Res. Lett. Inf. Math. Sci., (2001) **2**, 1–10
Available online at http://www.massey.ac.nz/~wwims/rlims/

On the q-analogue of Kummer's 24 solutions

Shaun Cooper
I.I.M.S., Massey University Albany Campus, Auckland, New Zealand
s.cooper@massey.ac.nz

Abstract
The ${}_3\phi_2$ transformations are used to derive q-analogues of the relations amongst Kummer's 24 solutions.

1 Introduction

The purpose of this article is to give q-analogues of the 20 relations between Kummer's 24 hypergeometric functions.

Standard notation for q-series is used throughout - see, for example, [1, Chapter 10] or [4, Chapter 1].

An interesting feature of the results is the occurrence of divergent series, namely the ${}_3\phi_1$ and ${}_2\phi_0$ functions. It is still an open question to assign a meaningful interpretation to these divergent series. They formally reduce to hypergeometric functions when $q \to 1$.

The paper is organised as follows. In section 2, the basic transformation formulas for the ${}_2\phi_1$ function are given. Three fundamental transformation properties of ${}_3\phi_2$ functions are given in section 3. These formulas are written using Sears' [6] q-analogue of Whipple's notation. Only two of these formulas will be used; the third is mentioned for completeness. The 20 q-analogues of the relations between Kummer's 24 solutions - the main results of this paper - are given in section 4. A proof of one of the 20 results is given in detail in section 5 and the remaining proofs, which are similar, are summarized in a table. The key idea in the proofs is to use limiting properties of ${}_3\phi_2$ functions, for example:

$$\lim_{f=abcx/e,\ c\to 0} {}_3\phi_2\left(\begin{matrix} a,b,c \\ e,f \end{matrix}; \frac{ef}{abc}\right) = {}_2\phi_1\left(\begin{matrix} a,b \\ e \end{matrix}; x\right).$$

2 Basic properties of the ${}_2\phi_1$ function

The following formulas will be used throughout.

$$\begin{aligned}
{}_2\phi_1\left(\begin{matrix} a,b \\ e \end{matrix}; \frac{e}{ab}\right) &= \frac{(e/a, e/b; q)_\infty}{(e, e/ab; q)_\infty}; && (1) \\
{}_2\phi_1\left(\begin{matrix} a,b \\ e \end{matrix}; x\right) &= \frac{(b, ax; q)_\infty}{(e, x; q)_\infty} {}_2\phi_1\left(\begin{matrix} e/b, x \\ ax \end{matrix}; b\right) && (2) \\
&= \frac{(e/b, bx; q)_\infty}{(e, x; q)_\infty} {}_2\phi_1\left(\begin{matrix} abx/e, b \\ bx \end{matrix}; e/b\right) && (3) \\
&= \frac{(abx/e; q)_\infty}{(x; q)_\infty} {}_2\phi_1\left(\begin{matrix} e/a, e/b \\ e \end{matrix}; abx/e\right) && (4) \\
&= \frac{(ax; q)_\infty}{(x; q)_\infty} {}_2\phi_2\left(\begin{matrix} a, e/b \\ e, ax \end{matrix}; bx\right) && (5)
\end{aligned}$$

$$\begin{aligned} &= \frac{(ax, bx; q)_\infty}{(e, x; q)_\infty} {}_2\phi_2\left(\begin{matrix} x, abx/e \\ ax, bx \end{matrix}; e\right) && (6) \\ &= \frac{(abx/e; q)_\infty}{(bx/e; q)_\infty} {}_3\phi_2\left(\begin{matrix} a, e/b, 0 \\ e, eq/bx \end{matrix}; q\right). && (7) \end{aligned}$$

Equation (1) is Heine's q-analogue of Gauss' theorem [1, p. 522], [2, p. 68], [4, p. 10]. Equations (2), (3) and (4) are Heine's transformation, the iterate of Heine's transformation, and the q-analogue of Euler's transformation, respectively [1, pp. 521–524], [4, pp. 9–10]. Equation (5) is Jackson's q-analogue of Pfaff's transformation [1, (10.10.12)], [4, (1.5.4)], and equation (6) appears in [6] as the function $Y(1,6)$ in Table IIA. Equation (7) is due to Jackson [4, p. 241, (III.5)].

3 Transformation properties of ${}_3\phi_2$ functions

Let r_0, r_1, r_2, r_3, r_4, r_5 be six parameters such that $r_0r_1r_2r_3r_4r_5 = 1$. With these parameters, associate numbers α_{lmn} and β_{mn} such that

$$\begin{aligned} \alpha_{lmn} &= q^{1/2} r_l r_m r_n \\ \beta_{mn} &= q r_m / r_n. \end{aligned}$$

Let $(i; j, k; l, m, n)$ be any permutation of $(0, 1, 2, 3, 4, 5)$. The semicolons indicate that permutations with the second and third entries interchanged are regarded as equivalent, as are permutations with the last three entries in any order. For example, $(2; 1, 4; 5, 0, 3)$ is equivalent to $(2; 4, 1; 5, 0, 3)$, and both are equivalent to $(2; 1, 4; 0, 3, 5)$. Consequently, it is only necessary to specify the first three entries, in which case $(i; j, k)$ can be used to represent $(i; j, k; l, m, n)$. Accordingly, let

$$F(i; j, k) = (\beta_{ji}, \beta_{ki}, \alpha_{lmn}; q)_\infty {}_3\phi_2\left(\begin{matrix} \alpha_{jkl}, \alpha_{jkm}, \alpha_{jkn} \\ \beta_{ji}, \beta_{ki} \end{matrix}; \alpha_{lmn}\right). \quad (8)$$

Set $\alpha_{145} = a$, $\alpha_{245} = b$, $\alpha_{345} = c$, $\beta_{40} = e$, $\beta_{50} = f$. Equivalently,

$$\begin{aligned} r_0^3 &= \frac{q^{5/2}abc}{e^2 f^2} \\ r_1^3 &= \frac{aef}{q^{1/2}b^2c^2} \\ r_2^3 &= \frac{bef}{q^{1/2}a^2c^2} \\ r_3^3 &= \frac{cef}{q^{1/2}a^2b^2} \\ r_4^3 &= \frac{abce}{q^{1/2}f^2} \\ r_5^3 &= \frac{abcf}{q^{1/2}e^2}. \end{aligned}$$

All of the α's and β's can be expressed in terms of a, b, c, d, e and q:

$\alpha_{012} = q/c$	$\alpha_{123} = ef/abc$	$\beta_{01} = q^2bc/ef$	$\beta_{20} = ef/ac$	$\beta_{40} = e$
$\alpha_{013} = q/b$	$\alpha_{124} = e/c$	$\beta_{02} = q^2ac/ef$	$\beta_{21} = qb/a$	$\beta_{41} = qbc/f$
$\alpha_{014} = qa/f$	$\alpha_{125} = f/c$	$\beta_{03} = q^2ab/ef$	$\beta_{23} = qb/c$	$\beta_{42} = qac/f$
$\alpha_{015} = qa/e$	$\alpha_{134} = e/b$	$\beta_{04} = q^2/e$	$\beta_{24} = qf/ac$	$\beta_{43} = qab/f$
$\alpha_{023} = q/a$	$\alpha_{135} = f/b$	$\beta_{05} = q^2/f$	$\beta_{25} = qe/ac$	$\beta_{45} = qe/f$
$\alpha_{024} = qb/f$	$\alpha_{145} = a$	$\beta_{10} = ef/bc$	$\beta_{30} = ef/ab$	$\beta_{50} = f$
$\alpha_{025} = qb/e$	$\alpha_{234} = e/a$	$\beta_{12} = qa/b$	$\beta_{31} = qc/a$	$\beta_{51} = qbc/e$
$\alpha_{034} = qc/f$	$\alpha_{235} = f/a$	$\beta_{13} = qa/c$	$\beta_{32} = qc/b$	$\beta_{52} = qac/e$
$\alpha_{035} = qc/e$	$\alpha_{245} = b$	$\beta_{14} = qf/bc$	$\beta_{34} = qf/ab$	$\beta_{53} = qab/e$
$\alpha_{045} = qabc/ef$	$\alpha_{345} = c$	$\beta_{15} = qe/bc$	$\beta_{35} = qe/ab$	$\beta_{54} = qf/e$

With this notation,

$$F(0;4,5) = (e,f,ef/abc;q)_\infty {}_3\phi_2\left(\begin{matrix} a,b,c \\ e,f \end{matrix}; \frac{ef}{abc}\right). \tag{9}$$

Two term transformations

Hall's [1, (10.10.8)], [4, (4.2.10)], [5], [6, (p. 173, statement I)] formula

$$\begin{aligned} &(e,f,ef/abc;q)_\infty {}_3\phi_2\left(\begin{matrix} a,b,c \\ e,f \end{matrix}; \frac{ef}{abc}\right) \\ = \quad &(a,ef/ab,ef/ac;q)_\infty {}_3\phi_2\left(\begin{matrix} e/a,f/a,ef/abc \\ ef/ab,ef/ac \end{matrix}; a\right) \end{aligned}$$

becomes in this notation simply

$$F(0;4,5) = F(0;2,3). \tag{10}$$

Interchanging r_4 and r_1, we find that

$$F(0;1,5) = F(0;2,3)$$

and thus

$$F(0;1,5) = F(0;4,5). \tag{11}$$

Accordingly all of the permutations of the indices 1 to 5 are legitimate, and we see that all ten expressions $F(0;v,w)$ are equal, and may be denoted by $F(0)$. For every value $u = 1,\cdots,5$, the ten functions $F(u;v,w)$ are equal and may be denoted by $F(u)$.

Three term relations

The three term relation [4, (3.3.1)], [6, (p. 173 II(a))] in this notation is equivalent to

$$\begin{aligned} F(i) \quad = \quad & \frac{(\alpha_{klm},\alpha_{kln},\alpha_{kmn},\beta_{ji};q)_\infty}{(\beta_{kj}/q;q)_\infty} {}_3\phi_2\left(\begin{matrix} \alpha_{jlm},\alpha_{jln},\alpha_{jmn} \\ \beta_{ji},\beta_{jk} \end{matrix}; q\right) \\ + \quad & \frac{(\alpha_{jlm},\alpha_{jln},\alpha_{jmn},\beta_{ki};q)_\infty}{(\beta_{jk}/q;q)_\infty} {}_3\phi_2\left(\begin{matrix} \alpha_{klm},\alpha_{kln},\alpha_{kmn} \\ \beta_{ki},\beta_{kj} \end{matrix}; q\right), \end{aligned} \tag{12}$$

where (i,j,k,l,m,n) is any permutation of $(0,1,2,3,4,5)$. Observe that the right hand side is symmetric in j and k, and also in l, m and n. Consequently, there are $6\times\binom{5}{2} = 60$ equations of this type.

Although we shall not need it, the three term relation [4, (3.3.3)], [6, (p. 173 III(b))], written in this notation, is

$$\begin{aligned} 0 \quad = \quad & (r_k - r_j)(\beta_{kj},\beta_{jk},\alpha_{ilm},\alpha_{iln},\alpha_{imn};q)_\infty F(i) \\ + \quad & (r_i - r_k)(\beta_{ik},\beta_{ki},\alpha_{jlm},\alpha_{jln},\alpha_{jmn};q)_\infty F(j) \\ + \quad & (r_j - r_i)(\beta_{ji},\beta_{ij},\alpha_{klm},\alpha_{kln},\alpha_{kmn};q)_\infty F(k). \end{aligned} \tag{13}$$

Any permutation of the indices $0, \cdots, 5$ is legitimate, and so there are $\binom{6}{3} = 20$ equations of this type relating any three of the functions $F(0), \cdots, F(5)$.

Equations (10) and (11) are q-analogues of [2, section 3.6, (1) and (2)], while (12) and (13) are q-analogues of [2, section 3.7, (2) and (3)], respectively.

4 The 20 relations

In this section, q-analogues of the 20 relations between Kummer's solutions are given. For the purpose of ease of reference, the same equation numbering system as [3, pp. 106 – 108] is used, i.e., our equation (25) is the q-analogue of equation (25) on page 106 of [3], etc.

$$\begin{aligned} &{}_3\phi_1\left(\begin{matrix} a, b, q/x \\ qab/e \end{matrix}; \frac{x}{e}\right) \\ = \;& \frac{(qa/e, e)_\infty}{(qab/e, e/b)_\infty} \frac{(x, q/x)_\infty}{(bx, q/bx)_\infty} {}_2\phi_0\left(\begin{matrix} 1/b, bx/e \\ - \end{matrix}; \frac{e}{x}\right) {}_2\phi_1\left(\begin{matrix} a, b \\ e \end{matrix}; x\right) \\ + \;& \frac{(b, qa/b)_\infty}{(qab/e, e/b)_\infty} \frac{(abx/e, qe/abx)_\infty}{(bx/q, q^2/bx)_\infty} {}_2\phi_0\left(\begin{matrix} q/e, x/q \\ - \end{matrix}; \frac{e}{x}\right) {}_2\phi_1\left(\begin{matrix} a, qa/e \\ qa/b \end{matrix}; \frac{qe}{abx}\right) \end{aligned} \tag{25}$$

$$\begin{aligned} &{}_3\phi_1\left(\begin{matrix} a, b, q/x \\ qab/e \end{matrix}; \frac{x}{e}\right) \\ = \;& \frac{(qb/e, e)_\infty}{(qab/e, e/a)_\infty} \frac{(x, q/x)_\infty}{(ax, q/ax)_\infty} {}_2\phi_0\left(\begin{matrix} 1/a, ax/e \\ - \end{matrix}; \frac{e}{x}\right) {}_2\phi_1\left(\begin{matrix} a, b \\ e \end{matrix}; x\right) \\ + \;& \frac{(a, qb/a)_\infty}{(qab/e, e/a)_\infty} \frac{(abx/e, qe/abx)_\infty}{(ax/q, q^2/ax)_\infty} {}_2\phi_0\left(\begin{matrix} q/e, x/q \\ - \end{matrix}; \frac{e}{x}\right) {}_2\phi_1\left(\begin{matrix} b, qb/e \\ qb/a \end{matrix}; \frac{qe}{abx}\right) \end{aligned} \tag{26}$$

$$\begin{aligned} &\frac{(abx/e)_\infty}{(x)_\infty} {}_3\phi_1\left(\begin{matrix} e/a, e/b, eq/abx \\ eq/ab \end{matrix}; \frac{abx}{e^2}\right) \\ = \;& \frac{(q/a, e)_\infty}{(qe/ab, b)_\infty} \frac{(abx/e, qe/abx)_\infty}{(ax, q/ax)_\infty} {}_2\phi_0\left(\begin{matrix} b/e, ax/e \\ - \end{matrix}; \frac{e^2}{abx}\right) {}_2\phi_1\left(\begin{matrix} a, b \\ e \end{matrix}; x\right) \\ + \;& \frac{(qb/a, e/b)_\infty}{(qe/ab, b)_\infty} \frac{(abx/e, qe/abx)_\infty}{(ax/q, q^2/ax)_\infty} {}_2\phi_0\left(\begin{matrix} q/e, abx/eq \\ - \end{matrix}; \frac{e^2}{abx}\right) {}_2\phi_1\left(\begin{matrix} b, bq/e \\ bq/a \end{matrix}; \frac{qe}{abx}\right) \end{aligned} \tag{27}$$

$$\begin{aligned} &{}_3\phi_1\left(\begin{matrix} qa/e, qb/e, q/x \\ qab/e \end{matrix}; \frac{ex}{q^2}\right) \\ = \;& \frac{(a, q^2/e)_\infty}{(qab/e, q/b)_\infty} \frac{(x, q/x)_\infty}{(qbx/e, e/bx)_\infty} {}_2\phi_0\left(\begin{matrix} e/qb, bx/q \\ - \end{matrix}; \frac{q^2}{ex}\right) {}_2\phi_1\left(\begin{matrix} qa/e, qb/e \\ q^2/e \end{matrix}; x\right) \\ + \;& \frac{(qb/e, qa/b)_\infty}{(qab/e, q/b)_\infty} \frac{(abx/e, qe/abx)_\infty}{(bx/e, qe/bx)_\infty} {}_2\phi_0\left(\begin{matrix} e/q, x/q \\ - \end{matrix}; \frac{q^2}{ex}\right) {}_2\phi_1\left(\begin{matrix} qa/e, a \\ qa/b \end{matrix}; \frac{qe}{abx}\right) \end{aligned} \tag{28}$$

$$\begin{aligned} &\frac{(abx/e)_\infty}{(x)_\infty} {}_3\phi_1\left(\begin{matrix} e/a, e/b, eq/abx \\ eq/ab \end{matrix}; \frac{abx}{e^2}\right) \\ = \;& \frac{(q/b, e)_\infty}{(qe/ab, a)_\infty} \frac{(abx/e, qe/abx)_\infty}{(bx, q/bx)_\infty} {}_2\phi_0\left(\begin{matrix} a/e, bx/e \\ - \end{matrix}; \frac{e^2}{abx}\right) {}_2\phi_1\left(\begin{matrix} a, b \\ e \end{matrix}; x\right) \\ + \;& \frac{(qa/b, e/a)_\infty}{(qe/ab, a)_\infty} \frac{(abx/e, qe/abx)_\infty}{(bx/q, q^2/bx)_\infty} {}_2\phi_0\left(\begin{matrix} q/e, abx/eq \\ - \end{matrix}; \frac{e^2}{abx}\right) {}_2\phi_1\left(\begin{matrix} a, aq/e \\ aq/b \end{matrix}; \frac{qe}{abx}\right) \end{aligned}$$

$$(29)$$

$$\begin{aligned}
&{}_3\phi_1\left(\begin{matrix} qa/e, qb/e, q/x \\ qab/e \end{matrix}; \frac{ex}{q^2}\right) \\
= \quad & \frac{(b, q^2/e)_\infty}{(qab/e, q/a)_\infty} \frac{(x, q/x)_\infty}{(qax/e, e/ax)_\infty} {}_2\phi_0\left(\begin{matrix} e/qa, ax/q \\ - \end{matrix}; \frac{q^2}{ex}\right) {}_2\phi_1\left(\begin{matrix} qa/e, qb/e \\ q^2/e \end{matrix}; x\right) \\
+ \quad & \frac{(qa/e, qb/a)_\infty}{(qab/e, q/a)_\infty} \frac{(abx/e, qe/abx)_\infty}{(ax/e, qe/ax)_\infty} {}_2\phi_0\left(\begin{matrix} e/q, x/q \\ - \end{matrix}; \frac{q^2}{ex}\right) {}_2\phi_1\left(\begin{matrix} qb/e, b \\ qb/a \end{matrix}; \frac{qe}{abx}\right)
\end{aligned}$$

$$(30)$$

$$\begin{aligned}
&{}_3\phi_1\left(\begin{matrix} q/a, q/b, qe/abx \\ qe/ab \end{matrix}; \frac{abx}{q^2}\right) \\
= \quad & \frac{(e/a, q^2/e)_\infty}{(qe/ab, qb/e)_\infty} \frac{(x, qe/abx)_\infty}{(qax/e, e/ax)_\infty} {}_2\phi_0\left(\begin{matrix} b/q, ax/q \\ - \end{matrix}; \frac{q^2}{abx}\right) {}_2\phi_1\left(\begin{matrix} qa/e, qb/e \\ q^2/e \end{matrix}; x\right) \\
+ \quad & \frac{(q/b, qb/a)_\infty}{(qe/ab, qb/e)_\infty} \frac{(x, qe/abx)_\infty}{(ax/e, qe/ax)_\infty} {}_2\phi_0\left(\begin{matrix} e/q, abx/eq \\ - \end{matrix}; \frac{q^2}{abx}\right) {}_2\phi_1\left(\begin{matrix} qb/e, b \\ qb/a \end{matrix}; \frac{qe}{abx}\right)
\end{aligned}$$

$$(31)$$

$$\begin{aligned}
&{}_3\phi_1\left(\begin{matrix} q/a, q/b, qe/abx \\ qe/ab \end{matrix}; \frac{abx}{q^2}\right) \\
= \quad & \frac{(e/b, q^2/e)_\infty}{(qe/ab, qa/e)_\infty} \frac{(x, qe/abx)_\infty}{(qbx/e, e/bx)_\infty} {}_2\phi_0\left(\begin{matrix} a/q, bx/q \\ - \end{matrix}; \frac{q^2}{abx}\right) {}_2\phi_1\left(\begin{matrix} qa/e, qb/e \\ q^2/e \end{matrix}; x\right) \\
+ \quad & \frac{(q/a, qa/b)_\infty}{(qe/ab, qa/e)_\infty} \frac{(x, qe/abx)_\infty}{(bx/e, qe/bx)_\infty} {}_2\phi_0\left(\begin{matrix} e/q, abx/eq \\ - \end{matrix}; \frac{q^2}{abx}\right) {}_2\phi_1\left(\begin{matrix} a, qa/e \\ qa/b \end{matrix}; \frac{qe}{abx}\right)
\end{aligned}$$

$$(32)$$

$$\begin{aligned}
&{}_2\phi_1\left(\begin{matrix} a, b \\ e \end{matrix}; x\right) \\
= \quad & \frac{(e/a, e/b)_\infty}{(e, e/ab)_\infty} {}_3\phi_2\left(\begin{matrix} a, b, abx/e \\ 0, qab/e \end{matrix}; q\right) \\
+ \quad & \frac{(a, b)_\infty}{(e, ab/e)_\infty} \frac{(abx/e)_\infty}{(x)_\infty} {}_3\phi_2\left(\begin{matrix} e/a, e/b, x \\ 0, eq/ab \end{matrix}; q\right)
\end{aligned}$$

$$(33)$$

$$\begin{aligned}
&{}_2\phi_1\left(\begin{matrix} a, b \\ e \end{matrix}; x\right) \\
= \quad & \frac{(e/a, b)_\infty}{(e, b/a)_\infty} \frac{(ax, q/ax)_\infty}{(x, q/x)_\infty} {}_2\phi_1\left(\begin{matrix} a, qa/e \\ qa/b \end{matrix}; \frac{qe}{abx}\right) \\
+ \quad & \frac{(a, e/b)_\infty}{(e, a/b)_\infty} \frac{(bx, q/bx)_\infty}{(x, q/x)_\infty} {}_2\phi_1\left(\begin{matrix} b, qb/e \\ qb/a \end{matrix}; \frac{qe}{abx}\right)
\end{aligned}$$

$$(34)$$

$$\begin{aligned}
&{}_3\phi_1\left(\begin{matrix} a, b, q/x \\ qab/e \end{matrix}; \frac{x}{e}\right) \\
= \quad & \frac{(qa/e, qb/e)_\infty}{(q/e, qab/e)_\infty} {}_2\phi_1\left(\begin{matrix} a, b \\ e \end{matrix}; x\right) \\
+ \quad & \frac{(a, b)_\infty}{(qab/e, e/q)_\infty} {}_2\phi_0\left(\begin{matrix} e/q, q/x \\ - \end{matrix}; \frac{x}{e}\right) {}_2\phi_1\left(\begin{matrix} qa/e, qb/e \\ q^2/e \end{matrix}; x\right)
\end{aligned}$$

$$(35)$$

$$
\begin{aligned}
& {}_3\phi_1\left(\begin{matrix} a, b, q/x \\ qab/e \end{matrix}; \frac{x}{e}\right) \\
= \quad & \frac{(qb/e, b)_\infty}{(b/a, qab/e)_\infty} {}_2\phi_0\left(\begin{matrix} 1/a, ax/e \\ - \end{matrix}; \frac{e}{x}\right) {}_2\phi_1\left(\begin{matrix} a, qa/e \\ qa/b \end{matrix}; \frac{qe}{abx}\right) \\
+ \quad & \frac{(a, qa/e)_\infty}{(a/b, qab/e)_\infty} {}_2\phi_0\left(\begin{matrix} 1/b, bx/e \\ - \end{matrix}; \frac{e}{x}\right) {}_2\phi_1\left(\begin{matrix} b, qb/e \\ qb/a \end{matrix}; \frac{qe}{abx}\right)
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
& {}_2\phi_1\left(\begin{matrix} a, qa/e \\ qa/b \end{matrix}; \frac{qe}{abx}\right) \\
= \quad & \frac{(q/b, qa/e)_\infty}{(q/e, qa/b)_\infty} \frac{(bx/e, qe/bx)_\infty}{(abx/e, qe/abx)_\infty} {}_2\phi_1\left(\begin{matrix} a, b \\ e \end{matrix}; x\right) \\
+ \quad & \frac{(a, e/b)_\infty}{(e/q, qa/b)_\infty} \frac{(bx/q, q^2/bx)_\infty}{(abx/e, qe/abx)_\infty} {}_2\phi_1\left(\begin{matrix} qa/e, qb/e \\ q^2/e \end{matrix}; x\right)
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
& {}_2\phi_1\left(\begin{matrix} a, qa/e \\ qa/b \end{matrix}; \frac{qe}{abx}\right) \\
= \quad & \frac{(a, qa/e)_\infty}{(ab/e, qa/b)_\infty} \frac{(q/x)_\infty}{(qe/abx)_\infty} {}_3\phi_2\left(\begin{matrix} q/b, e/b, qe/abx \\ 0, qe/ab \end{matrix}; q\right) \\
+ \quad & \frac{(q/b, e/b)_\infty}{(qa/b, e/ab)_\infty} {}_3\phi_2\left(\begin{matrix} a, qa/e, q/x \\ qab/e, 0 \end{matrix}; q\right)
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
& {}_2\phi_1\left(\begin{matrix} b, qb/e \\ qb/a \end{matrix}; \frac{qe}{abx}\right) \\
= \quad & \frac{(q/a, qb/e)_\infty}{(q/e, qb/a)_\infty} \frac{(ax/e, qe/ax)_\infty}{(abx/e, qe/abx)_\infty} {}_2\phi_1\left(\begin{matrix} a, b \\ e \end{matrix}; x\right) \\
+ \quad & \frac{(b, e/a)_\infty}{(e/q, qb/a)_\infty} \frac{(ax/q, q^2/ax)_\infty}{(abx/e, qe/abx)_\infty} {}_2\phi_1\left(\begin{matrix} qa/e, qb/e \\ q^2/e \end{matrix}; x\right)
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
& {}_2\phi_1\left(\begin{matrix} b, qb/e \\ qb/a \end{matrix}; \frac{qe}{abx}\right) \\
= \quad & \frac{(q/a, e/a)_\infty}{(qb/a, e/ab)_\infty} {}_3\phi_2\left(\begin{matrix} b, qb/e, q/x \\ qab/e, 0 \end{matrix}; q\right) \\
+ \quad & \frac{(b, qb/e)_\infty}{(ab/e, qb/a)_\infty} \frac{(q/x)_\infty}{(qe/abx)_\infty} {}_3\phi_2\left(\begin{matrix} q/a, e/a, qe/abx \\ 0, qe/ab \end{matrix}; q\right)
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
& {}_2\phi_1\left(\begin{matrix} qa/e, qb/e \\ q^2/e \end{matrix}; x\right) \\
= \quad & \frac{(q/a, q/b)_\infty}{(q^2/e, e/ab)_\infty} {}_3\phi_2\left(\begin{matrix} aq/e, bq/e, abx/e \\ 0, qab/e \end{matrix}; q\right) \\
+ \quad & \frac{(qa/e, qb/e)_\infty}{(q^2/e, ab/e)_\infty} \frac{(abx/e)_\infty}{(x)_\infty} {}_3\phi_2\left(\begin{matrix} q/a, q/b, x \\ 0, eq/ab \end{matrix}; q\right)
\end{aligned} \tag{41}
$$

$$
{}_2\phi_1\left(\begin{matrix} qa/e, qb/e \\ q^2/e \end{matrix}; x\right)
$$

$$
\begin{aligned}
&= \frac{(q/a, qb/e)_\infty}{(q^2/e, b/a)_\infty}\frac{(qax/e, e/ax)_\infty}{(x, q/x)_\infty}{}_2\phi_1\left(\begin{matrix}a, qa/e\\ qa/b\end{matrix}; \frac{qe}{abx}\right)\\
&+ \frac{(q/b, qa/e)_\infty}{(q^2/e, a/b)_\infty}\frac{(qbx/e, e/bx)_\infty}{(x, q/x)_\infty}{}_2\phi_1\left(\begin{matrix}b, qb/e\\ qb/a\end{matrix}; \frac{qe}{abx}\right)
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
&\frac{(abx/e)_\infty}{(x)_\infty}{}_3\phi_1\left(\begin{matrix}e/a, e/b, eq/abx\\ eq/ab\end{matrix}; \frac{abx}{e^2}\right)\\
&= \frac{(q/a, q/b)_\infty}{(q/e, qe/ab)_\infty}{}_2\phi_1\left(\begin{matrix}a, b\\ e\end{matrix}; x\right)\\
&+ \frac{(e/a, e/b)_\infty}{(qe/ab, e/q)_\infty}{}_2\phi_0\left(\begin{matrix}e/q, eq/abx\\ -\end{matrix}; \frac{abx}{e^2}\right){}_2\phi_1\left(\begin{matrix}aq/e, bq/e\\ q^2/e\end{matrix}; x\right)
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
&\frac{(q/x)_\infty}{(eq/abx)_\infty}{}_3\phi_1\left(\begin{matrix}e/a, e/b, eq/abx\\ eq/ab\end{matrix}; \frac{abx}{e^2}\right)\\
&= \frac{(e/a, q/a)_\infty}{(qe/ab, b/a)_\infty}{}_2\phi_0\left(\begin{matrix}b/e, ax/e\\ -\end{matrix}; \frac{e^2}{abx}\right){}_2\phi_1\left(\begin{matrix}a, qa/e\\ qa/b\end{matrix}; \frac{qe}{abx}\right)\\
&+ \frac{(q/b, e/b)_\infty}{(a/b, qe/ab)_\infty}{}_2\phi_0\left(\begin{matrix}a/e, bx/e\\ -\end{matrix}; \frac{e^2}{abx}\right){}_2\phi_1\left(\begin{matrix}b, qb/e\\ qb/a\end{matrix}; \frac{eq}{abx}\right)
\end{aligned} \tag{44}
$$

5 Proofs

Proof of (25)
Take $(i; j, k; l, m, n) = (3; 0, 2; 1, 4, 5)$ in (12) and use $F(3) = F(3; 4, 5)$, to get

$$
\begin{aligned}
&(\beta_{43}, \beta_{53}, \alpha_{012}; q)_\infty {}_3\phi_2\left(\begin{matrix}\alpha_{045}, \alpha_{145}, \alpha_{245}\\ \beta_{43}, \beta_{53}\end{matrix}; \alpha_{012}\right)\\
&= \frac{(\alpha_{124}, \alpha_{125}, \alpha_{245}, \beta_{03}; q)_\infty}{(\beta_{20}/q; q)_\infty}{}_3\phi_2\left(\begin{matrix}\alpha_{014}, \alpha_{015}, \alpha_{045}\\ \beta_{03}, \beta_{02}\end{matrix}; q\right)\\
&+ \frac{(\alpha_{014}, \alpha_{015}, \alpha_{045}, \beta_{23}; q)_\infty}{(\beta_{02}/q; q)_\infty}{}_3\phi_2\left(\begin{matrix}\alpha_{124}, \alpha_{125}, \alpha_{245}\\ \beta_{23}, \beta_{20}\end{matrix}; q\right).
\end{aligned}
$$

In terms of the parameters a, b, c, e, f and q, this is

$$
\begin{aligned}
&{}_3\phi_2\left(\begin{matrix}qabc/ef, a, b\\ qab/f, qab/e\end{matrix}; \frac{q}{c}\right)\\
&= \frac{(e/c, f/c, b, q^2ab/ef; q)_\infty}{(q/c, qab/f, qab/e, ef/qac; q)_\infty}{}_3\phi_2\left(\begin{matrix}qa/f, qa/e, qabc/ef\\ q^2ab/ef, q^2ac/ef\end{matrix}; q\right)\\
&+ \frac{(qa/f, qa/e, qabc/ef, qb/c; q)_\infty}{(q/c, qab/f, qab/e, qac/ef; q)_\infty}{}_3\phi_2\left(\begin{matrix}e/c, f/c, b\\ qb/c, ef/ac\end{matrix}; q\right).
\end{aligned}
$$

Set $f = abcx/e$ to get

$$
\begin{aligned}
&{}_3\phi_2\left(\begin{matrix}q/x, a, b\\ qe/cx, qab/e\end{matrix}; \frac{q}{c}\right)\\
&= \frac{(b, abx/e; q)_\infty}{(qab/e, bx/q; q)_\infty}\frac{(e/c, q^2/cx; q)_\infty}{(q/c, qe/cx; q)_\infty}{}_3\phi_2\left(\begin{matrix}qe/bcx, qa/e, q/x\\ q^2/cx, q^2/bx\end{matrix}; q\right)\\
&+ \frac{(qa/e, q/x; q)_\infty}{(qab/e, q/bx; q)_\infty}\frac{(qb/c, qe/bcx; q)_\infty}{(q/c, qe/cx; q)_\infty}{}_3\phi_2\left(\begin{matrix}e/c, abx/e, b\\ qb/c, bx\end{matrix}; q\right).
\end{aligned} \tag{45}
$$

Next, observe that as $c \to 0$,

$$
{}_3\phi_2\left(\begin{matrix} q/x, a, b \\ qe/cx, qab/e \end{matrix}; \frac{q}{c}\right) \to {}_3\phi_1\left(\begin{matrix} q/x, a, b \\ qab/e \end{matrix}; \frac{x}{e}\right) \tag{46}
$$

$$
\begin{aligned}
{}_3\phi_2\left(\begin{matrix} qe/bcx, qa/e, q/x \\ q^2/cx, q^2/bx \end{matrix}; q\right) &\to {}_2\phi_1\left(\begin{matrix} qa/e, q/x \\ q^2/bx \end{matrix}; \frac{e}{b}\right) \\
&= \frac{(qe/abx, qa/b; q)_\infty}{(q^2/bx, e/b; q)_\infty} {}_2\phi_1\left(\begin{matrix} a, qa/e \\ qa/b \end{matrix}; \frac{qe}{abx}\right)
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
{}_3\phi_2\left(\begin{matrix} e/c, abx/e, b \\ qb/c, bx \end{matrix}; q\right) &\to {}_2\phi_1\left(\begin{matrix} abx/e, b \\ bx \end{matrix}; \frac{e}{b}\right) \\
&= \frac{(x, e; q)_\infty}{(bx, e/b; q)_\infty} {}_2\phi_1\left(\begin{matrix} a, b \\ e \end{matrix}; x\right).
\end{aligned} \tag{48}
$$

The iterate of Heine's transformation (3) has been used to transform the ${}_2\phi_1$ functions in (47) and (48). Furthermore, using (1) we have

$$
\begin{aligned}
\frac{(e/c, q^2/cx; q)_\infty}{(q/c, qe/cx; q)_\infty} &= {}_2\phi_1\left(\begin{matrix} q/e, x/q \\ q/c \end{matrix}; \frac{qe}{cx}\right) \\
&\to {}_2\phi_0\left(\begin{matrix} q/e, x/q \\ - \end{matrix}; \frac{e}{x}\right)
\end{aligned} \tag{49}
$$

and similarly

$$
\begin{aligned}
\frac{(qb/c, qe/bcx; q)_\infty}{(q/c, qe/cx; q)_\infty} &= {}_2\phi_1\left(\begin{matrix} 1/b, bx/e \\ q/c \end{matrix}; \frac{qe}{cx}\right) \\
&\to {}_2\phi_0\left(\begin{matrix} 1/b, bx/e \\ - \end{matrix}; \frac{e}{x}\right).
\end{aligned} \tag{50}
$$

Equation (25) now follows by taking the limit as $c \to 0$ in (45) and using equations (46) – (50).

Remaining proofs

The remaining equations can be proved in the same way. Equations (26) – (32) can also be obtained from (25) just by change of variable. A summary of the proofs is as follows.

Eqn.	$(i;j,k;l,m,n)$	Alternative proof by change of variable
(25)	$(3;0,2;1,4,5)$	
(26)	$(3;0,1;2,4,5)$	$a \leftrightarrow b$ in (25)
(27)	$(5;0,1;2,3,4)$	$a \to e/a$, $b \to e/b$, $x \to abx/e$ in (25)
(28)	$(3;2,4;0,1,5)$	$a \to qa/e$, $b \to qb/e$, $e \to q^2/e$ in (25)
(29)	$(5;0,2;1,3,4)$	$a \to e/b$, $b \to e/a$, $x \to abx/e$ in (25)
(30)	$(3;1,4;0,2,5)$	$a \to qb/e$, $b \to qa/e$, $e \to q^2/e$ in (25)
(31)	$(5;1,4;0,2,3)$	$a \to q/a$, $b \to q/b$, $e \to q^2/e$, $x \to abx/e$ in (25)
(32)	$(5;2,4;0,1,3)$	$a \to q/b$, $b \to q/a$, $e \to q^2/e$, $x \to abx/e$ in (25)
(33)	$(0;3,5;1,2,4)$	
(34)	$(0;1,2;3,4,5)$	
(35)	$(3;0,4;1,2,5)$	
(36)	$(3;1,2;0,4,5)$	
(37)	$(2;0,4;1,3,5)$	
(38)	$(2;3,5;0,1,4)$	
(39)	$(1;0,4;2,3,5)$	$a \leftrightarrow b$ in (37)
(40)	$(1;3,5;0,2,4)$	$a \leftrightarrow b$ in (38)
(41)	$(4;3,5;0,1,2)$	$a \to qa/e$, $b \to qb/e$, $e \to q^2/e$ in (33)
(42)	$(4;1,2;0,3,5)$	$a \to qa/e$, $b \to qb/e$, $e \to q^2/e$ in (34)
(43)	$(5;0,4;1,2,3)$	$a \to e/a$, $b \to e/b$, $x \to abx/e$ in (35)
(44)	$(5;1,2;0,3,4)$	$a \to e/a$, $b \to e/b$, $x \to abx/e$ in (36)

6 Remarks

Equation (33) is due to Sears [6, (p. 178 II(a))]. It also can be obtained by taking $\theta_r = (abx/ep)^r$ in [6, Theorem 4] and then replacing all occurrences of p with q.

Equation (34) is due to Watson [7, (p. 285 eq. (7))]. It also appears in [6, (p. 178 III(d))] and [4, (p. 106, eq. (4.3.2))]. Equivalent forms of (34) are given in [6, (p. 178 II(c))] and [4, (p. 92, ex. 3.8)]; the equivalence follows immediately using the transformation formula (7).

Equation (35) is due to Sears [6, (p. 178 III(c))]. An equivalent form (use Heine's transformation (2)) is given in [4, (p. 64, eq. (3.3.5))].

In equation (39), first interchange a and b, then replace x with x/b and let $b \to \infty$, to get

$$\begin{aligned} & \frac{(qa/e)_\infty}{(q/e)_\infty} \frac{(x/e, qe/x)_\infty}{(ax/e, qe/ax)_\infty} {}_1\phi_1 \left(\begin{matrix} a \\ e \end{matrix} ; x \right) \\ + \quad & \frac{(a)_\infty}{(e/q)_\infty} \frac{(x/q, q^2/x)_\infty}{(ax/e, qe/ax)_\infty} {}_1\phi_1 \left(\begin{matrix} qa/e \\ q^2/e \end{matrix} ; x \right) \\ = \quad & {}_2\phi_1 \left(\begin{matrix} a, qa/e \\ 0 \end{matrix} ; \frac{qe}{ax} \right) \end{aligned} \tag{51}$$

This formula is a q-analogue of [1, (p. 192, eq. (4.1.13))]. Note however that the right hand side of equation (4.1.13) in [1]is an asymptotic expansion which does not converge, while the q-analogue (51) is an equality among convergent series.

References

[1] G. Andrews, R. Askey and R. Roy, Special Functions, (Cambridge University Press, 1999).

[2] W. N. Bailey, Generalized Hypergeometric Series, (Cambridge University Press, 1935).

[3] A. Erdelyi, ed. Higher Transcendental Functions, Vol. 1, (McGraw-Hill, 1953; reprinted Robert E. Krieger Publishing Company, Inc., 1981).

[4] G. Gasper and M. Rahman, Basic Hypergeometric Series, (Cambridge University Press, 1990).

[5] N. A. Hall, An algebraic identity, J. Lond. Math. Soc. **11**, (1936), 276.

[6] D. B. Sears, On the transformation theory of basic hypergeometric functions, Proc. Lond. Math. Soc. (2) **53**, (1951), 158 – 180.

[7] G. N. Watson, The Continuations of Functions Defined by Generalised Hypergeometric Series, Trans. Camb. Phil. Soc. **XXI**, (1910), 281 – 299.