GENERALISED KNOT GROUPS OF CONNECT SUMS OF TORUS KNOTS

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Abstract

Kelly (1990) and Wada (1992) independently identified and defined the generalised knot groups (G_n). The square (SK) and granny (GK) knots are two of the most well-known distinct knots with isomorphic knot groups. Tuffley (2007) confirmed Lin and Nelson's (2006) conjecture that $G_n(SK)$ and $G_n(GK)$ were non-isomorphic by showing that they have different numbers of homomorphisms to suitably chosen finite groups. He concluded that more information about K is carried by generalised knot groups than by fundamental knot groups. Soon after, Nelson and Neumann (2008) showed that the 2-generalised knot group distinguishes knots up to reflection. The goal of this study is to show that for certain square and granny knot analogues, the difference can be detected by counting homomorphisms into a suitable finite groups. This study extends Tuffley's work to analogues $SK_{a,b}$ and $GK_{a,b}$ of the square and granny knots formed from connect sums of (a, b)-torus knots. It gives further information about the generalised knot groups of the connect sum of two torus knots, which differ only in their orientation.

Acknowledgments

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I also acknowledge the financial support from the Saudi government, which has allowed me to undertake this research in New Zealand. New Zealand has opened my eyes to the person I can be, and I look forward to advancing my studies further here in the future.

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Chapter 1

Introduction

Knot theory is the study of mathematical knots, which are quite different from regular knots tied in string. In mathematics, a piece of rope with a knot tied and the two ends glued together makes a mathematician's knot. The two ends of a knot must be joined together so that it cannot be undone, because in knot theory, the knot must be a continuous loop to allow the knot to be deformed into new knot formations. The essential question in knot theory is how we prove whether two knots are the same or are different. This is difficult because knots can be drawn in many different ways by stretching the knot, deforming the knot or twisting the knot; it is still the same knot, but it could look extremely different. The primary concern of knot theory is to classify and distinguish between knots.

There is a family of methods to distinguish knots which are called invariants. A knot invariant is a mapping from the set of all knots to some other set, where the value of the map does not change, even if the picture of the knot does. In other words, a knot invariant is a test to distinguish one type of knot from another. One such invariant is the knot group (the fundamental group). This knot group is a powerful invariant; however, it has limitations. For example, it cannot distinguish between the square knot (SK) and granny knot (GK), two well-known distinct knots with isomorphic knot groups. Currently, no practical invariant is known that will always succeed at distinguishing all knots.

Kelly [13] and Wada [30] independently introduced several methods to define generalised knot groups (G_n). These may be defined by a Wirtinger presentation (see Section 3.2.1), with conjugation by the generator a_i replaced by conjugation by a_i^n for all i. A second definition of G_n shows that $\pi_1(K)$, which is the fundamental group of a knot (K), is a subgroup of $G_n(K)$ for each n. Generalised knot groups (G_n) were the focus of three key studies, starting with the conjecture of Lin and Nelson [16] who used granny and square knots to test G_n, continuing to Tuffley [29], who proved Lin and Nelson's conjecture. In the third study, Nelson and Neumann [21] took a topological view to show that generalised knot groups distinguish knots up to reflection.

1.1 Previous work

Generalised knot groups were first presented and introduced independently by Wada [30] and Kelly [13]. Wada's work searched for homomorphisms of the braid group B_n into $Aut(F_n)$, where F_n is the free group on n generators, while Kelly was working with knot racks or quandles and the Wirtinger presentation.

Lin and Nelson [16] challenged themselves to show that $G_n(GK)$ and $G_n(SK)$ are not isomorphic for all $n > 1$. They introduced generalised knot groups through the language of quandles. They suspected generalised knot groups held additional information about knot types that are not present in the usual fundamental group. They used the square knot (SK) and the granny knot (GK) as a test case. As $\pi_1(SK)$ and $\pi_1(GK)$ are isomorphic, they wanted to check whether $G_n(GK)$ and $G_n(SK)$ are isomorphic to each other for $n > 1$ by using a computer program to calculate the number of homomorphisms of $G_n(GK)$ and $G_n(SK)$ into chosen finite groups H. They did not succeed in detecting a difference, but, nevertheless, conjectured that $G_n(GK)$ and $G_n(SK)$ are not isomorphic to each other for all $n \geq 2$.

Tuffley [29] proved Lin and Nelson's conjecture. He also used the square and granny knots for his testing case, distinguishing $G_n(GK)$ and $G_n(SK)$ for all $n \geq 2$ by counting the number of homomorphisms into a chosen finite group. He proved the following theorem:

Theorem 1.1 (Tuffley [29]). *For each $n \geq 2$, there is a finite group H such that*

$$|Hom(G_n(GK), H)| < |Hom(G_n(SK), H)|.$$

Consequently, $G_n(GK)$ and $G_n(SK)$ are not isomorphic to each other for all $n \geq 2$.

His target groups were wreath products over $PSL(2, p)$,

$$H_p^{q,r} = D_{q,r} \wr PSL(2,p) = (\mathbb{Z}_q^{r-1} \rtimes \mathbb{Z}_r)^{P^1(\mathbb{F}_p)} \rtimes PSL(2,p),$$

where p, q and r are distinct primes and $PSL(2, p)$ acts on $P^1(\mathbb{F}_p)$, the projective line over the p-element field. This showed that the isomorphism types of the generalised knot groups carry more information than the isomorphism types of the fundamental group itself.

Nelson and Neumann [21] showed that, in fact, the 2-generalised knot group distinguishes knots up to reflection. They proved the following theorem from a topological perspective:

Theorem 1.2 (Nelson and Neumann [21]). *The 2-generalized knot group $G_2(K)$ determines the knot up to reflection.*

In addition, they sketched the proof that the result also holds for $G_n(K)$ for $n > 2$.

1.2 Goals of this study

The main goal of this thesis is to show that the difference between generalised knot groups for the analogues $SK_{a,b}$ and $GK_{a,b}$ of the square and granny knots made from (a, b)-torus knots can be detected by counting homomorphisms into suitably finite groups. This shows that, in principle, the difference between the groups can be detected algorithmically. We will use the wreath product of two different groups of the form $D_{p,q;\theta}$, which are described in Chapter 5, as the target groups and generalise Tuffley's strategy to provide the result.

The structure of this thesis is as follows. In Chapter 2, we will give a detailed overview of knot theory and provide basic definitions and examples of knots. Then, we will outline the fundamental group and knot groups in Chapter 3. In Chapter 4, we will define the generalised knot groups and look at the G_n for the granny knot and square knot and their analogues. Next, we will provide a concise summary of some of the concepts of group theory which are needed and related to the study in terms of dihedral groups, semidirect products and wreath products in Chapter 5. Then, we will describe the strategy and show our main result of this study which extends Tuffley's technique by using different target groups in Chapter 6. Finally, we will provide a summary of the thesis in Chapter 7.

Chapter 2

Fundamental Concepts of Knot Theory

In this chapter, we will review some basic concepts that have to do with knot theory. We will give a number of definitions and examples that will be used throughout the study. Knot theory is the mathematical study of knots. Mathematical knots are different from normal knots that we know and use in our daily lives. In mathematical knots, the two ends are joined together, so the knot cannot be undone. Knots can be defined in many ways; however, simply, a knot is a tangled piece of string in $\mathbb{R}^3$ which is a closed loop.

The definitions, theorems and lemmas in this chapter and chapter 3 are taken from: Adams [1], Alexander [2], Burde and Zieschang [5], Cromwell [7], Hatcher [9], Kauffman [11], Kawauchi [12], Lickorish [15], Livingston [17], Manturov [18], Murasugi [20], Reidemeister [22], Rolfsen [23] and Stillwell [28]. Most of the pictures in this chapter were created with KnotPlot [26].

2.1 Knots

Definition 2.1. A **knot** is a non-self-intersecting closed curve in $\mathbb{R}^3$.

In other words, a knot is a subset of $\mathbb{R}^3$ that is homeomorphic to a circle. Some examples of knots are shown in Figure 2.1. We now describe when two knots are considered to be the same.

Definition 2.2. A continuous function $F : \mathbb{X} \times I \to \mathbb{X}$ is said to be an **isotopy** if $F(\cdot, t)$ is a homeomorphism for all t.

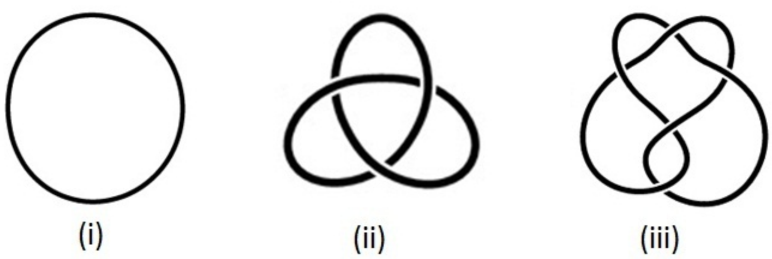

Figure 2.1: Examples of knots. (i) The unknot (trivial knot). (ii) The trefoil knot. (iii) The 5_2 knot.

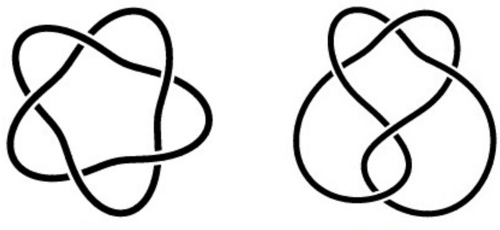

Figure 2.2: (i) and (ii) refer to 5_1 and 5_2, the first and second of the two 5-crossing knots. However, this ordering is completely arbitrary, being inherited from the earliest tables compiled.

Two knots K_1 and K_2 in $\mathbb{R}^3$ are **equivalent** if there is an isotopy of $\mathbb{R}^3$ that carries K_1 to K_2. Such an isotopy is said to be an **ambient isotopy** from K_1 to K_2.

Definition 2.3. An **unknot** (trivial knot) is a knot that is equivalent to the knot in Figure 2.1(i).

Some knots have specific names such as (i) the unknot and (ii) the trefoil knot (Figure 2.1); however, most are referred to by their numbers in the standard tables (Figure 2.2).

2.2 Knot diagrams

In order to study knots, it is useful to draw pictures of them called projections. The problem with projections is that they do not show over and under crossings, and, therefore, do not contain enough information to reconstruct the original knot. To solve this problem, we can use what is called a knot diagram. We will discuss these concepts in more detail.

Definition 2.4. A **knot diagram** is a way to picture and manipulate knots by projecting the knot on to a plane, so that the projection has no triple points or tangencies (Figure 2.3). Such a projection is called a **regular projection**. During the projection, crossings (double points) may occur. At each crossing, it is important to distinguish the over strand from the under strand by breaking the strand that goes underneath, as shown in Figure 2.4. So by giving this information, the original knot can be reconstructed. Knots can be represented by many possible diagrams, as seen in Figure 2.5. That leads to the essential question in knot theory: **when do two diagrams represent the same knot?**

A regular diagram of a knot has a finite number of crossing points, which is called the crossing number of the diagram. Now, it is relevant to define the crossing number of knots.

Definition 2.5. The **crossing number** $c(K)$ of a knot K is the minimal number of crossings in any diagram of that knot. A minimal diagram of K is one with $c(K)$ crossings.

A knot diagram without any crossing is a trivial knot (unknot). There are no knots with crossing number one or two. The two trefoils are the only knots with crossing number three. However, there are tables of knots up to about 16 crossings.

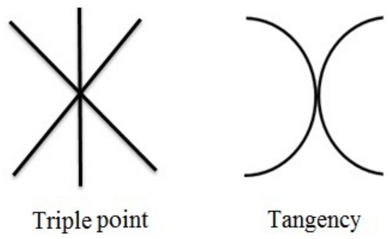

Figure 2.3: This picture shows a triple point and a tangency. Any projection of a knot can be perturbed to eliminate triple points and tangencies, so every knot has a regular projection.

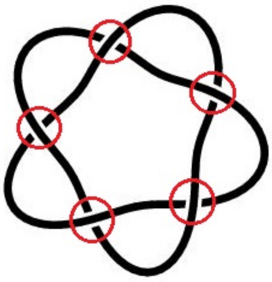

Figure 2.4: A knot diagram of 5_1, where the crossings are circled.

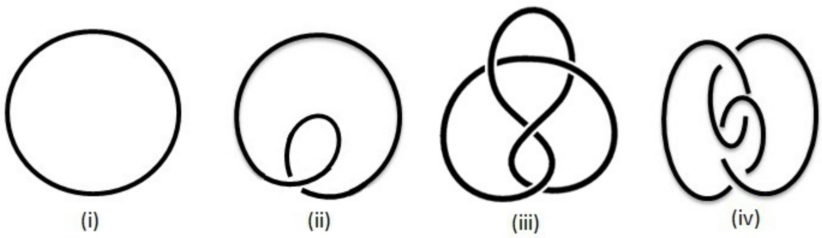

Figure 2.5: (i) and (ii) are two different diagrams of the unknot. (iii) and (iv) present different diagrams of the figure eight knot.

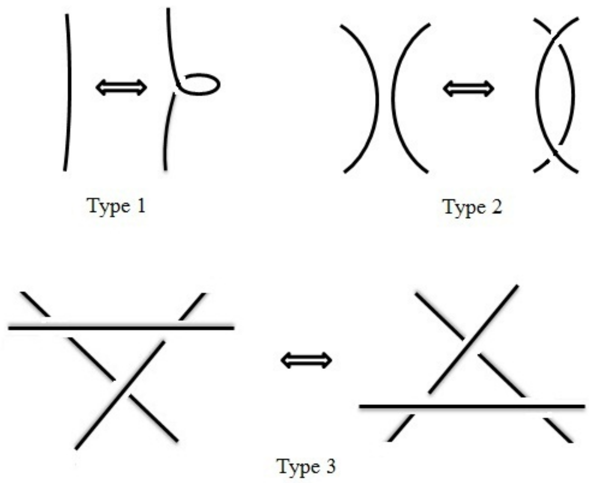

Figure 2.6: Pictures of the three types of Reidemeister moves.

2.3 The Reidemeister moves

Kurt Reidemeister (1927) developed what are known as the Reidemeister moves. A Reidemeister move is an operation that can be performed on the diagram of a knot. Simply, there are three types of Reidemeister moves that can be used to modify a knot diagram. Figure 2.6 shows the typical pictures that are used to define the Reidemeister moves: type (1) is the addition or removal of a twist; type (2) is the moving of a strand through a tangency; and type (3) is the moving of a strand through a triple point. These three moves can change one diagram of a knot to another. The three moves can be used together or just one or two moves can be used depending on what is needed.

The following theorem shows that these three moves plus planar isotopy are enough to transform any diagram of a knot into any other. See Reidemeister [22] for a proof.

Theorem 2.1 (Reidemeister [22])**.** *Two knot diagrams K_1 and K_2 represent equivalent knots if and only if there is a sequence of Reidemeister moves taking K_1 to K_2.*

2.4 Oriented knots and mirror images

It is important to consider how a knot is oriented. In this section, we will discuss orientation and mirror images of knots.

Definition 2.6. An **oriented knot** is one with a chosen direction of circulation along the string.

Orientation may be specified by putting an arrow somewhere on the knot to designate the direction (Figure 2.7). Oriented knots are important for many applications in knot theory. For example, giving knots orientations allows us to define the sum of oriented knots by taking the connect sum of the knots as oriented manifolds. Also, the orientation of the knot can be used to determine whether a crossing is a negative or positive crossing, and this can be easily identified using the right hand rule. We can point the thumb of our right hand along the over-strand in

Figure 2.7: An oriented trefoil knot.

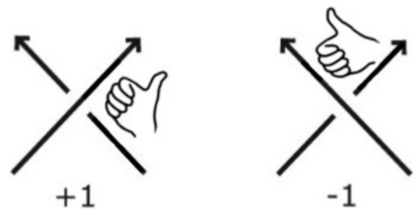

Figure 2.8: Positive and negative crossings. They are also known as right- and left- handed crossing respectively.

the direction of the arrow and curl our fingers into a letter-C shape. If our fingers curl in the direction of the under-strand, then it is a positive or right-handed crossing; if our fingers curl against the direction, then it is a negative or left-handed crossing. Figure 2.8 pictures how can we establish the sign of a crossing. Note that reversing the orientation of a knot reverses the orientation of both the over and under strand, and this means that the sign of a crossing does not depend on the choice of orientation.

We will define the reverse and mirror image of a knot. These involve changing the orientation of the knot, or of the ambient space $\mathbb{R}^3$.

Definition 2.7. The **reverse** rK of an oriented knot K has the same projection, but with the opposite orientation.

Before defining the mirror image, we will define crossing changes. As we can see in Figure 2.9 to change the crossing, we have to change the over-strand to be the under-strand.

Definition 2.8. The **mirror image** $\bar{K}$ of the knot K is obtained by reflecting it in a plane in $\mathbb{R}^3$. If the mirror is placed behind the knot and parallel to the plane of the projection, then

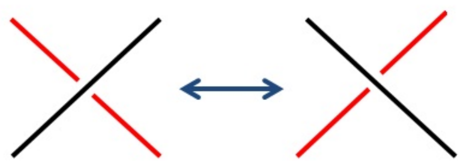

Figure 2.9: A crossing change. The portion of the knot outside the picture is unchanged.

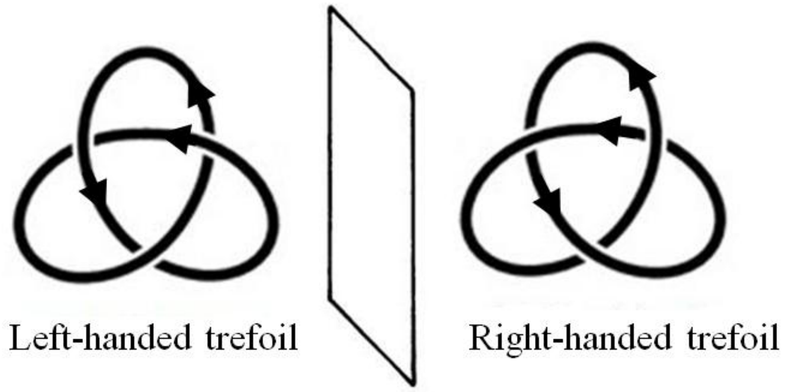

Figure 2.10: The trefoil and its mirror image. The rectangle represents the plane of reflection.

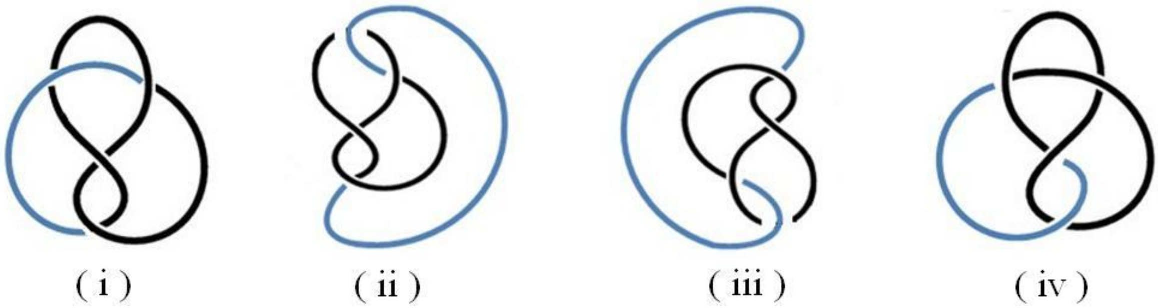

Figure 2.11: This picture shows that the figure eight knot can be deformed into its mirror image. (i) The figure eight knot with one arc coloured blue. (ii) Rearrange the blue arc. (iii) Rotate the figure eight 180°. (iv) Smooth deformation to get the mirror image of the figure eight.

the effect is to change all the crossings. The **inverse** $r\bar{K}$ is the composition of the reversal and mirror-image.

It is now relevant to define **knot chirality (handedness)**.

Definition 2.9. A knot is said to be **chiral** (handed) if it is not equivalent to its mirror image. An example is the trefoil and its mirror image in Figure 2.10. The right-handed trefoil can not be continuously deformed into the left-handed one, see Dehn [8] or Stillwell [28, pages 218-225]. Therefore, the trefoil knot is called **topologically chiral**. However, if a knot is equivalent to its mirror image, it is called **amphichiral** or **achiral**. The figure eight knot is an example of an amphichiral knot (Figure 2.11).

2.5 Links

In this section, we will briefly introduce links.

Definition 2.10. A **link** is a collection of disjoint closed curves in $\mathbb{R}^3$; each curve is called a **component** of the link.

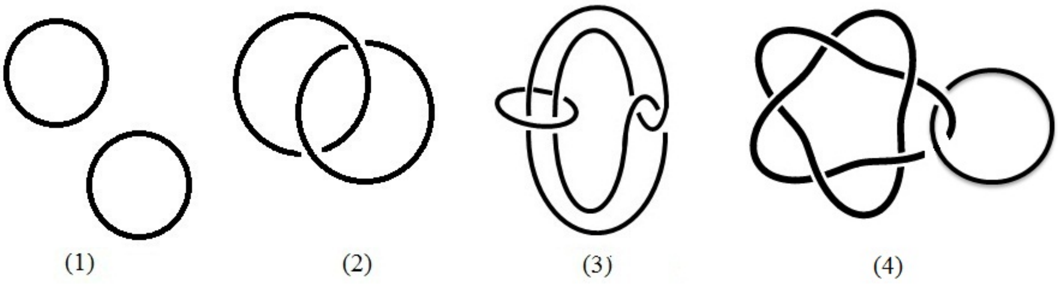

Figure 2.12: This figure shows different links. (1) is said to be *split*, because there is a plane in $\mathbb{R}^3$ that separates the components. (2) is a two component link, where both of the components are unknots. It is called the Hopf link. (3) is also a two component link of unknots. It is called the Whitehead link. (4) is a two component link, where one component is 5_1 and one is an unknot.

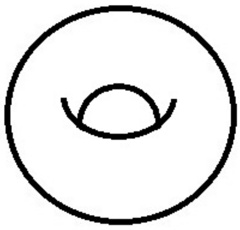

Figure 2.13: A torus.

A knot is a one-component link. It is important to notice that individual components may or may not be unknots. Some examples of links are shown in Figure 2.12.

2.6 Torus knots

There are many types of knots, for example, torus knots, satellite knots, hyperbolic knots and almost alternating knots. However, torus knots are being used for the purpose of this study, so will be the type outlined in more detail. The main reason for looking at torus knots is the flexibility of the presentation of their knot groups.

The concept of torus can be defined geometrically and represented topologically. From a geometric perspective, a torus is a surface of revolution generated by revolving a circle in $\mathbb{R}^3$ about an axis coplanar with the circle. Most of the time it is assumed that the axis does not touch the circle, resulting in a ring shape called a ring torus. Topologically, a ring torus is homeomorphic to the product of two circles: $S^1 \times S^1$.

Definition 2.11. A **torus** is a topological product of two circles (Figure 2.13).

Simply, a torus is a surface similar to that of a doughnut. It is a connected surface or shape. A torus can be constructed from a rectangle by gluing both pairs of opposite edges together with no twists (Figure 2.14).

Definition 2.12. A **torus knot** is a knot that lies on the surface of an unknotted torus in $\mathbb{R}^3$.

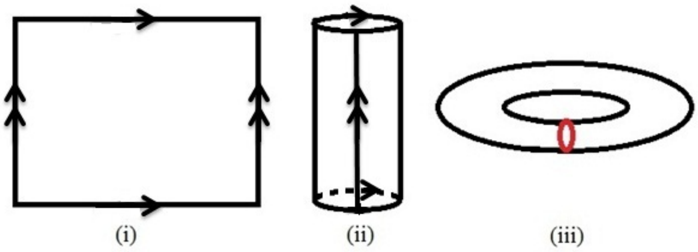

Figure 2.14: (i) is a rectangle. (ii) is after gluing the opposite vertical edges together. (iii) is the final shape of a torus after gluing the horizontal edges together with no twists.

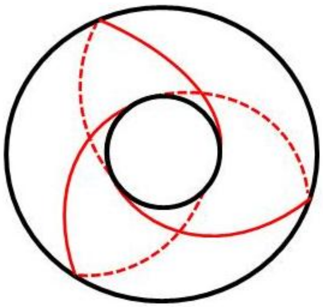

Figure 2.15: An example of a (2, 3)-torus knot (trefoil).

Every torus knot can be described with two integers (p, q), where p and q are co-prime. A (p, q)-torus knot is a simple closed curve on the torus that winds p times around the first S^1 factor and q times around the second. Let us denote this torus knot by $T_{p,q}$. The simplest nontrivial example of two co-prime integers is $p = 2, q = 3$ or $p = 3, q = 2$. In both of these cases, we obtain the trefoil knot which can be written as (2, 3)-torus knot or $T_{2,3}$ (Figure 2.15). If p and q are not relatively prime, then there will be a torus link with more than one component.

The following remark shows some properties of torus knots.

Remark 2.1. A $(p, -q)$-torus knot is the mirror image of a (p, q)-torus knot. A $(-p, -q)$-torus knot is equivalent to a (p, q)-torus knot. Also, a (p, q)-torus knot is equivalent to a (q, p)-torus knot.

2.7 The connect sum

The connect sum or knot sum is an operation that connects two knots together to make a single knot.

Definition 2.13. Two oriented knots K_1 and K_2 can be connected by breaking the two knots and joining them with straight bars, so that the orientation of both knots is retained (Figure 2.16). The operation is called a **connect sum** and is denoted by $\#$.

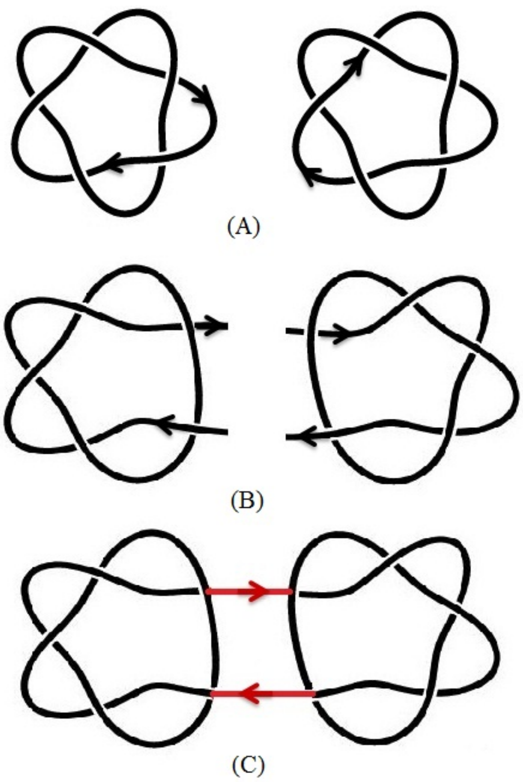

Figure 2.16: The knots in (A) are (2,5)-torus knots (the knot 5_1) that have the same handedness. (B) shows where the knots are broken to connect them together and (C) is a picture of the connect sum of the two knots $5_1\#5_1$. This knot is analogous to the granny knot, which is the connect sum $3_1\#3_1$ of two trefoils with the same handedness.

Remark 2.2. A knot can be connected with another copy of itself or its inverse or any other knot.

Chapter 3

Knot Groups

3.1 The fundamental group

In order to develop a knot group, it is important to first understand key definitions related to the fundamental group. The information in this section is taken from Hatcher [9].

Definition 3.1. Let X and Y be topological spaces. Let f and f' be continuous maps from X into Y. Then f is **homotopic** to f' if there is a continuous map $F : X \times [0,1] \to Y$ such that

$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = f'(x).$$

F is called a **homotopy** between f and f'.

In other words, two functions are homotopic if one can be continuously deformed into the other.

Definition 3.2. Let $x_0, x_1 \in X$. A **path** from x_0 to x_1 is a continuous map $f : [0,1] \to X$ such that

$$f(0) = x_0, \quad f(1) = x_1.$$

A **loop** is a path that begins and ends at the same point.

Definition 3.3. Let f and f' be paths with fixed end points x_0 and x_1. The paths f and f' are **homotopic rel endpoints** if there is a continuous map $F : [0,1] \times [0,1] \to X$ such that

$$\begin{aligned} F(s,0) &= f(s), \qquad & F(0,t) &= x_0, \\ F(s,1) &= f'(s), \qquad & F(1,t) &= x_1. \end{aligned}$$

Figure 3.1 shows a homotopy from f to f'. Homotopy rel endpoints is an equivalence relation on paths. We will write $[f]$ for the homotopy class of f.

Next, we will define a product operation on paths.

Definition 3.4. Let f be a path in X from x_0 to x_1 and let g be a path in X from x_1 to x_2. Then we can define the product $f \cdot g$ as the following;

$$(f \cdot g)(s) = \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ g(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

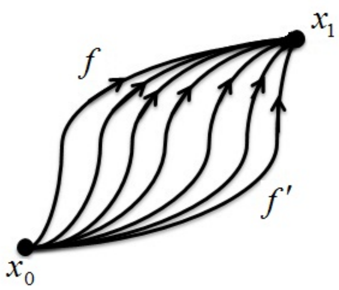

Figure 3.1: A picture of a homotopy from f to f'.

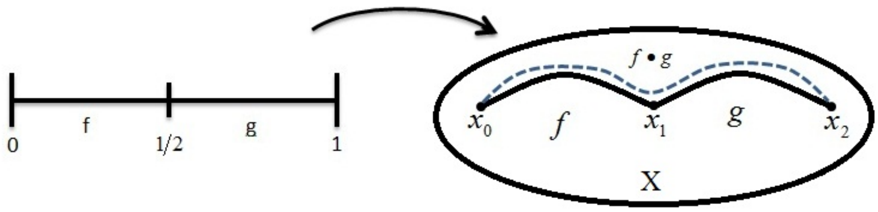

Figure 3.2: The product $f \cdot g$ of two paths f and g.

Geometrically, $f \cdot g$ is the concatenation of the paths f and g: in other words, it is the path from x_0 to x_2 that follows f from x_0 to x_1, then g from x_1 to x_2 (Figure 3.2).

The product operation on paths is a well-defined operation on homotopy classes, defined by:

$$[f] \cdot [g] = [f \cdot g]. \tag{3.1}$$

Theorem 3.1 (Hatcher [9]). *Let X be a topological space and let x_0 be a point of X. The set of all homotopy classes of loops f based at x_0 is a group under the operation* (3.1)*. It is called **the fundamental group** of X relative to x_0 and is denoted by $\pi_1(X, x_0)$.*

The following example illustrates the fact that the fundamental group of the circle S^1 is isomorphic to the additive group of the integers $\mathbb{Z}$.

Example 3.1. Let

$$S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\} = \{z \in \mathbb{C} : \|z\| = 1\},$$

and use 1 as the base point.

Every loop is equivalent to a unique path of the form $a_n(t) = e^{2n\pi it}$, for some $n \in \mathbb{Z}$. There are three cases;

$n > 0$ ($a_n(t)$ is a path that goes n times around S^1 in the counterclockwise direction and returns to 1),

$n < 0$ ($a_n(t)$ is a path that goes n times around S^1 in the clockwise direction and returns to 1),

$n = 0$ ($a_n(t)$ is the path that stays at 1).

Let $n > 0$ and $m > 0$. The path a_n travels n times around S^1 and a_m travels m times around S^1. Then $a_n \cdot a_m$ is a path that loops n times around S^1 and then loops m more times. This is equivalent to the loop a_{n+m}. The equation $[a_n \cdot a_m] = [a_{n+m}]$ holds even when n, m are not necessarily positive, and therefore $\pi_1(S^1, 1) \cong \mathbb{Z}$.

Although the base point is required for the definition, the following theorem shows that the resulting group typically does not depend on the choice of base point:

Theorem 3.2 (Hatcher [9, page 28]). *Let X be path-connected and let x_0 and x_1 be two points of X. Then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.*

The group $\pi_1(X, x_0)$ can be written as $\pi_1(X)$, if X is path-connected and $\pi_1(X, x_0)$ is up to isomorphism, independent of the choice of x_0.

Generally, if a space is path-connected and has a trivial fundamental group, then it is called **simply connected**.

We conclude this section with the following theorem which shows the fundamental group of a product of two path-connected spaces.

Theorem 3.3 (Hatcher [9, page 35]). *Let X and Y be path-connected. Then*

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y).$$

The torus is an example of the theorem above. The fundamental group of the torus is just the direct product of the fundamental group of the circle with itself:

$$\pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2.$$

3.2 Knot groups

In this section we will define the knot group and explain the presentation of the knot group.

Definition 3.5. Let K be a knot in $\mathbb{R}^3$. The fundamental group of the complement of K is called the **knot group** of K, and is denoted by $\pi_1(K)$.

Two elements a and b of the knot group $\pi_1(K)$ are paths in the complement of the knot K that begin and end at a fixed base point x_0, up to deformation, and multiplied by concatenation. We can give the formula, but the picture is more important (Figure 3.3).

3.2.1 The presentation of knot groups

Wilhelm Wirtinger (1925) proved that for every diagram of a knot there is a presentation of the knot group that has one generator for each arc of the knot diagram and a relation for each crossing. In an oriented knot, the generators start at a base point above the plane of the knot diagram, go around the specified arc in the positive direction, and return to the base point. The

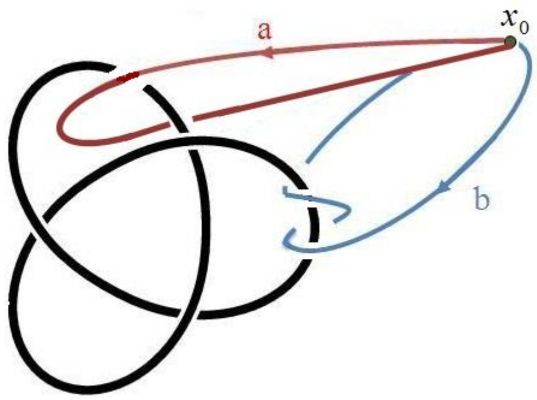

Figure 3.3: A product ab of two elements a and b of the knot group can be calculated by passing first along a and then along b.

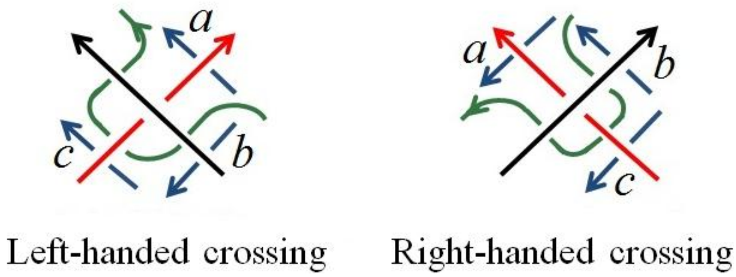

Figure 3.4: The Wirtinger relations at left- and right-handed crossings.

positive direction is determined by the right hand rule: if we hold the strand in our right hand with our thumb pointing in the direction of the orientation of the knot, then our fingers curl in the positive direction. We get the following relation at each crossing, as shown in Figure 3.4:

$$a = bcb^{-1} \qquad \text{(left-handed crossing)},$$
$$a = b^{-1}cb \qquad \text{(right-handed crossing)}.$$

This gives a presentation which is called **the Wirtinger presentation** of the knot group.

A diagram with $m \geq 2$ crossings will have m arcs. Then G will have the following presentation:

$$G = \langle g_1, g_2, g_3, \ldots, g_m | r_1, r_2, r_3, \ldots, r_m \rangle,$$

where G is the quotient of the free group on the Wirtinger generators $g_1, g_2, g_3, \ldots, g_m$ by the smallest normal subgroup generated by the Wirtinger relations $r_1, r_2, r_3, \ldots, r_m$. We can see that these relations hold; the significance of Wirtinger's work is that no other relations are required. This is the content of following theorem, which is proved using van Kampen's theorem; see Rolfsen [23] for the proof.

Theorem 3.4. *Let K be a knot. Then the fundamental group $\pi_1(K)$ is generated by $g_1, \ldots, g_m$ and has presentation*

$$\pi_1(K) = \langle g_1, \ldots, g_m | r_1, \ldots, r_m \rangle.$$

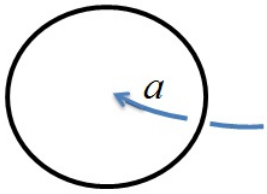

Figure 3.5: Generator for the knot group of K_0.

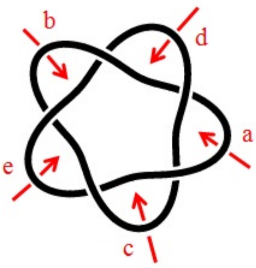

Figure 3.6: Generators for the knot group of a (2, 5)-torus knot.

Moreover, any one relation r_i can be eliminated and the presentation of $\pi_1(K)$ still holds.

Abelianising the Wirtinger presentation makes all generators equal, leading to the following theorem.

Theorem 3.5. *The abelianisation of $\pi_1(K)$ is isomorphic to $\mathbb{Z}$.*

The following example shows that the fundamental group of the trivial knot is isomorphic to $\mathbb{Z}$.

Example 3.2. The trivial knot K_0 has a diagram with one arc and no crossings. It has one generator a and no relations (Figure 3.5). So the knot group is given by

$$\pi_1(K_0) \cong \langle a \rangle \cong \mathbb{Z}.$$

We will show the Wirtinger presentation of (5, 2)-torus knot in the next example.

Example 3.3. Consider the (5, 2)-torus knot (Figure 3.6). Let a, b, c, d, e be the group generators. Then the Wirtinger presentation gives

$$\langle a, b, c, d, e \mid a = dbd^{-1}, b = ece^{-1}, c = ada^{-1}, d = beb^{-1}, e = cac^{-1} \rangle.$$

We can use the relations $a = dbd^{-1}, b = ece^{-1}, c = ada^{-1}, d = beb^{-1}$ and $e = cac^{-1}$ to write

$$\begin{aligned} a = dbd^{-1} &\Leftrightarrow ad = db, \\ c = ada^{-1} &\Leftrightarrow ca = ad, \\ b = ece^{-1} &\Leftrightarrow be = ec, \\ d = beb^{-1} &\Leftrightarrow db = be, \\ e = cac^{-1} &\Leftrightarrow ec = ca. \end{aligned}$$

We can rewrite the presentation as follows

$$\langle a, b, c, d, e \mid ad = db = be = ec = ca \rangle.$$

Then we can rewrite $ca = ec$ by using the above expressions for c, a and d to get

$$\begin{aligned}
(ada^{-1})a &= e(ada^{-1}) && \text{(substitute } c \text{ with } ada^{-1}\text{)} \\
ad &= eada^{-1} && \\
ada &= ead && \text{(multiply the right side by } a\text{)} \\
(dbd^{-1})d(dbd^{-1}) &= e(dbd^{-1})d && \text{(substitute } a \text{ with } dbd^{-1}\text{)} \\
dbdbd^{-1} &= edb && \\
dbdb &= edbd && \text{(multiply the right side by } d\text{)} \\
(beb^{-1})b(beb^{-1})b &= e(beb^{-1})b(beb^{-1}) && \text{(substitute } d \text{ with } beb^{-1}\text{)} \\
bebe &= ebebeb^{-1}. &&
\end{aligned}$$

Then we rearrange $bebe = ebebeb^{-1}$ to get $bebeb = ebebe$, and we also can express the other relations in terms of b and e in a similar way. Each relation leads to the same or a trivial relation in terms of b and e, and we find

$$\pi_1(T_{2,5}) \cong \langle b, e | bebeb = ebebe \rangle.$$

We can go from $\langle b, e | bebeb = ebebe \rangle$ to $\langle x, y | x^2 = y^5 \rangle$ by writing $x = bebeb$ and $y = be$, because b and e can also be written in terms of x and y as $b = y^{-2}x$ and $e = b^{-1}y = x^{-1}y^2y = x^{-1}y^3$.

In the following section, we will derive the presentation for the torus knot $\pi_1(T_{a,b}) \cong \langle x, y | x^a = y^b \rangle$ by using the van Kampen theorem.

3.3 Torus knots and the van Kampen theorem

In this section, we will compute the knot group of a torus knot by using the van Kampen theorem (see Hatcher [9, page 43]). We will briefly give an overview of the van Kampen theorem.

The van Kampen theorem expresses the fundamental group of a path-connected union $A_1 \cup A_2$ in terms of the fundamental groups of the path-connected spaces A_1, A_2 and $A_1 \cap A_2$. As seen in Figure 3.7, every element of $A_1 \cup A_2$ can be expressed as a product of loops in A_1 and A_2. Consequently, $\pi_1(A_1 \cup A_2)$ is generated by the generators of $\pi_1(A_1)$ and $\pi_1(A_2)$, and the relations for these groups hold in $\pi_1(A_1 \cup A_2)$ also. We get an additional relation for each generator of $\pi_1(A_1 \cap A_2)$ reflecting the fact that it can be expressed as a word in the generators of each of $\pi_1(A_1)$ and $\pi_1(A_2)$.

The method for computing the fundamental group of the torus knot is as follows. Let $T_{a,b}$ be an (a, b)-torus knot on a standard torus T in S^3. The torus T divides S^3 into two regions R_1 and R_2, such that the closures $\bar{R}_1$ and $\bar{R}_2$ are solid tori. If we set A_i to be $\bar{R}_i - T_{a,b}$, then $S^3 \backslash T_{a,b}$ can be written as $A_1 \cup A_2$.

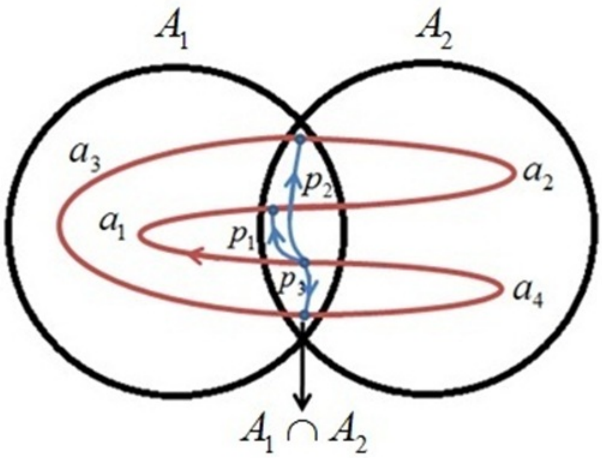

Figure 3.7: Given a loop a, break it into paths $a_1 \dots a_k$ entirely in A_1 or in A_2; join the base point to the end of a_i by a path p_i:

$$a = a_1a_2a_3a_4 \backsimeq \underbrace{a_1p_1^{-1}}_{\text{loop in } A_1} \underbrace{p_1a_2p_2^{-1}}_{\text{loop in } A_2} \underbrace{p_2a_3p_3^{-1}}_{\text{loop in } A_1} \underbrace{p_3a_4}_{\text{loop in } A_2},$$

which means that any loop in $A_1 \cup A_2$ can be expressed as a product of loops in A_1 and A_2.

The intersection of T with the complement of $T_{a,b}$ is a ribbon which turns around the torus a times in the direction of one factor and b times in the direction of the other. Let C denote this ribbon and let $x_0 \in A_1 \cap A_2$ be a base point. The space A_1 is essentially a solid torus, so its fundamental group is $\pi_1(A_1) \cong \mathbb{Z}$ with the single generator x, corresponding to a path that makes one circle around the hole in the torus. The space A_2 is also essentially a solid torus and its fundamental group is $\pi_1(A_2) \cong \mathbb{Z}$, with the single generator y corresponding to a path that passes once through the hole. The intersection $A_1 \cap A_2 = C$ is an annulus; the fundamental group is also $\pi_1(C) \cong \mathbb{Z}$ with one generator c representing the path which travels once around the annulus. From A_1 and A_2's points of view, c passes a and b times respectively around the torus, so it represents x^a and y^b respectively. Then, $\pi(A_1 \cup A_2) = \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} = \langle x, y | x^a = y^b \rangle$.

We summarise the above result in the following theorem:

Theorem 3.6. *The fundamental group π_1 of $T_{a,b}$ has the presentation $\langle x, y | x^a = y^b \rangle$.*

Figure 3.8 shows a (2,3)-torus knot embedded on the surface of a torus. According to the above discussion the fundamental group is $\pi_1(T_{2,3}) = \langle x, y | x^2 = y^3 \rangle$.

We will define $G_{a,b}$ as follows.

Definition 3.6. The group $G_{a,b}$ is the knot group of $(T_{a,b})$ generated by x and y, with one relation $x^a = y^b$.

The following theorem states that the knot groups of torus knots have centres. In fact, torus knots are the only knots whose knot groups have a nontrivial centre (see Stillwell [28]).

Theorem 3.7. *The centre of the knot group of the (a, b)-torus knot is infinite cyclic generated by $x^a = y^b$.*

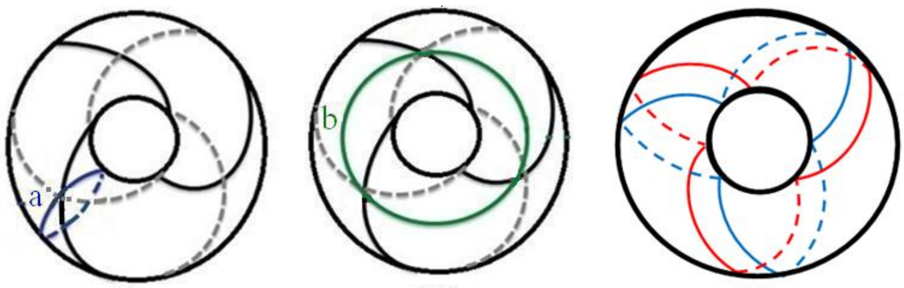

Figure 3.8: A (2,3)-torus knot on the surface of a torus. (i) and (ii) show that this knot cycles 2 times around the blue curve of the torus and 3 times around the green curve. (iii) shows the generator c with red colour which turns once around the annulus.

3.4 Expressing the van Kampen generators in terms of the Wirtinger generators

In the previous section we found that $\pi_1(T_{a,b}) = \langle x, y | x^a = y^b \rangle$. In this section, we will find expressions for the generators x and y in terms of Wirtinger generators.

Figure 3.9 shows a (3,5)-torus knot with Wirtinger generators $\omega_i, i = 0, 1, 2, 3, 4$. Any other Wirtinger generator can be expressed in terms of these; for instance $\omega' = \omega_0^{-1}\omega_4\omega_0$. Let x be the green core curve of the inner torus, and let y be the red core curve of the outer torus which can move around the diagram as shown. Then x and y can be written as

$$x = \omega_0\omega_1\omega_2\omega_3\omega_4,$$
$$y = \omega_0\omega_1\omega_2 = \omega_2\omega_3\omega_4 = \omega_4\omega_0\omega_1 = \omega_1\omega_2\omega_3 = \omega_3\omega_4\omega_0.$$

Then we can check that $x^3 = y^5$ as follows:

$$\begin{aligned}(\omega_0\omega_1\omega_2\omega_3\omega_4)^3 &= (\omega_0\omega_1\omega_2\omega_3\omega_4)(\omega_0\omega_1\omega_2\omega_3\omega_4)(\omega_0\omega_1\omega_2\omega_3\omega_4)\\ &= (\omega_0\omega_1\omega_2)(\omega_3\omega_4\omega_0)(\omega_1\omega_2\omega_3)(\omega_4\omega_0\omega_1)(\omega_2\omega_3\omega_4)\\ &= y^5.\end{aligned}$$

So x and y are generators giving the presentation

$$G_{3,5} = \langle x, y | x^3 = y^5 \rangle,$$

which is the knot group of $T_{3,5}$.

In general, for a, b relatively prime satisfying $0 < a < b$, $G_{a,b}$ has Wirtinger generators $\omega_i, i = 0, \dots, b-1$, and we may write x and y as follows

$$x = \omega_0 \cdots \omega_{b-1},$$
$$y = \omega_0 \cdots \omega_{a-1} = \omega_1 \cdots \omega_a = \cdots = \omega_{b-1}\omega_0 \cdots \omega_{a-2}.$$

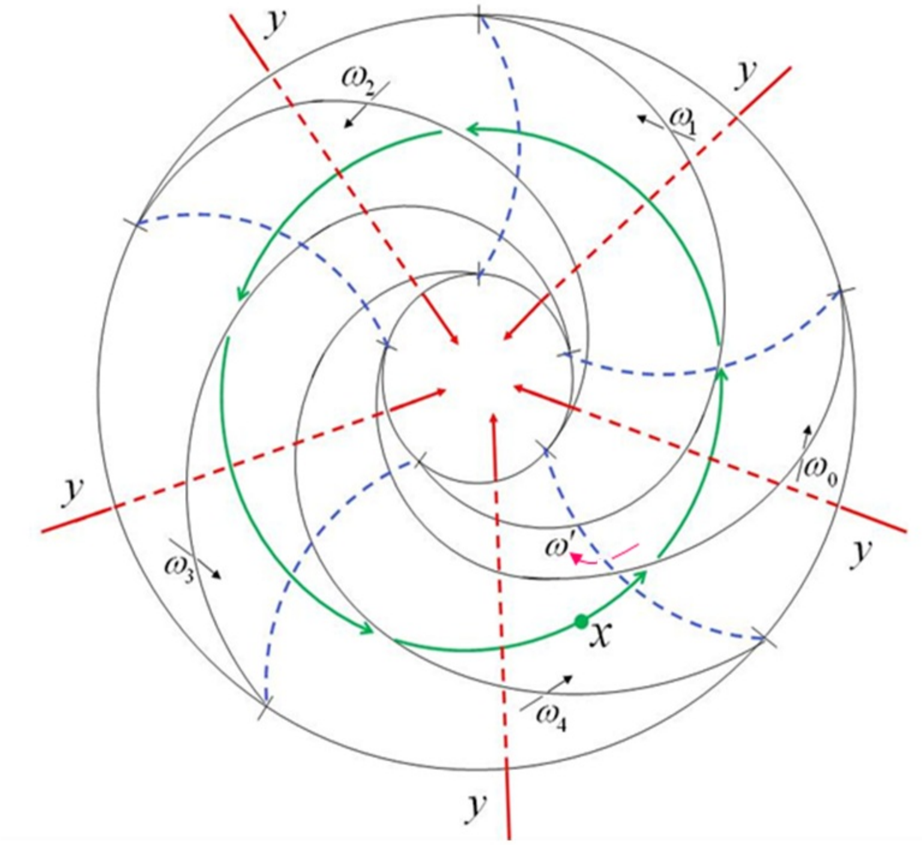

Figure 3.9: Generators for a $(3, 5)$-torus knot.

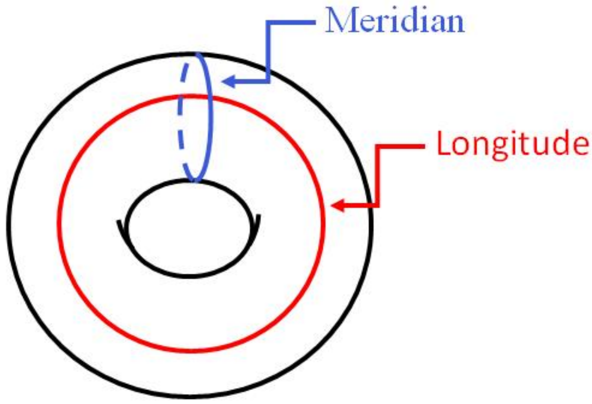

Figure 3.10: The meridian and longitude of a solid torus.

3.5 The meridian and the longitude

In this section, we will define the meridian and longitude of a solid torus and give an explanation for the meridian and the longitude of a knot group. Also, we will briefly explain the preferred longitude of a knot. Finally, we will identify the meridian of $G_{a,b}$ in terms of x and y.

Let K be a knot, then thicken the knot to a solid torus $L = S^1 \times D^2$, where S^1 is a circle and D^2 is a disc (Figure 3.10). A simple closed curve on ∂D that bounds a disc in L is called a **meridian** (μ),

$$\mu = \{x_0\} \times \partial D.$$

In other words, it is a simple closed curve encircling the width of L. A **longitude** (λ) is a simple closed curve that runs the entire length of the knot which follows the same orientation as the knot,

$$\lambda = S^1 \times \{x_0\},$$

where $x_0 \in \partial D$.

Any two meridians are equivalent, because they can be slid around the knot so that they coincide. However, two longitudes are not necessarily equivalent, because they can twist different numbers of times around the knot. The **preferred longitude** is the one that represents zero in the abelianization of the $\pi_1(K)$. Figure 3.11 shows the meridian and a longitude of a trefoil knot.

The meridian and a longitude are oriented simple closed curves that intersect at one point. We can think of the meridian μ and a longitude λ as elements of the knot group $\pi_1(K)$ by choosing a path from the base point to the point of intersection. They commute with each other in every knot group, because they give a subgroup isomorphic to $\pi(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$.

Remark 3.1. Any of the generators in the Wirtinger presentation can be used as a meridian of the knot. Consequently, Theorem 3.5 shows that the meridian generates the abelianization of $\pi_1(K)$; and it can be seen from the Wirtinger presentation that the meridian normally generates $\pi_1(K)$, since the remaining generators are conjugates of the chosen meridian.

The following example shows the way to find the meridian and the longitude of $G_{2,2k+1}$.

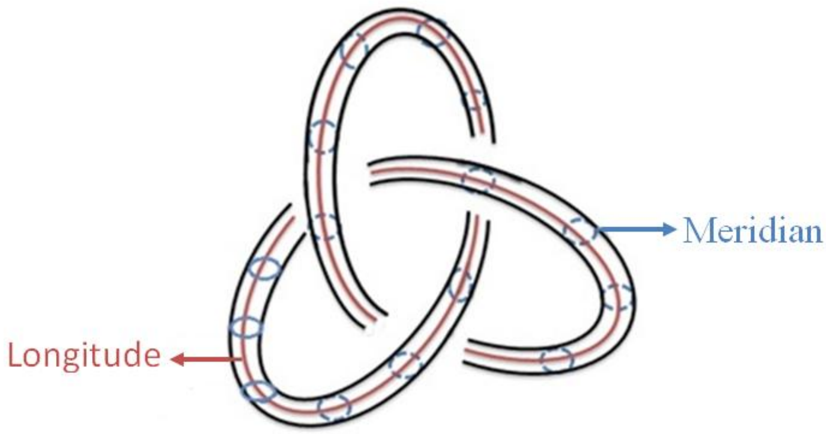

Figure 3.11: The meridian and longitude of a trefoil knot.

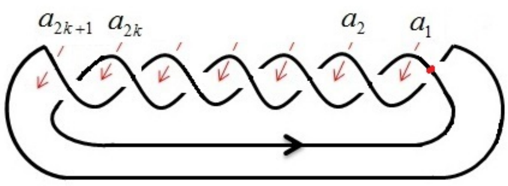

Figure 3.12: Calculation of the longitude which begins and ends at the point a_1.

Example 3.4. Let $G_{2,2k+1} = \langle a_1, \ldots, a_{2k+1} | a_i a_{i+1} = a_{i+1} a_{i+2}, \forall i \rangle = \langle x, y | x^2 = y^{2k+1} \rangle$. Let

$$x = a_1 a_2 a_3 \cdots a_{2k} a_{2k+1},$$
$$y = a_1 a_2 = a_2 a_3 = \cdots = a_{2k} a_{2k+1} = a_{2k+1} a_1.$$

Then

$$x = a_1 (a_2 a_3) \cdots (a_{2k} a_{2k+1}),$$
$$x = a_1 y^k \Longrightarrow a_1 = x y^{-k},$$

therefore $\mu = xy^{-k} = a_1$. As we can see in Figure 3.12, the longitude can be taken as

$$\lambda = a_2 a_4 \cdots a_{2k} a_1 a_3 \cdots a_{2k+1}.$$

The preferred longitude is $\lambda a_1^{-(2k+1)}$ which gives 0 in the abelianisation of the knot. Another choice of longitude is obtained by starting at a_1 and following the knot, twisting once around the knot in the positive direction before each of the crossings, then

$$\lambda' = a_1 a_2 \cdots a_{2k+1} a_1 a_2 \cdots a_{2k+1}.$$

Therefore, the longitude may be taken to be $\lambda' = x^2 = y^{2k+1}$.

The following lemma shows the meridian of $G_{a,b}$, where a and b are integers and co-prime.

Lemma 3.1. *Let $G_{a,b} = \langle x, y | x^a = y^b \rangle$ such that $0 < a < b$, and let c and d be elements of $\mathbb{N}$ such that $bc - ad = 1$. Then $\mu = x^c y^{-d} = \omega_0$ is a meridian of $T_{a,b}$, with corresponding longitude $\lambda = x^a = y^b$.*

Note that we always can find a solution with $c, d > 0$.

Proof. Let

$$x = \underbrace{\omega_0\omega_1\cdots\omega_{b-1}}_{b} \qquad \text{(which is of length } b\text{)},$$

$$y = \underbrace{\omega_0\omega_1\cdots\omega_{a-1}}_{a} \qquad \text{(which is of length } a\text{)},$$

such that $x^a = y^b$. There are b representations of y of length a such that

$$y = \underbrace{\omega_0\cdots\omega_{a-1}}_{a} = \underbrace{\omega_1\cdots\omega_a}_{a} = \underbrace{\omega_2\cdots\omega_{a+1}}_{a} = \cdots = \underbrace{\omega_{b-1}\omega_0\cdots\omega_{a-2}}_{a}.$$

Choose c and $d \in \mathbb{N}$, such that $bc - ad = 1$. Then

$$\begin{aligned} x^c y^{-d} &= (\omega_0\omega_1\cdots\omega_{b-1})^c y^{-d} \\ &= \underbrace{(\omega_0\omega_1\cdots\omega_{b-1})\cdots(\omega_0\omega_1\cdots\omega_{b-1})}_{c \quad \text{times}} y^{-d} \qquad = (\omega_0 \underbrace{\omega_1\cdots\omega_{b-1}\cdots\omega_0\omega_1\cdots\omega_{b-1}}_{ad \quad \text{factors}})y^{-d}. \end{aligned}$$

By using the fact that $y = \omega_i\omega_{i+1}\cdots\omega_{i+a-1}$ for any i, we see that the product $\omega_1\cdots\omega_{b-1}$ with ad factors is equal to y^d. Then

$$x^c y^{-d} = \omega_0 y^d y^{-d} = \omega_0,$$

which is a meridian of the knot. Then we may take $\mu = x^c y^{-d} = \omega_0$, as required. □

We end this section with an example that illustrates the above lemma.

Example 3.5. Let $G_{5,7} = \langle x, y | x^5 = y^7 \rangle$. Then

$$x = \omega_0\omega_1\omega_2\omega_3\omega_4\omega_5\omega_6,$$

and

$$\begin{aligned} y &= \omega_1\omega_2\omega_3\omega_4\omega_5 = \omega_2\omega_3\omega_4\omega_5\omega_6 = \omega_3\omega_4\omega_5\omega_6\omega_0 = \omega_4\omega_5\omega_6\omega_0\omega_1 \\ &= \omega_5\omega_6\omega_0\omega_1\omega_2 = \omega_6\omega_0\omega_1\omega_2\omega_3 = \omega_0\omega_1\omega_2\omega_3\omega_4, \end{aligned}$$

such that $x^5 = y^7$. We need to solve the equation $bc - ad = 1$, one solution is $c = 3$ and $d = 4$. Then

$$\begin{aligned} x^3 y^{-4} &= (\omega_0\omega_1\omega_2\omega_3\omega_4\omega_5\omega_6)(\omega_0\omega_1\omega_2\omega_3\omega_4\omega_5\omega_6)(\omega_0\omega_1\omega_2\omega_3\omega_4\omega_5\omega_6)(\omega_1\omega_2\omega_3\omega_4\omega_5)^{-4}, \\ &= (\omega_0 \underbrace{\omega_1\omega_2\omega_3\omega_4\omega_5\omega_6\omega_0\omega_1\omega_2\omega_3\omega_4\omega_5\omega_6\omega_0\omega_1\omega_2\omega_3\omega_4\omega_5\omega_6}_{y^4})(\omega_1\omega_2\omega_3\omega_4\omega_5)^{-4}. \end{aligned}$$

Then we get

$$\begin{aligned} x^3 y^{-4} &= \omega_0 y^4 y^{-4} = \omega_0, \\ \mu &= x^3 y^{-4} = \omega_0. \end{aligned}$$

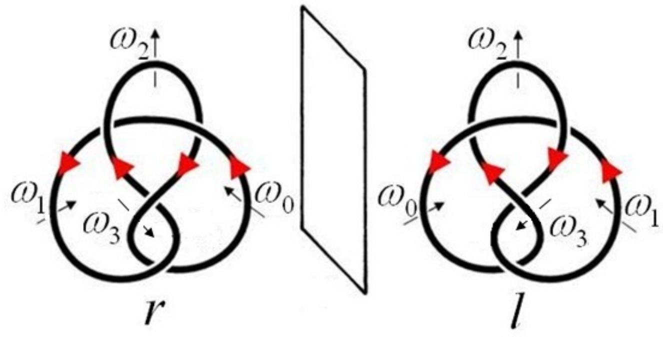

Figure 3.13: Figure eight knot and its mirror image. The rectangle represents the plane of reflection.

3.5.1 The effect of reflection on the meridian and the longitude

In this section we will show by example that a knot and its mirror image have the same knot group and meridian, but inverted longitudes.

The figure eight knot has the following presentation

$$G = \langle \omega_0, \omega_3 | \omega_0\omega_3\omega_0^{-1}\omega_3\omega_0 = \omega_3\omega_0\omega_3^{-1}\omega_0\omega_3 \rangle.$$

Figure 3.13 shows the figure eight knot and its mirror image. By reflecting in the given plane, we can choose generators for the two knot groups, so that we get identical presentations. However, to do so we must orient the two knots as shown, so that the generators go around each knot in the positive direction. The 0-framed longitude corresponding to ω_0 of the figure eight is represented by $\lambda_r = \omega_3^{-1}\omega_0\omega_1^{-1}\omega_2$, and the longitude of the mirror image of the figure eight is represented by $\lambda_l = \omega_2^{-1}\omega_1\omega_0^{-1}\omega_3$. Note that $\omega_3^{-1}\omega_0\omega_1^{-1}\omega_2$ and $(\omega_3^{-1}\omega_0\omega_1^{-1}\omega_2)^{-1} = \omega_2^{-1}\omega_1\omega_0^{-1}\omega_3$ represent longitudes corresponding to ω_0. Both the figure eight and its mirror image have the same meridian $\mu = \omega_0$.

3.5.2 The meridian and the longitude of a connect sum

In this section we will demonstrate the meridian and the longitude of a connect sum in the following remark.

Remark 3.2. Let K_1 and K_2 be knots, and let $K_1\#K_2$ be the connect sum of these knots. Let μ_1 and λ_1 be the meridian and the longitude of K_1 and let μ_2 and λ_2 be the meridian and the longitude of K_2. The meridian (μ) of the connect sum will be exactly the same as the meridian of K_1 and K_2, therefore $\mu = \mu_1 = \mu_2$. The longitude of the connect sum (λ) will be the multiplication of the longitude of K_1 and that of K_2, $\lambda = \lambda_1 \times \lambda_2$. Figure 3.14 shows the meridian and the longitude of the connect sum of two trefoil knots.

The following theorem calculates the presentation of the knot group of the connect sum of two knots, and the meridian and a longitude.

Theorem 3.8. *Let*

$$\pi_1(K_1) = \langle g_1, g_2, \ldots, g_l | r_1, r_2, \ldots, r_m \rangle, \quad \textit{and} \quad \pi_1(K_2) = \langle h_1, h_2, \ldots, h_s | q_1, q_2, \ldots, q_t \rangle,$$

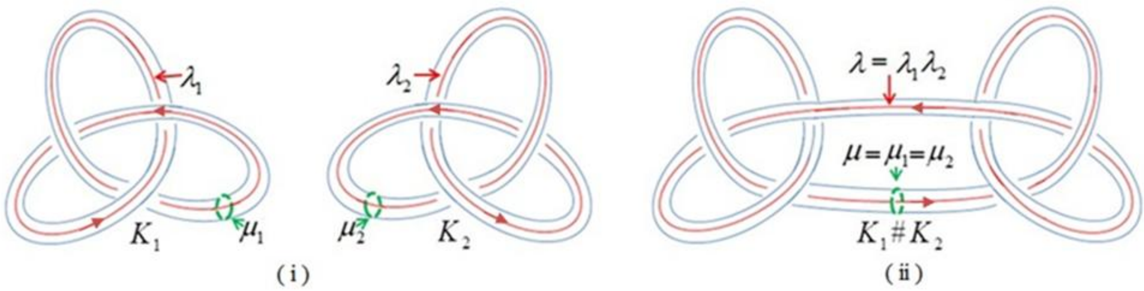

Figure 3.14: (i) shows the meridian and the longitude of two trefoil knots K_1 and K_2. (ii) shows the meridian and the longitude of the connect sum of K_1 and K_2.

with meridians μ_1 *and* μ_2 *and longitudes* λ_1 *and* λ_2 *respectively. Then*

$$\pi_1(K_1\#K_2) = \langle g_1, g_2, \ldots, g_l, h_1, h_2, \ldots, h_s | r_1, r_2, \ldots, r_m, q_1, q_2, \ldots, q_t, \mu_1 = \mu_2\rangle,$$

with meridian $\mu = \mu_1 = \mu_2$ *and longitude* $\lambda = \lambda_1\lambda_2$.

Chapter 4

Generalised Knot Groups

In this chapter, we will define and explain the generalised knot groups G_n, which were independently defined by Wada [30] and Kelly [13]. Also, we will define the granny knot (GK) and the square knot (SK) and look at π_1 for both of them. Then we will obtain G_n for the granny knot analogues ($GK_{a,b}$) and the square analogues ($SK_{a,b}$) from $\pi_1(GK_{a,b})$ and $\pi_1(SK_{a,b})$ respectively.

4.1 The group G_n

Assuming K is a knot, the generalised knot groups $G_n(K)$ can be defined in many different ways. We may define them by the Wirtinger presentation. The presentation of $G_n(K)$ has a generator for each arc of the knot diagram; and a relation $a_k = a_i^n a_j a_i^{-n}$ at a left- handed crossing and $a_k = a_i^{-n} a_j a_i^n$ at a right-handed crossing (Figure 4.1). Observe that if we substitute n with 1, then we will get $G_1 = \pi_1$. The group G_n can be shown to be a knot invariant using the Reidemeister moves.

If we set $A_i = a_i^n$ for all i, then we recover the Wirtinger presentation of π_1, so we can think of a_i as an nth root of the corresponding meridian. We can show that a_i commutes with the corresponding longitude by using the Wirtinger presentation, as shown below for the figure eight knot in Example 4.1. Moreover, the G_n crossing relations can be recovered from this and the π_1 relations, leading to the following presentation: if $\pi_1(K)$ has presentation

$$\pi_1(K) = \langle g_1, g_2, g_3, \ldots, g_p | r_1, r_2, r_3, \ldots, r_q \rangle,$$

and μ and λ are words in the generators representing the meridian and the longitude, then $G_n(K)$ has a presentation

$$G_n(K) = \langle g_1, g_2, g_3, \ldots, g_p, \nu | r_1, r_2, r_3, \ldots, r_q, \nu^n = \mu, \lambda\nu = \nu\lambda \rangle.$$

Therefore, $G_n(K)$ is obtained from π_1 by adding a new generator ν which is an nth root of the meridian and commutes with the corresponding longitude. This presentation can also be obtained directly using the van Kampen theorem from a topological definition of G_n (see Wada [30]).

The following example gives an explanation for the above discussion.

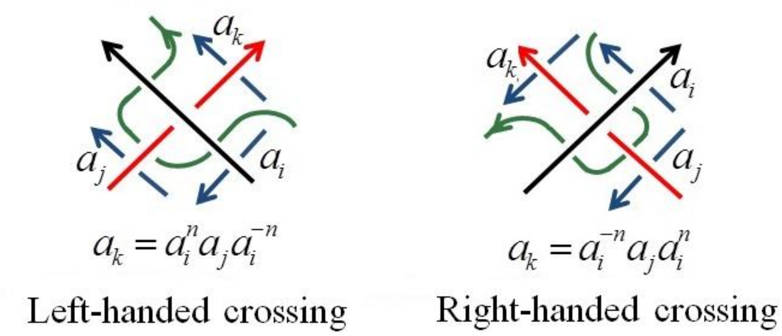

Figure 4.1: A G_n relation at a crossing of a diagram

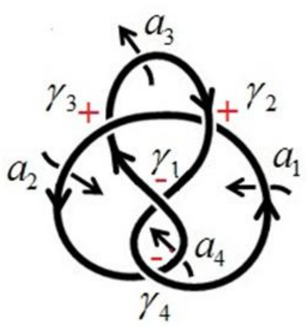

Figure 4.2: The figure eight knot, with labelled generators and relations.

Example 4.1. Let K be a figure eight knot (Figure 4.2). Then the generalised knot group $G_n(K)$ has generators a_i and relations γ_i for $i = 1, 2, 3, 4$ as follows:

$$\begin{aligned}
\gamma_1 &: a_1 = a_4^{-n} a_3 a_4^n \\
\gamma_2 &: a_2 = a_3^n a_1 a_3^{-n} \\
\gamma_3 &: a_3 = a_2^n a_4 a_2^{-n} \\
\gamma_4 &: a_4 = a_1^{-n} a_2 a_1^n.
\end{aligned}$$

By setting $A_i = a_i^n$ for $i = 1, 2, 3, 4$, then we can see that the A_i satisfy the knot group $\pi_1(K)$ relations:

$$\begin{aligned}
\gamma_1' &: A_1 = A_4^{-1} A_3 A_4 \\
\gamma_2' &: A_2 = A_3 A_1 A_3^{-1} \\
\gamma_3' &: A_3 = A_2 A_4 A_2^{-1} \\
\gamma_4' &: A_4 = A_1^{-1} A_2 A_1.
\end{aligned}$$

Now, we will show that the generator a_1 commutes with the longitude $\lambda = A_3^{-1} A_1 A_2^{-1} A_4$ as follows:

$$\begin{aligned} a_1\lambda &= a_1A_3^{-1}A_1A_2^{-1}A_4 \\ &= A_3^{-1}a_2A_1A_2^{-1}A_4 \\ &= A_3^{-1}A_1a_4A_2^{-1}A_4 \\ &= A_3^{-1}A_1A_2^{-1}a_3A_4 \\ &= A_3^{-1}A_1A_2^{-1}A_4a_1 \\ &= \lambda a_1. \end{aligned}$$

The generalised knot group relations can be written in terms of the generator a_1 and the generators of $\pi_1(K)$ as we now show:

$$\begin{aligned} \gamma_2 &: a_2 = A_3a_1A_3^{-1} \\ \gamma_3 &: a_3 = A_2A_1^{-1}A_3a_1A_3^{-1}A_1A_2^{-1} \\ \gamma_4 &: a_4 = A_1^{-1}A_3a_1A_3^{-1}A_1. \end{aligned}$$

Therefore the Wirtinger presentation of the group G_n can be written as follows:

$$G_n = \langle A_1, A_2, A_3, A_4, \nu | \gamma_1', \gamma_2', \gamma_3', \gamma_4', \nu^n = A_1, \nu A_3^{-1}A_1A_2^{-1}A_4 = A_3^{-1}A_1A_2^{-1}A_4\nu \rangle,$$

where ν is an nth root of the meridian A_1 and commutes with the corresponding longitude $\lambda = A_3^{-1}A_1A_2^{-1}A_4$.

4.2 The granny and square knots

The granny and square knots are both connected sums of trefoil knots. So we can define them as follows:

Definition 4.1. The **granny knot** is the connect sum of two left- or two right-handed trefoil knots, while the connect sum of two knots with one left- and one right-handed trefoil knot is the **square knot** (see Figure 4.3).

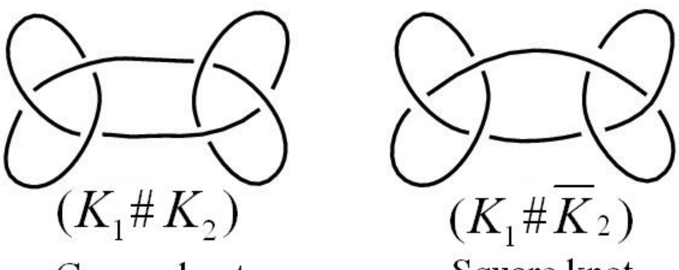

Figure 4.3: The granny and square knots.

Let K_1 and K_2 be two right-handed trefoils. If $\pi_1(K_1) \cong \langle x, y | x^3 = y^2 \rangle$, with $\mu_1 = yx^{-1}$ and $\lambda_1 = x^3$; and $\pi_1(K_2) \cong \langle w, z | w^3 = z^2 \rangle$, with $\mu_2 = zw^{-1}$ and $\lambda_2 = w^3$, then π_1 for the granny knot is

$$\pi_1(GK) \cong \langle x, y, w, z | x^3 = y^2, w^3 = z^2, yx^{-1} = zw^{-1} \rangle,$$

which has meridian and longitude $\mu = \mu_1 = \mu_2 = yx^{-1} = zw^{-1}, \lambda = x^3w^3$. Similarly, let K_1 and $\bar{K}_2$ be right- and left-handed trefoils respectively. If $\pi_1(K_1) \cong \langle x, y | x^3 = y^2 \rangle$, with $\mu_1 = yx^{-1}$ and $\lambda_1 = x^3$; and $\pi_1(\bar{K}_2) \cong \langle w, z | w^3 = z^2 \rangle$, with $\mu_2 = zw^{-1}$ and $\lambda_2 = w^{-3}$, then π_1 for the square knot is

$$\pi_1(SK) \cong \langle x, y, w, z | x^3 = y^2, w^3 = z^2, yx^{-1} = zw^{-1} \rangle,$$

which has meridian and longitude $\mu = \mu_1 = \mu_2 = yx^{-1} = zw^{-1}, \lambda = x^3w^{-3}$. Thus, $\pi_1(GK)$ and $\pi_1(SK)$ are both isomorphic to

$$\langle x, y, w, z | x^3 = y^2, w^3 = z^2, yx^{-1} = zw^{-1} \rangle.$$

Remark 4.1. The fundamental groups for the granny and square knots are both isomorphic:

$$\pi_1(K_1 \# K_2) \cong \pi_1(K_1 \# \bar{K}_2).$$

In other words, they have the same π_1.

The presentation for G_n for GK and SK is obtained from π_1 by adding a new generator ν which is an nth root of μ as follows:

$$G_n(GK) \cong \langle x, y, w, z, \nu | x^3 = y^2, w^3 = z^2, \nu^n = yx^{-1} = zw^{-1}, x^3w^3\nu = \nu x^3w^3 \rangle,$$
$$G_n(SK) \cong \langle x, y, w, z, \nu | x^3 = y^2, w^3 = z^2, \nu^n = yx^{-1} = zw^{-1}, x^3w^{-3}\nu = \nu x^3w^{-3} \rangle.$$

We can see that the difference between the presentations for $G_n(GK)$ and $G_n(SK)$ is just in the last relation, which states that the longitude commutes with an nth root of the meridian.

4.2.1 The granny and square knot analogues

The knots $GK_{a,b}$ and $SK_{a,b}$ are analogues of square and granny knots built from (a, b)-torus knots, where a and b are co-prime. We define $GK_{a,b}$ to be the connect sum of two right-handed (a, b)-torus knots or two left-handed (a, b)-torus knots, and $SK_{a,b}$ to be the connect sum of a right-handed (a, b)-torus knot and a left-handed (a, b)-torus knot. Recall from Lemma 3.1 that the meridian of $G_{a,b}$ is equal to x^cy^{-d}, where c and $d \in \mathbb{N}$ are a solution to $bc - ad = 1$. We can write the presentations for $\pi_1(GK_{a,b})$ and $\pi_1(SK_{a,b})$ as follows:

$$\pi_1(GK_{a,b}) \cong \langle x, y, w, z | x^a = y^b, w^a = z^b, x^cy^{-d} = w^cz^{-d} \rangle,$$
$$\pi_1(SK_{a,b}) \cong \langle x, y, w, z | x^a = y^b, w^a = z^b, x^cy^{-d} = w^cz^{-d} \rangle,$$

where both have the same meridian $x^cy^{-d} = w^cz^{-d}$, but different longitudes $\lambda_{GK_{a,b}} = x^aw^a$ and $\lambda_{SK_{a,b}} = x^aw^{-a}$. As we can see both of $GK_{a,b}$ and $SK_{a,b}$ have the same knot group. The presentation for $G_n(GK_{a,b})$ and $G_n(SK_{a,b})$ can be obtained in exactly the same way as those for $G_n(GK)$ and $G_n(SK)$:

$$G_n(GK_{a,b}) \cong \langle x, y, w, z, \nu | x^a = y^b, w^a = z^b, \nu^n = x^cy^{-d} = w^cz^{-d}, x^aw^a\nu = \nu x^aw^a \rangle,$$
$$G_n(SK_{a,b}) \cong \langle x, y, w, z, \nu | x^a = y^b, w^a = z^b, \nu^n = x^cy^{-d} = w^cz^{-d}, x^aw^{-a}\nu = \nu x^aw^{-a} \rangle.$$

By Nelson and Neumann [21], we know that $G_n(GK_{a,b})$ is not isomorphic to $G_n(SK_{a,b})$. The goal of this study is to show that the difference between these groups can be detected by counting homomorphisms into suitably chosen finite groups. We will give the proof in Chapter 6.

Chapter 5

Group Theory

In this chapter, we will narrow our focus to the types of groups that will appear in our study. We will briefly introduce these groups: dihedral groups, direct products, semidirect products and wreath products.

The definitions and theorems in this chapter are taken from: Aschbacher [3], Bogopolski [4], Chirikjian [6], Humphreys [10], Ledermann [14], Meldrum [19], Rose [24], Rotman [25], Scott [27] and Tuffley [29].

5.1 Dihedral groups

In this section, we will outline the dihedral groups with some examples.

Definition 5.1. For $n \geq 3$ the **dihedral group** D_n is the group of symmetries of a regular polygon with n sides. For $n = 1$ or $n = 2$, D_n is the group of symmetries of an interval or rectangle respectively. The order of the dihedral group D_n is $2n$, for $n \geq 1$.

There are two types of symmetries of the n-gon; exactly half of them are rotations and the other half are reflections as follows:

Rotations $\rho_0, \rho_{\frac{2\pi}{n}}, \rho_{\frac{4\pi}{n}}, \ldots, \rho_{\frac{2(n-1)\pi}{n}}$, where ρ_θ is an anti-clockwise rotation of angle θ.

Reflections $\mu_0, \mu_{\frac{\pi}{n}}, \mu_{\frac{2\pi}{n}}, \ldots, \mu_{\frac{(n-1)\pi}{n}}$, where μ_θ is reflection about the line through the origin meeting the vertical axis with an angle θ.

The dihedral group is generated by a rotation through $2\pi/n$ and a reflection. The dihedral group D_n has the presentation $\langle a, b | b^n = a^2 = e, aba^{-1} = b^{-1} \rangle$, where a represents a reflection and b a rotation.

We will clarify the above discussion in the following example.

Example 5.1. Let $\rho_0, \rho_{\frac{2\pi}{3}}$ and $\rho_{\frac{4\pi}{3}}$ be the rotations of D_3 and let $\mu_0, \mu_{\frac{\pi}{3}}$ and $\mu_{\frac{2\pi}{3}}$ be the reflections of the group. Then the group of symmetries of the triangle can be written as

$$D_3 = \{\rho_0, \rho_{\frac{2\pi}{3}}, \rho_{\frac{4\pi}{3}}, \mu_0, \mu_{\frac{\pi}{3}}, \mu_{\frac{2\pi}{3}}\}.$$

As we can see in Figure 5.1 rotating the triangle gives three symmetries: ρ_0 is the identity rotation, $\rho_{\frac{2\pi}{3}}$ is the $120°$ anti-clockwise rotation and $\rho_{\frac{4\pi}{3}}$ is the $240°$ anti-clockwise rotation.

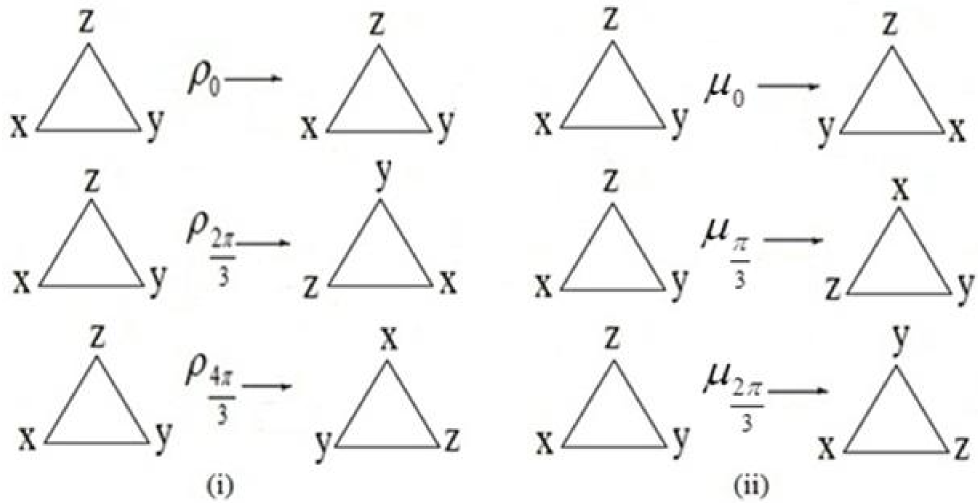

Figure 5.1: The elements of D_3, the symmetry group of an equilateral triangle. (i) is a diagram of the three rotations. (ii) is a diagram of the three reflections of D_3.

The reflections across the diagonal and vertical lines of symmetry give three more symmetries which are μ_0, the reflection through the vertical line; $\mu_{\frac{\pi}{3}}$, the reflection through the diagonal running northwest to southeast; and $\mu_{\frac{2\pi}{3}}$, the reflection through the diagonal running northeast to southwest.

We will conclude this section with the following remark which shows D_n may be realised as a subgroup of S_n.

Remark 5.1. By labeling the vertices of the regular polygon with $0, 1, \ldots, n-1$, D_n can be realised as a subgroup of S_n. It is generated by the rotation $\rho(i) = i + 1$ and the reflection $\sigma(0) = 0, \sigma(i) = n - i$, for $1 \leqslant i \leqslant n - 1$. Also, both ρ and σ can be written in cycle form as follows:

$$\rho = (0\ 1\ \ldots\ n-1)$$
$$\sigma = (0)(1\ n-1)\cdots(\lfloor n/2 \rfloor\ \lceil n/2 \rceil).$$

Note that $(\lfloor n/2 \rfloor\ \lceil n/2 \rceil)$ represents the two cycle $(k\ k+1)$ when $n = 2k+1$ is odd, and the one cycle (k) when $n = 2k$ is even.

Let us illustrate the above remark by an example.

Example 5.2. The group S_5 has a subgroup that is isomorphic to D_5 (Figure 5.2), which is generated by ρ and σ, where

$$\rho = (0\ 1\ 2\ 3\ 4),$$
$$\sigma = (0)(1\ 4)(2\ 3).$$

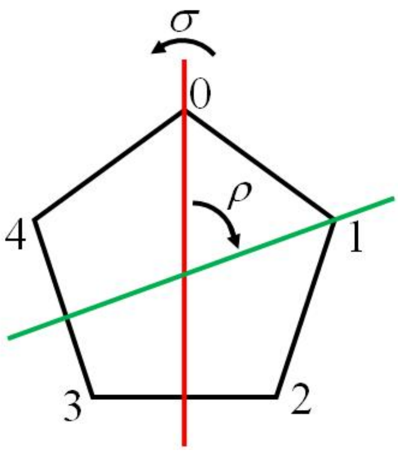

Figure 5.2: A pentagon with labelled vertices which realises D_5 as a subgroup of S_5.

5.2 Semidirect products

In this section, we will describe the semidirect product and some of its properties. To be able to define the semidirect product, the direct product needs to be defined first.

Definition 5.2. Let G be a group. Then G is to said to be a **direct product** of two groups H and K if and only if they are normal subgroups of G such that $H \cap K = \{e\}$ and $G = HK$.

Definition 5.3. G is said to be the **semidirect product** of H and K if and only if H and K are subgroups of G, which satisfy the following conditions:

$$\begin{aligned} H &\lhd G, \\ H \cap K &= \{e\}, \\ HK &= G. \end{aligned}$$

Let $G = HK$, where H and K are subgroups of G and satisfy the conditions in Definition 5.3. Each element of G can be uniquely expressed in the form hk, since $hk = h'k'$ implies $(h')^{-1}h = k'k^{-1} \in H \cap K = \{e\}$. Since $H \lhd G$, for each $k \in K$ there is an automorphism of H which is the inner automorphism of k restricted to H given by $h \mapsto khk^{-1}$. Moreover, the map $K \to Aut(H)$ given by $k \mapsto (h \mapsto khk^{-1})$ is a group homomorphism, as we now show. Suppose φ_k is the inner automorphism of k restricted to H. Then, for $t, k \in K$ and $h \in H$,

$$\begin{aligned} \varphi_{tk}(h) &= tkh(tk)^{-1} \\ &= tkhk^{-1}t^{-1} \\ &= t(khk^{-1})t^{-1} \\ &= \varphi_t(khk^{-1}) \\ &= \varphi_t(\varphi_k(h)) \\ &= (\varphi_t \circ \varphi_k)(h). \end{aligned}$$

For $h \in H$ and $k \in K$ we associate the ordered pair $(h, k) \in H \times K$ with the element $hk \in G = HK$. For $hk, h'k' \in G$ and the homomorphism φ, the multiplication on G can be written as follows:

$$\begin{aligned}(h,k)(h',k') &= (hkh'k^{-1}, kk')\\ &= (h\varphi_k(h'), kk').\end{aligned}$$

This is what called a semidirect product, and it is denoted by $H \rtimes_\varphi K$, or it can be written as $H \rtimes K$, when φ is understood.

We now give an example of the semidirect product.

Example 5.3. Let $H = \langle x | x^n = 1\rangle \cong \mathbb{Z}_n$, and $K = \langle y|y^2 = 1\rangle \cong \mathbb{Z}_2$. Let $\varphi : K \to Aut(H)$ be the homomorphism such that $\varphi_y(x) = x^{-1}$. Then the semidirect product of the normal subgroup H and the subgroup K is

$$G = H \rtimes_\varphi K.$$

We now show that $G \cong D_n$, the dihedral group of order $2n$. The element $(x, 1)$ has order n and $(1, y)$ has order 2. So,

$$\begin{aligned}(1,y)(x,1)(1,y) &= (\varphi_y(x), y)(1,y)\\ &= (\varphi_y(x), y^2)\\ &= (\varphi_y(x), 1)\\ &= (x^{-1}, 1)\\ &= (x,1)^{-1}.\end{aligned}$$

Hence if we set $v = (x, 1)$ and $u = (1, y)$ then v and u satisfy $v^n = u^2 = 1$ and $uvu = v^{-1}$. We can see that v and u generate G and $|G| = |K||H| = 2n$. So the relations of the group G are exactly the same as those of the dihedral group D_n and that implies G is isomorphic to D_n.

Remark 5.2. The example above shows that the dihedral group D_n is isomorphic to the semidirect product $G = \mathbb{Z}_n \rtimes \mathbb{Z}_2$.

Next, we will use the semidirect product to construct the group $D_{p,q;\theta}$, which will form the building blocks for our target groups in Chapter 6.

5.3 The construction of $D_{p,q;\theta}$

We will begin with the definition of Tuffley's $D_{p,q}$ [29]. For p and q distinct primes the group $D_{p,q}$ is a semidirect product

$$D_{p,q} = \mathbb{Z}_p^{q-1} \rtimes \mathbb{Z}_q.$$

To define multiplication in $D_{p,q}$ we regard $\mathbb{Z}_p^{q-1}$ as the additive group of the finite field $\mathbb{F}_{p^{q-1}}$. The multiplicative group $\mathbb{F}^\times_{p^{q-1}}$ is cyclic of order $p^{q-1} - 1$, and so contains an element ζ of order q, as q divides $p^{q-1} - 1$ by Fermat's Theorem. The multiplication in $D_{p,q}$ can be defined by the action of $i \in \mathbb{Z}_q$ on $\mathbb{Z}_p^{q-1}$ given by multiplication by ζ^i. We remark that $D_{p,q}$ may be regarded as a generalised dihedral group, in the sense that $D_{p,2} \cong D_p$.

If $F(x) = 1+x+x^2+\cdots+x^{q-1}$ factors over $\mathbb{Z}_p$, then the isomorphism type of $D_{p,q}$ depends on the choice of root ζ, and therefore to avoid this ambiguity we need to define $D_{p,q;\theta}$. To construct the semidirect product $D_{p,q;\theta}$, let $\theta(x)$ be an irreducible factor of $F(x) = 1 + x + x^2 + \cdots + x^{q-1}$ over $\mathbb{Z}_p$, and let ζ be a root of $\theta(x)$ in $\mathbb{F}_{p;\theta} = \mathbb{Z}_p[x]/\langle\theta(x)\rangle$ which is a finite field of order $p^{\deg\theta}$. The group $D_{p,q;\theta}$ is the semidirect product

$$D_{p,q;\theta} = \mathbb{V}_{p;\theta} \rtimes \mathbb{Z}_q,$$

where $\mathbb{V}_{p;\theta} = (\mathbb{Z}_p)^{\deg\theta}$ is the additive group of $\mathbb{F}_{p;\theta}$. The multiplicative group $\mathbb{F}^{\times}_{p;\theta}$ is cyclic of order $p^{\deg\theta} - 1$, and ζ is an element of multiplicative order q, because $\theta(x)$ divides $1 + x + x^2 + \cdots + x^{q-1}$, and so divides $x^q - 1 = (x-1)(1 + x + x^2 + \cdots + x^{q-1})$. Therefore, the multiplication in $D_{p,q;\theta}$ can be defined by

$$(v, i) \cdot (u, j) = (v + \zeta^i u, i + j).$$

Thus $D_{p,q;\theta}$ is a subgroup of Tuffley's $D_{p,q}$ [29] for a suitably chosen element ζ of order q, and $D_{p,q;\theta} \cong D_{p,q}$ when θ is irreducible over $\mathbb{Z}_p$.

Remark 5.3. In the special case where $q = 2$, we have $F(x) = x + 1$ which is irreducible over $\mathbb{Z}_p$. Then $\zeta = -1$ and we see that $D_{p,2;x+1} \cong D_p$, the dihedral group.

Since $\mathbb{V}_{p;\theta}$ is normal, there is a map from $D_{p,q;\theta}$ into $\mathbb{Z}_q$, which we will use in Section 5.5. If $f = (v, i) \in D_{p,q;\theta}$, then we will write $[f] = i$ for its image in $\mathbb{Z}_q$.

The next section proves some properties of the group $D_{p,q;\theta}$.

5.3.1 Properties of $D_{p,q;\theta}$

In this section, we will introduce some properties of $D_{p,q;\theta}$ and show that it is generated by any element of order p together with any element of order q. Also, we will define cyclic and noncyclic solutions to $x^a = y^b$, in a group G, and show when $D_{p,q;\theta}$ has a noncyclic solution to $x^a = y^b$.

The following lemma will present some facts about the elements of $D_{p,q;\theta}$. For a proof see Tuffley [29].

Lemma 5.1 (Tuffley, [29, Lemma 3.2]). *Let $D_{p,q;\theta} = \mathbb{V}_{p;\theta} \rtimes \mathbb{Z}_q$, such that $\mathbb{V}_{p;\theta} = (\mathbb{Z}_p)^{\deg\theta}$. Then*

1. *Elements in $D_{p,q;\theta}$ are of order 1, p, or q.*
2. *If two elements f and $g \in D_{p,q;\theta}$ commute with each other, then they belong to $\mathbb{V}_{p;\theta}$ or the same cyclic subgroup of order q.*
3. *If an element $f = (v, 0) \in \mathbb{V}_{p;\theta}$ has order p, then the conjugacy class of f is $(\zeta^i v, 0)$, where $0 \leqslant i \leqslant q - 1$.*
4. *If $f = (v, i)$ has order q, then the conjugacy class of f is $\{g : [g] = i\} = \{(\omega, i) : \omega \in \mathbb{V}_{p;\theta}\}$.*

Remark 5.4. Since p and q are prime, as a consequence of Lemma 5.1 we note that if f is a nontrivial element of $D_{p,q;\theta}$, then f has an nth root in $D_{p,q;\theta}$ if and only if $ord(f)$ is co-prime to n. Moreover, any such nth root belongs to $\langle f\rangle$.

We next will state our lemma which shows the generators of the group $D_{p,q;\theta}$.

Lemma 5.2. *Let α and β be elements of $D_{p,q;\theta}$ of orders p and q. Then the group $D_{p,q;\theta}$ is generated by α and β.*

Proof. The group $D_{p,q;\theta} = \mathbb{V}_{p;\theta} \rtimes \mathbb{Z}_q$. Suppose α is an element of order p such that $\langle \alpha \rangle = \{(ku, 0) | k = 0, \ldots, p-1\}$, and suppose β is an element of order q such that $\langle \beta \rangle = \{(v_j, j) = (v, 1)^j | j = 0, \ldots, q-1\}$. Then letting $\langle \beta \rangle$ act on $\alpha = (u, 0)$ by conjugation we see that

$$\begin{aligned}
\beta^j \alpha \beta^{-j} &= (v_j, j)(u, 0)(v_j, j)^{-1} \\
&= (v_j, j)(u, 0)(-\zeta^{-j} v_j, -j) \qquad \text{(as } (v_j, j)^{-1} = (-\zeta^{-j} v_j, -j)) \\
&= (v_j, j)(u - \zeta^{-j} v_j, -j) \\
&= (v_j + \zeta^j (u - \zeta^{-j} v_j), j - j) \\
&= (v_j - v_j + \zeta^j u, 0) \\
&= (\zeta^j u, 0),
\end{aligned}$$

so $(\zeta^j u, 0)$ belongs to $\langle \alpha, \beta \rangle$ for $j = 0, 1, \ldots, q-1$.
Let $d = \deg \theta$. We claim that $(u, 0), (\zeta u, 0), \ldots, (\zeta^{d-1} u, 0)$ are linearly independent, and therefore generate $\mathbb{V}_{p;\theta}$. We regard $\mathbb{V}_{p;\theta} = (\mathbb{Z}_p)^d$ as a vector space over $\mathbb{Z}_p$. Suppose that

$$c_0 u + c_1 \zeta u + \cdots + c_{d-1} \zeta^{d-1} u = 0, \tag{5.1}$$

where $c_i \in \mathbb{Z}_p$ for $i = 0, \ldots, d-1$. Factoring (5.1) we get

$$\left[\sum_{i=0}^{d-1} c_i \zeta^i \right] u = 0,$$

so $\sum_{i=0}^{d-1} c_i \zeta^i = 0 \in \mathbb{F}_{p;\theta}$. But then ζ is a root of $\sum_{i=0}^{d-1} c_i x^i$, a polynomial of degree less than the degree of θ, so we must have $c_i = 0$ for all i since θ is irreducible over $\mathbb{Z}_p$. Then $(\zeta^i u, 0), i = 0, \ldots, d-1$ are linearly independent and therefore generate $\mathbb{V}_{p;\theta}$ as claimed. □

Since we are interested in homomorphisms from $G_{a,b}$, we are interested in solutions to $x^a = y^b$ in a group G. The following definition gives more explanation.

Definition 5.4. If G has a solution to $x^a = y^b$, then the solution is **cyclic** if $\langle x, y \rangle$ is a cyclic subgroup of $D_{p,q;\theta}$. Otherwise, the solution is **noncyclic**.

Similarly, we will say that $\rho : G_{a,b} \to G$ is cyclic if $\langle \rho(x), \rho(y) \rangle$ is cyclic. Otherwise, $\rho : G_{a,b} \to G$ is noncyclic.

We now characterise when $D_{p,q;\theta}$ has a noncyclic solution to $x^a = y^b$.

Lemma 5.3. *Let a and b be co-prime. Any noncyclic solution to $x^a = y^b$ in $D_{p,q;\theta}$ satisfies $x^a = y^b = 1$. Consequently, such a solution exists if and only if $p|a$ and $q|b$, or $q|a$ and $p|b$.*

Proof. Suppose x and y are a solution to $x^a = y^b$ such that $x^a = y^b = g \neq 1$. Then g has order either p or q. The element g belongs to $\langle x \rangle$ and $\langle y \rangle$, where $\langle x \rangle$ and $\langle y \rangle$ are groups of order p or q which are primes. Therefore, g generates $\langle x \rangle$ and $\langle y \rangle$, and $\langle x \rangle$ and $\langle y \rangle$ are cyclic subgroups of $\langle g \rangle$. This implies that $\langle x, y \rangle = \langle g \rangle$ is cyclic, so any noncyclic solution must satisfy $x^a = y^b = 1$. We show that a noncyclic solution exists only if $p|a$ and $q|b$, or $q|a$ and $p|b$. Let $x^a = y^b$ be a noncyclic solution satisfying $x^a = y^b = 1$. Since a and b are co-prime and p and q are distinct

primes, either x is an element of order p and y is an element of order q or x is of order q and y is of order p. Then $p|a$ and $q|b$, or $q|a$ and $p|b$.
We now prove a noncyclic solution exists if $p|a$ and $q|b$, or $q|a$ and $p|b$. Without loss of generality, assume $p|a$ and $q|b$. In this case, choose x of order p and y of order q, and therefore $x^a = y^b = 1$ and $\langle x, y\rangle$ is noncyclic by Lemma 5.2, since it equals $D_{p,q;\theta}$. □

Remark 5.5. By Lemma 5.3, if $\langle x, y\rangle$ is a noncyclic solution, then $\langle x, y\rangle = D_{p,q;\theta}$.

5.4 Wreath products

The **wreath product** in group theory is a product of two groups based on a semidirect product. It is important in the classification of permutation groups. For our purposes we may define the wreath product as follows.

Definition 5.5. Let G be a group, and let X be a set, then $G^X = \prod_{i\in X} G_i$, where $G_i = G$ for $i \in X$. Suppose H is a group that acts on X on the right. Then there is a natural left action of H on G^X defined by

$$(h(g))_i = g_{i\cdot h},$$

where $g = (g_i)_{i\in X}$. Then the **wreath product** $G \wr H$ is the semidirect product $G^X \rtimes_\varphi H$, where φ is the homomorphism $\varphi : H \to Aut(G^X)$ defined as follows: for $(g_i)_{i\in X} \in G^X$ and $h \in H$, we have $(\varphi_h(g))_i = g_{i\cdot h}$.

We will illustrate the multiplication in the wreath product with a simple remark followed by an example.

Remark 5.6. Suppose $(g_1, g_2, \ldots, g_n), (k_1, k_2, \ldots, k_n) \in G^X, X = \{1, \ldots, n\}$ and $h_1, h_2 \in H$. Then the multiplication rule in $G \wr H = G^X \rtimes_\varphi H$ can be defined by

$$\big((g_1, g_2, \ldots, g_n), h_1\big)\big((k_1, k_2, \ldots, k_n), h_2\big) = (g_1k_{1\cdot h_1}, g_2k_{2\cdot h_1}, \ldots, g_nk_{n\cdot h_1}, h_1h_2).$$

If $g = \big((g_1, \ldots, g_n), h\big)$, then we define $\hat{g} = h$. In general, let $g, k \in G \wr H$ and $\hat{g} \in H$. Then $(gk)_i = g_ik_{i\cdot\hat{g}}$, and the multiplication in H is given by $\widehat{gk} = \hat{g}\hat{k}$. The multiplication in permutations is composed from left to right, as we will see in the following example.

Example 5.4. Let $(g_1, g_2, g_3), (k_1, k_2, k_3) \in G^3$ and let $(1\ 2), (1\ 3) \in S_3$. Then, the product of $g = \big((g_1, g_2, g_3), (1\ 2)\big)$ and $k = \big((k_1, k_2, k_3), (1\ 3)\big)$ in $G^3 \wr S_3$ is

$$\begin{aligned} gk &= \big((g_1, g_2, g_3), (1\ 2)\big) \cdot \big((k_1, k_2, k_3), (1\ 3)\big), \\ &= \big((g_1k_2, g_2k_1, g_3k_3), (1\ 2\ 3)\big). \end{aligned}$$

In the following section we will construct the group $\mathbb{W}^{h,k;\tau}_{s,t;\varrho}$ by modifying the construction of Tuffley's $\mathcal{H}^{q,r}_p$ in [29].

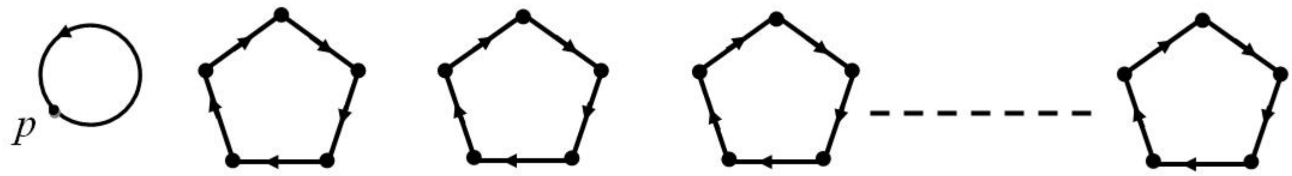

Figure 5.3: When $\hat{\omega}$ has order $t = 5$, its action on $\mathcal{F}$ has a single 1-cycle (p) and all other cycles are of length 5.

5.5 The construction of $\mathbb{W}^{h,k;\tau}_{s,t;\varrho}$

To construct our target group $\mathbb{W}^{h,k;\tau}_{s,t;\varrho}$ we will modify the construction of Tuffley's $\mathcal{H}^{q,r}_{p}$, which is a wreath product $D_{q,r} \wr PSL(2,p)$. We will do this by replacing $PSL(2,p)$ with $D_{s,t;\varrho}$ as follows.

Given distinct primes h, k, s and t and suitable polynomials $\tau \in \mathbb{Z}_k[x]$ and $\varrho \in \mathbb{Z}_t[x]$, the group $\mathbb{W}^{h,k;\tau}_{s,t;\varrho}$ is a wreath product of $D_{h,k;\tau}$ over $D_{s,t;\varrho}$,

$$\mathbb{W}^{h,k;\tau}_{s,t;\varrho} = D_{h,k;\tau} \wr D_{s,t;\varrho} = (D_{h,k;\tau})^{\mathcal{F}} \rtimes D_{s,t;\varrho},$$

where $\mathcal{F} = \{g \in D_{s,t;\varrho} | [g] = 1\}$ is a conjugacy class of $D_{s,t;\varrho}$ and we let $D_{s,t;\varrho}$ act on $\mathcal{F}$ by conjugation.

Elements of $\mathbb{W}^{h,k;\tau}_{s,t;\varrho}$ have the form

$$\omega = ((\omega_i)_{i \in \mathcal{F}}, \hat{\omega}),$$

where $\omega_i \in D_{h,k;\tau}$ for each $i \in \mathcal{F}$, so there will be $|\mathbb{V}_{s;\varrho}| = s^{\deg \varrho}$ entries. The element $\hat{\omega} \in D_{s,t;\varrho}$ has order $1, s$ or t.

Remark 5.7. We use Lemma 5.1 to describe the cycle structure of an element $\hat{\omega} \in D_{s,t;\varrho}$ acting on $\mathcal{F}$ by conjugation. Note that all elements of $\mathcal{F}$ have order t. If $\hat{\omega}$ has order s, then $\omega \in \mathbb{V}_{s;\varrho}$. Elements of order s and t do not commute, and therefore $\hat{\omega}$ has no fixed point and all cycles are of length s, because s is prime. If $\hat{\omega}$ has order t, then it commutes only with $\langle \hat{\omega} \rangle$. There is a unique element $\hat{\omega} \in \langle \hat{\omega} \rangle$ such that $\hat{\omega}^r \in \mathcal{F}$, $0 \leqslant r \leqslant t-1$. Then, $\hat{\omega}$ has a unique fixed point in $\mathcal{F}$ and therefore all other cycles have length t, because t is prime. Figure 5.3 pictures the above discussion, with $t = 5$.

The map $D_{h,k;\tau} \to \mathbb{Z}_k$ leads to a map $\mathbb{W}^{h,k;\tau}_{s,t;\varrho} \to \mathbb{Z}_k \wr D_{s,t;\varrho}$ which is given by

$$[\omega] = (([\omega_i])_{i \in \mathcal{F}}, \hat{\omega}).$$

It is convenient to factor the map $\mathbb{W}^{h,k;\tau}_{s,t;\varrho} \to \mathbb{Z}_k \wr D_{s,t;\varrho}$ into two maps to distinguish a subgroup of $\mathbb{W}^{h,k;\tau}_{s,t;\varrho}$ isomorphic to $\mathbb{Z}_k \wr D_{s,t;\varrho}$. Choose ξ in $D_{h,k;\tau}$ of order k such that $[\xi] = 1$, so $\langle \xi \rangle \cong \mathbb{Z}_k$. We define $\mathcal{A}^k_{s,t;\varrho} = \langle \xi \rangle \wr D_{s,t;\varrho} \subseteq \mathbb{W}^{h,k;\tau}_{s,t;\varrho}$.

Furthermore, we can get a well defined homomorphism $\mathbb{W}^{h,k;\tau}_{s,t;\varrho} \to \mathbb{Z}_k \wr D_{s,t;\varrho} \to \mathbb{Z}_k$ which is given by

$$[[\omega]] = \sum_{i\in F}[\omega_i],$$

as $\mathbb{Z}_k$ is abelian.

Similarly, the subgroup $\mathbb{V}_{h;\tau} \wr D_{s,t;\varrho}$ has the homomorphism $||\cdot|| : \mathbb{V}_{h;\tau} \wr D_{s,t;\varrho} \to \mathbb{V}_{h;\tau}$ given by

$$||(v_i, \hat{v})_{i\in F}|| = \sum_{i\in F} v_i.$$

To be able to use $\mathbb{W}^{h,k;\tau}_{s,t;\varrho}$, we need to restrict h, k, s and t to be distinct primes which will allow some flexibility in dealing with the processes in the following chapter in order to prove the main theorem.

We conclude this section with an example which illustrates the group operations in the wreath product $G \wr D_5$.

Example 5.5. Consider the wreath product $G \wr D_{5,2;x+1}$. Since $D_{5,2;x+1}$ is isomorphic to D_5, we therefore have $G \wr D_5$. The group D_5 is generated by $\rho = (0\ 1\ 2\ 3\ 4)$ and $\sigma = (0)(1\ 4)(2\ 3)$ (see example 5.2).
Elements of $G\wr D_s$ may be described using diagrams such as those in Figures 5.4 (i) and (ii). The arrows describe the permutation $\hat{g}$ and $\hat{h}$, and they are labelled by the corresponding element g_i, h_i. For example, let

$$g = \big((g_0, g_1, g_2, g_3, g_4), (0\ 1\ 2\ 3\ 4)\big), \quad h = \big((h_0, h_1, h_2, h_3, h_4), (0)(1\ 4)(2\ 3)\big).$$

Then

$$gh = \big((g_0h_1, g_1h_2, g_2h_3, g_3h_4, g_4h_0), (0\ 4)(1\ 3)(2)\big),$$

which is represented by Figure 5.4(iii). As we can see in Figure 5.4 (iv), the inverse of g is found by reversing all arrows and inverting all labels,

$$g^{-1} = \big((g_0, g_1, g_2, g_3, g_4), (0\ 1\ 2\ 3\ 4)\big)^{-1} = \big((g_4^{-1}, g_0^{-1}, g_1^{-1}, g_2^{-1}, g_3^{-1}), (0\ 4\ 3\ 2\ 1)\big).$$

If g is raised to the third power: $g \cdot g \cdot g$, then

$$\begin{aligned} g^3 &= \big((g_0g_1g_2, g_1g_2g_3, g_2g_3g_4, g_3g_4g_0, g_4g_0g_1), (0\ 1\ 2\ 3\ 4)^3\big) \\ &= \big((g_0g_1g_2, g_1g_2g_3, g_2g_3g_4, g_3g_4g_0, g_4g_0g_1), (0\ 3\ 1\ 4\ 2)\big). \end{aligned}$$

This is represented by Figure 5.4 (v).

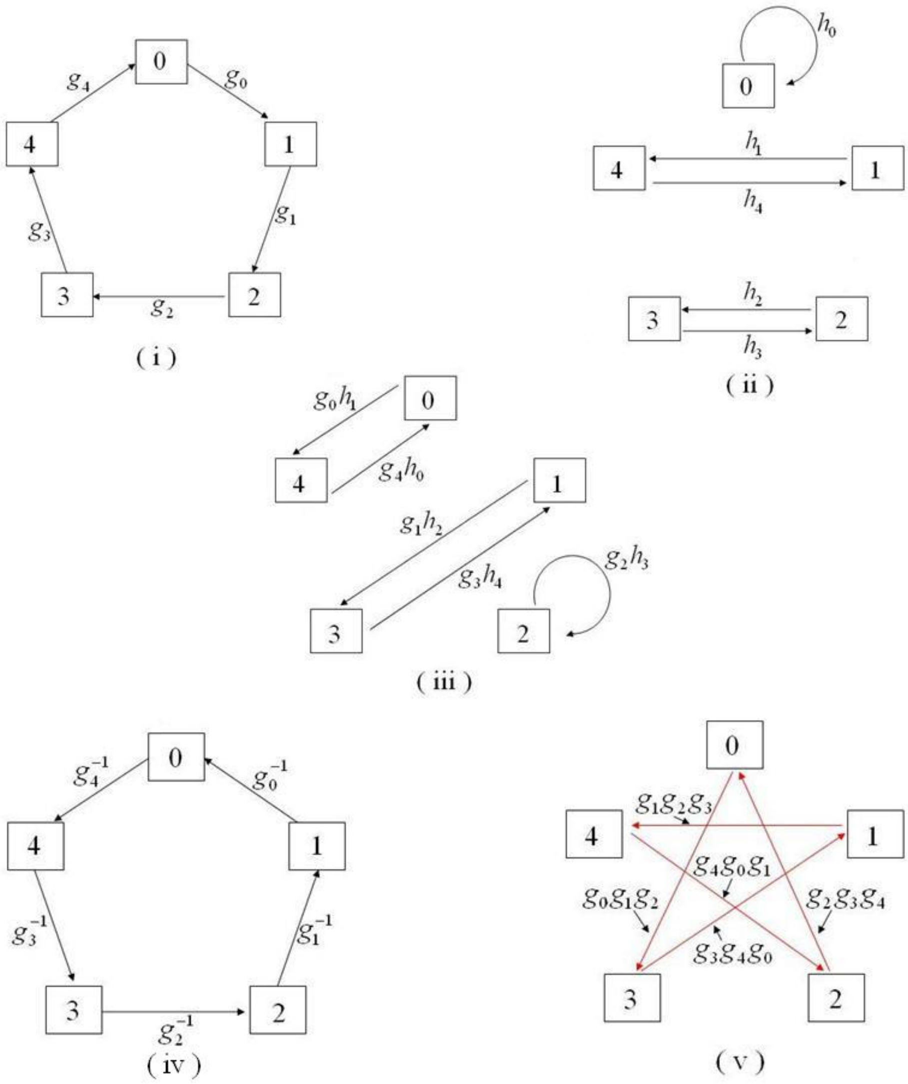

Figure 5.4: Visualising elements and group operations in $G \wr D_5$.

Chapter 6

The Main Result

In this section we will describe the strategy of this study which extends the work of Tuffley [29], and we will state some definitions and lemmas which we will use in order to show that the difference between $G_n(GK_{a,b})$ and $G_n(SK_{a,b})$ can be detected by counting homomorphisms into suitable finite groups. For simplicity we will restrict our attention to values of n satisfying a certain coprimality condition with respect to a and b. This avoids certain technical complications that arise in Tuffley's proof [29] that $G_n(GK)$ and $G_n(SK)$ are not isomorphic for $n \geq 2$.

6.1 Strategy

Let $K = GK_{a,b}$ or $SK_{a,b}$, and let μ, λ be the meridian and longitude of K. Then we will consider triples $(\mathbb{W}, \rho, \eta)$, where $\mathbb{W}$ is a finite group, ρ is a homomorphism which maps $\pi_1(K)$ into $\mathbb{W}$, and η is an nth root of $\rho(\mu)$. If $\eta^n = \rho(\mu)$, then we will called (ρ, η) a **map-root** pair for K in $\mathbb{W}$. Then we will have a homomorphism $\tilde{\rho} : G_n(K) \to \mathbb{W}$, when it satisfies the compatibility condition $\rho(\lambda)\eta = \eta\rho(\lambda)$. As we know $\pi_1(GK_{a,b})$ and $\pi_1(SK_{a,b})$ are both isomorphic to

$$H_{a,b} = \langle x, y, z, w | x^a = y^b, w^a = z^b, x^c y^{-d} = w^c z^{-d} \rangle,$$

where they have common meridian $\mu = x^c y^{-d} = w^c z^{-d} = \omega_0$, so the map-root pairs for both of them are the same. However, the compatibility conditions for $GK_{a,b}$ and $SK_{a,b}$ are $\rho(x^a w^a)\eta = \eta\rho(x^a w^a)$ and $\rho(x^a w^{-a})\eta = \eta\rho(x^a w^{-a})$ respectively. As $H_{a,b} = G_{a,b} *_{\langle \omega_0 \rangle} G_{a,b}$, we can think of $\rho : H_{a,b} \to \mathbb{W}$ as two homomorphisms (ρ_1, ρ_2) mapping $G_{a,b}$ into $\mathbb{W}$ such that $\rho_1(\omega_0) = \rho_2(\omega_0)$.

The following is our main theorem.

Theorem 6.1. ***(Main Theorem)*** *Suppose that $n \geq 2$, and a and b are co-prime integers such that $1 < a < b$. Choose primes $s|a$ and $t|b$ such that $\gcd(st, n) = 1$. Let h be a prime dividing n, k a prime co-prime to $2nab$, and let $\tau \in \mathbb{Z}_k[x]$ be an irreducible factor of $F(x) = 1 + \cdots + x^{k-1}$ over $\mathbb{Z}_h$, and $\varrho \in \mathbb{Z}_t[x]$ be an irreducible factor of $F(x) = 1 + \cdots + x^{t-1}$ over $\mathbb{Z}_s$. Then*

$$|Hom(G_n(GK_{a,b}), \mathbb{W}^{h,k;\tau}_{s,t;\varrho})| < |Hom(G_n(SK_{a,b}), \mathbb{W}^{h,k;\tau}_{s,t;\varrho})|.$$

Therefore, $G_n(GK_{a,b})$ is not isomorphic to $G_n(SK_{a,b})$ for $n \geq 2$ which satisfies the theorem hypotheses; in particular, this holds if $\gcd(ab, n) = 1$.

Note that any nontrivial torus knot is equivalent up to reflection to a torus knot $T_{a,b}$ satisfying $1 < a < b$.

The target group of this study is the wreath product $\mathbb{W}_{s,t;\varrho}^{h,k;\tau}$ defined earlier in Section 5.5. Each one of the divisibility conditions that is described in the hypotheses of the above theorem will play a crucial role in the proof of the theorem as we will see in the following sections. Now, let us describe the idea of the proof of Theorem 6.1 which is the same as that of Tuffley [29]. We will show that:

A1. Every compatible pair (ρ, η) for $GK_{a,b}$ in $\mathbb{W}_{s,t;\varrho}^{h,k;\tau}$ is also compatible for $SK_{a,b}$.

A2. Some pairs (ρ, η) are compatible for $SK_{a,b}$ in $\mathbb{W}_{s,t;\varrho}^{h,k;\tau}$, but are incompatible for $GK_{a,b}$.

Remark 6.1. Let $a = 2 \times 3$ and $b = 5 \times 7$, and let s and t be distinct primes such that $s|a$ and $t|b$. Therefore, we have four possibilities for s and t. If we choose

- $s = 2$ and $t = 5$, then the possibilities for n will be $3, 7, 9, 11, 13, 17, 19, 21, \ldots$;
- $s = 2$ and $t = 7$, then the possibilities for n will be $3, 5, 9, 11, 13, 15, 17, 19, 23, 25, \ldots$;
- $s = 3$ and $t = 5$, then the possibilities for n will be $2, 4, 7, 9, 11, 13, 16, 17, 19, 22, 23, \ldots$;
- $s = 3$ and $t = 7$, then the possibilities for n will be $2, 4, 8, 11, 13, 16, 17, 19, 20, 22, 23, \ldots$.

6.2 The cycle product and applications

In this section we will state some lemmas and definitions which will play an important role in the proof of our Main Theorem 6.1. Although the lemmas and definitions were proved for Tuffley's $H_p^{q,r} = D_{q,r} \wr PSL(2,p)$ in Tuffley [29], the proofs still apply for our group $\mathbb{W}_{s,t;\varrho}^{h,k;\tau} = D_{s,t;\varrho} \wr D_{h,k;\tau}$. First we will define the cycle product which is an important tool for finding the results.

Definition 6.1 (Tuffley [29]). Let $\omega \in \mathbb{W}_{s,t;\varrho}^{h,k;\tau}$, $i \in \digamma$ and let $l_i(\hat{\omega})$ be the length of the disjoint cycle of $\hat{\omega}$ that contains i. Then the cycle product of ω at i can be defined as follows:

$$\pi_i(\omega) = \prod_{r=0}^{l_i(\hat{\omega})-1} \omega_{i\cdot\hat{\omega}^r} = \omega_i \omega_{i\cdot\hat{\omega}} \cdots \omega_{i\cdot\hat{\omega}^{l_i(\hat{\omega})-1}}.$$

The cycle product is thus the ordered product, beginning at i, of ω_j for j in the disjoint cycle of $\hat{\omega}$ containing i. Notice that given a cycle, we can then see that the value of the cycle product is dependent on the beginning point i, while the conjugacy class is not, as $\pi_{i\cdot\hat{\omega}}(\omega) = \omega_i^{-1}\pi_i(\omega)\omega_i$.

Next, we will define elements in (reduced) standard form.

Definition 6.2 (Tuffley [29]). Let ω belong to $\mathbb{W}_{s,t;\varrho}^{h,k;\tau}$. Then ω is said to be in standard form if

$$\omega_{i\cdot\hat{\omega}} = \omega_i,$$

for each i. Moreover, ω is in reduced standard form if ω is in standard form and $\pi_i(\omega) = \omega_i^{l_i(\hat{\omega})} = 1$ if and only if $\omega_i = 1$.

The next lemma implies that an element of $\mathbb{W}_{s,t;\varrho}^{h,k;\tau}$ is conjugate to an element in reduced standard form.

Lemma 6.1 (Tuffley, [29, Lemma 3.3])**.** *Let $l_i(\hat{\omega})$ be co-prime to the order of $\pi_i(\omega)$ for all i. Then, ω is conjugate to an element α in reduced standard form such that $\hat{\alpha} = \hat{\omega}$ and $\pi_i(\alpha)$ is conjugate to $\pi_i(\omega)$ for all i.*

Note that in our case $l_i(\hat{\omega}) \in \{1, s, t\}$ is co-prime to $ord(\pi_i(\omega)) \in \{1, h, k\}$, so the hypotheses of Lemma 6.1 are always satisfied. It follows that every element of $\mathbb{W}_{s,t;\varrho}^{h,k;\tau}$ is conjugate to an element in reduced standard form.

Lemma 6.2 (Tuffley, [29, Lemma 3.4])**.** *Suppose ω is an element of $\mathbb{W}_{s,t;\varrho}^{h,k;\tau}$ in standard form such that ω_i has order 1 or k for all i. Then ω is conjugate to an element α of $\mathcal{A}_{s,t;\varrho}^{k}$ in standard form. Moreover, if ω is reduced then α may be chosen to be reduced.*

The next lemma gives a condition for an element in reduced standard form to commute with another element.

Lemma 6.3 (Tuffley, [29, Lemma 3.5])**.** *Suppose ω is in reduced standard form and let γ commute with ω. If ω_i is constant on orbits of $\hat{\gamma}$, then γ_i commutes with ω_i for each i and γ_i is constant on orbits of $\hat{\omega}$.*

We conclude this section with the following lemma which gives a necessary condition for an element ω of $\mathbb{W}_{s,t;\varrho}^{h,k;\tau}$ to be an nth power.

Lemma 6.4 (Tuffley, [29, Lemma 3.6])**.** *If $\omega = \gamma^n$ belongs to $\mathbb{W}_{s,t;\varrho}^{h,k;\tau}$, then $\hat{\gamma}^n = \hat{\omega}$ and*

$$\pi_i(\omega) = (\pi_i(\gamma))^{n/\gcd(l_i(\hat{\gamma}),n)}.$$

In particular, the conjugacy class of $\pi_i(\omega)$ is constant on orbits of $\hat{\gamma}$.

The proof of the main theorem requires the lemmas in the following section.

6.3 Images

In this section, we will identify the possibilities for images and roots of the meridian in the pair (ρ, η) by distinguishing up to conjugacy solutions to $\eta^n = \omega$ with $\hat{\omega} \neq 1$. Also, we will find the possible values of the longitude. The lemmas and proofs in this section generalise those used in Tuffley's proofs [29].

6.3.1 The meridian

Lemma 6.5. *Let h, k, s, t, τ and ϱ be as given in Theorem 6.1. If $\omega \in \mathbb{W}_{s,t;\varrho}^{h,k;\tau}$ is an nth power such that $\hat{\omega} \neq 1$, then ω is a conjugate to an element of $\mathcal{A}_{s,t;\varrho}^{k}$ that is in reduced standard form. Conversely, for such an element ω of $\mathcal{A}_{s,t;\varrho}^{k}$ the solutions to $\eta^n = \omega$ are described by*

1. *$\hat{\eta} = \hat{\omega}^r$, where $\hat{\omega}^r$ is the unique nth root of $\hat{\omega}$ in $\langle\hat{\omega}\rangle$;*

2. *$\eta_{i\cdot\hat{\omega}} = \eta_i$ and so also $\eta_{i\cdot\hat{\eta}} = \eta_i$ for all i;*

3. $\eta_i = \omega_i^{\frac{1}{n}} \in \langle \xi \rangle$ *if* $\omega_i \neq 1$*, where* $\dfrac{1}{n}$ *is the multiplicative inverse of* n *in* $\mathbb{Z}_k$*; and*

4. $\eta_i \in \mathbb{V}_{h;\tau}$ *if* $\omega_i = 1$.

Consequently, the solutions to $\eta^n = \omega$ *are parametrised by* $(\mathbb{V}_{h;\tau})^c$*, where* c *is the number of cycles of* $\hat{\omega}$ *on which* $\omega_i = 1$.

Proof. Suppose $\eta^n = \omega$. Then by Lemma 6.4, we get

$$\pi_i(\omega) = (\pi_i(\eta))^{n/\gcd(l_i(\hat{\eta}),n)}.$$

We have $\gcd(l_i(\hat{\eta}), n) = 1$, because $l_i(\hat{\eta}) \in \{1, s, t\}$ and $\gcd(st, n) = 1$. So $\pi_i(\omega) = \pi_i(\eta)^n$. Now h divides n, so $\pi_i(\omega) \in D_{h,k;\tau}$ is an hth power in $D_{h,k;\tau}$ and therefore $\pi_i(\omega)$ has order 1 or k for all i. Since $l_i(\hat{\omega})$ is co-prime to k for all i, Lemma 6.1 and Lemma 6.2 imply that ω is conjugate to an element of $\mathcal{A}^k_{s,t;\varrho}$ in reduced standard form, as claimed.

To prove part (1), suppose ω is an element of $\mathcal{A}^k_{s,t;\varrho}$ in reduced standard form, such that $\hat{\omega} \neq 1$. Then $\hat{\omega}$ has order either s or t, which are distinct primes and co-prime to n. Therefore $\hat{\omega}$ has a unique nth root $\alpha \in D_{s,t;\varrho}$, and α is of the form $\alpha = \hat{\omega}^r$ for some r. We show that ω has an nth root η such that $\hat{\eta} = \alpha = \hat{\omega}^r$. Suppose that $\omega = \eta^n$. The order of $\hat{\omega}$ is prime, so the orbits of $\hat{\eta} = \hat{\omega}^r$ and the orbits of $\hat{\omega}$ coincide. Therefore ω_i is constant on orbits of $\hat{\eta}$, because ω is in reduced standard form. Since $\omega = \eta^n$ commutes with η, Lemma 6.3 implies that ω_i commutes with η_i, and therefore $\eta_{i\cdot\hat{\omega}} = \eta_i$ and also $\eta_{i\cdot\hat{\eta}} = \eta_i$ as they have the same orbits, which proves part (2). Since $\eta_{i\cdot\hat{\eta}} = \eta_i$, η is in standard form (Definition 6.2) and we therefore have

$$\omega_i = (\eta^n)_i = (\eta_i)^n$$

in $D_{h,k;\tau}$. Because of our construction, n is divisible by h, but not k. Therefore, either ω_i is of order k and η_i is the unique nth root of ω_i in $\langle \omega_i \rangle = \langle \xi \rangle$, or ω_i is of order 1 and then we can choose any element $\eta_i \in \mathbb{V}_{h;\tau}$, which prove parts (3) and (4). These values do in fact define nth roots, so the nth roots of ω are parametrised by $(\mathbb{V}_{h;\tau})^c$, where c is the number of cycles of $\hat{\omega}$ on which $\omega_i = 1$. □

6.3.2 The longitude

Lemma 6.6. *Let* ω *belong to* $\mathcal{A}^k_{s,t;\varrho}$ *in reduced standard form such that* $\hat{\omega}$ *is nontrivial. Let* h, k, s, t, τ *and* ϱ *be as we described them earlier in Theorem 6.1. Assume* $\rho : G_{a,b} \longrightarrow \mathbb{W}^{h,k;\tau}_{s,t;\varrho}$ *is a homomorphism such that* $\rho(\omega_0) = \omega$ *and let* $\varepsilon = \rho(x^a)$*. If* $\langle \widehat{\rho(x)}, \widehat{\rho(y)} \rangle$ *is cyclic then* $\varepsilon = \omega^{ab}$*, and if* $\langle \widehat{\rho(x)}, \widehat{\rho(y)} \rangle$ *is noncyclic then*

1. $\hat{\varepsilon} = 1$*;*

2. ε_i *is constant on orbits of* $\hat{\omega}$*;*

3. *the conjugacy class of* ε_i *is constant on* $\digamma$*;*

4. $[\varepsilon_i] = \frac{ab}{s^{\deg \varrho}}[[\omega]]$ *for all* i*;*

5. $\varepsilon_i = \xi^{ab[[\omega]]/s^{\deg \varrho}}$ *if* $\omega_i \neq 1$.

Proof. Define $\chi = \rho(x)$ and $\psi = \rho(y)$, so $\varepsilon = \chi^a = \psi^b$. We consider two cases:

1. $\langle \hat{\chi}, \hat{\psi} \rangle$ is a noncyclic subgroup of $D_{s,t;\varrho}$,

2. $\langle \hat{\chi}, \hat{\psi} \rangle$ is a cyclic subgroup of $D_{s,t;\varrho}$.

Case 1: noncyclic case. We consider first the case that $\langle \hat{\chi}, \hat{\psi} \rangle$ is noncyclic. Since $x^a = y^b$ generates the centre of $G_{a,b}$ (by Theorem 3.7), ε commutes with ω. To prove part (1), we have $\hat{\varepsilon} = \hat{\chi}^a = \hat{\psi}^b$ in $D_{s,t;\varrho}$, and $\langle \hat{\chi}, \hat{\psi} \rangle$ is noncyclic, so by Lemma 5.3 $\hat{\varepsilon} = 1$. Therefore ω_i is constant on orbits of $\hat{\varepsilon}$ (since each orbit is a singleton) and by applying Lemma 6.3, ε_i is constant on orbits of $\hat{\omega}$ and commutes with ω_i for all i, which proves part (2). Since $\omega_i \in \langle \xi \rangle$ for all i, by applying Lemma 5.1 part (2) we have $\varepsilon_i \in \langle \xi \rangle$, whenever $\omega_i \neq 1$.

As we have $\varepsilon_i = \pi_i(\varepsilon)$ for all i and $\varepsilon = \chi^a = \psi^b$, so by Lemma 6.4 the conjugacy class of ε_i is constant on orbits of $\hat{\chi}$ and $\hat{\psi}$. Since $\hat{\chi}$ and $\hat{\psi}$ generate $D_{s,t;\varrho}$ (by Lemma 5.2) and $D_{s,t;\varrho}$ acts transitively on $\digamma$, the conjugacy class of ε_i is constant on $\digamma$, which proves part (3).

To prove part (4), since the conjugacy class of ε_i is constant on $\digamma$, the value of $[\varepsilon_i]$ is constant on $\digamma$. So we can calculate the common value $[\varepsilon_i]$ by using the abelianisation $\mathbb{W}_{s,t;\varrho}^{h,k;\tau} \to \mathbb{Z}_k$ as follows:

$$[[\varepsilon]] = \sum_{r \in \digamma} [\varepsilon_r] = s^{\deg \varrho}[\varepsilon_i]. \tag{6.1}$$

Since ω generates the abelianisation of $G_{a,b}$ (by Remark 3.1), we have

$$[[\varepsilon]] = [[\chi^a]] = a[[\chi]] = ab[[\omega]]. \tag{6.2}$$

From (6.1) and (6.2), we get

$$s^{\deg \varrho}[\varepsilon_i] = ab[[\omega]]. \tag{6.3}$$

We can divide (6.3) by $s^{\deg \varrho} \mod k$ as $s^{\deg \varrho}$ is co-prime to k. Then we have

$$[\varepsilon_i] = \frac{ab}{s^{\deg \varrho}}[[\omega]], \tag{6.4}$$

for all i as required.

When $\omega_i \neq 1$, we can use (6.4) and the fact that $\varepsilon_i \in \langle \xi \rangle$ to get

$$\varepsilon_i = \xi^{ab[[\omega]]/s^{\deg \varrho}}, \tag{6.5}$$

which proves part (5).

Case 2: cyclic case. Now, consider the case that $\hat{\chi}$ and $\hat{\psi}$ are powers of $\hat{\omega}$. Therefore ρ maps $G_{a,b}$ into the subgroup $D_{h,k;\tau} \wr \langle \hat{\omega} \rangle$. Since ω generates the abelianisation of $G_{a,b}$ (by Remark 3.1), we therefore have $\hat{\varepsilon} = \hat{\omega}^{ab}$, so ω_i is constant on orbits of $\hat{\varepsilon}$. By applying Lemma 6.3, ε_i is constant on orbits of $\hat{\omega}$ and commutes with ω_i for all i, so $\varepsilon_i \in \langle \xi \rangle$ if $\omega_i \neq 1$. Let $\sigma_1, \ldots, \sigma_m$ be the disjoint cycles of $\hat{\omega}$ of lengths $l_1, \ldots, l_m$, then the subgroup $D_{h,k;\tau} \wr \langle \hat{\omega} \rangle$ can be regarded as a subgroup of the direct product

$$\prod_{r=1}^{m} D_{h,k;\tau} \wr \langle \sigma_r \rangle = \prod_{r=1}^{m} (D_{h,k;\tau})^{l_r} \rtimes \sigma_r.$$

We will regard the map ρ as a product of maps ρ_r to each factor. In the rest of the proof, σ_r will denote both the cycle and the set of points in the orbit of this cycle of $\digamma$. For $\gamma \in \prod_{r=1}^{m}(D_{h,k;\tau})^{l_r} \rtimes \sigma_r$, we will write $\gamma|_{\sigma_r}$ for the projection of γ on the factor.

Since ω_i and ε_i are constant on σ_r, the abelianisation $[[\cdot]] : D_{h,k;\tau} \wr \langle\sigma_r\rangle \to \mathbb{Z}_k$ gives

$$[[\varepsilon|_{\sigma_r}]] = l_r[\varepsilon_i], \tag{6.6}$$

and

$$[[\varepsilon|_{\sigma_r}]] = ab[[\omega|_{\sigma_r}]]. \tag{6.7}$$

From (6.6) and (6.7), we get

$$[[\varepsilon|_{\sigma_r}]] = l_r[\varepsilon_i] = ab[[\omega|_{\sigma_r}]] = abl_r[\omega_i]. \tag{6.8}$$

As $l_i(\hat{\omega}) \in \{1, s, t\}$ is co-prime to k for all i, we have $[\varepsilon_i] = ab[\omega_i]$ on σ_r. If ω_i is nontrivial on σ_r, then we have

$$\varepsilon_i = \xi^{[\varepsilon_i]} = \xi^{ab[\omega_i]} = \omega_i^{ab} = (\omega^{ab})_i.$$

If ω_i is trivial on σ_r, then $[\omega_i] = 0$ so ε_i is in $\mathbb{V}_{h;\tau}$. We need to prove that ε_i is trivial. We assume that $\varepsilon_i = v \in \mathbb{V}_{h;\tau}$, and that $v \neq 1$, to lead to a contradiction. We claim that ρ_r maps $G_{a,b}$ into the subgroup $\mathbb{V}_{h;\tau} \wr \langle\sigma_r\rangle$. To prove this we need to show that the images of x and y lie in this subgroup, because x and y generate $G_{a,b}$ (by Definition 3.6). As we know, ε and χ commute and ε_i is constant on orbits of $\hat{\chi}$, because the orbits of $\hat{\chi}$ are contained in the orbits of $\hat{\omega}$. Applying Lemma 6.3 we can conclude that χ_i commutes with ε_i for all i. By applying Lemma 5.1 part (2), this means $\chi_i \in \mathbb{V}_{h;\tau}$ on σ_r. We can apply the same argument to y, and then we may conclude that $\psi_i \in \mathbb{V}_{h;\tau}$ for all i; as required.

We have the abelianisation $||\cdot|| : \mathbb{V}_{h;\tau} \wr \langle\sigma_r\rangle \to \mathbb{V}_{h;\tau}$, and applying this we get

$$||\varepsilon|_{\sigma_r}|| = l_r v = ab||\omega|_{\sigma_r}|| = 0.$$

Since l_r is co-prime to h, $v = 1$ is the identity. Therefore, $\varepsilon_i = 1 = \omega_i^{ab} = (\omega^{ab})_i$ on σ_r.

In both cases above we have $\varepsilon_i = (\omega^{ab})_i$, so $\varepsilon = \omega^{ab}$ as required. □

Let us now investigate Lemma 6.6 when $[[\omega]] = 0$, and $\varepsilon \neq \omega^{ab}$.

Lemma 6.7. *Suppose $\omega, \varepsilon, \rho$ and h, k, s, t, τ and ϱ are as described in Lemma 6.6, and let $[[\omega]] = 0$ and $\varepsilon \neq \omega^{ab}$. If ω_j is nontrivial for some j, then ε is trivial.*

Proof. Since $\varepsilon \neq \omega^{ab}$, ε is described by the noncyclic case of Lemma 6.6, and $\varepsilon_i \in \mathbb{V}_{h;\tau}$ for all i. If ω_j is nontrivial, then $\varepsilon_j = \xi^0 = 1$ by (5); and now by (3) the conjugacy class of ε_i is constant, so $\varepsilon_i = 1$ for all i. From statement (1), we have $\hat{\varepsilon} = 1$, so $\varepsilon = 1$. □

Remark 6.2. When $[[\omega]] = 0$, and $\varepsilon \neq \omega^{ab}$, it follows from Lemma 6.5 and Lemma 6.6, if $\omega_i = 1$ for all i, then η_j and ε_j belong to $\mathbb{V}_{h;\tau}$ for all j. Since the orbits of $\hat{\eta}$ coincide with the orbits of $\hat{\omega}$, ε_i is constant on orbits of $\hat{\eta}$. Therefore

$$(\eta\varepsilon)_i = \eta_i\varepsilon_{i\cdot\hat{\eta}} = \eta_i\varepsilon_i = \varepsilon_i\eta_i = (\varepsilon\eta)_i.$$

So ε and η commute as $\widehat{\eta\varepsilon} = \hat{\eta} = \widehat{\varepsilon\eta}$.

6.4 The proof of the main theorem

We devote this section to the proof of our Main Theorem 6.1. To prove the theorem we need to show (ρ, η), which is a map-root pair for $GK_{a,b}$ and $SK_{a,b}$ in $\mathbb{W}^{h,k;\tau}_{s,t;\varrho}$, is compatible for $GK_{a,b}$ only if it is compatible for $SK_{a,b}$. We will study the compatibility conditions for $GK_{a,b}$ and $SK_{a,b}$ in the two cases: when $\widehat{\rho(\omega_0)} = 1$, which is a trivial induced map to $D_{s,t;\varrho}$, and $\widehat{\rho(\omega_0)} \neq 1$, which is a nontrivial induced map to $D_{s,t;\varrho}$.

6.4.1 Case 1: Trivial induced maps

Let $\widehat{\rho(\omega_0)} = 1$. The induced homomorphism $\hat{\rho} : H_{a,b} \to D_{s,t;\varrho}$ is trivial, because the conjugacy class of ω_0 generates $H_{a,b}$ (by Remark 3.1). Then we may regard the homomorphism ρ as a homomorphism $H_{a,b} \to D^{F}_{h,k;\tau}$ which is a product of homomorphisms $H_{a,b} \to D_{h,k;\tau}$. As we have chosen h and k to be co-prime to a and b, each map $\rho : H_{a,b} \to D_{h,k;\tau}$ factors through $\mathbb{Z}$ (by Lemma 5.3). As a result, $\rho : H_{a,b} \to D^{F}_{h,k;\tau}$ factors through $\mathbb{Z}$ too. Then, we have $\rho(x^a) = \rho(w^a) = \rho(\omega_0)^{ab} = \omega^{ab}$. As η commutes with $\eta^n = \omega$, it commutes with both $\rho(x^a)$ and $\rho(w^a)$. Therefore, the pair (ρ, η) is compatible for $GK_{a,b}$ and $SK_{a,b}$.

6.4.2 Case 2: Nontrivial induced maps

In this case, let $\widehat{\rho(\omega_0)} \neq 1$. There is $\omega, \alpha \in \mathbb{W}^{h,k;\tau}_{s,t;\varrho}$ such that $\omega = \alpha\rho(\omega_0)\alpha^{-1} \in \mathcal{A}^{k}_{s,t;\varrho}$ is in reduced standard form (by Lemma 6.5). Then, we can prove statement (A.1) by assuming $\rho(\omega_0) = \omega$, because the pair (ρ, η), where ρ maps $G_{a,b}$ into $\mathbb{W}^{h,k;\tau}_{s,t;\varrho}$, is compatible for either $GK_{a,b}$ or $SK_{a,b}$ if and only if $(\rho', \eta') = (\alpha\rho\alpha^{-1}, \alpha\eta\alpha^{-1})$ is.

If $\varepsilon = \rho(x^a)$ and $\delta = \rho(w^a)$, then we can rewrite the compatibility conditions as follows:

$$(SK_{a,b}) : \qquad \varepsilon\delta^{-1}\eta = \eta\varepsilon\delta^{-1},$$
$$(GK_{a,b}) : \qquad \varepsilon\delta\eta = \eta\varepsilon\delta.$$

Referring to the possibilities for ε and δ in Lemma 6.6, we can study the compatibility conditions in two separate cases:

At least one of ε or δ equals ω^{ab}

We can easily show a pair is compatible for $GK_{a,b}$ if and only if it is compatible for $SK_{a,b}$ from the compatibility conditions by writing them in the form

$$\delta^{\mp 1}\eta\delta^{\pm 1} = \varepsilon^{-1}\eta\varepsilon.$$

As we can see at least one of ε and δ commutes with η, so we can conclude that each compatibility condition is satisfied if and only if the other commutes with η as well.

Neither ε nor δ equals ω^{ab}

In this section we will restrict our attention to the case $[[\omega]] \neq 0$, as when $[[\omega]] = 0$ it is clear both ε and δ commute with η (by Lemma 6.7 and Remark 6.2). Consequently, any such map-root pair is compatible for both $GK_{a,b}$ and $SK_{a,b}$. Now, let us assume that $[[\omega]] \neq 0$. Then

ω_i is constant on orbits of $\hat{\eta}$, and because $\hat{\eta}$ is a power of $\hat{\omega}$, we will investigate cycles of $\hat{\eta}$ on which ω_i is nontrivial and trivial in two separate cases.
In the first case, let $\omega_i \neq 1$. Then by using Lemma 6.5, η_i belongs to $\langle\xi\rangle$, and by Lemma 6.6 both of ε_i and δ_i satisfy Equation (6.5). Therefore,

$$\begin{aligned}(\eta\varepsilon\delta^{\mp1})_i &= \eta_i\varepsilon_{i\cdot\hat{\eta}}\delta^{\mp1}_{i\cdot\hat{\eta}} && \text{(using multiplication in wreath product)}\\ &= \eta_i\varepsilon_i\delta_i^{\mp1} && \text{(as } \varepsilon_i = \varepsilon_{i\cdot\hat{\eta}} \text{ and } \delta_i^{\mp1} = \delta^{\mp1}_{i\cdot\hat{\eta}} \text{ (by Lemma 6.5))}\\ &= \varepsilon_i\delta_i^{\mp1}\eta_i && \text{(both } \varepsilon_i \text{ and } \delta_i^{\mp1} \text{ belong to } \langle\xi\rangle\text{, and therefore they commute with } \eta_i)\\ &= (\varepsilon\delta^{\mp1}\eta)_i && \text{(since } \hat{\varepsilon} = \hat{\delta} = 1).\end{aligned}$$

In the second case, let $\omega_i = 1$. Then by Lemma 6.5 η_i belongs to $\mathbb{V}_{h;\tau}$. Therefore we can write the compatibility condition for $SK_{a,b}$ as follows:

$$\varepsilon_i\delta_i^{-1}\eta_i = \eta_i\varepsilon_{i\cdot\hat{\eta}}\delta^{-1}_{i\cdot\hat{\eta}} = \eta_i\varepsilon_i\delta_i^{-1}.$$

By Lemma 6.6, we get $[\varepsilon_i\delta_i^{-1}] = [\varepsilon_i] - [\delta_i] = \frac{ab}{s^{deg\varrho}}[[\omega]] - \frac{ab}{s^{deg\varrho}}[[\omega]] = 0$, which implies $\varepsilon_i\delta_i^{-1}$ belongs to $\mathbb{V}_{h;\tau}$, and therefore $\varepsilon_i\delta_i^{-1}$ and η_i commute.

On the other hand, the compatibility condition for $GK_{a,b}$ is

$$\varepsilon_i\delta_i\eta_i = \eta_i\varepsilon_{i\cdot\hat{\eta}}\delta_{i\cdot\hat{\eta}} = \eta_i\varepsilon_i\delta_i.$$

By Lemma 6.6, we get $[\varepsilon_i\delta_i] = [\varepsilon_i] + [\delta_i] = 2ab[[\omega]]/s^{\deg\varrho} \neq 0$, because k is co-prime to $2ab$. Thus, $\varepsilon_i\delta_i \neq \mathbb{V}_{h;\tau}$; and therefore $\varepsilon_i\delta_i$ and η_i do not commute (by Lemma 5.1) except when $\eta_i = 1$. Then we can conclude that the condition for $SK_{a,b}$ is satisfied, but the condition for $GK_{a,b}$ is not satisfied.

In the next section, we will complete the proof by showing that not all pairs that are compatible for $SK_{a,b}$ in $\mathbb{W}^{h,k;\tau}_{s,t;\varrho}$ are compatible for $GK_{a,b}$, by showing that we can realise the case above where $[[\omega]] \neq 0$, but $\omega_i = 1$ on some cycle of $\hat{\omega}$.

6.4.3 Realisation

Choose $\hat{\chi} \in D_{s,t;\varrho}$ of order s and $\hat{\psi} \in D_{s,t;\varrho}$ of order t. By Lemma 3.1 we solved $bc - ad = 1$ to get $\hat{\mu} = \hat{\chi}^c\hat{\psi}^{-d}$, where $c, d > 0$, to be the meridian. Since $s|a$ and $t|b$ and $bc - ad = 1$, we have $\gcd(s, c) = 1$. Furthermore, $\hat{\chi}^c$ belongs to $\mathbb{V}_{s;\varrho}$ and has order s. Because elements of order s and t do not commute, $\hat{\chi}$ has no fixed point in $\digamma$ and all cycles are of length s. Then $\hat{\chi}^c$ has no fixed point (by Remark 5.7). We have $\gcd(t, -d) = 1$ (because $t|b$ and $bc - ad = 1$), and then $\hat{\psi}^{-d}$ has order t. Therefore, $\hat{\chi}^c\hat{\psi}^{-d}$ has order t, so $\hat{\mu}$ has a fixed point. Choose f to be the unique fixed point of $\hat{\mu}$ in $\digamma$ which is given by Remark 5.7. Note that the fixed point f is not the fixed point of $\hat{\psi}$, because otherwise, f would be a fixed point of $\hat{\chi}^c = \hat{\mu}\hat{\psi}^d$. Define χ_j and ψ_j in $\mathbb{W}^{h,k;\tau}_{s,t;\varrho}$ such that $\chi_j = \xi^b$ for all j, and

$$\psi_j = \begin{cases}\xi^{a+1}, & \text{if } j = f;\\ \xi^{a-1}, & \text{if } j = f\cdot\hat{\psi}^{-1};\\ \xi^a, & \text{otherwise.}\end{cases}$$

Then it is seen (using Figure 6.1 (i) for the cycle of $\hat{\psi}$ containing f) that $\pi_j(\psi) = \xi^{at}$, for j not the fixed point of $\hat{\psi}$. It follows that

$$(\chi^a)_j = (\psi^b)_j = \xi^{ab}$$

for all j. Since also $\hat{\chi}^a = \hat{\psi}^b = 1$, we have $\chi^a = \psi^b$. Hence $\rho'(x) = \chi, \rho'(y) = \psi$ defines a homomorphism $\rho' : G_{a,b} \to \mathbb{W}^{h,k;\tau}_{s,t;\varrho}$. We may then obtain a homomorphism $H_{a,b} \to \mathbb{W}^{h,k;\tau}_{s,t;\varrho}$ by setting $\rho'(w) = \chi, \rho'(z) = \psi$ also.
We now compute $\rho'(\mu) = \chi^c\psi^{-d}$. To do this it will first be convenient to calculate $\psi^{-d} = (\psi^{-1})^d$. From above we obtain

$$(\psi^{-1})_j = \begin{cases} \xi^{-a+1}, & \text{if } j = f; \\ \xi^{-a-1}, & \text{if } j = f \cdot \hat{\psi}; \\ \xi^{-a}, & \text{otherwise.} \end{cases}$$

Let $d = qt+r$ with $0 < r < t-1$. Then $((\psi^{-1})^d)_j = (\pi_j(\psi^{-1}))^q \prod_{i=0}^{r-1} \psi^{-1}_{j\cdot\hat{\psi}^{-i}} = \xi^{-atq} \prod_{i=0}^{r-1} \psi^{-1}_{j\cdot\hat{\psi}^{-i}}$. Therefore, we will have the following cases:

On cycles that do not contain f

On such cycles $(\psi^{-1})_i = \xi^{-a}$, for all i, so

$$(\psi^{-d})_j = ((\psi^{-1})^d)_j = \xi^{-ad}.$$

On the cycle that contains f

We have $(\psi^{-1})_i = \xi^{-a}$ except at two exceptional points f and $f \cdot \hat{\psi}$. Then the value depends on whether $\prod_{i=0}^{r-1} \psi^{-1}_{j\cdot\hat{\psi}^{-i}}$ involves zero, one or two of these.
When it involves zero

$$\prod_{i=0}^{r-1} \psi^{-1}_{j\cdot\hat{\psi}^{-i}} = (\xi^{-a})^r = \xi^{-ar}.$$

When it involves two, the exceptional points contribute $\xi^{-a+1}\xi^{-a-1} = \xi^{-2a}$ and the others ξ^{-a}. Therefore we get ξ^{-ar}.
When it involves one, the exceptional points are adjacent, so the calculation only involves one if the product begins or ends at an exceptional point.
When it begins at f, we get

$$(\psi^{-d})_f = (\pi_f(\psi^{-1}))^q \prod_{i=0}^{r-1} \psi^{-1}_{f\cdot\hat{\psi}^{-i}} = \xi^{-atq} \prod_{i=0}^{r-1} \psi^{-1}_{f\cdot\hat{\psi}^{-i}} = \xi^{-aqt}\xi^{-ar}\xi = \xi^{-a(qt+r)+1} = \xi^{-ad+1}.$$

When it begins at $f \cdot \hat{\psi}^d$, we get

$$(\psi^{-d})_{f\cdot\hat{\psi}^d} = (\pi_{f\cdot\hat{\psi}^d}(\psi^{-1}))^q \prod_{i=0}^{r-1} \psi^{-1}_{(f\cdot\hat{\psi}^d)\cdot\hat{\psi}^{-i}} = \xi^{-atq} \prod_{i=0}^{r-1} \psi^{-1}_{(f\cdot\hat{\psi}^d)\cdot\hat{\psi}^{-i}} = \xi^{-aqt}\xi^{-ar}\xi^{-1} = \xi^{-a(qt+r)-1} = \xi^{-ad-1}.$$

We can rewrite $((\psi^{-1})^d)_j$ (Figure 6.1(ii)) as follows:

$$((\psi^{-1})^d)_j = \begin{cases} \xi^{-ad+1}, & \text{if } j = f; \\ \xi^{-ad-1}, & \text{if } j = f \cdot \hat{\psi}^d; \\ \xi^{-ad}, & \text{otherwise.} \end{cases}$$

We can write the image of the meridian as follows:

$$\begin{aligned}(\rho'(\mu))_j &= (\chi^c\psi^{-d})_j \\ &= (\chi^c)_j(\psi^{-d})_{j\cdot\hat{\chi}^c} \\ &= (\chi_j)^c(\psi^{-d})_{j\cdot\hat{\chi}^c} \\ &= \xi^{bc}(\psi^{-d})_{j\cdot\hat{\chi}^c}.\end{aligned}$$

To calculate $(\rho'(\mu))_{i,i\in F}$, we use the fact that

$$\begin{aligned}f\cdot\hat{\mu} &= f \\ f\cdot\hat{\chi}^c\hat{\psi}^{-d} &= f \\ f\cdot\hat{\chi}^c &= f\cdot\hat{\psi}^d.\end{aligned}$$

For $j\cdot\hat{\chi}^c \neq f, f\cdot\hat{\psi}^d$, we have

$$(\psi^{-d})_{j\cdot\hat{\chi}^c} = \xi^{-ad}.$$

Therefore,

$$\begin{aligned}(\rho'(\mu))_j &= \xi^{bc}(\psi^{-d})_{j\cdot\hat{\chi}^c} \\ &= \xi^{bc}\xi^{-ad} \\ &= \xi.\end{aligned}$$

The exceptional cases are $j = f\cdot\hat{\psi}^d\hat{\chi}^{-c} = f$ and $j = f\cdot\hat{\chi}^{-c}$.
When $j = f$, we have

$$\begin{aligned}(\rho'(\mu))_f &= \xi^{bc}(\psi^{-d})_{f\cdot\hat{\chi}^c} \\ &= \xi^{bc}(\psi^{-d})_{f\cdot\hat{\psi}^d} && (\text{as } f\cdot\hat{\chi}^c = f\cdot\hat{\psi}^d) \\ &= \xi^{bc}\xi^{-ad-1} && (\text{substitute } ((\psi^{-1})^d)_{f\cdot\hat{\psi}^d} = \xi^{-ad-1}) \\ &= \xi^{bc-ad-1} \\ &= \xi^0 = 1 && (\text{as } bc-ad=1).\end{aligned}$$

When $j = f\cdot\hat{\chi}^{-c}$, we have

$$\begin{aligned}(\rho'(\mu))_{f\cdot\hat{\chi}^{-c}} &= \xi^{bc}(\psi^{-d})_{(f\cdot\hat{\chi}^{-c})\cdot\hat{\chi}^c} \\ &= \xi^{bc}(\psi^{-d})_{f\cdot\hat{\chi}^{c-c}} \\ &= \xi^{bc}(\psi^{-d})_f && (\text{as } \hat{\chi}^{c-c} = \hat{\chi}^0 = 1) \\ &= \xi^{bc}\xi^{-ad+1} && (\text{substitute } ((\psi^{-1})^d)_f = \xi^{-ad+1}) \\ &= \xi^{bc-ad+1} = \xi^2 && (\text{as } bc-ad=1).\end{aligned}$$

Therefore $(\rho'(\mu))_j$ can be written as

$$(\rho'(\mu))_j = \begin{cases} 1, & \text{if } j = f; \\ \xi^2, & \text{if } j = f\cdot\hat{\chi}^{-c}; \\ \xi, & \text{otherwise.}\end{cases}$$

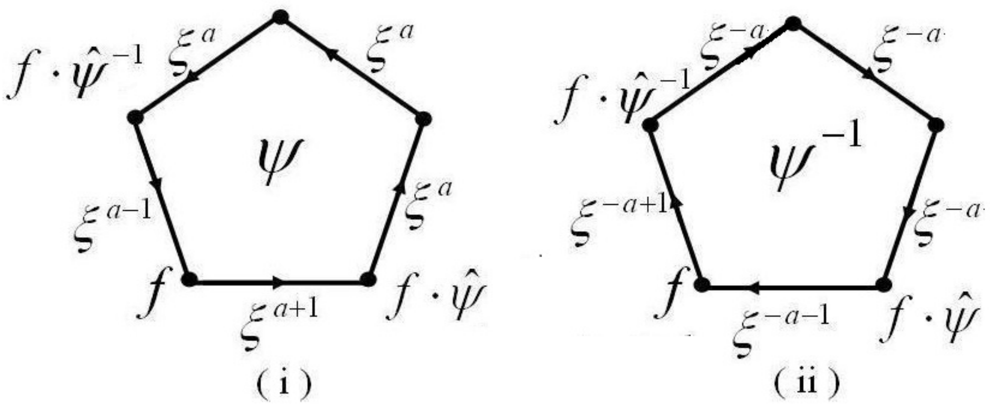

Figure 6.1: Diagram of the cycle of ψ containing f in the case where $t = 5$.

After conjugating ρ' so that ω is in reduced standard form, observe that on the cycle (f) of $\hat{\omega}$ we have $\omega_f = 1$. Since $[[\omega_0]] = s^{\deg \varrho} \neq 0$, this gives us the case where we have map-root pairs that are compatible for $SK_{a,b}$, but are not compatible for $GK_{a,b}$. This shows that the inequality is strict.

Chapter 7

Summary

This thesis has generated a new theorem, which shows that the difference between $G_n(GK_{a,b})$ and $G_n(SK_{a,b})$ can be detected by counting homomorphisms into suitably chosen finite groups. This shows that the difference can be detected by an algorithm.

Main Theorem (Theorem 6.1)**.** Suppose that $n \geq 2$, and a and b are co-prime integers such that $1 < a < b$. Choose primes $s|a$ and $t|b$ such that $\gcd(st, n) = 1$. Let h be a prime dividing n, k a prime co-prime to $2nab$, and let $\tau \in \mathbb{Z}_k[x]$ be an irreducible factor of $F(x) = 1 + \cdots + x^{k-1}$ over $\mathbb{Z}_h$, and $\varrho \in \mathbb{Z}_t[x]$ be an irreducible factor of $F(x) = 1 + \cdots + x^{t-1}$ over $\mathbb{Z}_s$. Then

$$|Hom(G_n(GK_{a,b}), \mathbb{W}^{h,k;\tau}_{s,t;\varrho})| < |Hom(G_n(SK_{a,b}), \mathbb{W}^{h,k;\tau}_{s,t;\varrho})|.$$

Therefore, $G_n(GK_{a,b})$ is not isomorphic to $G_n(SK_{a,b})$ for $n \geq 2$ which satisfies the theorem hypotheses; in particular, this holds if $\gcd(ab, n) = 1$.

In order to prove the resulting theorem, we defined and understood knots and knot groups. For the purpose of the study, we gave our attention to torus knots $T_{a,b}$. We have explained that two curves, a meridian (μ) and a longitude (λ), represent two elements of a knot group in a natural way. We have shown that the knot group $G_{a,b} = \langle x, y | x^a = y^b \rangle$ of the torus knot $T_{a,b}$ has a meridian $\mu = x^c y^{-d} = \omega_0$, where c and $d \in \mathbb{N}$ are a solution to $bc - ad = 1$ (see Lemma 3.1), with a corresponding longitude $\lambda = x^a = y^b$.

We defined generalised knot groups which are knot invariants; and also we gave a brief discussion about the square (SK) and granny (GK) knots and their analogues ($SK_{a,b}$) and ($GK_{a,b}$) made of (a, b)-torus knots. We discussed previous results. Tuffley proved $G_n(SK)$ and $G_n(GK)$ are non-isomorphic, then Nelson and Neumann showed that generalised knot groups distinguish knots up to reflection. We generalised and extended Tuffley's result to the granny and square knot analogues.

We also developed a subgroup $D_{p,q;\theta}$ from Tuffley's $D_{p,q}$ group, as we have found that if $F(x) = 1 + x + x^2 + \cdots + x^{q-1}$ factors over $\mathbb{Z}_p$, then the isomorphism type of $D_{p,q}$ depends on the choice of root of F. We proved that the group $D_{p,q;\theta}$ is generated by elements of order p and q and we also characterised when $D_{p,q;\theta}$ has cyclic and noncyclic solutions to $x^a = y^b$. We constructed our target groups for this study, which are the wreath products $\mathbb{W}^{h,k;\tau}_{s,t;\varrho}$ of the semidirect products $D_{h,k;\tau}$ over $D_{s,t;\varrho}$. We put some restrictions on $\mathbb{W}^{h,k;\tau}_{s,t;\varrho}$ to provide some flexibility in order to prove the result.

We generalised Tuffley's strategy and identified the images and the roots of the meridian and the images of the longitude which are important tools to prove the result. We considered pairs (ρ, η) where ρ is a homomorphism that maps the knot group of the analogues of the granny or square knots into the target group $\mathbb{W}^{h,k;\tau}_{s,t;\varrho}$, and η is an nth root of $\rho(\mu)$. Then, we proved that every compatible pair (ρ, η) for $GK_{a,b}$ in $\mathbb{W}^{h,k;\tau}_{s,t;\varrho}$ is also compatible for $SK_{a,b}$; however, some compatible pairs for $SK_{a,b}$ in $\mathbb{W}^{h,k;\tau}_{s,t;\varrho}$ are not compatible for $GK_{a,b}$.

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