

A Variation Equation for the Wave Forcing of Floating Thin Plates

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Abstract

A variational equation is derived for a floating thin plate subject to wave forcing. This variational equation is derived from the thin plate equations of motion by including the forcing due to the wave through the integral equation derived using the free surface Green's function. This equation combines the optimum method for solving the motion of a thin plate (the variational equation) with the optimum method for solving the wave forcing of a floating body (the Green's function method). Solutions of the variational equation are presented for some simple thin plate geometries using polynomial basis functions. The variational equation is extended to the case of plates of variable properties and to multiple plates and example solutions are presented.

1 Introduction

One of the most well studied hydroelastic problems is the floating thin plate subject to wave forcing. This is because a floating thin plate is a standard model for a number of physical problems such as a very large floating structure (VLFS) (Kashiwagi [5]), a sea ice floe (Meylan and Squire [6]) or a floating breakwater. The solution has been calculated by many authors (e.g. Kashiwagi [5], Meylan and Squire [6], Newman [9] and Wang et. al. [11]).

The solution methods used to solve for the response of a thin plate floating on water subject to wave forcing can be categorised into two broad classes. In the first class, commonly called the eigenfunction expansion method, the plate motion is expanded in the eigenfunctions of the free plate. These eigenfunctions of the free plate are sometimes called the dry modes of the plate. The eigenfunction expansion method requires the determination of the eigenfunctions of the plate separately, usually by the finite element method, since except for very simple geometries the eigenfunctions must be determined numerically. However once the eigenfunctions have been found the equation of motion for the thin plate has been essentially solved. The equation of motion for the water is solved by the free surface Green's function for the water (John [3] and [4]). The eigenfunction expansion method is the method used by Meylan and Squire [6], Newman [9] and Wang et. al. [11].

The second class of methods is based on expanding the thin plate motion in modes other than the eigenfunctions of the plate. The advantage being that the modes do not need to be calculated numerically but now the equations of motion for the thin plate must be solved as well as the equations of motion of the water. The thin plate equations of motion are normally solved by a Galerkin method. The equations of motion for the water are solved in exactly the same way as the eigenfunction method using the Green's function for the water. This solution method is used by Kashiwagi [5] and by Newman [9].

The best methods for solving for the motion of a thin plate are the variational methods, which are methods derived from the variational equation of motion. In the eigenfunction expansion method the eigenfunctions of the thin plate are usually calculated using a variational method, for example the finite element method. There are several reasons why it is preferable to solve the equations of motion for the thin plate using the variational equation. The variational equation includes the boundary conditions in a very simple way, the differentials which appear in the variational form are only second order and there is a very straight forward method to approximate the solution of the variational equation (the Rayleigh-Ritz method).

In this paper a variational equation for the plate-water system is derived. This is done by expressing the equations of motion of the water as a operator equation linking the velocity potential of the water to the displacement of the plate. This variational equation can then be solved by a number of methods. The advantages to rewriting the equations of motion of the system in this form are the following. The variation equation can be solved very straight forwardly, for example using simple polynomials to expand the plate motion. This is the solution method which is presented in this paper. The variational equation is also a good place to begin to develop numerically efficient schemes since it makes various questions about the order of approximations transparent. The variational equation is a theoretically important equation from which properties of the solution can be derived. Most importantly this variational equation contains the optimum method for both the thin plate (the variational equation) and the water (the Green's function method).

2 The problem

A floating thin plate is subject to wave forcing and the response is to be determined. The wave amplitude is assumed to be sufficiently small that linear water wave theory may be applied and the displacement of the plate is assumed to be small enough that linear thin plate theory can be used. The thin plate is assumed to float with negligible draft in water of constant depth. The water depth is sufficiently large that shallow water approximations cannot be made and the full linear water wave theory must be used. The wave forcing is assumed to be due to a plane wave of a single frequency.

3 Equations of motion for the water

The wave amplitudes are assumed sufficiently small that the boundary value problem for the velocity potential for the water, Φ , is the following,

$$\left. \begin{aligned} \nabla^2 \Phi &= 0, & -H < z < 0, \\ \frac{\partial \Phi}{\partial z} &= 0, & z = -H, \\ \rho g W + \rho \frac{\partial \Phi}{\partial t} &= P, & z = 0, \end{aligned} \right\} \quad (1)$$

where z is assumed to point vertically upward and the top water surface is at $z = 0$ and the bottom water surface is at $z = -H$. In (1) W is the displacement of the water surface, P is the pressure on the water surface, ρ is the density of the water and g is the acceleration due to gravity. Equation (1) requires appropriate boundary conditions to be met as $|\mathbf{x}| \rightarrow \infty$ where $\mathbf{x} = (x, y)$. These boundary conditions will be introduced later.

The boundary value problem given by equation (1) contains both the velocity potential for the water and the displacement of the water surface. The displacement of the water surface, W , can be substituted for using the kinematic condition, which at the water surface is

$$\frac{\partial \Phi}{\partial z} = \frac{\partial W}{\partial t}, \quad z = 0. \quad (2)$$

Substituting equation (2) into equation (1) gives the following boundary value problem for the potential only

$$\left. \begin{aligned} \nabla^2 \Phi &= 0, & -H < z < 0, \\ \frac{\partial \Phi}{\partial z} &= 0, & z = -H, \\ \rho g \frac{\partial \Phi}{\partial z} + \rho \frac{\partial^2 \Phi}{\partial t^2} &= \frac{\partial P}{\partial t}, & z = 0, \end{aligned} \right\} \quad (3)$$

The floating thin plate occupies the region Δ . The submergence of the plate is assumed negligible so that Δ is at the water surface, i.e. $z = 0$. Since the pressure at the water surface, except under the plate, can be assumed constant it follows that except under the plate $\frac{\partial P}{\partial t} = 0$. The time derivative in equation (3) is removed by considering wave forcing of a single frequency ω so

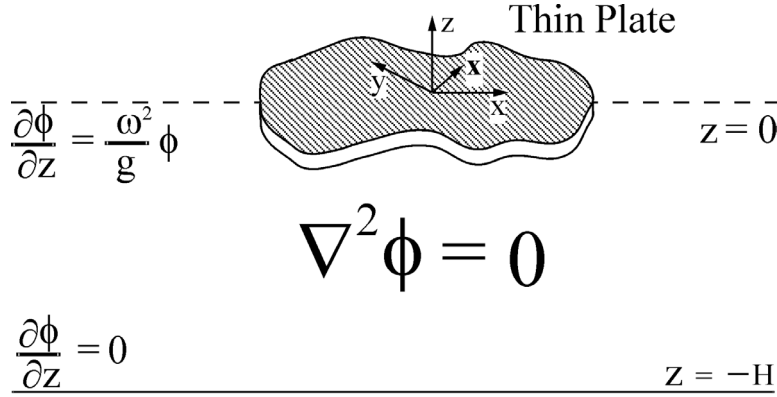


Figure 1: Schematic diagram showing the potential problem in the water and the co-ordinates

that from linearity all quantities have the same harmonic time dependence. Therefore the potential, displacement, and pressure may be written as $\Phi(\mathbf{x}, z, t) = \phi(\mathbf{x}, z)e^{-i\omega t}$, $W(\mathbf{x}, z, t) = w(\mathbf{x}, z)e^{-i\omega t}$, and $P(\mathbf{x}, z, t) = p(\mathbf{x}, z)e^{-i\omega t}$ respectively. The boundary value problem (3), under the assumption of harmonic time dependence, becomes,

$$\left. \begin{aligned} \nabla^2 \phi &= 0, & -H < z < 0, \\ \frac{\partial \phi}{\partial z} &= 0, & z = -H, \\ \rho \left(g \frac{\partial \phi}{\partial z} - \omega^2 \phi \right) &= 0, & z = 0, \mathbf{x} \notin \Delta \\ \rho \left(g \frac{\partial \phi}{\partial z} - \omega^2 \phi \right) &= i\omega p, & z = 0, \mathbf{x} \in \Delta. \end{aligned} \right\} \quad (4)$$

A schematic diagram of this boundary value problem and the coordinate system is shown in Figure 1.

As mentioned previously, the boundary value problem (4) is subject to a boundary condition as $|\mathbf{x}| \rightarrow \infty$ which is the Sommerfeld radiation condition (Wehausen and Laitone [12])

$$\lim_{|\mathbf{x}| \rightarrow \infty} \sqrt{|\mathbf{x}|} \left(\frac{\partial}{\partial |\mathbf{x}|} - i\omega^2 \right) (\phi - \phi^i) = 0. \quad (5)$$

where ϕ^i is the incident potential, that is the potential due to the incoming wave. The incident potential ϕ^i is assumed to be plane wave, of frequency ω and unit amplitude, travelling in the θ direction where θ is the angle from the x -axis. That is

$$\phi^i(x, y, z) = \frac{i\omega}{k \sinh(kH)} e^{ik(x \cos \theta + y \sin \theta)} \cosh k(z - H) \quad (6)$$

where k is the wavenumber which satisfies the dispersion equation

$$gk \tanh(kH) = \omega^2.$$

The standard solution method to the linear wave problem is to transform the boundary value problem (4) into an integral equation over the wetted surface of the body (John [3] and [4], Wehausen and Laitone [12], and Sarpkaya and Issacson [10]) by the free surface Green's function. This is the method used by almost all authors who have solved for the time harmonic motion of a floating thin plate (e.g. Kashiwagi [5], Meylan and Squire [6] and Newman [9]) and it captures the boundary value problem and the radiation condition at ∞ . Performing just such a transformation Equations (4) and (5) become,

$$\phi(\mathbf{x}) = \phi^i(\mathbf{x}) + \int \int_{\Delta} G_{\omega}(\mathbf{x}; \mathbf{y}) (\omega^2 \phi(\mathbf{x}) + i\omega w(\mathbf{x})) d\mathbf{y}. \quad (7)$$

where, because the submergence of the plate is assumed negligible, the shallow draft approximation (Meylan and Squire [6]) has been used. This substitutes for the normal derivative of the Green's function so that the integral equation only involves the Green's function itself. The Green's function $G_\omega(\mathbf{x}; \mathbf{y})$ is the restriction of the Green's function for the full water depth to the water surface, i.e.

$$G_\omega(\mathbf{x}; \mathbf{y}) = G_\omega(\mathbf{x}, z; \mathbf{y}, \xi)|_{z=0, \xi=0}$$

where $G_\omega(\mathbf{x}, z; \mathbf{y}, \xi)$ is the free surface Green's function which satisfies the equation

$$\left. \begin{aligned} \nabla^2 G_\omega(\mathbf{x}, z; \mathbf{y}, \xi) &= \delta(\mathbf{x} - \mathbf{y}) \delta(z - \xi), & -H < z, \xi < 0, \\ \frac{\partial G_\omega}{\partial \xi} &= 0, & \xi = -H, \\ g \frac{\partial G_\omega}{\partial \xi} - \omega^2 G_\omega &= 0, & \xi = 0, \mathbf{x} \notin \Delta. \end{aligned} \right\} \quad (8)$$

This Green's function is given in Wehausen and Laitone [12]. Efficient numerical methods for the calculation of the Green's function are described in Newman [8].

The calculations presented here will all be for water of infinite depth. In this case the Green's function becomes the infinite depth Green's function, which is denoted $G_\omega^\infty(\mathbf{x}, \mathbf{y})$, and which is given by

$$\begin{aligned} G_\omega^\infty(\mathbf{x}, \mathbf{y}) &= \frac{1}{4\pi} \left(\frac{2}{|\mathbf{x} - \mathbf{y}|} - \pi \frac{\omega^2}{g} \left(\mathbf{H}_0\left(\frac{\omega^2}{g}|\mathbf{x} - \mathbf{y}|\right) + Y_0\left(\frac{\omega^2}{g}|\mathbf{x} - \mathbf{y}|\right) \right) \right) \\ &+ \frac{i\omega^2}{2g} J_0\left(\frac{\omega^2}{g}|\mathbf{x} - \mathbf{y}|\right), \end{aligned} \quad (9)$$

(Wehausen and Laitone [12]). In equation (9) J_0 and Y_0 are respectively Bessel functions of the first and second kind of order zero, and \mathbf{H}_0 is the Struve function of order zero (Abramowitz and Stegun, [1]). An advantage of the shallow draft formulation is that the Greens function, G_ω , depends only on a single parameter, $|\mathbf{x} - \mathbf{y}|$, so that negligible computational effort is required to determine G_ω^∞ since it can be simply looked up in a table.

We now write equation (7) in operator notation as

$$\phi = \phi^i + \mathbf{G}_\omega(\omega^2 \phi + i\omega w) \quad (10)$$

where ϕ , w , and ϕ^i are now vectors in the Hilbert space of functions over the wetted surface of the plate, Δ . This rewriting is nothing more than notation by it will be used extensively in the derivations which follow. The solution of equation (10) is given formally by

$$\phi = \mathbf{H}_\omega \phi^i + i\omega \mathbf{H}_\omega \mathbf{G}_\omega w \quad (11)$$

where the operator \mathbf{H}_ω is given by

$$\mathbf{H}_\omega = (\mathbf{I} - \omega^2 \mathbf{G}_\omega)^{-1} \quad (12)$$

where \mathbf{I} is the identity operator. It should be remembered that in any practical calculation the operators will be approximated by matrices so that the calculation of \mathbf{H}_ω will be by a matrix inverse.

The most significant speed ups to any numerical scheme are to be made in calculating the inverse of the matrix $\mathbf{I} - \omega^2 \mathbf{G}_\omega$. Wang et. al. [11] and Wu et. al. [14] discuss the composite singularity distribution method which can be understood as follows. If the thin plate has a symmetry (a rectangular plate will have two) then the matrix $\mathbf{I} - \omega^2 \mathbf{G}_\omega$, with the appropriate ordering, can be written in block form as

$$\mathbf{I} - \omega^2 \mathbf{G}_\omega = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{pmatrix}.$$

Now the inverse of a matrix of this form can be calculated as

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{pmatrix} \begin{pmatrix} (\mathbf{A} + \mathbf{B})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{A} - \mathbf{B})^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{pmatrix}$$

where \mathbf{I} is an identity matrix of the appropriate size. Therefore the inverse of $\mathbf{I} - \omega^2 \mathbf{G}_\omega$ can be calculated by finding the inverse of two matrices of half the size. Further symmetries can also be exploited analogously. Wu et. al. [14] also suggest that for large structures the matrices can be approximated by sparse matrices and an iterative method to calculate the inverse can be used, leading to further savings of computational effort.

Since the Green's function in the inverse of a differential operator it is compact so that the operator \mathbf{H}_ω will be bounded. The only difficulty in calculating \mathbf{H}_ω is at the so called irregular frequencies which are the points at which $(\mathbf{I} - \omega^2 \mathbf{G}_\omega)$ is not invertible. However it turns out that for the shallow draft formulation the irregular frequencies do not exist for any values of ω . (Meylan and Squire [6]).

4 Equations of motion for the plate

Equation (7) links the potential under the plate and the plate displacement. To solve for the thin plate motion a further equation is required which connects the velocity potential of the water under the plate with the plate displacement. This equation will be derived from the equation of motion for a thin plate subject to a pressure force. This equation is the following

$$D\nabla^4 W + \rho_r h \frac{\partial^2 W}{\partial t^2} = P, \quad (13)$$

where W is the plate or surface displacement as before, ρ_r is the plate density, h is the plate thickness, D is the modulus of rigidity of the plate and P is the pressure on the wetted surface of the plate. Equation (13) is subject to the boundary conditions of a free plate at the plate edges which are

$$\frac{\partial^2 W}{\partial n^2} + \nu \frac{\partial^2 W}{\partial s^2} = 0, \quad \text{and} \quad \frac{\partial^3 W}{\partial n^3} + (2 - \nu) \frac{\partial^3 W}{\partial n \partial s^2} = 0, \quad (14)$$

where n and s denote the normal and tangential directions respectively and ν is Poisson's ratio. These boundary conditions are chosen because in most practical cases the edges of the plate will be free. However the derivation of the variational equation could be easily modified to include other boundary conditions. The inclusion of these boundary conditions often present a significant complication to any numerical scheme but they are solved for in a very simple manner in the variational formulation.

The pressure P is given by the linearised Bernoulli's equation at the water surface,

$$P = -\rho g W - \rho \frac{\partial \Phi}{\partial t}. \quad (15)$$

Restricting consideration to a single frequency ω as before and substituting equation (15) into equation (13) gives the following equation

$$D\nabla^4 w - \omega^2 \rho_r h w = i\omega \rho \phi - \rho g w, \quad (16)$$

which is the second equation linking ϕ and w that was required. Equations (7) and (16), together with the appropriate boundary conditions for the plate, represent two simultaneous equations for ϕ and w to solve. These equations will be solved by substituting the integral equation (7) into the variational equation for the thin plate. This is in contrast to the solution method of most authors who substitute the equations of motion of the thin plate (through the thin plate modes of vibration

for example) into the integral equation. Before transforming equation (16) to a variational equation (11) is substituted to give

$$D\nabla^4 w + (\rho g - \omega^2 \rho_r h) w = i\omega \rho (\mathbf{H}_\omega \phi^i + i\omega \mathbf{H}_\omega \mathbf{G}_\omega w), \quad (17)$$

which is an equation in the plate displacement only. Equation (17) is still subject to the boundary conditions of a free plate given by equation (14).

5 The Variational Equation

Equation (17) is now transformed into a variational equation by multiplying by a variation δw in the standard way (Hildebrand, [2]). This gives the following equation

$$\begin{aligned} \delta \int \int_{\Delta} \left\{ \frac{1}{2} D (w_{xx}^2 + w_{yy}^2 + 2\nu w_{xx} w_{yy} + 2(1-\nu) w_{xy}^2) + \frac{(\rho g - \omega^2 \rho_r h)}{2} w^2 \right\} dx \\ = \int \int_{\Delta} i\omega \rho (\mathbf{H}_\omega \phi^i + i\omega \mathbf{H}_\omega \mathbf{G}_\omega w) \delta w dx, \end{aligned} \quad (18)$$

The left hand side of equation (18) comes from integration by parts of the biharmonic operator ∇^4 and the substitution of the edge conditions (14) to eliminate the boundary terms which arise from the integration by parts. All that remains is to transform the right hand side of equation (18). The right hand side transform as follows

$$\int \int_{\Delta} i\omega \rho (\mathbf{H}_\omega \phi^i + i\omega \mathbf{H}_\omega \mathbf{G}_\omega w) \delta w dx = \delta \int \int_{\Delta} i\omega \rho \left(\mathbf{H}_\omega \phi^i + \frac{i\omega}{2} \mathbf{H}_\omega \mathbf{G}_\omega w \right) w dx \quad (19)$$

where the $\frac{1}{2}$ must be included because the second term depends linearly on w . Substituting equation (19) into equation (18) gives

$$\begin{aligned} \delta \int \int_{\Delta} \left\{ \frac{1}{2} D (w_{xx}^2 + w_{yy}^2 + 2\nu w_{xx} w_{yy} + 2(1-\nu) w_{xy}^2) + \frac{(\rho g - \omega^2 \rho_r h)}{2} w^2 + \frac{\rho \omega^2}{2} (\mathbf{H}_\omega \mathbf{G}_\omega w) w \right\} dx \\ = \delta \int \int_{\Delta} i\omega \rho \mathbf{H}_\omega \phi^i w dx. \end{aligned} \quad (20)$$

This is a variational equation for the thin plate - water system subject to an incident wave ϕ^i .

There are several advantages to constructing a solution to the problem by using this variational equation. Most of the methods for calculating the motion of thin plate are derived from the variational equation, for example the finite element method. The variational equation provides a unified setting in which the finite element method can be used in conjunction with the integral equation for the water, rather than using the finite element method to calculate the modes of the plate and then substituting these into the equations of motion. The variational equation also reveals information about the structure of the equations, for example that the inertia of the plate and the hydrostatic pressure make the same contribution to the equation and should be treated similarly. Also the variational equation is important theoretically and is a starting point to study many aspects of the problem rigorously.

6 Solution of the Variational Equation by the Rayleigh-Ritz Method

The standard numerical technique to solve variational equations is the Rayleigh-Ritz method. In this method the displacement is expanded as,

$$w(\mathbf{x}) = \sum_{i=0}^I c_i w_i(\mathbf{x}), \quad (21)$$

and this expansion is substituted into the variational equation which is then solved by seeking a minimum with respect to the coefficients c_i . The finite element method is of course a form of the Rayleigh-Ritz method in which the basis functions $w_i(\mathbf{x})$ are chosen to be functions which are defined only locally. Firstly the equations for the coefficients c_i will be derived without worrying about the form of the basis functions $w_i(\mathbf{x})$ except for assuming that they are complete as $i \rightarrow \infty$ and are ordered in some logical way. Then a solution will be presented with a particular choice of basis functions.

Substituting the expansion for w (21) into equation (7), deriving with respect to the coefficient c_j , and equating to zero the following simultaneous equations for \vec{c} (the vector of coefficients c_i) is derived

$$\left(\mathbf{K} + \mathbf{M} + \frac{1}{2} (\mathbf{L} + \mathbf{L}^T) \right) \vec{c} = \vec{f} \quad (22)$$

where the elements of the matrix \mathbf{K} (called the stiffness matrix in the finite element method) are

$$k_{ij} = \int \int_{\Delta} D \left\{ \frac{\partial^2 w_i}{\partial x^2} \frac{\partial^2 w_j}{\partial x^2} + \frac{\partial^2 w_i}{\partial y^2} \frac{\partial^2 w_j}{\partial y^2} \right. \\ \left. + \nu \left(\frac{\partial^2 w_i}{\partial x^2} \frac{\partial^2 w_j}{\partial y^2} + \frac{\partial^2 w_i}{\partial y^2} \frac{\partial^2 w_j}{\partial x^2} \right) + 2(1 - \nu) \frac{\partial^2 w_i}{\partial x \partial y} \frac{\partial^2 w_j}{\partial x \partial y} \right\} d\mathbf{x},$$

the elements of the matrix \mathbf{M} (called the mass matrix in the finite element method, although the hydrostatic force is included here as well as the inertia) are

$$m_{ij} = \int \int_{\Delta} (\rho g - \omega^2 \rho_r h) w_i(\mathbf{x}) w_j(\mathbf{x}) d\mathbf{x},$$

the elements of the matrix \mathbf{L} (which represents the interaction between the modes $w_i(\mathbf{x})$ due to the coupling through the water) are

$$l_{ij} = \int \int_{\Delta} \rho \omega^2 w_i(\mathbf{x}) (\mathbf{H}_\omega \mathbf{G}_\omega w_j(\mathbf{x})) d\mathbf{x},$$

and \vec{f} is the vector with elements

$$f_i = \int \int_{\Delta} i\omega \rho (\mathbf{H}_\omega \phi^i(\mathbf{x})) w_i(\mathbf{x}) d\mathbf{x}.$$

What is apparent from this derivation is that an efficient numerical scheme to solve for the plate motion will depend on two factors. The first is that the basis functions should be chosen to have suitable properties, such as the right order of continuity and some kind of orthogonality. The second is that the solution of the integral equation (the operator \mathbf{H}_ω and \mathbf{G}_ω) should be done in a way to make the solution as accurate as possible for the basis functions which are used in the expansion of the plate displacement.

7 Non-Dimensionalisation

The results which are present will be in non-dimensional coordinates. This is primarily because it gives a minimal number of variables. Also a floating thin plate on water models a wide variety of problems so that it is better to choose non-dimensional coordinates than to choose parameters appropriate for a particular problem. The variational equation could have been derived in the non-dimensional variables but since the dimensional form of the equations is more standard the derivation was presented in this form.

The non-dimensionalisation is as follows. The equations are non-dimensionalised using a length parameter a (the length of the plate say) and a time parameter $\sqrt{\frac{a}{g}}$. This later parameter being chosen so that the dispersion equation for the surface gravity waves in the non-dimensional coordinates (which are denoted by an over bar) is $\bar{\omega}^2 = \bar{k}$. Therefore the non-dimensional length and time parameters are

$$\bar{\mathbf{x}} = \frac{\mathbf{x}}{a}, \text{ and } \bar{t} = t\sqrt{\frac{g}{a}}.$$

The non-dimensional matrix elements become

$$\begin{aligned} \bar{k}_{ij} &= \int \int_{\bar{\Delta}} \beta \left\{ \frac{\partial^2 \bar{w}_i}{\partial \bar{x}^2} \frac{\partial^2 \bar{w}_j}{\partial \bar{x}^2} + \frac{\partial^2 \bar{w}_i}{\partial \bar{y}^2} \frac{\partial^2 \bar{w}_j}{\partial \bar{y}^2} \right. \\ &+ \nu \left(\frac{\partial_i^2 \bar{w}_i}{\partial \bar{x}^2} \frac{\partial^2 \bar{w}_j}{\partial \bar{y}^2} + \frac{\partial_i^2 \bar{w}_i}{\partial \bar{y}^2} \frac{\partial^2 \bar{w}_j}{\partial \bar{x}^2} \right) + 2(1 - \nu) \frac{\partial_i^2 \bar{w}_i}{\partial \bar{x} \partial \bar{y}} \frac{\partial_j^2 \bar{w}_j}{\partial \bar{x} \partial \bar{y}} \left. \right\} d\bar{\mathbf{x}}, \\ \bar{m}_{ij} &= \int \int_{\bar{\Delta}} (1 - \bar{\omega}^2 \gamma) \bar{w}_i(\bar{\mathbf{x}}) \bar{w}_j(\bar{\mathbf{x}}) d\bar{\mathbf{x}}, \\ \bar{l}_{ij} &= \int \int_{\bar{\Delta}} \bar{\omega}^2 \bar{w}_i(\bar{\mathbf{x}}) (\mathbf{H}_\omega \mathbf{G}_\omega \bar{w}_j(\bar{\mathbf{x}})) d\bar{\mathbf{x}}, \end{aligned}$$

and

$$\bar{f}_i = \int \int_{\bar{\Delta}} i\bar{\omega} (\mathbf{H}_\omega \phi^i(\bar{\mathbf{x}})) w_i(\bar{\mathbf{x}}) d\bar{\mathbf{x}}$$

where the operators \mathbf{H}_ω and \mathbf{G}_ω have been appropriately non-dimensionalised as well and where

$$\beta = \frac{D}{g\rho a^4} \text{ and } \gamma = \frac{\rho_r h}{\rho a}$$

It is clear that the in the non-dimensional variables only three parameters, besides the geometry of the plate, govern the solution. The first is the non-dimensional frequency $\bar{\omega}$, the second in the non-dimensional stiffness of the plate β and the third is the non-dimensional mass of the plate γ . However because the density of the plate is likely to be close to that of water and because the plate is thin γ may be assumed small and neglected. This approximation is becoming more standard (e.g. Namba and Ohkusu [7]) primarily because of the simplification of the result which must be considered if there is one less parameter. Interestingly in the variational formulation, since the inertia of the plate and the hydrostatic pressure enter the equations in the same manner, setting $\gamma = 0$ makes no simplification of the solution. For this reason a non-zero value of γ is chosen for the results presented here. However the value of γ is chosen to be realistic and hence it is small enough to make little difference to the solution.

8 Results

The basis functions used in the solutions presented are simple polynomials. This choice is made primarily for the simplicity of the resulting equations. Indeed this is one of the advantages of using the variational equation, good results can be calculated with simple basis functions. Since the basis functions are polynomials the matrices \mathbf{K} and \mathbf{M} can be calculated analytically. Also when we approximate the operators \mathbf{H}_ω and \mathbf{G}_ω by matrices integration rules are used which are designed specifically for the polynomial basis functions. Therefore, although polynomials are basis functions with poor orthogonality the approximation of the operators \mathbf{H}_ω and \mathbf{G}_ω is quite accurate. Furthermore, the results presented are for plates whose size is of the order of the wavelength. This means that the motion can accurately capture by a small number of basis elements. This means that the problems of the poor orthogonality of the basis functions do not manifest themselves. For

	$m = 10$	$m = 15$	$m = 20$	$m = 25$
$p = 6$	0.1974	0.2187	0.2224	0.2233
$p = 8$	0.2797	0.3108	0.3194	0.3228
$p = 10$	0.2855	0.3171	0.3260	0.3295
$p = 12$	0.2858	0.3175	0.3263	0.3299

Table 1: The convergence for a 2 by 2 plate

	$m = 20$	$m = 30$	$m = 40$	$m = 50$
$p = 8$	0.0566	0.0626	0.0646	0.0654
$p = 12$	0.2537	0.2637	0.2680	0.2702
$p = 16$	0.2698	0.2794	0.2837	0.2859
$p = 20$	0.2702	0.2804	0.2840	0.2861

Table 2: The convergence for a 4 by 4 plate

applications where the plate length is many times wavelength then different basis functions such as splines or finite elements should be used.

Since the plate is two dimensional, a slight change of notation is useful to express the basis functions as a double sum over the x and y polynomials as follows.

$$w_{ij}(\mathbf{x}) = \sum_{i=0}^p \sum_{j=0}^q x^i y^j.$$

Given that beyond the plate parameters, β and γ , the plate geometry and the wavelength may also be varied it is obviously not possible to present anything approaching a comprehensive survey of results. Instead a few examples of the sort of calculations which are possible are presented. The wavelength is fixed to be two, so that the non-dimensional frequency is $\bar{\omega} = \sqrt{\pi}$, the stiffness is fixed to be $\beta = 0.005$ and the mass is fixed to be $\gamma = 0.01$ for all our calculations.

Firstly a convergence study for a thin plate of length 2 by 2 is presented. The plate is discretised into m by m panels which are used to calculate the operators \mathbf{H}_ω and \mathbf{G}_ω (note that because the basis functions are global we have a different discretisation scheme for the operators \mathbf{H}_ω and \mathbf{G}_ω and the thin plate motion). Since the plate is square we set $p = q$. A synopsis of the convergence results are shown in table 1 which shows the absolute value of the solution at the point $(0, 0)$ for the values of m and p shown where the wave angle has been set to be $\theta = \frac{\pi}{3}$ (this value being chosen to avoid symmetries). Of course the solution at one point could be anomalous but these results summarise the conclusions of a more detailed convergence study. The result which is highlighted in bold is considered the optimum choice of accuracy versus work, remembering that all the matrices grow in size as m^4 and p^4 (that is the matrix which approximates \mathbf{H}_ω and \mathbf{G}_ω when $m = 10$ has $100 \times 100 = 10^4$ elements and when $m = 20$ has $400 \times 400 = 1.6 \times 10^5$ elements) and represents the values chosen in Figure 2) and Figure 3.

Figures 2 and 3 show the displacement of a 2 by 2 square plate for the times $t = 0$, $t = \frac{T}{8}$, $t = \frac{T}{4}$, and $t = \frac{3T}{8}$ where T is the wave period with an incident waveangle of $\theta = 0$ (Figure 2) and $\theta = \frac{\pi}{2}$ (Figure 3) respectively. The four values of time are chosen to give some indication of the dynamic response of the plate over one wave period. It is more usual to plot the real and imaginary parts of the displacement and these of course correspond to the displacement at $t = 0$ and $t = \frac{T}{4}$. It is worth noting that in the next picture in the sequence, which would be at $t = \frac{T}{2}$, the displacement is the same as that for $t = 0$ except that the sign of the displacement is reversed.

Table 2 is a convergence study for a 4 by 4 plate. As before the table shows the absolute value of the displacement at $(0, 0)$ and the wave angle is $\theta = \frac{\pi}{3}$. The result in bold is again for the values of p and m which will be used in subsequent calculations.

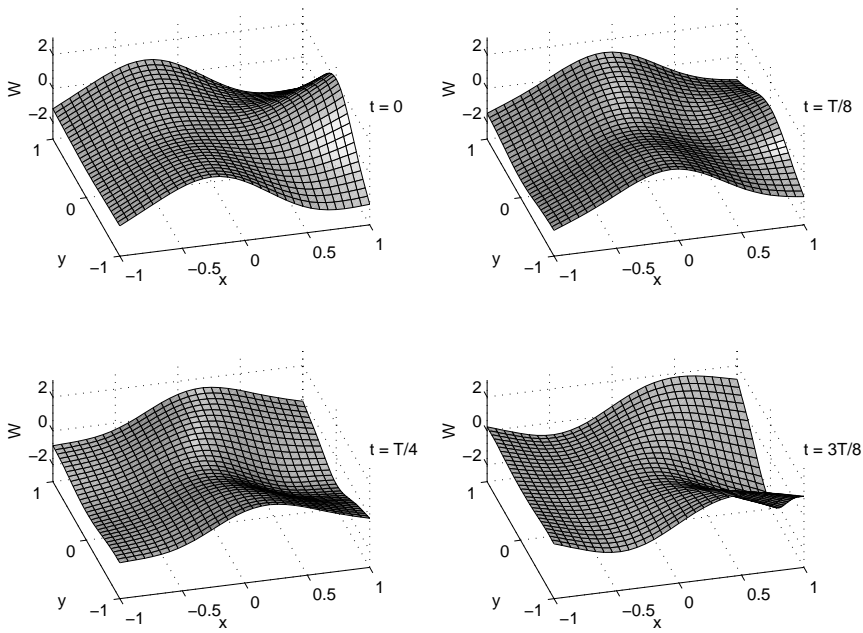


Figure 2: The motion of a 2 by 2 plate for the times shown where T is the wave period. The wavelength $\lambda = 2$, $\beta = 0.005$, $\gamma = 0.01$, and $\theta = 0$

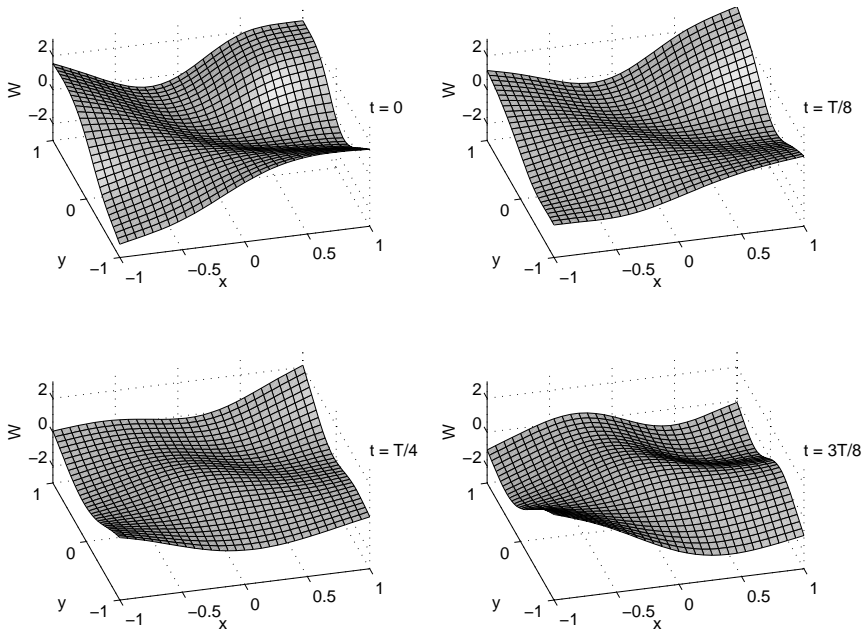


Figure 3: The motion of a 2 by 2 plate for the times shown where T is the wave period. The wavelength $\lambda = 2$, $\beta = 0.005$, $\gamma = 0.01$, and $\theta = \frac{\pi}{4}$

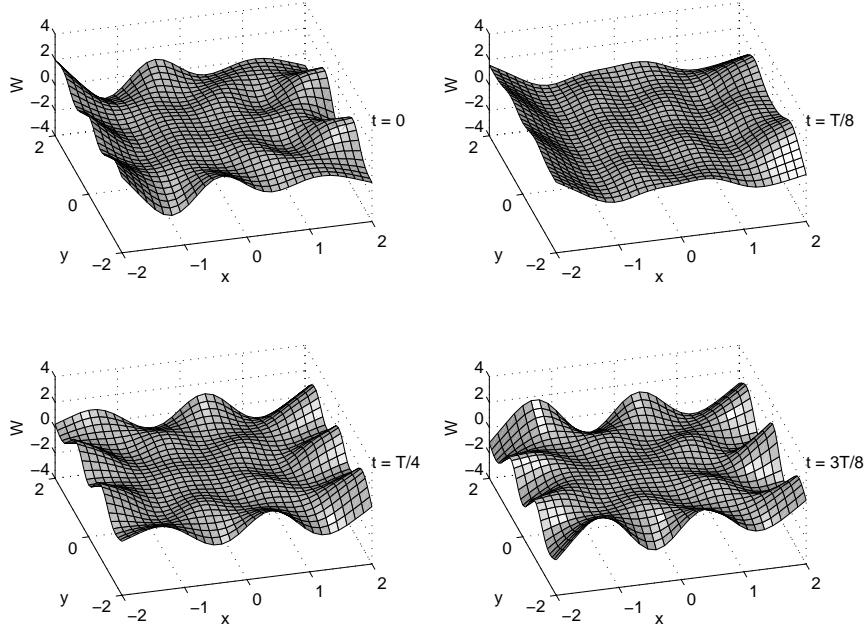


Figure 4: The motion of a 4 by 4 plate for the times shown where T is the wave period. The wavelength $\lambda = 2$, $\beta = 0.005$, $\gamma = 0.01$, and $\theta = 0$.

Figures 4 and 5 show the displacement of a 4 by 4 square plate for the times $t = 0$, $t = \frac{T}{8}$, $t = \frac{T}{4}$, and $t = \frac{3T}{8}$ where T is the wave period calculated with $p = q = 16$ and $m = n = 40$. The incident wave angle is $\theta = 0$ (Figure 4) and $\theta = \frac{\pi}{2}$ (Figure 5).

Figures 6 and 7 show the displacement of a 4 by 2 square plate for the times $t = 0$, $t = \frac{T}{8}$, $t = \frac{T}{4}$, and $t = \frac{3T}{8}$ where T is the wave period calculated with $p = 16$, $q = 10$, $m = 40$, and $n = 20$. As before the incident wave angle is $\theta = 0$ (Figure 6) and $\theta = \frac{\pi}{2}$ (Figure 7).

9 Variable Thickness

It is trivial to modify the variational equations to allow the properties of the plate, such as plate thickness to vary, over the plate. Essentially β and γ are now no longer constants and so must be included in the integration. However since the operators \mathbf{H}_ω and \mathbf{G}_ω do not depend on the properties of the plate they do not need to be recalculated. All that changes is the matrices \mathbf{K} and \mathbf{M} . That is the elements of these matrices become

$$k_{ij} = \int \int_{\Delta} D(\mathbf{x}) \left\{ \frac{\partial^2 w_i}{\partial x^2} \frac{\partial^2 w_j}{\partial x^2} + \frac{\partial^2 w_i}{\partial y^2} \frac{\partial^2 w_j}{\partial y^2} \right. \\ \left. + \nu \left(\frac{\partial_i^2 w_i}{\partial x^2} \frac{\partial^2 w_j}{\partial y^2} + \frac{\partial_i^2 w_i}{\partial y^2} \frac{\partial^2 w_j}{\partial x^2} \right) + 2(1 - \nu) \frac{\partial_i^2 w_i}{\partial x \partial y} \frac{\partial_j^2 w_j}{\partial x \partial y} \right\} d\mathbf{x},$$

and

$$m_{ij} = \int \int_{\Delta} (\rho g - \omega^2 \rho_r h(\mathbf{x})) w_i(\mathbf{x}) w_j(\mathbf{x}) d\mathbf{x},$$

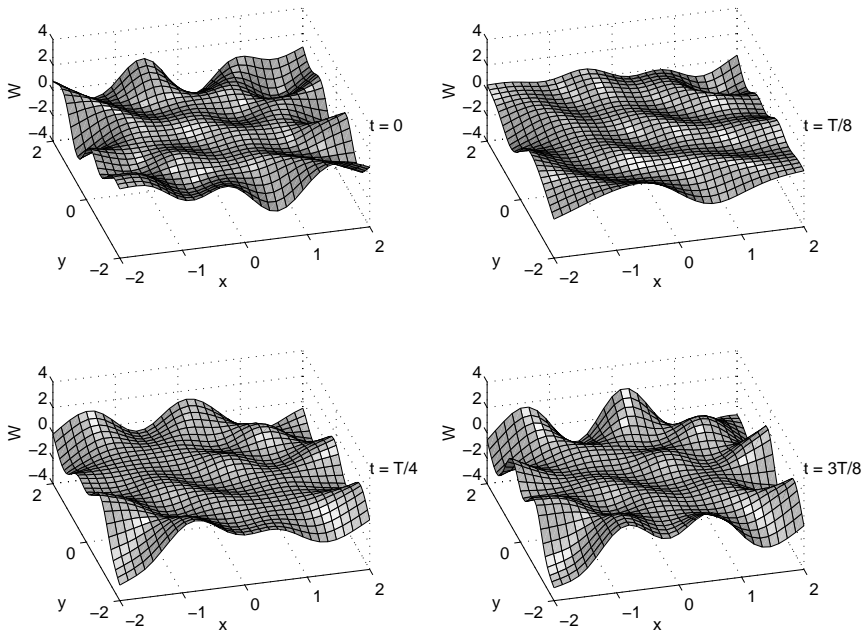


Figure 5: The motion of a 4 by 4 plate for the times shown where T is the wave period. The wavelength $\lambda = 2$, $\beta = 0.005$, $\gamma = 0.01$, and $\theta = \frac{\pi}{4}$

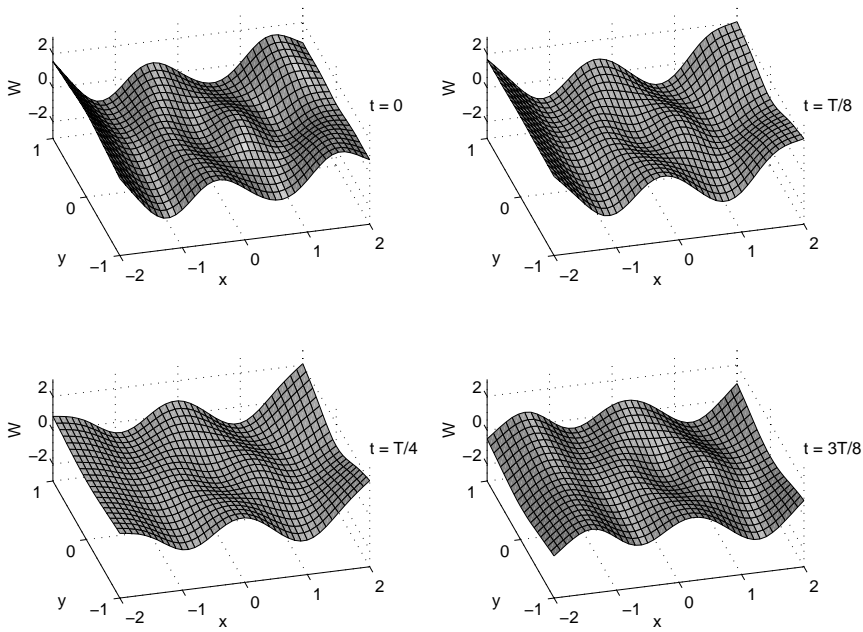


Figure 6: The motion of a 4 by 2 plate for the times shown where T is the wave period. The wavelength $\lambda = 2$, $\beta = 0.005$, $\gamma = 0.01$, and $\theta = 0$

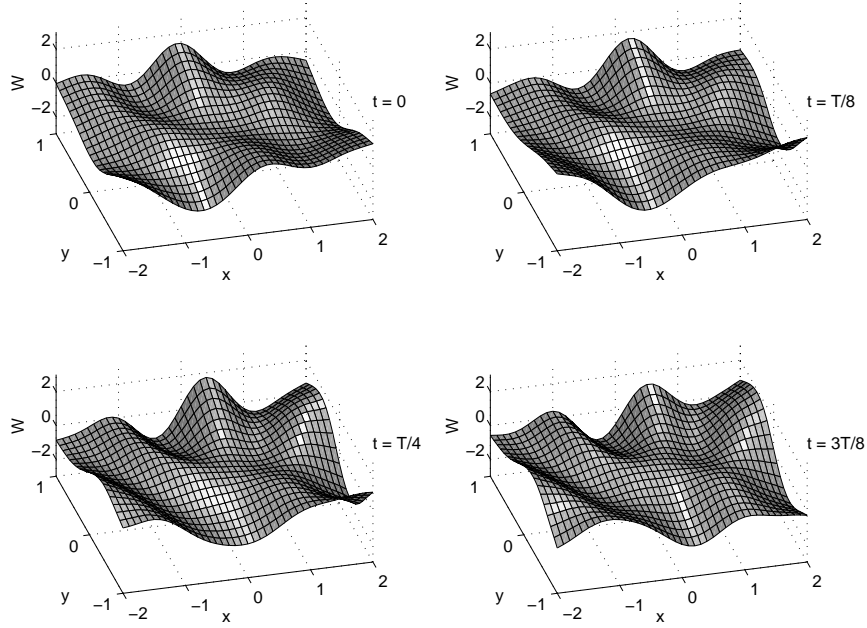


Figure 7: The motion of a 4 by 2 plate for the times shown where T is the wave period. The wavelength $\lambda = 2$, $\beta = 0.005$, $\gamma = 0.01$, and $\theta = \frac{\pi}{4}$

respectively. If the variation in the parameters of the plate is sufficiently simple then these matrices can still even be calculated analytically. This will be the case in the simple example calculations presented.

The results will be presented in non-dimensional coordinates exactly as before. Figures 8 and 9 show the displacement for a 2 by 2 thin plate with wavelength two, $\beta = 0.005(x+1)^3$ and $\gamma = 0.01(x+1)$ for the times $t = 0$, $t = \frac{T}{8}$, $t = \frac{T}{4}$, and $t = \frac{3T}{8}$ where T is the wave period for the angle $\theta = 0$ (Figure 2) and $\theta = \frac{\pi}{2}$ (Figure 9) respectively with $m = n = 20$ and $p = q = 10$. The values of β and γ were chosen because they correspond to a linear increase in the thickness of the plate across the x -coordinate.

10 Multiple Thin Plates

It is surprisingly straight forward to extend the formulation to the case of multiple thin plates. It is assumed that the plates are not tethered together in any way so that the free boundary conditions apply. However, inclusion of some other boundary conditions in the formulation would be a simple modification. Suppose there are n thin plates located at Δ_p respectively. The integral equation, derived from the Green's function (equation 7) becomes

$$\phi(\mathbf{x}) = \phi^i(\mathbf{x}) + \sum_{p=1}^n \int \int_{\Delta_p} G_\omega(\mathbf{x}; \mathbf{y}) (\omega^2 \phi(\mathbf{x}) + i\omega w(\mathbf{x})) dy \quad (23)$$

which is written in operator notation as

$$i\omega\rho \left(\tilde{\mathbf{H}}_\omega \phi^i + i\omega \tilde{\mathbf{H}}_\omega \tilde{\mathbf{G}}_\omega w \right) = \phi$$

where the tilde indicates that the operators are now acting on the space of the wetted surface of all the plates.

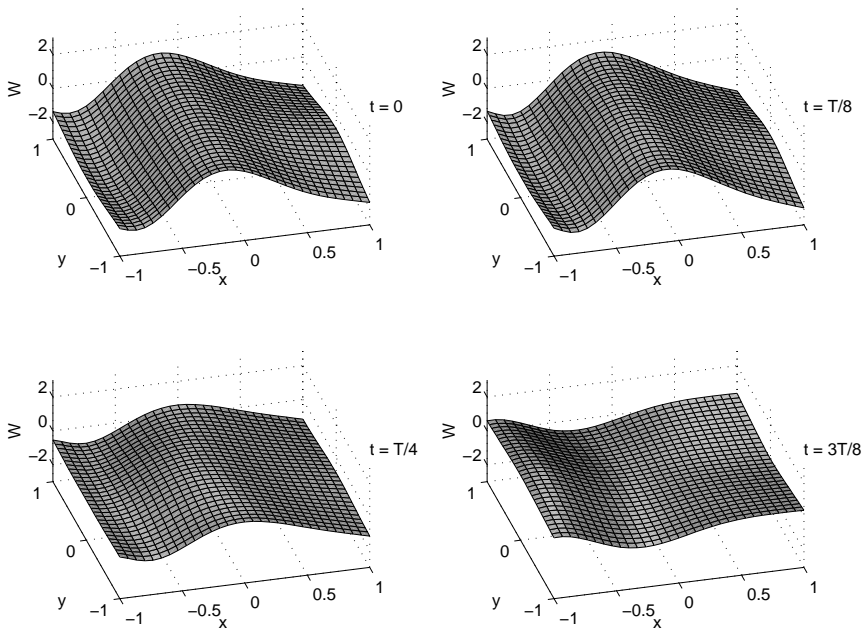


Figure 8: The motion of a 2 by 2 plate for the times shown where T is the wave period. The wavelength $\lambda = 2$, $\beta = 0.005$, $\beta = 0.005(x + 1)^3$, $\gamma = 0.01(x + 1)$, and $\theta = 0$

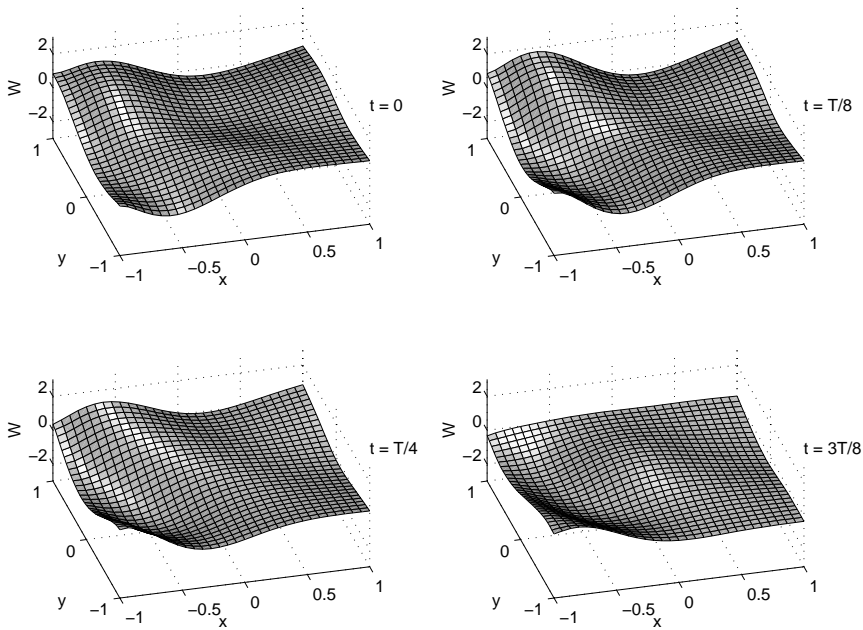


Figure 9: The motion of a 2 by 2 plate for the times shown where T is the wave period. The wavelength $\lambda = 2$, $\beta = 0.005$, $\beta = 0.005(x + 1)^3$, $\gamma = 0.01(x + 1)$, and $\theta = \frac{\pi}{4}$

Consider the variational equation for the p th plate which is

$$\begin{aligned} \delta \int \int_{\Delta_p} \left\{ \frac{1}{2} D \left((w_{xx}^p)^2 + (w_{yy}^p)^2 + 2\nu w_{xx}^p w_{yy}^p + 2(1-\nu) (w_{xy}^p)^2 \right) + \frac{(\rho g - \omega^2 \rho_r h)}{2} (w^p)^2 \right\} d\mathbf{x} \\ = \int \int_{\Delta_p} i\omega \rho \left(\tilde{\mathbf{H}}_\omega \phi^i + i\omega \tilde{\mathbf{H}}_\omega \tilde{\mathbf{G}}_\omega \sum_q w^q \right) \delta w^p d\mathbf{x}. \end{aligned} \quad (24)$$

The right-hand side transforms as follows

$$\delta \int \int_{\Delta_p} i\omega \rho \tilde{\mathbf{H}}_\omega \phi^i w^p d\mathbf{x} - \delta \int \int_{\Delta_p} \omega^2 \rho \left(\tilde{\mathbf{H}}_\omega \tilde{\mathbf{G}}_\omega \sum_{q \neq p} w^q \right) w^p d\mathbf{x} - \delta \int \int_{\Delta_p} \frac{\omega^2 \rho}{2} \left(\tilde{\mathbf{H}}_\omega \tilde{\mathbf{G}}_\omega w^p \right) w^p d\mathbf{x}.$$

The variational equation depends on not just the displacement of the p th plate, but through the operators $\tilde{\mathbf{H}}_\omega$ and $\tilde{\mathbf{G}}_\omega$ on all the other plate displacements as well.

As before the variational equation will be solve by the Rayleigh-Ritz method. The displacement is expand as

$$w(\mathbf{x}) = \sum_{p=1}^n \sum_{i=0}^I c_i^p w_i^p(\mathbf{x}) \quad (25)$$

where the $w_i^p(\mathbf{x})$ are chosen such that,

$$w_i^p(\mathbf{x}) = 0, \text{ if } \mathbf{x} \notin \Delta_p \quad (26)$$

(this condition is to simplify the derivation of the equations and means that the expansion modes are limited to the each plate). This gives the following equation for the coefficients in the expansion, written in vector form as \vec{c} ,

$$\left(\tilde{\mathbf{K}} + \tilde{\mathbf{M}} + \tilde{\mathbf{L}} \right) \vec{c} = \vec{f} \quad (27)$$

where

$$\begin{aligned} \tilde{\mathbf{K}} &= \begin{pmatrix} \mathbf{K}^1 & 0 & \cdots & 0 \\ 0 & \mathbf{K}^2 & & \vdots \\ \vdots & & & \\ 0 & 0 & \cdots & \mathbf{K}^n \end{pmatrix} \\ \tilde{\mathbf{M}} &= \begin{pmatrix} \mathbf{M}^1 & 0 & \cdots & 0 \\ 0 & \mathbf{M}^2 & & \vdots \\ \vdots & & & \\ 0 & 0 & \cdots & \mathbf{M}^n \end{pmatrix} \\ \tilde{\mathbf{L}} &= \begin{pmatrix} \frac{1}{2} \left(\mathbf{L}^{11} + (\mathbf{L}^{11})^T \right) & \mathbf{L}^{12} & \cdots & \mathbf{L}^{1n} \\ \mathbf{L}^{21} & \frac{1}{2} \left(\mathbf{L}^{11} + (\mathbf{L}^{11})^T \right) & & \vdots \\ \vdots & & & \\ \mathbf{L}^{n1} & & \cdots & \frac{1}{2} \left(\mathbf{L}^{nn} + (\mathbf{L}^{nn})^T \right) \end{pmatrix} \end{aligned}$$

and

$$\vec{f} = \begin{pmatrix} \vec{f}^1 \\ \vec{f}^2 \\ \vdots \\ \vec{f}^n \end{pmatrix}.$$

The elements of the matrix \mathbf{K}^p are

$$k_{ij}^p = \int \int_{\Delta_p} D \left\{ \frac{\partial^2 w_i^p}{\partial x^2} \frac{\partial^2 w_j^p}{\partial x^2} + \frac{\partial^2 w_i^p}{\partial y^2} \frac{\partial^2 w_j^p}{\partial y^2} \right. \\ \left. + \nu \left(\frac{\partial^2 w_i^p}{\partial x^2} \frac{\partial^2 w_j^p}{\partial y^2} + \frac{\partial^2 w_i^p}{\partial y^2} \frac{\partial^2 w_j^p}{\partial x^2} \right) + 2(1 - \nu) \frac{\partial^2 w_i^p}{\partial x \partial y} \frac{\partial^2 w_j^p}{\partial x \partial y} \right\} d\mathbf{x},$$

the elements of the matrix \mathbf{M}^p are

$$m_{ij}^p = \int \int_{\Delta_p} (\rho g - \omega^2 \rho_r h) w_i^p(\mathbf{x}) w_j^p(\mathbf{x}) d\mathbf{x},$$

the elements of the matrix \mathbf{L}^{pq} are

$$l_{ij}^{pq} = \int \int_{\Delta_p} \rho \omega^2 w_i^p(\mathbf{x}) (\mathbf{H}_\omega \mathbf{G}_\omega w_j^q(\mathbf{x})) d\mathbf{x},$$

and the elements of the vector \vec{f}^p are

$$f_i^p = \int \int_{\Delta_p} i\omega \rho (\mathbf{H}_\omega \phi^i(\mathbf{x})) w_i^p(\mathbf{x}) d\mathbf{x}.$$

Again the results are presented in non-dimensional form. Figures 10 and 11 show the displacement for four 2 by 2 thin plates arranged adjacent in a square. The wavelength is two, $\beta = 0.005$ and $\gamma = 0.01$ for each plate and the displacement is shown for the times $t = 0$, $t = \frac{T}{8}$, $t = \frac{T}{4}$, and $t = \frac{3T}{8}$ where T is the wave period. The wave angle is $\theta = 0$ (Figure 2) and $\theta = \frac{\pi}{2}$ (Figure 9) respectively. Each plate is expanded in polynomial basis functions as before and we use $m = n = 20$ and $p = q = 10$ for each plate.

11 Conclusion

A variational equation for a thin plate floating on the water surface subject to plane wave forcing has been derived. This variational equation has the advantage of capturing the optimum method for solving for the motion of a thin plate (the variational equation) with the optimum method of solving for the wave forcing of a floating body (the Green's function integral equation method). A solution to the variational equation has been presented using simple polynomials as the basis functions, these basis function being chosen primarily for the simplicity of the resulting equations. The extension of the solution to the case of variable plate properties and to multiple floating plates was also presented and the solutions were calculated for some example problems. The variational equation could be extended straightforwardly to the case of more complex elastic bodies, for example three dimensional elastic bodies, thick plates or multiple plates hinged in some fashion.

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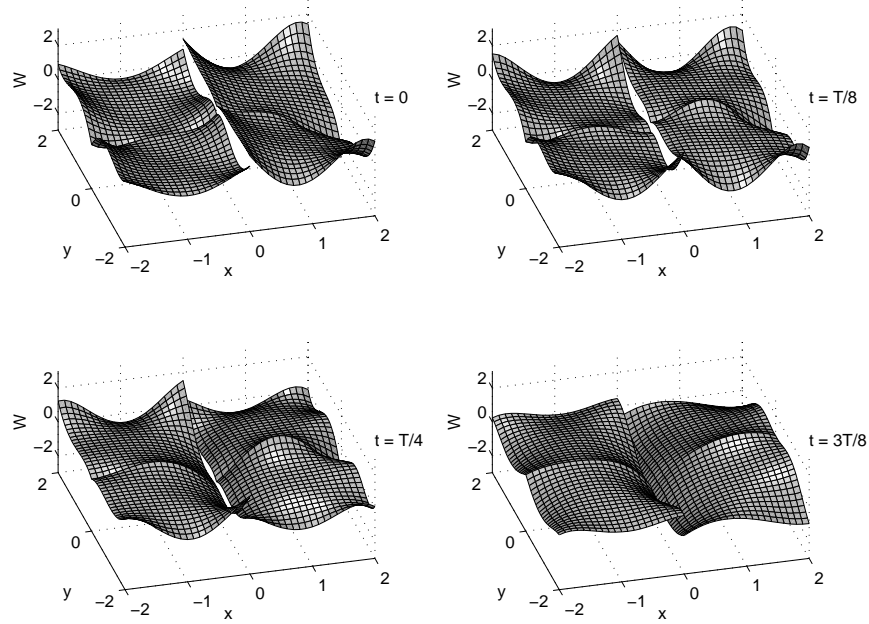


Figure 10: The motion of four 2 by 2 plates for the times shown where T is the wave period. $\alpha = \pi$ (the wavelength $\lambda = 2$), $\beta = 0.005$, $\gamma = 0.01$, and $\theta = 0$

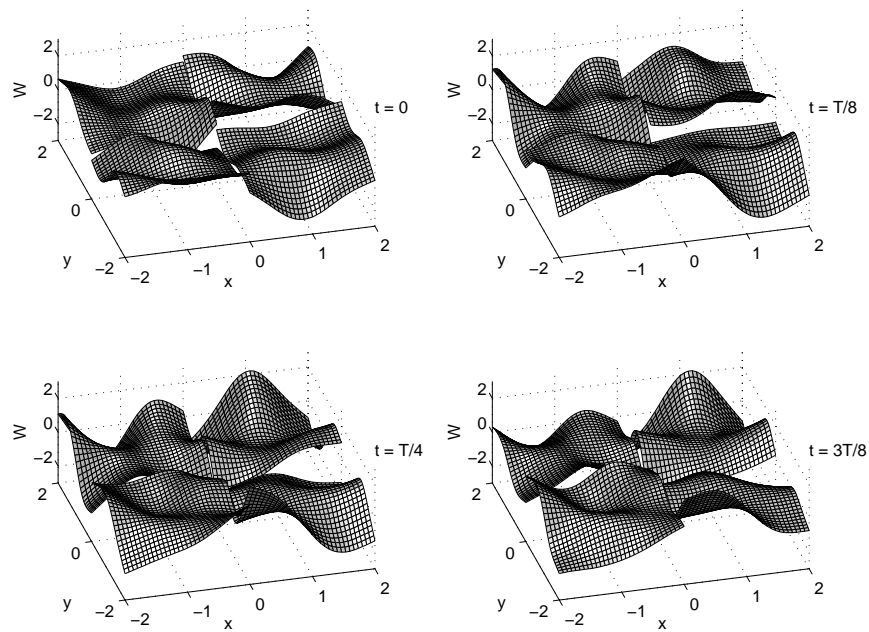


Figure 11: The motion of four 2 by 2 plates for the times shown where T is the wave period. The wavelength $\lambda = 2$, $\beta = 0.005$, $\gamma = 0.01$, and $\theta = \frac{\pi}{3}$

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