpha x)$ into equation (5.51) yields

$$xa_1\alpha^2E''(\alpha x) - a_1\alpha E'(\alpha x) - x^3a_1E(\alpha x) = x^3a_0E(\alpha x).$$

Since

$$\alpha xE''(\alpha x) - E'(\alpha x) = \alpha^3x^3E(\alpha x),$$

the above equation can be converted into the equation

$$x^3a_1(\alpha^4 - 1)E(\alpha x) = x^3a_0E(\alpha x),$$

and so $a_1 = \frac{a_0}{\alpha^4-1}$. Therefore,

$$y_1(x) = \frac{a_0}{\alpha^4 - 1}E(\alpha x),$$

so that the third component $y_2(x)$ is a solution to the equation

$$xy_2''(x) - y_2'(x) - x^3y_2(x) = x^3\frac{a_0}{\alpha^4 - 1}E(\alpha^2x).$$

Using the arguments similar to those used for getting $y_1(x)$, we have $a_2 = \frac{a_0}{(\alpha^4-1)(\alpha^8-1)}$ and thus

$$y_2(x) = \frac{a_0}{(\alpha^4 - 1)(\alpha^8 - 1)}E(\alpha^2x).$$

In general,

$$y_n(x) = \frac{a_0}{\prod_{m=1}^{n}(\alpha^{4m}-1)}E(\alpha^n x),$$

and consequently, we have the formal solution:

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty}(-\lambda)^n a_n E(\alpha^n x) \\ &= a_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{4m}-1)} e^{-\frac{1}{2}\alpha^{2n}x^2}, \end{aligned} \tag{5.52}$$

where a_0 can be determined by normalizing $y(x)$.

Clearly, the Dirichlet series (5.52) is uniformly convergent on $[0,\infty)$ so that it can be integrated/differentiated term by term.

Moreover, a substitution of the series (5.52) into $L(y) = xy''(x) - y'(x) - x^3y(x)$ yields

$$\begin{aligned} L(y) &= a_0 x \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{4m}-1)} \alpha^{2n} E''(\alpha^n x) \\ &\quad - a_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{4m}-1)} \alpha^n E'(\alpha^n x) \\ &\quad - a_0 x^3 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{4m}-1)} E(\alpha^n x), \end{aligned}$$

and since $\alpha^n x E''(\alpha^n x) - E'(\alpha^n x) = \alpha^{3n}x^3E(\alpha^n x)$,

$$\begin{aligned} L(y) + \lambda x^3 y(\alpha x) &= a_0 x^3 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{4m}-1)} (\alpha^{4n}-1)E(\alpha^n x) \\ &\quad + \lambda a_0 x^3 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{4m}-1)} E(\alpha^{n+1}x) \\ &= a_0 x^3 \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n-1}(\alpha^{4m}-1)} E(\alpha^n x) \\ &\quad - a_0 x^3 \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n-1}(\alpha^{4m}-1)} E(\alpha^n x) \\ &= 0. \end{aligned}$$

Therefore, the series (5.52) is a solution to Problem 5.3 and since

$$\begin{aligned} \int_0^{\infty} e^{-\frac{1}{2}\alpha^{2n}x^2}\,dx &= \frac{1}{\sqrt{2}\alpha^n}\int_0^{\infty} x^{1/2-1}e^{-x}\,dx \\ &= \frac{\Gamma(1/2)}{\sqrt{2}\alpha^n}, \end{aligned}$$

the coefficient a_0 is given by

$$a_0 = \frac{\sqrt{2}}{\Gamma(1/2)} \left(\sum_{n=0}^{\infty} \frac{(-\lambda/\alpha)^n}{\prod_{m=1}^{n}(\alpha^{4m}-1)} \right)^{-1}.$$

5.4.2 Uniqueness of Solutions

Theorem 5.4.1 *If $|\lambda| \le \alpha^4$, then for a given λ and α, the solution $y(x)$ to Problem 5.3, satisfying $x^3 y(x) \in L_1[0,\infty)$ is unique.*

Proof: Suppose y_1 and y_2 are distinct solutions to Problem 5.3 and let $z(x) = y_1(x) - y_2(x)$. Then $z(x)$ satisfies equation (5.50) and $\int_0^\infty z(t)\,dt = 0$. Integrating equation (5.50) for z instead of y from x to ∞ leads to

$$-xz'(x) + 2z(x) - \int_x^\infty t^3 z(t)\,dt + \frac{\lambda}{\alpha^4}\int_{\alpha x}^\infty t^3 z(t)\,dt = 0, \tag{5.53}$$

and using the transformation

$$\eta(x) = \int_x^\infty t^3 z(t)\,dt, \tag{5.54}$$

we get the equation

$$x\eta''(x) - 5\eta'(x) - x^3\eta(x) + \frac{\lambda}{\alpha^4}x^3\eta(\alpha x) = 0, \tag{5.55}$$

since $\eta'(x) = -x^3 z(x)$ and $\eta''(x) = -3x^2 z(x) - x^3 z'(x)$. It can be proven that $\eta'(x) \ge 0$ or $\eta'(x) \le 0$ in the same arguments as those used in the proof of Lemma 4.2.1 and so $z(x) \le 0$ or $z(x) \ge 0$. Hence the condition $\int_0^\infty z(x)\,dx = 0$ indicates that $z(x) = 0$.■

5.4.3 Qualitative Properties of Solutions

Without difficulty, the qualitative properties of the solutions to Problem 5.3 stated in the following theorems can be shown in the same manner as that used in the previous sections using the transformation (5.54) into equation (5.50) and so those are not discussed in detail here:

Theorem 5.4.2 *If $|\lambda| < \alpha^4$, then a solution $y(x)$ to Problem 5.3, satisfying $x^3 y(x) \in L_1[0,\infty)$ is positive for $x \ge 0$.*

Theorem 5.4.3 *A Dirichlet series solution $y(x)$ to Problem 5.3 cannot have a positive minimum nor a negative maximum.*

5.4.4 The Values λ When $y(0) = 0$

Now,

$$\begin{aligned} y(0) &= a_0(1+\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{4m}-1)}) \\ &= a_0\prod_{n=0}^{\infty}(1-\frac{\lambda}{\alpha^{4(n+1)}}) \\ &= a_0\bar{V}(\lambda), \end{aligned}$$

and $a_0 = \frac{\sqrt{2}}{\Gamma(1/2)}\bar{V}^{-1}(\lambda/\alpha)$ so that $y(0) = 0$ when $\lambda = \alpha^{4n}$, $n = 1, 2, \ldots$. It can be shown that the first eigenfunction y_1 corresponding to $\lambda = \alpha^4$ is positive for $x > 0$ in the same manner as that used in the previous sections using equation (5.55) and clearly, it has exactly one maximum critical point.

5.4.5 Bounds on the Maximum Critical Point of y_1

At $x = X_m$, $y_1'(X_m) = 0$ so that equation (5.53) for y_1 instead of z indicates that

$$y_1(X_m) = \frac{1}{2}\int_{X_m}^{\alpha X_m} t^3 y_1(t)\,dt,$$

and so

$$\begin{aligned} y_1(X_m) &\le \frac{1}{2}y_1(X_m)\int_{X_m}^{\alpha X_m} t^3\,dt \\ &\le \frac{1}{8}y_1(X_m)(\alpha^4-1)X_m^4, \end{aligned}$$

and consequently,

$$X_m \ge (\frac{8}{\alpha^4-1})^{1/4}. \tag{5.56}$$

On the other hand, $y_1(\alpha X_m)$ is the minimum of $y_1(x)$ in $[X_m, \alpha X_m]$ and $\alpha^4 y_1(\alpha X_m) \ge y_1(X_m)$ from equation (5.50) so that

$$y_1(X_m) \ge \frac{1}{8}y_1(\alpha X_m)(\alpha^4-1)X_m^4 \ge \frac{1}{8}\alpha^{-4}y_1(X_m)(\alpha^4-1)X_m^4,$$

and therefore,

$$X_m \le \alpha(\frac{8}{\alpha^4-1})^{1/4}. \tag{5.57}$$

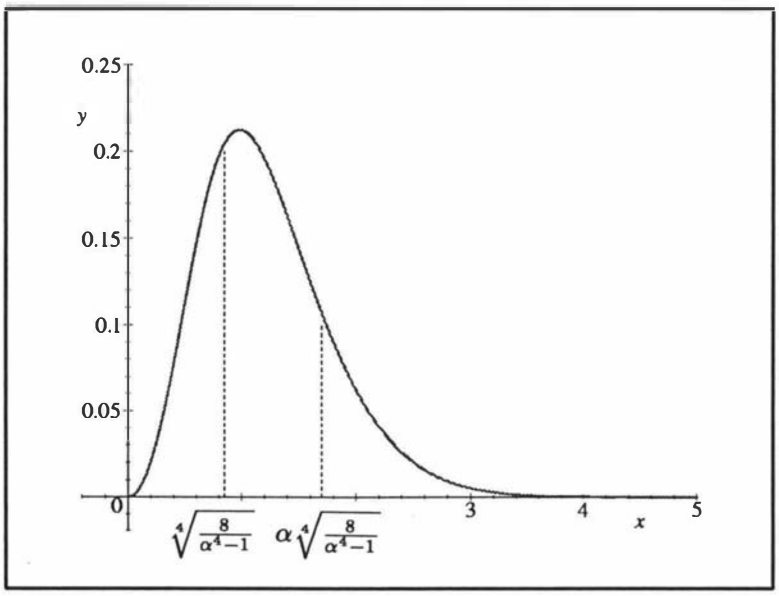

Figure 5.5.3: The bounds on X_m and the solution y of $xy''(x) - y'(x) - x^3y(x) + 16x^3y(2x) = 0$.

Combining two inequalities (5.56) and (5.57) leads to the bounds

$$(\frac{8}{\alpha^4-1})^{1/4} \leq X_m \leq \alpha(\frac{8}{\alpha^4-1})^{1/4}.$$

Figure 5.5.3 illustrates the bounds $(8/15)^{1/4} \leq X_m \leq 2(8/15)^{1/4}$ corresponding to the case $\alpha = 2$.

Note that a solution to the equation $xy_0''(x) - (k-1)y_0'(x) - x^{2k-1}y_0(x) = 0$ for $k > 0$, satisfying $y_0(\infty) = 0$ is $y_0 = e^{-\frac{1}{k}x^k}$ so that a solution to the equation $xy''(x) - (k-1)y'(x) - x^{2k-1}y(x) + \lambda x^{2k-1}y(\alpha x) = 0$ satisfying conditions (5.2) and (5.3) is given by

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{m=1}^{n}(\alpha^{4m}-1)} e^{-\frac{1}{k}\alpha^{kn}x^k},$$

where a_0 can be obtained from the condition (5.3) so that

$$a_0 = \left(k^{\frac{1-k}{k}}\Gamma(1/k)\sum_{n=0}^{\infty}\frac{(-\lambda/\alpha)^n}{\prod_{m=1}^{n}(\alpha^{4m}-1)}\right)^{-1}.$$

Here, the above equation can be treated as the same as that in this section.

5.5 A Solution Expressed by Green's Function

We have observed in the previous section three types of equations which have a series solution. Apart from the method to use a series solution, a method involving a Green's function and the contraction mapping theorem is used in this section to show the existence of a solution to the last case V.

We consider the equation

$$(x+k)^2y''(x)+a(x+k)y'(x)+by(x)+\lambda y(\alpha x)=0, \tag{5.58}$$

where $b<0$, $k>0$ and $\alpha>1$. We will refer to equation (5.58) along with conditions (5.2) and (5.3) as *Problem 5.4*.

5.5.1 Positivity of Solutions

Theorem 5.5.1 *Suppose a solution y to problem 5.4 satisfies $(x+k)^2y', (x+k)y \to 0$ as $x\to\infty$. If $a>b+2$ and $|\lambda|<\alpha(a-b-2)$, then the solution is positive and strictly decreasing for all $x>0$.*

Proof: Integrating equation (5.58) from x to ∞ yields the integro-differential equation

$$\begin{aligned}-(x+k)^2y'(x) &- (a-2)(x+k)y(x)\\ &- (a-b-2)\int_x^\infty y(t)\,dt+\frac{\lambda}{\alpha}\int_{\alpha x}^\infty y(t)\,dt=0,\end{aligned}$$

and using the transformation

$$\sigma(x)=\int_x^\infty y(t)\,dt,$$

we have the equation

$$\begin{aligned}(x+k)^2\sigma''(x) &+ (a-2)(x+k)\sigma'(x)\\ &- (a-b-2)\sigma(x)+\frac{\lambda}{\alpha}\sigma(\alpha x)=0.\end{aligned}$$

Using the same arguments as used those in the proof of Lemma 4.2.1, it can be proven that $\sigma'(x)\le 0$ and $\sigma(x)>0$ since $\sigma(0)=1>0$. Suppose $\sigma'(\hat{x})=0$ for some $\hat{x}\in[0,\infty)$, then $\sigma''(\hat{x})=0$ since σ cannot have a maximum critical point so that

$$(a-b-2)\sigma(\hat{x})=\frac{\lambda}{\alpha}\sigma(\alpha\hat{x}). \tag{5.59}$$

Now, $\frac{|\lambda|}{\alpha(a-b-2)}<1$ and so equation (5.59) cannot be satisfied since $\sigma(\hat{x})\ge\sigma(\alpha\hat{x})$; thus, $y(x)>0$ for all $x>0$. Moreover, $\sigma''(x)>0$ since $(a-2)(x+k)\sigma'(x)<0$ and $-(a-b-2)\sigma(x)+\lambda/\alpha\sigma(\alpha x)<0$ so that $y'(x)<0$ for all $x>0$.■

5.5.2 Existence of Solutions

We first find the Green's function associated with the operator g such that

$$-(x+k)^2\frac{d^2}{dx^2}g(x) - a(x+k)\frac{d}{dx}g(x) - bg(x) = 0. \tag{5.60}$$

Substituting $g(x) = (x+k)^r$ into equation (5.60) leads to the indicial equation

$$r^2 + (a-1)r + b = 0.$$

Since $b < 0$, there is a positive root and a negative root; let r_1 be the positive root and r_2 be the negative root, i.e.,

$$r_1 = \frac{1 - a + \sqrt{(a-1)^2 - 4b}}{2}, \quad r_2 = \frac{1 - a - \sqrt{(a-1)^2 - 4b}}{2}.$$

Let

$$g(x,s) = \begin{cases} A(x+k)^{r_1} & \text{if } 0 < x < s, \\ B(x+k)^{r_2} & \text{if } x > s. \end{cases}$$

Then, since $A(s+k)^{r_1} = B(s+k)^{r_2}$ and $Br_2(s+k)^{r_2-1} - Ar_1(s+k)^{r_1-1} = -1/(s+k)^2$, we have $A = \frac{1}{r_1-r_2}(s+k)^{-(r_1+1)}$ and $B = \frac{1}{r_1-r_2}(s+k)^{-(r_2+1)}$. Therefore,

$$g(x,s) = \begin{cases} \frac{1}{r_1-r_2}(s+k)^{-(r_1+1)}(x+k)^{r_1} & \text{if } 0 < x < s, \\ \frac{1}{r_1-r_2}(s+k)^{-(r_2+1)}(x+k)^{r_2} & \text{if } x > s. \end{cases}$$

Theorem 5.5.2 *If $a > b+2$ and $|\lambda| < \alpha(a-b-2)$, then there is a unique solution to Problem 5.3.*

Proof: We use the *Contraction Mapping Theorem* to show this result. Let

$$Ty(x) = \lambda \int_0^\infty g(x,s)y(\alpha s)\,ds, \tag{5.61}$$

where $y(x) \in L_1[0,\infty)$. Then

$$\begin{aligned} \|Ty\| &\leq |\lambda| \int_0^\infty \int_0^\infty g(x,s)|y(\alpha s)|\,ds\,dx \\ &\leq [\sup_{s\in[0,\infty)} \int_0^\infty g(x,s)dx]\frac{|\lambda|}{\alpha}\|y\|, \end{aligned}$$

where $\|\cdot\|$ is the L_1 norm. Since $a > b+2$, we have $r_2+1<0$ so that

$$\begin{aligned}
&\int_0^\infty g(x,s)\,dx \\
&= \frac{1}{r_1-r_2}\Big[\int_0^s (s+k)^{-(r_1+1)}(x+k)^{r_1}\,dx + \int_s^\infty (s+k)^{-(r_2+1)}(x+k)^{r_2}\,dx\Big] \\
&= \frac{1}{r_1-r_2}\Big[\frac{1}{r_1+1} - k^{r_1+1}\frac{1}{r_1+1}(s+k)^{-(r_1+1)} - \frac{1}{r_2+1}\Big].
\end{aligned}$$

Now, $0 < k^{r_1+1}\frac{1}{r_1+1}(s+k)^{-(r_1+1)} < \infty$ and therefore,

$$\begin{aligned}
\sup_{s\in[0,\infty)}\int_0^\infty g(x,s)dx &= \frac{-1}{(r_1+1)(r_2+1)} \\
&= \frac{1}{a-b-2}.
\end{aligned}$$

Consequently,

$$\|Ty\| \le \frac{|\lambda|}{\alpha(a-b-2)}\|y\|,$$

and this means that T maps $L_1[0,\infty)$ into $L_1[0,\infty)$. For $y_1, y_2 \in L_1[0,\infty)$,

$$\begin{aligned}
\|Ty_1 - Ty_2\| &\le |\lambda|\int_0^\infty\int_0^\infty g(x,s)|y_1(\alpha s)-y_2(\alpha s)|\,ds\,dx \\
&\le \Big[\sup_{s\in[0,\infty)}\int_0^\infty g(x,s)dx\Big]\frac{|\lambda|}{\alpha}\|y_1-y_2\| \\
&\le \frac{|\lambda|}{\alpha(a-b-2)}\|y_1-y_2\|,
\end{aligned}$$

and so *Contraction Mapping Theorem* indicates that if $\frac{|\lambda|}{\alpha(a-b-2)} < 1$, then there is the only one fixed point $y(x)$ in L_1 satisfying

$$y(x) = \lambda\int_0^\infty g(x,s)y(\alpha s)\,ds. \tag{5.62}$$

Clearly $y(\infty)=0$.

Let us show the existence of the limiting values for $(x+k)^2y'(x)$ and $(x+k)y(x)$ as $x\to\infty$. From (5.62),

$$(x+k)^2y'(x) = \lambda\int_0^\infty (x+k)^2\frac{dg(x,s)}{dx}y(\alpha s)\,ds,$$

and, as $x\to\infty$,

$$(x+k)^2\frac{dg(x,s)}{dx} \sim \frac{r_2}{r_1-r_2}(s+k)^{-(r_2+1)}(x+k)^{r_2+1} \to 0,$$

because $r_2+1<0$, and consequently $(x+k)^2y'(x)\to 0$. In the same manner, it can be proven that $(x+k)y(x)\to 0$ as $x\to\infty$.

Now,

$$y(0)=\frac{\lambda k^{r_1}}{r_1-r_2}\int_0^{\infty}(s+k)^{-(r_1+1)}y(\alpha s)\,ds,$$

and since $(x+k)^2y'(x)$, $(x+k)y(x)\to 0$ as $x\to\infty$, Theorem 5.5.1 indicates $y(x)>0$ for $x>0$; thus, $y(0)\neq 0$. This means that the solution is non-trivial and so we normalize $y(x)$ to satisfy the condition (5.3). ■

Conclusions

In this thesis we focused on advanced second order functional differential equations. Earlier studies for first order advanced equations with constant coefficients showed that these equations have Dirichlet series solutions. Motivated by these results we showed that advanced second order equations with constant coefficients also have Dirichlet series solutions for a range of coefficients. As an application of this theory we studied the second order equation arising in a cell growth model devised by Hall and Wake [1989] and showed there exists a unique solution which can be represented by a Dirichlet series. We proved the solution is positive, in agreement with the interpretation of the solution as a probability density function of the cell size, and that it has exactly one maximum.

In Chapter 3, we studied equations of the form

$$y''(x) + ay'(x) + by(x) + cy(\alpha x) = 0,$$

where a, b, c, α are real constants and $\alpha > 1$. It was shown that the general equations have a Dirichlet series solution provided there exists a non-trivial root r to the indicial equation

$$r^2 - ar + b = 0,$$

such that $Re(r) \geq 0$. We showed that a certain range of coefficients of the equations determines some qualitative properties of a solution. It was confirmed that the sign combination of the coefficients must be the same as that of the equation arising in a cell growth model in order to have a positive solution with one maximum. The shape of the graphs of real solutions has a distinct difference according to whether the indicial equation has a real root or a complex root. For a real root r, a Dirichlet solution is monotone for large x, while for a complex root r, we obtain real solutions from a Dirichlet series expression and the general solution is oscillatory ; if $Re(r) > 0$, then the solution is approaching zero as x goes to infinity and if $Re(r) = 0$, then the solution is not approaching zero, but bounded.

The qualitative properties of a solution to the equation arising in a cell growth model are consistent with the nature of a probability density function but there are no solutions such that $y(0) \neq 0$. This motivated the study of the eigenvalue problem. The fact that the initial value problem for an advanced functional differential equation needs not have a unique solution supports the possibility of the existence of non-trivial solutions, i.e. eigenfunctions. In Chapter 4 we investigated the eigenvalue problem. Using a Dirichlet series solution and partition functions, we obtained eigenvalues and the corresponding eigenfunctions.

In Chapter 5 we considerd the existence of solutions to more general equation with variable coefficients. For a first order equation, we converted the equation into a Fredholm equation of the second kind and showed the existence of a solution using the methods for the Fredholm equations. In principle, solutions can be expressed as a Neumann series; in practice, if it is too complicated we may obtain a basis for the solution and then deduce the coefficients directly from the equation. In fact, for the Dirichlet series solution to the first order equation derived by Hall and Wake [1990], the basis of the solution can be obtained by a Neumann series. For second order equations, we used two methods to show the existence of solutions. One involves the use of a Green's function and the contraction mapping theorem; the other is to seek a series type solution. Either method requires that for the equation $L(y)+r(x)y(\alpha x) = 0$, $L(y) = 0$ has a solution satisfying $y(\infty) = 0$. We showed that special classes of equations have a solution and their series solutions involve Airy or Bessel functions. These methods are not always effective even if the solutions to $L(y) = 0$ satisfy $y(\infty) = 0$. Consider, for example, the Hermite type equation

$$y''(x) + xy'(x) + y(x) + \lambda y(\alpha x) = 0.$$

Here, the equation $L(y) = 0$ has a solution $y(x) = e^{-1/2x^2}$ which decays to zero as x goes to infinity. However the associated Green's function is not in $L_1[0,\infty)$. It may be that there exists a solution in another function space. This is a future work along with finding a more general way to examine the existence of solutions to these equations. Another direction would be to investigate a solution decaying slower than exponentially since our solutions obtained in this thesis are mostly decaying exponentially. This would complement the work of T. Kato and J.B. McLeod [1971] on the first order case.

Appendix A

Inhomogeneous Functional Differential Equations

In this appendix we examine a special case of a problem involving an inhomogeneous equation, which illustrates a technique for investigating the solution existence, uniqueness, and construction. Let $f(x)$ be in $L_1[0,\infty)$. We consider the equation

$$y''(x) - y'(x) - y(x) + \lambda y(\alpha x) = f(x), \tag{A.1}$$

where $\alpha > 1$, satisfying the boundary condition

$$y(\infty) = 0, \tag{A.2}$$

and the normalizing condition

$$\int_0^\infty y(t)\,dt = 1. \tag{A.3}$$

We first find the Green's function associated with the operator g such that

$$-\frac{d^2}{dx^2}g(x) + \frac{d}{dx}g(x) + g(x) = 0. \tag{A.4}$$

Substituting $g(x) = e^{rx}$ into the equation (A.4) leads to the indicial equation

$$r^2 - r - 1 = 0,$$

and so there are two solutions

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}.$$

Let

$$g(x,s) = \begin{cases} A(e^{r_1 x} + ke^{r_2 x}) & \text{if } 0 < x < s, \\ Be^{r_2 x} & \text{if } x > s, \end{cases}$$

for $k \in R$. Note that k is determined by an initial condition $y(0)$. Since

$$A(e^{r_1 s} + ke^{r_2 s}) = Be^{r_2 s}, \quad Br_2 e^{r_2 s} - A(r_1 e^{r_1 s} + kr_2 e^{r_2 s}) = -1,$$

we have

$$A = \frac{1}{r_1 - r_2} e^{-r_1 s}, \quad B = \frac{1}{r_1 - r_2}(e^{-r_2 s} + ke^{-r_1 s});$$

therefore,

$$g(x,s) = \begin{cases} \frac{1}{r_1 - r_2} e^{-r_1 s}(e^{r_1 x} + ke^{r_2 x}) & \text{if } 0 < x < s, \\ \frac{1}{r_1 - r_2}(e^{-r_2 s} + ke^{-r_1 s})e^{r_2 x} & \text{if } x > s. \end{cases}$$

The Contraction Mapping Theorem will be used to show the existence of solutions. Define

$$Ty(x) = \lambda \int_0^\infty g(x,s) y(\alpha s)\, ds - \int_0^\infty g(x,s) f(s)\, ds,$$

for $y \in L_1[0,\infty)$. Then

$$\begin{aligned} \|Ty\| &\leq |\lambda| \int_0^\infty \int_0^\infty |g(x,s)||y(\alpha s)| ds\, dx + \int_0^\infty \int_0^\infty |g(x,s)||f(s)| ds\, dx \\ &\leq [\sup_s \int_0^\infty |g(x,s)|\, dx](\frac{|\lambda|}{\alpha}\|y\| + \|f\|), \end{aligned}$$

where $\|\cdot\|$ is the L_1 norm. Now,

$$\begin{aligned} &\int_0^\infty |g(x,s)|\, dx \\ \leq\ & \frac{1}{r_1 - r_2}[\int_0^s e^{-r_1 s}(e^{r_1 x} + |k|e^{r_2 x})\, dx + \int_s^\infty (e^{-r_2 s} + |k|e^{-r_1 s})e^{r_2 x}\, dx] \\ =\ & \frac{1}{r_1 - r_2}[(e^{-r_1 s}(\frac{1}{r_1}e^{r_1 s} + \frac{|k|}{r_2}e^{r_2 s} - (\frac{1}{r_1} + \frac{|k|}{r_2})) - (e^{-r_2 s} + |k|e^{-r_1 s})\frac{1}{r_2}e^{r_2 s}] \\ =\ & \frac{1}{r_1 - r_2}[\frac{1}{r_1} - (\frac{1}{r_1} + \frac{|k|}{r_2})e^{-r_1 s} - \frac{1}{r_2}]. \end{aligned}$$

Hence if $\frac{1}{r_1} + \frac{|k|}{r_2} \geq 0$, i.e. $|k| \leq \frac{\sqrt{5}-1}{1+\sqrt{5}}$, then

$$\begin{aligned} \sup_{s \in [0,\infty)} \int_0^\infty g(x,s) dx &= -\frac{1}{r_1 r_2} \\ &= 1. \end{aligned}$$

If $|k| > \frac{\sqrt{5}-1}{1+\sqrt{5}}$, then

$$\begin{aligned} \sup_{s \in [0,\infty)} \int_0^\infty g(x,s) dx &= \frac{1}{r_1 - r_2}[\frac{1}{r_1} - (\frac{1}{r_1} + \frac{|k|}{r_2}) - \frac{1}{r_2}] \\ &= -\frac{1}{r_1 r_2} - \frac{r_2 + |k| r_1}{r_1 r_2 (r_1 - r_2)} \\ &= 1 + L, \end{aligned}$$

where $L = \frac{1+|k|+(|k|-1)\sqrt{5}}{2\sqrt{5}} > 0$. Therefore,

$$\|Ty\| \leq (1+L)(\frac{\lambda}{\alpha}\|y\| + \|f\|) < \infty,$$

and so T maps $L_1[0,\infty)$ into $L_1[0,\infty)$. For $y_1, y_2 \in L_1[0,\infty)$,

$$\begin{aligned}
\|Ty_1 - Ty_2\| &\leq |\lambda| \int_0^\infty \int_0^\infty |g(x,s)||y_1(\alpha s) - y_2(\alpha s)| ds\, dx \\
&\leq \frac{|\lambda|}{\alpha}[\sup_s \int_0^\infty |g(x,s)|\, dx]\|y_1 - y_2\| \\
&= \frac{|\lambda|}{\alpha}(1+L)\|y_1 - y_2\|,
\end{aligned}$$

and the *Contraction Mapping Theorem* indicates that if $\frac{|\lambda|}{\alpha}(1+L) < 1$, then there exists a unique fixed point $y(x)$ in $L_1[0,\infty)$ satisfying

$$y(x) = \int_0^\infty g(x,s)(\lambda y(\alpha s) - f(s))\, ds. \tag{A.5}$$

Clearly $y(\infty) = 0$. Note that the solution exists for any real values of $y(0)$ because of $k \in R$. We can normalize non-trivial $y(x)$ to satisfy the condition $\int_0^\infty y(t)\, dt = 1$.

In fact, equation (A.5) can be expressed as a Fredholm equation of the second kind,

$$y(x) = F(x) + \lambda K y(x),$$

if we substitute

$$F(x) = -\int_0^\infty g(x,s) f(s)\, ds.$$

and

$$Ky(x) = \int_0^\infty K(x,s) y(s)\, ds,$$

where $K(x,s) = \frac{1}{\alpha} g(x, s/\alpha)$. Now, K is a bounded operator and satisfies the condition

$$\|Ky_1 - Ky_2\| \leq \frac{1+L}{\alpha}\|y_1 - y_2\|,$$

so that the well known results for Fredholm integral equations imply that there exists a solution in $L_1[0,\infty)$ for all $F \in L_1[0,\infty)$ and sufficiently small $|\lambda|$. The solution $y(x)$ can be expressed as a Neumann series so that

$$y = F + \lambda K F + \lambda^2 K^2 F + \ldots + \lambda^n K^n F + \ldots,$$

where

$$K^n F(x) = \int_0^\infty K_n(x,s)F(s)\,ds,$$

and

$$\begin{aligned} K_n(x,y) &= \int_0^\infty K(x,z)K_{n-1}(z,y)\,dz, \quad n = 2,3,\dots, \\ K_1(x,y) &= K(x,y). \end{aligned}$$

For example, let $f(x) = e^{-qx}$, where $q > 0$, then

$$\begin{aligned} F(x) &= -\int_0^\infty g(x,s)e^{-qs}\,ds \\ &= \frac{-1}{r_1 - r_2}[e^{r_2 x}\int_0^x (e^{-r_2 s} + ke^{-r_1 s})e^{-qs}\,ds + (e^{r_1 x} + ke^{r_2 x})\int_x^\infty e^{-r_1 s}e^{-qs}\,ds] \\ &= \frac{-1}{(r_1+q)(r_2+q)}[(r_1 + r_2 + 2q)e^{r_2 x} - e^{-qx}] \\ &= M(Ne^{r_2 x} - e^{-qx}), \end{aligned}$$

where $M = \frac{-1}{(r_1+q)(r_2+q)}$ and $N = r_1 + r_2 + 2q$. Hence

$$\begin{aligned} \int_0^\infty F(x)\,dx &= -M(N\frac{1}{r_2} + \frac{1}{q}) \\ &< \infty, \end{aligned}$$

and so the solution can be expressed as a Neumann series. Now,

$$\begin{aligned} KF(x) &= \frac{M}{r_1 - r_2}[e^{r_2 x}\int_0^x (e^{-r_2 s} + ke^{-r_1 s})(Ne^{\alpha r_2 s} - e^{-\alpha q s})ds \\ &\quad +(e^{r_1 x} + ke^{r_2 x})\int_x^\infty e^{-r_1 s}(Ne^{\alpha r_2 s} - e^{-\alpha q s})ds] \\ &= -\frac{U}{r_1 - r_2}e^{r_2 x} + \frac{-MN}{(\alpha-1)r_2(\alpha r_2 - r_1)}e^{\alpha r_2 x} + \frac{M}{(\alpha q + r_2)(\alpha q + r_1)}e^{-\alpha q x} \\ &= L_1 e^{r_2 x} + L_2 e^{\alpha r_2 x} + L_3 e^{-\alpha q x}, \end{aligned}$$

where $U = M(\frac{N}{(\alpha-1)r_2} + \frac{1}{\alpha q + r_2} + \frac{kN}{\alpha r_2 - r_1} + \frac{k}{\alpha q + r_1})$. Since

$$K_2(x,y) = \frac{1}{\alpha^2}\int_0^\infty g(x,s/\alpha)g(s,y/\alpha)ds,$$

we have

$$K_2(x,y) = \begin{cases} P[m_1 e^{r_2 x - \frac{r_2}{\alpha^2} y} + m_2(e^{\alpha r_2 x - \frac{r_2}{\alpha} y} + k e^{\alpha r_2 x - \frac{r_1}{\alpha} y} - e^{r_2 x - \frac{r_2}{\alpha^2} y}) \\ -m_3(e^{\alpha r_2 x - \frac{r_2}{\alpha} y} + k e^{\alpha r_2 x - \frac{r_1}{\alpha} y} + k e^{r_2 x - \frac{r_1}{\alpha^2} y}) \\ +m_4 k e^{r_2 x - \frac{r_1}{\alpha^2} y} + Q] \qquad \text{if } 0 < y < \alpha^2 x, \\ \\ P[m_1 e^{\alpha r_1 x - \frac{r_1}{\alpha} y} + m_2 k e^{\alpha r_2 - \frac{r_1}{\alpha} y} - m_3(k e^{\alpha r_2 x - \frac{r_1}{\alpha} y} \\ +e^{r_1 x - \frac{r_1}{\alpha^2} y} + k e^{r_2 x - \frac{r_1}{\alpha^2} y}) + m_4(e^{r_1 x - \frac{r_1}{\alpha^2} y} + k e^{r_2 x - \frac{r_1}{\alpha^2} y} \\ -e^{\alpha r_1 x - \frac{r_1}{\alpha} y}) + Q] \qquad \text{if } y > \alpha^2 x, \end{cases}$$

where

$$\begin{aligned} P &= \frac{1}{\alpha^2(r_1 - r_2)^2}, \quad m_1 = \frac{1}{r_1 - r_2/\alpha}, \quad m_2 = \frac{1}{r_2 - r_2/\alpha}, \\ m_3 &= \frac{1}{r_2 - r_1/\alpha}, \quad m_4 = \frac{1}{r_1 - r_1/\alpha}, \quad Q = -(m_1 + k m_2 + k m_4 + k^2 m_3). \end{aligned}$$

So,

$$K^2 F(x) = K_1 e^{r_2 x} + K_2 e^{\alpha r_2 x} + K_3 e^{\alpha^2 r_2 x} + K_4 e^{-q\alpha^2 x},$$

where

$$\begin{aligned} K_1 &= P(-m_1 n_1 - m_1 n_3 + m_2 n_1 + m_2 n_3), \\ K_2 &= P(m_2 n_2 + m_2 n_4 - m_3 n_2 - m_3 n_4), \\ K_3 &= P(m_1 n_1 - m_1 n_6 - m_2 n_1 + m_2 n_2 - m_3 n_2 + m_3 n_5 - m_4 n_5 + m_4 n_6), \\ K_4 &= P(m_1 n_3 - m_1 n_8 + m_2 n_4 - m_2 n_3 - m_3 n_4 + m_3 n_7 + m_4 n_8 - m_4 n_7), \\ n_1 &= N/(r_2 - r_2/\alpha^2), \quad n_2 = N/(r_2 - r_2/\alpha), \quad n_3 = 1/(q + r_2/\alpha^2), \\ n_4 &= 1/(q + r_2/\alpha), \quad n_5 = N/(r_2 - r_1/\alpha^2), \quad n_6 = N/(r_2 - r_1/\alpha), \\ n_7 &= 1/(q + r_1/\alpha^2), \quad n_8 = 1/(q + r_1/\alpha). \end{aligned}$$

Therfore, the solution can be expressd in the form

$$\begin{aligned} y(x) &= MNe^{r_2 x} - Me^{-qx} + \lambda(L_1 e^{r_2 x} + L_2 e^{\alpha r_2 x} + L_3 e^{-\alpha q x}) \\ &\quad + \lambda^2(K_1 e^{r_2 x} + K_2 e^{\alpha r_2 x} + K_3 e^{\alpha^2 r_2 x} + K_4 e^{-\alpha^2 q x}) + \dots . \end{aligned}$$

A basis for the solution is given by $\{e^{-\alpha^n q x}, e^{\alpha^n r_2 x}\}$ so that we have a series solution

$$y(x) = \sum_0^\infty a_n e^{\alpha^n r_2 x} + \sum_0^\infty b_n e^{-\alpha^n q x}.$$

Substituting the series into equation (A.1), noting that $f(x) = e^{-qx}$, leads to the equation

$$b_0(q^2 + q - 1) = 1,$$

so that $b_0 = \frac{1}{q^2+q-1}$. The recurrence relations are given by

$$(\alpha^{2n} r_2^2 - \alpha^n r_2 - 1) a_n = -\lambda a_{n-1},$$

and

$$(\alpha^{2n} q^2 + \alpha^n q - 1) b_n = -\lambda b_{n-1}.$$

Therfore,

$$\begin{aligned} y(x) &= a_0 \sum_0^\infty \frac{(-\lambda)^n}{\prod_{m=1}^n (\alpha^{2m} r_2^2 - \alpha^m r_2 - 1)} e^{\alpha^n r_2 x} \\ &+ \frac{1}{q^2+q-1} \sum_{n=0}^\infty \frac{(-\lambda)^n}{\prod_{m=1}^n (\alpha^{2m} q^2 + \alpha^m q - 1)} e^{-\alpha^n q x}, \end{aligned}$$

where a_0 is determined by the normalizing condition so that

$$a_0 = r_2 \Big(1 - \frac{1}{q(q^2+q-1)} \sum_{n=0}^\infty \frac{(-\lambda/\alpha)^n}{\prod_{m=1}^n (\alpha^{2m} q^2 + \alpha^m q - 1)}\Big) \left[\sum_0^\infty \frac{(-\lambda/\alpha)^n}{\prod_{m=1}^n (\alpha^{2m} r_2^2 - \alpha^m r_2 - 1)} \right]^{-1}.$$

Appendix B

Second Order Functional Differential Equations with Two Functional Terms

We examine solutions to the equation

$$y''(x) - y'(x) - y(x) + \lambda_1 y(\alpha x) + \lambda_2 y(\beta x) = 0, \tag{B.1}$$

where $\alpha > 1$ and $\beta > 1$, satisfying the boundary condition

$$y(\infty) = 0, \tag{B.2}$$

and the normalizing condition

$$\int_0^\infty y(t)\, dt = 1. \tag{B.3}$$

The method to investigate the equation (B.1) is to use a Green's function and the contraction mapping theorem. Note that the Green's function
and $\sup_s \int_0^\infty |g(x,s)|\, dx \le L + 1$ are the same as those in Appendix A . Let

$$Ty(x) = \lambda_1 \int_0^\infty g(x,s) y(\alpha s)\, ds + \lambda_2 \int_0^\infty g(x,s) y(\beta s)\, ds.$$

Then for $y \in L_1[0,\infty)$,

$$\|Ty\| \le (1+L)(\frac{|\lambda_1|}{\alpha} + \frac{|\lambda_2|}{\beta})\|y\| < \infty,$$

and for $y_1, y_2 \in L_1[0,\infty)$,

$$\|Ty_1 - Ty_2\| \le (1+L)(\frac{|\lambda_1|}{\alpha} + \frac{|\lambda_2|}{\beta})\|y_1 - y_2\|.$$

So, if

$$(1+L)(\frac{|\lambda_1|}{\alpha}+\frac{|\lambda_2|}{\beta})<1, \tag{B.4}$$

there exists a unique solution to equation (B.1) satisfying the conditions (B.2) and (B.3) so that

$$y(x)=\int_0^\infty g(x,s)(\lambda_1 y(\alpha s)+\lambda_2 y(\beta s))\,ds.$$

On the other hand, a solution to equation (B.1) satisfying the conditions (B.2) and (B.3) can be expressed in a form of a Dirichlet series

$$y(x)=\sum_{n=0}^\infty a_n e^{-l_1\alpha^n x}+\sum_{n=0}^\infty b_n e^{-l_2\beta^n x}.$$

Substituting the above series into the equation (B.1) leads to $l = l_1 = l_2$ and the indicial equation

$$l^2+l-1=0,$$

and the recurrence relations

$$\frac{a_n}{a_{n-1}}=\frac{-\lambda_1}{\alpha^{2n}l^2+\alpha^n l-1},$$

and

$$\frac{b_n}{b_{n-1}}=\frac{-\lambda_2}{\beta^{2n}l^2+\beta^n l-1}.$$

Hence the solution is given by

$$\begin{aligned} y(x) &= a_0\sum_{n=0}^\infty\frac{(-\lambda_1)^n}{\prod_{m=1}^n(\alpha^{2m}l^2+\alpha^m l-1)}e^{-\alpha^n lx} \\ &\quad +b_0\sum_{n=0}^\infty\frac{(-\lambda_2)^n}{\prod_{m=1}^n(\beta^{2m}l^2+\beta^m l-1)}e^{-\beta^n lx} \\ &= y_1(x)+y_2(x), \end{aligned}$$

where a_0 and b_0 are determined by the condition (B.3) and an initial condition at $x = 0$. It is clear that there exists the only one Dirichlet series solution to equation (B.1) provided that the condition (B.4) is satisfied.

When $\beta = \alpha^m$ for some $m \in N$, the solution can be expressed by a basis of the form $\{e^{-\alpha^n lx}\}$. As an example, let us examine the case when $\beta = \alpha^2$. Terms in $y_2(x)$ can be added in each even index term of $y_1(x)$, viz.,

$$\begin{aligned} y(x) &= (a_0+b_0)e^{-lx}+a_1e^{-\alpha lx}+(a_2+b_1)e^{-\alpha^2 lx}+a_3e^{-\alpha^3 lx}+\dots, \\ &= (a_0+b_0)e^{-lx}-\frac{a_0\lambda_1}{\alpha^2l^2+\alpha l-1}e^{-\alpha lx} \\ &\quad +\left(\frac{a_0\lambda_1^2}{(\alpha^4l^2+\alpha^2l-1)(\alpha^2l^2+\alpha l-1)}-\frac{b_0\lambda_2}{\alpha^4l^2+\alpha^2l-1}\right)e^{-\alpha^2 lx}+\dots. \end{aligned}$$

Appendix C

Equations with Variable Coefficients

We consider here second order functional differential equations with variable coefficients, which have more general form for the equations of Problem 5.1 and Problem 5.2 in Chapter 5.

C.1 The General Equation from Problem 5.1

Consider the functional differential equation of the form

$$y''(x) - x^m y(x) + x^m \lambda y(\alpha x) = 0, \tag{C.1}$$

where $0 \le m < \infty$, satisfying the conditions

$$y(\infty) = 0, \tag{C.2}$$

and

$$\int_0^\infty y(x)\, dx = 1. \tag{C.3}$$

Motivated by the method used in Chapter 5, we first obtain a solution to the equation

$$L(s(x)) = s''(x) - x^m s(x) = 0, \tag{C.4}$$

such that the solution satisfies the condition (C.2).

Note that the differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + p^2)x = 0, \tag{C.5}$$

has the general solution

$$y = AI_p(x) + BK_p(x),$$

where $I_p(x)$ is the Bessel function of the first kind and $K_p(x)$ is the Bessel function of the second kind. Since $I_p(x) \to \infty$ and $K_p(x) \to 0$ as $x \to \infty$, we take $A = 0$ in order to have a solution satisfying the condition (C.2).

We will make the equation (C.4) into the form of equation (C.5) using some transformations. Let $s(x) = \sqrt{x}u(x)$. Then

$$s'(x) = \sqrt{x}u'(x) + \frac{1}{2\sqrt{x}}u(x), \tag{C.6}$$

and

$$s''(x) = \sqrt{x}u''(x) + \frac{1}{\sqrt{x}}u'(x) - \frac{1}{4x\sqrt{x}}u(x). \tag{C.7}$$

Substituting (C.6) and (C.7) into equation (C.4) and multiplying by $x\sqrt{x}$ yield

$$x^2u''(x) + xu'(x) - (x^{m+2} + \frac{1}{4})u(x) = 0. \tag{C.8}$$

Let $z = \frac{2}{m+2}x^{\frac{m+2}{2}}$ and $v(z) = u(x)$, then

$$u'(x) = v'(z)x^{\frac{m}{2}}, \tag{C.9}$$

and

$$u''(x) = v''(z)x^m + v'(z)\frac{m}{2}x^{\frac{m}{2}-1}. \tag{C.10}$$

A substitution of (C.9) and (C.10) into equation (C.8) produces

$$(\frac{m+2}{2})^2z^2v''(z) + (\frac{m+2}{2})^2zv'(z) - ((\frac{m+2}{2})^2z^2 + \frac{1}{4})v(z) = 0,$$

so that

$$z^2v''(z) + zv'(z) - (\left(\frac{1}{m+2}\right)^2 + z^2)v(z) = 0. \tag{C.11}$$

Comparing equation (C.11) with equation (C.5) produces a solution

$$s(x) = \sqrt{x}u(x) = \sqrt{x}v(z) = \sqrt{x}K_{\frac{1}{m+2}}\left(\frac{2}{m+2}x^{\frac{m+2}{2}}\right),$$

and $s(\infty) = 0$ since $K_p(x)$ decays exponentially. We now seek a series solution to equation (C.1) of the form

$$y(x) = \sum_{n=0}^{\infty}(-\lambda)^n a_n s(\alpha^n x). \tag{C.12}$$

Substituting the series into equation (C.1) and the equation
$s''(\alpha^n x) - (\alpha^n x)^m s(\alpha^n x) = 0$ give the recurrence relation

$$(\alpha^{n(2+m)} - 1)a_n = a_{n-1}.$$

We will show that the series (C.12) is uniformly convergent and integrable on $[0, \infty)$. Now,

$$\begin{aligned}
D(x) &= \frac{d}{dx}\sqrt{\alpha^n x}K_{\frac{1}{m+2}}\left(\frac{2}{m+2}(\alpha^n x)^{\frac{m+2}{2}}\right) \\
&= \frac{\alpha^n}{2\sqrt{\alpha^n x}}K_{\frac{1}{m+2}}\left(\frac{2}{m+2}(\alpha^n x)^{\frac{m+2}{2}}\right) \\
&\quad +\sqrt{\alpha^n x}K'_{\frac{1}{m+2}}\left(\frac{2}{m+2}(\alpha^n x)^{\frac{m+2}{2}}\right)(\alpha^n x)^{\frac{m}{2}}\alpha^n \\
&= \frac{\alpha^n}{2\sqrt{\alpha^n x}}K_{\frac{1}{m+2}}\left(\frac{2}{m+2}(\alpha^n x)^{\frac{m+2}{2}}\right) \\
&\quad +\frac{\alpha^n}{\sqrt{\alpha^n x}}(\alpha^n x)^{\frac{m+2}{2}}K'_{\frac{1}{m+2}}\left(\frac{2}{m+2}(\alpha^n x)^{\frac{m+2}{2}}\right),
\end{aligned} \tag{C.13}$$

and using $xK'_p(x) = -pK_p(x) - xK_{p-1}(x)$ for $p \in R$ (cf. Gray [1922]),
$K_p(x) = K_{-p}(x)$ and $K_p(x) \geq 0$, we have

$$\begin{aligned}
D(x) &= \frac{\alpha^n}{2\sqrt{\alpha^n x}}K_{\frac{1}{m+2}}\left(\frac{2}{m+2}(\alpha^n x)^{\frac{m+2}{2}}\right) \\
&\quad +\frac{(m+2)\alpha^n}{2\sqrt{\alpha^n x}}\Big(-\frac{1}{m+2}K_{\frac{1}{m+2}}\left(\frac{2}{m+2}(\alpha^n x)^{\frac{m+2}{2}}\right) \\
&\quad -\frac{2}{m+2}(\alpha^n x)^{\frac{m+2}{2}}K_{\frac{m+1}{m+2}}\left(\frac{2}{m+2}(\alpha^n x)^{\frac{m+2}{2}}\right)\Big) \\
&= -\alpha^n(\alpha^n x)^{\frac{m+1}{2}}K_{\frac{m+1}{m+2}}\left(\frac{2}{m+2}(\alpha^n x)^{\frac{m+2}{2}}\right) \\
&\leq 0.
\end{aligned} \tag{C.14}$$

This implies that $\sqrt{\alpha^n x}K_{\frac{1}{m+2}}(\frac{2}{m+2}(\alpha^n x)^{\frac{m+2}{2}})$ is decreasing for all x. Now,

$$\begin{aligned}
K(0) &= \lim_{x\to\infty}\sqrt{\alpha^n x}K_{\frac{1}{m+2}}\left(\frac{2}{m+2}(\alpha^n x)^{\frac{m+2}{2}}\right) \\
&= \frac{\Gamma(\frac{1}{m+2})}{2}(m+2)^{\frac{1}{m+2}},
\end{aligned}$$

where we use $K_p(x) \sim \frac{\Gamma(p)}{2}(\frac{2}{x})^p$ as $x \to 0$ (cf. Andrews [1971]). Therefore,

$$\begin{aligned}
&\sum_{n=0}^{\infty} |\lambda^n||a_n|\sqrt{\alpha^n x} K_{\frac{1}{m+2}} \left(\frac{2}{m+2} (\alpha^n x)^{\frac{m+2}{2}} \right) \\
\leq \; &\sum_{n=0}^{\infty} |\lambda^n||a_n| K(0),
\end{aligned}$$

so that the series(C.24) is uniformly convergent, and $y(\infty) = 0$ since

$$\lim_{x\to\infty} \left(\sqrt{\alpha^n x} K_{\frac{1}{m+2}} \left(\frac{2}{m+2} (\alpha^n x)^{\frac{m+2}{2}} \right) \right) = 0,$$

for all n.
Furthermore,

$$\begin{aligned}
I &= \int_0^\infty \sqrt{\alpha^n x} K_{\frac{1}{m+2}} \left(\frac{2}{m+2} (\alpha^n x)^{\frac{m+2}{2}} \right) dx \\
&= \left(\frac{m+2}{2} \right)^{\frac{3}{m+2}-1} \alpha^{-n} \int_0^\infty x^{\frac{3}{m+2}-1} K_{\frac{1}{m+2}}(x)\, dx \\
&= \left(\frac{m+2}{2} \right)^{\frac{3}{m+2}-1} \alpha^{-n} \Gamma\left(\frac{2}{m+2} \right) \Gamma\left(\frac{1}{m+2} \right) \\
&< \infty,
\end{aligned}$$

for all n (cf. (5.17)) and so the uniform convergence of the series (C.12) implies that the series is integrable on $[0, \infty)$. Consequently, a solution to equation (C.1) satisfying the conditions (C.2) and (C.3) is given by

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{k=1}^n (\alpha^{k(2+m)} - 1)} \sqrt{\alpha^n x} K_{\frac{1}{m+2}} \left(\frac{2}{m+2} (\alpha^n x)^{\frac{m+2}{2}} \right), \tag{C.15}$$

where

$$a_0 = \frac{(\frac{m+2}{2})^{-\frac{3}{m+2}+1}}{\Gamma(\frac{2}{m+2})\Gamma(\frac{1}{m+2})} \left(\sum_{n=0}^{\infty} \frac{(-\lambda/\alpha)^n}{\prod_{k=1}^n (\alpha^{k(2+m)} - 1)} \right)^{-1}.$$

Before we finish this section, let us derive another form of the equation to have a solution (C.15) from the equation (C.1). Let $w(x) = x^m y(x)$. Then $w(\alpha x) = \alpha^m x^m y(\alpha x)$ and

$$w'(x) = m x^{m-1} y(x) + x^m y'(x),$$

so that

$$y'(x) = \frac{w'(x)}{x^m} - \frac{m y(x)}{x} = \frac{w'(x)}{x^m} - \frac{m w(x)}{x^{m+1}}.$$

Now,

$$w''(x) = m(m-1)x^{m-2}y(x) + 2mx^{m-1}y'(x) + x^m y''(x),$$

and so

$$y''(x) = \frac{w''(x)}{x^m} - \frac{2mw'(x)}{x^{m+1}} + \frac{(m^2+m)w(x)}{x^{m+2}}.$$

Substituting $y'(x)$ and $y''(x)$ into equation (C.1) and multiplying by x^m lead to

$$w''(x) - \frac{2m}{x}w'(x) + (\frac{m^2+m}{x^2} - 1)w(x) + \frac{\lambda}{\alpha^m}w(\alpha x) = 0.$$

C.2 The General Equation from Problem 5.2

Consider the functional differential equation of the form:

$$xy''(x) - y'(x) - x^m y(x) + x^m \lambda y(\alpha x) = 0, \tag{C.16}$$

where $0 \leq m < \infty$, along with the condition (C.2) and (C.3).

Now,

$$L(s(x)) = xs''(x) - s'(x) - x^m s(x) = 0. \tag{C.17}$$

Let $s(x) = xu(x)$. Then

$$s'(x) = xu'(x) + u(x), \tag{C.18}$$

and

$$s''(x) = xu''(x) + 2u'(x). \tag{C.19}$$

Substituting (C.18) and (C.19) into equation (C.17) yields

$$x^2u''(x) + xu'(x) - (x^{m+1} + 1)u(x) = 0. \tag{C.20}$$

Let $z = \frac{2}{m+1}x^{\frac{m+1}{2}}$ and $v(z) = u(x)$, then

$$u'(x) = v'(z)x^{\frac{m+1}{2}-1}, \tag{C.21}$$

and

$$u''(x) = v''(z)x^{m-1} + v'(z)\frac{m-1}{2}x^{\frac{m+1}{2}-2}. \tag{C.22}$$

By substituting (C.21) and (C.22) into equation (C.20), we have

$$\left(\frac{m+1}{2}\right)^2 z^2 v''(z) + \left(\frac{m+1}{2}\right)^2 z v'(z) - (\left(\frac{m+1}{2}\right)^2 z^2 + 1)v(z) = 0,$$

and so

$$z^2 v''(z) + z v'(z) - (\left(\frac{2}{m+1}\right)^2 + z^2)v(z) = 0. \qquad \text{(C.23)}$$

This implies thus that a solution to equation (C.16) is given by

$$s(x) = xu(x) = xv(z) = xK_{\frac{2}{m+1}}\left(\frac{2}{m+1}x^{\frac{m+1}{2}}\right),$$

and $y(\infty) = 0$ since $K_p(x)$ decays exponentially. Now, a substitution of a series solution of the form

$$y(x) = \sum_{n=0}^{\infty}(-\lambda)^n a_n s(\alpha^n x), \qquad \text{(C.24)}$$

into the equation (C.16) leads to the recurrence relation

$$(\alpha^{n(m+1)} - 1)a_n = a_{n-1}.$$

We show the uniform convergence and integrability of the series (C.24). Now,

$$\begin{aligned}
D(x) &= \frac{d}{dx}\alpha^n x K_{\frac{2}{m+1}}\left(\frac{2}{m+1}(\alpha^n x)^{\frac{m+1}{2}}\right) \\
&= \alpha^n K_{\frac{2}{m+1}}\left(\frac{2}{m+1}(\alpha^n x)^{\frac{m+1}{2}}\right) \\
&\quad + \alpha^n x K'_{\frac{2}{m+1}}\left(\frac{2}{m+1}(\alpha^n x)^{\frac{m+1}{2}}\right)(\alpha^n x)^{\frac{m-1}{2}}\alpha^n \\
&= \alpha^n K_{\frac{2}{m+1}}\left(\frac{2}{m+1}(\alpha^n x)^{\frac{m+1}{2}}\right) \\
&\quad + \alpha^n (\alpha^n x)^{\frac{m+1}{2}} K_{\frac{2}{m+1}}\left(\frac{2}{m+1}(\alpha^n x)^{\frac{m+1}{2}}\right) \\
&= \alpha^n K_{\frac{2}{m+1}}\left(\frac{2}{m+1}(\alpha^n x)^{\frac{m+1}{2}}\right) \\
&\quad + \frac{m+1}{2}\alpha^n (-\frac{2}{m+1}K_{\frac{2}{m+1}}\left(\frac{2}{m+1}(\alpha^n x)^{\frac{m+1}{2}}\right) \\
&\quad - \frac{2}{m+1}(\alpha^n x)^{\frac{m+1}{2}} K_{\frac{1-m}{m+1}}\left(\frac{2}{m+1}(\alpha^n x)^{\frac{m+1}{2}}\right)) \\
&= -\alpha^n (\alpha^n x)^{\frac{m+1}{2}} K_{\frac{1-m}{m+1}}\left(\frac{2}{m+1}(\alpha^n x)^{\frac{m+1}{2}}\right) \\
&\leq 0,
\end{aligned}$$

(cf. (C.13) and (C.14)) so that $\alpha^n x K_{\frac{2}{m+1}}(\frac{2}{m+1}(\alpha^n x)^{\frac{m+1}{2}})$ is decreasing for all x. This implies that

$$\begin{aligned}
&\sum_{n=0}^{\infty}|a_n|\alpha^n x K_{\frac{2}{m+1}}\left(\frac{2}{m+1}(\alpha^n x)^{\frac{m+1}{2}}\right) \\
\leq\ &\sum_{n=0}^{\infty}|a_n|K(0),
\end{aligned}$$

where $K(0) = \lim_{x\to 0}\alpha^n x K_{\frac{2}{m+1}}\left(\frac{2}{m+1}(\alpha^n x)^{\frac{m+1}{2}}\right) = \frac{\Gamma(\frac{2}{m+1})}{2}(m+1)^{\frac{2}{m+1}}$. Therefore, the series (C.24) is uniformly convergent and $y(\infty) = 0$ since

$$\lim_{x\to\infty}(\alpha^n x K_{\frac{2}{m+1}}\left(\frac{2}{m+1}(\alpha^n x)^{\frac{m+1}{2}}\right)) = 0,$$

for all n.

Moreover,

$$\begin{aligned}
I &= \int_0^{\infty}\alpha^n x K_{\frac{2}{m+1}}\left(\frac{2}{m+1}(\alpha^n x)^{\frac{m+1}{2}}\right)dx \\
&= \left(\frac{m+1}{2}\right)^{\frac{3-m}{m+1}}\alpha^{-n}\int_0^{\infty}x^{\frac{3-m}{m+1}}K_{\frac{2}{m+1}}(x)\,dx \\
&= \left(\frac{m+1}{2}\right)^{\frac{3-m}{m+1}}\alpha^{-n}\Gamma\left(\frac{3}{m+1}\right)\Gamma\left(\frac{1}{m+1}\right) \\
&< \infty,
\end{aligned}$$

for all n (cf. (5.17)) and so the uniform convergence of the series implies that the series (C.24) is integrable on $[0,\infty)$. Thus, a solution to equation (C.16) satisfying the conditions (C.2) and (C.3) is given by

$$y(x) = a_0\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\prod_{k=1}^{n}(\alpha^{k(m+1)}-1)}\alpha^n x K_{\frac{2}{m+1}}\left(\frac{2}{m+1}(\alpha^n x)^{\frac{m+1}{2}}\right), \tag{C.25}$$

where

$$a_0 = \frac{(\frac{m+1}{2})^{\frac{m-3}{m+1}}}{\Gamma(\frac{3}{m+1})\Gamma(\frac{1}{m+1})}\left(\sum_{n=0}^{\infty}\frac{(-\lambda/\alpha)^n}{\prod_{k=1}^{n}(\alpha^{k(m+1)}-1)}\right)^{-1}.$$

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