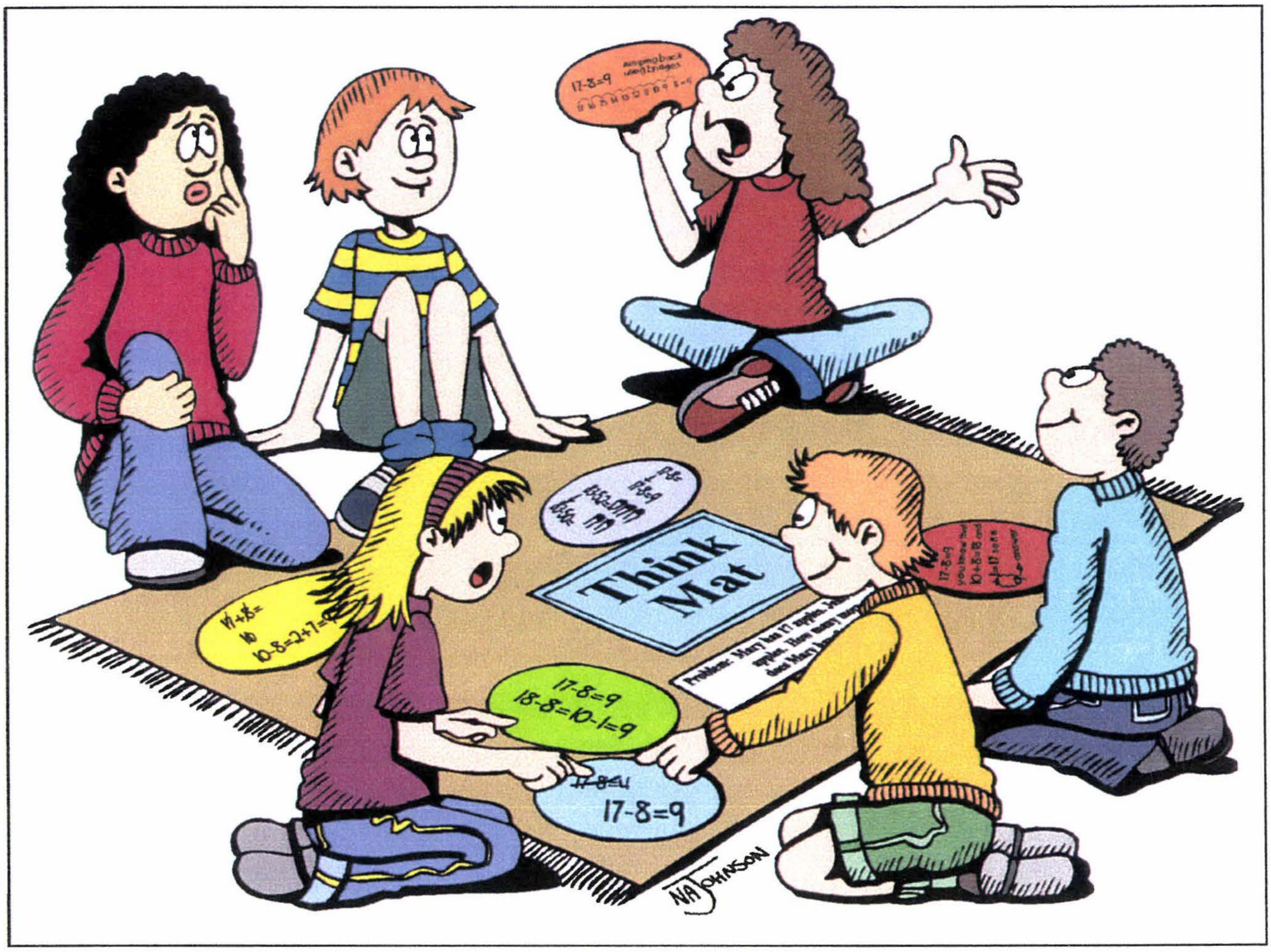
Children's Notation of Number Computations



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ABSTRACT

This study examines the development of children's notational schemes including their use of informal nonstandard notations and formal standard notations. A Year 5/6 class of students, their teacher and the researcher were involved in a collaborative teaching experiment in the context of qualitative developmental research. 'Experiment' refers not to untried or unusual instruction, but rather to collaborative analysis and planning of the students' mathematical activity. In order to gain information about children's notation of number computations data was gathered through interviewing, observing, and analyzing work samples of six case study students.

This research study documents the emergence and development of notational schemes from children's problem-solving activities. The ways of symbolizing that emerged in the classroom evolved from the need to clarify and communicate thinking. Children represented their mathematical ideas using a variety of notational forms, both informal and formal. Within the classroom children used notational schemes as a 'thinking device' to help them make sense of their developing mathematical knowledge.

Classroom practice intellectually engaged children with key mathematical ideas. Children increasingly became engaged in genuine dialogical encounters making reference to their own and others' explanations as captured by the notational schemes. As a result, notational schemes served to support shifts in children's mathematical understanding and development.

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CHAPTER 1

INTRODUCTION

1.1 Background to the Study

In many New Zealand primary school classrooms schemes of work have determined how and when number computations, such as addition and subtraction, were introduced. The focus tended to be on developing procedural knowledge rather than children's conceptual knowledge of computations. Teachers based instruction on pre-determined sequences rather than children's knowledge. Frequently symbolic notations representing algorithms were imposed on children by the pre-planned instructional sequence. As a result many children found themselves trying to understand and use external representations that were not experientially real to them (Lane, 2000).

In the early 1990s the changing contexts within New Zealand created the need to update the mathematics curriculum. The new curriculum statement (Ministry of Education, 1992) represented a significant shift from a content-based curriculum to one based on outcomes. However, the curriculum changes did not automatically translate into high achievement outcomes. Results from the Third International Mathematics and Science Study identified New Zealand students' achievement in mathematics as below international averages (Garden, 1997). In addition it was reported that classroom teachers, especially primary teachers, were experiencing difficulties in implementing the new curricula (Ministry of Education, 2001). In response to this the Mathematics and Science Taskforce, set up by the Minister of Education, established and recommended the provision of help for teachers of five- to nine-year olds, focusing firstly on number concepts.

Such a focus is in accord with world-wide attention on numeracy development highlighting the importance of high-quality mathematics programmes. Research, over the last decade, into children's understanding of number reveal that there are identifiable progressions in the development of number concepts (for example, Carpenter, Fennema, Franke, Levi, & Empson, 1999a; Clarke, Sullivan, Cheeseman, & Clarke, 2000; Wright, Martland, & Stafford, 2000; and Young-Loveridge, 1999). It also became evident that children's computational strategies were based on their intuitive understanding of number and the action needed. This highlighted the need to perceive the learning of multi-digit concepts and skills as a problem-solving activity *rather than as the acquisition of established rules and procedures* (Carpenter et al., 1999b, p. 45). These findings have led to the evolution of various models, or frameworks, of early number development (Thomas & Ward, 2002). As part of the current numeracy reforms the Ministry of Education has developed the *Numeracy Project* which emphasizes mental computations.

Mental arithmetic is more than instant recall of basic facts. Threlfall (2002) describes mental calculation strategies as *the application of known or quickly calculated number facts in combination with specific properties of the number system to find the solution of a calculation whose answer is not known* (p. 31). Within the *Numeracy Project* children are encouraged to do a lot more thinking in their head and to verbalize their ideas in order to develop their informal computation strategies. However whole class discussions often consist of selected children verbally sharing their computation strategies with all contributions equally accepted and valued. McClain and Cobb's (2001) recent analysis of US reform classes concluded that *the students' participation in these discussions appeared to involve waiting quietly for their turn to explain, but without listening to others' explanations* (p. 247). Significant change in teaching practice is required if children are to engage intellectually with key mathematical ideas (Higgins, 2003).

Anghileri (2000) recognizes that *there can be difficulties for teachers where much of the calculating is done orally and little written recording takes place.* She recommends that *recording some of the ideas will be necessary for clear communication and this can be the focus of follow-up work, rather than the driving force in the calculating process* (p. 130). Children should be expected to use written recording primarily to "think through" calculations (Ministry of Education, 2001). In addition *informal jottings of students are to be encouraged as a "way to capture their mental processes" so that their ideas can be shared with others* (Ministry of Education, 2002a, p. 9). This poses an interesting situation of how to 'bridge the gap' between mental computations and written algorithms, especially as they are structurally different. Plunket (1979, cited in Thompson, 1997) describes mental algorithms as fleeting, variable, flexible, iconic, holistic, and usually not generalizable; while standard written algorithms are portrayed as symbolic, automatic, contracted, efficient, analytic, and generalizable. The movement from informal jottings to standard written algorithms requires some time. Thompson (1999b) states *it is unrealistic to expect a smooth progression from idiosyncratic mental methods to standard written algorithms* (p. 170).

The focus of this study has been influenced by a number of key factors with links to the numeracy reforms: the role that notation plays in the formation of children's number concepts and relationships; the influence of social constructivist learning theories; and the roles of children and their teacher in the mathematics classroom. Given the implementation of the *Numeracy Project* (Ministry of Education, 2002a), this research study is timely and significant. The results of the study will directly inform ongoing developments and contribute to greater understanding and knowledge on children's mathematical learning.

1.2 Research Questions

The purpose of this study is to examine the development of notational schemes within a classroom unit of work related to the *Numeracy Project*. To investigate how students might come to use informal nonstandard notations and formal standard notations in powerful ways, it is important to document the ways in which they participate in practices that involve the development of ways of recording mathematical activity. The research is defined by the following questions with specific areas of interest noted as sub-questions:

1. How are recording conventions of children's thinking invented and established within a classroom?
    a) What influences methods of recording a mathematical activity?

2. In what ways does notation contribute to the productiveness of group and whole class discussions of computation strategies?
    a) How are discussions structured to allow report back of strategies?
    b) How are sociomathematical norms associated with recording notation established?

3. In what ways do notational schemes reflect a shift in children's reasoning?
    a) How does written recording allow children to organize and reflect on strategies for computation?
    b) In what ways can notation track the children's thinking and assist in the identification of the strategies they use?

1.3 Definition of Terms

For the purpose of this study the main terms are defined as follows:

In its purest form **notation** refers to *records that communicate about thinking* (Carpenter & Lehrer, 1999, p. 29).

Notation is a form of **external representation** which refers to *symbolic organizations that have as their objective to represent externally a certain mathematical 'reality'* (Dufour-Janvier, Bednarz, & Belanger, 1987, p. 109).

Notation does not usually occur in isolation but appears in highly structured schemes. A **notational scheme** is described as *a concretely realizable collection of characters together with more or less explicit rules for identifying and combining them* (Kaput, 1987, p. 162).

Informal nonstandard notation is *personal and idiosyncratic* while **formal standard notation** is *cultural and conventional* (Goldin & Kaput, 1996, p. 389).

1.4 Overview

Chapter 2 reviews the literature from both an international and New Zealand perspective providing a theoretical background from which this research can be viewed. It specifically summarizes relevant findings on the role of notation within the teaching of numeracy programmes.

Chapter 3 describes the research methodology used for this research. Data collection instruments are presented and justifications for the use of these instruments made.

The following two chapters report the results of the study. Chapter 4 discusses the recording of mathematical activity in the classroom and the contribution notation made to discussions of computation strategies and solutions. Chapter 5 presents and documents the development of notational schemes of the six case studies in the research sample.

In Chapter 6, the common themes and conclusions are discussed. The implications for teaching practice are presented as well as suggestions for further research.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

A study of the development of notational schemes must be based on the assumption that notation has a role to play in the development of children's numeracy knowledge and strategies. This review of literature traces the emergence of ways children record mathematical ideas based on their existing knowledge. It examines research related to instruction designed to bridge children's informal mental methods to more formal written methods. The central role of the teacher in instructional practices is highlighted. Strategies that children use to solve different types of number problems, the effectiveness of discourse as part of the learning environment, and their relationship to notation are discussed. Finally, it explores how notation supports children's sense making, and considers how misconceptions may be overcome.

2.2 Children's Mathematical Development

2.2.1 Sociocultural learning

The development of mathematical understanding is a dynamic process. It is not only influenced by contextual factors such as gender, ethnicity, and socioeconomic background of an individual but also determined by the social interactions occurring throughout the learning process. Although there are variations in the current socio-constructivist views it is generally agreed that knowledge is not passively received, but actively 'constructed' by the learner. According to Piaget (1963) children filter and interpret new information in terms of what they already understand. Mathematics learned in this manner should make sense. Children construct new knowledge through 'reflection' upon their physical and mental actions. Influenced by the work of Diene, it is maintained that abstract concepts and generalizations are developed and derived from children's reflections upon existing knowledge (Bobis, Mulligan, Lowrie, & Taplin, 1999).

Sociocultural theorists, such as Vygotsky, emphasize the role played by social interaction in the learning process. Proponents of social constructivism hold the view that learning is a 'social process' with children learning through interaction with others. For example, Cobb, Wood, and Yackel (1990) describe learning as both a 'constructive' and an 'interactive' activity. Opportunities for children to construct mathematical knowledge arise through the social context of the classroom as they interact with other children and with adults. Communication is viewed as 'mutual adaptation' as individuals negotiate mathematical meanings. Using this perspective, *learning is characterized as the personal reconstruction of societal means and models through negotiation in interaction* (McClain & Cobb, 1999, p. 352).

Therefore, mathematical learning involves two aspects: the individual actively constructing knowledge as well as an enculturation into the mathematical practices of the wider society. In this context, notation is seen to be both an individual and socially mediated convention.

2.2.2 Current reform practices

The development of intellectual and social autonomy is a major goal in the current reform movement in mathematics education (Yackel & Cobb, 1996). This involves a move away from the passive transmission of knowledge and procedures by way of the teacher. Traditional instruction revolved around committing facts and computational procedures to memory. As a result, children lacked number sense and an understanding of mathematical operations. In contrast, current reform focuses on learners being actively engaged with mathematical ideas. This involves establishing a community of inquirers building on children's ways of thinking about mathematics (Anthony & Walshaw, 2002). Children's intuitive or informal mathematical sense making is used as a base for developing ways of knowing (Heaton, 2000). Booker (1996) asserts that *knowledge is, in part, a product of the activity, context, and culture in which it is developed and used* (p. 386).

2.2.3 Mathematical understanding

Children begin school with a rich experience of, and informal knowledge about, number (Booker, 1996). They come to 'understand' mathematics conceptually when existing knowledge merges with new ideas, through their own thinking and reasoning (Wood, 2001). Children's learning involves *an interplay between their informal ways of knowing mathematics and the mathematical structure* (Beishuizen & Anghileri, 1998, p. 114). Understanding cannot be imposed upon children; it must develop gradually as they actively try to make sense of the new knowledge. Carpenter and Lehrer (1999) describe 'understanding' as *emerging* or *developing* rather than *presuming that someone does or does not understand a given topic, idea, or process* (p. 20). Mathematical understanding emerges as children construct relationships, extend and apply mathematical knowledge, reflect on experiences, articulate what they know, and make mathematical knowledge their own.

During classroom interactions the teacher and children construct taken-as-shared mathematical interpretations and understandings. The development of children's reasoning and sense-making processes cannot be separated from their participation in the interactive constitution of taken-as-shared mathematical meanings. The teacher serves as a representative of the mathematical community in the classroom where children develop their own personally meaningful ways of knowing. Social norms are collective understandings of the expectations and obligations that are constituted in the classroom. Classroom norms that are specific to the mathematical aspects of children's activity are known as 'sociomathematical norms' (Yackel & Cobb, 1996).

2.3 Language and Notation

2.3.1 Communication of thinking

Learning to communicate about and through mathematics is part of learning to become a mathematical problem solver and learning to think mathematically (Ministry of Education, 1992, p. 11). Children should be encouraged to share ideas, use their own words to explain ideas, and to record their thinking in a variety of different ways. Language is a central component in learning about numbers and it is through children's verbal or symbolic explanations that their current mathematical understanding is expressed (Anghileri, 2000; Pirie, 1998). Language communicates ideas, not only to other people, but also to ourselves by helping us to understand and clarify them in our minds (Bobis et al., 1999). These ideas become objects of reflection, refinement, discussion and amendment. *When students are challenged to think and reason about mathematics and to communicate the results of their thinking to others orally or in writing, they learn to be clear and convincing* (National Council of Teachers of Mathematics [NCTM], 2000, p. 60).

Understanding may be limited by the ways in which children try to express their thinking as others' interpretation is shaped by their own construction of the meanings expressed. Pirie (1998) comments that through a combination of personal experience and cultural tradition sense is made of a concept so that meaning can be constructed. Therefore a *link between semantics (the sense of the language) and semiotics (the symbolism) has to be created* (p. 10). There must be a common basis for communication to articulate one's thinking. Notation can provide that common basis for discussion, and may help children to reflect on and clarify their thinking (Carpenter & Lehrer, 1999).

Albert (2000) explored the relation between children's oral thought processes and written thought processes by building on the research of Vygotsky regarding the role of social interaction and the zone of proximal development in learning and development. In describing the act of writing as a mode for children to reflect on their thinking she contends that writing *is a device for mediating cognitive development, moving the learner through the zone of proximal development to the zone of proximal practice* (p. 111). Oral language is the tool used to shape the discourse in a collaborative situation; however at an independent level of learning and development, writing is the tool children can use to shape their thinking. Menon (1998) supports this, claiming that through the process of writing children's *thinking becomes clearer* (p. 19).

Effective communication in mathematics warrants specific attention. Children need to learn how to represent mathematical ideas accurately using natural language and mathematical symbols (Carpenter, Franke, & Levi, 2003). However Pirie (1998) noticed that a unique mathematical communication problem may arise as *the language used when talking about mathematics and that used when writing mathematics (as opposed to writing about mathematics) are completely different* (p. 9). She explains that verbal and symbolic forms do not always match, for example, a direct translation of '*subtract 2 from 3*' into symbolic notation may look like '*-23*' but in a standardized form appears as '*3 – 2*'. The interaction and communication that takes place around writing are also important. Through discussion of mathematical ideas children come to understand the importance of representing their thinking so it is comprehensible to others. McClain, Cobb, Gravemeijer, and Estes' (1999) study on developing mathematical reasoning found that children's representations made it possible for them to reflect on and compare not just different calculational processes but also different ways of interpreting and reasoning about the problems. The children became aware of the obligation to represent their thinking so that others might understand. Explaining their reasoning with reference to notation helped children to communicate their thinking and often resulted in solution methods becoming topics of conversation and investigation.

2.3.2 Classroom discourse

One major difference between traditional and reform classes is the different patterns of interaction that evolve through the social norms which are constituted among the participants. The traditional pattern of the teacher directly telling children the correct procedures and rules conveys a view that the mathematics to be learned rests solely within the authority of the teacher. A different pattern of interaction occurs in the classroom when children are expected to express their thinking (Wood, 1998). Using a 'focusing pattern' of interaction conveys to the children that what counts as mathematics in the classroom are the meanings and understandings that they have constructed for themselves. Although valuing and accepting a variety of solutions the teacher may draw attention to one of the solution methods to help children notice an idea rather than to reflect the teacher's predetermined solution. While not all aspects of the solution may be fully understood the teacher leaves the responsibility of solving the problem to the children. In this way the teacher turns control and ownership of the situation back to the children.

It is not easy for children to learn to consider, evaluate and build on the thinking of others, especially when peers are still developing their own mathematical understanding (NCTM, 2000). For those not making connections the teacher may need to 'fold back' to a 'taken-as-shared' stage, for example, if they are having difficulty explaining using number properties then they might revert to using imaging (McClain & Cobb, 1998;

Ministry of Education, 2002b; Pirie, 1998). Where such imagery, that is the 'taken-as-shared', has not been developed there is nothing to which the teacher and children can 'fold back' to. In these situations the teacher will need to 'drop back' to a stage that supports the children's development of imagery, for example, 'dropping back' to the level of acting out the scenario. *In doing so, the teacher can be seen to initiate the renegotiation of the sociomathematical norm of what counts as an acceptable explanation* (McClain & Cobb, 1998, p. 67).

Although many teachers find it easy to pose questions and ask children to describe their strategies, it is more challenging pedagogically to engage children in genuine mathematical inquiry and push them to go beyond what might come easily for them (Kazemi & Stipek, 2001). Carpenter and colleagues (1999b) suggest that children be encouraged to ask questions if they do not understand, to comment on solution methods, and to compare strategies to others they have used or shared. Through questioning and probing one another's thinking children are able to clarify underdeveloped ideas. Examining the methods and ideas of others will also assist with determining strategies' strengths and limitations (NCTM, 2000). Blote, Klein, and Beishuizen (2000) recommend that a third of class time is spent on discussion about the different strategies, suggesting that discussion can be facilitated when children's computation steps are written down. However very young children might be better to 'model' the problems to enhance discussion. By doing so children *have an 'instrument' to show others how they solved problems; it also facilitates discussion about which strategy is the best one to solve a certain problem* (p. 224).

Within the context of discussions children can consider how connections between alternative solutions are similar or different (Carpenter & Lehrer, 1999). Children are able to construct relationships among different strategies by *juxtaposing alternative strategies and discussing commonalities and differences among them*; and deciding *how their strategies can be applied to different problems in different ways* (Carpenter et al., 1999b, p. 59). The meaning of what constitutes a 'different' mathematical solution is negotiated by the teacher and her students through their interaction. This also provides an opportunity to compare and contrast the 'efficiency and elegance' of a variety of strategies (NCTM, 2000). Class members can then establish sociomathematical norms of what counts as an 'efficient' mathematical solution and as a 'sophisticated' mathematical solution (McClain & Cobb, 2001). Sociomathematical norms, such as these, support higher-level cognitive activity (Yackel & Cobb, 1996).

Less competent or confident children who have difficulty with recognizing and understanding number concepts for themselves often hold on too long to low-level procedures which are less efficient. Therefore it is necessary to expose them to more sophisticated strategies partly because these are often, in fact, simpler (Beishuizen & Angiherli, 1998; McIntosh, 1998). Interactive whole-class teaching can help children to solve problems in a smart and flexible way providing opportunities for differentiation (Menne, 2001).

In highly interactive situations class norms need to be specifically constituted for children as 'explainers and listeners' (Wood, 2001). Children's explanations need to entail not only the mathematical strategies and/or ideas but also the thinking and reasoning that led to their solution. Sufficient details should be given taking into consideration what the others might not know or understand. Children become critical thinkers by listening to and thinking about claims made by others (NCTM, 2000). If discussions about number computation strategies are to be effective for children to develop their own ideas then the *quality of listening is a key factor* (Coles, 2001, p. 281). 'Evaluative listening' occurs when judgments are made about what others say in terms of it being 'correct' or 'incorrect'. However children need to learn that what they hear may not be what the speaker intended which is a characteristic of 'interpretive listening'. Thus the speaker becomes open to questioning of assumptions made. 'Transformative listening' is indicated when children make a connection to a previous piece of work or link something that has been said before, and restructure their thoughts.

Teachers also need to be careful not to be constrained by 'listening for' something in particular, that is a mathematical explanation, rather than 'listening to' the speaker (Davis, 1997). This may easily occur in situations where a child's explanation of an invented strategy, although clearly indicating an intuitive understanding of the relationships between numbers, may be labelled as non-mathematical. Listening to, and appreciating the ways in which children express their thinking gives an insight into their understanding (Smith & Phillips, 2000). The teacher is able to ascertain their grasp of particular relationships and their ability to apply number sense. This will help identify children who have achieved efficiency in their approaches and those who persist in using inefficient strategies (Anghileri, 2000).

Lampert (2001) maintains that when the teacher interacts with the whole-class at once, she needs to retain overall coherence while drawing different kinds of individuals into a common experience of the content. In facilitating whole class discussions Lampert advocates using visual representations of the ideas being discussed *as a common record of the class journey* and as a referent for discussion (p. 174). Although the teacher may redescribe and notate children's verbal explanations, children should also work to devise notational schemes that express their thinking (McClain & Cobb, 1999). *Children who invent their own notation for communicating ideas about mathematics are taking crucial steps towards developing their understanding of mathematical concepts* (Bobis et al., 1999, p. 18).

2.4 Notational Schemes

2.4.1 Developing children's notation

When children represent their reality they represent their ideas about reality and not reality itself (Piaget, 1977, cited in Kamii, Kirkland, & Lewis, 2001). Children have *distinct modes of self-expression* (Stix, 1994, p. 268) which represent different ideas at different levels of abstraction (Kamii & Housman, 2000). According to Bruner (1968) children progress through three stages in representing their ideas: enactive, iconic, and symbolic. This is usually interpreted as involving apparatus, followed by pictorial, then symbolic representation (Gifford, 1997). Research studies carried out with young children, both preschool and in junior classes, on emergent notation have established various developmental forms (Thompson, 1994). For example, Hughes (1986) identified four categories of notation: *idiosyncratic* (irregular marks); *pictographic* (an indication of shape, position, colour or orientation); *iconic* (a system of using discrete marks); and *symbolic* (conventional symbols) (p. 56). Kamii and Housman (2000) classified children's use of notation into six different types: *global representation of quantity*; *representation of the object-kind*; *representation of the object-kind*; *one-to-one correspondence with numerals*; *cardinal value alone*; and *cardinal value and object* *kind* (p. 21). Children's interpretations and associated notational methods emerge as they act with experientially-real mathematical objects (Whitenack, Knipping, & Novinger, 2000).

Nearly every attempt to develop 'understanding' involves spontaneous invention and use of notations (Lehrer, Jacobson, Kemeny, & Strom, 1999). Kamii et al. (2001) claim that when children represent their ideas they *prefer to make their own drawings because they can think better with the symbols they make by externalizing their own ideas* (p. 34). Children's use of drawings frequently involve the creation and manipulations of symbols which may not look conventional, but are purposeful, intentional, and carry meaning (Mills et al., 1996, cited in Woleck, 2001). By inventing and using their own form of notation children experience the sense-making quality that should underlie the use of all symbols in mathematics (McClain & Cobb, 1999). Drawings facilitate children's reasoning because they come out of their own thinking (Kamii & Warrington, 1999). Cobb (2000a) emphasizes that children need to negotiate the meaning of the symbols they use in order to communicate their reasoning. *The focus is not on symbols and their meaning, but on the activity of symbolizing and meaning making* (Yackel, 2000, p. 5).

Cognitive research reveals that children do not simply imitate and adopt adult strategies or patterns of thought (Anghileri, 2001c; Baroody & Ginsburg, 1990; Booker, 1996). Children hold an enduring position in their intuitive thinking which may not easily be reconciled with teacher-taught procedures. To assist children in constructing new knowledge the teacher should use teaching strategies that are articulated by the representations developed by the children (Dufour-Janvier et al., 1987). Therefore teachers need to talk to children about their symbolization so that they can link the signs they use to their concepts of number (Munn, 1997).

Children's first external representations are tied to language with attempts to represent everything by writing and drawing. Dufour-Janvier et al. (1987) found that children considered a 'good representation' should contain everything presented with no information being lost. While pictorial representations may be suitable for representing amounts Gifford (1997) ascertains they are unhelpful for representing number operations. She suggests it may be more useful for children to record operations using abbreviated words rather than using pictures, making an easier transition to conventional symbols. Anghileri (2000) recommends that children learn to record their thinking by first *learning to use words to record results they can already talk about* (p. 42). Furthermore, this should include discussions about the way they themselves and their peers could record their findings with symbols introduced as a short-hand for the words they are using.

The focus of notating involves the teacher guiding children to find ways to express their ideas where *the resulting mathematical notation arises from and accords with the child's verbally based strategies* (Wright et al., 2000, p. 143). Kamii and Housman (2000) suggest that during whole class discussion when children explain their strategies the teacher tracks thinking on the board. This has a two-fold purpose: firstly, to let the speaker know what she has understood; and secondly, to enable other members of the class to follow what the speaker is saying. So if a child says '*1 and 2...that's 3*' the teacher writes '1 + 2 = 3'. If the child says '*then I added 3 to it*', the teacher writes '+ 3' leaving the following equation on the board as '1 + 2 = 3 + 3'. The researchers claim that a 'non-conventional' equation is appropriate in this particular situation because it facilitates children's thinking and exchange of ideas. However they do acknowledge that the teacher should also write to teach the social knowledge of equations by modelling their use in a meaningful way at the appropriate time in children's learning.

The symbols for addition and subtraction problems can be used in flexible ways, for example, 'arrows' are useful as an alternative notation (Anghileri, 2000). They do not require the 'equals' symbol which can add complications in interpreting written statements, such as 6 = 4 + 2:

$$(4 \text{ and } 2) \xrightarrow{\text{together make}} 6$$

The *Cognitively Guided Instruction* teaching programme (Carpenter et al., 1999a) also advocates using arrow notation, for example, $1 + 2 \rightarrow 3 + 3$; or better still

$$1 \xrightarrow{+2} 3 \xrightarrow{+3} 6$$

In McClain and Cobb's (1999) study, involving first-grade students, the teacher redescribed and notated children's solutions of number problems. She devised a simple method of notating children's reasoning by using an inverted "V" symbol which came to signify the partitioning or decomposing of a number. The teacher would typically follow the "V" notation with the number sentences that expressed the result of the partitioning. For example,

```
7 + 8 =
    / \
    7  1
 7 + 7 = 14
14 + 1 = 15
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This particular "V" notation does not necessarily fit with all of the children's thought processes, for example, a child may have conceptually partitioned the eight as '3 + 5'. Subsequently the research team acknowledged the need for children to discuss their own notational schemes. The children's role in developing notational schemes needs to be brought to the fore more prominently by asking students how 'they' might notate the problem. Within the context of their study McClain and Cobb concede that the absence of individual children's work and lack of direct evidence of the children's actual interpretation of number sentences prevent definitive claims about how they use notational schemes. Lamon (2001) emphasizes that *if children individually construct knowledge, then there should be something unique about their representations and explanations—they should not look and sound exactly like those presented in instruction* (p. 155). Progress is indicated by children using original representations rather than mimicking ones presented in instruction. Sometimes they 'reinvent' methods where they begin with a conventional form, as established within the classroom community, but then adapt it to fit with their own informal strategies (Whitenack et al., 2000).

There is a tension involving how and when to utilize the children's symbolic initiative, and when to require that certain conventional representations are used, that is between the children's own notation and other people's symbols, and how to work between the two (Pimm, 1992). To deal with this arising tension Stephan, Cobb, Gravemeijer, and Estes (2001) suggest that notational schemes can either be *created by students with the skilful guidance of the teacher; or introduced by the teacher as a natural solution to a dilemma with which the students were grappling* (p. 63).

Research on prescribed notation is mixed in its findings. The importance of conventional number notation is emphasized in Munn's (1997) research where she found that symbols with conventional meaning had more power than symbols with personal meaning in the context of children thinking about number concepts. Kamii and Housman (2000) contend that working with pictures is not necessarily a step toward being able to deal with mathematical symbols. They maintain, unlike Bruner's assumption, there is not a natural progression. Rather mathematical symbols are a form of social knowledge that has to be transmitted to the children therefore *pictures and mathematical symbols have different sources* (p. 19). However, van Oers (2000) stresses that culturally established meanings cannot be transmitted readymade; instead children have to decide with others' help the generally accepted meaning of conventional ways of using mathematical symbols, thereby making their own meaning. In other words, children actively construct meaning as they participate in increasingly substantial ways in the reenactments of established mathematical practices.

Common to these theories is the tenet that the use of standard mathematical symbols is something that develops slowly in young children (Thompson, 1997). Children *need to be exposed to symbols but not obligated to use them* until they feel comfortable with them (ibid., p. 98). Children need opportunities *to play at using abstract forms as with a new genre of writing* (Gifford, 1997, p. 85). Rather than an early imposition of conventional mathematical symbolism children should be allowed to represent their own invention, even if understood only by themselves (Sierpinska, 1998). Notational schemes are not predetermined or imposed by instructional sequence but emerge from the children's attempts to explain and justify their thinking.

The use of mediating or emergent models to serve as a catalyst in which formal knowledge evolves from informal knowledge is supported by several researchers (Angihileri, 2001b; Beishuizen, 1999; Gravemeijer, 1998; van den Heuvel-Panhuizen, 2001). The term 'model' can *concern a model situation, a scheme, a description, or a way of notating* (Gravemeijer, 1998, p. 286). Ideally, the children reinvent the models on their own. Yackel (2000) uses children's models to forecast *a vision of how the teacher and students might collectively progress through a series of social negotiations, towards the more conventionally accepted ways of notating* (p. 9). In cases where a model is presented to the children a requirement is that it fits in with informal strategies demonstrated by them. The model must adapt to the children's thought processes rather than expecting them to adapt their solution procedures to the model. To fulfil the bridging function between the informal and the formal level *models have to shift from a 'model of' a particular situation to a 'model for' all kinds of other, but equivalent situations* (van den Heuvel-Panhuizen, 2001, p. 51). The evolving notational schemes can then be used to generalize and formalize informal knowledge and strategies.

Herscovics (1996) suggests that children may benefit from an 'intermediate' notation. Diagrams may be an appropriate form because they provide a 'structural representation' of children's thinking and are recognized as being different to pictures which provide 'surface details' (Diezmann & English, 2001). One such diagram that has been introduced to primary schools is the 'empty number line' (ENL). All calibration has been removed from the number line to enable children to use it flexibly for 'jumps' of any size, in either direction, providing imagery to encourage and support mental strategies (Beishuizen,1999). In the process of 'jumping' number structures are analyzed and the relationships between jumps and numbers can be explored (Menne, 2001). Beishuizen (1999) purports that the ENL is *a more natural and transparent model for number operations*; and provides *support for children to develop more formal and efficient strategies* (p. 160). Written work on the ENL is seen as supporting or recording strategies chosen as mental decisions in the first place. Menne (2001) maintains that *arrow notation follows naturally from jumping on the empty number line* (p. 99). For example,

$$43 - 18 = \square$$

$$43 \xrightarrow{(-20)} 23 \xrightarrow{(+2)} 25$$

However Yackel (2001) disagrees that the ENL is a 'transparent model'; she advocates that because individuals bring prior experiences to any situation, their interpretations are constrained by those experiences. *Adults who already 'know' mathematical concepts interpret diagrams and models in ways that are consistent with their understandings. On the other hand, children who do not yet have those conceptual understandings do not 'see' the diagrams and models in the same way as the knowing adults* (p. 29).

Teachers need to guide children to greater efficiency and effective ways to record their number computations without reducing their understanding. When performing calculations Buys (2001) encourages children to initially write down every step they carry out mentally to prevent them from losing track of their thinking. This eventually changes to the *notation of purely mental steps in formal number language* (p. 115). Thompson (1999b) advocates that it is a good idea to develop children's 'jottings' into informal written methods. He presents an alternative classification for the development of recording number computations (p. 172):

a. *Informal non-standard algorithms*, for example, using 15 – 8, the child writes "First I took five away to get ten... then I took the three away... so it's seven".
b. *Formal non-standard algorithms*, for example, $67 + 28 = 80 + 15 = 95$
c. *Formal standard algorithms*, for example,

$$\begin{array}{r} {}^{1} \\ 67 \\ \underline{+28} \\ 95 \end{array}$$

While Anghileri (2001c) agrees this progression is certainly necessary, as algorithms are dependent on mental calculations and often jottings, she also argues that the development be in both directions. She qualifies this by claiming that *learning written methods should strengthen mental calculations and facilitate jottings by helping to structure a calculation* (p. 80). Recording long-winded and inefficient methods reflect a stage of understanding that is the starting point for gains in efficiency. *At this stage it is important that pupils are helped to organize their recording without losing the personal nature of these strategies* (ibid., p. 90).

2.4.2 Recording in numeracy programmes

Varying approaches are used in mathematics classes to assist with the development of communicating children's thinking. In *Cognitively Guided Instruction* classrooms it is advocated that each child keep a mathematical journal. Children use drawings, numerals, words, and other symbols they have acquired or invented as ways to express their ideas (Wisconsin Center for Education Research [WCER], 2001). Gervasoni (1999) suggests using a 'Think Board' as a useful way to cater for the different learning and thinking styles of individual children. The 'Think Board' is partitioned into four sections with language, physical, visual, and symbolic representations of number computations being explored allowing children to represent their ideas in a flexible manner. Meanwhile Vale (1999) uses 'Thinking Clouds' to track children's thinking and identify strategies they have utilized. Lampert (2001) recommends that norms be established for written communication. Children in her class used notebooks to keep a running record of their daily work, demonstrating their experimentation and reasoning. This documentation was then used as a shared reference for talking about mathematics; selecting points of interest for whole class discussion from the records; and comparing with others what they were doing. Individual insights, knowledge and abilities lead to the constitution and acceptance of 'taken-as-shared' practice within the classroom (Gravemeijer, 2001).

In the *Numeracy Project* written recordings of children's addition and subtraction number computations are encouraged (see Figure 2.1). Children's notation is perceived as 'knowledge' to be taught alongside the strategy outcomes (Ministry of Education, 2002c).

STRATEGY STAGE	WRITTEN RECORDING
Stages 1,2,3 Counting from One • one-to-one • on materials • by imaging	*The student records:* o the results of counting and operations using symbols, pictures, and diagrams.
Stage 4 Advanced Counting	*The student records:* o the results of mental addition and subtraction, using equations, for example, $4 + 5 = 9$, $12 - 3 = 9$
Stage 5 Early Additive Part-Whole	*The student records:* o the results of addition and subtraction calculations using equations, for example, $35 + 24 = 59$, and diagrams, for example, an empty number line.
Stage 6 Advanced Additive Part-Whole	*The student records:* o the results of calculations using addition and subtraction equations, for example, $349 + 452 = 350 + 451 = 801$, and diagrams, for example, an empty number line. *The student performs:* o column addition and subtraction with whole numbers up to four digits.
Stages 7, 8 • Advanced Multiplicative Part-Whole • Advanced Proportional Part-Whole	*The student records:* o the results of calculations using equations and, diagrams, for example, the empty number line. *The student performs:* o column addition and subtraction for whole numbers.

Figure 2.1 Overview of written recording in 'The Number Framework' (Ministry of Education, 2002a, pp. 13-15)

2.5 Number Computations

2.5.1 Contextual and numerical problems

In the initial stages of number work priority is given to contextual problems over bare number problems due to the link to children's informal knowledge. By making a connection to prior knowledge, problems posed in meaningful contexts can motivate children's learning (Carroll & Porter, 1997). Word problems or contextual problems elicit informal knowledge and form a starting point for 'mathematization', that is translating real-life situations into mathematical terms. Since word problems are like everyday situations the questions develop children's logic while trying to answer them (Kamii & Housman, 2000).

With contextual problems the task becomes one of interpreting the meaning of the question and identifying ways of how to solve them. Carpenter et al. (1999a) classify addition and subtraction word problems into three main semantic categories: change; combine; and compare. Gibbs and Orton (1994, p. 102, cited in Menon, 1998, p. 27) remark that the level of difficulty of a word problem is a function of not only the mathematical content of the problem but also of '*its linguistic form and semantic structure*'. Blote et al. (2000) found that the problem type makes a difference to which strategy is chosen, for example, 'change-type' problems evoke an addition or subtraction strategy, 'combine-type' problems evoke addition, and 'compare-type' problems are solved utilizing a variety of strategies. As a result of non-attending to 'meaning' in early instruction some primary school children respond to the cue given by the words used rather than engage with the logical structure of the problem. For example, when children encounter the language related to a greater quantity, such as 'more than', 'longer than', and 'heavier than', these particular words often signal addition rather than subtraction (Haylock & Cockburn, 1997).

Symbolic representation of number problems may help establish connections between numbers. *Children will need to use and understand some complex mathematical language when 'reading' or 'putting into words' the relationships in 'missing number' problems as such problems can help establish that symbols can be interpreted in flexible ways* (Anghileri, 2000, p. 57). For example,

$$3 + \square = 8 \qquad \square + 3 = 8 \qquad 3 = 8 - \square$$
$$5 = \square - 3 \qquad \square - 5 = 3 \qquad 8 = 5 + \square$$

Blote et al. (2000) found in their study the results of the type of strategies used on contextual problems differed from those on numerical-expression problems. Children were more flexible in their preferences for various computation procedures when the problems presented to them were in a contextual form. Kamii and Housman (2000) highlight the fact that subtraction problems are often harder to solve than addition problems. Part-whole relationships are very difficult for children, especially in subtraction when they have to think in two opposite directions simultaneously. Although subtraction word problems may be given to encourage children to make part-whole relationships they do not have to use subtraction to solve the problems so *they tend to use addition whenever possible* (ibid., p. 91).

Providing clear reasons for calculating assists children in making sense of number. Anghileri (2000) suggests there are two distinct purposes: firstly, to solve individual problems where the aim is to find and interpret the solution to a particular problem; and secondly, to explore the structure of the number system focusing on as many possible ways to calculate in order to highlight the mathematical relationships and processes involved. Through notating the logical sequence of solving a problem children come to understand the process of getting an answer. Teachers can look at children's solutions and examine their progress, for example, ascertaining whether a child knows how numbers can be partitioned into 'tens' and 'ones' then recombined to formulate answers. When problems increase in complexity, expressing ideas on paper lessens the details a child must keep in their memory about a problem (WCER, 2001). Through keeping track of their thinking children can use their writing to decide what to do next (Kamii & Housman, 2000).

2.5.2 Conceptual structures of number

There are many similarities and interrelationships in the variety of numeracy frameworks being developed. Within these frameworks, it is generally accepted that there are two broad areas of understanding, and hence approaches, to teaching addition and subtraction with one digit and two digit numbers. They are: "counting-based" which is grounded on a unitary concept of number and counting strategies; and "collections-based" which is grounded on multi-unit knowledge of number and partitioning strategies (Thomas & Ward, 2001, 2002).

As children progress from using 'counting-based' to 'collections-based' strategies they begin to use part-whole thinking to solve number problems (Ministry of Education, 2002a). The most common 'invented' procedural way, British and American, that children use to solve multi-digit computations is where in both numbers the 'tens' are 'split off' and added or subtracted (given the acronym *1010*). This *partitioning* or *split* method proceeds mostly by adding or subtracting the 'ones', for example: $46 + 38 = \square$ $40 + 30 = 70$, $6 + 8 = 14$, $70 + 14 = 84$ (Beishuizen, 1999; Kamii & Housman, 2000; Thompson, 1997). Blote et al. (2000) suggest that the dominance of this method is due to the teaching of place value number structure. The *Numeracy Project* refers to this strategy as *standard place value partitioning* (Ministry of Education, 2002a, p. 4). In the Netherlands another strategy (given the acronym *N10*) is widely used. The first number is not split up but kept intact while the 'tens' are added or subtracted through counting tens, for example: $57 - 34 = \square$ $57 - 30 = 47$, $47 - 4 = 43$ (Beishuizen, 1999). However, when children are free to do their own thinking there is invariably a range of other 'creative' procedures used to solve problems, for example *sequential, compensation,* and *reversibility* (Buys, 2001; Ministry of Education, 2002a).

2.6 Inquiry-Based Classroom

2.6.1 Learning environment

A problem solving environment allows for 'genuine inquiry' to take place. Romberg and Kaput (1999) describe the inquiry process as *raising and evaluating questions grounded in experience, proposing and developing alternative explanations, marshalling evidence from various sources, representing and presenting that information to a larger community, and debating the persuasive power of that information with respect to various claims* (p. 11). Carpenter and Lehrer (1999) emphasize that the tasks or activities that children engage in and the problems that they solve must be for the purpose of fostering understanding, not simply for the purpose of completing the task. Schifter (1999) supports this, arguing that placing children's thinking at the centre of instruction gives them a chance to articulate their own reasoning as well as encouraging them to build on their own ideas. She sees lessons on calculation as opportunities for children to devise a variety of appropriate computational procedures and thus develop a deeper understanding of place value and the properties of operations. Menne (2001) found that when children do their 'own productions' they have the opportunity to think about the number computations with the result that *many structures in the counting sequence are discovered as well as relationships that exist between numbers* (p. 104). However for an inquiry-based approach to be successful children need to both 'listen to' and 'give explanations for' their problem-solving strategies (Higgins, 2003).

A classroom in which children are encouraged to present their thinking provides a rich environment for learning mathematical reasoning (NCTM, 2000). In order to advance children's mathematical thinking the teacher needs to provide learning opportunities in a safe environment so that children feel comfortable in sharing their solution methods, and where their conceptual understanding is supported (Fraivillig, 2001). Cobb et al. (1990) explain that the sharing of ideas involves a certain amount of risk-taking by children as it is one thing to think privately about how to solve a problem but quite another to express these thoughts to peers. Open to public scrutiny and evaluation children risk feelings of incompetence and embarrassment. If children are to express their thoughts then the teacher needs to show that she respects their thinking and should place all children in the classroom under the same obligation.

2.6.2 The teacher's role

The teacher is both a participant and a commentator within the classroom. She has to facilitate and engage children in problem posing and problem solving together with promoting reasoning and conjecture about the relationships between numbers (Rittenhouse, 1998). Higgins (2003) emphasizes that opportunities must be provided for children to explain their thinking. Explanations are often aided or mediated by written recording of the solution methods. In order to support children's mathematical development the teacher may have to initiate the use of *symbols and notations to communicate students' ways of reasoning* (McClain & Cobb, 1999, p. 352). The development of ways of symbolizing problem situations and *progressive formalization*, that is *the transition from informal to formal semiotics*, are important aspects of classroom instruction (Romberg & Kaput, 1999, p. 11).

Children need to make a connection between their mental methods of calculation and personal written recording which reflect these methods (Thompson, 1994). To assist children in making this link the teacher needs to have a sound knowledge of their external representations; the conceptions attached to a concept corresponding to the external representations of certain reality; the obstacles encountered by the child of a representation; and how representations evolve (Dufour-Janvier et al., 1987). Steffe and Wiegel (1996) claim that it is not until the teacher actually constructs schemes to model the children's mathematical knowledge that she can legitimately claim to understand their representational structures. The teacher can gain an insight into the children's understanding and thinking through observing the language children attach to their notational schemes.

Due to the fact that children represent their mathematical ideas in a variety of ways they, too, should be encouraged to analyze and understand the various representations. Initially the teacher may model this process with the children when they share their own representations while collaboratively working through a problem. Children are then able to view the problem from different perspectives as well as the different ways of thinking about the problem (NCTM, 2000). Using the class as an audience for the children's own mathematical writing promotes the 'publishing' of their mathematical theories; provides an opportunity for feedback; and also allows the sharing of ideas with other classes (Gifford, 1997).

2.6.3 Sense making

Mathematical knowledge is viewed as dynamic, constructed and reconstructed through an ongoing process of sense making by the learner (Heaton, 2000, p. 4). Lampert (2001) emphasizes the importance of children learning how to make sense of problem-solving situations rather than relying on others. To develop their sense-making skills children use several 'tools' which include various symbolic representations of, for example, addition of two-digit numbers; language for talking about the meaning of the addition; and a culture in which one is able to publicly change one's mind. Klein (2002) found that when children are *authorized to make sense of the mathematics in ways that* *are meaningful to them they have a sense of themselves as able to go beyond the given to forge new ways-of-being in mathematics* (p. 68). Likewise, Kamii and Warrington (1999) note that when children are encouraged to rely on their own thinking *they will become able to go on to create ever higher levels of reasoning* (p. 91). As a result children develop 'ownership' in making decisions and deriving meanings from their actions. According to Anghileri (2000) this ownership helps children to develop confidence in their thinking and an inclination to work with numbers. Under the guidance of the teacher children share their own ideas about the way calculations can be recorded allowing them to be exposed to many ways of writing mathematics while still retaining ownership of their personal ideas.

By developing relationships between objects in the world, and rendering those relationships in mathematical notations, *children integrate mathematics with experience and enhance their understandings of both* (Lehrer et al., 1999, p. 70). Thinking is promoted when children select and organize their key ideas about a problem and its solution (WCER, 2001). Notation allows mathematically important ideas to be lifted out, selected, and discussed thus providing an opportunity to develop further understanding.

Children's ability to express and clarify their mathematical thinking to others also encourages reflection on their own understanding and reasoning (Wood, 1998). Resnick (1987, cited in Steen, 1999) recognized that successful mathematics learners were more likely to engage in reflective or metacognitive activity. Metacognition ability, thinking about one's own thinking, is a complex skill requiring both reflection on thought processes as well as being able to describe these to another person. It is an elementary component of children's reasoning processes (Tang & Ginsburg, 1999). Beishuizen and Anghileri (1998) suggest that the development of metacognitive skills is stimulated by emphasizing children's written recording of their own solution methods from the beginning for the purpose of children's reflection and interaction in whole-class discussion. Since notation is a more permanent record it encourages children to reflect on, revise what is written, and to later recall their process and thinking. Thus written recording provides a basis through which children's numerical knowledge can be advanced making connections between prior knowledge and new concepts (Morgan, 1998; Wright et al., 2000). Thompson (1996) also highlights that the written expression of an idea gives a child the opportunity to reflect on what has been said. It allows the child *to consider if what she said was what she intended to say and if what she intended to say is what she said* (p. 276).

When children monitor their own or someone else's explorations, implementations, or verifications they are involved in another aspect of reasoning, that is, the 'justification' of ideas (Artzt & Yaloz-Femia, 1999). Children learn far more than arithmetic when justification is a central component of instruction - they learn what constitutes a mathematical argument (Yackel, 2001). They come to understand what counts as evidence and justification for a particular point of view, that is, the structure of a mathematical argument. This leads to an agreement on what is acceptable as an adequate argument (Cobb et al., 1990; NCTM, 2000). Krummheuer (1998, p. 228) states that *a format for collective argumentation regulates the steps of the interaction in the mathematics classroom* as it sets out an appropriate sequence for statements made. Formatted argumentation assists in orientating the children's processes of constructing new meanings as well as increasing the chances of creating these new meanings.

Children come to realize they are expected to not only explain but also justify how they solve problems through the renegotiation of social norms in whole class discussions. By involving children in discussions where they justify their solutions, especially in the face of disagreement, they gain better mathematical understanding when they work to convince peers about differing points of view (Hatano & Inagaki, 1991, cited in NCTM, 2000). The aim is to construct powerful and reasonable understandings of why particular solutions and problem-solving methods make sense. It is anticipated that children will use and create mathematical representations to construct and demonstrate their understanding. Thus, notation provides a vehicle not only for communication but also for argumentation (McClain & Cobb, 1999).

An obligation to make sense of problems is facilitated when children work, not only in whole class situations, but also in small groups (Cobb et al., 1990). Children are able to evaluate their own work through privately reflecting on what they are doing and whether it makes sense; talking about it in a local community in which they are sitting; and by assessing their own work and that of others in whole-class discussions under the teacher's guidance (Lampert, 2001). When children take responsibility for their own learning they realize that mathematics can make sense and that they can make sense (Carpenter et al., 1999b).

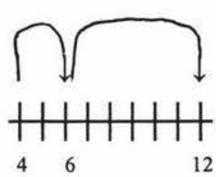
2.6.4 Errors and misconceptions

Written recording is one way of making children's thinking visible, providing a window for others to view how number problems were solved. Children are able to reflect upon how and why they got an answer, not just that they found an answer (WCER, 2001). It also assists them with identifying if and where an error has been made. Errors provide useful insights into children's thinking and mathematical understanding with persistent mistakes often highlighting gaps in their knowledge. Therefore, there is a need to create a class culture where mistakes and misconceptions are seen as acceptable and useful. Koshy (2000) describes a 'mistake' as a *wrong idea or wrong action*, and a 'misconception' as a *misunderstanding* (p. 172). An error can often be the result of a misconception but this is not always the case as there are many other factors contributing to this. She emphasizes that if children are to develop logic and reasoning they need the confidence to follow leads which may prove to be the wrong ones without fear of failure.

Achievement and long-term retention of mathematical skills and concepts is improved by addressing misconceptions during teaching. Anghileri (2000) contends that drawing attention to a misconception before giving examples is less effective than letting children solve number problems and then having the discussion. Children will learn to reason about their strategies and follow the reasoning of others through sharing misconceptions thus developing important skills in the communication of mathematical thinking.

Many teachers who encounter a child having difficulty in resolving a number problem often focus on the lack of understanding of the mathematical concept involved rather than on the representations that have been utilized. There have been attempts to provide children with notational models some of which have emerged from the teacher's understanding rather than from the children's own interpretations and ways of notating and symbolizing (Lampert, 1989, cited in Whitenack et al., 2000). In these situations the emerging symbolization exists 'outside' the children's actions with mathematical objects, but 'inside' the mathematical practices of the wider society. In such cases children may syntactically manipulate symbols without reference to their meaning (Dufour-Janvier et al., 1987). Maher and Davis (1990) claim that if the teacher fails to recognize the way a child is thinking about a problem then mutual misunderstandings may occur.

Negative consequences may be caused by introducing notations prematurely or by using an inappropriate context leading children to develop erroneous conceptions that subsequently become obstacles to learning (Dufour-Janvier et al., 1987), for example, introducing conventional signs, such as ‘<’, ‘>’, while children are learning to write numerals and letters. When children have not yet developed appropriate conceptual understandings they do not ‘see’ diagrams and models in the same way as knowing adults. Numerous misconceptions can occur with the number line which is often used in an attempt to give sense to equations. Children may view each step in the number line as a ‘stepping stone’ with ‘holes’ in between, hence creating the impression that there are no numbers between whole numbers. The use of arrows on the number line may cause confusion, for example, interpreting the following notation as:



$$2 + 6 = 12$$

Such interpretations can lead children to develop misconceptions that will hinder them in later learning. Furthermore, Dufour-Janvier et al. (1987) discovered that children were not at all disturbed to find that the same problem solved by someone else using a different representation got a different answer. From the children's point of view it is quite natural that a problem done in different ways may lead to different answers. Therefore the researchers contend *the focus needs to be placed on the way mathematics is presented or taught* (p. 114).

2.7 Summary

Mathematical understanding is developed by children actively engaging in tasks which build upon their existing knowledge. The teacher has a proactive role in establishing a classroom culture where children construct and develop number concepts and relationships through social interactions. Literature suggests that the key aspects of instruction which enhance understanding are the activities children engage in; the representations of their ideas; and the normative practices that are negotiated within the classroom. In an inquiry-based classroom explanations, listening, modelling, and questioning are promoted while children work collaboratively to reach a consensus through mathematical argumentation.

Communication plays an important role in helping children construct links between their informal, intuitive notions and the abstract language and symbolism of mathematics. It also plays a key role in helping children make important connections among idiosyncratic, pictorial, iconic, symbolic notations and mental representations of mathematical ideas (NCTM, 1989, cited in Ministry of Education, 1995).

Mathematical activity can be recoded through developing meaningful notational schemes which fit with children's current understandings about number. Notation records children's ideas and strategies used for problem solving. Indeed written recording does more than this by providing a 'tool' for clarification, reflection, justification and argumentation. When children's notations become increasingly more prominent in number computation instruction it is imperative that not only are their interpretations considered but also the intention of using them (Yackel, 2001).

Children's strategic thinking and reasoning about number is now the focal point of many numeracy programmes. A focus on, and the promotion of, mathematical thinking is far more challenging than traditional teaching approaches. Such a paradigm shift in the teaching of numeracy emphasizes the development of 'flexible' thinking for problem solving.

CHAPTER 3

RESEARCH DESIGN

3.1 Introduction

Qualitative research is a form of social inquiry that explores phenomenon in their natural setting, and focuses on the way people interpret and make sense of their experiences (Arsenault & Anderson, 1998; Holloway, 1997). A fundamental assumption of the qualitative paradigm is that a deep understanding of the world can be gained through conversation and observation within these settings. Burns (1997) states that the role of the qualitative researcher is to portray what people say and do as they interpret their complex world, and to understand events from the participants' viewpoints.

The foundation of qualitative research lies in the interpretive approach to the social reality (Holloway, 1997). In this context education is considered to be a process and school a lived experience (Merriam, 1998). The qualitative researcher uses multi-methods to explore, understand and explain the behaviour, perspectives and experiences of the people they study. Interpretivists claim that the experiences of people are basically context-bound, that is, they cannot be free from time and location or the mind of the participants. Researchers must understand the socially constructed nature of the situation and realize that values and interests are part of the research process (Holloway, 1997).

While qualitative research might appear to be more subjective than results that involve statistical analyses, there are nevertheless systematic ways to gather evidence that converge to a result and hence a conclusion (McKnight, Magrid, Murphy & McKnight, 2000, p. 59). Qualitative research best addresses questions of why and how something is happening. Although it is difficult to find precise distinctions between the different qualitative approaches and strategies it is obvious that all of them focus on the everyday life, interaction and language of people. The underlying rationale and framework of ideas and theories determines the approaches, methods and strategies to be adopted by the researcher (Holloway, 1997). To achieve their aims qualitative researchers choose from a variety of approaches and procedures, including 'developmental research'.

3.1.1 Developmental research

'Developmental research' consists of *curriculum development and educational research in which the development of instructional activities is used as a means to elaborate and test an instructional theory* (Gravemeijer, 1998, p. 277). It is seen as a form of basic research that lays the foundations for the work of professional curriculum developers. A key feature of the *Numeracy Project* is its dynamic and evolutionary approach to implementation which ensures that it can be informed by developing understandings about mathematics learning (Ministry of Education, 2003). Within the context of the current *Numeracy Project* this study has the potential to provide information to inform the modification and further development of the programme. In this sense, the *Numeracy Project* can be viewed as 'developmental research'. Current reform in mathematics education is shifting away from 'teaching by telling', and replacing it by 'students constructing' or 'inventing'. This shift in emphasis highlights the problem of *how to direct this learning process, or 'how we can make children invent what we want them to invent'*? (Gravemeijer, 2001, p. 147). In this particular study the focus is on how children 'invent' notational schemes.

Gravemeijer (1998) describes developmental research as being *at the heart of an 'innovation process'* (p. 292). The development of a domain-specific instructional theory is based on a method of elaborating an instructional theory in a cyclic process of developing and testing instructional activities (see Figure 3.1). Developmental researchers have to address how to design instructional activities that link with the informal knowledge of children enabling them to develop more sophisticated, formal knowledge. The base level of the developmental research cycle is 'concrete' in which instructional activities are planned and tried out in the classroom on a day-to-day basis. The analysis of what happens in the classroom informs the planning of the next instructional activity. At a broader level, the developmental research cycle centers on the entire instructional sequence guiding instructional theory towards a theoretical basis.

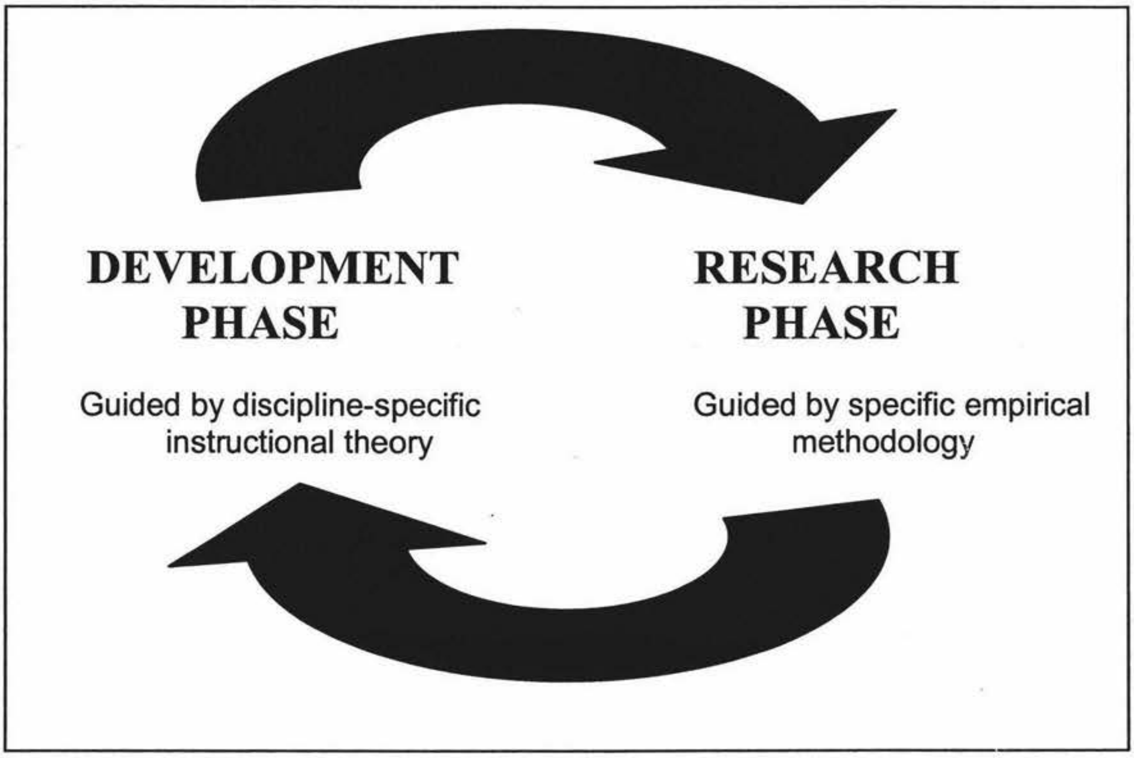


Figure 3.1 Aspects of the developmental research cycle (Gravemeijer, 1995, cited in Cobb, 2000b, p. 315)

In this study a 'teaching experiment' in the context of developmental research was implemented in collaboration with the classroom teacher. The primary purpose for using 'teaching experiment' methodology *is for researchers to experience firsthand children's learning and reasoning* (Steffe & Thompson, 2000, p. 267). 'Experiment' refers not to untried or unusual instruction, but rather to collaborative analysis and planning of the children's mathematical activity. A teaching experiment involves a sequence of teaching 'episodes' with each one consisting of a teaching agent; one or more children; a witness of the teaching episodes, in this case the researcher; and a method of recording what transpires. These records are used in preparing subsequent episodes and conducting retrospective conceptual analysis of the teaching experiment (Steffe & Thompson, 2000).

This methodology involved the researcher performing a 'thought experiment', as she designed instructional activities, in which she envisioned how the teaching-learning process would proceed. *Instructional sequences should contain activities in which children create and elaborate symbolic models of their informal mathematical activity.* This modelling might *entail developing informal notations or using conventional* *mathematical notations* (Cobb, 2000b, p. 319). This was discussed with the teacher and modifications were made where necessary. The 'instruction experiment', that is the actual activity, was then tried out in the classroom with the researcher searching for signs that confirmed or rejected the expectations of the thought experiment. Moreover, the researcher kept her eyes open for new possibilities. The feedback from empirical data continued the cyclic process of deliberating and testing (see Figure 3.2).

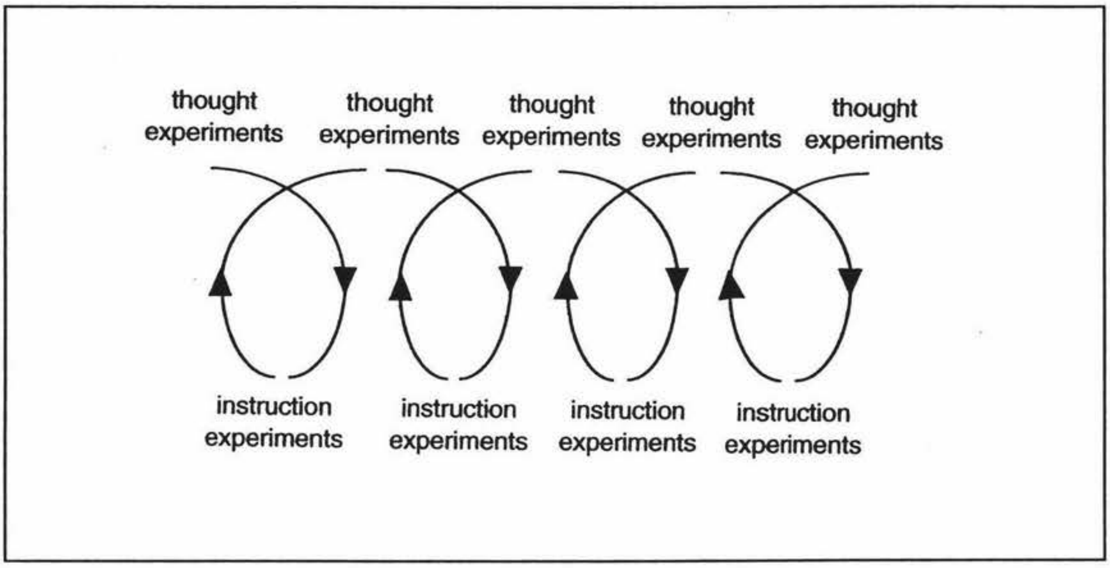


Figure 3.2 Developmental research: a cumulative cyclic process (Gravemeijer, 2001, p. 153)

Although the researcher relied on theory and tasks developed by others, there was still room, and a need, for her to construe her own learning trajectories. Activities had to be adapted to the specific situation of the teacher and her goals, as well as with her students, at a particular moment in time. Daily analysis was required with regular meetings held with the teacher to discuss and plan for children's mathematical learning. A critical part of the methodology is the retrospective analysis of all records made (Steffe & Thompson, 2000).

3.2 Data Collection Methods

Data collection involves the gathering of information through a variety of data sources (Holloway, 1997). In order to gain information about children's notation of number computations the data collection instruments deemed appropriate for this study were the case study, interview, and the researcher.

3.2.1 Case study

A case study in research examines in detail a single entity or phenomenon and is bounded by time and activity (Cresswell, 1994). These boundaries are clarified in terms of questions asked, the data sources used and the setting and person(s) involved (Holloway, 1997). Information is gathered, during a sustained period of time, from a wide variety of sources to present a description of the phenomenon or experience from the perspective of the participants (Ertmer, 1997). The case study focuses on process rather than outcome, on discovery rather than confirmation (Burns, 1997). This form of data collection was chosen as it provides a rich descriptive real life holistic account that offers insights and clarifies meanings.

An advantage of the case study is that the effects within real contexts can be observed, *recognizing that context is a powerful determinant of both causes and effects* (Cohen, Manion & Morrison, 2000, p. 181). The researcher can observe and understand the phenomenon *as it is experienced by the participants* (Ertmer, 1997, p. 158). There are two main types of observation in the case study: firstly, participant observation where the researcher engages in the activities she set out to observe; and, secondly, non-participant observation where the researcher stands away from the activities she is investigating (Burns, 1997). Merriam (1998) maintains that, in reality, researchers are rarely 'total participants' or 'total observers' as the balance of observation and participation is likely to change when the researcher gains familiarity with the phenomenon being studied. The data collected from observation enables the researcher to understand the content of mathematical programmes; be open-ended and inductive; discover things that participants might not freely talk about in interview situations; move beyond perception-based data; and access the participant's knowledge (Cohen et al., 2000). The case study was used in this research to gain an understanding of the children's development of notational schemes and how mathematical activities are recoded for communicative purposes.

The case study is less readily generalizable than other qualitative research as it is used mainly to investigate cases which are tied to a specific situation and locality (Holloway, 1997). However the generalizability of the research may be enhanced through a multi-case study, that is, adding two or more cases. This also allows the results to be compared and contrasted for comparative purposes (Wiersma, 2000). In an attempt to address this issue the research study used six case studies. Fieldwork included the researcher's 'non-participant observation' of the case studies as they worked individually, in a group, as well as within whole class situations.

3.2.2 Interview

Holloway (1997) describes the qualitative interview *as a 'conversation with a purpose' in which the interviewer aims to obtain perspectives, feelings and perceptions from the participant(s) in the research* (p. 94). Qualitative interviews are based on a set of questions asked of a sample of participants, and can differ in their degree of structure (Holloway, 1997; McKnight et al., 2000). Drew, Hardman and Weaver-Hart (1996) point out that depending on the purpose of the interview, the researcher's familiarity with and knowledge about the setting, and the nature of the study, the interview 'format' can be open-ended or highly structured. An unstructured interview begins with a broad, open-ended question within the topic area with the interviewer having minimal control. Ideas of the participants are followed up with prompts and questions being reformulated throughout the interview. On the other hand, a semi-structured interview is more focused as it has a specific research agenda. It allows the researcher to collect all important information about the research topic while still giving participants the opportunity to report on their own thoughts and feelings. Goldin (1998) recommends a structured interview be used to observe mathematical behaviour, and to draw inferences from the observations made.

A task based interview explores children's approaches to and thinking about problem situations. It involves the child being asked to solve a particular problem, explaining out loud what he or she is thinking and doing. The interviewer may ask probing, or even prompting, questions in an effort to understand the child's thought processes (McKnight et al., 2000). Goldin (1998) suggests that the interview guide for this type of interview be 'flexible' to allow the researcher to *pursue a variety of avenues of inquiry with the problem solver*; and is 'reproducible' permitting *the same interview to be administered by different researchers to different children in different contexts* (p. 53).

This study utilized a task based interview as one of the data gathering methods. While the actual format was detailed (see Appendix A), the researcher felt that a semi-structured approach was more flexible to gather data. This allowed follow-up questions to be asked which are often are the most important part of the interview process (Drew et al., 1996). The initial interview was used to obtain information about children's numerical knowledge and strategies. The final interview enabled the researcher to compare the children's numerical development, especially with regards to notational schemes, prior to and after the teaching of the numeracy unit. Audio-taping and note taking was undertaken by the researcher with samples of children's notation collected.

The individual interview has some distinct advantages as the researcher can gather information and perspectives that are unique to each participant without being affected by other class members. Dialogue between the interviewer and the participant makes it possible to clarify questions and responses, ask for more information, or follow up with probes that elicit additional information (Drew et al., 1996). Despite having clear advantages, some disadvantages in the interview process have been identified. Interviewing requires intense concentration, ability to listen, write, anticipate a future question all at the same time, and strong human relations skills (Drew et al., 1996).

When involving children as participants the researcher needs to be sensitive to the scheduling of the interview. In this study interviews were scheduled at a time deemed appropriate by both teacher and children. The researcher ensured that the children were willing, informed participants and a conscious effort was made to put them at ease.

3.2.3 The researcher

The researcher is the principal instrument in qualitative research, and cannot be separated from the phenomenon under study (Merriam, 1998). As such she can respond to the situation by maximizing opportunities for collecting and producing meaningful information. Conversely, *complete objectivity and neutrality are impossible to achieve; the values of researchers and participants can become an integral part of the research* (Smith, 1983, cited in Holloway, 1997, p. 2). Additionally, Merriam (1998) considers that the qualitative researcher should have the following important attributes: a tolerance for ambiguity, sensitivity to context and data, and good communication skills. The researcher can influence the study either negatively or positively depending on a number of factors which can interfere, including her biases, values and judgments. Therefore the researcher must explicitly state these taking into account her own stance and assumptions. This provides an openness that can be considered helpful and positive.

The researcher is an experienced primary trained teacher who has recently taught in pre-service primary and early years teacher education programmes specializing in mathematics education. In addition, she has worked under contract as a facilitator to implement new curriculum documents, as well as being an adviser to schools. The study reflects the researcher's interest in children's number development and the implementation of the *Numeracy Project* within New Zealand schools.

A number of assumptions are brought to the research study by the researcher as result of her experience in mathematics education, specifically:

(i) number development is critical for children's future confidence in mathematics

(ii) notating of children's ideas is an key aspect of being able to communicate in mathematics

(iii) children actively construct their knowledge through social interaction therefore purposeful discussion in whole class situations, in small groups, and with individuals should be encouraged

(iv) children should be challenged to think by explaining (both verbally and in written form), listening, and problem solving

(v) children should be able to select and use strategies that are efficient and effective

(vi) numerical reasoning is fostered when children's thinking becomes the central focus of the mathematics classroom

(vii) teachers have a vital role in the development of notational schemes.

3.3 The Research Study: Settings, Sample, and Schedule

This section outlines the setting for the study, details of those who participated in the study are discussed, and finally the phases of the study and the data analysis methods are considered.

3.3.1 The setting and the sample

The research was conducted at a New Zealand provincial primary school during Term One of the 2003 school year. Puru School[1] has a decile rating of 4[2] with a roll of approximately 350 children. In consultation with the Principal, one teacher and her class of Year 5/6 students was selected for the research. The teacher has had three years experience of working with the *Numeracy Project* - two years of the *Early Numeracy Project* and one year with the *Advanced Numeracy Project*. Distribution of Information Sheets and Consent Forms to children and their parents/caregivers in the class preceded selection of case study students. From the returned Consent Forms the teacher assisted the researcher by providing advice to select a purposeful sample of six students, representing a range of mathematical abilities, to participate as case studies.

3.3.2 The research study schedule

Phase One

This phase included a preliminary literature review, consultation with mathematics colleagues, teachers and numeracy facilitators.

Phase Two

The initial interview questions were drafted, using the *Numeracy Project Assessment (NumPA)* (Ministry of Education, 2002d) as a guide, and piloted at another primary school. Towards the end of the 2002 school year the researcher trialled the set of questions, including number computation problems, with four Year 4 and two Year 5 children. When each interview was completed the children provided constructive feedback about the questions and subsequent modifications were made. As this school was not involved in the *Numeracy Project* the modified Interview Guide was then trialled with a Year 5 child who was part of the numeracy programme during 2002. This enabled the researcher to finalize appropriate questions for the initial interview of the case study participants (see Appendix A).

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[1] A pseudonym was used to protect the identity of the school.

[2] Each state and integrated school is ranked into deciles, low to high, on the basis of an indicator. The indicator used measures the extent to which schools draw from low socio-economic communities.

Phase Three

The researcher and teacher met to discuss the organization required to implement the research study during her classroom teaching. To realize this, the teaching of the Numeracy Unit was delayed until Week 4 of Term I, 2003.

During Week 3 the children who were to be case studies were named. Letters were sent out to parents/caregivers and children informing them whether or not they were to be a case study. The six case studies were interviewed by the researcher prior to the teaching of the Numeracy Unit. The interviews which were audio-taped, took place in a room selected by the teacher, at a time suitable to her and the children. Transcriptions of each interview were made and analyzed by the researcher. Following this the researcher met with the teacher to discuss and collaboratively plan activities for the commencement of the unit.

Phase Four

The study was undertaken over a four week period with the researcher observing sixteen mathematics lessons during February/March, 2003. The sessions were timetabled for an hour in the morning; however, such was the enthusiastic response from participants these sessions were often extended. There were five mathematics groups in the classroom, four of which were ability based[3], while the fifth group (consisting of the case studies) were of mixed ability. A 'taskboard' was introduced to guide organization and routines. Typically a mathematics lesson was structured to begin with a class activity, followed by the rotation of group activities which included a teaching session for three of the groups. Group activities of the case studies were audio-taped and transcribed by the researcher. Samples of the teacher's and children's written recording were collected daily and analyzed. Although the teacher conducted her usual mathematics programme based on the *Numeracy Project* this was modified as she and the researcher considered children's learning; discussed and planned instructional activities.

[3] Children were grouped according to strategy stages determined by the *Numeracy Project Assessment (NumPA)* (Ministry of Education, 2002d).

Phase Five

This involved final interviews of the case studies at the conclusion of the Numeracy Unit. The children were individually interviewed with four of the six questions being the same as in the initial interview. Two further questions were modified as a result of the teaching of the unit, and an additional two questions were presented to the more competent case study students (see Appendix B).

Phase Six

Through a retrospective analysis of data collected, categories and themes were identified and the information sorted accordingly. Details of these are given in subsequent chapters of findings and reflect the purpose of the research study.

3.4 Quality Criteria

As in any research reliability, validity, and ethics are major concerns. Qualitative research studies, in particular, are difficult to replicate as they occur in natural settings (Wiersma, 2000). These issues need to be addressed as every researcher wants to contribute results that are believable and trustworthy (Merriam, 1998).

3.4.1 Reliability

Reliability of research concerns the replicability and consistency of the methods, conditions, and results (Wiersma, 2000, p. 9). Although the same procedures and techniques may be followed and adopted, qualitative research can never be completely replicated, as the relationship between the researcher and the participants in the research is unique (Holloway, 1997). Consistency is difficult to achieve because the researcher is the main research instrument. Bogdan and Biklen (1992) contend that qualitative researchers *tend to view reliability as a fit between what they record as data and what actually occurs in the setting under study, rather than the literal consistency across different observations* (p. 48).

Merriam (1998) claims that reliability is enhanced when the researcher explains the assumptions and theory underlying the study; triangulates data; and leaves an audit trail, that is, by describing in detail how the study was conducted and how the findings were derived from the data. An audit trail, open to public scrutiny, is provided to enable subsequent researchers to inspect and, where appropriate, replicate aspects of this study.

3.4.2 Validity

Validity is an important component that establishes the truth and authenticity of a piece of research. It is the extent to which an instrument measures what it is supposed to measure. More specifically, in qualitative research, it is the extent to which the researcher's findings accurately reflect the purpose of the study and represent reality (Holloway, 1997). Validity of qualitative research for the most part is established on a logical basis, and thus requires well-documented research and a comprehensive description providing a sound argument. It must be *based on fact or evidence, that is, "capable of being justified"* (Wiersma, 2000, p. 4). Validity involves two concepts simultaneously: internal validity and external validity.

Internal validity, the *extent to which research findings are congruent with reality*, relies on logical analysis of the results (Merriam, 1998, p. 218). There is not an option of controlling the variables because research is done in natural settings, often with complex phenomenon (Wiersma, 2000). Internal validity is achieved when the researcher can demonstrate that there is evidence for the statements and descriptions made (Holloway, 1997). *Verifying results and conclusions from two or more sources or perspectives enhances internal validity* (Wiersma, 2000, p. 211). This is addressed by using triangulation; checking interpretations with case studies and teacher; being on-site at the school over a period of time; asking peers to comment on emerging findings; involving participants in all phases of the research; and clarifying the researcher's biases and assumptions.

Triangulation is *the use of two or more methods of data collection in the study of some aspect of human behaviour* (Burns, 1997, p. 324). It is argued that if different methods of assessment or investigation produce the same results then the data is likely to be valid. Findings based on conclusions suggested by different data sources are far stronger than those implied by one alone (Anderson, 1998). Triangulation is used to interpret findings, test alternative ideas, identify negative cases and direct the analysis towards a clear conclusion based on the evidence collected. This study has used a 'between methods' approach to triangulation which adopts different strategies but stays within a single paradigm; for instance case studies and interviews (Holloway, 1997). The use of multiple sources or instruments ensures a depth and richness of data as well as reliability and validity of results (McKnight et al., 2000).

External validity, the extent to which the findings can be generalized to other situations, is more difficult to establish as qualitative research is often very specific to a particular location and place (Holloway, 1997; Merriam, 1998). Wiersma (2000) states that the qualitative researcher is more concerned with the 'comparability' and the 'translatability' of the research study, rather than with generalizability, which enables others to understand the results. This highlights the central importance of 'rich thick description' so that the reader has the knowledge on which to base judgments (Holloway, 1997; Merriam, 1998). External validity is enhanced when a situation and setting which is typical of its kind is selected by the researcher (Schofield, 1993, cited in Holloway, 1997). By using multiple cases, especially those that maximize diversity in the phenomenon of interest, allow results to be applied by the reader to a greater range of other situations (Anderson, 1998; McMillan & Schumacher, 1997). This variation can be achieved through purposeful or random sampling (Merriam, 1998). Using a naturalistic setting and adherence to the strategies suggested above enhance the external validity of this study.

3.4.3 Ethical considerations

Ethical concerns have to be considered in all research methods and at each stage of the research design (Holloway, 1997, p. 55). Researchers apply the principles to protect the rights of the research participants and to conduct the research in an ethical manner (Wiersma, 2000). The major ethical principles need to be interpreted and discussed in an open and informed way with participants involved in the study (Massey University, 2000).

In conducting the research the following steps were taken to ensure that ethical principles were applied:

(i) Approval was given by College of Education Ethics Committee, Massey University.

(ii) Approval was obtained from the Principal to enter the school for research purposes.

(iii) Informed consent was obtained from the Board of Trustees to conduct the research study after they had been given an Information Sheet and had time to consider implications of the school's involvement.

(iv) Informed consent was obtained from the teacher after she had been given an Information Sheet about the research study and had time to consider the implications of her involvement.

(v) Informed consent was gained in writing from the children's parents/caregivers after they had been given an Information Sheet about the research study.

(vi) Informed consent was gained in writing from the children after they had been given an Information Sheet about the study and had time to consider the implications of granting consent.

(vii) Participants were informed about the researcher's credentials and why the study was being conducted. They were given a comprehensive explanation of the nature and purpose of the activities, their rights to decline participation, to withdraw from the study at any time, to have privacy and confidentiality protected, to have a recording device turned off at any time, to decline to allow copies of written samples to be taken, and to receive information about the outcome of the study in an appropriate form. Care was taken to ensure that all participants were provided with this information in a manner and form which they could understand.

(viii) To ensure confidentiality the information was handled in a way which protected the confidentiality of the participants and safe custody of the data was maintained. The audio-recordings were transcribed by the researcher only.

(ix) Assurance was given that all efforts would be taken to maximize the confidentiality and anonymity of the participants and the school.

3.5 Summary

A qualitative research design was adopted to explore children's notation of number computations by Year 5/6 students in a primary school. In order to examine the complexity of issues relating to numeracy, and to inform ongoing evolution of the *Numeracy Project*, a teaching experiment in the context of developmental research was conducted. The major data collection techniques used were case studies and interviews. Mediated information was gathered from the participating children with regular discussion of findings between the teacher and the researcher. The results of this data collection are presented in Chapters 4 and 5.

CHAPTER 4

RECORDING MATHEMATICAL ACTIVITY

4.1 Introduction

This chapter combines the information gathered from the children and the teacher during the four cycles of the developmental research. A summary of each cycle undertaken in this study is provided in Appendix C. Presented descriptions are based on the researcher's observations, samples of students' and teacher's work[4], and audiotape evidence.

A description of how recording conventions of children's thinking become established within the classroom is provided. Presentation of the six case study students solving number computations during classroom practice is submitted. The ways in which they recoded mathematical activities, and the influences which affected this are highlighted.

The contribution notation made to the productiveness of group and whole class discussions of strategies and solutions is considered. An exploration of how discussions were structured to allow the report back of strategies is presented. The ways in which children used notation to analyze their thinking, and the thinking of others, in order to compare and contrast strategies and solutions is described.

4.2 Establishing Recording Conventions

During group and class activities Ms. Vine often introduced ways of symbolizing in an attempt to clarify a child's thinking either for herself or for other children. In contrast, during Independent Time when children completed individual activities they were asked to record their own thinking so that others might understand their reasoning. The ways

[4] Pseudonyms were used to protect the identity of participants.

of symbolizing that emerged in Ms. Vine's classroom evolved from the need to clarify and communicate the children's thinking.

4.2.1 Redescribing and notating explanations

The teacher took a proactive role in redescribing and notating children's explanations of their mathematical activity. These were recorded in the 'Modelling Book' (see Figure 4.1) during both group Teaching Time and in whole class settings, where various strategies were discussed and shared.

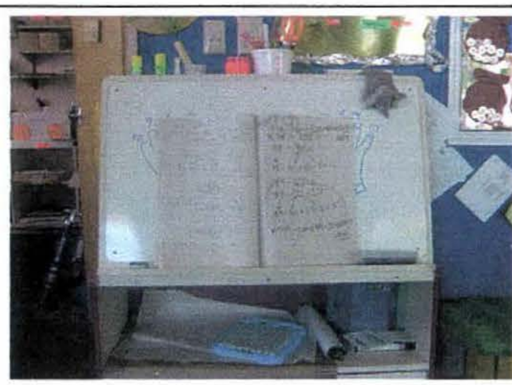


Figure 4.1 Modelling Book

This was particularly important when different answers were obtained. For example, for the problem: 24 + 26 = □, the answers varied from 36, 48, 50, to 51.

Ms. Vine: *How did we get these answers?*

Jack: *25 and 25 is 50.*

Rob: *You take the 1 off the 6 and add it to the 4...equals 5.*

Ms. Vine: *That makes it 25.* [The teacher notates in the Modelling Book.]

$$2\cancel{4} + 2\overset{1}{6} = 50$$

$$25 + 25 = 50$$

Sue: *I did it a different way. 20 and 20 is 40* [Teacher notates and verbalizes child's statement] *plus 6 plus 4 equals 50.*

$$24 + 26 = 50$$

$$20 + 20 + 10 = 40 + \overbrace{6 + 4}^{10} = 50$$

Ms. Vine: *Equals 50. These 2 numbers make 10, don't they* [pointing to the notation].

In order to assist those children who were having difficulty with place value the teacher used the notated solution to highlight and discuss these concepts. For example, in the problem below Ms. Vine guided the discussion to finding 'smart tens' by grouping numbers using counters. She asked if there was a quicker way than 'counting all' or 'counting on' to find the answer:

Problem:	*Tina catches 6 fish, Merriam catches 7 fish, and Liam catches 4 fish. How many did they catch altogether?*

Rob: *You can…6 plus 4 equals 10 plus 7 equals 17.*

Ms. Vine: [The teacher redescribes and notates Rob's verbal explanation.] *We can make groups of 10.*

$$6 + 7 + 4 = 17$$
$$10 + 7 = 17$$

The teacher encouraged the children to work collaboratively during problem solving activities. An example of this was when she introduced the game "Can you make it?" to mixed ability groups. Using any combination of the numbers 6, 4 and 3, and the operations of addition and subtraction, they made numbers from 1 to 20. The children discussed how they would do this in their groups then verbally shared solutions with the whole class:

Maggie: *To make 18, you go 6 and 6…3 and 3.*

Ms. Vine redescribed Maggie's explanation recording the number equation on the whiteboard. When everyone agreed that this was acceptable she notated it on the sheet and placed it in the Modelling Book:

$$18 = (6 + 6) + (3 + 3)$$

4.2.2 Children's recording ideas

During group Teaching Time the case study students were encouraged to consider how they might record their thinking as they discussed possible solutions. This provided an opportunity to find out what external representations the children used to express their ideas. The teacher gave them a problem to solve which they talked about as a group. By doing this, the children came up with a range of different strategies which the teacher redescribed, and, with the help of the children, notated:

Problem:	$13 + 9 = \square$

Rob: *You can go in 'bridges'.*

Ms. Vine: *13 plus 9...so you'd start at 13. How would you 'bridge' it?*

Rob: *You'd go 14 - write 14 down – write 1 on the top.*

Ms. Vine: *So you go* [notates using 'bridging' symbols] *15, 16, 17...21, 22. Count them until you got to 22.*

Written recording reflected strategies that were familiar to the children:

Simon: *Count it into little fives, like this* [demonstrates with imagery strokes].

Ms. Vine: *They are called tally marks.* [Notates using tally marks in the Modelling Book.]

One child picked up on a strategy that the teacher had used the previous day:

Jack: *Yesterday you taught us about 'tens'. 'Taking down'* [uses the teacher's terminology].

Ms. Vine: *Excellent, you could try grouping them.* [Redescribes and notates strategy in Modelling Book.]

$$13 + 9 =$$

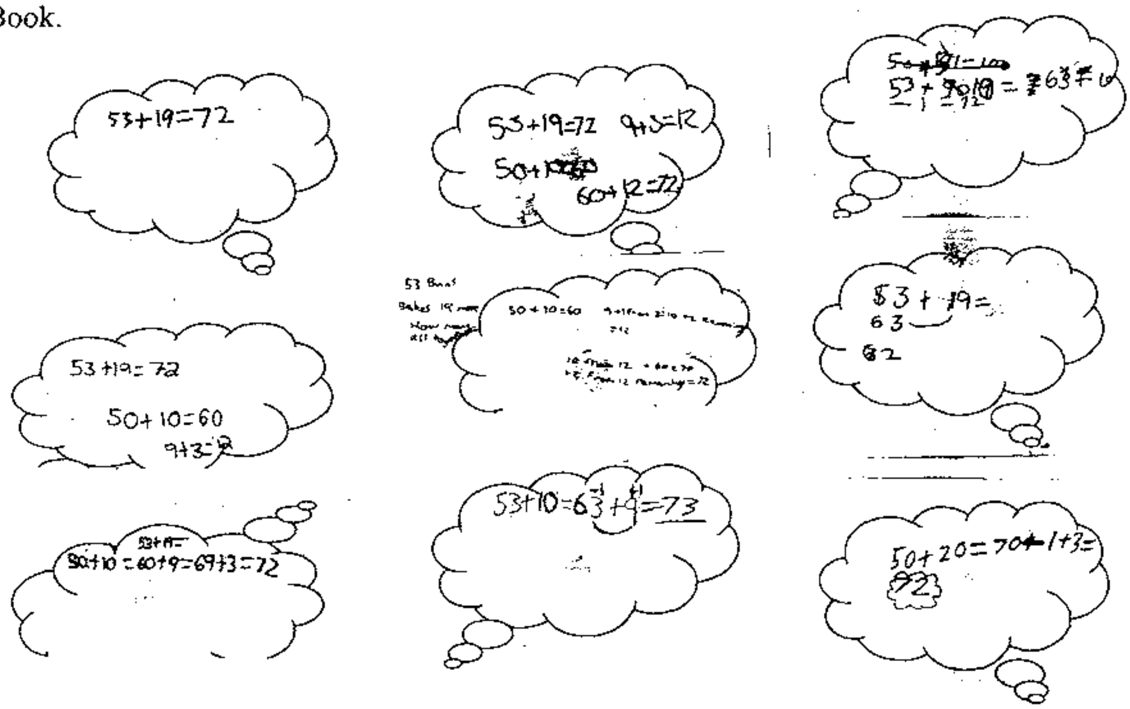
$$+ \ 3 + 9 = 12 + 10 = 22$$

4.2.3 Making connections

Vale's (1999) 'Thinking Clouds' was adapted to ascertain the recording methods of class members. The following mental computation word problem was posed to the children in the class:

Problem:	*There are 53 buns in a bakers shop. The Baker cooks 19 more. How many are there altogether?*

They were given time to think about the solution. Each child received a 'Thinking Bubble' to record their thinking and was invited to place their bubble into the Modelling Book.



The teacher commented: *Look here – there are 27 different ways of thinking. Can we see any that are the same?*

At this stage of the teaching unit, while a range of strategies had been presented, the children responded by grouping solutions that recorded the number equation and answer only, that is '53 + 19 = 72'.

The teacher took the opportunity to ask some children how they had solved the problem. As Sue began to explain her strategy, the teacher stated: *Let's begin with the equation.* She wrote '53 + 19 = ' as this provided clarification, for the children, of where numbers came from when Sue described her solution method. The teacher used a 'pull-down' notational scheme to highlight these connections:

53 + 19 = 72

53 + 10 = 63

9 - 2 = 7

63 + 7 = 70

70 + 2 = 72

Ms. Vine praised the children stating: *You've never been asked to notate your thinking before. I can see some really good thinking.* Jess commented, without prompting, that she thought that Child X's notation was 'good'. Another child agreed justifying: *He explains what he has done... shown it with 'buns'* [refers to the contextual problem where a combination of words and numbers have been used to notate thinking].

4.2.4 So what makes good recording?

After the first week the teacher asked the case study students this question. The replies included:

- *Neat work.*
- *Write it clearly.*
- *Write down everything you can think of.*
- *You can see your mistakes.*
- *Use it for talking. You can discuss it with another buddy.*
- *Use different ways to work (the problem) out.*

The teacher and the children used the Modelling Book to reflect on how the group had solved and notated problems to this point. This validated the use of children's own strategies, highlighting what was acceptable and taken-as-shared by the group:

Ms. Vine: *Oh look, we rounded it up to tens – we made tens. That's an easy number to do. We can do our 'bridges' or 'seagulls'* [same notation but labelled differently by some children]. *We can 'take them down' to make groups of tens* [referring to 'pull-down' notation]. *You can use your fingers, tally marks, dots, or you could do your own way* [pointing to children's pictures and symbols]. *Now this is something different* [noticing a descriptive sentence]...*I don't want you to write sentences. When you are doing reading or language that's when you write sentences.*

The teacher encouraged the children to use the recorded strategies in the Modelling Book to assist them with solving problems. This also provided an opportunity for them to reflect on the various notational 'forms' used:

Ms. Vine: *I want you to work the problem out by yourself. We've talked about the different strategies, and they are all in this Book. So you might need to take the Book with you...*

4.3 The Development of Notational Schemes

4.3.1 Teacher's notational scheme

During Teaching Time with the case study students the use of 'grouping tens and ones' (*1010* strategy) emerged as a taken-as-shared way of solving tasks. Ms. Vine devised a simple method of notating this reasoning by using a "V" symbol that came to signify the partitioning of numbers. Ms. Vine would typically follow the "V" notation, referred to as 'pulling-down', with the number sentences that expressed the result of the partitioning (see Figure 4.2).

T|O T|O
24 + 16 =
\ /
20 + 10 = 30

Ms. Vine: *Add your tens. This is a 'ten' and this is a 'ten'* [circles the 'tens']. *When you pull them down this is what it looks like – 20 plus 10. This is the 'ten' and this is the 'one'.* [Points to notation and writes 'T' and 'O' over the tens and ones.] *When you bring them down you need to make sure what are 'tens' and what are 'ones'.* [The teacher then pulls-down the 'ones'.]

Figure 4.2 Notating partitioning of numbers (*1010* strategy)

The "V" did not necessarily fit with the children's activity as some continued to use counting-based solutions which were less sophisticated than the collection-based solution.

The teacher employed a similar recording method of 'pulling-down numbers' when using the *N10* strategy to solve problems. She had noticed, during a class activity, that some children used this strategy to solve addition problems. Recognizing the difficulties that had arisen when children used the *1010* strategy for subtraction, in particular with the 'smaller-from-larger bug' (Baroody, 1987, cited in Blote et al., 2000, p. 223), she decided to model the *N10* strategy as an alternative method (see Figure 4.3). Using a number problem as an example, the teacher verbalized and notated each step in the Modelling Book with the case study students.

$$26 - 13 =$$

$$26 - 10 = 16$$

$$16 - 3 = 13$$

Ms. Vine: *You are only to 'bring down' the first number like this* [records steps]. *26, and then you are going to take away the ten from here* [points to notation]. *That leaves 16. Then 'bring down' this 3 ... you take away the 3 and that leaves 13.*

Figure 4.3 Notating 'First Number – tens – ones' (*N10* strategy)

Some children found it difficult to use this new strategy as it involved thinking differently about numbers. Jack and Rob referred to using 'doubles' and the *1010* approach for solving the problem:

Jack: *Everyone knows that 3 and 3 is 6.*

Rob: *And 10 and 10 is 20... and 3 and 3 is 6.*

The teacher gave the group another problem to try using the new strategy. Maggie continued to use the *1010* strategy, even though she thought that she was using the *N10* strategy. She could not understand how they were different. Only Rob's way of notating was entirely consistent with that of the teacher. Although the other children used elements of the teacher's scheme, they adapted them in original ways, devising notational schemes that expressed their thinking. The teacher's notational terms of 'pulling-down', and 'bringing down' were used by some children in their verbal description, as well as being adapted to fit with their own solution. For example, Jack explained: *It's dragging down, then you pull along and turn the second number into a 'ten'.* Even when expressed in terms of the teacher's notational scheme, most children continued to solve problems using a range of different, personally meaningful notation schemes.

4.3.2 Mental vs written algorithms

Not all of the case study students took to recording their thinking. Jess often wrote the number equation and then verbally explained her solution by pointing to the equation.

Problem:	*Hemi has 39 sweets. He buys a packet of 20 sweets. How many does he have altogether?*

To solve the above problem Jess wrote the equation '39 + 20 =', mentally solved it then explained: *20 and 30 ... just take out the 9. 20 and 30, add these together. It equals 50 and you put the 9 back on – it's 59.* The teacher's redescription and notation of Jess' solution immediately prompted others in the group to compare methods.

$$39 + 20 =$$

$$30 + 20 = 50 + 9 = 59$$

Maggie: *I've got another way. The 20 onto the 30 is 50 plus 9.* [She explains pointing to notated solution in her NUMP Book, not recognizing that her strategy is the same.]
Sue: *Same way!* [Recognizes that Jess and Maggie's strategies are the same].

Jack explained his way: *All you need to do is 40 and 20 would equal 60 and because it's 39 you take 1 away from the 60. But I didn't need to write it down cos it just came up in my head.*

When the teacher suggested that he record his strategy Jack insisted: *But I didn't do it that way! It just came up in my head.* Sue stepped in at this stage and asked him: *What were the numbers you used?* Jack suddenly realized that the others wanted him to show how he had solved it: *Oh... you mean the numbers I used for the problem?*

4.3.3 Using iconic symbols to solve problems

During an Independent Card session Rob had a question about whether '11 – 10 = □' was the same as '10 – 11 = □'. He decided that they were; Jack agreed with him. Rob justified this by saying: *you can't take a higher number away from a lower one.* On reflection he decided that '11 – 10 = 1', while '10 – 11 = 0'. Jack drew eleven circles (iconic symbols) and suggested that they could use the 'crocodile mouth' (wrote down the conventional symbols of '>', and '<') to show which one was bigger. There was a discussion about whether there were any numbers less than zero. Jack said: *there's infinity*; while Rob recognized there were numbers less than zero he could not remember what they were called. At this stage Simon joined in and stated: *It's a minus.*

Jack: *So the answer would be minus 11.*

Simon: *No, minus one.*

Starting with eleven circles Jack counted the iconic symbols, marking off ten, which left one circle: *So it would equal 1 ... minus 1 left!*

4.3.4 Standardizing notation

Sometimes the children's invented notation did not clearly portray their thinking. For example, the teacher observed that Jess had drawn a picture of a 'smiley face and five fingers', and asked her to describe her solution method:

Problem:	*27 + 17 = □*

Jess: *I went 27 plus 10 and put on 7.*

Ms. Vine: *So you grouped them. You tell me how you did it, and I'll write down exactly what you say.*

Jess: *10... oh 27 plus 10 equals 37, plus 7 equals 44.* [The teacher records this using conventional notation.] *I used fingers for 'plus 7'.*

$$27 + 17 = 44$$

$$27 + 10 = 37 + 7 = 44$$

The teacher introduced standardized notation when she saw that some children were confused with place value concepts. For example, Simon had recorded '8' instead of '80'.

Ms. Vine: *Is it 'tens' or is it 'ones'?* [Pause – Simon shows confusion]. *Okay this is what I'm going to do. When you bring down the 'ten' and the 'one' this is what you've got to do. You've got to write 70 plus 10.* [Notates '70 + 10 = 80'.] *And that says 80.*

Jack observed this interchange and exclaimed: *Oh, I'm getting confused cos I'm **not** putting in the 'plus'.* The teacher confirmed this and reinforced: *So what you have to do is 'bring it down' and say '70 **plus** 10 is 80'.*

4.3.5 Errors and misconceptions

At one stage while redescribing Sue's solution to '53 + 19 = □' the teacher mistakenly recorded that '3' and '7' had been added. The child quickly pointed out that she had added '63' and '7' together. As the teacher acknowledged this and corrected her notation children were able to see that it was acceptable to make errors, and that notation allowed for mistakes to be recognized. A supportive community of learners was established as children began to respond positively when errors occurred. For example, during one episode Rob encouragingly said: *You were close;* while another child helpfully commented: *You were only 10 off.*

Notation allowed the children to reflect on their thinking and to follow-up where errors were made. When sharing solutions for the problem below Jack inquired of a peer: *What's the 'tens' answer?*

Problem:	*89 – 52 =* □

Following the discussion he remained unclear where he had gone wrong. He recognized that the error was located in the 'tens' position. Jack showed his notated solution to Ms. Vine and commented: *My 'tens' are wrong. How did I get the 'tens'?*

89 − 52 =
↓
89 − 50 = 38 − 2 = 28 37

Jack had attempted to use the 'new' *N10* strategy, and the teacher quickly 'spotted' his mistake. She pointed to his notation: *Look – 89 takeaway 50 is 39.* Jack realized that he had used '30' instead and made the appropriate changes.

As well as supporting individual reflection notation allowed children to discuss their ideas, frequently resulting in new understanding:

Problem:	*13 + 9 =* □

Maggie: [Points to '9' in the Modelling Book]. *Just say that was a 10.*

Ms. Vine: *So you can change it?*

Maggie: *You takeaway the 1 so it's a 3, plus 1 so it's a 4. You cross that out* [assists the teacher with changing '9' to '10']. *Ten... then you cross the one out...oh no...you plus the...*

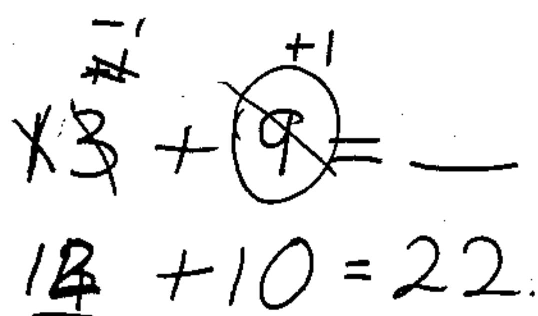
Ms. Vine: *We've made this into a 10, then you are going to change it.* [Teacher re-voices statement.]

Maggie: *You got to take one off.*

Rob: *You add one.*

Maggie: *I thought it would be takeaway.*

Rob: *It's rounding it up.* [Explains the strategy used.]

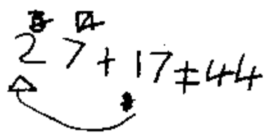


4.3.6 Window into thinking

Written recording also provided an additional 'window' into children's thinking. Information conveyed in notated form was especially useful for the teacher when verbal explanations were unclear.

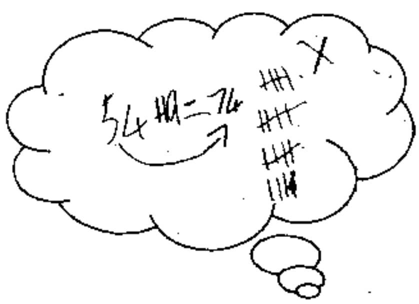
Problem:	*27 + 17 =* □

For example, in response to Maggie's description of her solution to the above problem: *I used... plus...* [pauses], Ms. Vine looked at the child's work to assist her in figuring out how the problem was solved. She recognized that Maggie had used this strategy previously: *You like that one, don't you.*



Maggie's notation indicated that she had used the *1010* strategy.

Conversely, the importance of verbalizing a strategy in conjunction with written recording, was highlighted during one episode where Jess had jotted down tally marks in her 'Thinking Bubble'.



Using her notation she reported back to the class that to find the answer to '54 + 19' she had added '1' to '54', and subtracted '1' from '19' to make the equation '55 + 18' (*compensation*). Now she had made a 'tidy number' so could 'skip count' in fives using tally marks (as a guide) then 'added 4 more': *55, 60, 65, 70, 74.* This demonstrated she was using a more sophisticated strategy than the 'counting on' that had previously been assumed.

4.3.7 Children redescribing and notating others' solutions

The case study students had difficulty redescribing and notating others' solutions if the number knowledge of one child was too advanced. For example, Maggie's explanation of her solution method was difficult for her peer to understand:

Problem:	*73 – 28 =* □

Maggie: *You take the 2 off the 7 so it's 5.*

Simon: *I don't know how to...*

Maggie: *You put the 8 with the 5.*

Simon: *That's 58.*

Maggie: *No, no, no. You got 53 and then I used my fingers - 52, 51, 50...*

Simon: *Oh...* [Jots down numbers as he hears them but clearly does not follow her thinking.] *I would just make it... take off that* [Starts to explain how he would solve it pointing to notation]. *Put the 20 on to that one cos it would equal one big number.*

73 – 28 = 45
58 50 48 47 45 c45

Simon's notation

~~7~~3 – ~~2~~8 = 49
~~5~~ 45

Maggie's notation

However, if their number knowledge was similar the children were able to assist each other as they redescribed and notated strategies as demonstrated below:

Problem:	*45 + 19 =* □

Rob: *It's 40 – you said 55.* [Rob reflects back on notation.]

Maggie: *Did I? Well, it's 45.* [Rob jots down '45'.] *You take 1... put the 1 on to the 4, that's 55.*

Rob: *So we take the 1...* [notates]... *the 10, that's the 10.*

Sometimes the familiar strategy interfered with children's thinking. For example, as Rob verbally explained to Maggie how he had solved the following problem she started to notate her own strategy:

Problem:	*39 + 66 =* □

Rob: *I went 39 plus 66. I got the 3 and the 6 and I put them together. I put the...*

Maggie: [Notating as she redescribes Rob's method]. *So you cross out that and that, and you add them together... 3, 4, 5, 6, 7, 8, 9. 99 plus 6* [Takes over - recording her own thinking rather than listening to Rob].

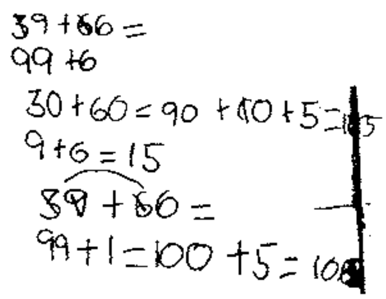
Rob: *No, not that way... 30 plus 60. I went like that! 9 plus 6 is 15... Then you put the 10 on to the 90... plus 5.*

This episode provided an opportunity for the two students to discuss the two methods and consider which was more efficient:

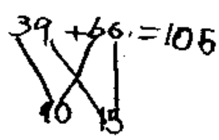
Maggie: *I reckon this one's quicker* [pointing to her own way]. *You know 3 and 3 is 6 plus 3 is 9, so it's 90. So it's 99 plus 1, plus 5.*

Rob: [Indicating that his way is better.] *It's the way the teacher said – 'by tens'. It's quicker because you can make 'tens'. It's an easier number to work from.*

When redescribing and notating their partner's solutions the case study students often used differing forms of symbolization:



Maggie's notation



Rob's notation

4.4 Addition Problems

4.4.1 Recognizing 'same' mathematical solutions

The teacher would, at times, record a child's thinking as a 'non-conventional' equation as the following example demonstrates:

Problem:	*There were 30 people at a bus stop waiting for the bus. Along came the bus and there were another 34 on the bus. How many people are now on the bus altogether?*

Sue: *I did 30 plus 30 is 60 plus 4 is 64.* [Teacher notates the method as it is verbalized.]

Ms. Vine: *Is that how you wrote it down?*

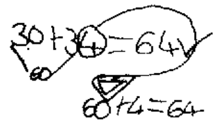
$$30 + 30 = 60 + 4 = 64$$

Sue: *Yes.*

The teacher's tracking of 'thinking' was intended to communicate to the speaker what she had understood, and enable other group members to follow what the speaker had said (Kamii & Housman, 2000). Maggie indicated agreement with the presented method. The teacher then inquired if there was a different way.

Jack: *30 plus 30 equals 60 plus 4.* [Does not recognize that it is the 'same' strategy.]

In providing a different method Jess suggested: *Take the 4 out, put a circle around it* [referring to notation]. *Plus 30, plus 30 equals 64.*



Sue quickly pointed out: *It's just the same but it's around the other way.*

During the group's Teaching Time it became obvious that some of the case study students could not recognize which strategies were the 'same' or 'different'. A similar result occurred when an activity using 'Thinking Bubbles' was implemented for the whole class:

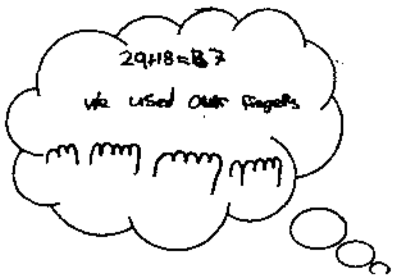
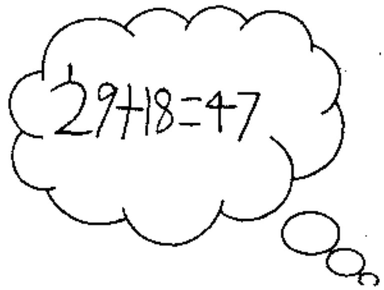
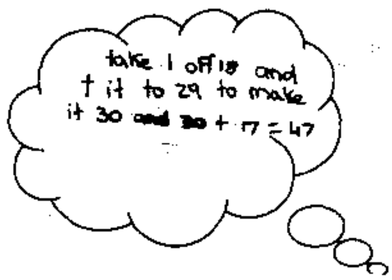
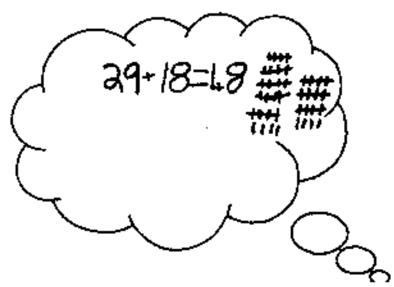
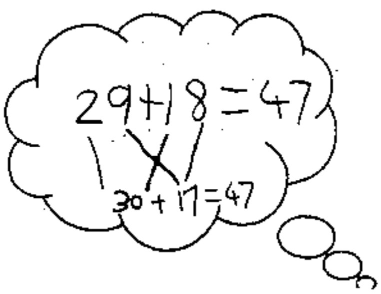
Problem:	*29 + 18* = □

After they had recorded their thinking in a 'Thinking Bubble' the children were then required to find others who had solved the problem in the 'same' way. The children moved around the classroom seeking others to form groups. There were three ways that were used to ascertain whether solutions were the 'same':

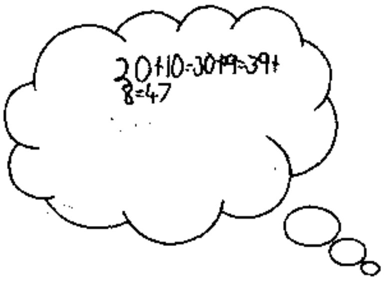
1) showed 'Thinking Bubble', comparing notation to see if it was written **exactly** as their one;
2) verbally explained their strategy; and
3) discussed and 'read' each other's notation.

A large number of children were **unable** to identify others who had used a similar strategy. The teacher circulated asking questions to assist children in deciding whether strategies were alike. Eventually seven groups were formed, and group members compared their notated 'Thinking Bubbles' to decide how best to notate the common strategy for class presentation. When they shared their strategy with the whole class they were asked by the teacher what 'label' would best describe the strategy they used (see Table 4.1).

Table 4.1 Problem-solving Strategies and Selected Notation

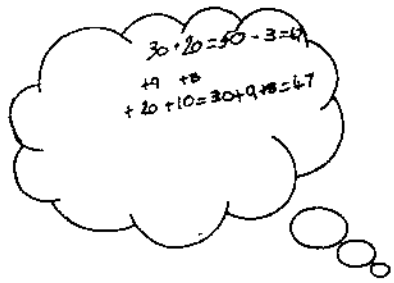
Group 1: Number of students – 3 Label: FINGERS Strategy: Counted on using fingers *Notation selected:* 	***Group 2:*** Number of students – 2 Label: VERTICAL ALGORITHM Strategy: Carry-over using vertical or horizontal layout *Notation selected:* 
Group 3: Number of students – 1 Label: ROUNDED TO '0's Strategy: Rounding to the nearest decade and compensation *Notation selected:* 	***Group 4:*** Number of students – 4 Label: TALLY MARKS Strategy: Counted on using tally marks (2 children) Counted all from one using tally marks (2 children) *Notation selected:* 
Group 5: Number of students – 2 Label: GROUP TENS Strategy: Standard place value partitioning (*1010*) (1 child) First number, add tens (*N10*) (1 child) *Notation selected:* 	*NB: The next two groups were split as they had originally gathered as one group indicating that they had used the same method.*

Group 6
Number of students – 4
Label: GROUP TENS
Strategy: Standard place value partitioning (*1010*) - with variations of adding 'ones' separately or together
Notation selected:



Group 7:
Number of students – 6
Label: GROUPS OF TENS
Strategy: There was a mixture of strategies in this group including:
- Standard place value partitioning (*1010*) (2 children)
- First number, add tens (*N10*) (1 child)
- Rounding to the nearest decade and compensation (1 child)
- Place Value Misconceptions (2 children)

Notation selected: The children chose two different ways to notate their strategies.



An analysis of the 'Thinking Bubbles' revealed that there were six main strategies:

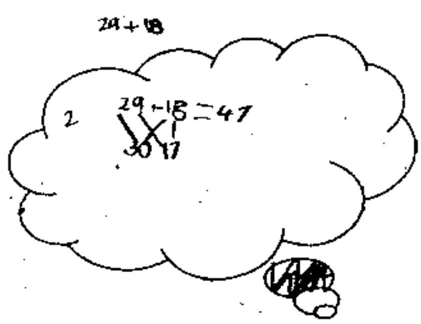
1. Counting from one (2 children)
   - using tally marks
2. Counting on (5 children)
   - using tally marks
   - using fingers
3. Grouping tens (9 children)
   - *1010*: add tens, add ones then add sums together
   - *1010*: add tens, add on each one
4. *N10* (2 children)
   - take first number, add tens then add second ones
5. Rounded to 0's (2 children)
   - round to the nearest decade (add/subtract appropriate amount). The two solutions given were slightly different:
     (a) subtracts '1' and adds to the other to made a decade
     (b) adds on to make a decade of both numbers, then subtracts

6. Vertical Algorithm (2 children)

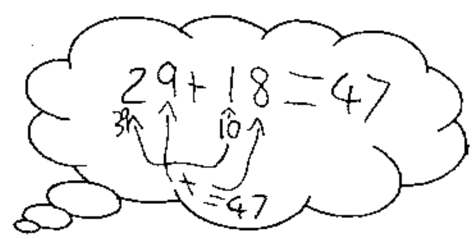
- uses standard procedure to solve (a) mentally; and (b) written form.

The activity highlighted other aspects of children's notation of addition problems. In accord with Thompson's (1994) findings, the children displayed a preference for using a horizontal layout and for working from left to right, with notation reflecting their mental calculation strategy. Even those children who set out their work vertically still tended to work left to right using their own computation method. The few who used the standard written algorithm for addition appeared to have been influenced by parents, siblings, or peers.

The influence of peers' notational schemes was also evident. This was observed when Rob and another child compared their recorded solutions to see if they were the same:

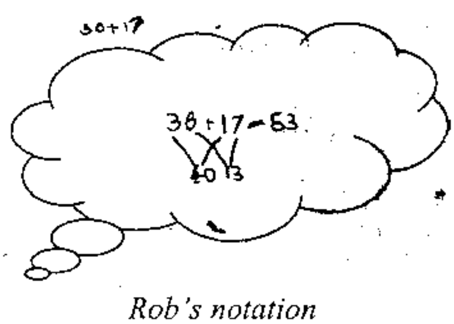


Rob's notation

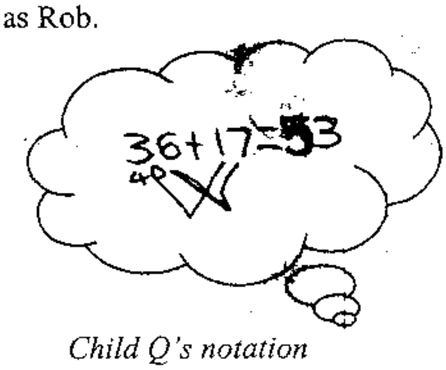


Child Q's notation

Rob did not think that the solution methods were the same but did not clarify his reasons; while the other child clearly thought that they were the same. Indeed the notational form was similar (using 'pull-downs') but while Rob had used the *1010* strategy, the other child had used *N10*. During the next class activity Child Q used exactly the same strategy and notational layout as Rob.



Rob's notation



Child Q's notation

When reporting back to the whole class some children had difficulty in verbally communicating which strategy they had used. The teacher was observed to scaffold more descriptive language for the children, as illustrated below:

Problem:	*29 + 18 =* □

Ms. Vine: *What did you do?*

Child G: *Plus.*

Ms. Vine: *But what did you 'plus'? I can see that you didn't just do '29 + 18'*... [looking at the notated solution].

Child G: *Tens.*

Ms. Vine: *You 'grouped the tens'.*

At the conclusion of this activity the children's notated solutions in the 'Thinking Bubbles' were displayed complete with strategy 'labels' (see Figure 4.4).

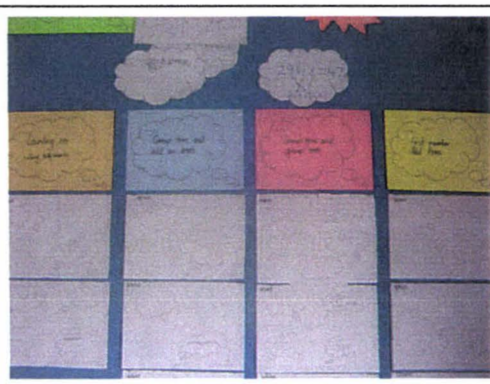


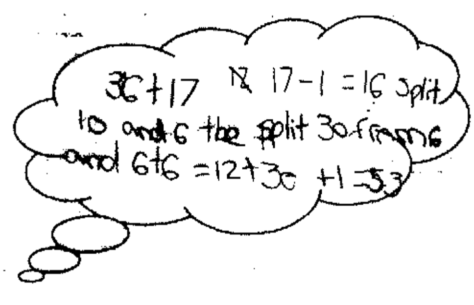
Figure 4.4 Wall display of strategies

4.4.2 Comparing and contrasting mathematical solutions

This class activity was designed to assist children to identify the 'same' strategies and to compare 'different' strategies. Children now found it easier to find others who had solved the addition number problem in a similar way. In groups they verbally shared solutions then gathered around the wall display. A discussion about 'different' ways to solve problems was initiated by the teacher. Individual children were asked if they had solved it using a particular strategy, to which they either agreed or disagreed. Each child pinned their 'Thinking Bubble' under the 'strategy label' that they thought best described their method. In order to do this they had to compare and contrast their solution with examples under the 'labels'.

Four children were unsure where their 'Thinking Bubble' fitted with the strategies displayed so placed them under the '?' label. For each of these cases the child was asked to explain their method so that the rest of the class could discuss and consider where it might be placed. For example, Rob suggested that one solution would fit with: *Group tens. Group ones.* By examining the structure of the written solution it was pointed out that: *You've just done the 6 + 7 first, then the tens, 30 + 10.* After agreement that grouping 'ones' then the 'tens' was the same solution method her 'Thinking Bubble' was placed under the 'Group 10s. Group 1s' label.

An analysis of the individual placement of 'Thinking Bubbles' revealed that only six children incorrectly identified their solution strategy. This time no one used 'counting on' with tally marks, or 'rounding to the nearest decade'. The following solution was identified by the children as a 'new' strategy which they labelled 'splitting numbers'.



Splitting Numbers

The children noted that the most popular strategy was 'Group tens and group ones' (see Table 4.2). This was not unexpected as the teacher often used this method in the Maths Group Teaching Time. She explained to the children: *It is the one I find most efficient however it is not the only way. This* [referring to the notated solutions under 'Group tens and group ones'] *is a step-by-step way showing how thinking goes.* Jess commented that by being able to view what other people had done had changed her thinking because she could see that there were: *9 different ways to solve problems.*

Table 4.2 Summary of Strategies Used by Class Members

Strategy	Number of Children
Counting from one (using tally marks)	2
Counting on (using fingers)	5
Group tens and add on ones	2
Group tens and group ones	8
First number and add tens	2
Splitting numbers	1
Vertical algorithm	2
?	4

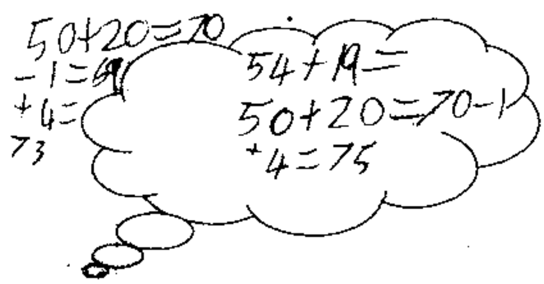
4.4.3 What counts as a 'different' mathematical solution?

To introduce the class session on identifying 'different' strategies the teacher asked one child to explain to the class how her method differed from the one used previously. Child H pointed to her notated 'Thinking Bubble' on the wall display in order to highlight the difference between the two selected strategies (that is, 'rounding up' and 'splitting numbers'). As she compared the notated solutions she commented how she had originally thought she had done them the 'same' but now recognized as she talked about them, that the numbers were structured differently.

The children reflected on their own notated solution from the previous day and compared it to the next strategy on the wall display. Each child discussed with their partner how it differed by contrasting the solutions. The children then reported back to the whole class. For example, Jess commented: *I used fingers yesterday. There's not much difference with the next one (tally marks). One uses lines and the other fingers.* Jack had used 'rounding to the nearest decade', and thought that the strategy of 'splitting numbers' would be 'quicker': *cos you split the numbers in half and add it on.*

The teacher gave the class the number problem: $54 + 19 = \square$, and challenged them to use a 'different' strategy to their usual method. A number of children gathered around the wall display to reflect on the various notated solutions (in 'Thinking Bubbles') to assist with the understanding of another strategy. Most of the children felt 'safe' in their environment to 'risk-take' by attempting to use 'different' strategies which showed more sophisticated thinking. For example, while Simon had previously used 'counting on' with fingers, this time he tried the *1010* method. There was great excitement when children successfully solved problems using different strategies. An example of this was when Rob recognized that he had found an answer using the *N10* strategy. He exclaimed: *Yeh Baby*; then proudly showed his notated solution to the teacher to demonstrate how he had done it.

Some children became so motivated that they wanted to re-visit strategies to clarify solution processes. An instance of this was demonstrated by Jack when he recognized that his answer to the class problem was incorrect. During Independent Time he returned to the problem, reflected on his notation, keen to solve it again using the 'same' strategy of 'rounding'. The solution was re-recorded in the corner of the 'Thinking Bubble' so his original thinking was still visible for reflection.



4.5 Subtraction Problems

4.5.1 Recognizing 'same' mathematical solutions

To ascertain the range of strategies used for subtraction problems by the children in the class the teacher scribed the following number equation on the whiteboard:

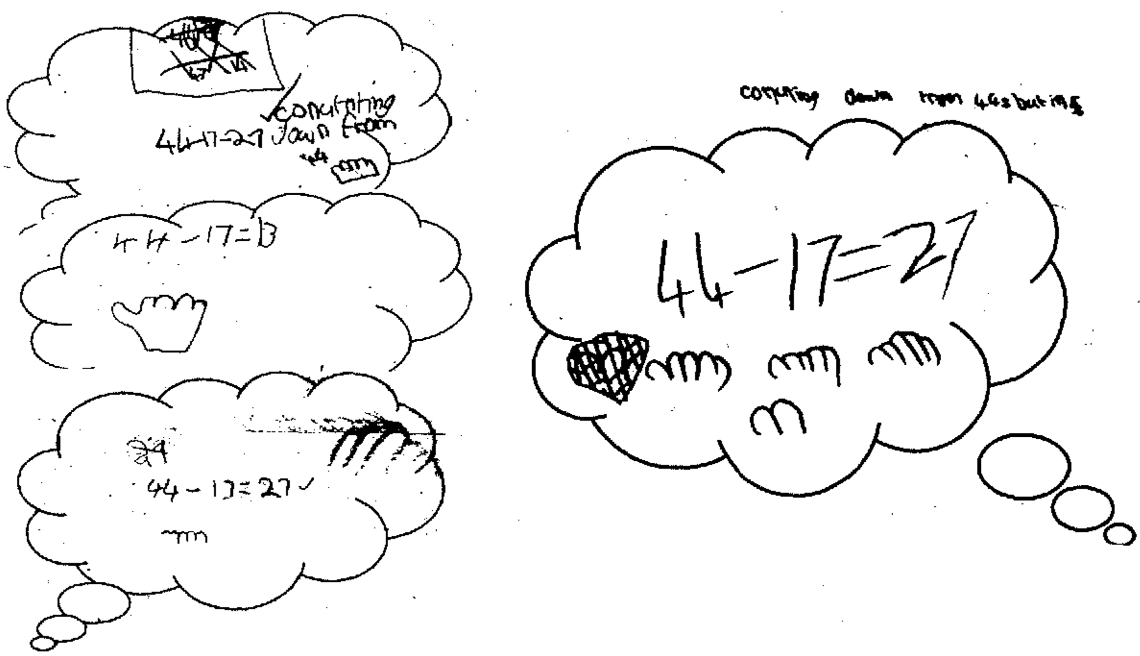
Problem:	*44 – 17* = □

Ms. Vine reinforced: *I want you to do it the way you feel comfortable with. I want to see what you can tell us about solving subtraction problems.* Some children jotted down numbers in their 'Thinking Bubble' during 'think time' while others recorded after they had mentally solved the problem. The children moved around the class forming groups as they found others who had solved it in a similar way. Ms. Vine assisted three individuals who were unsure which group they should join.

Each group was given a large 'Thinking Bubble' and a vivid pen to notate the group's solution. After discussing how best to notate their solution seven of the nine groups selected one child's written recording which was copied on to the big 'Thinking

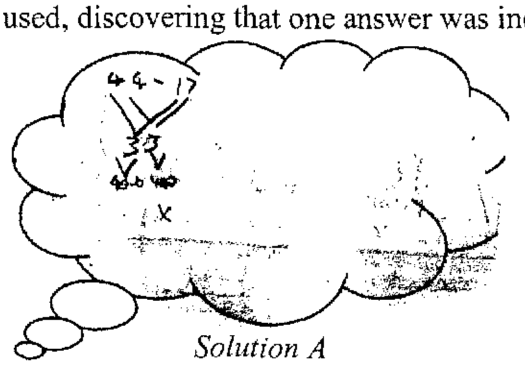
Bubble'. Meanwhile the other two groups adapted individual 'Thinking Bubbles', combining ideas, to represent the group's thinking.

The teacher pointed to the wall display: *Above your notation you are to give your strategy a label... like the ones on the wall for 'addition'.* The children worked on this by verbalizing their solutions; discussing which notation was most suitable; and deciding how best to describe the strategy for the label. A representative from each group adhered their big 'Thinking Bubble' (with the individual 'Thinking Bubbles' attached) to the whiteboard and explained how they had completed this task.

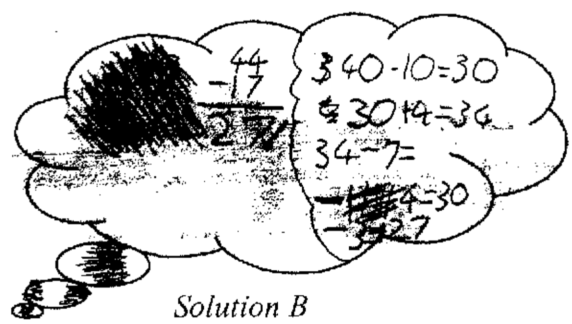


Individual 'Thinking Bubbles' and the group's notated solution

Much discussion was generated as children compared and contrasted solutions within the group and between groups. They frequently pointed to the notation to highlight differences, as well as similarities. For example, the following two solutions had been grouped together which provided an opportunity for children to discuss the methods used, discovering that one answer was incorrect.

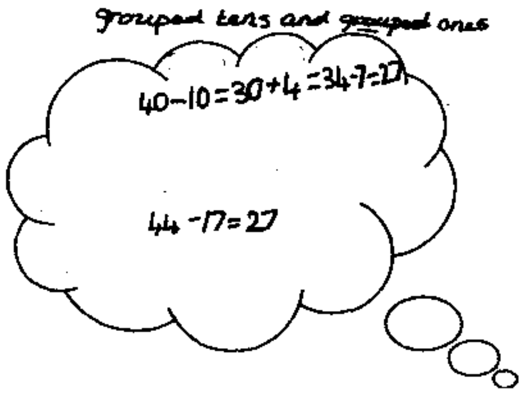


Solution A



Solution B

When some children recognized that Solution B was similar to another group's the class decided to place the two solutions together under the label 'Grouped tens and grouped ones'.



Grouped tens and grouped ones

Children gained confidence through sharing their different solutions and allowed misconceptions to be discussed in whole class situations. For example, when one group selected Jack's notation to represent their thinking (*40 – 10 = 30 – 11 = 29)* the question was asked about how they got '11'. Jack immediately came up to the front of the class, and without being asked, gave his explanation. Listening to his reasoning Maggie commented: *It's kind of the same as 'Grouped tens and group ones'.* However it was pointed out no one else had added the '7' and '4'.

Overall, the subtraction problem was recorded using a variety of different notational schemes. It was observed that:

- three children drew pictures;
- two used tally marks;
- five attempted to use a 'vertical algorithm'- four of which were incorrectly solved using a 'vertical layout', while one used a 'horizontal' layout which had visible signs of 'carry-overs' using the standard procedure;
- there was evidence of five children using a notational scheme similar to the teacher's form of 'pull-downs' for addition, however it was noted that they had difficulty in obtaining a correct answer; and
- the other children used a 'horizontal layout' to track their thinking solving the problem from 'left to right'.

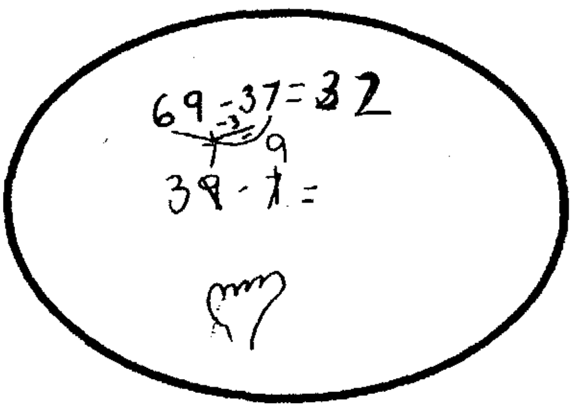
4.5.2 Comparing and contrasting mathematical solutions

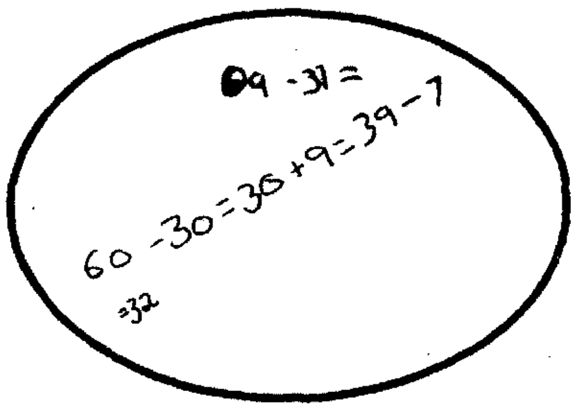
In another class activity the children explored the following problem:

Problem:	*69 – 37 =* □

Gervasoni's (1999) 'Think Board' was adapted to devise group 'Think Mats'. After recording their thinking on a 'Mat Spot' each child verbally explained to peers in their Maths Group how they had solved the problem. Such discussion enabled children to clarify their thoughts and share ideas. The children were encouraged to identify strategies that were the 'same' and 'different'.

Maggie explained her solution to the problem: *Take away 60 equals 30... and then I did 7 takeaway 9 to get 27* [refers to her notation as she explains but does not show it others]. She thought that her solution was 'different' from the rest of the group. Sue pointed out in her solution she had taken: *the 60 off the 69.*

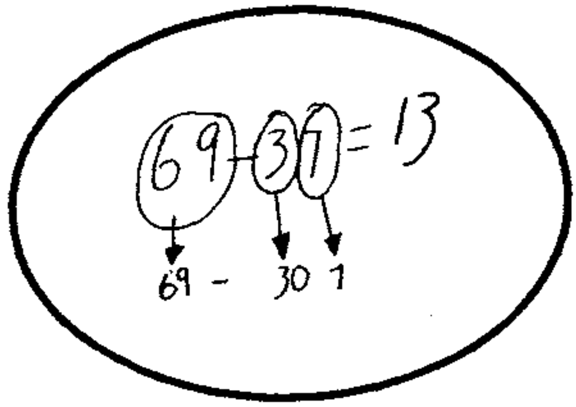




Maggie's notated solution *Sue's notated solution*

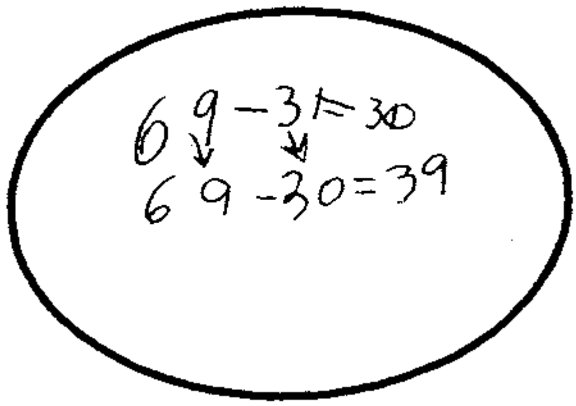
The two students compared their notated solutions to see if they **had** used the 'same' method. Maggie clarified her explanation by re-verbalizing her thinking: *I did 60 takeaway 30 is 30. Then I did 9 takeaway 7 is 2. I put the 3 and the 2* [to get an answer of 32]. Sue commented: *it's sort of the same ... oh it **is** the same strategy.*

Jess, using a different partitioning method, relied on her notated solution to explain her method: *What I done was – got 69 and put an arrow down and just put 69. I put an arrow down below the first ten which was 30. So I took away 30 from 69 and that left 21, I think. Then I took away 7 and that gave me 13.*



Jess' notated solution

Most group members quickly saw that Simon's solution was similar as he explained: *I bring the 69 down then I bring the 3 down. That makes 39 then I just took away the 1, which is 30.* [He has mistakenly copied the equation as '69 – 31'.]



Simon's notated solution

The children recognized it as: *the same strategy Ms. Vine taught us yesterday.* However Maggie did not agree: *cos Jess doesn't have the same answer as Simon.* Sue commented: *but the strategy is the same (even though) the answers are different.*

The children classified the solution methods using the group's 'Think Mat' (see Figure 4.5) as they discussed and compared which ones were the 'same' and 'different'. Each Maths Group presented their 'Think Mat' to the class to explain why they had grouped them in this way. All Maths Groups correctly identified and classified strategies as being the 'same' or 'different'.

Figure 4.5 Think Mats

The two 'most able' mathematics groups took the initiative to question their peers if they were unsure of any aspect to clarify misunderstandings. They also used the wall display to compare solutions, and assist in naming their strategies. The 'middle' group fought with each other after they had written a title on their 'Think Mat' and thereafter little discussion occurred. The 'less able' mathematics group worked under the guidance of the teacher to establish similarities and differences.

4.6 Reasoning

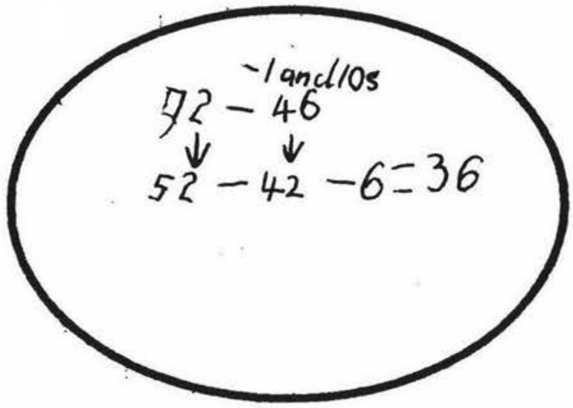
4.6.1 Is there a 'better' strategy?

Each Maths Group was given a "How many are left" word problem adapted to their respective levels. In the case study group one student read the problem aloud then they decided what they were being asked to find:

Problem:	*You have 57 people on the bus and 46 people get off at different stops along the way. How many are left on the bus at the end?*

Jess suggested: *So it's more like 57 equal... 57 takeaway 46. How much is left at the end.* When each child had recorded their thinking on a 'Mat Spot', the strategies were discussed and a label to describe their methods was attached. A number of children, including Jess, went over to the wall display to compare their solution in order to find out which strategy had been used.

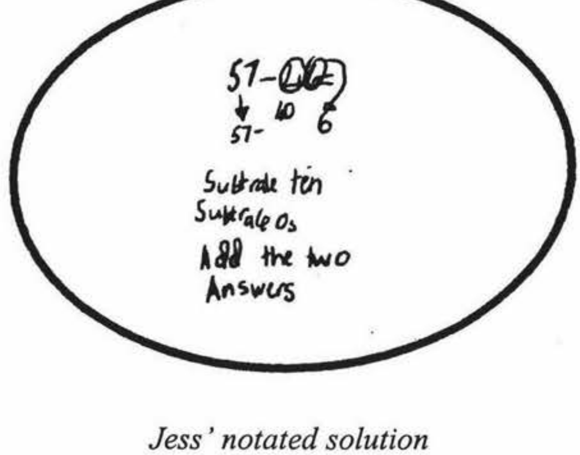
The case study students confidently shared their solutions, using their notation to point out what they had done. Simon immediately started the discussion with a description of his method.



Simon's notated solution

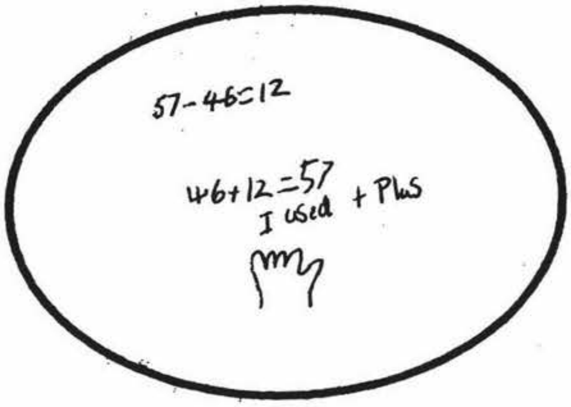
Maggie realized that she had solved it using a 'different' operation, and questioned: *Oh? ...it is 'plus'.* Rob counteracted this explaining: *it's takeaway* then described his method. Maggie was confused as to whether it was 'takeaway' or 'plus' but listened to other group members. She verbally 'echoed' Sue's strategy to herself and reflected on her notation as she did so. When it was her turn she explained her solution and came up with an answer of 'zero'.

Jess shared her solution: *I took 40 away from 57 which makes... I don't know... I put 6 and that makes 11... Oh no...*Sue stepped in to assist her by confirming that Jess had solved it correctly: *You put the 'ones' together.* Jess now recognized what she had done and emphasized: *So my answer* ***is*** *11.* Rob inquired: *What's that way?* Jess replied, using her interpretation of the strategy label on the wall display: *It's "Subtraction 10, subtraction zeros* [referring to the 'ones'], *add the two answers".*



Jess' notated solution

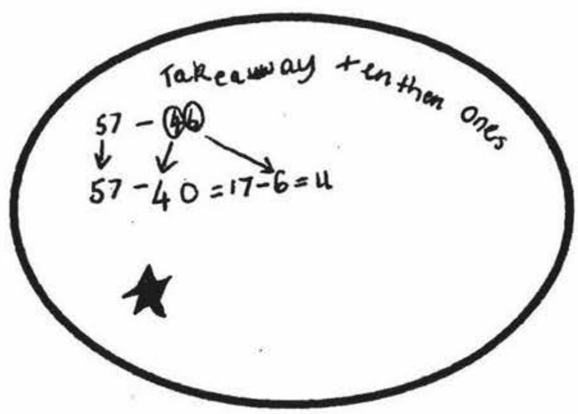
When Jack explained that he had used: *'plus' and 'equals'*, there was disagreement in the group that both operations could be used to find an answer for the problem. However Jack reinforced: *I **used** 'plus' to get the answer... It's 'plus'. See look* [uses his notation to assist his explanation].



Jack's notated solution

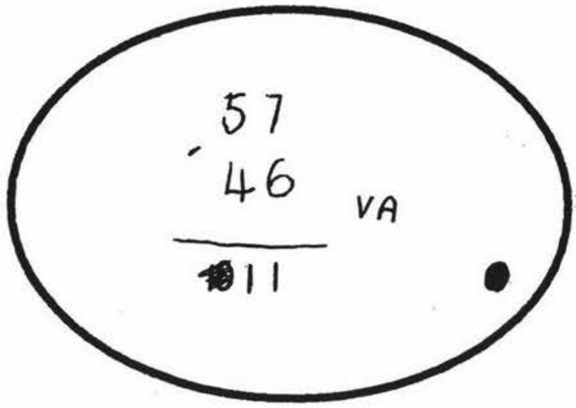
After listening to other group members' solutions Maggie made some changes to her notation and commented: *It's one. No... it's 11.* Jess pointed out that Maggie had not originally got that answer but she accounted for this by replying: *I did it again.*

Meanwhile Rob compared and contrasted his solution to Jess': *Yeh! You got the **same** one as me. It's 'tens and ones'.*

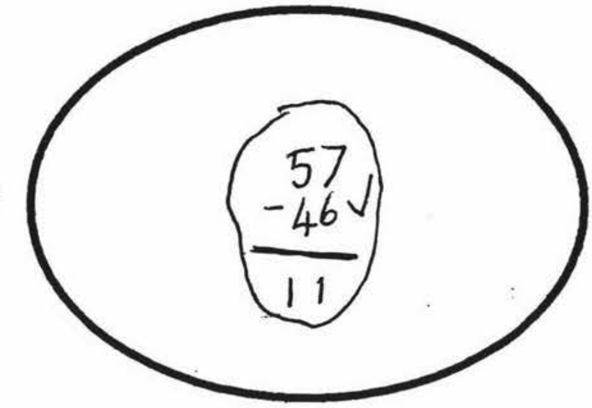


Rob's notated solution

Jack questioned them: *How do you do that one?* [referring to this strategy]. Jess, Rob, and Sue clarified the steps employed to solve the problem. Jack then attempted to use a standard vertical algorithm and began subtracting the 'ones'. Maggie interrupted and suggested: *Do it this way first... So that's ten* [notates the first step using the *1010* method]. *There you are* [shows Jack]. There was further discussion between these two as they questioned and recorded each step.



Maggie's second notated solution



Jack's second notated solution

In order to promote mathematical thinking about numbers and number relationships the Maths Groups were asked to decide which strategy they felt was 'better' for solving the problem. Maggie had recorded two solutions and told the group that: *the top one's not in it on my Mat Spot*; and crossed out her first notated solution. Jack also had two solutions and asked for the group's advice: *Which one do you reckon I should cross off – that one there or that one there?* Sue replied: *Whichever one you like.* Whereas Simon suggested: *I think that one* [pointing to the vertical algorithm]. The group 'voted' to decide which was the 'better' strategy. There was no discussion or reasons given as to why they thought that certain solutions were better than others. They felt that two of the strategies were good and indicated this by placing a 'dot' on Maggie's second solution and a 'star' on Rob's one.

Each Maths Group selected a representative to share the group's conclusions with the rest of the class using their 'Think Mat' to support claims made. Only one group agreed unanimously on one choice, while other groups were split in their decision but selected the preferred solution (except for the case study students who chose both). Terms, such as 'quick', and 'easier' were used to justify choices made. The case study students' representative, Simon, pointed out that one method (vertical algorithm) was: *quicker to do* while the other *(N10)* was: *easy to learn.* Rob supported the selection of the *N10* solution stating: *It's the way you (Ms. Vine) taught us.* Another group's spokesperson noted: *It isn't hard to follow the answer – when you 'read' it.* Children listened attentively throughout the sharing session; and some, like Jack, quietly moved to positions where they could clearly see the 'Think Mats' when solutions were discussed. At the end of the session Jack perceptively pointed out: *people choose their one (solution) cos it's easier for them but it may not be for others.*

4.6.2 Comparison problems

It had been observed earlier that some children had difficulty with solving and notating comparison problems. For example, Jack in an Independent Card activity struggled with the notation of the following problem:

Problem:	*Jason has 17 music cassettes. Justin has 8. How many more cassettes than Justin does Jason have?*

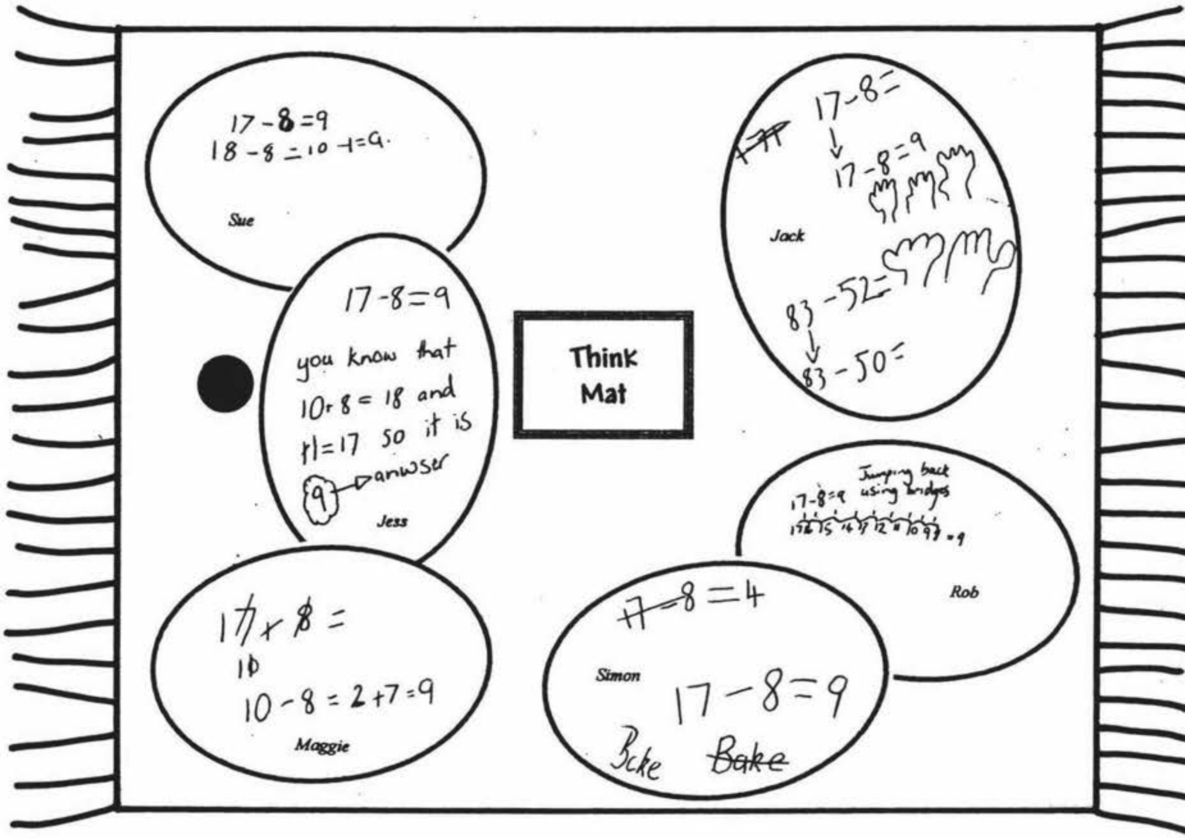
After thinking about the problem he recorded the number equation as '17 + 8' but realized that this was incorrect as he would get an answer of '25'. Jack knew that the difference was '9' by 'counting on' from '8' so he crossed out the number equation and just wrote the answer as he did not know how to write this as a number equation.

It was decided to explore this type of problem further within Maths Groups using the 'Think Mat'. Each Maths Group was given a comparison word problem (see Figure 4.6) that was at an appropriate level, by the teacher.

Case Study:	*Mary has 17 apples. Jake has 8 apples. How many more apples does Mary have?*
Group A:	*Frank has 321 lollies. Mary has 76 lollies. How many more lollies does Frank have?*
Group B:	*Jo has 374 marbles and John has 234 marbles. How many more marbles does Jo have?*
Group C:	*Zak has 29 cars. Donny has 16 cars. How many more cars does Zak have?*
Group D:	*Sean has 17 muffins. Bob has 6 muffins. How many more muffins does Sean have?*

Figure 4.6 Group word problems

After considering how they might solve the problem and record their thinking, each group discussed their solutions. Initially some case study students thought that it was 'plus' problem because, as Jack pointed out, it asked: *how many more...**more**!* He suggested that the question should be written as: *How many 'less than' ... it sounds like a subtraction then.*



Case study students' Think Mat

Children reflected on their written recording to assist with describing their solution to others. For example, Jess explained: *I just knew that 10 plus 8 is 18. So that means...* [checks her notated ideas]*...and takeaway 1 is 17 so the answer is 9.*

The group worked co-operatively assisting peers as they reflected on solutions.

Simon: *In my brain I went 17 plus 8...*

Jess: *It's not 'plus' it's 'takeaway'.*

Maggie: *Do you want to change it?*

Simon: *Yes* [begins working on it].

The children patiently waited for Simon to solve the problem and as they did so discussed alternative strategies. Maggie explained: *I went 17...um... 10 takeaway 8 equals 2, and... takeaway 7 equals 9.* Sue expressed her solution as: *I know that 18 takeaway 8 is 10, takeaway 1 is 9.* After these two students conferred Maggie started to make changes to her notated solution. Rob, who had been carefully listening to their verbal explanations, turned to Maggie stating: *it's supposed to be 'plus 7' so it equals 9.* Sue supported this, and pointed to Maggie's notated solution to highlight her error. Maggie looked puzzled and Rob said: *We're talking about the 2 takeaway 7.* Other group members joined in the discussion:

Jess: *You can't take 7 away from 2...*

Jack: *Put a 'plus' there* [indicates to use the conventional operational sign to overcome the confusion].

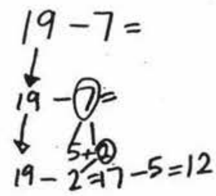
Maggie: *Oh, I thought you meant plus that one there* [points to another number].

Rob: [Leans forward across the group to assist Maggie]. *So it equals **nine**. Cos 2 plus 7 equals 9!*

In unfamiliar circumstances, Rob's own solution showed that he had reverted to using his reliable method of 'bridges': *I went down by 'ones' in bridges. You take the highest number which was 17 and then you go down by 'ones' until you get to 8. Then you add up all the numbers along the top...which equals 9.* The children recognized that this was similar to Simon's amended method of 'counting back' using fingers whereas Rob had employed a notated model.

Jack attempted to use the *N10* strategy demonstrated by the teacher rather than use his own way: *I was going to solve it the way we were taught – the new way. But it didn't work ...because the last number wasn't a double digit so I used my 'fingers'.* Maggie claimed that she had used: *the strategy that Ms. Vine taught us.* Jack queried this: *How - when there's no double digits?* He began to notate another example to demonstrate to the group what he meant. Meantime Maggie, while comparing notated solutions, realized she had used the same strategy as Sue who confirmed this: *we got the same sort of numbers.*

Jack followed up his query about the *N10* strategy during the group's Teaching Time. The teacher redescribed and notated alternative ways of 'splitting numbers' as suggested by the case study students in order to find an efficient way for solving the problem:

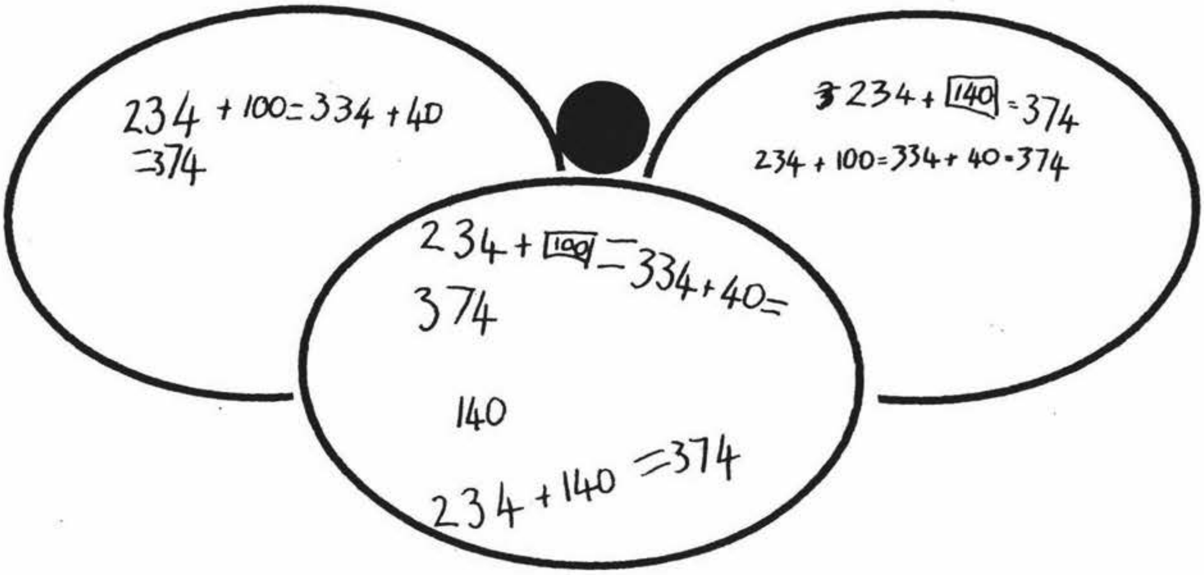


4.6.3 Justification

The teacher established the need for children to think about what counts as an 'efficient' strategy by asking them to justify their choice(s). She explained to the class: *You can't just say "Mine's the best". You need to justify that. That means you need to explain why you think it's the best way. If you think that yours is the most 'efficient' way you must have a good reason.* She provided an example of one Maths Group whom she had observed discussing different solutions which they had then compared to those on the wall display. This enabled the children to provide reasons as to which strategy they considered 'best'.

Rob wanted to 'vote' but the other the case study students disagreed saying that they needed to discuss '*whose was the bestest*'. Jack said he thought: *Sue and Maggie's was the most effective way because they are using a nice easy way. They are using the way Ms. Vine taught us.* Rob counteracted: *That doesn't mean it's the most* ***effective*** *way.* When asked which way he thought was best by group members he chose his own but he was unable to provide a reason. Jess acknowledged that she had chosen her own way because: *it's easy to me.* A discussion arose as to whether it would be easy for others to use. Jess emphasized that people: *know 'pluses' ... 'plus' is easier.* As they debated the issue they came to recognize that Jess' strategy was similar to the other two girls'. After further discussion the group selected Jess, Sue and Maggie's solution (who had used their number knowledge of 'tens') as being the most efficient way, justifying: *it's short...looks easier... and it 'might' be quite easy for other people to do.*

The class gathered to share each group's selection of the most efficient strategy for solving the word problem. One group chose 'using equipment', that is unifix blocks: *to take 6 off, and count the rest.* They considered it was the best method as it was: *the* *first way they had worked it out*. Group B (comparing 374 and 234) used their 'Think Mat' to demonstrate how they had classified different strategies then selected the 'checking' method of addition (that is, *reversibility*) because they found it was easier than subtraction. They discovered that by beginning with 'hundreds' then adding on 'groups of ten' they were able to make estimations to get closer to the answer. Although three group members had solved it this way each had notated the strategy differently:



Another group selected 'counting on' with 'fingers' as it was an 'easy way', recognizing it was a 'sort of addition' problem. The last group commented that: *every strategy (in the group) is a good one so we decided to place a 'dot' in the middle of our 'Think Mat'*. Their representative explained the different strategies that had been used.

4.7 What counts as a 'sophisticated' mathematical solution?

As efficient ways of solving problems became highlighted during discussions, 'grouping-based' solutions were seen not only as different from 'counting-based' solutions but also as more 'sophisticated' (McClain & Cobb, 2001).

A number of children began to make a distinction between 'counting' and 'grouping'.

Ms. Vine: *Which strategy have you used?*

Jess: *Counting on with my fingers, and I used tally marks.*

Ms. Vine: *Which way is easier?*

Jess: *Tally marks because they are in 'fives'.*

Notational schemes supported the development of grouping, rather than counting solutions.

Ms. Vine: *Now you used to be the 'Bridge King'* [speaking to Rob]. *You used to do 'bridges' all the time but now you're not. Why?*

Rob: *'Pull-downs'* [referring to the *1010* strategy] *are faster than 'bridges'* [referring to 'counting on'].

Written recording of children's thinking provided opportunities for them to differentiate between solution types, and to monitor each other's contributions.

4.8 Summary

Notational schemes came to serve as thinking devices as children participated in individual, group and class activities. They began to use the written records in their own numeracy book, the class Modelling Book, and on the wall display as a means of reflecting on and comparing their own and other children's mathematical activity. By symbolizing solutions other children were given the opportunity to clarify for themselves how these compared to their own and to each other. This necessitated the need for children to discuss and listen to other's solutions. Thus the children were empowered to judge for themselves whether their mathematical solution was the 'same' or 'different'. As a result, the notational schemes came to provide a means of highlighting key aspects of different solution strategies.

The teacher played a central role in initiating the development of notational schemes within the classroom. By redescribing and notating explanations she implicitly communicated that children's contributions were valued (Whitenack et al., 2000). This created a learning environment where children felt comfortable to offer insights, which made sense to them, during group and whole-class discussions. The use of notation contributed to these discussions allowing individual children's ideas to become explicit topics of conversation which could be compared and contrasted (McClain & Cobb, 1999). These forms of recording enabled reflection on and analysis of the children's prior mathematical activity. Consequently written recording provided opportunities for shifts in discourse, as well as thinking, to occur.

Examination of written records revealed diversity in children's notational schemes. As they participated in discussions the teacher guided the transition from informal non-standard notation to conventional, yet personally-meaningful, recording of solution processes. In doing so, mathematical thinking was promoted as children were expected to reason in ways that led to more sophisticated strategies.

CHAPTER 5

CASE STUDIES

5.1 Introduction

From the seventeen valid consent forms returned six children were selected as case studies (see Chapter 3.3 for selection process). Jess was a ten-year-old Year 6 student. Sue, Jack, Rob, Maggie, and Simon were all nine-year-old Year 5 students. The students, except for Jack, had had two years within the *Numeracy Project*. Jack was new to the school and had not previously been involved with this numeracy programme.

Initial and final interviews (see Appendices A and B) were used to establish what strategies the students used to solve computation problems; how they recorded their processes of thinking; and how notation affected their skills of reasoning. Student work samples were collected daily, over four weeks, in order to document the development of notational schemes and the ways in which notation tracked children's thinking.

In this chapter the researcher identifies the ways that notational schemes reflect a shift in children's reasoning. Similar aged children perform identical tasks differently and can explain and justify their reasoning with differing success due to social, cultural, intellectual, and pedagogical reasons (Anthony & Walshaw, 2002). The case study students' responses were located at varying developmental levels.

5.2 Jess

5.2.1 Summary of strategies and justifications

According to information from the previous year teacher Jess was working at Stage 4, Advanced Counting of 'The Number Framework' (see Appendix D).

During the initial interview the strategy she mainly used was 'counting on and back'. Jess firstly used mathematical apparatus but then resorted to using her fingers to solve some of the problems.

Problem:	*I have 6 counters under here, and I'm putting some more counters under here* [screen the counters]. *Altogether there are 14 counters now. How many are under here* [circling above unknown collection]*?*

Jess: *I got 6 (beaNZ[5]) in one pile and then from 6 I added some more to add up to 14. Then I counted them up to 8.*

There was evidence of part-whole thinking when Jess used her knowledge of 'splitting' numbers to 'make ten'.

Problem:	$65 + 8 = \square$

Jess: *If you add 65 and add 5 would equal 70. Add 3, the rest of 8, that would equal 73.*

She also attempted to group 'tens' and 'ones'.

Problem:	$86 - 32 = \square$

Jess: *8 takeaway 3 would be 5, 50 takeaway 2 would be 3 – that's 30. Takeaway 6 would be...*[counts back from 30] *... 24. It's 24 – I used my fingers.*

The comparison problem caused confusion.

Problem:	*At the Inter-school Athletics Competition Puru School won 72 medals and Tahi School won 25 medals. How many more medals did Puru win?*

Jess: *I don't get it. Is it like 72 plus 25?*
She proceeded to solve it as an addition computation.

When asked why she had used these methods her responses included: *It just popped up into my head; that's the first one I thought of;* and *I don't know.*

[5] BeaNZ is a commercial mathematical product consisting of bean-shaped counters.

Jess was able to suggest another strategy for solving the problems: *There's always another way.* Her most usual response was to reflect on her first notated solution and change the order of numbers in the equation to an alternative solution. However, her lack of confidence was demonstrated when she hesitated and inquired of the Interviewer: *Is that right?*

When asked which strategy was 'better' Jess usually selected her first method as the preferred method. Words used to justify her choice included: *easier*; *quicker*; and *not as complicated.* Twice she mentioned *you start with the highest number* which had been *taught by a teacher.*

During the final interview Jess employed similar strategies to those used to solve problems in the initial interview. However her solution to the following problem indicated that Jess was beginning to use more sophisticated thinking, in particular, applying the *1010* method.

Problem:	*At the school camp there are 58 boys and 35 girls. How many children are there altogether?*

Jess: *Well, it's 58 plus 35 equals 93. What I done was add the 30 and 50 together which made 80 then 8 and 5 made 13 and... 80 and 13... 80 plus 13 equals 93.*

Jess continued to find subtraction difficult.

Problem:	*86 – 32 =*□

Jess: *I took 3 away from the 80 which equaled 50. Take away 2 which equaled 48. Take away 6 is 42.*

When presented with the comparison problem Jess recognized she could solve it by 'counting all' using materials.

Problem:	*At the Inter-school Athletics Competition Puru School won 41 medals and Tahi School won 33 medals. How many more medals did Puru win?*

Jess: [Draws 41 iconic strokes.] *There is 33* [points to notation] *and I just counted on... 2, 3, 4, 5, 6, 7, 8. That's 8.*

When questioned about why she had used these strategies she still reasoned *it was the first one that came to my head.* Her frequent notation of alternative solutions indicated her awareness of other possible solution strategies. However alternative solutions were often created by 'splitting numbers' rather than offering an alternative strategy.

Problem:	*You have 47 lollies and you eat 9 of them. How many have you got left?*

Jess: *47 takeaway 7 equals 40, takeaway 2 equals 38* [Reads her notation]. *And my other answer is 47 takeaway 6 equals 1... ohhh... 47 takeaway 6, takeaway 1 equals 40, takeaway 2 equals 38.*

Jess did not necessarily select her first solution as the preferred strategy. She regularly referred to the efficiency of the solution, justifying: *It's easier – I know my 'tens' and 'ones'. I know how to add them.*

5.2.2 Notation

At the initial interview Jess recorded her thinking as a written recount of the actions taken.

Problem: *I have 6 counters under here, and I'm putting some more counters under here* [screen the counters]. *Altogether there are 14 counters now. How many are under here* [circling above unknown collection]*?*	1. you siad there is six under one pice of paper and more under one ourther how mach dose it al add up to. and I pulld out sit beans mole in outher paper and addid up to 14

This approach was adapted as she translated the recount into a number equation which included invented symbols to indicate 'splitting numbers'.

Problem: *You have 47 lollies and you eat 9 of them. How many have you got left?*	First solution: 2. ther is 47 lollies and c eat 9 of them How meny have you got left. 47 − 9, 47 − 7 − 2 38,

Jess frequently used a 'spew' horizontal layout (Thompson, 1994) whereby the information is written in one continuous stream.

Problem: *65 + 8 =* □	First solution: 65+5=70+3=73

When problems contained 'large' numbers Jess made informal jottings to keep track of her thinking.

Problem: *There were 298 sheep in the stockyard, and the farmer brought in another 143 sheep. How many sheep are there altogether?*	300 430 441 300+130=430+11=441

During the series of classroom lessons Jess initially recorded her thinking of addition problems as a number equation alongside words and pictures to indicate how she solved them. Mental steps were verbalized as she shared her solutions with class members.

Problem: *74 + 8 =* □	74+8=83 I uses my head and fingers

Recording of subtraction problems indicated the range of strategies used by Jess. For example, pictures of 'fingers' for 'counting back'; tally marks for 'skip counting'; as well as explanations consisting of words and numbers.

Problem: *45 – 20 =* □	45-20=25 I know that 20 and 20 = 40 then that means that 40-40=20 so 45-20=25

When Jess was introduced to the *1010* strategy for addition problems she adapted the teacher's 'pull-down' notational scheme.

Problem: *39 + 20 =* □	39+20=59 50 59

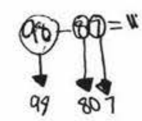
However it was observed that this notational form was used only when working with the teacher during group sessions. While she attempted to use 'pull-down' notation for one particular class activity she became confused and resorted to 'counting back' with fingers.

Problem: *36 + 17* = □	

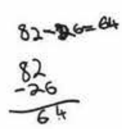
When computations involved large numbers she jotted figures down and crossed them out as she worked.

Problem: *943 + 72* = □	

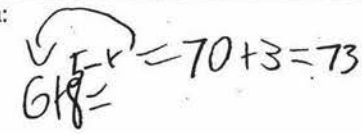
Following the introduction of the *N10* strategy for subtraction problems Jess again adapted the teacher's notational form. Operational signs were often missing in notation with these steps carried out mentally.

Problem: *98 - 87* = □	

Some subtraction problems were set out in either a 'vertical' format or 'spew' horizontal layout, and incorrectly solved using the *1010* strategy.

Problem: *82 - 26* = □	

At the final interview invented symbols were used to clarify verbal explanations.

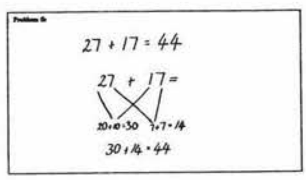
Problem: *65 + 8* = □	First solution: 

Jess: *What I would do is take the 5 away from the 8, add it on to the 65.* [Draws an arrow to show direct action of subtracting and adding 5.] *Which makes 70 and then add 3 more. That makes 73.*

There was no indication of her using the teacher's 'pull-down' notational scheme; rather, she used a 'spew' horizontal layout. An exception was the comparison problem where iconic symbols were drawn with the corresponding number equation recording her direct action.

Problem: *At the Inter-school Athletics Competition Puru School won 41 medals and Tahi School won 33 medals. How many more medals did Puru win?*	

Jess was able to identify whether strategies were the 'same' or 'different' by comparing and contrasting notated solutions.

Problem: *Hayley had 27 marbles and she won another 17 marbles. How many marbles does she have now?*	Jess' notated solution: 27–7=20+10=30+7=37+7=44	Interviewer's card: 

Jess: [Looks at Interviewer's Card and ponders.] *I don't know how to do this. Did they take the 7 away?* [Compares it to her method.] *Oh yes, it is the same… it hasn't got a 7 there. 20 plus 10 is 30… I done 27 takeaway… but that's still the same… plus 10 equals 30, plus 7 equals 37 plus 7 equals 44.*

5.3 Sue

5.3.1 Summary of strategies and justifications

Teacher information from the previous year placed Sue as 'transitional' between Stage 5, Early Additive Part-Whole, and Stage 6, Advanced Additive Part-Whole.

During the initial interview Sue mainly focused on 'making 10' and 'grouping tens' when solving problems.

Problem:	$65 + 8 = \square$

Sue: *You take the 5 off the 8 and stick it on the 65 plus ...3 more equals 73.*

The *1010* strategy was used for solving two-digit addition problems. The 'tens' would be calculated first, with the 'ones' being calculated in a variety of ways, depending on the numbers in the problem.

Problem:	*At the school camp there are 58 boys and 35 girls. How many children are there altogether?*

Sue: *50 plus 30 equals 80. You take the 5 out of the 8 and stick it onto the other 5 equals 90, and there will be 3 left. And you stick that on to the 90, and that will equal 93.*

Problem:	*Hayley had 79 stickers in her book, and she was given 22 more. How many stickers does she have now?*

Whereas for this problem she explained: *70 plus 20 equals 90 plus 9 equals 99 plus 2 more equals 101.*

The *1010* strategy was incorrectly applied for solving two-digit subtraction problems.

Problem:	*86 – 32 =*□

Sue: *If you take 30 off the 8* [80], *that's 50. Take away another 8* [adding the 'ones' together] *will be 51.*

The word 'more' in the comparison problem indicated to Sue that the numbers were to be added together.

Problem:	*At the Inter-school Athletics Competition Puru School won 72 medals and Tahi School won 25 medals. How many more medals did Puru win?*

Sue: *I stuck the 2* [20] *on the 7* [70] *which is 90... plus 7 is 97.*

When asked why she had used these methods to solve the problems Sue responded: *It was the only way I could work it out.*

Sue reflected on the notation of her first solution for each problem to assist with providing an alternative solution. Number patterns were generated by recombining the 'ones' in different ways.

Problem:	*You have 47 lollies and you eat 9 of them. How many have you got left?*

For the first solution Sue explained: *I know that 47 takeaway 7 is 40, and then I took away 2 more. And that equals 38.* She then described an alternative solution: *By getting 47 and taking off 6, and that's 41. Take...* [pause as she works out 6 + □ = 9] *3 more and that would equal 38.*

Sue considered that her first solutions were 'better', justifying: *they look easier and sound easier.* When asked how it was 'easier' she responded: *Cos it's shorter* [referring to the physical length of the equation]; and *won't take as long to solve.*

During the final interview Sue employed similar strategies to those used in the initial interview with a prominent use of the *1010* strategy. She now recognized how to subtract the 'ones' in a two-digit problem.

Problem:	*86 – 32 =*□

Sue: *50 plus 6... takeaway 6... oh no - plus 6 equals 56 takeaway 2 equals 54.*

But she still exhibited having a problem with the 'smaller-from-larger bug' (Baroody, 1987, cited in Blote et al., 2000, p. 223).

Problem:	*At the Inter-school Athletics Competition Puru School won 41 medals and Tahi School won 33 medals. How many more medals did Puru win?*

Sue: *Take 30 off 40 equals 10... plus 1 off the 3 which is 2... 10 plus 2 is 12.*

Initially Sue attempted to solve the above comparison problem by adding the numbers but changed her mind after placing 'herself' in the context: *That means it must be 'takeaway'... because, like, say, I had 43 medals and you had 21 medals... how many more would **I** have?*

When questioned about why she had used these strategies for the problems Sue reasoned: *It's the only way I could think of* making reference to the fact that *most people know that way. It wouldn't take that long – it doesn't have much numbers to remember.*

Contrary to the initial interview Sue's range of alternative solutions for the problems were not based on 'pattern making'. She used the *N10* strategy for some subtraction problems; as well as attempting to use 'vertical algorithms' incorporating the *1010* strategy.

Problem:	*Hayley had 27 marbles and she won another 17 marbles. How many marbles does she have now?*

Sue: [Records the equation in a vertical format. Works from left to right, adding the 'tens'.] *I can't do it with two 7s, with the doubles on top. I can't do the doubles like that. Because it* **seems** *different. I can do it like 20 plus 20... Oh, I've got it! ... I think... Can I take ten off the 7 and 7? You know 7 plus 7 equals 14. Can I take the ten off the 14 and stick it on there?*

For the majority of problems Sue decided that the second solution was 'better'. A key aspect of her choice was based on 'efficiency' as she compared and contrasted methods, justifying: *That's just wasting half of your time* [pointing to one solution]. *You got to take one off that and add on to that.* She circled the 'pull-down' notational form exclaiming: *I really like this one;* and then used counterexamples to highlight how she found some solutions 'confusing'.

5.3.2 Notation

At the initial interview Sue recorded her thinking using a 'spew' horizontal layout working from left to right. She verbalized her ideas as she wrote, with notation often mirroring the explanation.

Problem:	First solution:
You have 47 lollies and you eat 9 of them. How many have you got left?	47-7=40 ∓ 2=38.

Sue: *I know that 47 takeaway 7 is 40, and then I took away 2 more. And that equals 38.*

Even after recording her verbal description she was unable to recognize some solutions did not make sense.

Problem: *86 – 32 =□*	First solution: 80 – 30 – 8 = 51

Sue focused on creating number patterns to provide alternative solutions.

Problem: *65 + 8 = □*	First solution: 65+5=70 +3 = 73	Alternative solution: 65+7=72+1 = 73

Sue's inconsistent attendance at school inevitably led to missing some teaching sequences. During the series of classroom lessons she initially wrote the number equation followed by recording her thinking with a 'spew' horizontal layout.

Problem: *24 + 26 =□*	24 + 26 = 50 20+20 =40+ 6+4 = 10+40 50

Notation demonstrated that she did not have an understanding of the commutative law.

Problem: *□ – 30 = 20*	$\boxed{10}$ – 30 = 20 30 – 10 = 20

Following a group discussion on ways to record thinking Sue used words to describe her strategy rather than symbolic notation.

Problem: *74 + 8 = □*	74+8= 82 78+4= I use my fingers To break 82.

Sue's absence when the teacher introduced the *1010* strategy for addition problems using 'pull-down' notation probably contributed to her continued use of a horizontal notation format.

An examination of Sue's notation clearly showed that she had difficulty with subtraction, making a consistent error of adding the 'ones'.

Problem: *82 - 26 =* □	82-26=58 80-20=60 add 6 and 2= 58.

After another week's absence Sue was introduced to the *N10* strategy during a group teaching session. She perceived this to be similar to 'making groups of 10' and notated the subtraction problem using the *1010* strategy.

Problem: *98 - 87 =* □	98-87=11 10

At the final interview Sue's notational scheme illustrated inconsistent use of arrows and circles to indicate numbers being 'pulled-down' and split. A combination of various notational forms resulted.

Problem: *65 + 8 =* □	First solution: 65+8=73. (65+5=70+3=73.) 65 +8=73

Problem: *At the Inter-school Athletics Competition Puru School won 41 medals and Tahi School won 33 medals. How many more medals did Puru win?*	First solution: 41+33=74. 41-33=12. 40-10=30+2=12 40-10=30-20=10 13-1=12

Problem: *86 – 32 =*□	First solution: 86-32=54. 30 66-30=50+6=56-2=54

Notation did not always mirror the strategy which had been verbally explained.

Problem: *At the school camp there are 58 boys and 35 girls. How many children are there altogether?*	First solution: 80 50+30=80+5=~~84~~85 85+8=93 13

Sue: *50 plus 3* [referring to '30'] *is 80 and 8 plus 5 is 13. So 80 plus 13 is 93.*

Sue evaluated her work as she reflected on her notation to see if it made sense (Lampert, 2001). When she felt that a solution was incorrect she took time to reconsider the problem, redescribing her thinking.

Problem: *At the school camp there are 58 boys and 35 girls. How many children are there altogether?*	Alternative solution: 80+8=88+5=93

Sue: [Notates her thinking.] *I think I've done it wrong. I done 80... I added 50 and 30 together which is 80* [points to notation]. *Plus 8 is 88, plus 5 is ... 93. I've done it in different numbers and it's shorter in length* [compares it to her first solution]. Sue recognized the alternative solution as being correct and an efficient way to solve the problem.

When Sue focused on notating the solution in a similar way to the teacher the actual process of solving a problem became lost.

Problem: *You have 47 lollies and you eat 9 of them. How many have you got left?*	Alternative solution: 47−9=38 47−9=38

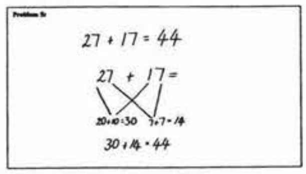
Sue: *I pulled down the 47, the biggest number first, and then I pulled down the lower number second, and then that gave me the answer.* When the Interviewer pointed out that this was exactly the same as the original equation Sue replied: *Yes but it's just 'pulling down'.*

Sue then adapted Ms. Vine's notational form for the *N10* strategy to solve another subtraction problem.

Problem: *86 – 32 =□*	Alternative solution: 86 -30 = 56 - 2 = 54

Sue: *I've just took the 30 off the 32, and sort of brought it down.*

The notational 'form' of a solution affected Sue's perception of whether her strategy was the 'same' or 'different'.

Problem: *Hayley had 27 marbles and she won another 17 marbles. How many marbles does she have now?*	Sue's notated solution: 20 +10=30+7=37+[illegible]=40+[illegible]=[illegible]	Interviewer's card: 

Sue: [Compares and contrasts the solutions.] *They pulled down the 20 and then the 10, then the 7s…the 'tens' and the 'ones'…I didn't do anything similar, the same as that… I just started adding other stuff. I started adding up the 'tens'. Then I started adding up the 'ones'. But they did a different strategy. I didn't do the 'pulling down' way, and they did do the 'pulling down' way.*

5.4 Jack

5.4.1 Summary of strategies and justifications

As Jack was new to the school he was assessed at the beginning of the year by Ms. Vine as being 'transitional' between Stage 4, Advanced Counting and Stage 5, Early Additive Part-Whole.

During the initial interview he used his knowledge of 'ten' to solve problems in a variety of ways.

Problem: *65 + 8 = □*

Jack: *65 plus 10 is 75 takeaway 2 is… would be 73.*

He split numbers to 'make a decade'.

Problem:	*You have 47 lollies and you eat 9 of them. How many have you got left?*

Jack: *Because you know 7 plus 2 is 9, you go down to 40, and takeaway 2 to make 38.*

The *1010* strategy was used for some of the addition problems.

Problem:	*At the school camp there are 58 boys and 35 girls. How many children are there altogether?*

Jack: *Cos you know 5 and 3 add up to 8, add on zeros, and it adds up to 80. Then you add on the 5, and add on 8* [splits the '8' into '5' and '3'].

He referred back to the above solution as he tried unsuccessfully to solve a two-digit subtraction problem.

Problem:	$86 - 32 = \square$

Jack: *It's pretty much the same as this one. Cos this has a 30 in it then I added up to 80... then takeaway 80 then it would be in the 50s. Takeaway 6, 2 is 8* [added the 'ones']... *I think it's 48.*

Jack solved the comparison problem by adding the two numbers together using a partially 'remembered' procedure of the standard algorithm to which he had been introduced the previous year.

Problem:	*At the Inter-school Athletics Competition Puru School won 72 medals and Tahi School won 25 medals. How many more medals did Puru win?*

Jack: [Points to numbers]. *Out of the 72 and the 25. 2 and 2 is 4. 7 and 5 is 12. So the answer is 124.*

When questioned about why he had used these strategies for the problems Jack's most common response was: *I don't know ... it just came into my head.*

Jack was only able to provide alternative solutions for three of the problems, re-phrasing his explanation of the first method. Consequently Jack had difficulty in deciding which solution was 'better'.

During the final interview there was a shift in the way Jack solved problems compared to his initial interview. *Standard place value with compensation* (Ministry of Education, 2002a) was used to solve some problems.

Problem:	*You have 47 lollies and you eat 9 of them. How many have you got left?*

Jack: *Takeaway 10 off 47 and it turns into 37 and then because it's a 9 you plus 1 and equals 38. If you takeaway 10 you got to plus 1 to make it into a 9 so it equals 38.*

The *N10* strategy which had been taught in class was used.

Problem:	*86 – 32 =□*

Jack: *86 takeaway 32 equals... 86 down there... takeaway 30 equals 46...takeaway 2.*

'Counting on' was often used in conjunction with other strategies.

Problem:	*Hayley had 27 marbles and she won another 17 marbles. How many marbles does she have now?*

Jack: *You just go 27 plus 10...equals 37...37, 37, 37.* [Subvocalizes and counts on his fingers]. *37 plus 7 equals 38, 39, 40, 41, 42, 43, 44. It's 44.*

When presented with the comparison problem Jack stated: *it's 'plus'...addition... because 'how many* ***more*** [emphasis on this word] *did Puru School get?'*

Problem:	*At the Inter-school Athletics Competition Puru School won 41 medals and Tahi School won 33 medals. How many more medals did Puru win?*

Jack: *3 and 1 is 4 ... 14 more. There's 3 and there's the 1 there.* [Points to the 'ones' digits in the numbers 3<u>3</u> and 4<u>1</u>.]

When questioned why he had used these strategies for the problems he responded: *It's what I normally do for these*, clarifying that for numbers below 10 he used one method but *if it's like 65 plus 20 I might do it another way.* He recognized that particular strategies were more suitable to solve certain problems. His willingness to take risks was demonstrated as he attempted to use strategies that were unfamiliar: *I just did that - I didn't know if it was going to work or not.*

Jack was able to provide alternative strategies for all problems using a variety of methods. For example, he combined 'counting on' with vertical algorithm, as well as with 'splitting numbers'.

In deciding which was the 'better' strategy for solving problems Jack preferred his first method. He justified his preference, often referring to the efficiency of the strategy as he compared and contrasted solutions. For example: *That is a longer way - doing 20 plus 10 - they are just wasting their time ... They might not have known that 10 out of 27 is just the next number up and it's 37. It's a quicker way by doing it that way.*

5.4.2 Notation

At the initial interview Jack had difficulty notating his thinking, commonly responding: *How would I write it down?* or *I don't know how to write it down.* He did not attempt to record thinking for some of the problems. Early endeavors to record thinking saw him verbalizing and notating the number equation.

Problem:	First solution:
You have 47 lollies and you eat 9 of them. How many have you got left?	47 - 9 = 38

Jack: *47 takeaway 9 is 38.*

Mental intermediary steps were not notated, rather there was a tendency to 'jot down' key numbers.

Problem: *Hayley had 79 stickers in her book, and she was given 22 more. How many stickers does she have now?*	First solution: 79+22= 79+1-1=

Jack: *Well if that was an 80, 80 plus 20...it would be 91...cos if it um wasn't 80, 80 plus 20 is 90 but because it's 79 takeaway 1 from there and it would add up to 71. I mean 91. I just added 1, and took away 1.*

Jack reflected on his first notated solution to assist with providing an alternative way using different number partitions.

Problem: *65 + 8 =* □	First solution: 65+10=75-2=73	Second solution: 65+5+5=75-2

The comparison problem was written as an addition number equation with key computations recorded prior to redescribing mental steps.

Problem: *At the Inter-school Athletics Competition Puru School won 72 medals and Tahi School won 25 medals. How many more medals did Puru win?*	First solution: 72+25= 2+2=~~4~~5 7+5=12

During the series of classroom lessons Jack initially recorded answers only. This made it difficult to ascertain how he had solved the problems, especially as many were incorrect. After discussing in a group teaching session the various ways to record thinking, Jack was able to clarify his thoughts.

Jack notated his strategy of 'counting on' to solve problems in three different ways using:

i) words	ii) drawings	iii) symbols
74+8= 74+8= Fingers 83	29-13=16I Knowe	37-25=12 + The '+' sign indicates 'counting on' from 25.

Sometimes arrows and words were included in his recording.

Problem: *13 – 4 =* □	13-4=9 → ~~14+~~4=10 so 13-4=9

Although Jack had solved problems using standard formal algorithms at his previous school he less frequently used this as the unit progressed. Instead a 'spew' horizontal layout tracked his thinking.

Problem: *39 + 20 =* □	30+20=50+9=59

When Jack was introduced to the 'pull-down' notational scheme of the *1010* strategy place value was disregarded as he tried to 'make tens' and add numbers together.

Problem: *27 + 17 =* □	27+17=17 10 27+17=17 2+7+1=10 7+10=7

He did not recognize that the answer to this number equation did not make sense even though he had both verbally explained and notated it. In contrast, when the problem was situated in a real-life context he solved it by drawing iconic symbols to 'count on'.

Problem: *If you have had 27 ice creams and you were given 17 more. How many have you got altogether?*	17+17 = 34 = 44

Jack attempted to use the *N10* strategy, following Ms. Vine's notational scheme, but often returned to using his own informal nonstandard notation.

At the final interview Jack appeared to be influenced by other people's notational schemes hence his personal notational scheme was a combination of vertical and horizontal layouts. The influence of the teacher's notation can be seen in some problems with the introduction of 'pull-down' strokes and arrows.

Problem:	Alternative solution:
At the school camp there are 58 boys and 35 girls. How many children are there altogether?	58 35 [illegible] 100

He also adapted the teacher's form of notation for other problems using a strategy that he felt was more efficient for 'single' digits.

Problem:	First solution:
You have 47 lollies and you eat 9 of them. How many have you got left?	47 – 9 = ↓ 47 – 10 = 37 + 1 = 38

Jack referred to previously notated solutions to explain strategies used.

Problem: *86 – 32 =□*	Problem: *Hayley had 27 marbles and she won another 17 marbles. How many marbles does she have now?*
First solution: 86 – 32 = ↓ 86 – 30 = 46 – 2 = 44	First solution: 27 + 10 = 37 + 7 = 44

Jack: *It's sort of like this one* [points to his solution of the subtraction problem]. *It's just doing the same as that but just doing it across and not doing all that.* [Points to the solutions comparing the horizontal layout to the 'pull-down' of numbers.]

Jack reflected on notated solutions in order to articulate similarities of strategies.

Problem:	Jack's notated solution:	Interviewer's card:
Hayley had 27 marbles and she won another 17 marbles. How many marbles does she have now?	27 + 10 = 37 + 7 = 44	27 + 17 = 44 27 + 17 = 20 + 10 = 30 7 + 7 = 14 30 + 14 = 44

Jack: *It's the same. Take all that* [points to 'pull-down' notation] *and it's nearly... it's pretty much... nearly all of it's the same. The 10 and 20 is pretty much the same because ... I just did it with the 27.*

However he had difficulty comparing different strategies when solutions not only had differing answers but also used inverse number operations.

Problem:	Jack's notated solution:	Interviewer's card:
At the Inter-school Athletics Competition Puru School won 41 medals and Tahi School won 33 medals. How many more medals did Puru win?	$33+14=41$	$41-33=8$ $41 - 33 =$ $41 - 30 = 11$ $11 - 3 = 8$

Jack: *I don't know what they've done – it's confusing.*

5.5 Rob

5.5.1 Summary of strategies and justifications

Teacher information from the previous year placed Rob as 'transitional' between Stage 5, Early Additive Part-Whole, and Stage 6, Advanced Additive Part-Whole.

During the initial interview he used his knowledge of known number facts to 'make tens' to solve addition problems.

Problem:	*65 + 8 =* □

Rob: *I took that one off* [points to '5'] *and put the 8 on which made it 68* [changing the equation to '68 + 5'] *and take the 2 out of the 5 which makes it 70 and you've got 3 left.*

These methods were often used in conjunction with the *1010* strategy.

Problem:	*At the school camp there are 58 boys and 35 girls. How many children are there altogether?*

Rob: *Add both the 'tens' together which is 5 and 3 which is 80. Then you got the 8 and 5 – joined it together which would equal 13. So you add the 13 on to the 80... equals 93. (To work out 8 and 5) you take 2 off the 5 and put that on the 8 so it makes 10 and then you've got 3 left... and it equals 13.*

Subtraction problems were solved by 'skip counting back' in 'tens' as well as using inconsistent groupings of numbers to 'count back' with the assistance of materials - both fingers and 'bridges'.

Problem:	*86 – 32 =*□

Rob: *80, 70, 60, 50...*[uses his fingers] *and then you get that* [40]. *You takeaway the 6, and then you takeaway 2... equals 4... 44.*

Problem:	*You have 47 lollies and you eat 9 of them. How many have you got left?*

Rob: ['Counts back' in 1s and 3s using notation]... *38. I've gone in bridges. You just count down from the highest, and you just count down 9.*

'Bridges' were used to solve the comparison problem.

Problem:	*At the Inter-school Athletics Competition Puru School won 72 medals and Tahi School won 25 medals. How many more medals did Puru win?*

Rob: [Notates then explains] *I went from 25, 30 - which is 5. Then I went 40, 50, 60, and 72. I added them all up – I went 10, 20, 30, and 40 then I added the one and two together. That equaled 47.*

Rob's rationale for using these strategies was based on three distinct reasons. Firstly, it was *easier* or *faster* to create 'even' numbers, for example: *68 is an even number and it's easier to work from. [It's easier] because you can just go up in twos with it.* Secondly, Rob chose to 'count on' or 'back' in 'tens' as it was *quicker*. Lastly, the 'bridges' method was used as *my teacher last year teached it.*

Rob had difficulty in being able to suggest alternative strategies to solve the problems. When he did provide another solution it was usually in the form of 'skip counting', for example: *By using the beaNZ. I would go in twos... Oh no... fives.*

A preference for using number facts rather than counters to solve addition problems was indicated by Rob: *You can work a lot faster – you don't have to count all of the beaNZ.*

For subtraction problems he was indecisive and tentatively said: *going in 'twos' might be a bit quicker than 'bridges'.*

During the final interview Rob used strategies that were different from the initial interview. He displayed confidence in his choice of methods to solve the problems. Part-whole thinking was evident in the solutions with prominent use of the *1010* strategy (four of the six problems were solved this way).

Problem:	*65 + 8 = □*

Rob: *You got to pull the 5 and the 8 down – put a 'plus' in the middle of it. Then you put an 'equal' sign which equals 13. Then you just bring the 'tens' down which is 60. Then 60 plus 13 equals 73.*

When he elected to do a 'vertical algorithm' the problem was actually solved using the *1010* strategy.

Problem:	*At the school camp there are 58 boys and 35 girls. How many children are there altogether?*

Rob: [Records the equation in a vertical format, and works from left to right.] *5 plus 3 equals 80, and 5 plus 8 equals 13.*

The *N10* strategy was attempted for some subtraction problems incorporating 'splitting numbers', and 'counting back' in 'tens'.

Problem:	*You have 47 lollies and you eat 9 of them. How many have you got left?*

Rob: *I've taken the 'tens' down first... I took down the 'first number' first and then I split the 9 up to make it easier... into 3s. Because it's kind of hard to takeaway that... it just the same if you just going to write the same down like you did up there. There's no use writing that again.* [Referring to the fact that 'First Number – Tens – Ones' would repeat '47 – 9']. *And the teacher told us to break the number up.*

Problem:	*86 – 32 =□*

Rob: *I was counting back from 86... oh no I counted back from 80 in 'tens'. I went 86 takeaway 30 equals 56...And I took away the 2 which equals 54.*

When presented with the comparison problem Rob used the *1010* strategy but made the common error of taking the smaller number from the larger number.

Problem:	*At the Inter-school Athletics Competition Puru School won 41 medals and Tahi School won 33 medals. How many more medals did Puru win?*

Rob: *I've gone down in 'tens' and 'ones'. It's 12.*

When questioned about why Rob had used these strategies he referred to the 'speed' of the method, for example the notational form of 'pulling-down tens and ones' was seen as a *quick* way. When asked to clarify this further he used terminology such as *effective* and *smart thinking*, but had difficulty explaining what these meant.

Unlike the initial interview Rob was able to give alternative ways for all problems. He used 'bridges' for some addition and subtraction problems. For other problems he often used strategies learned in class, explaining: *I've done it the way our teacher taught us. It's called 'Taking down the tens first then the ones'.*

Rob preferred the first method he had used to solve the problems. His justification clearly demonstrated the influence of teacher authority: *It's the way our teacher taught us.*

5.5.2 Notation

At the initial interview Rob notated 'bridges' as a form of 'support material' for 'counting on and back'.

Problem:	First solution:
You have 47 lollies and you eat 9 of them. How many have you got left?	$47 - 9$ $47 \xrightarrow{1} 46 \xrightarrow{1} 45 \xrightarrow{1} 44 \xrightarrow{3} 41 \xrightarrow{3} 38 = 38$

When asked why he had 'counted back' in 'ones then threes' Rob replied: *to try to write it down quicker.*

The alternative strategy of 'skip counting' to solve problems was demonstrated by Rob as steps were recorded and used as 'support material'.

Problem:	First solution:	Alternative solution:
At the Inter-school Athletics Competition Puru School won 72 medals and Tahi School won 25 medals. How many more medals did Puru win?	5 10 10 10 12 25 30 40 50 60 72=47	72 70 68 66 64 62 60 58 56 54 52 50 48 46 44 42 40 38 36 34 32 28 26 25

Invented symbols of curved lines above numbers were used to signify an interchange of the 'ones' digits. These were then erased and a conventional number equation was recorded.

Problem: *65 + 8 =* □	First solution:
	68+5=73

At times Rob used a 'spew' horizontal layout to record his thinking.

Problem:	First solution:
At the school camp there are 58 boys and 35 girls. How many children are there altogether?	30 + 50=80+13=93

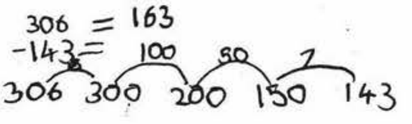
While notation often mirrored verbal explanations there was at times little reflection on the recorded solution to detect any computational errors.

Problem: *86 – 32 =*□	First solution:
	80-30=40-6-2=44

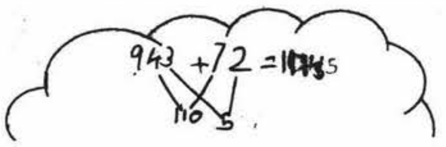
During the series of classroom lessons Rob frequently used 'bridges' to solve problems as evident in his initial interview.

Problem: *74 + 8 =* □	
	74 +8 =82 2 2 2 2 74 76 78 80 82

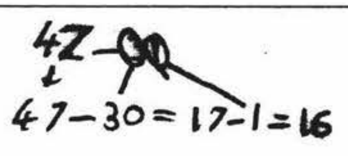
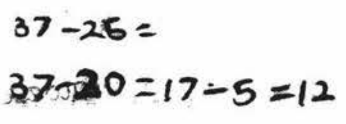
A further example of this was observed when he selected an Independent Card which had three-digit subtraction problems to solve.

Problem: *306* *- 143*	

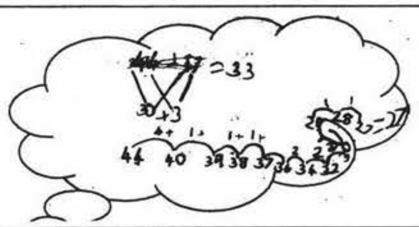
However this changed after Ms. Vine demonstrated using the *1010* strategy and notated it as 'pull-downs'. Rob quickly assimilated Ms. Vine's method and adopted a similar notational scheme.

Problem: *943 + 72 =* □	

Initially Rob copied the teacher's notational scheme for the *N10* strategy, including the circling of digits and placement of arrows; then adapted this notation.

Problem: *47 – 31 =* □	Problem: *37 – 26 =* □
	

If Rob encountered any difficulties he reverted to 'counting on and back' using 'bridges'.

Problem: *44 – 17 =* □	

At the final interview Rob would immediately notate his thoughts as he solved the problem. He continually reflected on his notation to assist with deciding which step to take next. The 'circling of numbers' and 'drawing of arrows' were similar to the teacher's notational scheme.

Problem: *86 – 32 =□*	First solution: 86-32 86-30=56-2=54

Rob adapted the notational scheme to clarify his method of 'splitting numbers'.

Problem: *You have 47 lollies and you eat 9 of them. How many have you got left?*	First solution: 47-9 3+3+1=7-47=40-2=38 47-3-3-3=38

An 'idiosyncratic vertical' layout of the traditional written algorithm (Thompson, 1994) was used. In this instance Rob worked left to right using the *1010* strategy.

Problem: *At the school camp there are 58 boys and 35 girls. How many children are there altogether?*	First solution: 58 +35 93

'Bridges' were used to provide an alternative solution for some problems.

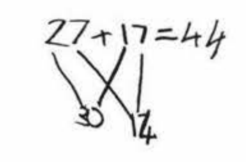
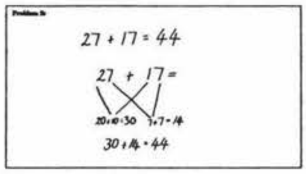
Problem: *65 + 8 = □*	Problem: *You have 47 lollies and you eat 9 of them. How many have you got left?*
Alternative solution: 1+1+1+1+1+1+1+1=73 65 66 67 68 69 70 71 72 73	Alternative solution: 7+ 2=9 47 40 38 =38

Rob referred to previously notated solutions to assist with solving problems.

Problem: *Hayley had 27 marbles and she won another 17 marbles. How many marbles does she have now?*	Problem: *At the Inter-school Athletics Competition Puru School won 41 medals and Tahi School won 33 medals. How many more medals did Puru win?*
First solution: 27+17=44 30 14	First solution: 41-33=12 10+12=8

Rob: *I was copying off that one* [points to the notated solution for the problem '27 + 17'].

Rob was able to identify whether strategies were the 'same' or 'different' by comparing and contrasting notated solutions.

Problem:	Rob's notated solution:	Interviewer's card:
Hayley had 27 marbles and she won another 17 marbles. How many marbles does she have now?		

Rob: *He went 20 plus 10, and I went that and that equals 30* [referring to the fact that he mentally solved it, instead of writing all his thinking out]. *And then they went 7 plus 7 equals 14, and I just wrote 14. It's the same. They've just 'taken it down'.*

5.6 Maggie

5.6.1 Summary of strategies and justifications

Teacher information from the previous year placed Maggie at Stage 5, Early Additive Part-Whole.

During the initial interview she used her number knowledge of 'tens' to solve problems, including 'making a decade' through 'splitting numbers'. Maggie often used her fingers to 'count on'.

Problem:	*65 + 8 =* □

Maggie: *You'd say 8 was a 5 so that's 70 and...5, 6, 7, 8* [counts on to find 5 + □ = 8]. *So add on 3... you'd get 73.*

Standard place value with compensation was used, although unsuccessfully.

Problem:	*You have 47 lollies and you eat 9 of them. How many have you got left?*

Maggie: *47 takeaway 10, plus... No, takeaway 1 equals... 36.*

Informal knowledge and a partially remembered 'procedure' were employed to solve some two-digit addition problems using a standard vertical algorithm.

Problem:	*At the school camp there are 58 boys and 35 girls. How many children are there altogether?*

Maggie: *You go 58 plus 35 equals.* [Writes the equation in a vertical layout.] *So you'd go ...5 and 5* [adds '50' and '5']. *That's ten. You go 8 and 5 ... 8, 9, 10, 11, 12, 13* [uses fingers to 'count on' the 'ones']. *So that's um...13 so 3, then 1 up there... 6, 7, 8, 9* ['counts on' the 'tens']. *It looks wrong. Put the 8 there and the 5 there.* [Interchanges the 'tens' and 'ones' digits of '58' to make '85'.] *So that's 85... 85 plus 95, 105... 110* [skip counts]... *You go 5, 6, 60, 70, 80* ['counts on' the 'tens']. *That's 80* [records '80']*...that's 88... 89, 90, 91, 92, 93. There!* [Solves the original problem.]

The *1010* strategy was used for some subtraction problems.

Problem:	*86 – 32 =*□

Maggie: *You go 60 - cos it's always the first ones that are 'tens'. So I took 3 away from the 8 so that's 60. And 6 - cos there's 6 there... and takeaway 2 from the 6 is 64.*

Initially Maggie thought the comparison problem could be solved through addition but then decided to use division.

Problem:	*At the Inter-school Athletics Competition Puru School won 72 medals and Tahi School won 25 medals. How many more medals did Puru win?*

Maggie: *You go... 72 plus... oh no... It could be in groups... the one that goes like this.* [Writes '÷' division sign.] *Like that. Just say there was 5...um...10 you could 5 and 5 like that in a group.* [She tries to separate the '72' into equal groups of 'ten'.]

When asked why she had used these methods Maggie's frequently responded they were *quick*, *easy*, or *used a ten*. In relation to her use of the vertical algorithm she said: *that had got taught at home.*

Maggie provided alternative solutions for most of the problems. She used a variety of strategies, including 'counting back' using fingers.

Problem:	*You have 47 lollies and you eat 9 of them. How many have you got left?*

Maggie: *I'd go 47, 46 ,45, 44, 43, 42 ,41, 40... like that.*

Some solutions highlighted misconceptions, for example, the commutative law.

Problem:	*86 – 32 =□*

Maggie: *You could do it back-to-front. You go 32 takeaway 86* [notates and verbalizes]. *That would be 66 – just turn them around.*

Confusion was shown as she attempted to 'make tens'.

Problem:	*65 + 8 = □*

Maggie: *Like if that was a ten, you could just put that* [changes the '8' into a '10'] *so that would be 80, and then you take off 5, and add on 3.*

Without exception Maggie thought that the first solution was 'better' than her alternative method. Justification of her choices included: *The other way is 'longer'*; *last year we got told the higher number should go first*; and *it's easier to count in 'tens'.*

During the final interview there was prominent use of the *1010* strategy.

Problem:	*At the school camp there are 58 boys and 35 girls. How many children are there altogether?*

Maggie: *You can plus the 'tens', you go 5, 6, 7, 8* ['counting on' using single digits] *so its 80...88 and 2 more equals 90. You takeaway 2 from the 5 equals 90 and you put 3 in... equals 93.*

There was evidence of Maggie's stance to 'start with the highest number first', as demonstrated in the following two problems.

Problem:	*65 + 8 = □*

Maggie: *You swap them so it's 68 plus 5.* [Records '68 + 5'.] *Cos 8 is bigger than 5.*

Problem:	*You have 47 lollies and you eat 9 of them. How many have you got left?*

Maggie: *47 takeaway 9 equals ... I change that into 9 and that into a 7* [interchanges the 'ones' digits]. *Because the 9 is higher than the 7.*

Maggie experienced some difficulty with solving the comparison problem. In thinking how the solution could be found she relied on cue words to signify which operation to use: *It's plus - how many* ***more*** *is 'plus'.*

Problem:	*At the Inter-school Athletics Competition Puru School won 41 medals and Tahi School won 33 medals. How many more medals did Puru win?*

Maggie: *The 4 and 3 are 'tens' so it equals 7...But they can't have won 74 more medals!* [Referring back to the real-life context]. This encouraged her to use subtraction for solving the problem: *Take 3 from the 4 is 1* [subtracting the 'tens'] ... *and 1 and 3, takeaway 1, is 14.*

As Maggie redescribed her first solution she reflected on her notation, made changes and began to use the *N10* strategy: *Takeaway 3 as 'tens' and cross out them... so it's 1 and the 1 there.* [Notates '11-3'.] *Takeaway 3 equals 8...* [pause]. *It's not 8, it's 14.*

When asked why she had used these methods Maggie reasoned they were 'quick' and 'easy' compared to other strategies she knew. For example: *It's easy for other people if they don't know how to do 'pulling down the tens' and the 'vertical algorithm'.* She often continued to clarify the strategy used. For example: *It's just adding 'tens and ones'. The first numbers are the 'tens', and the second numbers are the 'ones'. So you just add the 'tens' and add the two 'ones' together.* Maggie also identified the *1010* strategy as: *quite easy for 'double-digit-takeaways'.*

Maggie chose to do 'vertical algorithms' as an alternative strategy for the problems. She set the equation out in a vertical format, interchanged the 'ones' digits to 'make a higher number' if necessary, and used the *1010* strategy. As well as showing confusion with these procedures she also attempted more complex methods involving compensation.

Problem:	*You have 47 lollies and you eat 9 of them. How many have you got left?*

Maggie: *You take 1 off there, takeaway 1 off 7 equals 10... the 6 up there, the 6 from the top and then you go swap them around... go 50... no - it's takeaway....*

In selecting which was the 'better' strategy Maggie preferred either her first method or thought that both ways were good. She did not consider vertical algorithms as 'better'; rather she stated: *if you don't know how to do vertical algorithms you shouldn't do them because if you get stuck it's quite hard to get out.* Maggie discovered that *'pluses' is easier with vertical algorithms* whereas using this strategy for 'takeaway' made it 'confusing'.

5.6.2 Notation

At the initial interview Maggie used a range of signs and symbols, both nonstandard and standard, to record her thinking. Operational signs were employed to indicate when *compensation* had occurred. 'Counting back' was notated as a series of 'takeaways' using numerals and the operational sign.

Problem:	First solution:	Alternative solution:
You have 47 lollies and you eat 9 of them. How many have you got left?	47 − 10 + 1 = 38	47 − 1 − 1 − 1

'Splitting numbers' was shown by 'crossing out' numerals with changes notated above.

Problem: $65 + 8 = \square$	First solution:
	$\overset{70}{\cancel{65}} + \overset{5}{\cancel{8}} + 3 = 73$

Maggie did not reflect on notation to ascertain whether her solution made sense.

Problem: $65 + 8 = \square$	Alternative solution:
	65 + 10 = 80 − 5 + 3 = 73

An 'idiosyncratic vertical' layout of the traditional written algorithm (Thompson, 1994) was used. The written recording of the following solution portrayed four notational aspects:

- the standard formal algorithm with the operational sign to the right of the first number;
- 'dots' showed the 'counting on' of numbers;
- lines above the first number indicated the swapping of digits to make it 'look right'; and
- '80' recorded on the left side of the original solution to keep track of thinking.

Problem:	First solution:
At the school camp there are 58 boys and 35 girls. How many children are there altogether?	58 + 80 35 93

Arrows were used to indicate which numbers were being operated on, while lines indicated a change in the order of numbers.

Problem: $86 - 32 = \square$	First solution:	Alternative solution:
	86 − 32 = 68	32 − 86 =

Maggie did not know what the division sign was called but wrote it down referring to it as 'making groups'.

Problem:	First solution:
At the Inter-school Athletics Competition Puru School won 72 medals and Tahi School won 25 medals. How many more medals did Puru win?	72 ÷ 25 = 38 45 56 575

Sometimes written recording involved a combination of words and numerals mirroring the verbal explanation.

Problem:	First solution:
Hayley had 79 stickers in her book, and she was given 22 more. How many stickers does she have now?	79 + 22 7+29 1 on to the 9 is 80 + 1 91

During the series of classroom lessons Maggie initially wrote the number equation without any indication of the strategy used. After discussing in a group teaching session the various ways to record thinking, she soon began to include:

i) words	ii) drawings	iii) iconic symbols
11 + 7 = 18 I now it	16 − 7 = 9	13 − 58
Indicating 'instant recall'.	Indicating 'counting back'.	Indicating 'counting back'.

Problems solved using the *1010* strategy were recorded in a horizontal layout. Some digits were 'crossed-off' after they had been used in the computation.

Problem: *34 + 25 =* □	~~3~~4 + ~~2~~5 = 59 50

Maggie then adapted this notation to include 'arrows' or 'lines' in order to illustrate the *1010* strategy more clearly to others during class discussion.

Problem: *54 + 19 =* □	54 + 19 = 73 ✓ add 10 and ~~4~~ one's

Notation of some solutions revealed her application of the standard algorithm.

Problem: *943 + 73 =* □	943 + 73 +10 73 1016

Maggie became confused when she tried to imitate Ms. Vine's 'pull-down' notational scheme for the *1010* and *N10* strategies.

Problem: *943 + 73* =□	Problem: *91 – 78* = □
2 943 + 73 7 10 10	91 − 78 = 20 = 21 − 1 = 20 90 − 70 = ~~20~~ − 1 − 9 = +80 7060 5040 30 20 91 − 78 = 14 91 − 78 − 70 − 9

Therefore Maggie continued to use her own form of notation.

Problem: *47 – 31* = □	47 − 31 = 16 18 40 − 30 = 10 17 7 − 1 = 6

At the final interview Maggie used a combination of invented signs and symbols as well as standard forms to record her thinking. Lines were used to signify digits that had been interchanged to make a higher number.

Problem: *65 + 8* = □	Alternative solution: 65 +88 68 +8 3−1=2= 70+3=73

Maggie recognized that the verbal answer for the subtraction problem did not make sense when she referred back to the real-life context. Therefore she resorted to using iconic symbols to keep track of thinking as she 'counted back'.

Problem: *You have 47 lollies and you eat 9 of them. How many have you got left?*	First solution: 9 7 ~~||||~~ |||| 47 − 9 = 38

Alternative solutions recorded as 'vertical algorithms' employed the same strategy as the first solution. Inclusion of symbols, such as '+10' and '-2', clarified her thinking.

Problem:	First solution:	Alternative solution:
At the school camp there are 58 boys and 35 girls. How many children are there altogether?	58 + 35 = +10 −2 88 90 = 93	+ 88 85 −2 88 − 2 = 90 + 3 = 93

When Maggie reflected on her notation of the two solutions, for the above problem, she suddenly realized they were *exactly the same but I did a vertical algorithm here instead of that* [pointing to written recording]. She recognized that instead of writing it horizontally she had 'gone up and down'.

Maggie developed other forms of symbolization to denote mental steps taken:

- a curved line to indicate adding or subtracting of 'tens';

Problem:	First solution:	Revised solution:
At the Inter-school Athletics Competition Puru School won 41 medals and Tahi School won 33 medals. How many more medals did Puru win?	41 + 33 = +10 71 + 3 = 74	41 + 33 = 10 − 1 = 14 −3 11 − 3 = 8 41 − 33 = −3

- the 'crossing-off' of digits as she finished calculating them.

Problem: 86 – 32 =□	First solution: 86 − 32 = 54

Maggie identified whether strategies were the 'same' or 'different' as she compared and contrasted solutions.

Problem:	Maggie's notated solution:	Interviewer's card:
Hayley had 27 marbles and she won another 17 marbles. How many marbles does she have now?	27 + 17 = 30 44	27 + 17 = 44 27 + 17 = 20+10=30 7+7=14 30+14=44

Maggie: *I haven't pulled down the 'tens' and the 'ones'... I grouped them together - the 'tens' and the 'ones'. It's kind of the same, eh? But I didn't do 'pulling down'. It's got like the same work – like the same letters* [referring to the numerals].

Maggie continued to explain why her notational scheme was 'better': *This way* [points to her notated solution] *is quite easy to learn. This one* [refers to 'pull-down' notation on the Interviewer's Card] *might get quite hard with 3-digits.* She then provided an example to demonstrate this.

Maggie's example:	'Pull-down' notation:	Preferred notation:
343 + 202 = □	343+202= 545 4 500 5	343+202= =500+4=504 2+3=5=545

5.7 Simon

5.7.1 Summary of strategies and justifications

Teacher information from the previous year placed Simon as 'transitional' between Stage 3, Counting from One by Imaging, and Stage 4, Advanced Counting. It was not apparent until the research study was underway that Simon was dyslexic. As a consequence some of the interview questions were simplified. He was insistent that he be given the original questions as well.

During the initial interview Simon used the strategy of 'counting' with the support of materials, including counters, fingers, and notation. With the 'join-change-unknown' question Simon asked if he could use equipment as this was how he 'usually' solved problems.

Problem:	*I have 6 counters under here, and I'm putting some more counters under here* [screen the counters]. *Altogether there are 14 counters now. How many are under here* [circling above unknown collection]*?*

Simon: *I'll just use beaNZ.* [Collects 6 beaNZ and gathers some more. Subvocalizes as he 'counts all' the beaNZ.] *There's 14.* [Counts out 8 beaNZ, leaving 6 beaNZ in the pile.] *8.*

Other addition problems were solved by 'counting on'.

Problem:	*At the school camp there are 58 boys and 35 girls. How many children are there altogether?*

Simon: *What I'm going to do – start there* [referring to '58'] *and count up 35. 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 69, 70, 70, 13, 14...74, 75.* ['Counts on' with fingers, then checks to see how many fingers he has used.] *So that's 20. 20, 21...oh...74, 75, 76, 78, 79, 79, 80, 81, 82, 83, 84, 85 - 85! Never done that hard before.*

Various forms of notation were used to assist with 'counting on'.

Problem:	*65 + 8 =* □

Simon: [Writes the numerals 1 to 8, and 'counts on' using one-to-one correspondence with the notated digits.] *65, 66...67, 68, 69...*[pause]*...69, 69* [Has difficulty going over to the next decade]. *60, 71, 72, 73...73.* [Writes '73'.] *73. Now I can't forget it.*

Similar methods were employed to solve subtraction problems.

Problem:	*86 – 32 =*□

Simon: *I'll start at that* [points to '86'] *and count down. I need to use my fingers. 85, 84, 83, 82, 81...*[Continues 'counting back' then checks to see how many fingers he has used.] *That's 20 – no, 10 and 5.*

Problem:	*You have 47 lollies and you eat 9 of them. How many have you got left?*

Simon: [Begins to use his fingers.] *47 take off 9... 1 that makes 46.* [Writes the numerals 1 to 9, and 'counts back' using one-to-one correspondence with the notated digits.] *45, 43, 42, 40... it's 32.*

Simon was able to relate to the real-life situation of the comparison word problem which had been simplified.

Problem:	*At the Inter-school Athletics Competition Puru School won 28 medals and Tahi School won 21 medals. How many more medals did Puru win?*

Simon immediately replied: *7 cos you just takeaway 8, equals 7.* He was given a different pair of numbers, that is '35 and 20', and solved it as follows: *15 cos it says 20... and you count up. 15 is 5. 15 is three 5s. I always count up in fives.*

When asked why he had used these methods Simon responded: *it's easier. I learned it very quick and I know it works.*

Simon suggested that there was another way to solve the problems: *I'd just get equipment.* He demonstrated how he would use counters to either 'count all': *I'd just like, count out 40 beaNZ... of these...and then I'd just count everything and take out 9;* or 'skip count': *If it was like an even number, I'd just go 5, 10, 15, 20, 25, 30.*

When asked which strategy was 'better' Simon preferred the challenge of not using mathematical apparatus: *Cos you use your brain more than equipment.*

During the final interview there was continued use of 'counting on and back', using support materials.

Problem:	*You have 47 lollies and you eat 9 of them. How many have you got left?*

Simon: *I'll just count back. Where's those pencils?* [Takes two pencils and moves them around each other as he counts.] *39... oh... 76, 75...46, 45, 44, 43, 42, 41, 39, 38.*

He experimented with variations to this strategy, including attempts to 'skip count'.

Problem:	$65 + 8 = \square$

Simon: *65 plus 2, plus 2, plus 2, plus 2...that makes 8* [Verbalizes as he writes the number equation.] *65... 65, 66, 67, 68, 69, 40, 41, 42, 43.* [At each '+ 2' he 'counts on' 2 more; then begins counting in the 'forties'.]

Simon attempted to solve some two-digit addition problems using 'grouping' strategies which had been discussed in class.

Problem:	*At the school camp there are 58 boys and 35 girls. How many children are there altogether?*

Simon: *I'd 'take down'. So* [notates and verbalizes] *58 plus 35. Bring down the 5 that's 50 plus 8 so that 58 plus 30... 70. So 50 plus 8 equals 58... plus 60...plus 30. Equals 70 plus 5. So that's 75.*

He specifically stated for some strategies, it was *what Ms. Vine taught us.*

Problem:	*86 – 32 =*□

Simon: *Bring down the 80... Bring down 30* [Notates and verbalizes.] *What's 6 and 2? It's 8 so I'll just* [subvocalizes] *6... and 2... 8... so put 8 up here. 80, 80, 90, 100... that's 3, eh?* ['Counts on' in 'tens' using fingers.] *100...equals 100 and that one there equals 108.* [Simon has added the numbers.]

The real-life context of the comparison problem assisted Simon to solve it.

Problem:	*At the Inter-school Athletics Competition Puru School won 41 medals and Tahi School won 33 medals. How many more medals did Puru win?*

Simon: *We've won more. How much* ***more****? So it's...* [Thinks, then uses his fingers to 'count on' from '33'.] *7 more.* When he redescibed his strategy he realized that the answer was '8'.

When asked why he had used these methods Simon acknowledged that 'counting on' was his *favourite strategy.* For the comparison problem he indicated that there was no other way: *you had to count from that one up to that one. I had to do it that way.* He was also prepared to take risks by trying new strategies, commenting: *I never really used it (before)*; and *it's the one Ms. Vine taught us.*

Simon suggested using a number line for 'counting on' as an alternative way for solving most problems. However for the problem '58 + 35', he described a confused method of 'grouping tens and ones'.

In selecting the 'better' strategy Simon thoughtfully reviewed each solution and chose the one which he considered to be 'quicker'. An example of his justification was: *the second way was quicker* [pointing to the number line] *cos that* [referring to using 'pencils'] *made me think a lot - you got to keep track.* He also recognized that 'grouping numbers' was an 'efficient' way to solve problems: *It's easy if you know your 'plus' table on the 10s and 70s and your 8s.*

5.7.2 Notation

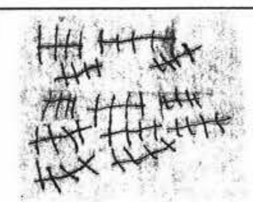
At the initial interview notation played an important role as a form of support material for 'counting on and back'. Listed digits were used for one-to-one matching as he counted.

Problem: *You have 47 lollies and you eat 9 of them. How many have you got left?*	Problem: *65 + 8 =* □
First solution: 47 − 9 = 32 48 1234 56 789	First solution: 65 + 8 = 73 123456 78 73

When Simon used 'fingers' to solve a problem the number equation would be recorded only. He pointed to his solution notated in the vertical layout and stated: *the answer's upside down.*

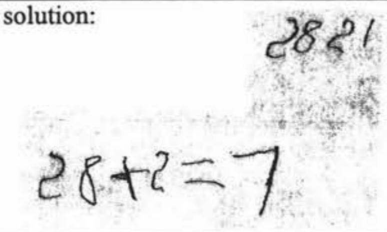
Problem: *At the school camp there are 58 boys and 35 girls. How many children are there altogether?*	Problem: *86 – 32 =*□
First solution: 35 + 58 11 85	First solution: 8632 = 73

Iconic symbols were drawn to record 'skip counting' with counters.

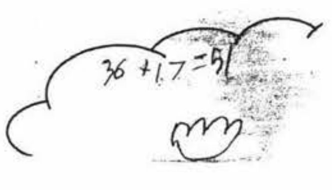
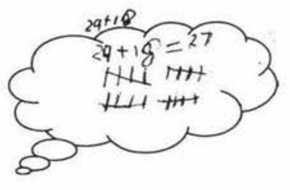
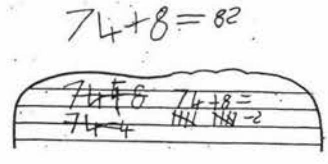
Problem: *At the school camp there are 58 boys and 35 girls. How many children are there altogether?*	Alternative solution: 

Simon: *I've got to start at 35...so 35...plus...5...*[notates with tally marks]. *Just got to get this 5... then we got 35, 35, 35, 44... I'd say 40.* [Draws more tally marks]. *45...*

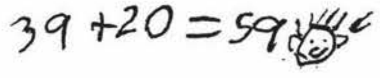
As Simon listened to the comparison problem he jotted down the two numbers as they were read aloud. After the problem was solved Simon recorded the corresponding number equation.

Problem: *At the Inter-school Athletics Competition Puru School won 28 medals and Tahi School won 21 medals. How many more medals did Puru win?*	First solution: 

During the series of classroom lessons Simon initially copied number equations from the board then used 'counting on' with fingers and wrote the answer. After discussing in a group teaching session the various ways to record thinking, he began to use a range of ways to record his preferred strategy of 'counting on':

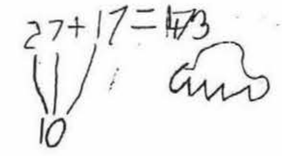
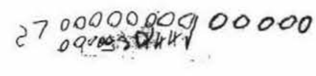
i) drawings 	ii) iconic symbols 	iii) combination of symbols 

Simon drew pictures of 'heads' to indicate problems had been solved mentally.

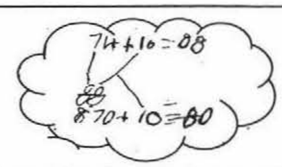
Problem: $39 + 20 = \square$	

This notation was adapted to include 'words' in order to clarify his problem-solving strategies to others during class discussions. For example, he wrote '*bck*' to indicate that he had 'counted back' to find the solution.

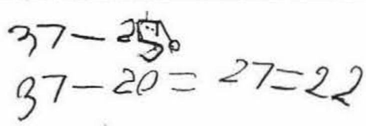
Due to Simon's limited number knowledge, especially with place value, he had difficulty in comprehending the teacher's notational scheme. After observing peers he resorted to using iconic symbols to support his 'counting' strategy.

Problem: *27 + 17 =* □	Initial solution: 	Revised solution: 

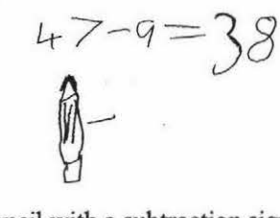
As a result of Simon's developing mathematical knowledge a change was observed over the last two weeks of the classroom unit. 'Pull-down' strokes, distinguishing 'tens' and 'ones', began to appear in his notation.

Problem: *74 + 16 =* □	

When notating the *N10* strategy he initially used arrows, similar to the teacher's model which was then adapted to fit with his own notational scheme. He often combined 'counting' with these new 'grouping' strategies to assist with solving problems.

Problem: *37 – 25 =* □	

At the final interview a variety of invented notational forms were used to record how he had solved problems.

Problem: *You have 47 lollies and you eat 9 of them. How many have you got left?*	First solution:  Drawing of pencil with a subtraction sign.

Simon drew a number line, which was referred to as his 'checker', for alternative solutions.

Problem: $65 + 8 = \square$	Problem: *You have 47 lollies and you eat 9 of them. How many have you got left?*
Alternative solution: 1 2 3 4 5 6 7 8	Alternative solution: 1 2 3 4 5 6 7 8 9

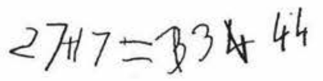
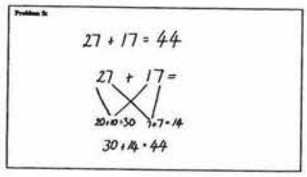
The influence of the teacher's notational scheme can be seen as Simon attempted to 'group' numbers to solve two-digit problems. A variation of this notational form was used for the alternative solutions.

Problem: *At the school camp there are 58 boys and 35 girls. How many children are there altogether?*	Problem: $86 - 32 = \square$
First solution: 58+35 10 50+8=58+30 70=75	First solution: 86-32 80 30=108 6+2=8
Alternative solution: 58 + 35 10815 8+30 (Symbol for "tens and ones")	Alternative solution: 806 86-32 80-30=50=41

Some recorded 'number equations' represented the direct actions taken to solve problems. For example, the equation for the comparison problem indicated the number Simon had to 'count on' to, that is '41'; the 'plus' sign indicated he had to 'add more'; and the number counted on, that is '8'.

Problem: *At the Inter-school Athletics Competition Puru School won 41 medals and Tahi School won 33 medals. How many more medals did Puru win?*	Simon's notated solution: I gos my  41+8	Interviewer's card: 41-33=8 41 - 33 = 41 - 30 = 11 11 - 3 = 8

Simon was beginning to identify whether strategies were the 'same' or 'different' by comparing and contrasting notated solutions.

Problem:	Simon's notated solution:	Interviewer's card:
Hayley had 27 marbles and she won another 17 marbles. How many marbles does she have now?	 'Counted on' using fingers.	

For example, Simon initially thought he had solved the above problem *the same but mine's using my brain not 'pulling-down'*. He began to explain the strategy notated on the Interviewer's Card: *He's tooken the first number 30... Hey, that is the same as...* [looks back at his written recording.] *Is it the 'taking down' one? That one* [points to his notated solution for '58 + 35'].

5.8 Summary of Case Studies

Uncovering the case study students' mental strategies involved finding out how they solved number problems. Initially some children used 'counting on', often with support material; while others used their number knowledge to 'make tens'. Only two of the six children used the real-life context of the comparison problem to successfully solve it. Three children did not know why they had used a particular strategy to solve problems. Although others identified the strategy as being 'quick' or 'easy' they were often unable to articulate why this was so. The children found it difficult to provide alternative strategies for the problems. This is consistent with the latest New Zealand mathematics assessment results (Crooks & Flockton, 2002) which showed that children have difficulty with explaining more than one strategy for computation problems. The methods suggested by the case study students' were frequently a rephrased, often long-winded, version of their first solution. Consequently children preferred their first method for solving the problems.

Changes were observed in the final interview as all case study students' attempted to 'group numbers', with the *1010* strategy being prominently applied. This caused difficulty in calculating the correct answer for some subtraction problems. The children often referred to the cue word 'more' and the real-life situation to make sense of the comparison problem. Most were able to reason why they had used a particular strategy to solve a problem often comparing it with other methods. They were able to explain alternative ways to solve problems which included a variety of methods. The children did not necessarily select their first solution as the preferred method; and sometimes thought both ways were 'good'. Most referred to the efficiency of the strategy to justify their choice; while one child rationalized that it was the teacher's method.

The case study students' notational schemes combined both informal nonstandard and formal standard symbolizations. Drawings, iconic symbols, words, invented signs and symbols, as well as conventional notation were used. It was observed that most children seemed to move among notational forms in an integrated, fluid process to represent and communicate mathematics (Tang & Ginsburg, 1999; Woleck, 2001). However if they became confused or unsure of how to solve a problem they often reverted to using a representation that was familiar to them (Dufour-Janvier et al., 1987). The most common form of notation was in a 'spew' horizontal layout with children working left to right (Thompson, 1994).

Some children were influenced by the teacher's 'pull-down' notational form which they either copied or adapted to fit with their own notational scheme. Other children continued to use their preferred form of written recording. It was apparent that peers and 'home' had an influence on some children's notational style.

Written recording often mirrored the children's verbal explanation or their direct action for solving problems. Sometimes the children focused on key numbers and operations; as a result, not all mental steps were recorded. Extra signs and symbols were occasionally inserted to clarify their thinking to others during discussion. Notation helped the children keep track of their thinking, especially if large numbers were involved, or if the number computation became too complex. In such instances they frequently 'jotted down' numbers. While some children used notation as a way to record their thinking, others reflected on partially recorded calculations before deciding their next step. Notation was also employed as a form of support material for 'counting', for example, 'bridges'.

Notation was used as a tool to support children's mathematical learning. Most children reflected on their written recording to see if solutions made sense. Notation provided a window to their thinking highlighting misconceptions and mistakes. When the case study students' compared and contrasted notated solutions they were able to distinguish strategies as 'same' or 'different'. Justifying their use of a particular strategy was made easier when there were two solutions to compare. As noted in earlier studies, notation appeared to play *a critical role in initiating shifts in students' ways of reasoning* (McClain & Cobb, 2001, p. 251).

CHAPTER 6

DISCUSSION AND CONCLUSION

6.1 Introduction

The major goal of this study was to examine the development of notational schemes within a unit of work in the *Numeracy Project*. A particular focus was on the way in which notation contributed to the productiveness of group or whole class discussions of computation strategies. A further focus was to determine the extent to which notational schemes reflected a shift in children's reasoning. In this chapter the role of notation in children's mathematical learning is examined. Arising issues and tensions about the topic are discussed highlighting the complex nature of teaching and learning. Implications of this study and suggestions for further research are outlined. Finally conclusions from this study are presented.

6.2 The Role of Notation in Children's Mathematical Learning

The ways of symbolizing that emerged in the classroom evolved from the need to clarify and communicate children's thinking. Initially the children worked individually, recorded the number equation and results, and made frequent appeals to the teacher as they asked: *Is that right? What's the answer?* The teacher played a proactive role in redescribing and notating children's explanations which externally represented their mathematical activity. The children and teacher continually negotiated their expectations to 'record thinking' so that notation became established as a normative practice within the classroom. It is not an easy task to establish taken-as-shared 'social norms' with mathematical activities often being re-visited to reinforce common understandings. Over the four week study notation was increasingly used as a 'tool' to articulate and support children's informal knowledge, strategies and reasoning. Written recording became an integral part of group and whole class discussions.

Written recording made children's thinking visible assisting them to reflect on the solution method rather than just the answer. The process of communicating their ideas promoted further thinking especially when the children became aware of the need for others to understand their ideas. Children's notation provided a 'window' to their thinking for others, including peers, teachers and parents. Simon remarked: *When you are notating you are thinking harder*. Rob agreed, explaining: *You are not only thinking of how to solve the problem but also have to think how to write it down*. This study supports research (WCER, 2001) which found that in deciding what to write the children had to select and organize their essential ideas about a problem and its solution. In formulating what to put on paper children had to consider and choose the most important parts of their thoughts.

Notated solutions acted as a common referent for discussion, focusing children's attention on specific mathematical ideas or misconceptions which is similar to findings in other studies (Carpenter et al., 2003; Lampert, 2001). Notation provided a visual support as ideas were shared and discussed. This was particularly helpful when some children had difficulty in expressing their ideas verbally. In such instances the children would often show their notated ideas when they were unable to explain further. The teacher supported children's problem-solving through 'scaffolding' ideas, prompting other class members to examine the solution with a particular focus on numerical concepts and relationships. Children became exposed to explicit mathematical language when they reviewed the steps taken to solve problems. Written recording revealed errors and misconceptions which were discussed and clarified in a safe learning environment.

At the outset it was obvious that the children were not accustomed to in-depth examination and discussion of problem-solving strategies. Classroom episodes (see Chapter 4.6.2) illustrated a lot of 'backwards and forwards talk' as the children fluctuated from 'certainty' to then raising questions of 'uncertainty'. Gradually children's ideas became explicit topics of conversation as solution methods were explained. The children referred to notated solutions to reflect on others' thinking providing an opportunity to question the validity of solution methods for the problem-type given. Thus, notation provided a powerful vehicle for developing children's sense-making skills and in the process retain their ownership of personal ideas. To engage children in such dialogue required time and practice, including regular renegotiation of taken-as-shared expectations.

Collaborative discourse and efforts to symbolize children's thinking contributed to the emergence of sociomathematical norms. Discussion facilitated the children's understanding about key mathematical aspects of the various solution strategies. Notational schemes drew attention to these features assisting children to recognize and reason why solution methods were mathematically different, efficient, or more sophisticated. Explanations and solutions were viewed by the children as 'generators of meaning' (McClain & Cobb, 2001, p. 255). Wall displays of recorded solutions provided an opportunity for children's 'thinking' to be open to public scrutiny and reflection. The Modelling Book proved to be an effective vehicle for children to clarify their thinking and review different solution strategies. Children articulated and justified their own and others' mathematical ideas providing a rationale for solution processes. Children increasingly became engaged in genuine dialogical encounters making reference to their own and others' explanations as captured by the notational schemes.

Written recording supported children's mathematical learning in other ways. Expressing ideas on paper reduced the details a child had to keep in their memory about a problem. Some children 'jotted down' numbers, especially if large numbers were involved, to assist with keeping track of their thinking. Others would reflect on partially recorded calculations to think more about the problem before deciding their next step. Notation was also used as a 'physical' form of support material for 'counting all' and 'counting on' to solve problems.

As the study progressed the children's advocacy for recording mathematical activity increased. Their responses included:

- *You can see how you think*
- *So I can understand*
- *The teacher can get more information*
- *We can see mistakes*
- *I learned a bit more maths - like the strategies*

- *By looking at other people's you can see how they did it. Each person did it a different way, and you can try it*
- *You can use a 'smarter' way of solving problems*
- *As you are working on it you might think of another way of doing it than the one you're using*
- *If you forget how you did it, as you try other strategies, you can look back to see how you solved it.*

The classroom community was exposed to a variety of notational schemes through the sharing and display of notated solutions during group and class discussions. In presenting their thinking children were encouraged to review their peers' notational schemes providing rich opportunities for individuals to consider and develop their own recording methods for communicative purposes.

6.3 The Complex Nature of Teaching and Learning

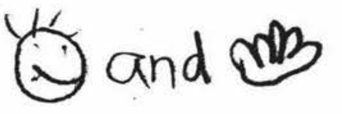
A number of arising issues and tensions in relation to the role of notation have emerged from this study that highlight the complexity of teaching and learning mathematics.

6.3.1 Forms of notation

Recording of mathematical activities exemplified the diverse ways in which children notate their thinking. The 'Thinking Bubbles', 'Think Mats', and individual maths books revealed that children used drawings, numerals, words, and other symbols to express their ideas. Some children invented unique notations to represent thinking, as the idiosyncratic symbol demonstrates in the example below:

Problem: $37 - 25 = \square$	$37 - 25 = 12 +$ The '+' sign indicates 'counting on' from 25.

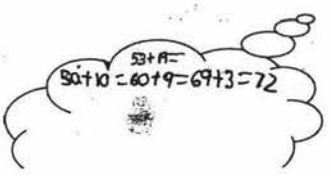
While some notation is ingenious and can be easily transferred to more conventional forms at an appropriate time, others were very idiosyncratic, bearing little connection to the mathematical ideas or problem being explored:

Problem: *27 + 17 = □*	

Some literature on emergent notation (Gifford,1997) suggests that to make an easier transition to conventional symbols it would more useful for children to record number operations using abbreviated words rather than pictures. This is illustrated by the following notation used for 'counting back':

Problem: *Mary has 17 apples. Jake has 8 apples. How many more apples does Mary have?*	17 − 8 = 9 Jcke ~~Bake~~

Many children recorded their thinking with conventional symbols in a non-mathematical sentence using a 'spew' horizontal format. The information was written as one long continuous mathematical statement working from left to right, as illustrated by the example:

Problem: *There are 53 buns in a bakers shop. The Baker cooks 19 more. How many are there altogether?*	

Research by Thompson (1994) analyzed the 'directionality' of children's written solutions, and the emergence or otherwise of standard or idiosyncratic written algorithms. The nine- and ten-year old children in his study had not been taught traditional standard algorithms providing an opportunity to investigate their responses to solve additive word problems. He found that the two main layout arrangement categories of 'horizontal' and 'vertical' could be further divided into 'spew' or 'punctuated horizontal' and 'idiosyncratic' or 'standard vertical'. Almost three quarters of the children in his study recorded their thinking in a horizontal format.

While the 'spew' layout tracks children's thinking this 'faulty' form of notation does not develop an appropriate conception of 'equality'. It was observed that, at times, Ms. Vine also used this format to track children's thinking. Literature (Carpenter et al., 2003) cautions against the inappropriate use of the 'equals' sign which may cause children to think of equality in terms of calculating answers rather than as a relation. Children need to understand that the equal sign signifies a relation between two numbers. Misconceptions of what the equal sign means is one of the major stumbling blocks in learning algebra. Therefore rather than use the equal sign to represent a series of calculations it would be preferable to use some other notation. An alternative notation that could be modelled by the teacher is the use of an arrow (→) to designate the sequence of operations. Some research studies (for example, Anghileri, 2000; and Menne, 2001) advocate the employment of such a symbol, for example:

Problem: $3 + 2 = \square$	$(3 \text{ and } 2) \xrightarrow{\text{together make}} 5$
Problem: $16 + 47 = \square$	$10 + 40 \rightarrow 50 + 6 \rightarrow 56 + 7 \rightarrow 63$
Problem: $43 - 18 = \square$	$43 \xrightarrow{(-20)} 23 \xrightarrow{(+2)} 25$

However Carpenter and colleagues (2003) contend it is preferable for children to write a longer version that emphasizes the correct use of the equal sign, as shown below:

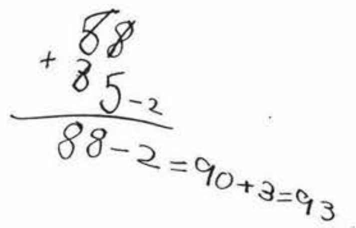
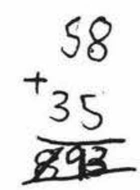
Problem: $27 + 38 = \square$	$20 + 30 = 50$ $50 + 7 = 57$ $57 + 8 = 65$

Children's natural inclination to use 'mathematically' incorrect notation to record thinking raises further concerns. The example below not only shows a 'spew' horizontal format but also erroneous recording of place value:

Problem: $343 + 202 = \square$	343+202= =500+4=5042+3=5=545

The acceptance of incorrect notation has the potential to foster misunderstanding and create confusion particularly when discussing number concepts and the properties of number operations. Number sentences provide children with a tool to represent mathematical ideas. To represent these mathematical ideas using correct symbolization significantly enhances children's ability to communicate mathematical ideas clearly and precisely (Carpenter et al., 2003). Given that accurate recording will assist the teaching and learning of more complex operations on mathematical expressions the teacher must decide when it is appropriate to intervene with correct sentence structure.

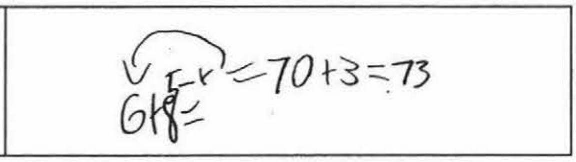
Several children attempted to use standardized notation and referred to their solution method as a 'vertical algorithm':

Problem: *At the school camp there are 58 boys and 35 girls. How many children are there altogether?*	
	

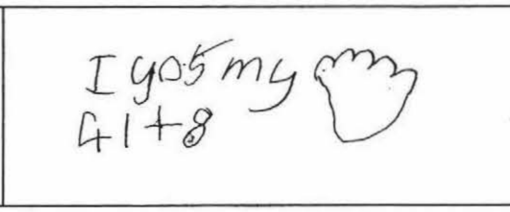
There was evidence that some parents encouraged their children to use standard vertical algorithms, for example, Maggie explained: *I got taught this at home.* However in accordance with the *Numeracy Project* the children were not exposed to this form of notation until mental strategies were sufficiently advanced. Thompson (1994) revealed that there is an increasing amount of kudos associated with the ability to use the 'proper method'. Not surprisingly, some children in Ms. Vine's class recorded an 'idiosyncratic vertical' layout that reflected sophisticated compensation strategies (see example above).

6.3.2 Introducing notational schemes

The study clearly confirmed previous research findings (McClain & Cobb, 1999) that children invent and use unconventional or individual notation as a thinking device to help them reason and make sense. Research literature (Sierpinska, 1998; WCER, 2001) supports children's 'invention' as a part of learning mathematics, encouraging the activity of symbolizing to be a creative thinking tool. The children are able to reason with their own symbols which make sense to them (Cobb, 2000a; Yackel, 2000). However consideration needs to be given to how long children are left to develop and use their own idiosyncratic notation before introducing conventional notation. The following example illustrates a child's use of invented symbols to clarify thinking:

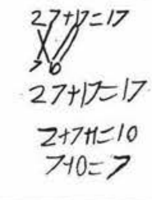
Problem: *65 + 8 = □*	

There is no definitive answer to the issue of 'when' and 'how much' a teacher should insist that conventional notation be employed by the children. Rather research findings (Thompson, 1999b) make the recommendation that children's 'jottings' are developed into informal written recordings before introducing standard algorithms. The teacher has to judge when it would be appropriate to model conventional notation. Such an instance arises with the following notated solution which represented the direct action taken to solve the problem but indicates that the child was unaware of an alternative way to record this:

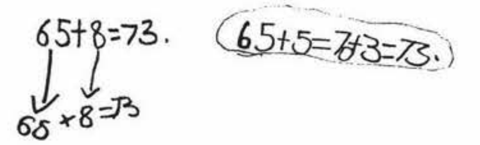
Problem: *At the Inter-school Athletics Competition Puru School won 41 medals and Tahi School won 33 medals. How many more medals did Puru win?*	

Diverse views of whether conventional notation should be 'transmitted readymade' to children are evident. Some studies (for example, Kamii & Housman, 2000) indicate that because mathematical symbols belong to a specific knowledge system they must be socially transmitted; while others (for example, van Oers, 2000) insist that children have to decide for themselves the generally accepted meaning of the conventional ways. A more moderate view proposed by Sierpinska (1998) and Thompson (1997) suggests that children be exposed to conventional symbols without being obligated to use them until they feel comfortable with them.

Although teachers may present a notational scheme as a 'natural solution' (Stephan et al., 2001), there is a concern about introducing notations prematurely (Dufour-Janvier et al., 1987). The symbolization may exist 'outside' the children's actions with mathematical objects causing some children to syntactically manipulate symbols without making reference to their meaning. Such concerns were evident in this current study. In the following example the child focused on notation as a 'procedure' rather than on its 'sense-making' skills:

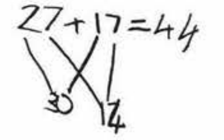
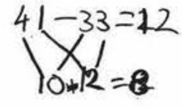
Problem: *27 + 17 =* □	

In this study there were also examples of children notating for 'notation-sake' where extraneous signs and symbols were recorded, as depicted below:

Problem: *65 + 8 =* □	

By emphasizing the need to notate too early children may feel compelled to write 'something' down. Notation should not be the result of pressure to write rather children need to be aware of the purpose for notating. Therefore consideration should be given to what notation children 'need' to do and what teachers 'want' them to do.

Research studies (for example, Angihileri, 2001b; Beihuizen, 1999; Gravemeijer, 1998; and van den Heuvel-Panhuizen, 2001) support the implementation of intermediary models to assist children in making connections between their informal and formal knowledge. Consideration needs to be given to the model employed to do this. It was observed that several children transferred the notational 'form' used for addition problems over to subtraction, as indicated by the following 'pull-down' solutions:

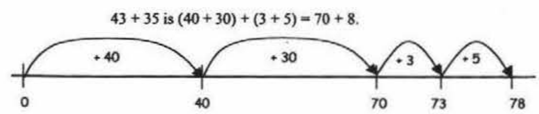
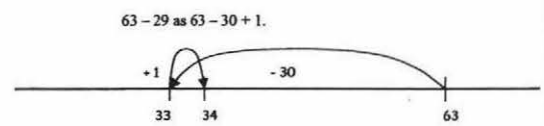
Problem: *Hayley had 27 marbles and she won another 17 marbles. How many marbles does she have now?*	Problem: *At the Inter-school Athletics Competition Puru School won 41 medals and Tahi School won 33 medals. How many more medals did Puru win?*
	

To transfer from addition notation to subtraction was not as straightforward as imagined. The 'pull-down' notation revealed erroneous thinking in subtraction with the 'smaller-from-larger bug' occurring (Baroody, 1987, cited in Blote et al., 2000, p. 233).

In this case, the notational form does not provide any further support than the standard vertical algorithm in terms of subtraction, as exemplified:

$$\begin{array}{r} 41 \\ \underline{-33} \\ \underline{12} \end{array}$$

This raises the issue of whether a notational form that works easily for addition problems but not subtraction should be introduced. An alternative model may be the 'empty number line' which works for both addition and subtraction, for example:

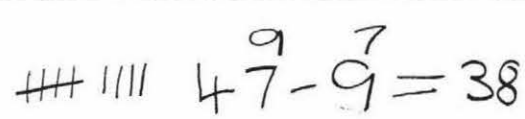
Problem: *43 + 35 = □*	
Problem: *63 – 29 = □*	

(Ministry of Education, 2002a, pp. 4-5).

It is essential that teachers endeavour to ensure that notational schemes are not a barrier to mathematical understanding. Notational 'forms' must not act as distracters to children learning problem-solving strategies. An awareness of the different ways children represent their thinking will assist the teacher in bridging knowledge from an informal to a formal level. Research studies (Steffe & Wiegel, 1996; Yackel, 2000) endorse the viewpoint that notational schemes constructed by the teacher need to model the children's mathematical knowledge providing a structural mathematical and, as far as possible, supportive representation of their thinking.

6.3.3 Modelling

It is important to provide models to assist children develop and make connections with mathematical ideas. There is potential for a child's misconceptions to remain if concrete referents or images are removed in favour of notated support:

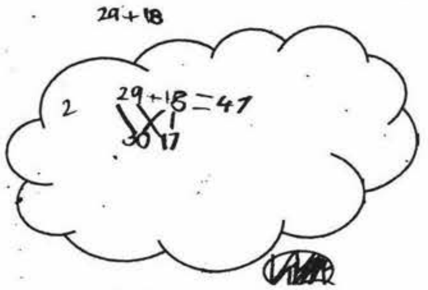
Problem: *You have 47 lollies and you eat 9 of them. How many have you got left?*	

Maggie: [Uncertain whether to interchange the 'ones' digits justifies]. *You have to take the 9 away from 47 because 7 is most probably the wrong way. You should do it the way it comes. Sometimes if you swap the ones around it can be wrong... sometimes.*

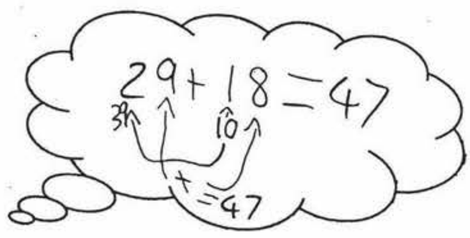
Carpenter et al., (1999a) ascertained that having children explain how they have solved problems with equipment played a significant role in extending the physical modelling strategies to more abstract symbolic procedures. By making more use of equipment over a longer period of time provides opportunities for children to connect emerging number concepts and procedures to prior knowledge. The chance of writing insensibly is increased if notation is precipitated. The implication is that children should continue to model with equipment until they are comfortable with the mathematical conception. Notation could then be introduced alongside equipment so that children make connections between and within the representational forms.

6.3.4 Differentiating strategies and notational schemes

When identifying 'mathematical' solutions that were the 'same' some children focused on and classified notational 'forms' rather than the solution or strategic solution processes. This was observed in the 'grouping' of the two notated solutions below:



Rob's notation



Child Q's notation

It became apparent that this was more likely to occur when children showed their notated solution to others without verbal explanation. Therefore children were encouraged to talk about their strategies alongside their written recording to determine 'same' and 'different' mathematical solutions. During discussions children were exposed to a range of different strategies and notational 'forms'. Several children adopted a 'new' notational form by copying their peers. It was difficult to ascertain if this was because they considered it a 'better' way or due to a lack of confidence on their part.

Furthermore, the exposure to a variety of strategies and notational 'forms' had both positive and negative consequences. Simon appeared to benefit from working in a mixed ability group where he became aware of more sophisticated ways to solve problems. Conversely, Maggie's attempts at problem solving exemplified the situation of 'over-exposure' as she continually attempted to use the 'right jargon' but in the process became more confused. Maggie's predicament showed that she needs to work in small manageable steps to clarify thinking and could possibly benefit from working in a homogeneous group. Grouping for instruction in the classroom presents a further dilemma.

6.3.5 The teacher and the numeracy programme

The teacher's own personal mathematical beliefs and values, together with her mathematical knowledge and understanding, have a significant effect on the microculture within the classroom (Yackel & Cobb, 1996). While Ms. Vine's guidance was purposely mediated some children viewed the teacher as an 'authority figure'. This issue of 'authority' influenced them in several ways. Children often made reference to some notational 'forms' and/or strategies as being: *Ms. Vine's way*. Frequently discourse of justification indicated that the 'teacher's way' was perceived as endorsing particular strategies. For example, Rob reasoned that he used 'bridges' because: *my teacher last year teached it*. Reliance on teacher-taught methods may impede children from using their own strategies and could cause them to increasingly 'fall-back' to employ the teacher's way. Consequently children may have difficulty in recognizing and establishing their own method as being efficient. A further concern is that some children relied on authority rather than their own sense-making skills as evidenced by Maggie's comment: *last year we got told the higher number should go first*.

Greater familiarity with one strategy, perhaps from an emphasis on it in group Teaching Time, can lead to that approach being the inevitable method of choice to solve problems. Teaching towards flexible thinking is best pursued within a learning environment where children calculate and describe to one another how they solved the problem with the teacher pointing out facts and connections, supporting descriptions with explicit language and revealing forms of recording (Threlfall, 2002). The children's strategy choice would then be related to the problem context and the numbers involved rather than selecting the teacher's 'supposedly preferred' strategy. Flexibility in thinking develops through an emphasis on considering possibilities for numbers presented in a variety of problem-types rather than focusing on teacher-taught strategies.

Teachers need to be open to and aware of the different forms of notation children use to express their mathematical ideas. Children often move among various forms of notation at different times while learning new mathematical concepts. Through open-ended questioning and listening the teacher can elicit children's thinking to foster their growth in mathematical understanding. It is important that the teacher 'listens to' the children valuing their contribution rather than 'listening for' an assumed outcome (Davis, 1997). In doing so, the teacher can assist children construct links between their informal, intuitive notions and the abstract language and symbolism of mathematics.

This research study provides an open forum to raise teachers' awareness of emerging understandings about how children might come to develop and use informal nonstandard notations and standard conventional notations for communicative purposes. The described issues related to notation and the implications for teaching numeracy should provoke further discussion and insights about developing children's mathematical thinking.

6.4 Further Research

Within the constraint of the four-week numeracy unit the development of notational schemes was the central focus so it is not surprising that the study has raised many unresolved issues. The following areas related to notation warrant further investigation:

- In this study the focus was on Year 5/6 children therefore it would be appropriate to examine the emergence of notational schemes that represent number operations in earlier class levels. A comparison of the development of notational schemes from Year 1 to Year 6 would greatly contribute to understanding children's external representation of mathematical ideas.

- The *Numeracy Project* encourages the use of a variety of concrete manipulatives to assist with the development of children's strategy and number knowledge. An exploration of the use of equipment and its relationship to the development of notational schemes would be beneficial. In particular, a focus on the types of equipment and associated classroom activities that supports the link between mathematical notation and thinking needs further research.

- Flexible thinking is developed through the teacher facilitating discussions assisting children to make mathematical connections with the support of 'revealing forms of recording'. An examination of notational schemes constructed by teachers to model children's mathematical knowledge is required. This would provide additional information on ways to record and track children's ideas using structural mathematical and supportive representations of their thinking.

- During this study children adopted and/or adapted notational schemes moving among different written forms to represent their thinking. They often reverted to using familiar forms of notation with the 'spew' horizontal layout being a prominent format. An examination of reasons for using particular forms of notation would help clarify the children's purpose for written recording.

- When children learn arithmetic with understanding, they implicitly use many of the unifying properties of number operations. These were often made explicit when children reflected on their prior activity as supported by notational schemes. This has the potential to initiate the first steps in the transition from arithmetical to algebraic reasoning. The existence of such a possible generative connection warrants further investigation.

- Additionally, this study highlighted the importance of the social nature of learning within the classroom. An examination of classroom discourse focusing on the way children engage in productive mathematical discourse would be valuable. This would also positively contribute to understanding the role of notation when negotiating and establishing sociomathematical norms.

6.5 Concluding Thoughts

This study highlights the complexity of the teaching and learning of mathematics within the classroom. There are many issues and tensions the teacher has to resolve in daily practice. The consequence of decisions made can give rise to other issues requiring further attention. By placing 'learning ways of thinking' at the centre of classroom instruction teachers will be able to judge what is best for students. Conceptual development is simultaneously an individual and a social process in which children construct or build knowledge within the mathematics classroom. Each child has a unique way of understanding and expressing mathematical ideas. Notational schemes are 'thinking devices' which make implicit knowledge explicit. Fostering mathematical thinking in a supportive intellectual community develops children's confidence and a positive disposition towards mathematics. During the final session of the numeracy unit Simon spontaneously exclaimed: *I used to 'suck' at Maths but now I'm better!* Ms. Vine inquired: *Who else has 'done better' at Maths?* Hands were raised... Simon had two hands up.

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APPENDIX A: Interview Questions (Pre-Unit)

Problem 1	**I have 6 counters under here, and I'm putting some more counters under here** (*screen the counters*). **Altogether there are 14 counters now. How many are under here** (*circling above unknown collection*)?

(*Have equipment/ pencil and paper available for child to use as required.*)
Would you explain to me how you worked it out?
Show me how you would record what thinking was happening in your head.

Show the Problem Card and read the problem aloud to the child.

- **For each of the problems ask the following questions:**

As we do not have a calculator I need you to think of **one** way you could work the problem out, and **explain** it to me.

Show me how you would record what thinking was happening in your head.
(*Provide child with pencil and paper.*)
Why did you choose this way to solve the problem?

Is there another way you could work it out?
Explain to me how you would do it.
Show me what thinking was happening in your head.
(*Provide child with pencil and paper.*)
Which way is better for solving this problem? Give your reason.

- **After reading Problems 6 and 7 to the child first ask:**

Show me how you would write this as a number equation. (*Provide child with pencil and paper.*) **Then proceed with the other questions.**

Problem 2	**You have 47 lollies and you eat 9 of them. How many have you got left?**

Problem 3	$65 + 8 = \square$

Problem 4	**At the school camp there are 58 boys and 35 girls. How many children are there altogether?**

Problem 5	$86 - 32 = \square$

Problem 6	**At the Inter-school Athletics Competition Puru won 72 medals and Tahi won 25 medals. How many more medals did Puru win?**

Problem 7	**Hayley had 79 stickers in her book, and she was given 22 more. How many stickers does she have now?**

Problem 8	$98 + 43 = \square$

EXTRA PROBLEM:
Use this problem if appropriate.

Problem 9	**There were 298 sheep in the stockyard, and the farmer brought in another 143 sheep. How many sheep are there altogether?**

APPENDIX B: Interview Questions (Post-Unit)

Show the Problem Card and read the problem aloud to the child.

- **For problems 1-4 ask the following questions:**

As we do not have a calculator I need you to think of **one** way you could work the problem out, and **explain** it to me.

Show me how you would record what thinking was happening in your head.
(*Provide child with pencil and paper.*)
Why did you choose this way to solve the problem?

Is there another way you could work it out?
Explain to me how you would do it.
Show me what thinking was happening in your head.
(*Provide child with pencil and paper.*)

Which way is better for solving this problem?
Justify why you think this.

- **For problems 5 and 6 ask the following questions:**

Think of **one** way you could work the problem out, and **explain** it to me.

Show me how you would record what thinking was happening in your head.
(*Provide child with pencil and paper.*)
Why did you choose this way to solve the problem?

Another person solved it this way. (*Show card with notated example.*)
Is it the same as yours? Explain how it is the same or different.

If the child responds that it is different ask:
Which is the better way for solving this problem?
Justify why you think this.

OR
If the child responds that it is the same ask:
Is there a better way for solving this problem?
Explain and justify why you think this.

Problem 1	***65 + 8 =*** □

Problem 2	**At the school camp there are 58 boys and 35 girls. How many children are there altogether?**

Problem 3	**You have 47 lollies and you eat 9 of them. How many have you got left?**

Problem 4	***86 – 32 =*** □

Problem 5	**Hayley had 27 marbles and she won another 17 marbles. How many marbles does she have now?**

Problem 6	**At the Inter-school Athletics Competition Puru won 41 medals and Tahi won 33 medals. How many more medals did Puru win?**

EXTRA PROBLEMS:
For those capable of 3-digit computation ask the following problems to ascertain notational form used.

Problem 7	***786 – 254 =*** □

Problem 8	**There were 298 sheep in the stockyard, and the farmer brought in another 143 sheep. How many sheep are there altogether?**

INTERVIEWER'S CARDS:

Problem 5:

$$27 + 17 = 44$$

$$27 + 17 =$$

$$20 + 10 = 30 \qquad 7 + 7 = 14$$

$$30 + 14 = 44$$

Problem 6:

$$41 - 33 = 8$$

$$41 - 33 =$$

$$41 - 30 = 11$$

$$11 - 3 = 8$$

APPENDIX C: Developmental Research Cycle

Developmental Research Cycle
WEEK 1

DEVELOPMENT PHASE

- The teacher's proactive role in guiding the development of notational schemes supports students' mathematical development.
- Students devise their own notational schemes to express their mathematical thinking.

RESEARCH PHASE

- The teacher redescribes and notates student's verbal explanation of the strategies used. Compare teacher / student notations.
- Students redescribe and notate other children's thinking. Compare notations.
- Students compare notation samples to ascertain what makes "good recording".

Developmental Research Cycle
WEEK 2

DEVELOPMENT PHASE

- Notational schemes provide a way of highlighting critical aspects of different solution strategies contributing to the productiveness of group and whole class discussions.
- Notation allows students' to compare and contrast solutions thereby initiating shifts in discourse.

RESEARCH PHASE

- Students find others in the class who have used the same strategy to solve a problem. Group explains their method, recording steps taken, so others can track their thinking.
- Students use 'Thinking Bubbles' to identify strategies utilised.

Developmental Research Cycle
WEEK 3

DEVELOPMENT PHASE

- Sociomathematical norms form the basis for the way notation is used for learning, and contribute to the development of what is 'taken-as-shared' by the classroom community.

RESEARCH PHASE

- Use the Modelling Book and 'Think Mats' to discuss and decide on what counts as a
  - *different*
  - *sophisticated*
  - *efficient*

  mathematical solution.

Developmental Research Cycle
WEEK 4

DEVELOPMENT PHASE

- Written records allow students to reflect on what they are doing.
- Notational schemes emerge as students attempt to explain and justify their mathematical thinking.

RESEARCH PHASE

- Students use critical mathematical thinking in their argumentation of explanations and justifications for solving problems - both contextual and numerical problems.

APPENDIX D: The Number Framework

(Ministry of Education, 2002a, pp. 10-16)

The Number Framework - Strategies

Global Stage		Addition and Subtraction (Operational Domains)	Multiplication and Division (Operational Domains)	Proportions and Ratios (Operational Domains)
Counting	Zero: Emergent	**Emergent** The student is unable to count a given set or form a set of up to ten objects.		
Counting	One: One-to-one Counting	**One-to-one Counting** The student is able to count a set of objects but is unable to form sets of objects to solve simple addition and subtraction problems.	**One-to-one Counting** The student is able to count a set of objects but is unable to form sets of objects to solve simple multiplication and division problems.	**Unequal Sharing** The student is unable to divide a region or set into two or four equal parts.
Counting	Two: Counting from One on Materials	**Counting - from One** The student solves simple addition and subtraction problems by counting all the objects, e.g., 5 + 4 as 1, 2, 3, 4, 5, 6, 7, 8, 9. The student needs supporting **materials**, like fingers.	**Counting - from One** The student solves multiplication and division problems by counting one to one with the aid of **materials.**	**Equal Sharing** The student is able to divide a region or set into two or more equal parts using **materials**.
Counting	Three: Counting from One by Imaging	**Counting - from One** The student **images** all of the objects and counts them. The student does not see ten as a unit of any kind and solves multi-digit addition and subtraction problems by counting all the objects.	**Counting - from One** The student **images** the objects in simple multiplication and division problems, e.g., 4 × 2 as 1, **2**, 3, **4**, 5, **6**, 7, **8**.	**Equal Sharing** The student is able to share a region or set into two or more equal parts by using **materials** or by **imaging** the materials for simple problems, e.g., $\frac{1}{2}$ of 8.
Counting	Four: Advanced Counting	**Counting On** The student uses counting on or counting back to solve simple addition or subtraction tasks, e.g., 8 + 5 by 8, 9, 10, 11, 12, 13 or 52 – 4 as 52, 51, 50, 49, 48. Initially, the student needs supporting **materials** but later **images** the objects and counts them. The student sees 10 as a completed count composed of 10 ones. The student solves addition and subtraction tasks by incrementing in ones (38, 39, 40, …), tens counts (13, 23, 33, …), and/or a combination of tens and ones counts (27, 37, 47, 48, 49, 50, 51).	**Skip-counting** On multiplication tasks, the student uses skip-counting (often in conjunction with one-to-one counting), e.g., 4 × 5 as 5, 10, 15, 20.	

		Operational Domains		
Global Stage		Addition and Subtraction	Multiplication and Division	Proportions and Ratios
Part-Whole	Five: Early Additive Part-Whole	**Early Addition and Subtraction** The student uses a limited range of mental strategies to estimate answers and solve addition or subtraction problems. These strategies involve deriving the answer from known basic facts, e.g., $8 + 7$ is $8 + 8 - 1$ (doubles) or $5 + 3 + 5 + 2$ (fives) or $10 + 5$ (making tens). Their strategies with multi-digit numbers involve using tens and hundreds as abstract units that can be partitioned, e.g., $43 + 25 = (40 + 20) + (3 + 5) = 60 + 8 = 68$ (standard partitioning) or $39 + 26 = 40 + 25 = 65$ (rounding and compensation).	**Multiplication by Repeated Addition** On multiplication tasks, the student uses a combination of known multiplication facts and derivation from addition fact knowledge, e.g., 4×6 as $(6 + 6) + (6 + 6) = 12 + 12 = 24$. The student uses known multiplication and repeated addition facts to anticipate the result of division, e.g., $20 \div 4 = 5$ because $5 + 5 = 10$ and $10 + 10 = 20$.	**Fraction of a Number by Addition** The student finds a fraction of a number mentally using addition fact knowledge, e.g., $\frac{1}{3}$ of 12 is 4 because $3 + 3 + 3 = 9$, so $4 + 4 + 4 = 12$ or $5 + 5 + 2 = 12$, so $4 + 4 + 4 = 12$. The student estimates answers and solves proportion and ratio problems by replicating the proportion or ratio repeatedly with the support of materials.
	Six: Advanced Additive (Early Multiplicative) Part-Whole	**Advanced Addition and Subtraction** The student can estimate answers and solve addition and subtraction tasks involving whole numbers mentally by choosing appropriately from a broad range of advanced mental strategies, e.g., $63 - 39 = 63 - 40 + 1 = 24$ (rounding and compensating) or $39 + 20 + 4 = 63$, so $63 - 39 = 24$ (reversibility).	**Derived Multiplication** The student uses a combination of known facts and a limited range of mental strategies to derive answers to multiplication and division problems, e.g., $4 \times 8 = 2 \times 16 = 32$ (doubling and halving), or 9×6 is $(10 \times 6) - 6 = 54$ (rounding), or $63 \div 7 = 9$ because $9 \times 7 = 63$ (reversibility).	**Fraction of a Number by Multiplication** The student derives from known multiplication and division facts to estimate answers and solve fractions and proportions problems, e.g., $\frac{1}{3}$ of 36, $3 \times 10 = 30$, $36 - 30 = 6$, $6 \div 3 = 2$, $10 + 2 = 12$ (compensating from a known fact). In the absence of a known related fact, the student will use strategies based on adding or skip-counting.

	Operational Domains		
Global Stage	Addition and Subtraction	Multiplication and Division	Proportions and Ratios
Part-Whole **Seven: Advanced Multiplicative (Early Proportional) Part-Whole**	**Decimal Addition and Subtraction** The student can estimate answers and solve addition and subtraction tasks involving decimal numbers mentally by choosing appropriately from a broad range of advanced mental strategies, e.g., $3.2 + 1.95 = 3 + 2 + 0.2 - 0.05$ $= 5.2 - 0.05$ $= 5.15$ (compensation); $8.65 - 4.2 = (8 - 4) + (0.6 - 0.2) + 0.05$, or $8.65 - 4 = 4.65$, then $4.65 - 0.2 = 4.45$ (place value); $6.03 - 5.8 = \square$ as $5.8 + \square = 6.03$ (reversibility); $\square + 3.98 = 7.04$ as $3.98 + \square = 7.04$ (commutativity).	**Advanced Multiplication and Division** The student is able to choose appropriately from a broad range of mental strategies to estimate answers and solve multiplication and division problems. These strategies involve partitioning one or more of the factors, e.g., $24 \times 6 = (20 \times 6) + (4 \times 6)$ (place value partitioning); $25 \times 6 - 6$ (rounding and compensating); $3 \times 27 = 9 \times 9 = 81$ (trebling and dividing by three); $96 \div 4 = 25 - 1$, since $25 \times 4 = 100$ (reversibility).	**Early Fractions, Ratios, and Proportions** The student uses a range of mental strategies based on multiplication and division to estimate answers and solve problems with fractions, proportions, and ratios. These strategies involve finding equivalent fractions and using unit fractions, e.g., $\frac{3}{4}$ of $24 = (24 \div 4) \times 3 = 6 \times 3 = 18$; e.g., 3:5 as $\square$:40, $8 \times 5 = 40$, $8 \times 3 = 24$ so $\square = 24$. The student renames fractions as decimals and percentages using multiplication and division, e.g., 3 out of 4 is equivalent to 75 out of 100 (multiplying by 25).
Eight: Advanced Proportional Part-Whole		**Multiplication and Division of Fractions and Decimals** The student can estimate answers and solve problems involving the multiplication and division of fractions and decimals using mental strategies. These strategies involve recognising the effect of number size on the answer and converting decimals to fractions where appropriate, e.g., $3.6 \times 0.75 = \frac{3}{4} \times 3.6 = 2.7$ (conversion and commutativity); e.g., $7.2 \div 0.4$ as $7.2 \div 0.8 = 9$ so $7.2 \div 0.4 = 18$ (doubling and halving with place value).	**Advanced Fractions, Ratios, and Proportions** The student chooses appropriately from a broad range of mental strategies to estimate answers and solve problems involving fractions, proportions, and ratios. These strategies involve finding relationships between units of different quantities and converting between fractions, decimals, and percentages, e.g., 6:9 as $\square$:24, $6 \times 1\frac{1}{2} = 9$, $\square \times 1\frac{1}{2} = 24$, $\square = 16$ (between unit multiplying); e.g., 65% of 24: 50% of 24 is 12, 10% of 24 is 2.4 so 5% is 1.2, $12 + 2.4 + 1.2 = 15.6$ (partitioning percentages).

The Number Framework - Knowledge

Stage	Number Identification	Number Sequence and Order	Grouping/Place Value	Basic Facts	Written Recording
Stage Zero: Emergent	The student identifies: • all of the numbers in the range 0–10.	The student says: • the number word sequences, forwards and backwards, in the range 0–10 at least; • the number before and after a given number in the range 0–10. The student orders: • numbers in the range 0–10.	The student instantly recognises: • patterns to 5, including finger patterns.		
Stages One, Two, Three: Counting from One	The student identifies: • all of the numbers in the range 0–20.	The student says: • the number word sequences, forwards and backwards, in the range 0–20; • the number before and after a given number in the range 0–20; • the skip-counting sequences, forwards and backwards, in the range 0–20 for twos and fives. The student orders: • numbers in the range 0–20.	The student knows: • groupings within 5, e.g., 2 and 3, 4 and 1; • groupings with 5, e.g., 5 and 1, 5 and 2 , …; • groupings within 10, e.g., 5 and 5, 4 and 6, … etc. The student instantly recognises: • patterns to 10 (doubles and 5-based), including finger patterns.	The student recalls: • addition and subtraction facts to five, e.g., 2 + 1, 3 + 2, 4 – 2, … etc; • doubles to 10, e.g., 3 + 3, 4 + 4, … etc.	The student records: • the results of counting and operations using symbols, pictures, and diagrams.

Stage	Number Identification	Number Sequence and Order	Grouping/Place Value	Basic Facts	Written Recording
Stage Four: Advanced Counting	The student identifies: • all of the numbers in the range 0–100; • symbols for halves, quarters, thirds, and fifths.	The student says: • the number word sequences, forwards and backwards, in the range 0–100; • the number before and after a given number in the range 0–100; • the skip-counting sequences, forwards and backwards, in the range 0–100 for twos, fives, and tens. The student orders: • numbers in the range 0–100.	The student knows: • groupings with 10, e.g., 10 and 2, 10 and 3, … and the pattern of "-teens"; • groupings within 20, e.g., 12 and 8, 6 and 14; • the number of tens in decades, e.g., tens in 40, in 60.	The student recalls: • addition facts to 10, e.g., $4 + 3$, $6 + 2$, …; • doubles to 20 and corresponding halves, e.g., $6 + 6$, $7 + 7$, $\frac{1}{2}$ of 14; • "ten and facts", e.g., $10 + 4$, $7 + 10$ • multiples of 10 that add to 100, e.g., $30 + 70$, $40 + 60$.	The student records: • the results of mental addition and subtraction, using equations, e.g., $4 + 5 = 9$, $12 - 3 = 9$.
Stage Five: Early Additive	The student identifies: • all of the numbers in the range 0–1000; • symbols for the most common fractions, including at least halves, quarters, thirds, fifths, and tenths; • symbols for improper fractions, e.g., $\frac{5}{4}$.	The student says: • the number word sequences, forwards and backwards, by ones, tens, and hundreds in the range 0–1000; • the number 1, 10, 100 before and after a given number in the range 0–1000; • the skip-counting sequences, forwards and backwards, in the range 0–100 for twos, threes, fives, and tens. The student orders: • numbers in the range 0–1000; • fractions with like denominators, e.g., $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, … etc.	The student knows: • groupings within 100, e.g., 49 and 51 (particularly multiples of 5, e.g., 25 and 75); • groupings of two that are in numbers to 20, e.g., 8 groups of 2 in 17; • groupings of five in numbers to 50, e.g., 9 groups of 5 in 47; • groupings of ten that can be made from a three-digit number, e.g., tens in 763 is 76; • the number of hundreds in centuries and thousands, e.g., hundreds in 800 is 8 and in 4000 is 40. The student rounds: • three-digit whole numbers to the nearest 10 or 100 e.g., 561 rounded to the nearest 10 is 560 and to the nearest 100 is 600.	The student recalls: • addition facts to 20, e.g., $7 + 5$, $8 + 7$; • multiplication facts for the 2, 5, and 10 times tables and the corresponding division facts; • multiples of 100 that add to 1000, e.g., 400 and 600, 300 and 700.	The student records: • the results of addition, subtraction, and multiplication calculations using equations, e.g., $35 + 24 = 59$, $4 \times 5 = 20$, and diagrams, e.g., an empty number line.

Stage	Number Identification	Number Sequence and Order	Grouping/Place Value	Basic Facts	Written Recording
Stage Six: Advanced Additive	The student identifies: • all of the numbers in the range 0–1 000 000; • decimals to three places; • symbols for any fraction including tenths, hundredths, thousandths, and improper fractions.	The student says: • the whole number word sequences, forwards and backwards, by ones, tens, hundreds, and thousands in the range 0–1 000 000; • the number 1, 10, 100, 1000 before and after a given whole number in the range 0–1 000 000; • forwards and backwards word sequences for halves, quarters, thirds, fifths, and tenths, e.g., $\frac{1}{3}$, $\frac{2}{3}$, 1, $\frac{4}{3}$, $\frac{5}{3}$, etc. • the decimal number word sequences, forwards and backwards, in tenths and hundredths. The student orders: • whole numbers in the range 0–1 000 000; • unit fractions for halves, thirds, quarters, fifths, and tenths.	The student knows: • groupings within 1000, e.g., 240 and 760, 498, and 502, …; • groupings of two, three, five, and ten that are in numbers to 100 and finds the resulting remainders, e.g., threes in 17 is 5 with 2 remainder, fives in 48 is 9 with 3 remainder. • groupings of 10 and 100 that can be made from a four-digit number, e.g., tens in 4562 is 456 with 2 remainder, hundreds in 7894. • tenths and hundredths in decimals to two places, e.g. tenths in 7.2 is 72, hundredths in 2.84 is 284. The student rounds: • whole numbers to the nearest 10, 100, or 1000. • decimals with up to two decimal places to the nearest whole number, e.g., rounds 6.49 to 6, rounds 19.91 to 20.	The student recalls: • addition and subtraction facts up to 20, e.g., $9 + 5$, $13 - 7$; • multiplication facts for the two, three, five, and ten times tables and the corresponding division facts; • multiplication facts for squares to 100, e.g., 4×4, 6×6, … etc.	The student: • records the results of calculations using addition, subtraction, multiplication, and division equations, e.g., $349 + 452$ $= 350 + 451$ $= 801$, e.g., $45 \div 9 = 5$, • demonstrates the calculation on a number line or with a diagram. The student performs: • column addition and subtraction with whole numbers of up to four digits.

Stage	Number Identification	Number Sequence and Order	Grouping/Place Value	Basic Facts	Written Recording
Stage Seven: Advanced Multiplicative		The student says: • the decimal word sequences, forwards and backwards, by thousandths, hundredths, tenths, ones, tens, etc.; • the number one-thousandth, one-hundredth, one-tenth, one, ten, etc. before and after any given whole number. The student orders: • decimals to three places, e.g., 6.25 and 6.3; • fractions, including halves, thirds, quarters, fifths, tenths.	The student knows: • the groupings of numbers to 10 that are in numbers to 100 and finds the resulting remainders, e.g., sixes in 38, nines in 68; • the groupings of ten, one hundred, and one thousand that can be made from a number of up to seven digits, e.g., tens in 47 562, hundreds in 782 894, thousands in 2 785 671; • equivalent fractions for halves, thirds, quarters, fifths, and tenths with denominators up to 100 and up to 1000, e.g., 1 in 4 is equivalent to 25 in 100 or 250 in 1000. The student rounds: • whole numbers and decimals with up to two places to the nearest whole number or $\frac{1}{10}$, e.g., rounds 6.49 to 6.5 (nearest tenth).	The student recalls: • multiplications facts to 10×10 and the corresponding division facts; • fraction ↔ decimal ↔ percentage conversions for halves, thirds, quarters, fifths, and tenths, e.g., $\frac{3}{4} = 0.75 = 75\%$.	The student records: • the results of calculations using equations, e.g., $6 \times 28 = 168$, and diagrams, e.g., empty number line. The student performs: • column addition and subtraction for whole numbers; • short multiplication and division of a three-digit whole number by a single-digit number.
Stage Eight: Advanced Proportional		The student says: • the decimal word sequences, forwards and backwards, by thousandths, hundredths, tenths, ones, tens, etc., starting at any decimal number; • the number one-thousandth, one-hundredth, one-tenth, one, ten, etc. before and after any given decimal number. The student orders: • fractions, decimals, and percentages.	The student knows: • the number of tenths, hundredths, and one-thousandths that are in numbers of up to three decimal places, e.g., tenths in 45.6 is 456, hundredths in 9.03 is 903, thousandths in 8.502 is 8502; • what happens when a whole number or decimal is multiplied or divided by a power of 10, e.g., 4.5×100, $67.3 \div 10$. The student rounds: • decimals to the nearest 100, 10, 1, $\frac{1}{10}$, or $\frac{1}{100}$, e.g., rounding 5234 to the nearest 100 gives 5200.	The student recalls: • fraction ↔ decimal ↔ percentage conversions for given fractions and decimals, e.g., $\frac{9}{8} = 1.125 = 112.5\%$.	The student records: • the results of calculations using equations, e.g., $\frac{3}{4} \times 28 = 21$, and diagrams, e.g., double number line. The student performs: • column addition and subtraction for whole numbers, and decimals to three places; • short multiplication and division of whole numbers and decimals by single-digit numbers; • multiplication of three- or four-digit whole numbers by two-digit whole numbers.