Res. Lett. Inf. Math. Sci., 2004, Vol.**6**, pp157-161
Available online at http://iims.massey.ac.nz/research/letters/

General relations between sums of squares and sums of triangular numbers

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Let $\lambda = (\lambda_1, \cdots, \lambda_m)$ be a partition of k. Let $r_\lambda(n)$ denote the number of solutions in integers of $\lambda_1 x_1^2 + \cdots + \lambda_m x_m^2 = n$, and let $t_\lambda(n)$ denote the number of solutions in non negative integers of $\lambda_1 x_1(x_1+1)/2 + \cdots + \lambda_m x_m(x_m+1)/2 = n$. We prove that if $1 \le k \le 7$, then there is a constant c_λ, depending only on λ, such that $r_\lambda(8n+k) = c_\lambda t_\lambda(n)$, for all integers n.

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1 Introduction

Let $\lambda = (\lambda_1, \cdots, \lambda_m)$ be a partition of k. That is, $\lambda_1, \cdots, \lambda_m$ are integers satisfying $\lambda_1 \ge \cdots \ge \lambda_m > 0$, $\lambda_1 + \cdots + \lambda_m = k$. For any integer n, let $r_\lambda(n)$ denote the number of solutions in integers of

$$\lambda_1 x_1^2 + \cdots + \lambda_m x_m^2 = n,$$

and let $t_\lambda(n)$ denote the number of solutions in non negative integers of

$$\lambda_1 \frac{x_1(x_1+1)}{2} + \cdots + \lambda_m \frac{x_m(x_m+1)}{2} = n.$$

The generating functions for $r_\lambda(n)$ and $t_\lambda(n)$ are

$$\begin{aligned} \sum_{n=0}^{\infty} r_\lambda(n) q^n &= \phi(q^{\lambda_1}) \cdots \phi(q^{\lambda_m}), \\ \sum_{n=0}^{\infty} t_\lambda(n) q^n &= \psi(q^{\lambda_1}) \cdots \psi(q^{\lambda_m}), \end{aligned}$$

where

$$\phi(q) = \sum_{j=-\infty}^{\infty} q^{j^2}, \quad \psi(q) = \sum_{j=0}^{\infty} q^{j(j+1)/2}.$$

Observe that if $\lambda_1 = \cdots = \lambda_m = 1$, then $r_\lambda(n)$ (resp. $t_\lambda(n)$) is the number of representations of n as a sum of m squares (resp. m triangular numbers).

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The purpose of this article is to present the following result.

Theorem 1
If $1 \leq k \leq 7$ and λ is a partition of k, then there exists a constant c_λ, depending only on λ, such that

$$r_\lambda(8n+k) = c_\lambda t_\lambda(n),$$

for all integers n. Setting $n = 0$ we see that the value of c_λ is given by $c_\lambda = r_\lambda(k)$.

2 Examples

1. Let $d_{i,j}(n)$ denote the number of divisors d of n with $d \equiv i \pmod{j}$. ¿From (7) we have

$$\begin{aligned} r_{(2,1)}(n) &= 2(d_{1,8}(n) + d_{3,8}(n) - d_{5,8}(n) - d_{7,8}(n)), \\ r_{(3,1)}(n) &= 2(d_{1,3}(n) - d_{2,3}(n)) + 4(d_{4,12}(n) - d_{8,12}(n)), \end{aligned}$$

while (1) gives

$$\begin{aligned} t_{(2,1)}(n) &= d_{1,8}(8n+3) - d_{7,8}(8n+3), \\ t_{(3,1)}(n) &= d_{1,6}(2n+1) - d_{5,6}(2n+1). \end{aligned}$$

Observe that $d_{1,8}(8n+3) = d_{3,8}(8n+3)$ and $d_{5,8}(8n+3) = d_{7,8}(8n+3)$. This implies

$$\begin{aligned} & r_{(2,1)}(8n+3) \\ &= 2(d_{1,8}(8n+3) + d_{3,8}(8n+3) - d_{5,8}(8n+3) - d_{7,8}(8n+3)) \\ &= 4(d_{1,8}(8n+3) - d_{7,8}(8n+3)) \\ &= 4t_{(2,1)}(n). \end{aligned}$$

This is Theorem 1 for the partition $\lambda = (2,1)$, and $c_{(2,1)} = 4$.

Similarly, observe that

$$d_{4,12}(8n+4) = d_{1,3}(2n+1), \quad d_{8,12}(8n+4) = d_{2,3}(2n+1),$$

$$d_{1,3}(8n+4) - d_{2,3}(8n+4) = d_{1,3}(2n+1) - d_{2,3}(2n+1),$$

and

$$d_{1,3}(2n+1) = d_{1,6}(2n+1), \quad d_{2,3}(2n+1) = d_{5,6}(2n+1).$$

Therefore

$$\begin{aligned} & r_{(3,1)}(8n+4) \\ &= 2(d_{1,3}(8n+4) - d_{2,3}(8n+4)) + 4(d_{4,12}(8n+4) - d_{8,12}(8n+4)) \\ &= 2(d_{1,3}(2n+1) - d_{2,3}(2n+1)) + 4(d_{1,3}(2n+1) - d_{2,3}(2n+1)) \\ &= 6(d_{1,3}(2n+1) - d_{2,3}(2n+1)) \\ &= 6(d_{1,6}(2n+1) - d_{5,6}(2n+1)) \\ &= 6t_{(3,1)}(n). \end{aligned}$$

This is Theorem 1 for the partition $\lambda = (3,1)$, and $c_{(3,1)} = 6$.

These examples motivated us to discover Theorem 1.

2. Let $\lambda = (1, \cdots, 1)$ be the partition consisting of k 1's. In this case $r_\lambda(n) = r_k(n)$ and $t_\lambda(n) = t_k(n)$, where $r_k(n)$ and $t_k(n)$ are the number of representations of n as a sum of k squares, and as a sum of k triangular numbers, respectively. Then it was shown in (2), (3) that
$$r_k(8n+k) = 2^{k-1}\left\{2 + \binom{k}{4}\right\} t_k(n)$$
for all n, provided $1 \le k \le 7$. Thus $c_{(1,\cdots,1)} = 2^{k-1}\left\{2 + \binom{k}{4}\right\}$.

3. We conclude with tables listing the values of the constants c_λ:

λ	(1)
c_λ	2

λ	(2)	$(1,1)$
c_λ	2	4

λ	(3)	$(2,1)$	$(1,1,1)$
c_λ	2	4	8

λ	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
c_λ	2	6	4	12	24

λ	(5)	$(4,1)$	$(3,2)$	$(3,1,1)$	$(2,2,1)$	$(2,1,1,1)$	$(1,1,1,1,1)$
c_λ	2	4	4	16	8	40	112

λ	(6)	$(5,1)$	$(4,2)$	$(4,1,1)$	$(3,3)$	$(3,2,1)$	$(3,1,1,1)$	$(2,2,2)$	$(2,2,1,1)$	$(2,1,1,1,1)$	$(1,1,1,1,1,1)$
c_λ	2	4	4	8	4	12	40	8	32	144	544

λ	(7)	$(6,1)$	$(5,2)$	$(5,1,1)$	$(4,3)$	$(4,2,1)$	$(4,1,1,1)$	$(3,3,1)$	$(3,2,2)$	$(3,2,1,1)$	$(3,1,1,1,1)$	$(2,2,2,1)$	$(2,2,1,1,1)$	$(2,1,1,1,1,1)$	$(1,1,1,1,1,1,1)$
c_λ	2	4	4	8	4	8	16	16	8	40	112	16	128	544	2368

3 Technique of proof

We illustrate the technique of proof by proving Theorem 1 for the case $\lambda = (3,2,1,1)$. Proofs for all the other partitions are similar, and in most cases simpler. The proofs all make use of various parts of the following lemma.

Lemma

$$\phi(q) = \phi(q^4) + 2q\psi(q^8),$$
$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2,$$
$$\phi(q)\psi(q^2) = \psi(q)^2,$$
$$\psi(q)\psi(q^3) = \phi(q^6)\psi(q^4) + q\psi(q^{12})\phi(q^2).$$

Proof
The first three parts can be obtained by combining various results in (5, p. 40, Entry 25). See (2) for the specific details. A proof of the fourth part is given in (5, p. 69 , Eq. (36.8)) or (6,

Preliminary lemmas, part (xxxiii)).

□

Proof of Theorem 1 in the case $\lambda = (3,2,1,1)$
Using the generating function for $r_{(3,2,1,1)}(n)$ and the first two parts of the Lemma, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} r_{(3,2,1,1)}(n)q^n\\
=\;& \phi(q^3)\phi(q^2)\phi(q)^2\\
=\;& \left[\phi(q^{12}) + 2q^3\psi(q^{24})\right]\left[\phi(q^8) + 2q^2\psi(q^{16})\right]\left[\phi(q^4) + 2q\psi(q^8)\right]^2\\
=\;& \left[\phi(q^{48}) + 2q^{12}\psi(q^{96}) + 2q^3\psi(q^{24})\right]\left[\phi(q^8) + 2q^2\psi(q^{16})\right]\\
& \times\left[\phi(q^4)^2 + 4q\phi(q^4)\psi(q^8) + 4q^2\psi(q^8)^2\right]\\
=\;& \left[\phi(q^{48}) + 2q^{12}\psi(q^{96}) + 2q^3\psi(q^{24})\right]\left[\phi(q^8) + 2q^2\psi(q^{16})\right]\\
& \times\left[\phi(q^8)^2 + 4q^4\psi(q^{16})^2 + 4q(\phi(q^{16}) + 2q^4\psi(q^{32}))\psi(q^8) + 4q^2\psi(q^8)^2\right].
\end{aligned}$$

Extract the terms in which the power of q is congruent to $7 \pmod 8$, divide by q^7 and replace q^8 by q, to obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} r_{(3,2,1,1)}(8n+7)q^n &= 16\phi(q^6)\psi(q^4)\psi(q^2)\psi(q)\\
&+16q\psi(q^{12})\phi(q^2)\psi(q^2)\psi(q) + 8\psi(q^3)\psi(q^2)^2\phi(q) + 16\psi(q^3)\psi(q^2)\psi(q)^2.
\end{aligned}$$

Now use the third and fourth parts of the Lemma to obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} r_{(3,2,1,1)}(8n+7)q^n\\
=\;& 16\left[\phi(q^6)\psi(q^4) + q\psi(q^{12})\phi(q^2)\right]\psi(q^2)\psi(q) + 8\psi(q^3)\psi(q^2)\left[\phi(q)\psi(q^2)\right]\\
& \quad +16\psi(q^3)\psi(q^2)\psi(q)^2\\
=\;& 16\psi(q^3)\psi(q^2)\psi(q)^2 + 8\psi(q^3)\psi(q^2)\psi(q)^2 + 16\psi(q^3)\psi(q^2)\psi(q)^2\\
=\;& 40\psi(q^3)\psi(q^2)\psi(q)^2\\
=\;& 40\sum_{n=0}^{\infty} t_{(3,2,1,1)}(n)q^n.
\end{aligned}$$

This proves Theorem 1 for the partition $\lambda = (3,2,1,1)$, and we see that
$c_{(3,2,1,1)} = 40$.

□

4 Concluding remarks

If $\lambda = (\lambda_1, \cdots, \lambda_m)$ is a partition of $k = 8$ and $\gcd(\lambda_1, \cdots, \lambda_m) = 1$, then it is straightforward to verify, by checking each partition one at a time, that there does not exist a constant c_λ such that $r_\lambda(8n+8) = c_\lambda t_\lambda(n)$ for all n. We conjecture that Theorem 1 does not hold for any partition $\lambda = (\lambda_1, \cdots, \lambda_m)$ of k, for which $k \geq 8$ and $\gcd(\lambda_1, \cdots, \lambda_m) = 1$. This conjecture is known to be true when $\lambda_1 = \cdots \lambda_m = 1$, $m \geq 8$; see (4).

Acknowledgement

The second author thanks the Department of Studies in Mathematics, University of Mysore, for warm hospitality during his visit.

References

[1] Chandrashekar Adiga and K. R. Vasuki, *On sums of triangular numbers,* The Mathematics Student, **70**, (2001), 185–190.

[2] P. Barrucand, S. Cooper and M. Hirschhorn, *Relations between squares and triangles,* Discrete Math. **248** (2002), no. 1-3, 245–247.

[3] P. T. Bateman and M. I. Knopp, *Some new old-fashioned modular identities,* Ramanujan J. **2** (1998), no. 1-2, 247–269.

[4] P. T. Bateman, B. A. Datskovsky and M. I. Knopp, *Sums of squares and the preservation of modularity under congruence restrictions,* Symbolic computation, number theory, special functions, physics and combinatorics (Gainesville, FL, 1999), 59–71, Dev. Math., **4**, Kluwer Acad. Publ., Dordrecht, 2001.

[5] B. C. Berndt, *Ramanujan's Notebooks, Part III,* Springer-Verlag, New York, 1991.

[6] S. Cooper and M. Hirschhorn, *Results of Hurwitz type for three squares,* Discrete Math. **274**, (2004), 9–24.

[7] M. Hirschhorn, *Partial fractions and four classical theorems of number theory,* Amer. Math. Monthly **107** (2000), no. 3, 260–264.