# COMPARISON OF THE EUCLIDEAN AND LINEAR DISCRIMINANT FUNCTIONS IN STATISTICAL DISCRIMINANT ANALYSIS

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# Abstract

*It is known that in the problem of statistical discriminant analysis, the linear discriminant function performs poorly when the dimension of the data, p, is large. It has been demonstrated by Marco, Young and Turner (1987) that the much simpler Euclidean distance classifier may out-perform the usual linear discriminant function under certain conditions. Their conclusions were arrived at from a simulation experiment which compared the probabilities of misclassification associated with the Euclidean distance classifier with those of the linear discriminant function, under certain conditions. In this dissertation, the asymptotic expansions of the probabilities of misclassification (the expected actual and expected plug-in error rates) associated with the two discriminant functions are obtained. These error rates are then used to investigate the relative performances of the two methods.*

*Chapter 1 introduces the problem of discriminant analysis and describes the two competing procedures for discriminant analysis and some associated error rates. Then Chapter 2 reviews previous results, in the literature which show that the Euclidean distance classifier can perform better than the linear discriminant function. Chapter 3 gives the asymptotic expansions of the error rates, i.e. the expected actual error rate, and the expected plug-in error rate. The relative performances of the two methods on the basis of the asymptotic expansions are discussed in Chapter 4. The results show that in general the plug-in error rates for the*

*Euclidean distance classifier give better estimates of the actual error rates for all dimensions of p which were considered, when compared to the linear discriminant function. Furthermore, the actual error rates for the Euclidean distance classifier also seem to give better estimates of the true error rates at large dimensions of p, when compared to the linear discriminant function. Certain situations where the linear discriminant function performs better than the Euclidean distance classifier are also identified. Final conclusions, discussions and recommendations for further work are given in Chapter 5.*

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# CHAPTER 1

# INTRODUCTION

## Section 1.1 : Introduction

The basic problem of statistical discriminant analysis is to assign an object, **x**, of unknown origin to one of two (or more) distinct groups on the basis of a set of measurements on the object. It is also important that the classification of an unknown observation to a group be carried out with a low probability of misclassification (which is usually referred to as the "error rate").

In this project we consider only two distinct groups, $\Pi_1$ and $\Pi_2$, which have multivariate normal distributions. Then, as shown in Figure1.1, our basic problem is to classify an object with observation **x**, of unknown origin, to population $\Pi_1$ with probability density function $f_1(\mathbf{X})$ or to population $\Pi_2$ with probability density function $f_2(\mathbf{X})$. In this dissertation $f_i(\mathbf{X})$ denotes a multivariate normal distribution of dimension p with mean vector $\boldsymbol{\mu}_i$ and covariance matrix $\Sigma$ i.e. $N_p(\boldsymbol{\mu}_i, \Sigma)$ for i=1,2.

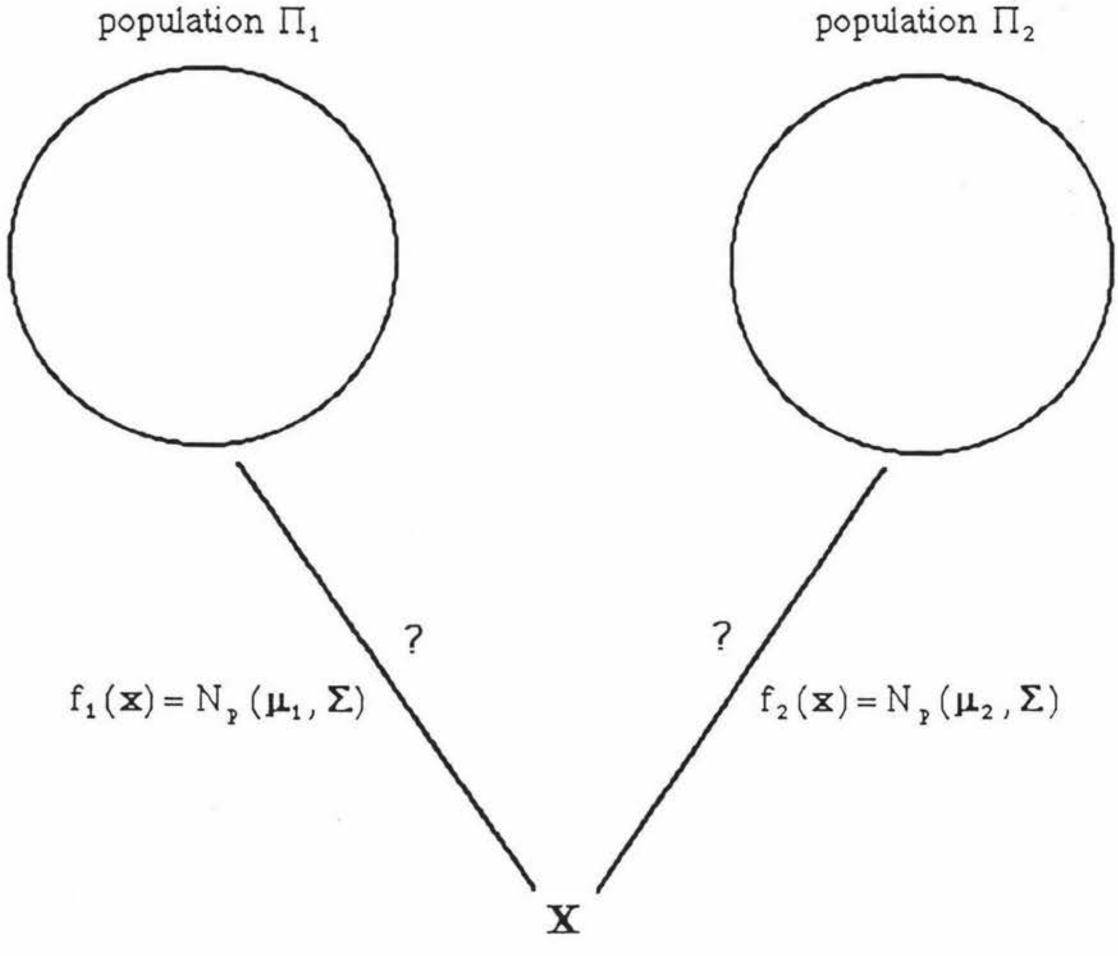


**Figure 1.1** : Illustration of a basic problem of statistical discriminant analysis with two populations.

The following example illustrates the fundamental features of discriminant analysis.

Example 1.1

Consider two populations $\Pi_1$ and $\Pi_2$ of two distinct varieties of wheat. Suppose that for each member (plant) of these two populations, the following observations are made

$x_1$ plant heights (cm)

$x_2$ number of effective tillers

$x_3$ length of ear (cm)

$x_4$ number of fertile spikelets per 10 ears

$x_5$ number of grains per 10 ears

$x_6$ weight of grains per 10 ears (gm).

Assuming that the two groups (populations) are six dimensional normal populations with different (unknown) mean vectors $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$ and the same (unknown) covariance matrix $\Sigma$, discriminant analysis considers the problem of classifying a plant with observation $\mathbf{x}=(x_1,x_2,x_3,x_4,x_5,x_6)^T$ to one of these two varieties. In this report $\mathbf{x}^T$ denotes the transpose of vector or matrix $\mathbf{x}$. To do this we usually develop a classification rule (or discriminant function) which, in most cases, is a function of the measurements on the particular plant $\mathbf{x}$, and of the parameters of the distributions, namely $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$ and $\Sigma$. Naturally, the classification should be done with some acceptably small probability of misclassification.

Section 1.2 : Classification Rules

As mentioned earlier, in order to classify the observation, **x**, of unknown origin into population $\Pi_1$ or $\Pi_2$, we need a classification rule. There are several classification rules which have been developed on the basis of different optimality criteria. For example, the quadratic discriminant function, the standard linear discriminant function and the Euclidean distance classifier are three of the many classification rules (see, for example Raudys and Pikelis (1980), Seber (1984, Chapter 6)). In this dissertation we focus on only two classification procedures, namely the linear discriminant function and the Euclidean distance classifier. We shall refer to these procedures as the LDF and the EDC respectively. Of these two rules, the linear discriminant function is used more widely .

It was mentioned earlier that we consider the situation where there are only two populations. Thus, if **R** denotes the p-dimensional space of all possible values of **X**, then forming a rule of classification involves partitioning **R** into two mutually exclusive regions $\mathbf{R}_1$ and $\mathbf{R}_2$. Suppose we classify the object with observation **X** to $\Pi_1$ if it falls in $\mathbf{R}_1$ and to $\Pi_2$ if it falls in $\mathbf{R}_2$. Sometimes an object will be classified to the wrong group and a good classification rule aims to keep the probability of misclassification as small as possible. Let P(i/j) be the probability of misclassifying an object into $\Pi_i$ when it actually belongs to $\Pi_j$ $(i \neq j)$. Then

$$P(i / j) = \int_{\mathbf{R}_i} f_j(\mathbf{X})\, d\mathbf{X} \qquad (i \neq j = 1, 2).$$

Let $q_i$ be the prior probability that the observation comes from $\Pi_i$, i.e. the proportion of the ith group in the general population. Then, the total probability of misclassification is given by

$$T(\mathbf{R},f) = q_1 P(2/1) + q_2 P(1/2)$$

$$= q_1 \int_{\mathbf{R}_2} f_1(\mathbf{X}) \ d\mathbf{X} + q_2 \int_{\mathbf{R}_1} f_2(\mathbf{X}) \ d\mathbf{X}$$

$$= q_1 [1 - \int_{\mathbf{R}_1} f_1(\mathbf{X}) \ d\mathbf{X}] + q_2 \int_{\mathbf{R}_1} f_2(\mathbf{X}) \ d\mathbf{X}$$

$$\therefore T(\mathbf{R},f) = q_1 + \int_{\mathbf{R}_1} [q_2 f_2(\mathbf{X}) - q_1 f_1(\mathbf{X})] \ d\mathbf{X} \qquad (1.1)$$

To minimise the total probability of misclassification we choose $\mathbf{R}_1$ so that $[q_2 f_2(\mathbf{X}) - q_1 f_1(\mathbf{X})] < 0$ for all $\mathbf{X}$ in $\mathbf{R}_1$. Thus the classification rule is to classify $\mathbf{X}$ to $\Pi_1$ if $(f_1(\mathbf{X})/f_2(\mathbf{X})) > (q_2/q_1)$. Otherwise it is classified to $\Pi_2$; see, for example Seber (1984, Chapter 6).

For multivariate normal populations, the classification rule which is obtained following the minimization of the probability of misclassification is the linear discrimination function, given by :

Classify individual with observation $\mathbf{x}$ to population $\Pi_1$ if

$$D_L(\mathbf{x}) > k ,$$

or classify individual with observation $\mathbf{x}$ to population $\Pi_2$ if

$$D_L(\mathbf{x}) \leq k ,$$

where $D_L(\mathbf{x}) = (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)^T \boldsymbol{\Sigma}^{-1} [\mathbf{x} - 1/2(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)]$. *(1.2)*

The subscript "L" denotes the Linear discriminant function. The constant k is usually taken to be log of $(q_2 / q_1)$. Details are available in Anderson (1984, Chapter 6), Lachenbruch (1975) and Seber (1984, Chapter 6), for example. In this project, we choose k to be zero which means that the two populations are assumed to occur in equal proportions. In this case, the linear discriminant function follows a Bayes procedure.

An alternative classification rule is the Euclidean distance classifier. In this rule we classify an individual with observation $\mathbf{x}$ to population $\Pi_1$ if

$$D_E(\mathbf{x}) > k,$$

or classify individual with observation $\mathbf{x}$ to population $\Pi_2$ if

$$D_E(\mathbf{x}) \leq k,$$

where $D_E(\mathbf{x}) = (\mu_1 - \mu_2)^T [\mathbf{x} - 1/2(\mu_1 + \mu_2)]$. *(1.3)*

The subscript "E" denotes the Euclidean distance classifier. Again, we choose k to be zero.

Comparing the algebraic expressions for $D_L(\mathbf{x})$ and $D_E(\mathbf{x})$ we can see that the Euclidean distance classifier is a simpler discriminant function than the linear discriminant function since it requires no matrix inversion. It thus avoids (i) the difficulties of inverting the covariance matrix when the data dimension is large relative to the training sample size and (ii) the problem associated with getting reliable estimates of the covariance matrix when sample sizes are small (when we use sample-based classification rules). In terms of performance, the linear discriminant function is known to perform poorly when dimension p becomes large (relative to sample size, n).

Section 1.3 : Error rates

The discriminant functions $D_L(\mathbf{x})$ and $D_E(\mathbf{x})$ as given in expressions *(1.2)* and *(1.3)* are the population linear discriminant function and the population Euclidean distance classifier repectively. They are obtained by assuming complete knowledge of the parameters of the probability density functions $f_1(\mathbf{X})$ and $f_2(\mathbf{X})$. As is well known, in most practical situations the parameters are not known.

Consider the situation when all population parameters are known. In this case the overall error rate (probability of misclassification) associated with the linear discriminant function, which is denoted here by $p^L$, is given by

$$p_L = q_1 p_{21}^{(L)} + q_2 p_{12}^{(L)} \quad , \qquad (1.4)$$

where $p_{ij}$ is the probability of misclassifying an observation $\mathbf{x}$ into population i when it actually belongs to population j, $(i \neq j = 1, 2)$. Since it is assumed that $q_1 = q_2 = 1/2$, $p_{21}^{(L)}$ is given by

$$p_{21}^{(L)} = \Pr[D_L(\mathbf{x}) < 0 \mid \mathbf{x} \in \Pi_1]$$

$$p_{21}^{(L)} = \Phi\left(-\frac{1}{2}[(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)]^{1/2}\right) \qquad (1.5)$$

$$p_{21}^{(L)} = \Phi(-\Delta / 2)$$

where $$\Delta = \left\{(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\right\}^{\frac{1}{2}} \qquad (1.5a)$$

is the so called "Mahalanobis distance" between the two populations, and $\Phi(\cdot)$ is the distribution function of a standard normal variable. In an obvious notation, the subscript or superscript "L" in expressions *(1.4)* and *(1.5)* refer to the Linear discriminant function. Note that $p_{21}^{(L)} = p_{12}^{(L)}$, so that $p_{21}^{(L)} = p_L$ in this case.

For the Euclidean distance classifier, the error rate is given by

$$p_E = q_1 p_{21}^{(E)} + q_2 p_{12}^{(E)} , \qquad (1.6)$$

where
$$p_{12}^{(E)} = \Phi[-\frac{1}{2}\frac{(\mu_1 - \mu_2)^T(\mu_1 - \mu_2)}{[(\mu_1 - \mu_2)^T \Sigma(\mu_1 - \mu_2)]^{1/2}}]. \qquad (1.7)$$

In this case $p_{12}^{(E)} = p_{21}^{(E)}$, and $p_{12}^{(E)} = p_E$.

The subscript or superscript "E" in expressions *(1.6)* and *(1.7)* refer to the Euclidean distance classifier.

When the parameters, $\mu_1$, $\mu_2$ and $\Sigma$ in the expressions *(1.5)* and *(1.7)* are unknown, they are replaced by $\bar{x}_1$, $\bar{x}_2$ and S respectively, where $\bar{x}_1$, $\bar{x}_2$, and S are the sample means and pooled sample covariance matrix respectively. After this replacement of $\mu_1$, $\mu_2$ and $\Sigma$ by $\bar{x}_1$, $\bar{x}_2$, and S, the sample linear discriminant function, $D_{SL}(x)$ and the sample Euclidean distance classifier, $D_{SE}(x)$ are obtained. These functions are

$$D_{SL}(x) = (\bar{x}_1 - \bar{x}_2)^T S^{-1}[x - 1/2(\bar{x}_1 + \bar{x}_2)] \qquad (1.8)$$

and

$$D_{SE}(x) = (\bar{x}_1 - \bar{x}_2)^T [x - 1/2(\bar{x}_1 + \bar{x}_2)]. \qquad (1.9)$$

It is clear that the sample Euclidean distance classifier ( just like the population Euclidean distance classifier ) is a simpler function since it does not require the inversion of a matrix.

In the situation where the parameters are unknown we consider two error rates, namely the "actual" error rates and the "plug-in" error rates (see, for example, Seber (1984, Chapter 6)) which are both associated with the sample linear discriminant function and the sample Euclidean distance classifier. The actual error rate is the error rate associated with the sample discriminant function as it will perform in future samples. For the sample linear discriminant function and the sample Euclidean distance classifier the actual error rates are given by

$$p_{A}^{(L)} = \frac{1}{2} p_{12A}^{(L)} + \frac{1}{2} p_{21A}^{(L)} \tag{1.10}$$

and

$$p_{A}^{(E)} = \frac{1}{2} p_{12A}^{(E)} + \frac{1}{2} p_{21A}^{(E)} \tag{1.11}$$

respectively, where

$$p_{21A}^{(L)} = \Phi\left( -\frac{[\boldsymbol{\mu}_1 - 1/2(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2)]^T \mathbf{S}^{-1} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)}{[(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T \mathbf{S}^{-1} \boldsymbol{\Sigma} \mathbf{S}^{-1} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)]^{1/2}} \right), \tag{1.10a}$$

$$p_{12A}^{(L)} = \Phi\left( \frac{[\boldsymbol{\mu}_2 - 1/2(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2)]^T \mathbf{S}^{-1} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)}{[(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T \mathbf{S}^{-1} \boldsymbol{\Sigma} \mathbf{S}^{-1} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)]^{1/2}} \right), \tag{1.10b}$$

$$p_{21A}^{(E)} = \Phi\left( -\frac{[\boldsymbol{\mu}_1 - 1/2(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2)]^T (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)}{[(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T \boldsymbol{\Sigma} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)]^{1/2}} \right), \tag{1.11a}$$

and

$$p_{12A}^{(E)} = \Phi\left( \frac{[\boldsymbol{\mu}_2 - 1/2(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2)]^T (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)}{[(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T \boldsymbol{\Sigma} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)]^{1/2}} \right). \qquad (1.11b)$$

The subscript "A" refers to the actual error rate and the superscripts "L" and "E" refer to the Linear discriminant function and the Euclidean distance classifier, respectively.

The plug-in error rate is the error rate obtained by replacing the unknown parameters in the actual error rate by their estimators. It is thus considered as an estimator of the actual error rate. The plug-in error rates for the sample linear discriminant function and the sample Euclidean distance classifier are

$$p_P^{(L)} = \frac{1}{2} p_{12P}^{(L)} + \frac{1}{2} p_{21P}^{(L)} \qquad (1.12)$$

and

$$p_P^{(E)} = \frac{1}{2} p_{12P}^{(E)} + \frac{1}{2} p_{21P}^{(E)} \qquad (1.13)$$

respectively, where

$$p_{12P}^{(L)} = p_{21P}^{(L)} = \Phi\left( -\frac{1}{2} [(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T \mathbf{S}^{-1} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)]^{1/2} \right) \qquad (1.12a)$$

and

$$p_{12P}^{(E)} = p_{21P}^{(E)} = \Phi\left( -\frac{1}{2} \frac{(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)}{[(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T \mathbf{S} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)]^{1/2}} \right). \qquad (1.13a)$$

Here, the subscript "P" refers to the plug-in error rates.

It should be noted that the unconditional distribution of the sample linear discriminant function, $D_{SL}(\mathbf{x})$, is unknown so that the exact unconditional error rates are unobtainable. However, an asymptotic expansion of the unconditional error rates can be obtained leading to the asymptotic expected actual error rates (see Okamoto, 1963 and McLachlan, 1974).

It is also possible to get exact conditional error rates (the plug-in error rates) for the linear discriminant function since the distribution of $D_{SL}(\mathbf{x})$ given $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ and S is normal. It can be seen from *(1.13a)* that the expressions for $p_{12P}^{(E)}$ and $p_{21P}^{(E)}$ contain the unknown $\Sigma$ .

Section 1.4 : Aim of study

The aim of this project is to compare the performances of the linear discriminant function and the Euclidean distance classifier. Their relative performances will be assessed using the expected actual error rates and the expected plug-in error rates associated with them. It was noted in the previous sections that the Euclidean distance classifier is a simpler function when compared to the linear discriminant function. It is of interest to investigate how well this simpler classification procedure performs against the linear discriminant function under various situations.

# *CHAPTER 2*

## REVIEW OF PREVIOUS RELATED WORK

### Section 2.1 : Introduction

The articles which have been used as the main references for this project are Marco, Young and Turner (1987), Raudys and Pikelis (1980) and Peck and Van Ness (1982). The main point about these articles is that they compared the Euclidean distance classifier and the linear discriminant function (plus other discriminant functions, in some cases) under various conditions and assumptions. The main results in these articles are summarized in this chapter. It was the work and results of these authors that motivated the work in this project.

### Section 2.2 : S. Raudys and V. Pikelis (1980)

Raudys and Pikelis compared the sample Euclidean distance classifier with three other discriminant functions when allocating individuals from two spherical normal populations. The three other discriminant functions were the quadratic discriminant function, the usual linear discriminant function and the linear discriminant function for independent measurements where the sample covariance matrix is replaced by the diagonal matrix of the sample covariance matrix. Furthermore, all the discriminant functions used follow the Bayes rule for normal populations and differ in assumptions on the structures of the covariance matrices. They represented the sample Euclidean distance classifier as the difference between two independent non-central chi-square random variables. In order to calculate the expected probability of misclassification, they used an inversion formula of Imhof (1961) for the distribution function of a quadratic form of normal variables. Thus, the error rates are obtained through numerical integration. From their simulation results they concluded that the sample Euclidean distance classifier outperforms the sample linear discriminant function when p (the dimension of **x**) is large relative to the training sample size. Furthermore, they noted that the sample Euclidean distance classifier performs as well as or superior to the sample linear discriminant function, even for nonspherical covariance configurations.

## Section 2.3 : R. Peck and J. Van Ness (1982)

In their paper Peck and Van Ness noted that the linear discriminant function has been shown to frequently behave poorly in high dimensions relative to other discriminant functions, even on suitable Gaussian data. This was due to the poor quality of the sample estimates of the means and covariance matrix used in the discriminant functions. Therefore, they used a shrinkage estimator (see, for example, B in expression *(2.1)* ) of the covariance matrix in the linear discriminant function ( given in expression *(1.2)* ). The idea is that **B** is more stable than **S** when the dimension of the data is large. Thus the linear discriminant function becomes $D_L(\mathbf{x}) = (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)^T \mathbf{B}\,(\mathbf{x} - \frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2))$.

There are several shrinkage estimators, for example the characteristic roots (see Stein (1975)), the correlation matrix method (see Lin (1978)) and the empirical Bayes method (see Haff (1979,1980)). In order to compare the performances using shrinkage estimators and the standard linear discriminant function, Peck and Van Ness chose the empirical Bayes method where the sample estimates of the population covariance matrix was replaced by a function of the sample covariance matrix (called the Bayes estimator),

$$\mathbf{B} = [1 - t(U)][2N - p - 3]\mathbf{S}^{-1} + [t(U)b / TR(S)]\mathbf{I}, \qquad (2.1)$$

where b is a positive constant, $U = [p\ \det(S)^{1/p}] / TR(S)$ is a measure of disparity among the sample eigenvalues ( it is the geometric mean divided by the arithmetic mean ) and the function t is a non-decreasing solution to $(2N-p-3)t^2-4t+(4U/p)t'<0$ where $TR(S) \equiv trace(S)$ and $0 \le t(U) \le 1$. They assumed that $\boldsymbol{\Sigma} = \sigma^2\,\mathbf{I}$, so that $[b / TR(S)]\mathbf{I}$ is a natural estimator of $\boldsymbol{\Sigma}^{-1}$ and chose b=p(2N-2)-2 because it yielded the unbiased estimate of $\boldsymbol{\Sigma}^{-1}$ .

Their simulation results showed that the discriminant function using shrinkage estimators outperformed the standard linear discriminant function in most cases. However, this performance is highly dependent on the Mahalanobis distance between the two populations. They also concluded that if the Euclidean distance between the means is small then the shrinkage estimator is of little effect because estimating the means is more damaging to the probability of correct classification when compared to the damage caused by the poor estimation of the covariance matrix. They have some other relevant results which are discussed and compared with the results of Marco, Young and Turner (1987) in the next section.

## Section 2.4 : V. R. Marco, D. M. Young and D.W. Turner (1987)

In their article, the authors compared the performances of the linear discriminant function and the Euclidean distance classifier via a simulation study in the cases of "equivalence" or "non-equivalence" of the linear discriminant function and the Euclidean distance classifier. "Equivalence" here means that the error rates of the Euclidean distance classifier and the linear discriminant function are the same, and so both are Bayes procedures. In order to perform the comparison, they derived conditions for which the two classifiers are equivalent when all parameters are known and then performed a Monte Carlo simulation experiment.

A trivial case of "equivalence" is when $\mathbf{\Sigma} = \mathbf{I}$ , which means that the linear discriminant function is indeed the Euclidean distance classifier. For the non-trivial case of equivalence of the linear discriminant function and the Euclidean distance classifier, consider the situation of known parameters and let $\mathbf{F}^+$ be the pseudoinverse of $\mathbf{F}_{(p\times 1)}$. If

$(\mu_1 - \mu_2)(\mu_1 - \mu_2)^+$ and $\Sigma^{-1}$ are commutative then it can be established that $p_E = p_L$, i.e. the error rates of the Euclidean distance classifier and the linear discriminant function are equal. Therefore from expressions *(1.5)* and *(1.7)*, we have

$$\frac{(\mu_1 - \mu_2)^T(\mu_1 - \mu_2)}{[(\mu_1 - \mu_2)^T \Sigma(\mu_1 - \mu_2)]^{1/2}} = [(\mu_1 - \mu_2)^T \Sigma^{-1}(\mu_1 - \mu_2)]^{1/2}. \qquad (2.2)$$

It follows from expression *(2.2)* that, if

(i) we choose $\mu_1 = (m, \ldots \ldots \ldots, m)^T$ and $\mu_2 = (0, \ldots \ldots \ldots, 0)^T$ where m is some scalar quantity, then we get the situation of "equivalence" of the linear discriminant function and the Euclidean distance classifier, and

(ii) we choose $\mu_1 = (m^*, 0, \ldots \ldots, 0)^T$ and $\mu_2 = (0, \ldots \ldots \ldots, 0)^T$, where $m^*$ is some other scalar quantity (not necessarily equal to m), then we get the situation of "non-equivalence" of the linear discriminant function and the Euclidean distance classifier.

Their simulation results showed that the Euclidean distance classifier performed better (with respect to the probability of correct classification) than the linear discriminant function under several situations. However, this was not the case in all situations. Infact their results showed that the relative performance is dependent on the ratio of the Mahalanobis distance to the Euclidean distance. The Euclidean distance classifier outperformed the linear discriminant function when the ratio is small and the linear discriminant function outperformed the Euclidean distance classifier when the ratio is large.

Their simulation results also showed that in most cases the sample Euclidean distance classifier outperformed (or did as well as) the sample linear discriminant function when the underlying parameter configurations are such that the Euclidean distance classifier is equivalent to or non-equivalent to the linear discriminant function (with all parameters known).

Comparison of their results with those of Peck and Van Ness (1982) show that the simpler sample Euclidean distance classifier performs as well as or better than the linear discriminant function using a shrinkage estimator (expression *2.1* ) of $\mathbf{\Sigma}$, except for the case when the Euclidean distance classifier is equivalent to the linear discriminant function and $\rho = -0.06$, where $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$, $-1/(p-1) \leq \rho \leq 1$.

It is important to note that their conclusions were arrived at through a simulation study.

Section 2.5 : Motivation for this project

As mentioned in the previous sections the expansion of the error rates given by Raudys and Pikelis (1980) involves numerical integration, which is not always easy to do and they consider the trivial case of equivalence of the linear discriminant function and the Euclidean distance classifier, where $\mathbf{\Sigma} = \mathbf{I}$. Also the results of Marco, Young and Turner (1987) for comparing the performances of the linear discriminant function and the Euclidean distance classifier were obtained through simulations only. It is therefore of interest to compare the performances of the Euclidean distance classifier and the linear discriminant function via asymptotic expansions (of error rates) under the same conditions as those used by Marco, Young and Turner (1987). In other words we do not use numerical integration, do not carry out simulation and consider the nontrivial case of equivalence, where $\mathbf{\Sigma} \neq \mathbf{I}$. We are interested in investigating if the simpler Euclidean distance classifier is "better" than the linear discriminant function on the basis of these asymptotic expansions of the error rates. It is also of interest to determine if the results and deductions arrived at from the asymptotic expansions are consistent with the simulation results of Marco, Young and Turner (1987).

# CHAPTER 3

## ASYMPTOTIC EXPANSIONS OF ERROR RATES

### Section 3.1 : Introduction

It was mentioned in Chapter 2 that our interest is in comparing the performances of the linear discriminant function and the Euclidean distance function via asymptotic expansions. These asymptotic expansions are used to obtain the expectations of the error rates, or unconditional error rates. The asymptotic expansions used is the Taylor series expansion. In particular, if we let H be a function of parameters $\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_s$ then, following the Taylor series expansion about the point $(\beta_1, \beta_2, \ldots, \beta_s)$, E(H) can be expressed as

$$E(H) \cong H(\beta_1, \beta_2, \ldots, \beta_s) + \sum_{j=1}^{s} \frac{\partial H}{\partial \hat{\beta}_j} E(\hat{\beta}_j - \beta_j)$$

$$+ \frac{1}{2} \sum_{i,j}^{s} \frac{\partial^2 H}{\partial \hat{\beta}_i \partial \hat{\beta}_j} E(\hat{\beta}_i - \beta_i)(\hat{\beta}_j - \beta_j) . \qquad (3.1)$$

In our expansion, $H = \Phi(\cdot)$ and the parameters which are used ( i.e. $\hat{\boldsymbol{\beta}}_1, \hat{\boldsymbol{\beta}}_2, \ldots, \hat{\boldsymbol{\beta}}_s$ ) are the elements of $\bar{\mathbf{x}}_1$, $\bar{\mathbf{x}}_2$ and $\mathbf{S}$. Note that the expression is evaluated at the point $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$ and $\Sigma$.

To obtain the partial derivatives terms, some results obtained by Dwyer (1967) and Okamoto (1963) were used. For example when the covariance matrix $\Sigma$ is symmetric and invertible, and we let $\boldsymbol{\Sigma}^{-1} = \{\sigma^{ij}\}$, where $\{\sigma^{ij}\}$ is a function of $\sigma_{rs}$, then the result

$$\frac{\partial(\sigma^{ij})}{\partial\sigma_{rs}} = -\frac{1}{(1+\delta_{rs})}(\sigma^{ir}\sigma^{sj} + \sigma^{is}\sigma^{rj}), \qquad (r \le s) \qquad (3.2)$$

where $\delta_{rs}$ is the Kronecker delta, is given in Okamoto (1963). In expression *(3.2)* $\sigma_{rs}$ represents the (r,s)th element of $\boldsymbol{\Sigma}$ .

In comparing the Euclidean distance classifier and the linear discriminant function, we consider the actual error rates and the plug-in error rates separately. They are considered in sections 3.2 and 3.3 respectively. For each error rate (i.e. actual or plug-in) the comparison is done under four different categories, namely cases A1, A2, A3 and A4 for the actual error rates and cases P1, P2, P3 and P4 for the plug-in error rates. Cases A1, A2, P1 and P2 consider the situation of "non-equivalence" of the linear discriminant function and the Euclidean distance classifier. Cases A3, A4, P3 and P4 consider the situation of "equivalence" of the linear discriminant function and the Euclidean distance classifier. As stated in Section 2.4, "non-equivalence" of the linear discriminant function and the Euclidean distance classifier arises when $\boldsymbol{\mu}_1 = (m^*, 0, \ldots\ldots, 0)^T$ and $\boldsymbol{\mu}_2 = (0, \ldots\ldots, 0)^T$, and "equivalence" occurs when $\boldsymbol{\mu}_1 = (m,\ldots\ldots,m)^T$ and $\boldsymbol{\mu}_2 = (0,\ldots\ldots,0)^T$. For each situation of "equivalence" or "non-equivalence" we consider two distinct structures of $\boldsymbol{\Sigma}$, namely (i) $\boldsymbol{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$, $-1/(p-1) \le \rho \le 1$, and (ii) $\boldsymbol{\Sigma}$ representing the autocorrelation structure of an autoregressive process of order 1, i.e.

$$\boldsymbol{\Sigma} = AR(1) = \begin{bmatrix} 1 & \rho & \cdots & \rho^{p-1} \\ \rho & 1 & \cdots & \rho^{p-2} \\ \vdots & & & \vdots \\ \rho^{p-1} & \cdots & \rho & 1 \end{bmatrix} .$$

Details of each category (case) are explained in the following sections. together with their respective asymptotic expansions. Note that for clarity, in the expressions of the asymptotic expansions m* in the case of equivalence is just denoted as m. This also occurs in the appendices. In the following sections, $\phi(\cdot)$ is the probability density function of a standard normal variable and $\Phi(\cdot)$ is as defined in after expression *(1.5)* .

## Section 3.2 : Asymptotic expansions for the actual error rate

The actual error rates associated with the linear discriminant function and the Euclidean distance classifier are given in expressions *(1.10)* and *(1.11)* respectively. From these functions, the asymptotic expansions were obtained using the Taylor series expansion as given in expression *(3.1)*. The following sections give the asymptotic expansions under the four different categories stated in section 3.1. Note that in all these expansions $n_1$ and $n_2$ denote the sizes of samples from populations (groups) $\Pi_1$ and $\Pi_2$ respectively, and $\sigma_{ij}$ denotes the (i,j)th element of the matrix $\mathbf{\Sigma}$ .

Section 3.2.1 : Case A1

Here we consider the case of non-equivalence of the linear discriminant function and the Euclidean distance classifier under the following conditions:

$$\boldsymbol{\mu}_1 = (m^*, 0, \ldots\ldots, 0)^T$$

$$\boldsymbol{\mu}_2 = (0, \ldots\ldots\ldots, 0)^T$$

$$\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J} = \begin{bmatrix} 1 & \rho & \cdots\cdots & \rho \\ \rho & 1 & & \rho \\ \vdots & & & \vdots \\ \rho & & \cdots\cdots\cdots & 1 \end{bmatrix} \quad , \quad -1/(p-1) \le \rho \le 1.$$

The asymptotic expansion of the expected actual error rate associated with the Euclidean distance classifier is given by

$$p_A^{(E)} = \Phi\left(-\frac{m}{2}\right) + \frac{1}{2n_1}\sum_{i,j=1}^{p} \frac{\partial^2 \Phi}{\partial \bar{x}_{1i}\partial \bar{x}_{1j}}\sigma_{ij} + \frac{1}{2n_2}\sum_{i,j=1}^{p} \frac{\partial^2 \Phi}{\partial \bar{x}_{2i}\partial \bar{x}_{2j}}\sigma_{ij} \qquad (3.3)$$

where

$$\frac{\partial^2 \Phi}{\partial \bar{x}_{1i}\partial \bar{x}_{1j}} = \frac{1}{2}\phi\left(-\frac{m}{2}\right) \times \begin{cases} \frac{1}{m}\left[\sigma_{ji} - 3\sigma_{1i}\sigma_{j1} + 2\right] + \frac{1}{4}m\sigma_{j1}\sigma_{i1} & \text{if} \quad i = j \\ \frac{1}{m}\left[\sigma_{ji} - 3\sigma_{1i}\sigma_{j1}\right] + \frac{1}{4}m\sigma_{j1}\sigma_{i1} & \text{if} \quad i \ne j \end{cases}$$

and

$$\frac{\partial^2 \Phi}{\partial \bar{x}_{2i}\partial \bar{x}_{2j}} = -\phi\left(-\frac{m}{2}\right)\begin{cases} \frac{1}{m}\left[\frac{3}{2}\left[\sigma_{1i}\sigma_{j1} - \sigma_{j1}\right] + 1 - \sigma_{1i}\right] - \frac{m}{8} & \text{if } i = j = 1 \\ \frac{1}{2m}\left[3\sigma_{1i}\sigma_{j1} - 3\sigma_{j1}\right] + \frac{m}{8}\sigma_{j1} & \text{if } i = 1, j \neq 1, i \neq j \\ \frac{1}{m}\left[\frac{1}{2}\left[3\sigma_{1i}\sigma_{j1} - \sigma_{ji}\right] - \sigma_{1i}\right] + \frac{m}{8}\sigma_{i1} & \text{if } i \neq 1, j = 1, i \neq j \\ \frac{1}{m}\left[\frac{1}{2}\left[3\sigma_{1i}\sigma_{j1} - \sigma_{ji}\right] + 1\right] - \frac{m}{8}\sigma_{i1}\sigma_{j1} & \text{if } i \neq 1, j \neq 1, i = j \\ \frac{1}{m}\left[\frac{1}{2}\left[3\sigma_{1i}\sigma_{j1} - \sigma_{ji}\right]\right] - \frac{m}{8}\sigma_{i1}\sigma_{j1} & \text{if } i \neq 1, j \neq 1, i \neq j \end{cases}$$

The corresponding asymptotic expansion for the linear discriminant function is given by

$$\begin{aligned} p_A^{(L)} = \Phi&\left(-\frac{m}{2}s^{11}\{\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\}^{-\frac{1}{2}}\right) \\ &+ \frac{1}{2n_1}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}\sigma_{ij} + \frac{1}{2n_2}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}\sigma_{ij} \\ &+ \frac{1}{2}\frac{n_1+n_2}{(n_1+n_2-2)^2}\sum_{k,l,j,i=1}^{p}A_{klij}\times(\sigma_{ik}\sigma_{jl}+\sigma_{il}\sigma_{jk}) \end{aligned} \qquad (3.4)$$

In expression *(3.4)* , $\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}$, $\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}$ and $A_{klij}$ are defined in expressions *(A1.26)* , *(A1.31)* and *(A1.36)* respectively (i.e. in appendix A1.2).

Section 3.2.2 : Case A2

Here we consider the case of non-equivalence of the linear discriminant function and the Euclidean distance classifier under the following conditions:

$$\boldsymbol{\mu}_1 = (m^*, 0, \ldots\ldots, 0)^T$$

$$\boldsymbol{\mu}_2 = (0, \ldots\ldots\ldots, 0)^T$$

$$\boldsymbol{\Sigma} = AR(1) = \begin{bmatrix} 1 & \rho & \cdots\cdots & \rho^{p-1} \\ \rho & 1 & & \rho^{p-2} \\ \vdots & & & \vdots \\ \rho^{p-1} & \cdots\cdots & \rho & 1 \end{bmatrix} .$$

The asymptotic expansion of the expected actual error rate for the Euclidean distance classifier is denoted by $p_A^{(E)}$ and is identical algebraically to expression *(3.3)* except that the structure of $\boldsymbol{\Sigma}$ is now as given for this case. The asymptotic expansion associated with the linear discriminant function is

$$\begin{aligned} p_A^{(L)} = \Phi\left( -\frac{m}{2} s^{11} \{ \sum_{v=1}^{p} [ s^{v1} \sum_{u=1}^{p} ( s^{1u} \sigma_{uv} ) ] \}^{-\frac{1}{2}} \right) \\ + \frac{1}{2n_1} \sum_{i,j=1}^{p} \frac{\partial^2 \Phi}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}} \sigma_{ij} + \frac{1}{2n_2} \sum_{i,j=1}^{p} \frac{\partial^2 \Phi}{\partial \bar{x}_{2i} \partial \bar{x}_{2j}} \sigma_{ij} \\ + \frac{1}{2} \frac{n_1 + n_2}{(n_1 + n_2 - 2)^2} \sum_{k,l,j,i=1}^{p} A_{klij} \times ( \sigma_{ik} \sigma_{jl} + \sigma_{il} \sigma_{jk} ) \end{aligned} \qquad (3.5)$$

The expressions $\frac{\partial^2 \Phi}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}}, \frac{\partial^2 \Phi}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}}$ and $A_{klij}$ are as defined in expressions *(A1.26)* , *(A1.31)* and *(A1.41)* respectively (i.e. in appendix A1.2).

Section 3.2.3 : Case A3

In this case of equivalence of the linear discriminant function and the Euclidean distance classifier, the following conditions hold :

$$\boldsymbol{\mu}_1 = (m^*, 0, \ldots\ldots, 0)^T$$

$$\boldsymbol{\mu}_2 = (0, \ldots\ldots\ldots, 0)^T$$

$$\boldsymbol{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J} = \begin{bmatrix} 1 & \rho & \cdots\cdots & \rho \\ \rho & 1 & & \rho \\ \vdots & & & \vdots \\ \rho & \cdots & \cdots\cdots & 1 \end{bmatrix} \quad , \quad -1/(p-1) \le \rho \le 1.$$

The asymptotic expansion of the expected actual error rate for the Euclidean distance classifier is given by

$$p_A^{(E)} = \Phi\left( -\frac{mp}{2}\left( \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma^{uv} \right)^{-\frac{1}{2}} \right) + \frac{1}{2n_1} \sum_{i,j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}} \sigma_{ij} + \frac{1}{2n_2} \sum_{i,j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}} \sigma_{ij} \qquad (3.6)$$

where

$$\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}=\phi\left(-\frac{1}{2}mp(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})^{-\frac{1}{2}}\right)\times\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{7}{2}}$$

$$\times\begin{cases}\frac{1}{2m}(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})\left[p(\sigma_{ij}\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}-3\sum_{u=1}^{p}\sigma_{ui}\sum_{u=1}^{p}\sigma_{ju})\right. \\ \qquad\left.+2(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})^2\right]+\frac{1}{8}mp\sum_{u=1}^{p}\sigma_{ju}\sum_{u=1}^{p}\sigma_{iu} & \text{if} \quad i=j \\ \frac{1}{2m}(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})\left[p(\sigma_{ij}\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}-3\sum_{u=1}^{p}\sigma_{ui}\sum_{u=1}^{p}\sigma_{ju})\right] \\ +\frac{1}{8}mp\sum_{u=1}^{p}\sigma_{ju}\sum_{u=1}^{p}\sigma_{iu} & \text{if} \quad i\neq j\end{cases}$$

$$\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}=-\frac{1}{2}\phi\left(-\frac{1}{2}mp(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})^{-\frac{1}{2}}\right)\times\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{7}{2}}$$

$$\times\begin{cases}\frac{1}{m}\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\left[3p\sum_{u=1}^{p}\sigma_{ui}\sum_{u=1}^{p}\sigma_{ju}-(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})(2\sum_{u=1}^{p}\sigma_{ju}+\sigma_{ji}p)\right. \\ \left.+2(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}-\sum_{u=1}^{p}\sigma_{ui})\right] \\ -mp(\frac{1}{2}\sum_{u=1}^{p}\sigma_{ju}-\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})(\frac{1}{2}\sum_{u=1}^{p}\sigma_{iu}-\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}) & \text{if} \quad i=j \\ \frac{1}{m}\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\left[3p\sum_{u=1}^{p}\sigma_{ui}\sum_{u=1}^{p}\sigma_{ju}-(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})(2\sum_{u=1}^{p}\sigma_{ju}+\sigma_{ji}p)\right. \\ \left.-2(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})\sum_{u=1}^{p}\sigma_{ui}\right] \\ -mp(\frac{1}{2}\sum_{u=1}^{p}\sigma_{ju}-\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})(\frac{1}{2}\sum_{u=1}^{p}\sigma_{iu}-\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}) & \text{if} \quad i\neq j\end{cases}$$

and $\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}=p\left[1+\rho(p-1)\right].$

Section 3.2.4 : Case A4

Here we consider the case of equivalence of the linear discriminant function and the Euclidean distance classifier under the following conditions:

$$\boldsymbol{\mu}_1 = (m, \ldots \ldots\ldots, m)^T$$

$$\boldsymbol{\mu}_2 = (0, \ldots\ldots\ldots, 0)^T$$

$$\boldsymbol{\Sigma} = AR(1) = \begin{bmatrix} 1 & \rho & \cdots\cdots & \rho^{p-1} \\ \rho & 1 & & \rho^{p-2} \\ \vdots & & & \vdots \\ \rho^{p-1} & \cdots\cdots & \rho & 1 \end{bmatrix} .$$

The asymptotic expansion of the expected actual error rate associated with the Euclidean distance classifier is

$$p_A^{(E)} = \Phi\left( -\frac{mp}{2}\left( \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma^{uv} \right)^{-\frac{1}{2}} \right) + \frac{1}{2n_1} \sum_{i,j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}} \sigma_{ij} + \frac{1}{2n_2} \sum_{i,j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}} \sigma_{ij} \qquad (3.7)$$

where

$$\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}=\phi\left(-\frac{1}{2}mp(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})^{-\frac{1}{2}}\right)\times\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{7}{2}}$$

$$\times\begin{cases}\frac{1}{2m}(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})\left[p(\sigma_{ij}\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}-3\sum_{u=1}^{p}\sigma_{ui}\sum_{u=1}^{p}\sigma_{ju})\right. \\ \left.\qquad+2(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})^2\right]+\frac{1}{8}mp\sum_{u=1}^{p}\sigma_{ju}\sum_{u=1}^{p}\sigma_{iu} & \text{if} \quad i=j \\ \frac{1}{2m}(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})\left[p(\sigma_{ij}\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}-3\sum_{u=1}^{p}\sigma_{ui}\sum_{u=1}^{p}\sigma_{ju})\right] \\ +\frac{1}{8}mp\sum_{u=1}^{p}\sigma_{ju}\sum_{u=1}^{p}\sigma_{iu} & \text{if} \quad i\neq j\end{cases}$$

$$\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}=-\frac{1}{2}\phi\left(-\frac{1}{2}mp(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})^{-\frac{1}{2}}\right)\times\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{7}{2}}$$

$$\times\begin{cases}\frac{1}{m}\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\left[3p\sum_{u=1}^{p}\sigma_{ui}\sum_{u=1}^{p}\sigma_{ju}-(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})(2\sum_{u=1}^{p}\sigma_{ju}+\sigma_{ji}p)\right. \\ \left.+2(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}-\sum_{u=1}^{p}\sigma_{ui})\right] \\ -mp(\frac{1}{2}\sum_{u=1}^{p}\sigma_{ju}-\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})(\frac{1}{2}\sum_{u=1}^{p}\sigma_{iu}-\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}) & \text{if} \quad i=j \\ \frac{1}{m}\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\left[3p\sum_{u=1}^{p}\sigma_{ui}\sum_{u=1}^{p}\sigma_{ju}-(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})(2\sum_{u=1}^{p}\sigma_{ju}+\sigma_{ji}p)\right. \\ \left.-2(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})\sum_{u=1}^{p}\sigma_{ui}\right] \\ -mp(\frac{1}{2}\sum_{u=1}^{p}\sigma_{ju}-\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})(\frac{1}{2}\sum_{u=1}^{p}\sigma_{iu}-\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}) & \text{if} \quad i\neq j\end{cases}$$

and $\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}=\sum_{v=1}^{p}\sum_{u=1}^{p}\rho^{|j-i|}$.

The asymptotic expansions for the actual error rate associated with the linear discriminant function in cases A3 and A4 ( sections 3.2.3 and

3.2.4 ) are not obtained for this project due to the complexity of the differentiation, and evaluation of the final expression.

## Section 3.3 : Asymptotic expansions for the plug-in error rate

The plug-in error rates associated with the linear discriminant function and the Euclidean distance classifier are given in expressions *(1.12)* and *(1.13)* respectively. From these functions, the asymptotic expansions are obtained using the Taylor series expansion given in expression *(3.1)*. The following sections give the asymptotic expansions under the four different categories (cases) as stated in section 3.1.

### Section 3.3.1 : Case P1

Here we consider the case of non-equivalence of the linear discriminant function and the Euclidean distance classifier under the following conditions:

$$\boldsymbol{\mu}_1 = (m^*, 0, \ldots\ldots, 0)^T$$

$$\boldsymbol{\mu}_2 = (0, \ldots\ldots\ldots, 0)^T$$

$$\boldsymbol{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J} = \begin{bmatrix} 1 & \rho & \cdots\cdots & \rho \\ \rho & 1 & & \rho \\ \vdots & & & \vdots \\ \rho & \cdots\cdots & \cdots & 1 \end{bmatrix} \quad , \quad -1/(p-1) \le \rho \le 1.$$

The asymptotic expansion of the expected plug-in error rate for the Euclidean distance classifier is given by

$$p_p^{(E)} = \Phi\left(-\frac{m}{2}\right) + \frac{1}{2n_1}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}\sigma_{ij} + \frac{1}{2n_2}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}\sigma_{ij}$$
$$+\frac{1}{2}\frac{n_1+n_2}{(n_1+n_2-2)^2}\sum_{k,l,i,j=1}^{p}\frac{\partial^2\Phi}{\partial s_{kl}\partial s_{1j}}\left(\sigma_{ik}\sigma_{jl}+\sigma_{il}\sigma_{jk}\right) \qquad (3.8)$$

where

$$\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}} = \frac{1}{2}\phi\left(-\frac{m}{2}\right)\begin{cases}\frac{1}{m}\left[3(s_{jl}-s_{1i}s_{jl})-2(1-s_{1i})\right]+\frac{m}{4} & \text{if } i=l, j=l, i=j\\ \frac{3}{m}(s_{jl}-s_{1i}s_{jl})-\frac{m}{4}s_{jl} & \text{if } i=l, j\neq l, i\neq j\\ \frac{1}{m}\left[s_{ji}-3s_{1i}s_{jl}+2s_{1i}\right]-\frac{m}{4}s_{il} & \text{if } i\neq l, j=l, i\neq j\\ \frac{1}{m}\left[s_{ji}-3s_{1i}s_{jl}-2\right]+\frac{m}{4}s_{jl}s_{il} & \text{if } i\neq l, j\neq l, i=j\\ \frac{1}{m}(s_{ji}-3s_{1i}s_{jl})+\frac{m}{4}s_{jl}s_{il} & \text{if } i\neq l, j\neq l, i\neq j\end{cases}$$

and

$$\frac{\partial^2\Phi}{\partial s_{kl}\,\partial s_{ij}} = -\frac{m}{8}\phi\left(-\frac{m}{2}\{s_{11}\}^{-\frac{1}{2}}\right)\left[3-\frac{m^2}{4}\right].$$

The corresponding asymptotic expansion for the linear discriminant function is given by

$$p_p^{(L)} = \Phi\left(-\frac{m}{2}\{s^{11}\}^{-\frac{1}{2}}\right) + \frac{1}{2n_1}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}\sigma_{ij} + \frac{1}{2n_2}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}\sigma_{ij}$$
$$+\frac{1}{2}\frac{n_1+n_2}{(n_1+n_2-2)^2}\sum_{k,l,i,j=1}^{p}\frac{\partial^2\Phi}{\partial s_{kl}\partial s_{1j}}\left(\sigma_{ik}\sigma_{jl}+\sigma_{il}\sigma_{jk}\right) \qquad (3.9)$$

where

$$\frac{\partial^2 \Phi}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}} = -\frac{1}{2}\phi\left(-\frac{1}{2}m\left(s^{11}\right)^{\frac{1}{2}}\right)\left(s^{11}\right)^{-\frac{3}{2}}\left[\frac{1}{m}\left[s^{11}s^{ji} - s^{1i}s^{j1}\right] - \frac{m}{4}s^{11}s^{j1}s^{i1}\right]$$

$$\frac{\partial^2 \Phi}{\partial s_{kl}\,\partial s_{ij}} = -\frac{m}{2}w_0 w_1\left\{s^{11}\right\}^{-\frac{3}{2}}\phi\left(-\frac{m}{2}\left\{s^{11}\right\}^{-\frac{1}{2}}\right)\times\left[s^{11}\left\{s^{1i}(s^{jk}s^{l1} + s^{jl}s^{k1})\right.\right.$$

$$\left.\left. + s^{jl}(s^{1k}s^{li} + s^{l1}s^{ki})\right\} - (1 + \frac{m^2}{4})s^{1i}s^{j1} \times s^{1k}s^{l1}\right]$$

$$w_0 = \begin{cases} -\frac{1}{2} & \text{if } i = j \\ -1 & \text{if } i \neq j \end{cases} \qquad \text{and} \qquad w_1 = \begin{cases} -\frac{1}{2} & \text{if } k = l \\ -1 & \text{if } k \neq l \end{cases} .$$

Section 3.3.2 : Case P2

In this situation of non-equivalence of the linear discriminant function and the Euclidean distance classifier the parameters take the following values:

$$\boldsymbol{\mu}_1 = (m^*, 0, \ldots\ldots, 0)^T$$

$$\boldsymbol{\mu}_2 = (0, \ldots\ldots\ldots, 0)^T$$

$$\boldsymbol{\Sigma} = AR(1) = \begin{bmatrix} 1 & \rho & \cdots\cdots & \rho^{p-1} \\ \rho & 1 & & \rho^{p-2} \\ \vdots & & & \vdots \\ \rho^{p-1} & \cdots\cdots & \rho & 1 \end{bmatrix} .$$

The asymptotic expansion of the expected plug-in error rate for the Euclidean distance classifier is denoted by $p_P^{(E)}$ and is identical to expression *(3.8)* except that now $\mathbf{\Sigma}$ has the structure given for this case. The corresponding expansion for the linear discriminant function is denoted by $p_P^{(L)}$ and is identical algebraically to expression *(3.9)* except that the structure of $\mathbf{\Sigma}$ is now as given for this case.

Section 3.3.3 : Case P3

Here we consider the situation of equivalence of the linear discriminant function and the Euclidean distance classifier with the parameters defined as follows :

$$\boldsymbol{\mu}_1 = (m, \ldots\ldots\ldots, m)^T$$

$$\boldsymbol{\mu}_2 = (0, \ldots\ldots\ldots\ 0)^T$$

$$\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J} = \begin{bmatrix} 1 & \rho & \cdots\cdots & \rho \\ \rho & 1 & & \rho \\ \vdots & & & \vdots \\ \rho & & \cdots\cdots\cdots & 1 \end{bmatrix} \quad , \quad -1/(p-1) \le \rho \le 1.$$

The asymptotic expansion of the expected plug-in error rate for the Euclidean distance classifier is given by

$$p_P^{(E)} = \Phi\left(-\frac{mp}{2}\{\sum_{v=1}^{p}\sum_{u=1}^{p} s_{uv}\}^{-\frac{1}{2}}\right) + \frac{1}{2}\left[\frac{1}{n_1} + \frac{1}{n_2}\right]\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}\sigma_{ij}$$
$$+\frac{1}{2}\frac{n_1+n_2}{(n_1+n_2-2)^2}\sum_{k,l,i,j=1}^{p}\frac{\partial^2\Phi}{\partial s_{kl}\partial s_{1j}}\left(\sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}\right) \qquad (3.10)$$

where

$$\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}} = -\frac{1}{2}\phi\left(-\frac{1}{2}mp(\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv})^{-\frac{1}{2}}\right)\times\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\}^{-\frac{7}{2}}$$

$$\times\begin{cases} \frac{1}{m}\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\left\{\left[-\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\left(2\sum_{u=1}^{p}s_{ju}+ps_{ji}\right)+3p\sum_{u=1}^{p}s_{ui}\sum_{u=1}^{p}s_{ju}\right]\right. \\ \left.+2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\left(-\sum_{u=1}^{p}s_{ui}+\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right)\right\} \\ -\frac{1}{4}mp\left\{-\sum_{u=1}^{p}s_{ju}+2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\}\left\{2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}-\sum_{u=1}^{p}s_{iu}\right\} & \text{if } i=j \\ \frac{1}{m}\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\left\{\left[-\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\left(2\sum_{u=1}^{p}s_{ju}+ps_{ji}\right)+3p\sum_{u=1}^{p}s_{ui}\sum_{u=1}^{p}s_{ju}\right]\right. \\ -2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\sum_{u=1}^{p}s_{ui} \\ -\frac{1}{4}mp\left\{-\sum_{u=1}^{p}s_{ju}+2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\}\left\{2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}-\sum_{u=1}^{p}s_{iu}\right\} & \text{if } i\neq j \end{cases}$$

$$\frac{\partial^2\Phi}{\partial s_{kl}\partial s_{ij}} = -\frac{mp}{8}\phi\left(-\frac{mp}{2}\{\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}\}^{-\frac{1}{2}}\right)\left\{\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}\right\}^{-\frac{7}{2}}\left[3\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}-\frac{m^2p^2}{4}\right].$$

The asymptotic expansion of the expected plug-in error rate for the linear discriminant function is given by

$$p_p^{(L)} = \Phi\left(-\frac{m}{2}\{\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\}^{\frac{1}{2}}\right)+\frac{1}{2}\left[\frac{1}{n_1}+\frac{1}{n_2}\right]\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}\sigma_{ij}$$
$$+\frac{1}{2}\frac{n_1+n_2}{(n_1+n_2-2)^2}\sum_{k,l,i,j=1}^{p}\frac{\partial^2\Phi}{\partial s_{kl}\partial s_{1j}}\left(\sigma_{ik}\sigma_{jl}+\sigma_{il}\sigma_{jk}\right) \qquad (3.11)$$

where

$$\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}} = -\frac{1}{2}\phi\left(-\frac{1}{2}m\left(\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv}\right)^{\frac{1}{2}}\right)\left(\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv}\right)^{-\frac{3}{2}}$$

$$\times\left[\frac{1}{m}\left[s^{ji}\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv} - \sum_{u=1}^{p}s^{ui}\sum_{u=1}^{p}s^{ju}\right] - \frac{m}{4}\left(\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv}\right)\sum_{u=1}^{p}s^{iu}\sum_{u=1}^{p}s^{ju}\right]$$

$$\frac{\partial^2\Phi}{\partial s_{kl}\,\partial s_{ij}} = -\frac{m}{4}w_0 w_1\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv}\right\}^{-\frac{3}{2}}\phi\left(-\frac{m}{2}\left(\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv}\right)^{\frac{1}{2}}\right)\times\left[\left(\{(\sum_{u=1}^{p}\sum_{v=1}^{p}s^{uv})\right.\right.$$

$$\times\sum_{u=1}^{p}\sum_{v=1}^{p}[s^{ui}(s^{jk}s^{lv}+s^{jl}s^{kv})+s^{jv}(s^{uk}s^{li}+s^{ul}s^{ki})$$

$$+s^{uj}(s^{ik}s^{lv}+s^{il}s^{kv})+s^{iv}(s^{uk}s^{lj}+s^{ul}s^{kj})]\}$$

$$\left.\left.-\{\frac{1}{2}\left(1+\frac{m^2}{8}\sum_{u=1}^{p}\sum_{v=1}^{p}s^{uv}\right)\sum_{u=1}^{p}\sum_{v=1}^{p}(s^{ui}s^{jv}+s^{uj}s^{iv})\sum_{u=1}^{p}\sum_{v=1}^{p}(s^{uk}s^{lv}+s^{ul}s^{kv})\}\right)\right]$$

$$w_o = \begin{cases} -\frac{1}{2} & \text{if} \quad i=j \\ -1 & \text{if} \quad i\neq j \end{cases} \qquad \text{and} \qquad w_1 = \begin{cases} -\frac{1}{2} & \text{if} \quad k=l \\ -1 & \text{if} \quad k\neq l \end{cases}.$$

## Section 3.3.4 : Case P4

Here we consider the situation of equivalence of the linear discriminant function and the Euclidean distance classifier when $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$ and $\boldsymbol{\Sigma}$ take the following values :

$$\boldsymbol{\mu}_1 = (m, \ldots\ldots\ldots, m)^T$$

$$\boldsymbol{\mu}_2 = (0, \ldots\ldots\ldots, 0)^T$$

$$\boldsymbol{\Sigma} = AR(1) = \begin{bmatrix} 1 & \rho & \cdots\cdots & \rho^{p-1} \\ \rho & 1 & & \\ \vdots & & & \vdots \\ \rho^{p-1} & \cdots\cdots & \rho & 1 \end{bmatrix} .$$

The asymptotic expansion of the expected plug-in error rate for the Euclidean distance classifier is denoted by $p_P^{(E)}$ and is identical algebraically to expression *(3.10)* except that $\boldsymbol{\Sigma}$ here is as given in this case. The corresponding expansion for the linear discriminant function is denoted by $p_P^{(L)}$ and is identical algebraically to expression *(3.11)* except that the structure of $\boldsymbol{\Sigma}$ is now as defined in this case.

# CHAPTER 4

## COMPUTATIONAL RESULTS AND DISCUSSION

### Section 4.1 : Introduction

As mentioned in section 1.4, the aim of this project is to compare the performances of the linear discriminant function and the Euclidean distance classifier and their relative performances are assessed using the expected actual error rates and the expected plug-in error rates associated with them. These error rates are defined in expressions *(1.10)* , *(1.11)*, *(1.12)* and *(1.13)* .

As mentioned earlier, under "equivalence" of the linear discriminant function and the Euclidean distance classifier, $\mu_1 = (m,\ldots\ldots\ldots,m)^T$ and $\mu_2 = (0,\ldots\ldots,0)^T$ whereas under "non-equivalence" $\mu_1 = (m^*,0,\ldots,0)^T$ and $\mu_2 = (0,\ldots\ldots\ldots,0)^T$. Some values of m, $m^*$ and $\Sigma$ are chosen to be the same as the values from Marco, Young and Turner (1987) so that direct comparison of these results with their simulated values can be made. However, additional values were used for further investigation. The values of m and $m^*$ are chosen so that the

Mahalanobis distance, $\Delta^2$, is the same in both cases of equivalence and non-equivalence of the linear discriminant function and the Euclidean distance classifier. The five values of the Mahalanobis distances chosen are $\Delta^2 = 0.5, 1.0, 1.5, 2.0, 2.5$. The covariance matrices are obtained by choosing different values of the parameter, $\rho$ , namely $\rho$ =-0.06,0.00, 0.01,0.20,0.40,0.65. In some cases (when possible) for $\mathbf{\Sigma} = \text{AR}(1)$, negative $\rho$ is chosen for further investigation, namely $\rho$=-0.20,-0.40, -0.65. These values are not used when $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ since, for $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ to be positive definite, $\rho$ must be greater than -1/(p-1). For example, John Van Ness (1982) chose the value of $\rho$ =-0.06 since, at dimension p=16, $\Sigma$ is not positive definite for $\rho \leq -\frac{1}{15}$. For most combinations of values of $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$, comparison of the performance of the linear discriminant function and the Euclidean distance classifier was carried out for four different values of p, namely p=4,8,12,16 .

The computation of all the error rates was done using PC-MATLAB ( The MathWorks. Inc., 1989 ). The computation time for the error rates of the Euclidean distance classifier was about 50 times faster than the computation time for the error rates of the linear discriminant function. Therefore, due to time constraint, not all values of the error rates for cases A1, A2, P1, P2, P3 and P4 of the linear discriminant function were obtained for p=12 and p=16. Note however that some values for p=12 were obtained. The computational results for the error rates are shown in Table 5 to Table 23. The values of m and $m^*$ for different combinations of $\rho$, $\Delta^2$ and p which are considered are given in Table 1 to Table 4. The sizes of samples, $n_1$ and $n_2$, from populations (groups) $\Pi_1$ and $\Pi_2$ respectively are chosen to be $n_1$=$n_2$=50, unless stated otherwise in the tables.

| ρ | $\Delta^2$ | $m^*$ | | | |
|---|---|---|---|---|---|
| | | p=4 | p=8 | p=12 | p=16 |
| -0.06 | 0.5 | 0.7028 | 0.6930 | 0.6712 | 0.5755 |
| | 1.0 | 0.9938 | 0.9801 | 0.9492 | 0.8139 |
| | 1.5 | 1.2172 | 1.2004 | 1.1625 | 0.9969 |
| | 2.0 | 1.4055 | 1.3861 | 1.3434 | 1.1511 |
| | 2.5 | 1.5714 | 1.5497 | 1.5008 | 1.2870 |
| 0.00 | 0.5 | 0.7071 | 0.7071 | 0.7071 | 0.7071 |
| | 1.0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| | 1.5 | 1.2247 | 1.2247 | 1.2247 | 1.2247 |
| | 2.0 | 1.4142 | 1.4142 | 1.4142 | 1.4142 |
| | 2.5 | 1.5811 | 1.5811 | 1.5811 | 1.5811 |
| 0.01 | 0.5 | 0.7071 | 0.7069 | 0.7068 | 0.7066 |
| | 1.0 | 0.9999 | 0.9997 | 0.9995 | 0.9993 |
| | 1.5 | 1.2246 | 1.2243 | 1.2241 | 1.2239 |
| | 2.0 | 1.4140 | 1.1437 | 1.4142 | 1.4133 |
| | 2.5 | 1.5809 | 1.5806 | 1.5811 | 1.5801 |
| 0.2 | 0.5 | 0.6761 | 0.6606 | 0.6532 | 0.6489 |
| | 1.0 | 0.9562 | 0.9342 | 0.9238 | 0.9177 |
| | 1.5 | 1.1711 | 1.1442 | 1.1314 | 1.1239 |
| | 2.0 | 1.3522 | 1.3212 | 1.3064 | 1.2978 |
| | 2.5 | 1.5119 | 1.4771 | 1.4606 | 1.4510 |
| 0.4 | 0.5 | 0.6055 | 0.5790 | 0.5692 | 0.5641 |
| | 1.0 | 0.8563 | 0.8189 | 0.8050 | 0.7977 |
| | 1.5 | 1.0488 | 1.0029 | 0.9859 | 0.9770 |
| | 2.0 | 1.2111 | 1.1581 | 1.1384 | 1.1282 |
| | 2.5 | 1.3540 | 1.2948 | 1.2728 | 1.2613 |
| 0.65 | 0.5 | 0.4738 | 0.4452 | 0.4361 | 0.4316 |
| | 1.0 | 0.6700 | 0.6296 | 0.6167 | 0.6103 |
| | 1.5 | 0.8206 | 0.7711 | 0.7553 | 0.7475 |
| | 2.0 | 0.9475 | 0.8904 | 0.8722 | 0.8632 |
| | 2.5 | 1.0594 | 0.9955 | 0.9751 | 0.9650 |

**Table 1** : Values of $m^*$ under the case of "non-equivalence" with $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ ( i.e.for cases A1 and P1 ).

| | $\Delta^2$ | | | | |
|---|---|---|---|---|---|
| ρ | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 |
| -0.65 | 0.5374 | 0.7599 | 0.9307 | 1.0747 | 1.2016 |
| -0.4 | 0.6481 | 0.9165 | 1.1225 | 1.2961 | 1.4491 |
| -0.2 | 0.6928 | 0.9798 | 1.2000 | 1.3856 | 1.5492 |
| -0.06 | 0.7058 | 0.9982 | 1.2225 | 1.4117 | 1.5783 |
| 0 | 0.7071 | 1.0000 | 1.2247 | 1.4142 | 1.5811 |
| 0.01 | 0.7071 | 0.9999 | 1.2247 | 1.4142 | 1.5811 |
| 0.2 | 0.6928 | 0.9798 | 1.2000 | 1.3856 | 1.5492 |
| 0.4 | 0.6481 | 0.9165 | 1.1225 | 1.2961 | 1.4491 |
| 0.65 | 0.5374 | 0.7599 | 0.9307 | 1.0747 | 1.2016 |

**Table 2 :** Values of $m^*$ under the case of "non-equivalence" with $\Sigma = AR(1)$ ( i.e.for cases A2 and P2 ).

Note : Same values of m* for all values of p.

| ρ | $\Delta^2$ | m | | | |
|---|---|---|---|---|---|
| | | p=4 | p=8 | p=12 | p=16 |
| -0.06 | 0.5 | 0.3202 | 0.1904 | 0.1190 | 0.0559 |
| | 1.0 | 0.4528 | 0.2693 | 0.1683 | 0.0791 |
| | 1.5 | 0.5545 | 0.3298 | 0.2062 | 0.0968 |
| | 2.0 | 0.6403 | 0.3808 | 0.2380 | 0.1118 |
| | 2.5 | 0.7159 | 0.4257 | 0.2661 | 0.1250 |
| 0.00 | 0.5 | 0.3536 | 0.2500 | 0.2041 | 0.1768 |
| | 1.0 | 0.5000 | 0.3536 | 0.2887 | 0.2500 |
| | 1.5 | 0.6124 | 0.4330 | 0.3536 | 0.3062 |
| | 2.0 | 0.7071 | 0.5000 | 0.4082 | 0.3536 |
| | 2.5 | 0.7906 | 0.5590 | 0.4564 | 0.3953 |
| 0.01 | 0.5 | 0.3588 | 0.2586 | 0.2151 | 0.1896 |
| | 1.0 | 0.5074 | 0.3657 | 0.3041 | 0.2681 |
| | 1.5 | 0.6215 | 0.4479 | 0.3725 | 0.3283 |
| | 2.0 | 0.7176 | 0.5172 | 0.4301 | 0.3791 |
| | 2.5 | 0.8023 | 0.5783 | 0.4809 | 0.4239 |
| 0.2 | 0.5 | 0.4472 | 0.3873 | 0.3651 | 0.3536 |
| | 1.0 | 0.6325 | 0.5477 | 0.5164 | 0.5000 |
| | 1.5 | 0.7746 | 0.6708 | 0.6325 | 0.6124 |
| | 2.0 | 0.8944 | 0.7746 | 0.7303 | 0.7071 |
| | 2.5 | 1.0000 | 0.8660 | 0.8165 | 0.7906 |
| 0.4 | 0.5 | 0.5244 | 0.4873 | 0.4743 | 0.4677 |
| | 1.0 | 0.7416 | 0.6892 | 0.6708 | 0.6614 |
| | 1.5 | 0.9083 | 0.8441 | 0.8216 | 0.8101 |
| | 2.0 | 1.0488 | 0.9747 | 0.9487 | 0.9354 |
| | 2.5 | 1.1726 | 1.0897 | 1.0607 | 1.0458 |
| 0.65 | 0.5 | 0.6072 | 0.5890 | 0.5827 | 0.5796 |
| | 1.0 | 0.8588 | 0.8329 | 0.8241 | 0.8197 |
| | 1.5 | 1.0518 | 1.0201 | 1.0093 | 1.0039 |
| | 2.0 | 1.2145 | 1.1779 | 1.1655 | 0.1592 |
| | 2.5 | 1.3578 | 1.3170 | 1.3030 | 1.2960 |

**Table 3** : Values of m under the case of " equivalence " with $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ ( i.e. for cases A3 and P3 ).

| ρ | $\Delta^2$ | m | | | |
|---|---|---|---|---|---|
| | | p=4 | p=8 | p=12 | p=16 |
| 0.00 | 0.5 | 0.3536 | 0.2500 | 0.2041 | 0.1768 |
| | 1.0 | 0.5000 | 0.3536 | 0.2887 | 0.2500 |
| | 1.5 | 0.6124 | 0.4330 | 0.3536 | 0.3062 |
| | 2.0 | 0.7071 | 0.5000 | 0.4082 | 0.3536 |
| | 2.5 | 0.7906 | 0.5590 | 0.4564 | 0.3953 |
| 0.01 | 0.5 | 0.3562 | 0.2522 | 0.2060 | 0.1784 |
| | 1.0 | 0.5038 | 0.3567 | 0.2913 | 0.2524 |
| | 1.5 | 0.6170 | 0.4368 | 0.3568 | 0.3091 |
| | 2.0 | 0.7124 | 0.5044 | 0.4120 | 0.3569 |
| | 2.5 | 0.7965 | 0.5639 | 0.4606 | 0.3990 |
| 0.2 | 0.5 | 0.4082 | 0.2970 | 0.2449 | 0.2132 |
| | 1.0 | 0.5774 | 0.4201 | 0.3464 | 0.3015 |
| | 1.5 | 0.7071 | 0.5145 | 0.4243 | 0.3693 |
| | 2.0 | 0.8165 | 0.5941 | 0.4899 | 0.4264 |
| | 2.5 | 0.9129 | 0.6642 | 0.5477 | 0.4767 |
| 0.4 | 0.5 | 0.4677 | 0.3536 | 0.2958 | 0.2594 |
| | 1.0 | 0.6614 | 0.5000 | 0.4183 | 0.3669 |
| | 1.5 | 0.8101 | 0.6124 | 0.5123 | 0.4494 |
| | 2.0 | 0.9354 | 0.7071 | 0.5916 | 0.5189 |
| | 2.5 | 1.0458 | 0.7906 | 0.6614 | 0.5801 |
| 0.65 | 0.5 | 0.5528 | 0.4486 | 0.3873 | 0.3458 |
| | 1.0 | 0.7817 | 0.6344 | 0.5477 | 0.4890 |
| | 1.5 | 0.9574 | 0.7770 | 0.6708 | 0.5989 |
| | 2.0 | 1.1055 | 0.8971 | 0.7746 | 0.6916 |
| | 2.5 | 1.2360 | 1.0030 | 0.8660 | 0.7732 |

**Table 4 :** Values of m under the case of " equivalence " when $\Sigma = AR(1)$ ( positive ρ in cases A4 and P4 ).

| ρ | $\Delta^2$ | m | | | |
|---|---|---|---|---|---|
| | | p=4 | p=8 | p=12 | p=16 |
| -0.65 | 0.5 | 0.1817 | 0.1213 | 0.0973 | 0.0835 |
| | 1.0 | 0.2570 | 0.1715 | 0.1375 | 0.1181 |
| | 1.5 | 0.3147 | 0.2100 | 0.1685 | 0.1446 |
| | 2.0 | 0.3634 | 0.2425 | 0.1945 | 0.1670 |
| | 2.5 | 0.4063 | 0.2712 | 0.2175 | 0.1867 |
| -0.4 | 0.5 | 0.2500 | 0.1698 | 0.1369 | 0.1179 |
| | 1.0 | 0.3536 | 0.2402 | 0.1936 | 0.1667 |
| | 1.5 | 0.4330 | 0.2942 | 0.2372 | 0.2041 |
| | 2.0 | 0.5000 | 0.3397 | 0.2739 | 0.2357 |
| | 2.5 | 0.5590 | 0.3798 | 0.3062 | 0.2635 |
| -0.2 | 0.5 | 0.3015 | 0.2085 | 0.1690 | 0.1459 |
| | 1.0 | 0.4264 | 0.2949 | 0.2390 | 0.2063 |
| | 1.5 | 0.5222 | 0.3612 | 0.2928 | 0.2526 |
| | 2.0 | 0.6030 | 0.4170 | 0.3381 | 0.2917 |
| | 2.5 | 0.6742 | 0.4663 | 0.3780 | 0.3262 |
| -0.06 | 0.5 | 0.3378 | 0.2371 | 0.1931 | 0.1671 |
| | 1.0 | 0.4777 | 0.3353 | 0.2731 | 0.2363 |
| | 1.5 | 0.5850 | 0.4107 | 0.3345 | 0.2894 |
| | 2.0 | 0.6755 | 0.4742 | 0.3863 | 0.3341 |
| | 2.5 | 0.7552 | 0.5302 | 0.4319 | 0.3736 |

**Table 4a :** Values of m under the case of " equivalence " with $\Sigma = AR(1)$ (negative ρ in cases A4 and P4 ).

Note that when $\boldsymbol{\mu}_1 = (m^*, 0, \ldots, 0)^T$, $\boldsymbol{\mu}_2 = (0, \ldots\ldots, 0)^T$ and $\boldsymbol{\Sigma} = AR(1)$, the values of $m^*$ are independent of p (see Table 2). This is due to the fact that $\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$ does not involve p when $\boldsymbol{\Sigma}$ is as denoted by the AR(1) structure.

Section 4.2 : Discussion

The relative performances of the linear discriminant function and the Euclidean distance classifier are compared using two main categories:

(i) category 1 consisting of comparisons for the actual error rates as defined in sections 3.2.1, 3.2.2, 3.2.3 and 3.2.4, and

(ii) category 2 consisting of comparisons for the plug-in error rates as defined in sections 3.3.1, 3.3.2, 3.3.3 and 3.3.4. Thus the discussion of the results will be separated in the above manner in the following sections.

From here on, the error rates when the parameters $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$ and $\boldsymbol{\Sigma}$ are known is referred to as the "true" error rate. The terms $p_A^{(E)}$, $p_A^{(L)}$, $p_p^{(E)}$ and $p_p^{(L)}$ in the tables are as defined in Chapter 3 ( expressions *3.3 - 3.11* ). The compiled results which are presented in Table 5 to Table 8a are extracted from Table 9 to Table 23 (Appendix A4). From now on we also refer to the asymptotic expectation of the actual error rate and the asymptotic expectation of the plug-in error rate as "the expected actual error rate" and "the expected plug-in error rate" respectively.

Section 4.2.1 : Actual error rates (category 1)

In all the cases associated with the actual error rates, such as cases A1, A2, A3 and A4 ( as defined in chapter 3 ), the expected actual error rate decreases as the Mahalanobis distance, $\Delta^2$, increases ( see Table 5 to Table 8a ). This is expected since it becomes easier to distinguish between two populations as the Mahalanobis distance increases. Furthermore, the expected actual error rates for the Euclidean distance classifier seem to overestimate the true error rates, although exceptions occur in some instances (see underlined values in Table 5). However, the value with $\rho=0.40$ is only slightly underestimated. The rest of the values are consistent with previous work ( see Chapter 2 ).

Non-equivalence, $\Sigma = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ (Table 5)

Consider the case of non-equivalence of the linear discriminant function and the Euclidean distance classifier. When $\Sigma = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ ( see Table 5 ), the expected actual error rates for the Euclidean distance classifier increases as p increases. Note, however that there are situations, especially when the level of correlation is high (i.e. $\rho=0.65$), when this does not happen; see, for example, the underlined values. For the linear discriminant function, the expected actual error rate increases as p increases when the correlation is small or zero (e.g. $\rho=0.00,0.20$) but it decreases as p increases when $\rho$ increases to 0.40 and 0.65. When both the correlation and the Mahalanobis distance are moderate to large (e.g. $\rho \geq 0.40$ and $\Delta^2 \geq 1$ in the Table 5), the expected actual error rates of the linear discriminant function underestimate the true error rates. For small

| $\Delta^2$ | ρ | p=4 | | | p=8 | | | p=12 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | true **EDC** *LDF* | actual **EDC** *LDF* | plug-in **EDC** *LDF* | true **EDC** *LDF* | actual **EDC** *LDF* | plug-in **EDC** *LDF* | true **EDC** *LDF* | actual **EDC** *LDF* | plug-in **EDC** *LDF* |
| 0.5 | 0.0 | **0.3618** | **0.3784** | **0.3456** | **0.3618** | **0.3996** | **0.3244** | **0.3618** | **0.4208** | **0.3032** |
| | | *0.3618* | *0.3857* | *0.3373* | *0.3618* | *0.3954* | *0.3051* | *0.3618* | *0.4133* | *0.2729* |
| | 0.2 | **0.3677** | **0.3823** | **0.3491** | **0.3706** | **0.4075** | **0.3293** | **0.3720** | **0.4346** | **0.3108** |
| | | *0.3618* | *0.3766* | *0.3370* | *0.3618* | *0.3899* | *0.3047* | *0.3618* | *0.4477* | *0.2724* |
| | 0.4 | **0.3810** | **0.3847** | **0.3515** | **0.3861** | **0.3891** | **0.3093** | **0.3880** | **0.3844** | **0.2573** |
| | | *0.3618* | *0.3685* | *0.3360* | *0.3618* | *0.3673* | *0.3036* | *0.3618* | *0.4359* | *0.2712* |
| | 0.65 | **0.4064** | **0.3611** | **0.3309** | **0.4119** | **0.2032** | **0.1293** | **0.4137** | ** | ** |
| | | *0.3618* | | *0.3312* | *0.3618* | | *0.2984* | *0.3618* | | *0.2660* |
| 1.0 | 0.0 | **0.3085** | **0.3200** | **0.2976** | **0.3085** | **0.3341** | **0.2835** | **0.3085** | **0.3481** | **0.2694** |
| | | *0.3085* | *0.3148* | *0.2866* | *0.3085* | *0.3242* | *0.2579* | *0.3085* | | *0.2291* |
| | 0.2 | **0.3163** | **0.3266** | **0.3037** | **0.3202** | **0.3456** | **0.2924** | **0.3221** | **0.3650** | **0.2811** |
| | | *0.3085* | *0.3135* | *0.2862* | *0.3085* | *0.3206* | *0.2573* | *0.3085* | | *0.2285* |
| | 0.4 | **0.3343** | **0.3375** | **0.3141** | **0.3411** | **0.3450** | **0.2895** | **0.3437** | **0.3449** | **0.2567** |
| | | *0.3085* | *0.3073* | *0.2850* | *0.3085* | *0.3008* | *0.2559* | *0.3085* | | *0.2270* |
| | 0.65 | **0.3688** | **0.3389** | **0.3169** | **0.3765** | **0.2371** | **0.1839** | **0.3789** | ** | ** |
| | | *0.3085* | | *0.2788* | *0.3085* | | *0.2493* | *0.3085* | | *0.2204* |
| 2.0 | 0.0 | **0.2398** | **0.2474** | **0.2328** | **0.2398** | **0.2562** | **0.2240** | **0.2398** | **0.2650** | **0.2153** |
| | | *0.2398* | *0.2277* | *0.2191* | *0.2398* | *0.2319* | *0.1920* | *0.2398* | | *0.1649* |
| | 0.2 | **0.2495** | **0.2566** | **0.2414** | **0.2544** | **0.2714** | **0.2367** | **0.2568** | **0.2853** | **0.2309** |
| | | *0.2398* | *0.2280* | *0.2187* | *0.2397* | *0.2308* | *0.1914* | *0.2398* | | *0.1642* |
| | 0.4 | **0.2724** | **0.2754** | **0.2592** | **0.2813** | **0.2861** | **0.2483** | **0.2846** | **0.2901** | **0.2303** |
| | | *0.2398* | *0.2252* | *0.2172* | *0.2397* | *0.2155* | *0.1897* | *0.2398* | | *0.1625* |
| | 0.65 | **0.3174** | **0.2995** | **0.2831** | **0.3281** | **0.2405** | **0.2018** | **0.3314** | **0.1252** | **0.0639** |
| | | *0.2398* | | *0.2101* | *0.2398* | | *0.1823* | *0.2397* | | *0.1550* |
| 2.5 | 0.0 | **0.2146** | **0.2213** | **0.2088** | **0.2146** | **0.2287** | **0.2014** | **0.2146** | **0.2361** | **0.1940** |
| | | *0.2146* | *0.1925* | *0.1944* | *0.2146* | *0.1953* | *0.1678* | *0.2146* | *0.1994* | *0.1412* |
| | 0.2 | **0.2248** | **0.2311** | **0.2180** | **0.2301** | **0.2448** | **0.2151** | **0.2326** | **0.2573** | **0.2107** |
| | | *0.2146* | *0.1937* | *0.1939* | *0.2146* | *0.1956* | *0.1671* | *0.2146* | *0.1984* | *0.1405* |
| | 0.4 | **0.2492** | **0.2522** | **0.2379** | **0.2587** | **0.2639** | **0.2307** | **0.2623** | **0.2690** | **0.2167** |
| | | *0.2146* | *0.1927* | *0.1924* | *0.2146* | *0.1826* | *0.1655* | *0.2146* | *0.1699* | *0.1388* |
| | 0.65 | **0.2982** | **0.2830** | **0.2680** | **0.3093** | **0.2357** | **0.2006** | **0.3129** | **0.1390** | **0.0836** |
| | | *0.2146* | | *0.1853* | *0.2146* | | *0.1580* | *0.2146* | | *0.1312* |

**Table 5 :** The 'true', expected actual and expected plug-in error rates of the EDC and LDF under the case of 'non - equivalence' with $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$.

Mahalanobis distance, (for example, $\Delta^2$=0.5 in Table 5), the expected actual error rate of the linear discriminant function overestimates the true error rate considerably as p increases. When the Mahalanobis distance is large, (i.e. $\Delta^2$= 2.5 in Table 5), the expected actual error rate of the linear discriminant function underestimates the true error rate considerably. Comparing the actual error rates of the linear discriminant function and the Euclidean distance classifier, the expected actual error rate for the Euclidean distance classifier is considerably closer to the true value than the expected actual error rate for the linear discriminant function as $\rho$ and p increase (e.g. $\rho \geq 0.40$ and $p \geq 8$ in Table 5). Note that the true error rate of the linear discriminant function is independent of p because of the structure of the function for the linear discriminant function (see equation *1.10a* ). This is due to the fact that only the first elements of $\mu_1$ and $\Sigma^{-1}$ are involved. Furthermore, $m^* = \sqrt{\Delta/\sigma^{11}}$ where $\sigma^{11}$ depends on p. Thus when p varies while $\Delta^2$ and $\rho$ remain the same, the true error rates remain the same too. The true error rates are also the same for all values of $\rho$ due to the way the values of m* are chosen and the structure of the linear discriminant function, i.e. $\Delta^2 = (\mu_1 - \mu_2)^T \Sigma^{-1} (\mu_1 - \mu_2)$ and $D_L(x) = \Phi(-\Delta/2)$ .

It can be seen from Table 5 that in general the expected actual error rate for the Euclidean distance classifier initially increases as $\rho$ increases but decreases again especially when the correlation coefficient and the dimension p become large (i.e. $\rho \geq 0.40$ and $p \geq 8$). The expected actual error rate is relatively small when the correlation is large (e.g. $\rho$=0.65) for all p. This shows that the asymptotic expansion gives a bad approximation to the true error rate as the correlation between elements gets stronger.

In Table 5 it can be seen that the expected actual error rate for the linear discriminant function may decrease as $\rho$ increases when $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$. This ( rather surprising ) result is consistent with some result ( Cochran, 1962 ) which states that it is possible for correlation (among elements of the vector $\mathbf{x}$) to improve discrimination. It is not possible to always predict what will happen (i.e. improvement or worsening) since it is the combined effects of $\rho$, p, $\Delta^2$ and the asymptotic approximations.

Non-equivalence, $\Sigma$ = AR(1) (Table 6)

Under the case of non-equivalence of the linear discriminant function and the Euclidean distance classifier when $\Sigma$ = AR(1) (see Table 6), the true error rate of the Euclidean distance classifier remains the same for all dimension p as the values of m are the same for all values of p (as explained in Section 4.1). For both the linear discriminant function and the Euclidean distance classifier, the expected actual error rate increases as p increases. The expected actual error rate increases too as $\rho$ increases. These strong patterns of behaviour of the expected actual error rates are shown in this case but not when $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ because for given values of $\rho$ the correlation among the elements of $\mathbf{x}$ for $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ is much stronger when compared to the corresponding correlations when $\Sigma = \mathrm{AR}(1)$. From Table 6a, the results show that as $|\rho|$ increases, the expected actual error rate increases, and as p increases the expected actual error rate increases too. From Table 6, it can be seen that the expected actual error rates for the linear discriminant function is very large when compared to the true error rates. Thus, it overestimates the true error rates considerably. On the other hand the expected actual error

| $\Delta^2$ | ρ | p=4 | | | p=8 | | | p=12 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | true **EDC** *LDF* | actual **EDC** *LDF* | plug-in **EDC** *LDF* | true **EDC** *LDF* | actual **EDC** *LDF* | plug-in **EDC** *LDF* | true **EDC** *LDF* | actual **EDC** *LDF* | plug-in **EDC** *LDF* |
| 0.5 | 0.0 | **0.3618** | **0.3784** | **0.3456** | **0.3618** | **0.3996** | **0.3244** | **0.3618** | **0.4208** | **0.3032** |
| | | *0.3618* | *0.3906* | *0.3373* | *0.3618* | *0.4228* | *0.3051* | *0.3618* | *0.4550* | *0.2729* |
| | 0.2 | **0.3645** | **0.3812** | **0.3478** | **0.3645** | **0.4047** | **0.3270** | **0.3645** | **0.4282** | **0.3062** |
| | | *0.3618* | *0.3969* | *0.3372* | *0.3618* | *0.4389* | *0.3049* | *0.3618* | *0.4844* | *0.2727* |
| | 0.4 | **0.3730** | **0.3886** | **0.3537** | **0.3730** | **0.4207** | **0.3352** | **0.3730** | **0.4530** | **0.3163** |
| | | *0.3618* | *0.4064* | *0.3365* | *0.3618* | *0.4669* | *0.3043* | *0.3618* | *0.5169* | *0.2720* |
| | 0.65 | **0.3941** | **0.3931** | **0.3565** | **0.3941** | **0.4523** | **0.3471** | **0.3941** | **0.5222** | **0.3414** |
| | | *0.3618* | | *0.3320* | *0.3618* | | *0.2998* | *0.3618* | | *0.2675* |
| 1.0 | 0.0 | **0.3085** | **0.3200** | **0.2976** | **0.3085** | **0.3341** | **0.2835** | **0.3085** | **0.3481** | **0.2694** |
| | | *0.3085* | *0.3292* | *0.2866* | *0.3085* | *0.3579* | *0.2579* | *0.3085* | | *0.2291* |
| | 0.2 | **0.3121** | **0.3236** | **0.3008** | **0.3121** | **0.3393** | **0.2870** | **0.3121** | **0.3549** | **0.2731** |
| | | *0.3085* | *0.3380* | *0.2864* | *0.3085* | *0.3796* | *0.2577* | *0.3085* | | *0.2289* |
| | 0.4 | **0.3234** | **0.3344** | **0.3102** | **0.3234** | **0.3559** | **0.2978** | **0.3234** | **0.3776** | **0.2852** |
| | | *0.3085* | *0.3517* | *0.2855* | *0.3085* | *0.4167* | *0.2568* | *0.3085* | | *0.2280* |
| | 0.65 | **0.3520** | **0.3521** | **0.3262** | **0.3520** | **0.3927** | **0.3199** | **0.3520** | **0.4404** | **0.3161** |
| | | *0.3085* | | *0.2797* | *0.3085* | | *0.2510* | *0.3085* | | *0.2222* |
| 2.0 | 0.0 | **0.2398** | **0.2474** | **0.2328** | **0.2398** | **0.2562** | **0.2240** | **0.2398** | **0.2650** | **0.2153** |
| | | *0.2398* | *0.2447* | *0.2191* | *0.2398* | *0.2719* | *0.1920* | *0.2398* | | *0.1649* |
| | 0.2 | **0.2442** | **0.2520** | **0.2370** | **0.2442** | **0.2618** | **0.2283** | **0.2442** | **0.2717** | **0.2196** |
| | | *0.2398* | *0.2558* | *0.2189* | *0.2398* | *0.2981* | *0.1918* | *0.2398* | | *0.1647* |
| | 0.4 | **0.2585** | **0.2661** | **0.2499** | **0.2585** | **0.2799** | **0.2420** | **0.2585** | **0.2936** | **0.2340** |
| | | *0.2398* | *0.2733* | *0.2178* | *0.2398* | *0.3414* | *0.1908* | *0.2398* | | *0.1637* |
| | 0.65 | **0.2955** | **0.2967** | **0.2784** | **0.2955** | **0.3236** | **0.2744** | **0.2955** | **0.3550** | **0.2719** |
| | | *0.2398* | | *0.2111* | *0.2398* | | *0.1840* | *0.2398* | | *0.1569* |
| 2.5 | 0.0 | **0.2146** | **0.2213** | **0.2088** | **0.2146** | **0.2287** | **0.2014** | **0.2146** | **0.2361** | **0.1940** |
| | | *0.2146* | *0.2102* | *0.1944* | *0.2146* | *0.2368* | *0.1678* | *0.2146* | *0.2634* | *0.1412* |
| | 0.2 | **0.2193** | **0.2261** | **0.2132** | **0.2193** | **0.2344** | **0.2059** | **0.2193** | **0.2426** | **0.1986** |
| | | *0.2146* | *0.2218* | *0.1941* | *0.2146* | *0.2638* | *0.1675* | *0.2146* | *0.3059* | *0.1409* |
| | 0.4 | **0.2344** | **0.2411** | **0.2271** | **0.2344** | **0.2528** | **0.2204** | **0.2344** | **0.2645** | **0.2135** |
| | | *0.2146* | *0.2402* | *0.1931* | *0.2146* | *0.3075* | *0.1665* | *0.2146* | *0.3747* | *0.1399* |
| | 0.65 | **0.2740** | **0.2755** | **0.2591** | **0.2740** | **0.2988** | **0.2558** | **0.2740** | **0.3258** | **0.2536** |
| | | *0.2146* | | *0.1862* | *0.2146* | | *0.1596* | *0.2146* | | *0.1330* |

**Table 6 :** The 'true', expected actual and expected plug-in error rates of the EDC and LDF under the case of 'non - equivalence' with $\Sigma = AR(1)$.

| $\Delta^2$ | ρ | p=4 | | | p=8 | | | p=12 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | true EDC *LDF* | actual EDC *LDF* | plug-in EDC *LDF* | true EDC *LDF* | actual EDC *LDF* | plug-in EDC *LDF* | true EDC *LDF* | actual EDC *LDF* | plug-in EDC *LDF* |
| 0.5 | - 0.2 | **0.3645** | **0.3812** | **0.3477** | **0.3645** | **0.4047** | **0.3269** | **0.3645** | **0.4282** | **0.3061** |
| | | *0.3618* | | *0.3372* | *0.3618* | | *0.3049* | *0.3618* | | *0.2727* |
| | - 0.4 | **0.3730** | **0.3886** | **0.3525** | **0.3730** | **0.4207** | **0.3336** | **0.3730** | **0.4530** | **0.3147** |
| | | *0.3618* | | *0.3365* | *0.3618* | | *0.3043* | *0.3618* | | *0.2720* |
| | -0.65 | **0.3941** | **0.3931** | **0.3498** | **0.3941** | **0.4523** | **0.3325** | **0.3941** | **0.5222** | **0.3245** |
| | | *0.3618* | | *0.3320* | *0.3618* | | *0.2998* | *0.3618* | | *0.2675* |
| 1.0 | - 0.2 | **0.3121** | **0.3236** | **0.3007** | **0.3121** | **0.3393** | **0.2869** | **0.3121** | **0.3549** | **0.2730** |
| | | *0.3085* | | *0.2824* | *0.3085* | | *0.2577* | *0.3085* | | *0.2289* |
| | - 0.4 | **0.3234** | **0.3344** | **0.3095** | **0.3234** | **0.3559** | **0.2968** | **0.3234** | **0.3776** | **0.2841** |
| | | *0.3085* | | *0.2855* | *0.3085* | | *0.2568* | *0.3085* | | *0.2280* |
| | -0.65 | **0.3520** | **0.3521** | **0.3216** | **0.3520** | **0.3927** | **0.3099** | **0.3520** | **0.4404** | **0.3041** |
| | | *0.3085* | | *0.2797* | *0.3085* | | *0.2510* | *0.3085* | | *0.2222* |
| 2.0 | - 0.2 | **0.2442** | **0.2520** | **0.2369** | **0.2442** | **0.2618** | **0.2283** | **0.2442** | **0.2717** | **0.2196** |
| | | *0.2398* | | *0.2189* | *0.2398* | | *0.1918* | *0.2398* | | *0.1647* |
| | - 0.4 | **0.2585** | **0.2661** | **0.2494** | **0.2585** | **0.2799** | **0.2414** | **0.2585** | **0.2936** | **0.2333** |
| | | *0.2398* | | *0.2178* | *0.2398* | | *0.1908* | *0.2398* | | *0.1637* |
| | -0.65 | **0.2955** | **0.2967** | **0.2754** | **0.2955** | **0.3236** | **0.2678** | **0.2955** | **0.3550** | **0.2643** |
| | | *0.2398* | | *0.2111* | *0.2398* | | *0.1840* | *0.2398* | | *0.1569* |
| 2.5 | - 0.2 | **0.2193** | **0.2661** | **0.2131** | **0.2193** | **0.2344** | **0.2058** | **0.2193** | **0.2426** | **0.1985** |
| | | *0.2146* | | *0.1941* | *0.2146* | | *0.1675* | *0.2146* | | *0.1409* |
| | - 0.4 | **0.2344** | **0.2441** | **0.2266** | **0.2344** | **0.2528** | **0.2198** | **0.2344** | **0.2645** | **0.2129** |
| | | *0.2146* | | *0.1931* | *0.2146* | | *0.1665* | *0.2146* | | *0.1399* |
| | -0.65 | **0.2740** | **0.2755** | **0.2565** | **0.2740** | **0.2988** | **0.2501** | **0.2740** | **0.3258** | **0.2470** |
| | | *0.2146* | | *0.1862* | *0.2146* | | *0.1596* | *0.2146* | | *0.1330* |

**Table 6a :** The 'true', expected actual and expected plug-in error rates of the EDC and LDF under the case of 'non-equivalence' with $\Sigma = AR(1)$.

rate for the Euclidean distance classifier is fairly close to the true error rate. Therefore, it gives a better estimate of the true error rate, when compared with the expected actual error rate from the linear discriminant function.

Thus from Table 6, two general results can be deduced, namely (i) the expected actual error rate of the Euclidean distance classifier is a better estimate of the true error rate when compared to the expected error rate of the linear discriminant function and (ii) the true and expected actual error rates for the Euclidean distance classifier are smaller than the true and expected actual error rates for the linear discriminant function. It can be seen that for the combinations of parameters considered and under the situation of non-equivalence with $\mathbf{\Sigma}$ = AR(1), the Euclidean distance classifier can be said to perform better than the linear discriminant function.

Equivalence, $\Sigma = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ (Table 7) and $\mathbf{\Sigma}$ = AR(1) (Table 8)

Under the case of equivalence of the linear discriminant function and the Euclidean distance classifier, only the results of the expected actual error rate for the Euclidean distance classifier were obtained because the evaluation of the asymptotic expansion of the linear discriminant function was found to be too complex for the time allocated for it. From the results ( see Tables 7 and 8 ) for the Euclidean distance classifier, the expected actual error rate appears to give a good estimate to the "true" error rate. The results show that there is not much difference in the error rates for different values of p, especially when $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$, and the level of correlation is low or moderate

| $\Delta^2$ | $\rho$ | p=4 | | | p=8 | | | p=12 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | true **EDC** *LDF* | actual **EDC** *LDF* | plug-in **EDC** *LDF* | true **EDC** *LDF* | actual **EDC** *LDF* | plug-in **EDC** *LDF* | true **EDC** *LDF* | actual **EDC** *LDF* | plug-in **EDC** *LDF* |
| 0.5 | 0.0 | **0.3618** | **0.3788** | **0.3470** | **0.3618** | **0.4001** | **0.3261** | **0.3619** | **0.4214** | **0.3050** |
| | | *0.3618* | | *0.3373* | *0.3618* | | *0.3037* | | | |
| | 0.2 | **0.3618** | **0.3669** | **0.3510** | **0.3618** | **0.3671** | **0.3426** | **0.3619** | **0.3667** | **0.3378** |
| | | *0.3618* | | *0.3378* | *0.3618* | | *0.3046* | | | |
| | 0.4 | **0.3618** | **0.3641** | **0.3554** | **0.3618** | **0.3639** | **0.3524** | **0.3618** | **0.3638** | **0.3511** |
| | | *0.3618* | | *0.3381* | *0.3618* | | *0.3052* | | | |
| | 0.65 | **0.3618** | **0.3631** | **0.3593** | **0.3618** | **0.3631** | **0.3586** | **0.3618** | **0.3632** | **0.3584** |
| | | *0.3618* | | *0.3384* | *0.3618* | | *0.3059* | | | |
| 1.0 | 0.0 | **0.3085** | **0.3205** | **0.2994** | **0.3085** | **0.3347** | **0.2857** | **0.3085** | **0.3489** | **0.2718** |
| | | *0.3085* | | *0.2867* | *0.3085* | | *0.2562* | | | |
| | 0.2 | **0.3085** | **0.3125** | **0.3020** | **0.3085** | **0.3128** | **0.2967** | **0.3085** | **0.3126** | **0.2936** |
| | | *0.3085* | | *0.2873* | *0.3085* | | *0.2574* | | | |
| | 0.4 | **0.3085** | **0.3107** | **0.3050** | **0.3085** | **0.3107** | **0.3032** | **0.3085** | **0.3106** | **0.3024** |
| | | *0.3085* | | *0.2877* | *0.3085* | | *0.2580* | | | |
| | 0.65 | **0.3085** | **0.3101** | **0.3076** | **0.3085** | **0.3102** | **0.3074** | **0.3085** | **0.3102** | **0.3073** |
| | | *0.3085* | | *0.2882* | *0.3085* | | *0.2590* | | | |
| 2.0 | 0.0 | **0.2398** | **0.2481** | **0.2351** | **0.2397** | **0.2571** | **0.2268** | **0.2398** | **0.2660** | **0.2182** |
| | | *0.2398* | | *0.2196* | *0.2397* | | *0.1902* | | | |
| | 0.2 | **0.2398** | **0.2431** | **0.2367** | **0.2397** | **0.2434** | **0.2336** | **0.2397** | **0.2433** | **0.2318** |
| | | *0.2398* | | *0.2204* | *0.2397* | | *0.1917* | | | |
| | 0.4 | **0.2398** | **0.2420** | **0.2386** | **0.2397** | **0.2421** | **0.2377** | **0.2397** | **0.2421** | **0.2373** |
| | | *0.2398* | | *0.2209* | *0.2397* | | *0.1926* | | | |
| | 0.65 | **0.2398** | **0.2416** | **0.2402** | **0.2397** | **0.2418** | **0.2403** | **0.2397** | **0.2418** | **0.2403** |
| | | *0.2398* | | *0.2214* | *0.2397* | | *0.1939* | | | |
| 2.5 | 0.0 | **0.2146** | **0.2219** | **0.2112** | **0.2146** | **0.2296** | **0.2043** | **0.2146** | **0.2370** | **0.1971** |
| | | *0.2146* | | *0.1951* | *0.2146* | | *0.1661* | | | |
| | 0.2 | **0.2146** | **0.2178** | **0.2126** | **0.2146** | **0.2181** | **0.2101** | **0.2146** | **0.2180** | **0.2085** |
| | | *0.2146* | | *0.1959* | *0.2146* | | *0.1677* | | | |
| | 0.4 | **0.2146** | **0.2168** | **0.2141** | **0.2146** | **0.2170** | **0.2135** | **0.2146** | **0.2170** | **0.2131** |
| | | *0.2146* | | *0.1964* | *0.2146* | | *0.1686* | | | |
| | 0.65 | **0.2146** | **0.2165** | **0.2155** | **0.2146** | **0.2167** | **0.2156** | **0.2146** | **0.2168** | **0.2157** |
| | | *0.2146* | | *0.1970* | *0.2146* | | *0.1699* | | | |

**Table 7 :** The 'true', expected actual and expected plug-in error rates of the EDC and LDF under the case of 'equivalence' with $\Sigma = (1-\rho)\mathbf{I} + \rho\mathbf{J}$.

| $\Delta^2$ | ρ | p=4 | | | p=8 | | | p=12 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | true<br>**EDC**<br>*LDF* | actual<br>**EDC**<br>*LDF* | plug-in<br>**EDC**<br>*LDF* | true<br>**EDC**<br>*LDF* | actual<br>**EDC**<br>*LDF* | plug-in<br>**EDC**<br>*LDF* | true<br>**EDC**<br>*LDF* | actual<br>**EDC**<br>*LDF* | plug-in<br>**EDC**<br>*LDF* |
| 0.5 | 0.0 | **0.3618** | **0.3788** | **0.3470** | **0.3618** | **0.4001** | **0.3261** | **0.3619** | **0.4214** | **0.3050** |
| | | *0.3618* | | *0.3373* | *0.3618* | | *0.3037* | | | |
| | 0.2 | **0.3624** | **0.3704** | **0.3495** | **0.3623** | **0.3806** | **0.3316** | **0.3622** | **0.3907** | **0.3135** |
| | | *0.3618* | | *0.3376* | *0.3618* | | *0.3042* | | | |
| | 0.4 | **0.3634** | **0.3673** | **0.3541** | **0.3636** | **0.3725** | **0.3417** | **0.3633** | **0.3776** | **0.3287** |
| | | *0.3618* | | *0.3378* | *0.3618* | | *0.3044* | | | |
| | 0.65 | **0.3642** | **0.3659** | **0.3597** | **0.3660** | **0.3692** | **0.3549** | **0.3660** | **0.3712** | **0.3482** |
| | | *0.3618* | | *0.3378* | *0.3618* | | *0.3046* | | | |
| 1.0 | 0.0 | **0.3085** | **0.3205** | **0.2994** | **0.3085** | **0.3347** | **0.2857** | **0.3085** | **0.3489** | **0.2718** |
| | | *0.3085* | | *0.2867* | *0.3085* | | *0.2562* | | | |
| | 0.2 | **0.3092** | **0.3153** | **0.3014** | **0.3092** | **0.3221** | **0.2897** | **0.3090** | **0.3288** | **0.2777** |
| | | *0.3085* | | *0.2871* | *0.3085* | | *0.2567* | | | |
| | 0.4 | **0.3106** | **0.3139** | **0.3052** | **0.3109** | **0.3176** | **0.2973** | **0.3105** | **0.3208** | **0.2885** |
| | | *0.3085* | | *0.2873* | *0.3085* | | *0.2570* | | | |
| | 0.65 | **0.3117** | **0.3135** | **0.3095** | **0.3140** | **0.3170** | **0.3079** | **0.3140** | **0.3183** | **0.3032** |
| | | *0.3085* | | *0.2874* | *0.3085* | | *0.2573* | | | |
| 2.0 | 0.0 | **0.2398** | **0.2481** | **0.2351** | **0.2397** | **0.2571** | **0.2268** | **0.2398** | **0.2660** | **0.2182** |
| | | *0.2398* | | *0.2196* | *0.2397* | | *0.1902* | | | |
| | 0.2 | **0.2406** | **0.2453** | **0.2368** | **0.2405** | **0.2496** | **0.2296** | **0.2404** | **0.2537** | **0.2221** |
| | | *0.2398* | | *0.2200* | *0.2397* | | *0.1908* | | | |
| | 0.4 | **0.2424** | **0.2452** | **0.2400** | **0.2428** | **0.2479** | **0.2355** | **0.2422** | **0.2497** | **0.2298** |
| | | *0.2398* | | *0.2203* | *0.2398* | | *0.1913* | | | |
| | 0.65 | **0.2437** | **0.2457** | **0.2433** | **0.2467** | **0.2495** | **0.2439** | **0.2466** | **0.2503** | **0.2412** |
| | | *0.2398* | | *0.2204* | *0.2398* | | *0.1917* | | | |
| 2.5 | 0.0 | **0.2146** | **0.2219** | **0.2112** | **0.2146** | **0.2296** | **0.2043** | **0.2146** | **0.2370** | **0.1971** |
| | | *0.2146* | | *0.1951* | *0.2146* | | *0.1661* | | | |
| | 0.2 | **0.2155** | **0.2198** | **0.2127** | **0.2154** | **0.2234** | **0.2208** | **0.2152** | **0.2269** | **0.2006** |
| | | *0.2146* | | *0.1951* | *0.2146* | | *0.1667* | | | |
| | 0.4 | **0.2173** | **0.2201** | **0.2158** | **0.2178** | **0.2225** | **0.2122** | **0.2172** | **0.2239** | **0.2074** |
| | | *0.2146* | | *0.1959* | *0.2146* | | *0.1671* | | | |
| | 0.65 | **0.2188** | **0.2208** | **0.2189** | **0.2219** | **0.2246** | **0.2201** | **0.2219** | **0.2254** | **0.2178** |
| | | *0.2146* | | *0.1959* | *0.2146* | | *0.1676* | | | |

**Table 8 :** The 'true', expected actual and expected plug-in error rates of the EDC and LDF under the case of 'equivalence' with $\Sigma = AR(1)$.

(i.e. $\rho$=0.20 and $\rho$=0.40). In general, when $\mathbf{\Sigma} = (1\text{-}\rho)\mathbf{I}+\rho\mathbf{J}$ (see Table 7), as p increases the expected actual error rates for the Euclidean distance classifier increases and, as $\rho$ increases the expected actual error rates decrease. Since the true eror rate remains the same (for a given combination of parameters) as $\rho$ increases, the decrease in the expected actual error rates as $\rho$ increases means that better estimates of the true error rates are obtained with increasing $\rho$.

When $\Sigma = AR(1)$ (see Table 8), the results show that when p increases, the actual error rate increases. As $\rho$ increases, the expected actual error rate decreases until $\rho$ becomes high when it increases again slightly (compare the results for $\rho$=0.4 with those for $\rho$=0.65). Furthermore, all the actual error rates overestimate the true error rates. From Table 8a the results show that as p increases, the expected actual error rate increases, and as $|\rho|$ increases, the expected actual error rate increases too. The results also show that the expected actual error rate obtained do not give good estimates of the true error rate when $\rho$ is negative and p is moderate to large (i.e. p=8,12 in Table 8a). Also, the expected actual error rates are out of bound when p is large and $\rho$ is negative and small.

It can be seen from Table 13 (in Appendix A4) that the expected actual error rates when the correlation is negative and small (i.e. $\rho$=-0.06) and $\Sigma = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ are incredibly large (out of bound at large p, i.e. p=16). This behaviour can be explained by the structure of $\mathbf{\Sigma}^{-1}$ when the correlation is negative and small (see Appendix E1) .

| $\Delta^2$ | ρ | p=4 | | | p=8 | | | p=12 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | true EDC *LDF* | actual EDC *LDF* | plug-in EDC *LDF* | true EDC *LDF* | actual EDC *LDF* | plug-in EDC *LDF* | true EDC *LDF* | actual EDC *LDF* | plug-in EDC *LDF* |
| 0.5 | - 0.2 | 0.3626 | 0.3994 | 0.3522 | 0.3624 | 0.4505 | 0.3400 | 0.3623 | 0.5021 | 0.3280 |
| | | *0.3618* | | *0.3367* | *0.3618* | | *0.3031* | | | |
| | - 0.4 | 0.3654 | 0.4549 | 0.3841 | 0.3644 | 0.6034 | 0.4346 | 0.3637 | 0.7601 | 0.4925 |
| | | *0.3618* | | *0.3350* | *0.3618* | | *0.3009* | | | |
| | -0.65 | 0.3721 | 0.7889 | 0.6551 | 0.3706 | ** | ** | 0.3683 | ** | ** |
| | | *0.3618* | | *0.3250* | *0.3618* | | *0.2875* | | | |
| 1.0 | - 0.2 | 0.3096 | 0.3347 | 0.3034 | 0.3092 | 0.3686 | 0.2953 | 0.3091 | 0.4028 | 0.2873 |
| | | *0.3085* | | *0.2859* | *0.3085* | | *0.2552* | | | |
| | - 0.4 | 0.3133 | 0.3736 | 0.3264 | 0.3119 | 0.4717 | 0.3595 | 0.3110 | 0.5756 | 0.3977 |
| | | *0.3085* | | *0.2837* | *0.3085* | | *0.2524* | | | |
| | -0.65 | 0.3222 | 0.6022 | 0.5125 | 0.3203 | ** | 0.9497 | 0.3175 | ** | ** |
| | | *0.3085* | | *0.2790* | *0.3085* | | *0.2350* | | | |
| 2.0 | - 0.2 | 0.2411 | 0.2576 | 0.2382 | 0.2407 | 0.2787 | 0.2332 | 0.2404 | 0.2999 | 0.2282 |
| | | *0.2398* | | *0.2186* | *0.2398* | | *0.1891* | | | |
| | - 0.4 | 0.2457 | 0.2844 | 0.2549 | 0.2439 | 0.3451 | 0.2751 | 0.2427 | 0.4093 | 0.2984 |
| | | *0.2398* | | *0.2161* | *0.2398* | | *0.1856* | | | |
| | -0.65 | 0.2570 | 0.4359 | 0.3789 | 0.2546 | 0.7905 | 0.6548 | 0.2509 | ** | 0.9991 |
| | | *0.2398* | | *0.2011* | *0.2398* | | *0.1646* | | | |
| 2.5 | - 0.2 | 0.2160 | 0.2302 | 0.2141 | 0.2155 | 0.2479 | 0.2099 | 0.2153 | 0.2657 | 0.2056 |
| | | 0.2146 | | 0.1941 | *0.2146* | | *0.1648* | | | |
| | - 0.4 | 0.2209 | 0.2539 | 0.2291 | 0.2190 | 0.3046 | 0.2458 | 0.2178 | 0.3584 | 0.2652 |
| | | 0.2146 | | 0.1915 | *0.2146* | | *0.1612* | | | |
| | -0.65 | 0.2328 | 0.3848 | 0.3365 | 0.2302 | 0.6843 | 0.5695 | 0.2264 | ** | 0.8592 |
| | | 0.2146 | | 0.1762 | *0.2146* | | *0.1394* | | | |

**Table 8a :** The 'true', expected actual and expected plug-in error rates of the EDC and LDF under the case of 'equivalence' with $\Sigma = AR(1)$.

Section 4.2.2 : Plug-in error rates ( category 2 )

As defined in Section 1.3, the plug-in error rate is the error rate obtained by replacing the unknown parameters in the actual error rate by their estimators. Therefore, it is of interest to investigate how the expected plug-in error rate estimates the expected actual error rate.

In all cases of the plug-in error rates, such as cases P1, P2, P3 and P4 (as defined in Section 3.3), the expected plug-in error rates decrease as the Mahalanobis distance increases (see Table 5 to Table 8a). As mentioned in Section 4.2.1, this result is expected. In all cases the plug-in error rate underestimates the expected actual error rates.

Non-equivalence, $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ ( Table 5 )

Under the case of non-equivalence of the linear discriminant function and the Euclidean distance classifier, the expected plug-in error rate for the Euclidean distance classifier decreases as the dimension p increases when $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ ( see Table 5 ). The results also show that the expected plug-in error rate underestimates the expected actual error rate. In general as $\Delta^2$ increases the expected plug-in error rate decreases except at a particular combination of parameters,(see the entry for p=12, $\rho$=0.4, $\Delta^2$=0.5 and $n_1$=$n_2$=50 , in the Table 5). However, when the sample size is large relative to the dimension size, p, (i.e. $n_1$=$n_2$=100; see Table 15a), the expected plug-in error rate decreases (in all situations) as $\Delta^2$ increases. This is due to the fact that when p is large (i.e. at p=16) the values of $n_1$ and $n_2$ chosen must be reasonably large relative to the dimension p in order to obtain reasonable results and estimates. The expected plug-in error rate for the linear discriminant function decreases ( considerably ) as the dimension p increases (see Table 5). For both the linear discriminant function and the Euclidean distance classifier, it was noted earlier that the expected actual error rate increases as p increases when the level of correlation is low (i.e. $\rho \leq 0.20$). The overall effect is that with zero or low correlation (i.e. $\rho \leq 0.20$) as p increases, the plug-in error rates of the linear discriminant function and the Euclidean distance classifier estimate the corresponding actual error rates poorly . Initially when p, $\Delta^2$ and $\rho$ are small (i.e. p=4, $\Delta^2$=0.5,1.0 and $\rho$=0.00,0.20), the difference between the expected plug-in error rate and the expected actual error rate of the Euclidean distance classifier is generally smaller than the corresponding difference for the linear discriminant function. The previous statement is also valid in general for moderate values of p (i.e. p=8) when the Mahalanobis distance is small (i.e. $\Delta^2$=0.5,1.0) for $\rho$=0.00,0.20,0.40. However, from the tables, when p=12 this difference for the Euclidean distance classifier is smaller than the corresponding difference for the linear discriminant function (for most values of $\Delta^2$ and $\rho$). Thus the results in Table 5 show that as p increases the plug-in error rate for the Euclidean distance classifier provides a better estimate of the actual error rate than that for the linear discriminant function, under the case of non-equivalence and $\Sigma = (1-\rho)\mathbf{I} + \rho\mathbf{J}$.

The results in Table 5 also show that at small values of p (i.e. p=4) the expected plug-in error rate for the Euclidean distance classifier increases except when the Mahalanobis distance is small and $\rho$ is large (i.e. $\Delta^2$=0.5, $\rho$=0.65). At moderate and large dimension (e.g. p=8,12) the expected plug-in error rate decreases as $\rho$ increases to moderate or high (e.g. $\rho$=0.40,0.65). This behaviour is consistent with the simulation results of Marco , Young and Turner (1987). On the other hand, the expected plug-in error rate for the linear discriminant function seems to always decrease as $\rho$ increases. This behaviour was referred to earlier, with reference to Cochran's result (Cochran, 1962).

Non-equivalence, $\Sigma$ = AR(1) ( Table 6 )

Under the case of non-equivalence of the linear discriminant function and the Euclidean distance classifier, and when $\Sigma = AR(1)$, the expected plug-in error rates of the linear discriminant function and the Euclidean distance classifier both decrease as p increases (see Table 6). These expected plug-in error rates are also generally less than the corresponding expected actual error rates considerably when p increases . The results also show that the expected plug-in error rate for the linear discriminant function decreases as $\rho$ increases, which is consistent with results by Cochran (1962). Futhermore, for all p, the expected plug-in error rate for the Euclidean distance classifier increases as $\rho$ increases which is generally consistent with results by Marco, Young and Turner (1987). Results for negative $\rho$ (see Table 6a) show that as p increases, the expected plug-in error rate decreases for both the linear discriminant function and the Euclidean distance classifier. Also, as $|\rho|$ increases the expected plug-in error rate of the Euclidean distance classifier increases except when the Mahalanobis distance is small, $|\rho|$ is large and p is small to moderate (e.g. $\Delta^2$=0.5, $\rho$=-0.65 and p=4,8). Meanwhile, the expected plug-in error rate of the linear discriminant function decreases as $|\rho|$ increases. Comparing the expected plug-in error rates of the linear discriminant function and the Euclidean distance classifier for positive $\rho$ (as in Table 6), the difference between the expected plug-in error rate and the expected actual error rate of the Euclidean distance classifier is considerably smaller than the corresponding difference for the linear discriminant function, especially when $\Delta^2$, $\rho$ and p are large. This result shows that the plug-in error rate associated with the Euclidean distance classifier gives a better estimate of its actual error rate, when compared to the performance of the plug-in error rate of the linear discriminant function.

Equivalence, $\boldsymbol{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ (Table 7) and $\boldsymbol{\Sigma} = AR(1)$ (Table 8)

Under the case of equivalence of the linear discriminant function and the Euclidean distance classifier, we cannot make a comparison of performances of the linear discriminant function and the Euclidean distance classifier as the results for the expected actual error rate of the linear discriminant function were not obtained. The reason for this was given earlier.

By considering the results of the expected plug-in error rates of the Euclidean distance classifier when $\boldsymbol{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ ( see Table 7 ), it can be seen that the expected plug-in error rate underestimates the expected actual error rate. The results also show that as $\rho$ increases, the expected plug-in error rate increases whereas the expected actual error rate decreases. Furthermore, the expected plug-in error rates decrease as p increases, except when $\Delta^2$ and $\rho$ are large (i.e. $\Delta^2$=2.0,2.5 and $\rho$=0.65), whereas the expected actual error rate increases when p is small or moderate (i.e. p=4,8) . Note however, that the expected actual error rate decreases slightly when p increases to 12. Therefore, the plug-in error rate estimates the en $\boldsymbol{\Sigma} = AR(1)$. However, note that the result for the expected plug-in error rate is out of bound when the correlation is small and negative (i.e. when $\rho$=-0.06; see Table 19). This is caused by the structure of $\Sigma^{-1}$ when $\rho$ is small and negative (as explained earlier). However when considering only negative $\rho$ (see Table 8a) the results show that as p increases and $|\rho|$ is moderate or large (i.e. $|\rho|$=0.4,0.65), the expected plug-in error rates increase for the Euclidean distance classifier. However, when $|\rho|$ and p are small (i.e. $|\rho|$=0.2 and p=4) the expected plug-in error rate decreases as p increases. The results also show that as $|\rho|$ increases the estimation of the true error rate by the expected actual error rate is quite bad (especially at high correlation, i.e. $\rho$=0.65). Furthermore, the plug-in error rate also gives bad estimates of the actual error rate.

For the linear discriminant function under the case of equivalence when $\Sigma = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ or $\Sigma = \mathrm{AR}(1)$ (see Table 7 and Table 8), the expected plug-in error rate decreases considerably (compared to the corresponding value for the Euclidean distance classifier) as p increases. By considering only negative $\rho$, under the case of equivalence and $\Sigma$ = AR(1), Table 8a shows that the expected plug-in error rate decreases as $|\rho|$ increases or as p increases. Since the results of the expected actual error rates for the linear discriminant function have not been obtained, a comparison cannot be made about how well these expected plug-in error rates estimate the expected actual error rates.

For all cases of the expected plug-in error rates (i.e. P1, P2, P3 and P4) for the linear discriminant function and the Euclidean distance classifier the general result is that as p increases the expected plug-in error rates decrease but the expected actual error rates increase. This shows that for both the linear discriminant function and the Euclidean distance classifier the estimation of the actual error rate by the plug-in error rate worsens as p increases. This result is consistent with the results of Murray (1977), which were obtained in practical applications.

It can be deduced from the analysis so far that the plug-in error rates for the Euclidean distance classifier give better estimates of the actual error rates for all dimensions of p which were considered, when compared to the corresponding estimates from the linear discriminant function (whose estimates worsen as p increases).

# CHAPTER 5

## SUMMARY AND CONCLUSION

As discussed in the previous chapter the expected actual error rates overestimates the "true" error rates for both the linear discriminant function and the Euclidean distance classifier, although it does so more significantly for the linear discriminant function. Exceptions occur for the case of non-equivalence with $\mathbf{\Sigma} = (1\text{-}\rho)\mathbf{I} + \rho\mathbf{J}$ and large Mahalanobis distances. The situation for the plug-in error rate is that for both the linear discriminant function and the Euclidean distance classifier it underestimates the actual error rate, and the situation is worse for the linear discriminant function at high dimensions of p. In the case of non-equivalence with $\mathbf{\Sigma} = (1\text{-}\rho)\mathbf{I} + \rho\mathbf{J}$ and small dimensions of p, the estimation of the actual error rate by the plug-in error rate is better for the Eucldean distance classifier when the Mahalanobis distance and $\rho$ are small. Both the expected actual error rate and the expected plug-in error rate for the linear discriminant function are very much affected by the dimension p, especially in the case of non-equivalence of the linear discriminant function and the Euclidean distance classifier at small Mahalanobis distance. Meanwhile, the expected actual error rate and the expected plug-in error rate of the Euclidean distance classifier are not affected as much by the dimension p, especially when $\rho$ is small or moderate (i.e. $\rho$=0.0,0.2,0.4) and $\Sigma = AR(1)$.

Therefore it is of interest to investigate how these results compare with the results obtained by the authors as described in Chapter 2. The results obtained here are supported by the results obtained by all the authors of the three articles to the effect that the Euclidean distance classifier outperforms the linear discriminant function at higher dimensions. This is explained by the results of J. Van Ness ( 1982 ) that at larger dimensions the sample estimate of the covariance matrix is poor. This affects only the sample linear discriminant function since it requires the estimation of $\Sigma$ . Furthermore, the Euclidean distance classifier has simpler actual error rate and plug-in error rate functions compared to the linear discriminant function and it is therefore easier to obtain the asymptotic expansions for the plug-in and actual error rates (see Appendix A1.2 and Appendix A2.2) associated with it. Thus the computation of the asymptotic error rates for the Euclidean distance classifier is considerably faster than for the linear discriminant function.

It is clear that the linear discriminant function involves the inverse matrix $\Sigma^{-1}$ while the Euclidean distance classifier does not involve this matrix for both cases of the plug-in error rate and the actual error rate. We can conclude that the poor performance of the linear discriminant function ( as the dimension p increases ) maybe due, to some extent, to the more complex structure of the function of the linear discriminant function (which involves $\Sigma^{-1}$). This is supported by the work of J. Van Ness (1982).

The expected actual error rate of the Euclidean distance classifier gave better estimates of the "true" actual error rate than the corresponding error rate for the linear discriminant function. Also, the expected plug-in error rate were closer to the expected actual error rate at large values of p. The dimension p does not have a great effect on the expected actual error rate for the Euclidean distance classifier in the case of equivalence with large Mahalanobis distance. Meanwhile, for the linear discriminant function, the dimension p affects the expected actual error rate and the expected plug-in error rate greatly.

As p increases the plug-in error rates for both the linear discriminant function and the Euclidean distance classifier estimate the actual error rate poorly. While this is true, the plug-in error rate for the Euclidean distance classifier gives better estimate of the actual error rate in general (compared to the linear discriminant function). Under the same situation (i.e. p increasing) however, the estimation of the true error rate by the expected actual error rate seems to improve for both the linear discriminant function and the Euclidean distance classifier. Therefore, we can conclude that if the factors of "ease of application" and evaluation of performance by the actual and plug-in error rates only are considered, the Euclidean distance classifier is a better method than the linear discriminant function as p increases.

This research work has extended the results of previous work in the sense that (i) comparison of the linear discriminant function and the Euclidean distance classifier has been done under the situation of (non-trivial) equivalence and non-equivalence of the linear discriminant function and the Euclidean distance classifier, (ii) the comparison has been done via asymptotic expansions rather than simulation (and the results obtained support previous work), and (iii) $\mathbf{\Sigma}$ is also chosen as an autoregressive process of order 1 ( $\mathbf{\Sigma} = \mathrm{AR}(1)$ ) and not just $\mathbf{\Sigma} = \mathbf{I}$ or $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$.

Further work that could be done are

(i) to obtain the asymptotic expansion for the expected actual error rate of the linear discriminant function under the case of equivalence of the linear discriminant function and the Euclidean distance classifier when $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ or $\mathbf{\Sigma} = \mathrm{AR}(1)$,

(ii) examine the performance of the two discriminant functions when the training data are correlated,

(iii) perform simulation study for the case when $\mathbf{\Sigma} = \mathrm{AR}(1)$,

(iv) perform simulation or work on smaller or bigger values of Mahalanobis distances, i.e. $\Delta^2 < 0.5$ or $\Delta^2 > 2.5$, and

(v) use bootstrap methods to estimate the error rates.

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## *Appendix A1.1*

# *ASYMPTOTIC EXPANSION FOR EUCLIDEAN DISTANCE CLASSIFIER*

## *Actual error rate*

We have $p_{21}^{(A)} = \Phi[\dfrac{-(\overline{x}_1 - \overline{x}_2)^T [\mu_1 - 1/2(\overline{x}_1 + \overline{x}_2)]}{[(\overline{x}_1 - \overline{x}_2)^T \Sigma(\overline{x}_1 - \overline{x}_2)]^{1/2}}] = \Phi[-A]$.

Using Taylor series expansion (to second order approximation), we obtain

$$
\begin{aligned}
\Phi(\overline{x}_1, \overline{x}_2) = \Phi(\mu_1, \mu_2) &+ \sum_{j=1}^{p} \frac{\partial \Phi}{\partial \overline{x}_{1j}}(\overline{x}_{1j} - \mu_{1j}) + \sum_{j=1}^{p} \frac{\partial \Phi}{\partial \overline{x}_{2j}}(\overline{x}_{2j} - \mu_{2j}) \\
&+ \frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}}(\overline{x}_{1i} - \mu_{1i})(\overline{x}_{1j} - \mu_{1j}) \\
&+ \frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}}(\overline{x}_{2i} - \mu_{2i})(\overline{x}_{2j} - \mu_{2j}) \\
&+ \frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{1i} \partial \overline{x}_{2j}}(\overline{x}_{1i} - \mu_{1i})(\overline{x}_{2j} - \mu_{2j}) \quad .
\end{aligned}
$$

Where $\sum_{i=1}^{p} \sum_{j=1}^{p} \dfrac{\partial^2 \Phi}{\partial \overline{x}_{1i} \partial \overline{x}_{2j}} = 0$.

Taking expectations gives :

$$E[\Phi(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2)] = \Phi(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2) + \sum_{j=1}^{p} \frac{\partial \Phi}{\partial \bar{x}_{1j}} E(\bar{x}_{1j} - \mu_{1j}) + \sum_{j=1}^{p} \frac{\partial \Phi}{\partial \bar{x}_{2j}} E(\bar{x}_{2j} - \mu_{2j})$$
$$+ \frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}} \operatorname{cov}(\bar{x}_{1,ij})$$
$$+ \frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \bar{x}_{2i} \partial \bar{x}_{2j}} \operatorname{cov}(\bar{x}_{2,ij})$$

where $\sum_{j=1}^{p} \frac{\partial \Phi}{\partial \bar{x}_{1j}} E(\bar{x}_{1j} - \mu_{1j}) = \sum_{j=1}^{p} \frac{\partial \Phi}{\partial \bar{x}_{2j}} E(\bar{x}_{2j} - \mu_{2j}) = 0$

$$\therefore E[\Phi(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2)] = \Phi(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2) + \frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}} \operatorname{cov}(\bar{x}_{1,ij})$$
$$+ \frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \bar{x}_{2i} \partial \bar{x}_{2j}} \operatorname{cov}(\bar{x}_{2,ij}) \quad .$$

Consider cases 1A and 1B where the conditions are as follow (respectively) :

$$\boldsymbol{\mu}_1 = (m, 0, \ldots \ldots, 0)^T \qquad\qquad \boldsymbol{\mu}_1 = (m, 0, \ldots \ldots, 0)^T$$

$$\boldsymbol{\mu}_2 = (0, \ldots \ldots \ldots, 0)^T \qquad \text{and} \qquad \boldsymbol{\mu}_2 = (0, \ldots \ldots \ldots, 0)^T$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho & \rho & \ldots & \rho \\ \rho & 1 & \rho & \ldots & \rho \\ \vdots & & & & \vdots \\ \rho & \ldots & \ldots \ldots & \rho & 1 \end{bmatrix} \qquad\qquad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho & \rho^2 & \ldots & \rho^{p-1} \\ \rho & 1 & \rho & \ldots & \rho^{p-2} \\ \vdots & & & & \vdots \\ \rho^{p-1} & \ldots \ldots & & \rho & 1 \end{bmatrix}$$

Need to find $\dfrac{\partial^2 \Phi}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}}$ :

$$\frac{\partial^2 \Phi}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}} = \frac{\partial}{\partial \bar{x}_{1i}}\left[ -\phi(-A)\frac{\partial A}{\partial \bar{x}_{1j}} \right] = -\left[ \phi(-A)\frac{\partial^2 A}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}} + \frac{\partial \phi}{\partial \bar{x}_{1i}} \frac{\partial A}{\partial \bar{x}_{1j}} \right]$$

where $$\frac{\partial \phi}{\partial \bar{x}_{1i}} = -A\phi(-A)\frac{\partial A}{\partial \bar{x}_{1i}}$$

$$\therefore \quad \frac{\partial^2 \Phi}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}} = -\left[ \phi(-A)\frac{\partial^2 A}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}} - A\phi(-A)\frac{\partial A}{\partial \bar{x}_{1i}} \frac{\partial A}{\partial \bar{x}_{1j}} \right]$$

$$= -\phi(-A)\left[ \frac{\partial^2 A}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}} - A\frac{\partial A}{\partial \bar{x}_{1i}} \frac{\partial A}{\partial \bar{x}_{1j}} \right] \qquad (A1.1)$$

Need to find $\dfrac{\partial A}{\partial \bar{x}_{1i}}$ ,$\dfrac{\partial A}{\partial \bar{x}_{1j}}$ and $\dfrac{\partial^2 A}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}}$ :

Under cases 1A and 1B ,

let $$A = (\bar{\mathbf{x}}_1^T \boldsymbol{\Sigma} \bar{\mathbf{x}}_1)^{-\frac{1}{2}} [\boldsymbol{\mu}_1 - \frac{1}{2}\bar{\mathbf{x}}_1]^T \bar{\mathbf{x}}_1$$

$$\frac{\partial A}{\partial \bar{\mathbf{x}}_1} = -\frac{1}{2}(\bar{\mathbf{x}}_1^T \boldsymbol{\Sigma} \bar{\mathbf{x}}_1)^{-\frac{3}{2}} 2\boldsymbol{\Sigma}\bar{\mathbf{x}}_1 \times (\boldsymbol{\mu}_1 - \frac{1}{2}\bar{\mathbf{x}}_1)^T \bar{\mathbf{x}}_1$$
$$+(\bar{\mathbf{x}}_1^T \boldsymbol{\Sigma} \bar{\mathbf{x}}_1)^{-\frac{1}{2}} \frac{\partial}{\partial \bar{x}_{1i}}\left[ (\boldsymbol{\mu}_1 - \frac{1}{2}\bar{\mathbf{x}}_1)^T \bar{\mathbf{x}}_1 \right]$$

where $$\frac{\partial}{\partial \bar{x}_{1i}}\left[ (\boldsymbol{\mu}_1 - \frac{1}{2}\bar{\mathbf{x}}_1)^T \bar{\mathbf{x}}_1 \right] = \frac{\partial}{\partial \bar{x}_{1i}}\left[ \boldsymbol{\mu}_1^T \bar{\mathbf{x}}_1 - \frac{1}{2}\bar{\mathbf{x}}_1^T \bar{\mathbf{x}}_1 \right]$$

$$= \boldsymbol{\mu}_1 - \bar{\mathbf{x}}_1$$

$$\therefore \quad \frac{\partial A}{\partial \bar{\mathbf{x}}_1} = -(\bar{\mathbf{x}}_1^T \boldsymbol{\Sigma} \bar{\mathbf{x}}_1)^{-\frac{3}{2}} \boldsymbol{\Sigma} \bar{\mathbf{x}}_1 (\boldsymbol{\mu}_1 - \frac{1}{2}\bar{\mathbf{x}}_1)^T \bar{\mathbf{x}}_1 + (\bar{\mathbf{x}}_1^T \boldsymbol{\Sigma} \bar{\mathbf{x}}_1)^{-\frac{1}{2}} (\boldsymbol{\mu}_1 - \bar{\mathbf{x}}_1) \quad \textit{(A1.2)}$$

where $$\bar{\mathbf{x}}_1^T \boldsymbol{\Sigma} \bar{\mathbf{x}}_1 = m^2 \sigma_{11}$$

$$\boldsymbol{\Sigma} \bar{\mathbf{x}}_1 (\boldsymbol{\mu}_1 - \frac{1}{2}\bar{\mathbf{x}}_1)^T \bar{\mathbf{x}}_1 = \frac{1}{2} m^3 \begin{bmatrix} \sigma_{11} \\ \vdots \\ \sigma_{p1} \end{bmatrix}$$

$$\bar{\mathbf{x}}_1^T \bar{\mathbf{x}}_1 = m^2$$

$$A = \frac{1}{2} m (\sigma_{11})^{-\frac{1}{2}} = \frac{1}{2} m$$

$$\therefore \quad \frac{\partial A}{\partial \bar{\mathbf{x}}_1} = -\left(m^2 \sigma_{11}\right)^{-\frac{3}{2}} \times \frac{1}{2} m^3 \begin{bmatrix} \sigma_{11} \\ \vdots \\ \sigma_{p1} \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} \sigma_{11} \\ \vdots \\ \sigma_{p1} \end{bmatrix}$$

$$\therefore \quad \frac{\partial A}{\partial \bar{x}_{1j}} = -\frac{1}{2} \sigma_{j1} \qquad \text{and} \qquad \frac{\partial A}{\partial \bar{x}_{1i}} = -\frac{1}{2} \sigma_{i1}$$

Aside:

$$\bar{\mathbf{x}}_1^T \boldsymbol{\Sigma} \bar{\mathbf{x}}_1 = \sum_{v=1}^{p} [\, \bar{x}_{1v} ( \sum_{u=1}^{p} \bar{x}_{1u} \sigma_{uv} ) ]$$

$$\boldsymbol{\Sigma}\overline{\mathbf{x}}_1 (\boldsymbol{\mu}_1 - \frac{1}{2}\overline{\mathbf{x}}_1)^T \overline{\mathbf{x}}_1 = \begin{bmatrix} \sum_{u=1}^{p} \overline{x}_{1u}\sigma_{1u} \\ \vdots \\ \sum_{u=1}^{p} \overline{x}_{1u}\sigma_{pu} \end{bmatrix} \times \sum_{u=1}^{p}[\overline{x}_{1u}(\mu_{1u} - \frac{1}{2}\overline{x}_{1u})]$$

Rewrite *(A1.2)* as:

$$\frac{\partial A}{\partial \overline{\mathbf{x}}_1} = -\left\{ \sum_{v=1}^{p} [\overline{x}_{1v}(\sum_{u=1}^{p} \overline{x}_{1u}\sigma_{uv})] \right\}^{-\frac{3}{2}} \begin{bmatrix} \sum_{u=1}^{p} \overline{x}_{1u}\sigma_{1u} \\ \vdots \\ \sum_{u=1}^{p} \overline{x}_{1u}\sigma_{pu} \end{bmatrix} \times \sum_{u=1}^{p}[\overline{x}_{1u}(\mu_{1u} - \frac{1}{2}\overline{x}_{1u})]$$
$$+\left\{ \sum_{v=1}^{p} [\overline{x}_{1v}(\sum_{u=1}^{p} \overline{x}_{1u}\sigma_{uv})] \right\}^{-\frac{1}{2}} \begin{bmatrix} \mu_{11} - \overline{x}_{11} \\ \vdots \\ \mu_{1p} - \overline{x}_{1p} \end{bmatrix}$$

$$\therefore \quad \frac{\partial A}{\partial \overline{x}_{1j}} = -\left\{ \sum_{v=1}^{p} [\overline{x}_{1v}(\sum_{u=1}^{p} \overline{x}_{1u}\sigma_{uv})] \right\}^{-\frac{3}{2}} \times \sum_{u=1}^{p} \overline{x}_{1u}\sigma_{ju} \times \sum_{u=1}^{p}[\overline{x}_{1u}(\mu_{1u} - \frac{1}{2}\overline{x}_{1u})]$$
$$+\left\{ \sum_{v=1}^{p} [\overline{x}_{1v}(\sum_{u=1}^{p} \overline{x}_{1u}\sigma_{uv})] \right\}^{-\frac{1}{2}} (\mu_{1j} - \overline{x}_{1j})$$

$$\therefore \quad \frac{\partial^2 A}{\partial \overline{x}_{1i}\partial \overline{x}_{1j}} = -\frac{\partial}{\partial \overline{x}_{1i}}\left[ \left\{ \sum_{v=1}^{p} [\overline{x}_{1v}(\sum_{u=1}^{p} \overline{x}_{1u}\sigma_{uv})] \right\}^{-\frac{3}{2}} \sum_{u=1}^{p} \overline{x}_{1u}\sigma_{ju} \right.$$
$$\left. \times \sum_{u=1}^{p}[\overline{x}_{1u}(\mu_{1u} - \frac{1}{2}\overline{x}_{1u})] \right]$$
$$+\frac{\partial}{\partial \overline{x}_{1i}}\left[ \left\{ \sum_{v=1}^{p} [\overline{x}_{1v}(\sum_{u=1}^{p} \overline{x}_{1u}\sigma_{uv})] \right\}^{-\frac{1}{2}} (\mu_{1j} - \overline{x}_{1j}) \right]$$

where

$$\frac{\partial}{\partial \overline{x}_{1i}}\left[\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{ju})\sum_{u=1}^{p}[\overline{x}_{1u}(\mu_{1u}-\frac{1}{2}\overline{x}_{1u})]\right]$$

$$=\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}}\times\frac{\partial}{\partial \overline{x}_{1i}}\left[(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{ju})\sum_{u=1}^{p}[\overline{x}_{1u}(\mu_{1u}-\frac{1}{2}\overline{x}_{1u})]\right]$$
$$+\frac{\partial}{\partial \overline{x}_{1i}}\left[\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}}\right]\times(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{ju})\sum_{u=1}^{p}[\overline{x}_{1u}(\mu_{1u}-\frac{1}{2}\overline{x}_{1u})]$$

$$=\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}}\times\left[(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{ju})\frac{\partial}{\partial \overline{x}_{1i}}\left[\sum_{u=1}^{p}[\overline{x}_{1u}(\mu_{1u}-\frac{1}{2}\overline{x}_{1u})]\right]\right.$$
$$\left.+\frac{\partial}{\partial \overline{x}_{1i}}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{ju})\times\sum_{u=1}^{p}[\overline{x}_{1u}(\mu_{1u}-\frac{1}{2}\overline{x}_{1u})]\right]$$
$$-3\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{uv})]\right\}^{-\frac{5}{2}}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{ui})(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{ju})\sum_{u=1}^{p}[\overline{x}_{1u}(\mu_{1u}-\frac{1}{2}\overline{x}_{1u})]$$

$$=\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}}$$
$$\times\left[(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{ju})(\mu_{1i}-\overline{x}_{1i})+\sigma_{ji}\sum_{u=1}^{p}[\overline{x}_{1u}(\mu_{1u}-\frac{1}{2}\overline{x}_{1u})]\right]$$
$$-3\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{uv})]\right\}^{-\frac{5}{2}}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{ui})(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{ju})\sum_{u=1}^{p}[\overline{x}_{1u}(\mu_{1u}-\frac{1}{2}\overline{x}_{1u})]$$

$$=\left\{m^2\sigma_{11}\right\}^{-\frac{3}{2}}\times\frac{1}{2}\sigma_{ji}m^2 \quad - \quad 3\left\{m^2\sigma_{11}\right\}^{-\frac{5}{2}}\times m\sigma_{1i}\times m\sigma_{j1}\times\frac{1}{2}m^2$$

$$=\frac{1}{2m}\sigma_{ji} \quad - \quad \frac{3}{2m}\sigma_{1i}\sigma_{j1}$$

$$=\frac{1}{2m}\left[\sigma_{ji}-3\sigma_{1i}\sigma_{j1}\right]$$

and $$\frac{\partial}{\partial \overline{x}_{1i}}\left[\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{uv})]\right\}^{-\frac{1}{2}}(\mu_{1j}-\overline{x}_{1j})\right]$$

$$=\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{uv})]\right\}^{-\frac{1}{2}}\times\begin{cases}-1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}$$

$$+\frac{\partial}{\partial \overline{x}_{1i}}\left[\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{uv})]\right\}^{-\frac{1}{2}}\right]\times(\mu_{1j}-\overline{x}_{1j})$$

$$=\begin{cases}-\left\{m^2\sigma_{11}\right\}^{-\frac{1}{2}} & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}\qquad=\begin{cases}-\dfrac{1}{m} & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}$$

$$\therefore\quad \frac{\partial^2 A}{\partial \overline{x}_{1i}\partial \overline{x}_{1j}}=-\frac{1}{2m}\left[\sigma_{ji}-3\sigma_{1i}\sigma_{j1}\right]+\begin{cases}-\dfrac{1}{m} & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}$$

$$=-\frac{1}{2m}\begin{cases}\sigma_{ji}-3\sigma_{1i}\sigma_{j1}+2 & \text{if } i=j\\ \sigma_{ji}-3\sigma_{1i}\sigma_{j1} & \text{if } i\neq j\end{cases}\qquad (A1.3)$$

$$A\frac{\partial A}{\partial \overline{x}_{1i}}\frac{\partial A}{\partial \overline{x}_{1j}}=\frac{1}{2}m\left\{\sigma_{11}\right\}^{-\frac{1}{2}}\times-\frac{1}{2}\sigma_{j1}\times-\frac{1}{2}\sigma_{i1}=\frac{1}{8}m\sigma_{j1}\sigma_{i1}\qquad (A1.4)$$

$$\therefore\frac{\partial^2 \Phi}{\partial \overline{x}_{1i}\partial \overline{x}_{1j}}=\phi\left(-\frac{m}{2}\right)\times\begin{cases}\dfrac{1}{2m}\left[\sigma_{ji}-3\sigma_{1i}\sigma_{j1}+2\right]+\dfrac{1}{8}m\sigma_{j1}\sigma_{i1} & \text{if } i=j\\ \dfrac{1}{2m}\left[\sigma_{ji}-3\sigma_{1i}\sigma_{j1}\right]+\dfrac{1}{8}m\sigma_{j1}\sigma_{i1} & \text{if } i\neq j\end{cases}$$

*expression (A1.5)*

Now need to find $\dfrac{\partial^2 \Phi}{\partial \overline{x}_{2i}\partial \overline{x}_{2j}}$ :

$$\frac{\partial^2 \Phi}{\partial \bar{x}_{2i} \partial \bar{x}_{2j}} = -\phi(-A)\left[\frac{\partial^2 A}{\partial \bar{x}_{2i} \partial \bar{x}_{2j}} - A \frac{\partial A}{\partial \bar{x}_{2i}} \frac{\partial A}{\partial \bar{x}_{2j}}\right] \qquad (A1.6)$$

Need to find $\dfrac{\partial A}{\partial \bar{x}_{2j}}$ , $\dfrac{\partial A}{\partial \bar{x}_{2i}}$ , $\dfrac{\partial^2 A}{\partial \bar{x}_{2i} \partial \bar{x}_{2j}}$ :

Let $A = \left[(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T \boldsymbol{\Sigma}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)\right]^{-\frac{1}{2}} \left[\boldsymbol{\mu}_1 - \frac{1}{2}(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2)\right]^T (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)$

$$\begin{aligned}\frac{\partial A}{\partial \bar{\mathbf{x}}_2} = &-\left[(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T \boldsymbol{\Sigma}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)\right]^{-\frac{3}{2}} \boldsymbol{\Sigma}(\bar{\mathbf{x}}_2 - \bar{\mathbf{x}}_1) \\ &\times \left[\boldsymbol{\mu}_1 - \frac{1}{2}(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2)\right]^T (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) \\ &+\left[(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T \boldsymbol{\Sigma}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)\right]^{-\frac{1}{2}} (\bar{\mathbf{x}}_2 - \boldsymbol{\mu}_1)\end{aligned} \qquad (A1.7)$$

where
$$(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T \boldsymbol{\Sigma}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) = m^2\sigma_{11} = m^2$$
$$\boldsymbol{\Sigma}(\bar{\mathbf{x}}_2 - \bar{\mathbf{x}}_1)\left[\boldsymbol{\mu}_1 - \frac{1}{2}(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2)\right]^T (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) = -\frac{1}{2} m^3 \begin{bmatrix}\sigma_{11} \\ \vdots \\ \sigma_{p1}\end{bmatrix}$$

$$\therefore \quad \frac{\partial A}{\partial \bar{\mathbf{x}}_2} = \frac{1}{2}\left\{m^2\sigma_{11}\right\}^{-\frac{3}{2}} m^3 \begin{bmatrix}\sigma_{11} \\ \vdots \\ \sigma_{p1}\end{bmatrix} + \left\{m^2\sigma_{11}\right\}^{-\frac{1}{2}} \begin{bmatrix}-m \\ 0 \\ \vdots \\ 0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1 \\ \sigma_{21} \\ \vdots \\ \sigma_{p1}\end{bmatrix}$$

$$\therefore \frac{\partial A}{\partial \bar{x}_{2j}} = \begin{cases} -\frac{1}{2} & \text{if} \quad j = 1 \\ \frac{1}{2}\sigma_{j1} & \text{if} \quad j \neq 1 \end{cases} \qquad \text{and} \qquad \frac{\partial A}{\partial \bar{x}_{2i}} = \begin{cases} -\frac{1}{2} & \text{if} \quad i = 1 \\ \frac{1}{2}\sigma_{i1} & \text{if} \quad i \neq 1 \end{cases}$$

Aside :

$$(\overline{\mathbf{x}}_1-\overline{\mathbf{x}}_2)^T\,\boldsymbol{\Sigma}(\overline{\mathbf{x}}_1-\overline{\mathbf{x}}_2)=(m-\overline{x}_{21})\left[(m-\overline{x}_{21})\sigma_{11}-\sum_{u=2}^{p}\overline{x}_{2u}\,\sigma_{u1}\right]$$
$$-\sum_{v=2}^{p}\overline{x}_{2v}\left[(m-\overline{x}_{21})\sigma_{1v}-\sum_{u=2}^{p}\overline{x}_{2u}\,\sigma_{uv}\right]$$
$$=C1$$

$$\boldsymbol{\Sigma}(\overline{\mathbf{x}}_2-\overline{\mathbf{x}}_1)\left[\boldsymbol{\mu}_1-\frac{1}{2}(\overline{\mathbf{x}}_1+\overline{\mathbf{x}}_2)\right]^T(\overline{\mathbf{x}}_1-\overline{\mathbf{x}}_2)$$
$$=-\frac{1}{2}\begin{bmatrix}(m-\overline{x}_{21})\sigma_{11}-\sum_{u=2}^{p}\overline{x}_{2u}\sigma_{1u}\\ \vdots\\ (m-\overline{x}_{21})\sigma_{p1}-\sum_{u=2}^{p}\overline{x}_{2u}\sigma_{pu}\end{bmatrix}\times\left\{(m-\overline{x}_{21})^2+\sum_{u=2}^{p}(\overline{x}_{2u})^2\right\}$$
$$(=-\frac{1}{2}\mathbf{S}\mathbf{y}\mathbf{y}^T\mathbf{y})$$

Using the above expressions rewrite *(A1.7)* as :

$$\frac{\partial A}{\partial\overline{\mathbf{x}}_2}=\frac{1}{2}\{C1\}^{-\frac{3}{2}}\begin{bmatrix}(m-\overline{x}_{21})\sigma_{11}-\sum_{u=2}^{p}\overline{x}_{2u}\sigma_{1u}\\ \vdots\\ (m-\overline{x}_{21})\sigma_{p1}-\sum_{u=2}^{p}\overline{x}_{2u}\sigma_{pu}\end{bmatrix}\times\left\{(m-\overline{x}_{21})^2+\sum_{u=2}^{p}(\overline{x}_{2u})^2\right\}$$
$$-\{C1\}^{-\frac{1}{2}}\begin{bmatrix}m-\overline{x}_{21}\\ -\overline{x}_{22}\\ \vdots\\ -\overline{x}_{2p}\end{bmatrix}$$

$$\therefore\frac{\partial A}{\partial\overline{x}_{2j}}=\frac{1}{2}\{C1\}^{-\frac{3}{2}}\left\{(m-\overline{x}_{21})\sigma_{j1}-\sum_{u=2}^{p}\overline{x}_{2u}\sigma_{ju}\right\}\left\{(m-\overline{x}_{21})^2+\sum_{u=2}^{p}(\overline{x}_{2u})^2\right\}$$
$$-\{C1\}^{-\frac{1}{2}}\begin{cases}(m-\overline{x}_{21}) & \text{if}\quad j=1\\ -\overline{x}_{2j} & \text{if}\quad j\neq 1\end{cases}$$

$$=a+b$$

$$\therefore \quad \frac{\partial^2 A}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}} = \frac{\partial}{\partial \overline{x}_{2i}} \left[ \frac{\partial A}{\partial \overline{x}_{2j}} \right] = \frac{\partial a}{\partial \overline{x}_{2i}} + \frac{\partial b}{\partial \overline{x}_{2i}} \qquad (A1.8)$$

(1) Find $\frac{\partial a}{\partial \overline{x}_{2i}}$ :

$$a = \frac{1}{2}\{C1\}^{-\frac{3}{2}} \times \left(\mathbf{S}\mathbf{y}\mathbf{y}^{T}\mathbf{y}\right)_j$$

$$\therefore \quad \frac{\partial a}{\partial \overline{x}_{2i}} = \frac{1}{2}\{C1\}^{-\frac{3}{2}} \frac{\partial}{\partial \overline{x}_{2i}}\left[\left(\mathbf{S}\mathbf{y}\mathbf{y}^{T}\mathbf{y}\right)_j\right] + \frac{1}{2}\frac{\partial}{\partial \overline{x}_{2i}}\left[\{C1\}^{-\frac{3}{2}}\right] \times \left(\mathbf{S}\mathbf{y}\mathbf{y}^{T}\mathbf{y}\right)_j$$

where
$$\frac{\partial}{\partial \overline{x}_{2i}}\left[\left(\mathbf{S}\mathbf{y}\mathbf{y}^{T}\mathbf{y}\right)_j\right] = \begin{cases} -3\sigma_{j1} m^2 & \text{if} \quad i = 1 \\ -\sigma_{ji} m^2 & \text{if} \quad i \neq 1 \end{cases}$$

and
$$\frac{\partial}{\partial \overline{x}_{2i}}\left[\{C1\}^{-\frac{3}{2}}\right] = -\frac{3}{2}\{C1\}^{-\frac{5}{2}} \times -2m\sigma_{1i} = 3m\sigma_{1i}\{C1\}^{-\frac{5}{2}}$$

$$\therefore \frac{\partial a}{\partial \overline{x}_{2i}} = \frac{1}{2}\{C1\}^{-\frac{3}{2}} \begin{cases} -3\sigma_{j1} m^2 & \text{if} \quad i = 1 \\ -\sigma_{ji} m^2 & \text{if} \quad i \neq 1 \end{cases} + \frac{3}{2} m\sigma_{1i}\{C1\}^{-\frac{5}{2}} \times (m^3 \sigma_{j1})$$

$$= \frac{1}{2m} \begin{cases} -3\sigma_{j1} + 3\sigma_{1i}\sigma_{j1} & \text{if} \quad i = 1 \\ -\sigma_{ji} + 3\sigma_{1i}\sigma_{j1} & \text{if} \quad i \neq 1 \end{cases} \qquad (A1.8a)$$

(2) Find $\frac{\partial b}{\partial \overline{x}_{2i}}$ :

$$b = -\{C1\}^{-\frac{1}{2}} \begin{cases} m - \overline{x}_{21} & \text{if} \quad j = 1 \\ -\overline{x}_{2j} & \text{if} \quad j \neq 1 \end{cases}$$

$$\frac{\partial b}{\partial \overline{x}_{2i}} = -\{C1\}^{-\frac{1}{2}} \times \begin{cases} -1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$
$$- \frac{\partial}{\partial \overline{x}_{1i}} \left[ \{C1\}^{-\frac{1}{2}} \right] \times \begin{cases} m - \overline{x}_{21} & \text{if} \quad j = 1 \\ -\overline{x}_{2j} & \text{if} \quad j \neq 1 \end{cases}$$

$$\frac{\partial b}{\partial \overline{x}_{2i}} = \begin{cases} \{C1\}^{-\frac{1}{2}} & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} - \{C1\}^{-\frac{3}{2}} m\sigma_{1i} \begin{cases} m - \overline{x}_{21} & \text{if} \quad j = 1 \\ -\overline{x}_{2j} & \text{if} \quad j \neq 1 \end{cases}$$

$$= \begin{cases} \left\{m^2 \sigma_{11}\right\}^{-\frac{1}{2}} & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} - \left\{m^2 \sigma_{11}\right\}^{-\frac{3}{2}} m\sigma_{1i} \begin{cases} m - \overline{x}_{21} & \text{if} \quad j = 1 \\ -\overline{x}_{2j} & \text{if} \quad j \neq 1 \end{cases}$$

$$= \frac{1}{m} \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} - \frac{1}{m} \sigma_{1i} \begin{cases} 1 & \text{if} \quad j = 1 \\ 0 & \text{if} \quad j \neq 1 \end{cases}$$

$$= \frac{1}{m} \begin{cases} 1 - \sigma_{1i} & \text{if} \quad i = j = 1 \\ 1 & \text{if} \quad i = j, j \neq 1 \\ -\sigma_{1i} & \text{if} \quad i \neq j, j = 1 \\ 0 & \text{if} \quad i \neq j, j \neq 1 \end{cases} \qquad (A1.8b)$$

Substitute (A1.8a) and (A1.8b) into (A1.8) gives $\frac{\partial^2 A}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}}$

$$= \frac{1}{2m} \begin{cases} -3\sigma_{j1} + 3\sigma_{1i}\sigma_{j1} & \text{if} \quad i = 1 \\ -\sigma_{ji} + 3\sigma_{1i}\sigma_{j1} & \text{if} \quad i \neq 1 \end{cases} + \frac{1}{m} \begin{cases} 1 - \sigma_{1i} & \text{if} \quad i = j = 1 \\ 1 & \text{if} \quad i = j, j \neq 1 \\ -\sigma_{1i} & \text{if} \quad i \neq j, j = 1 \\ 0 & \text{if} \quad i \neq j, j \neq 1 \end{cases}$$

$$
= \frac{1}{m}\begin{cases} \frac{1}{2}\left[-3\sigma_{j1} + 3\sigma_{1i}\sigma_{j1}\right] + 1 - \sigma_{1i} & \text{if} \quad i = j = 1 \\ \frac{1}{2}\left[-3\sigma_{j1} + 3\sigma_{1i}\sigma_{j1}\right] & \text{if} \quad i = 1,\, j \neq 1,\, i \neq j \\ \frac{1}{2}\left[-\sigma_{ji} + 3\sigma_{1i}\sigma_{j1}\right] + 1 & \text{if} \quad i \neq 1,\, i = j,\, j \neq 1 \\ \frac{1}{2}\left[-\sigma_{ji} + 3\sigma_{1i}\sigma_{j1}\right] - \sigma_{1i} & \text{if} \quad i \neq 1,\, i \neq j,\, j = 1 \\ \frac{1}{2}\left[-\sigma_{ji} + 3\sigma_{1i}\sigma_{j1}\right] & \text{if} \quad i \neq 1,\, j \neq 1,\, i \neq j \end{cases} \qquad \textit{(A1.9)}
$$

$$
A \frac{\partial A}{\partial \overline{x}_{2i}} \frac{\partial A}{\partial \overline{x}_{2j}} = \frac{m}{8}\begin{cases} 1 & \text{if} \quad i = j = 1 \\ -\sigma_{j1} & \text{if} \quad i = j,\, j \neq 1 \\ -\sigma_{i1} & \text{if} \quad i \neq j,\, j = 1 \\ \sigma_{i1}\sigma_{j1} & \text{if} \quad i \neq j,\, j \neq 1 \end{cases} \qquad \textit{(A1.10)}
$$

Substitute (A1.9) and (A1.10) into (A1.6) gives $\frac{\partial^2 \Phi}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}}$

$$
= -\phi\left(-\frac{m}{2}\right)\begin{cases} \frac{1}{m}\left[\frac{1}{2}\left[3\sigma_{1i}\sigma_{j1} - 3\sigma_{j1}\right] + 1 - \sigma_{1i}\right] - \frac{m}{8} & \text{if} \quad i = j = 1 \\ \frac{1}{2m}\left[3\sigma_{1i}\sigma_{j1} - 3\sigma_{j1}\right] + \frac{m}{8}\sigma_{j1} & \text{if} \quad i = 1,\, j \neq 1,\, i \neq j \\ \frac{1}{m}\left[\frac{1}{2}\left[3\sigma_{1i}\sigma_{j1} - \sigma_{ji}\right] - \sigma_{1i}\right] + \frac{m}{8}\sigma_{i1} & \text{if} \quad i \neq 1,\, j = 1,\, i \neq j \\ \frac{1}{m}\left[\frac{1}{2}\left[3\sigma_{1i}\sigma_{j1} - \sigma_{ji}\right] + 1\right] - \frac{m}{8}\sigma_{i1}\sigma_{j1} & \text{if} \quad i \neq 1,\, j \neq 1,\, i = j \\ \frac{1}{m}\left[\frac{1}{2}\left[3\sigma_{1i}\sigma_{j1} - \sigma_{ji}\right]\right] - \frac{m}{8}\sigma_{i1}\sigma_{j1} & \text{if} \quad i \neq 1,\, j \neq 1,\, i \neq j \end{cases}
$$

*expression (A1.11)*

$$\therefore E[\Phi(\overline{x}_1,\overline{x}_2)] = \Phi(\mu_1,\mu_2) + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{1i}\,\partial\overline{x}_{1j}}\operatorname{cov}(\overline{x}_{1,ij}) + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{2i}\,\partial\overline{x}_{2j}}\operatorname{cov}(\overline{x}_{2,ij})$$

$$= \Phi(\mu_1,\mu_2) + \frac{1}{2n_1}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{1i}\,\partial\overline{x}_{1j}}\times\sigma_{ij} + \frac{1}{2n_2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{2i}\,\partial\overline{x}_{2j}}\times\sigma_{ij}$$

where $\dfrac{\partial^2\Phi}{\partial\overline{x}_{1i}\,\partial\overline{x}_{1j}}$ and $\dfrac{\partial^2\Phi}{\partial\overline{x}_{2i}\,\partial\overline{x}_{2j}}$ are as in (A1.5) and (A1.11) respectively.

Consider cases 2A and 2B where the conditions are as follow (respectively) :

$\mu_1 = (m,\dots\dots\dots,m)^T$ $\qquad\qquad$ $\mu_1 = (m,\dots\dots\dots,m)^T$

$\mu_2 = (0,\dots\dots\dots,0)^T$ $\qquad$ and $\qquad$ $\mu_2 = (0,\dots\dots\dots,0)^T$

$$\Sigma = \begin{bmatrix} 1 & \rho & \rho & \dots & \rho \\ \rho & 1 & \rho & \dots & \rho \\ \vdots & & & & \vdots \\ \rho & \dots & \dots & \rho & 1 \end{bmatrix} \qquad\qquad \Sigma = \begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^{p-1} \\ \rho & 1 & \rho & \dots & \rho^{p-2} \\ \vdots & & & & \vdots \\ \rho^{p-1} & \dots & \dots & \rho & 1 \end{bmatrix}$$

Need to find $\dfrac{\partial^2 \Phi}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}}$ :

$$\frac{\partial^2 \Phi}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}} = \frac{\partial}{\partial \overline{x}_{1i}} \left[ -\phi(-A) \frac{\partial A}{\partial \overline{x}_{1j}} \right] = -\left[ \phi(-A) \frac{\partial^2 A}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}} + \frac{\partial \phi}{\partial \overline{x}_{1i}} \frac{\partial A}{\partial \overline{x}_{1j}} \right]$$

$$= -\phi(-A) \left[ \frac{\partial^2 A}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}} - A \frac{\partial A}{\partial \overline{x}_{1i}} \frac{\partial A}{\partial \overline{x}_{1j}} \right] \qquad (A1.12)$$

As in cases 1A and 1B :

$$\frac{\partial A}{\partial \overline{\mathbf{x}}_1} = -(\overline{\mathbf{x}}_1^T \boldsymbol{\Sigma} \overline{\mathbf{x}}_1)^{-\frac{3}{2}} \boldsymbol{\Sigma} \overline{\mathbf{x}}_1 (\boldsymbol{\mu}_1 - \frac{1}{2} \overline{\mathbf{x}}_1)^T \overline{\mathbf{x}}_1 + (\overline{\mathbf{x}}_1^T \boldsymbol{\Sigma} \overline{\mathbf{x}}_1)^{-\frac{1}{2}} (\boldsymbol{\mu}_1 - \overline{\mathbf{x}}_1) \qquad (A1.13)$$

where $\overline{\mathbf{x}}_1^T \boldsymbol{\Sigma} \overline{\mathbf{x}}_1 = m^2 \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv}$

$$\boldsymbol{\Sigma} \overline{\mathbf{x}}_1 (\boldsymbol{\mu}_1 - \frac{1}{2} \overline{\mathbf{x}}_1)^T \overline{\mathbf{x}}_1 = \frac{m^3}{2} \begin{bmatrix} \sum_{u=1}^{p} \sigma_{1u} \\ \vdots \\ \sum_{u=1}^{p} \sigma_{pu} \end{bmatrix}$$

$$\therefore \quad \frac{\partial A}{\partial \overline{\mathbf{x}}_1} = -\frac{1}{2} (\sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv})^{-\frac{3}{2}} \begin{bmatrix} \sum_{u=1}^{p} \sigma_{1u} \\ \vdots \\ \sum_{u=1}^{p} \sigma_{pu} \end{bmatrix}$$

$$\therefore \quad \frac{\partial A}{\partial \overline{x}_{1j}} = -\frac{1}{2} (\sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv})^{-\frac{3}{2}} \sum_{u=1}^{p} \sigma_{ju}$$

and $\dfrac{\partial A}{\partial \overline{x}_{1i}} = -\dfrac{1}{2}\left(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right)^{-\frac{3}{2}}\sum_{u=1}^{p}\sigma_{iu}$

As in cases 1A and 1B :

$$\frac{\partial^2 A}{\partial \overline{x}_{1i}\partial \overline{x}_{1j}} = -\frac{\partial}{\partial \overline{x}_{1i}}\left[\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}}\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{ju}\sum_{u=1}^{p}\overline{x}_{1u}(\mu_{1u}-\frac{1}{2}\overline{x}_{1u})\right]$$
$$+\frac{\partial}{\partial \overline{x}_{1i}}\left[\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{uv})]\right\}^{-\frac{1}{2}}\times(\mu_{1j}-\overline{x}_{1j})\right]$$

where $\dfrac{\partial}{\partial \overline{x}_{1i}}\left[\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}}\times\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{ju}\times\sum_{u=1}^{p}\overline{x}_{1u}(\mu_{1u}-\frac{1}{2}\overline{x}_{1u})\right]$

$$=\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}}\left[(\mu_{1i}-\overline{x}_{1i})\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{ju}+\sigma_{ij}\sum_{u=1}^{p}\overline{x}_{1u}(\mu_{1u}-\frac{1}{2}\overline{x}_{1u})\right]$$

$$-3\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{uv})]\right\}^{-\frac{5}{2}}\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{ui}\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{ju}\sum_{u=1}^{p}[\overline{x}_{1u}(\mu_{1u}-\frac{1}{2}\overline{x}_{1u})]$$
$$=\frac{1}{2}\left\{m^2\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{3}{2}}(\sigma_{ij}m^2p)$$
$$-3\left\{m^2\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{5}{2}}\times m^4\sum_{u=1}^{p}\sigma_{ui}\times\sum_{u=1}^{p}\sigma_{ju}\times\frac{1}{2}p$$
$$=\frac{1}{2m}p\sigma_{ij}\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{3}{2}}-\frac{3}{2m}p\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{5}{2}}\times\sum_{u=1}^{p}\sigma_{ui}\times\sum_{u=1}^{p}\sigma_{ju}$$
$$=\frac{p}{2m}\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{5}{2}}\left[\sigma_{ij}\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}-3\sum_{u=1}^{p}\sigma_{ui}\times\sum_{u=1}^{p}\sigma_{ju}\right]$$

and $$\frac{\partial}{\partial \overline{x}_{1i}}\left[\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{uv})]\right\}^{-\frac{1}{2}}\times(\mu_{1j}-\overline{x}_{1j})\right]$$

$$=\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}\sigma_{uv})]\right\}^{-\frac{1}{2}}\times\begin{cases}-1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}$$

$$=\begin{cases}-\left\{m^2\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})]\right\}^{-\frac{1}{2}} & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}$$

$$\therefore \quad \frac{\partial^2 A}{\partial \overline{x}_{1i}\partial \overline{x}_{1j}}=-\frac{p}{2m}\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{5}{2}}\left[\sigma_{ij}\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}-3\sum_{u=1}^{p}\sigma_{ui}\times\sum_{u=1}^{p}\sigma_{ju}\right]$$

$$-\frac{1}{m}\begin{cases}-\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{1}{2}} & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}$$

$$=-\frac{1}{2m}\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{5}{2}}$$

$$\times\begin{cases}p(\sigma_{ij}\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}-3\sum_{u=1}^{p}\sigma_{ui}\times\sum_{u=1}^{p}\sigma_{ju}) & \\ +2(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})^2 & \text{if } i=j \\ p(\sigma_{ij}\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}-3\sum_{u=1}^{p}\sigma_{ui}\times\sum_{u=1}^{p}\sigma_{ju}) & \text{if } i\neq j\end{cases} \quad (A1.14)$$

$$A\frac{\partial A}{\partial \overline{x}_{1i}}\frac{\partial A}{\partial \overline{x}_{1j}}=\frac{1}{8}mp\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{7}{2}}\times\sum_{u=1}^{p}\sigma_{ju}\times\sum_{u=1}^{p}\sigma_{iu} \quad (A1.15)$$

Substitute *(A1.14)* and *(A1.15)* into *(A1.12)* gives $\dfrac{\partial^2 \Phi}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}}$

$$= \phi\left( -\frac{1}{2} mp \left( \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} \right)^{-\frac{1}{2}} \right) \times \left\{ \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} \right\}^{-\frac{7}{2}}$$

$$\times \begin{cases} \dfrac{1}{2m} \left( \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} \right) \left[ p\left( \sigma_{ij} \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} - 3 \sum_{u=1}^{p} \sigma_{ui} \sum_{u=1}^{p} \sigma_{ju} \right) + 2\left( \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} \right)^2 \right] \\ + \dfrac{1}{8} mp \sum_{u=1}^{p} \sigma_{ju} \sum_{u=1}^{p} \sigma_{iu} & \text{if } i = j \\ \dfrac{1}{2m} \left( \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} \right) \left[ p\left( \sigma_{ij} \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} - 3 \sum_{u=1}^{p} \sigma_{ui} \sum_{u=1}^{p} \sigma_{ju} \right) \right] \\ + \dfrac{1}{8} mp \sum_{u=1}^{p} \sigma_{ju} \sum_{u=1}^{p} \sigma_{iu} & \text{if } i \neq j \end{cases}$$

*expression (A1.16)*

Now we need to find $\dfrac{\partial^2 \Phi}{\partial \bar{x}_{2i} \partial \bar{x}_{2j}}$:

$$\frac{\partial^2 \Phi}{\partial \bar{x}_{2i} \partial \bar{x}_{2j}} = -\phi(-A) \left[ \frac{\partial^2 A}{\partial \bar{x}_{2i} \partial \bar{x}_{2j}} - A \frac{\partial A}{\partial \bar{x}_{2i}} \frac{\partial A}{\partial \bar{x}_{2j}} \right] \qquad (A1.17)$$

As in cases 1A and 1B :

$$\begin{aligned} \frac{\partial A}{\partial \bar{\mathbf{x}}_2} = & -\left[ (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T \boldsymbol{\Sigma} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) \right]^{-\frac{3}{2}} \boldsymbol{\Sigma} (\bar{\mathbf{x}}_2 - \bar{\mathbf{x}}_1) \\ & \times \left[ \boldsymbol{\mu}_1 - \frac{1}{2} (\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2) \right]^T (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) \\ & + \left[ (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T \boldsymbol{\Sigma} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) \right]^{-\frac{1}{2}} (\bar{\mathbf{x}}_2 - \boldsymbol{\mu}_1) \end{aligned} \qquad (A1.18)$$

where $\quad (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T \boldsymbol{\Sigma} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) = m^2 \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv}$

$$\boldsymbol{\Sigma}(\overline{\mathbf{x}}_2 - \overline{\mathbf{x}}_1)\left[\boldsymbol{\mu}_1 - \frac{1}{2}(\overline{\mathbf{x}}_1 + \overline{\mathbf{x}}_2)\right]^T(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = -\frac{1}{2}m^3\begin{bmatrix}\sum_{u=1}^{p}\sigma_{1u}\\ \vdots \\ \sum_{u=1}^{p}\sigma_{pu}\end{bmatrix}$$

$$\therefore \quad \frac{\partial A}{\partial \overline{\mathbf{x}}_2} = \frac{1}{2}\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{3}{2}}\begin{bmatrix}\sum_{u=1}^{p}\sigma_{1u}\\ \vdots \\ \sum_{u=1}^{p}\sigma_{pu}\end{bmatrix} - \left\{\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{1}{2}}\begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix}$$

$$\therefore \quad \frac{\partial A}{\partial \overline{x}_{2j}} = \frac{1}{2}\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{3}{2}}\left(\sum_{u=1}^{p}\sigma_{ju}\right) - \left\{\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{1}{2}}$$

and

$$\frac{\partial A}{\partial \overline{x}_{2i}} = \frac{1}{2}\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{3}{2}}\left(\sum_{u=1}^{p}\sigma_{iu}\right) - \left\{\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{1}{2}}$$

Aside:

$$(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T\boldsymbol{\Sigma}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = \sum_{v=1}^{p}\left[(m - \overline{x}_{2v})\sum_{u=1}^{p}(m - \overline{x}_{2u})\sigma_{uv}\right] = C1$$

$$\boldsymbol{\Sigma}(\overline{\mathbf{x}}_2 - \overline{\mathbf{x}}_1)\left[\boldsymbol{\mu}_1 - \frac{1}{2}(\overline{\mathbf{x}}_1 + \overline{\mathbf{x}}_2)\right]^T(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)$$

$$= -\frac{1}{2}\begin{bmatrix}\sum_{u=1}^{p}(m - \overline{x}_{2u})\sigma_{1u}\\ \vdots \\ \sum_{u=1}^{p}(m - \overline{x}_{2u})\sigma_{pu}\end{bmatrix} \times \sum_{u=1}^{p}(m - \overline{x}_{2u})^2$$

$$\left(= -\frac{1}{2}\mathbf{S}\mathbf{y}\mathbf{y}^T\mathbf{y}\right)$$

Using the above expressions rewrite expression *(A1.18)* as :

$$\frac{\partial A}{\partial \overline{\mathbf{x}}_2}=\frac{1}{2}\{C1\}^{-\frac{3}{2}}\begin{bmatrix}\sum_{u=1}^{p}(m-\overline{x}_{2u})\sigma_{1u}\\ \vdots \\ \sum_{u=1}^{p}(m-\overline{x}_{2u})\sigma_{pu}\end{bmatrix}\times\sum_{u=1}^{p}(m-\overline{x}_{2u})^2-\{C1\}^{-\frac{1}{2}}\begin{bmatrix}m-\overline{x}_{21}\\ \vdots \\ m-\overline{x}_{2p}\end{bmatrix}$$

$$\frac{\partial A}{\partial \overline{x}_{2j}}=\frac{1}{2}\{C1\}^{-\frac{3}{2}}\sum_{u=1}^{p}(m-\overline{x}_{2u})\sigma_{ju}\times\sum_{u=1}^{p}(m-\overline{x}_{2u})^2-\{C1\}^{-\frac{1}{2}}(m-\overline{x}_{2j})$$

$$=a+b$$

$$\frac{\partial^2 A}{\partial \overline{x}_{2i}\partial \overline{x}_{2j}}=\frac{\partial}{\partial \overline{x}_{2i}}\left[\frac{\partial A}{\partial \overline{x}_{2j}}\right]=\frac{\partial a}{\partial \overline{x}_{2i}}+\frac{\partial b}{\partial \overline{x}_{2i}} \qquad (A1.19)$$

(1) find $\frac{\partial a}{\partial \overline{x}_{2i}}$ :

$$a=\frac{1}{2}\{C1\}^{-\frac{3}{2}}\sum_{u=1}^{p}(m-\overline{x}_{2u})\sigma_{ju}\times\sum_{u=1}^{p}(m-\overline{x}_{2u})^2$$

$$\frac{\partial a}{\partial \overline{x}_{2i}}=\frac{1}{2}\{C1\}^{-\frac{3}{2}}\times\frac{\partial}{\partial \overline{x}_{2i}}\left[\sum_{u=1}^{p}(m-\overline{x}_{2u})\sigma_{ju}\times\sum_{u=1}^{p}(m-\overline{x}_{2u})^2\right]$$
$$+\frac{1}{2}\frac{\partial}{\partial \overline{x}_{2i}}\left[\{C1\}^{-\frac{3}{2}}\right]\times\sum_{u=1}^{p}(m-\overline{x}_{2u})\sigma_{ju}\times\sum_{u=1}^{p}(m-\overline{x}_{2u})^2$$

where

$$\frac{\partial}{\partial \overline{x}_{2i}}\left[\sum_{u=1}^{p}(m-\overline{x}_{2u})\sigma_{ju}\times\sum_{u=1}^{p}(m-\overline{x}_{2u})^2\right]$$
$$=\sum_{u=1}^{p}(m-\overline{x}_{2u})\sigma_{ju}\times 2(m-\overline{x}_{2i})(-1)-\quad\sigma_{ji}\sum_{u=1}^{p}(m-\overline{x}_{2u})^2$$

$$= -2m^2 \sum_{u=1}^{p} \sigma_{ju} \quad - \quad m^2 \sigma_{ji} p$$

and

$$\frac{\partial}{\partial \overline{x}_{2i}} \left[ \{C1\}^{-\frac{3}{2}} \right] = -\frac{3}{2} \frac{\partial (C1)}{\partial \overline{x}_{2i}} \{C1\}^{-\frac{5}{2}}$$
$$= -\frac{3}{2} \{C1\}^{-\frac{5}{2}} \times -2 \times \sum_{u=1}^{p} (m - \overline{x}_{2u}) \sigma_{ui}$$
$$= 3 \{C1\}^{-\frac{5}{2}} \times \sum_{u=1}^{p} (m - \overline{x}_{2u}) \sigma_{ui}$$

$$\therefore \quad \frac{\partial a}{\partial \overline{x}_{2i}} = \frac{1}{2} \{C1\}^{-\frac{3}{2}} \times -m^2 (2 \sum_{u=1}^{p} \sigma_{ju} + \sigma_{ji} p)$$
$$+ \frac{3}{2} \{C1\}^{-\frac{5}{2}} \sum_{u=1}^{p} (m - \overline{x}_{2u}) \sigma_{ui} \times \sum_{u=1}^{p} (m - \overline{x}_{2u}) \sigma_{ju} \times \sum_{u=1}^{p} (m - \overline{x}_{2u})^2$$
$$= -\frac{1}{2} \left\{ m^2 \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} \right\}^{-\frac{3}{2}} \times m^2 (2 \sum_{u=1}^{p} \sigma_{ju} + \sigma_{ji} p)$$
$$+ \frac{3}{2} \left\{ m^2 \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} \right\}^{-\frac{5}{2}} \times m \sum_{u=1}^{p} \sigma_{ui} \times m \sum_{u=1}^{p} \sigma_{ju} \times m^2 p$$
$$= -\frac{1}{2m} \left\{ \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} \right\}^{-\frac{3}{2}} \times (2 \sum_{u=1}^{p} \sigma_{ju} + \sigma_{ji} p)$$
$$+ \frac{3p}{2m} \left\{ \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} \right\}^{-\frac{5}{2}} \times \sum_{u=1}^{p} \sigma_{ui} \times \sum_{u=1}^{p} \sigma_{ju}$$
$$= \frac{1}{2m} \left\{ \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} \right\}^{-\frac{3}{2}} \left[ 3p \sum_{u=1}^{p} \sigma_{ui} \sum_{u=1}^{p} \sigma_{ju} - (\sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv})(2 \sum_{u=1}^{p} \sigma_{ju} + \sigma_{ji} p) \right] \qquad (A1.19a)$$

(2) find $\frac{\partial b}{\partial \overline{x}_{2i}}$ :

$$b = -\{C1\}^{-\frac{1}{2}} (m - \overline{x}_{2j})$$

$$\frac{\partial b}{\partial \overline{x}_{2i}} = -\{C1\}^{-\frac{1}{2}} \begin{cases} -1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} + \frac{1}{2}\{C1\}^{-\frac{3}{2}} \frac{\partial (C1)}{\partial \overline{x}_{2i}} (m - \overline{x}_{2j})$$

$$= \{C1\}^{-\frac{1}{2}} \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} - \{C1\}^{-\frac{3}{2}} (m - \overline{x}_{2j}) \sum_{u=1}^{p} (m - \overline{x}_{2u}) \sigma_{ui}$$

$$= \left\{ m^2 \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} \right\}^{-\frac{1}{2}} \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} - \left\{ m^2 \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} \right\}^{-\frac{3}{2}} m^2 \sum_{u=1}^{p} \sigma_{ui}$$

$$= \frac{1}{m} \left\{ \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} \right\}^{-\frac{3}{2}} \begin{cases} \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} - \sum_{u=1}^{p} \sigma_{ui} & \text{if} \quad i = j \\ -\sum_{u=1}^{p} \sigma_{ui} & \text{if} \quad i \neq j \end{cases} \qquad (A1.19b)$$

Subsitute *(A1.19a)* and *(A1.19b)* into *(A1.19)* gives $\frac{\partial^2 A}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}}$

$$= \frac{1}{2m} \left\{ \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} \right\}^{-\frac{5}{2}} \left[ 3p \sum_{u=1}^{p} \sigma_{ui} \sum_{u=1}^{p} \sigma_{ju} - \left( \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} \right) \left( 2 \sum_{u=1}^{p} \sigma_{ju} + \sigma_{ji} p \right) \right]$$

$$+ \frac{1}{m} \left\{ \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} \right\}^{-\frac{3}{2}} \begin{cases} \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} - \sum_{u=1}^{p} \sigma_{ui} & \text{if} \quad i = j \\ -\sum_{u=1}^{p} \sigma_{ui} & \text{if} \quad i \neq j \end{cases}$$

$$= \frac{1}{2m} \left\{ \sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv} \right\}^{-\frac{5}{2}}$$

$$\times\begin{cases} 3p\sum_{u=1}^{p}\sigma_{ui}\sum_{u=1}^{p}\sigma_{ju}-(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})(2\sum_{u=1}^{p}\sigma_{ju}+\sigma_{ji}p) \\ +2(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}-\sum_{u=1}^{p}\sigma_{ui}) & \text{if} \quad i=j \\ 3p\sum_{u=1}^{p}\sigma_{ui}\sum_{u=1}^{p}\sigma_{ju}-(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})(2\sum_{u=1}^{p}\sigma_{ju}+\sigma_{ji}p) \\ -2(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})\sum_{u=1}^{p}\sigma_{ui} & \text{if} \quad i\neq j \end{cases} \tag{A1.20}$$

$$\therefore \frac{\partial^2\Phi}{\partial\overline{x}_{2i}\partial\overline{x}_{2j}}=-\frac{1}{2}\phi\left(-\frac{1}{2}mp(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})^{-\frac{1}{2}}\right)\times\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{7}{2}}$$

$$\times\begin{cases} \frac{1}{m}\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\Big[3p\sum_{u=1}^{p}\sigma_{ui}\sum_{u=1}^{p}\sigma_{ju}-(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})(2\sum_{u=1}^{p}\sigma_{ju}+\sigma_{ji}p) \\ +2(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}-\sum_{u=1}^{p}\sigma_{ui})\Big] \\ -mp(\frac{1}{2}\sum_{u=1}^{p}\sigma_{ju}-\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})(\frac{1}{2}\sum_{u=1}^{p}\sigma_{iu}-\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}) & \text{if} \quad i=j \\ \frac{1}{m}\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\Big[3p\sum_{u=1}^{p}\sigma_{ui}\sum_{u=1}^{p}\sigma_{ju}-(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})(2\sum_{u=1}^{p}\sigma_{ju}+\sigma_{ji}p) \\ -2(\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})\sum_{u=1}^{p}\sigma_{ui}\Big] \\ -mp(\frac{1}{2}\sum_{u=1}^{p}\sigma_{ju}-\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv})(\frac{1}{2}\sum_{u=1}^{p}\sigma_{iu}-\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}) & \text{if} \quad i\neq j \end{cases}$$

*expression (A1.21)*

$$\therefore E[\Phi(\overline{x}_1,\overline{x}_2)]=\Phi(\mu_1,\mu_2)+\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{1i}\,\partial\overline{x}_{1j}}\text{cov}(\overline{x}_{1,ij})$$
$$+\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{2i}\,\partial\overline{x}_{2j}}\text{cov}(\overline{x}_{2,ij})$$

$$= \Phi(\mu_1, \mu_2) + \frac{1}{2n_1} \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{1i} \, \partial \overline{x}_{1j}} \times \sigma_{ij}$$
$$+ \frac{1}{2n_2} \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{2i} \, \partial \overline{x}_{2j}} \times \sigma_{ij}$$

where $\frac{\partial^2 \Phi}{\partial \overline{x}_{1i} \, \partial \overline{x}_{1j}}$ and $\frac{\partial^2 \Phi}{\partial \overline{x}_{2i} \, \partial \overline{x}_{2j}}$ are as in *(A1.16)* and *(A1.21)* respectively.

# *Appendix A1.2*

## *ASYMPTOTIC EXPANSION FOR LINEAR DISCRIMINANT FUNCTION*

### *Actual error rate*

We have $p_{21}^{(A)} = \Phi[-\dfrac{[\,\mu_1 - 1/2(\,\overline{x}_1 + \overline{x}_2)]^T S^{-1}(\,\overline{x}_1 - \overline{x}_2)}{[\,(\,\overline{x}_1 - \overline{x}_2)^T S^{-1}\,\Sigma\, S^{-1}(\,\overline{x}_1 - \overline{x}_2)]^{1/2}}] = \Phi[-D]$.

Using the Taylor series expansion (to second order approximation), we obtain :-

$$
\begin{aligned}
\Phi(\overline{x}_1, \overline{x}_2, S) = \Phi(\mu_1, \mu_2, \Sigma) &+ \sum_{j=1}^{p} \frac{\partial \Phi}{\partial \overline{x}_{1j}}(\overline{x}_{1j} - \mu_{1j}) + \sum_{j=1}^{p} \frac{\partial \Phi}{\partial \overline{x}_{2j}}(\overline{x}_{2j} - \mu_{2j}) \\
&+ \sum_{i=1}^{p}\sum_{j=1}^{p} \frac{\partial \Phi}{\partial s_{ij}}(s_{ij} - \Sigma_{ij}) \\
&+ \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{1i}\partial \overline{x}_{1j}}(\overline{x}_{1i} - \mu_{1i})(\overline{x}_{1j} - \mu_{1j}) \\
&+ \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{2i}\partial \overline{x}_{2j}}(\overline{x}_{2i} - \mu_{2i})(\overline{x}_{2j} - \mu_{2j}) \\
&+ \frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{p}\sum_{i=1}^{p}\sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial s_{kl}\partial s_{ij}}(s_{kl} - \Sigma_{kl})(s_{ij} - \Sigma_{ij}) \\
&+ \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{1i}\partial \overline{x}_{2j}}(\overline{x}_{1i} - \mu_{1i})(\overline{x}_{2j} - \mu_{2j}) \\
&+ \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{1i}\partial s_{ij}}(\overline{x}_{1i} - \mu_{1i})(s_{ij} - \Sigma_{ij}) \\
&+ \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{2i}\partial s_{ij}}(\overline{x}_{2i} - \mu_{2i})(s_{ij} - \Sigma_{ij})
\end{aligned}
$$

where $\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{1i}\partial\overline{x}_{2j}}=0,\ \sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{1i}\partial s_{ij}}=0$ and $\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{2i}\partial s_{ij}}=0$ .

Taking expectations gives:-

$$
\begin{aligned}
E[\Phi(\overline{\mathbf{x}}_1,\overline{\mathbf{x}}_2,S)] = \Phi(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\boldsymbol{\Sigma}) &+ \sum_{j=1}^{p}\frac{\partial\Phi}{\partial\overline{x}_{1j}}E(\overline{x}_{1j}-\mu_{1j}) \\
&+\sum_{j=1}^{p}\frac{\partial\Phi}{\partial\overline{x}_{2j}}E(\overline{x}_{2j}-\mu_{2j})+\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial\Phi}{\partial s_{ij}}E(s_{ij}-\Sigma_{ij}) \\
&+\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{1i}\,\partial\overline{x}_{1j}}\operatorname{cov}(\overline{x}_{1,ij}) \\
&+\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{2i}\,\partial\overline{x}_{2j}}\operatorname{cov}(\overline{x}_{2,ij}) \\
&+\frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{p}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial s_{kl}\,\partial s_{ij}}\operatorname{cov}(s_{kl},s_{ij})
\end{aligned}
$$

where $\sum_{j=1}^{p}\frac{\partial\Phi}{\partial\overline{x}_{1j}}E(\overline{x}_{1j}-\mu_{1j})=\sum_{j=1}^{p}\frac{\partial\Phi}{\partial\overline{x}_{2j}}E(\overline{x}_{2j}-\mu_{2j})=0$

and $\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial\Phi}{\partial s_{ij}}E(s_{ij}-\Sigma_{ij})=0.$

$$
\begin{aligned}
\therefore E[\Phi(\overline{\mathbf{x}}_1,\overline{\mathbf{x}}_2,S)] = \Phi(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\boldsymbol{\Sigma}) &+\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{1i}\,\partial\overline{x}_{1j}}\operatorname{cov}(\overline{x}_{1,ij}) \\
&+\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{2i}\,\partial\overline{x}_{2j}}\operatorname{cov}(\overline{x}_{2,ij}) \\
&+\frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{p}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial s_{kl}\,\partial s_{ij}}\operatorname{cov}(s_{kl},s_{ij})
\end{aligned}
$$

So now we want to find $\frac{\partial\Phi}{\partial\overline{x}_{1j}},\ \frac{\partial\Phi}{\partial\overline{x}_{2j}},\ \frac{\partial^2\Phi}{\partial\overline{x}_{1j}\,\partial\overline{x}_{1i}},\ \frac{\partial^2\Phi}{\partial\overline{x}_{2j}\,\partial\overline{x}_{2i}},\ \frac{\partial^2\Phi}{\partial s_{kl}\,\partial s_{ij}}$ .

Under cases 1A and 1B, let

$$A=[\bar{\mathbf{x}}_1^T S^{-1}\boldsymbol{\Sigma} S^{-1}\bar{\mathbf{x}}_1]^{-\frac{1}{2}}(\boldsymbol{\mu}_1-\frac{1}{2}\bar{\mathbf{x}}_1)^T S^{-1}\bar{\mathbf{x}}_1$$

now we need to find $\dfrac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}$:

$$\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}=-\phi(-A)\left[\frac{\partial^2 A}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}-A\frac{\partial A}{\partial\bar{x}_{1i}}\frac{\partial A}{\partial\bar{x}_{1j}}\right] \quad (A1.22)$$

$$\frac{\partial A}{\partial\bar{\mathbf{x}}_1}=-[\bar{\mathbf{x}}_1^T S^{-1}\boldsymbol{\Sigma} S^{-1}\bar{\mathbf{x}}_1]^{-\frac{3}{2}}S^{-1}\boldsymbol{\Sigma} S^{-1}\bar{\mathbf{x}}_1(\boldsymbol{\mu}_1-\frac{1}{2}\bar{\mathbf{x}}_1)^T S^{-1}\bar{\mathbf{x}}_1$$

$$+[\bar{\mathbf{x}}_1^T S^{-1}\boldsymbol{\Sigma} S^{-1}\bar{\mathbf{x}}_1]^{-\frac{1}{2}}S^{-1}(\boldsymbol{\mu}_1-\bar{\mathbf{x}}_1) \quad (A1.23)$$

where $\bar{\mathbf{x}}_1^T S^{-1}\Sigma S^{-1}\bar{\mathbf{x}}_1=m^2\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})]$

$$S^{-1}\boldsymbol{\Sigma} S^{-1}\bar{\mathbf{x}}_1(\boldsymbol{\mu}_1-\frac{1}{2}\bar{\mathbf{x}}_1)^T S^{-1}\bar{\mathbf{x}}_1=\frac{1}{2}m^3 s^{11}\begin{bmatrix}\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\\ \vdots \\ \sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{pu}\sigma_{uv})]\end{bmatrix}$$

$$\therefore\frac{\partial A}{\partial\bar{\mathbf{x}}_1}=-\frac{1}{2}s^{11}\times\left\{\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}}\times\begin{bmatrix}\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\\ \vdots \\ \sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{pu}\sigma_{uv})]\end{bmatrix}$$

$$\therefore\frac{\partial A}{\partial\bar{x}_{1j}}=-\frac{1}{2}s^{11}\times\left\{\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}}\times\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{ju}\sigma_{uv})] \quad (A1.23a)$$

and

$$\frac{\partial A}{\partial\bar{x}_{1i}}=-\frac{1}{2}s^{11}\times\left\{\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}}\times\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{iu}\sigma_{uv})] \quad (A1.23b)$$

Aside:

$$\overline{\mathbf{x}}_1^{\mathrm{T}}\mathbf{S}^{-1}\boldsymbol{\Sigma}\mathbf{S}^{-1}\overline{\mathbf{x}}_1 = \sum_{w=1}^{p}\{\ \overline{x}_{1w}\sum_{v=1}^{p}[\ (\sum_{u=1}^{p}\overline{x}_{1u}s^{uv})(\sum_{u=1}^{p}\sigma_{vu}s^{uw})\ ]\} = C1$$

$$\mathbf{S}^{-1}\boldsymbol{\Sigma}\mathbf{S}^{-1}\overline{\mathbf{x}}_1(\boldsymbol{\mu}_1 - \frac{1}{2}\overline{\mathbf{x}}_1)^{\mathrm{T}}\mathbf{S}^{-1}\overline{\mathbf{x}}_1$$

$$= \left\{(m - \frac{1}{2}\overline{x}_{11})\sum_{u=1}^{p}\overline{x}_{1u}s^{1u} - \frac{1}{2}\sum_{v=2}^{p}[\ \overline{x}_{1v}\sum_{u=1}^{p}\overline{x}_{1u}s^{vu}\ ]\right\} \times \begin{bmatrix} \sum_{w=1}^{p}\{\ \overline{x}_{1w}\sum_{v=1}^{p}[\ s^{vw}(\sum_{u=1}^{p}s^{1u}\sigma_{uv})\ ]\} \\ \vdots \\ \sum_{w=1}^{p}\{\ \overline{x}_{1w}\sum_{v=1}^{p}[\ s^{vw}(\sum_{u=1}^{p}s^{pu}\sigma_{uv})\ ]\} \end{bmatrix}$$

Using the above expressions rewrite *(A1.23)* as :

$$\frac{\partial A}{\partial \overline{\mathbf{x}}_1} = -\{C1\}^{-\frac{3}{2}} \times \left\{(m - \frac{1}{2}\overline{x}_{11})\sum_{u=1}^{p}\overline{x}_{1u}s^{1u} - \frac{1}{2}\sum_{v=2}^{p}[\ \overline{x}_{1v}\sum_{u=1}^{p}\overline{x}_{1u}s^{vu}\ ]\right\} \times \begin{bmatrix} \sum_{w=1}^{p}\{\ \overline{x}_{1w}\sum_{v=1}^{p}[\ s^{vw}(\sum_{u=1}^{p}s^{1u}\sigma_{uv})\ ]\} \\ \vdots \\ \sum_{w=1}^{p}\{\ \overline{x}_{1w}\sum_{v=1}^{p}[\ s^{vw}(\sum_{u=1}^{p}s^{pu}\sigma_{uv})\ ]\} \end{bmatrix} + \{C1\}^{-\frac{1}{2}}\begin{bmatrix} ms^{11} - \sum_{u=1}^{p}s^{1u}\overline{x}_{1u} \\ \vdots \\ ms^{p1} - \sum_{u=1}^{p}s^{pu}\overline{x}_{1u} \end{bmatrix}$$

$$\therefore \frac{\partial A}{\partial \overline{x}_{1j}} = -\left[\{C1\}^{-\frac{3}{2}} \times \left\{(m - \frac{1}{2}\overline{x}_{11})\sum_{u=1}^{p}\overline{x}_{1u}s^{1u} - \frac{1}{2}\sum_{v=2}^{p}[\ \overline{x}_{1v}\sum_{u=1}^{p}\overline{x}_{1u}s^{vu}\ ]\right\} \times \sum_{w=1}^{p}\{\ \overline{x}_{1w}\sum_{v=1}^{p}[\ s^{vw}(\sum_{u=1}^{p}s^{ju}\sigma_{uv})\ ]\}\right] + \left[\{C1\}^{-\frac{1}{2}}\left(ms^{j1} - \sum_{u=1}^{p}s^{ju}\overline{x}_{1u}\right)\right]$$

$$= a + b$$

$$\therefore \frac{\partial^2 A}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}} = \frac{\partial}{\partial \overline{x}_{1i}} \left[ \frac{\partial A}{\partial \overline{x}_{1j}} \right] = \frac{\partial a}{\partial \overline{x}_{1i}} + \frac{\partial b}{\partial \overline{x}_{1i}} \qquad \textit{(A1.24)}$$

(1) Find $\frac{\partial a}{\partial \overline{x}_{1i}}$:

$$a = -\left[ \{C1\}^{-\frac{3}{2}} \times \left\{ (m - \frac{1}{2}\overline{x}_{11}) \sum_{u=1}^{p} \overline{x}_{1u} s^{1u} - \frac{1}{2} \sum_{v=2}^{p} [\, \overline{x}_{1v} \sum_{u=1}^{p} \overline{x}_{1u} s^{vu} \,] \right\} \times \sum_{w=1}^{p} \{\, \overline{x}_{1w} \sum_{v=1}^{p} [\, s^{vw} (\sum_{u=1}^{p} s^{ju} \sigma_{uv})\,]\,\} \right]$$

$$= (a1) \times (a2) \times (a3)$$

$$= (a1)(a2)\frac{\partial (a3)}{\partial \overline{x}_{1i}} + (a1)(a3)\frac{\partial (a2)}{\partial \overline{x}_{1i}} + (a2)(a3)\frac{\partial (a1)}{\partial \overline{x}_{1i}}$$

where

$$a1 = -\{C1\}^{-\frac{3}{2}}$$

$$\frac{\partial (a1)}{\partial \overline{x}_{1i}} = \frac{3}{2}\{C1\}^{-\frac{5}{2}} \frac{\partial (C1)}{\partial \overline{x}_{1i}}$$

where $\frac{\partial (C1)}{\partial \overline{x}_{1i}} = m\left\{ \sum_{v=1}^{p} [s^{iv} \sum_{u=1}^{p} (\sigma_{uv} s^{u1})] + \sum_{v=1}^{p} [s^{1v} \sum_{u=1}^{p} (\sigma_{uv} s^{ui})] \right\}$

$$\therefore \quad \frac{\partial (a1)}{\partial \overline{x}_{1i}} = \frac{3}{2}\{C1\}^{-\frac{5}{2}} \times m\left\{ \sum_{v=1}^{p} [s^{iv} \sum_{u=1}^{p} (\sigma_{uv} s^{u1})] + \sum_{v=1}^{p} [s^{1v} \sum_{u=1}^{p} (\sigma_{uv} s^{ui})] \right\}$$

$$a2 = (m - \frac{1}{2}\overline{x}_{11}) \sum_{u=1}^{p} \overline{x}_{1u} s^{1u} - \frac{1}{2} \sum_{v=2}^{p} [\, \overline{x}_{1v} \sum_{u=1}^{p} \overline{x}_{1u} s^{vu} \,]$$

$$\frac{\partial (a2)}{\partial \overline{x}_{1i}} = \frac{\partial}{\partial \overline{x}_{1i}} \left[ (m - \frac{1}{2}\overline{x}_{11}) \sum_{u=1}^{p} \overline{x}_{1u} s^{1u} \right] - \frac{1}{2} \frac{\partial}{\partial \overline{x}_{1i}} \left[ \sum_{v=2}^{p} [\, \overline{x}_{1v} \sum_{u=1}^{p} \overline{x}_{1u} s^{vu} \,] \right]$$

$$= -\begin{cases} ms^{11} & \text{if} \quad i = 1 \\ \frac{m}{2} s^{1i} & \text{if} \quad i \neq 1 \end{cases} \quad - \quad \frac{1}{2} \begin{cases} 0 & \text{if} \quad i = 1 \\ ms^{i1} & \text{if} \quad i \neq 1 \end{cases}$$

$$= -m\begin{cases} s^{11} & \text{if} \quad i = 1 \\ s^{1i} & \text{if} \quad i \neq 1 \end{cases}$$

$$= -ms^{1i}$$

$$a3 = \sum_{w=1}^{p}\{ \bar{x}_{1w} \sum_{v=1}^{p}[ s^{vw} ( \sum_{u=1}^{p} s^{ju}\sigma_{uv} ) ] \}$$

$$\frac{\partial(a3)}{\partial\bar{x}_{1i}} = \sum_{v=1}^{p}[ s^{vi} ( \sum_{u=1}^{p} s^{ju}\sigma_{uv} ) ]$$

$$\begin{aligned} \therefore \frac{\partial a}{\partial\bar{x}_{1i}} &= -\frac{1}{2}\frac{s^{11}}{m}\left\{ \sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}} \sum_{v=1}^{p}[s^{vi} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})] \\ &+\frac{s^{1i}}{m}\left\{ \sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}} \sum_{v=1}^{p}[s^{vi} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})] \\ &+\frac{3}{4}\frac{s^{11}}{m}\left\{ \sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\right\}^{-\frac{5}{2}} \sum_{v=1}^{p}[s^{vi} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})] \\ &\quad \times\left\{ \sum_{v=1}^{p}[s^{iv} \sum_{u=1}^{p}(\sigma_{uv}s^{u1})] + \sum_{v=1}^{p}[s^{1v} \sum_{u=1}^{p}(\sigma_{uv}s^{ui})]\right\} \end{aligned} \qquad (A1.24a)$$

(2) Find $\frac{\partial b}{\partial\bar{x}_{1i}}$ :

$$b = \{C1\}^{-\frac{1}{2}}\left( ms^{j1} - \sum_{u=1}^{p} s^{ju}\bar{x}_{1u} \right)$$

$$\begin{aligned} \frac{\partial b}{\partial\bar{x}_{1i}} &= \{C1\}^{-\frac{1}{2}} \frac{\partial}{\partial\bar{x}_{1i}}\left[ ms^{j1} - \sum_{u=1}^{p} s^{ju}\bar{x}_{1u} \right] + \frac{\partial}{\partial\bar{x}_{1i}}\left[ \{C1\}^{-\frac{1}{2}} \right](ms^{j1} - \sum_{u=1}^{p} s^{ju}\bar{x}_{1u}) \\ &= -\{C1\}^{-\frac{1}{2}} \times s^{ji} \quad - \quad \frac{1}{2}\{C1\}^{-\frac{3}{2}} \frac{\partial(C1)}{\partial\bar{x}_{1i}}(ms^{j1} - \sum_{u=1}^{p} s^{ju}\bar{x}_{1u}) \\ &= -s^{ji}\{C1\}^{-\frac{1}{2}} \\ &= -s^{ji}\left\{ m^2 \sum_{v=1}^{p}[ s^{v1} ( \sum_{u=1}^{p} s^{1u}\sigma_{uv} ) ] \right\}^{-\frac{1}{2}} \end{aligned} \qquad (A1.24b)$$

Substitute *(A1.24a)* and *(A1.24b)* into *(A1.24)* gives

$$\therefore \frac{\partial^2 A}{\partial \bar{x}_{1i}\partial \bar{x}_{1j}} = \frac{1}{2}\frac{s^{11}}{m}\left\{\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}}\sum_{v=1}^{p}[s^{vi}\sum_{u=1}^{p}(s^{ju}\sigma_{uv})]$$

$$+\frac{s^{1i}}{m}\left\{\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}}\sum_{v=1}^{p}[s^{vi}\sum_{u=1}^{p}(s^{ju}\sigma_{uv})]$$

$$+\frac{3}{4}\frac{s^{11}}{m}\left\{\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\right\}^{-\frac{5}{2}}\sum_{v=1}^{p}[s^{vi}\sum_{u=1}^{p}(s^{ju}\sigma_{uv})]$$

$$\times\left\{\sum_{v=1}^{p}[s^{iv}\sum_{u=1}^{p}(\sigma_{uv}s^{u1})] + \sum_{v=1}^{p}[s^{1v}\sum_{u=1}^{p}(\sigma_{uv}s^{ui})]\right\}$$

$$-s^{ji}\left\{m^2\sum_{v=1}^{p}[s^{v1}(\sum_{u=1}^{p}s^{1u}\sigma_{uv})]\right\}^{-\frac{1}{2}} \qquad (A1.25)$$

Substitute *(A1.23a)*, *(A1.23b)* and *(A1.25)* into *(A1.22)* gives $\frac{\partial^2 \Phi}{\partial \bar{x}_{1i}\partial \bar{x}_{1j}}$

$$= -\phi\left(-\frac{1}{2}ms^{11}\left\{\sum_{v=1}^{p}[s^{v1}(\sum_{u=1}^{p}s^{1u}\sigma_{uv})]\right\}^{-\frac{1}{2}}\right)$$

$$\times\left\{\left(\left(\frac{1}{2}\frac{s^{11}}{m}\left\{\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}}\sum_{v=1}^{p}[s^{vi}\sum_{u=1}^{p}(s^{ju}\sigma_{uv})]\right.\right.\right.$$

$$+\frac{s^{1i}}{m}\left\{\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}}\sum_{v=1}^{p}[s^{vi}\sum_{u=1}^{p}(s^{ju}\sigma_{uv})]$$

$$+\frac{3}{4}\frac{s^{11}}{m}\left\{\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\right\}^{-\frac{5}{2}}\sum_{v=1}^{p}[s^{vi}\sum_{u=1}^{p}(s^{ju}\sigma_{uv})]$$

$$\times\left\{\sum_{v=1}^{p}[s^{iv}\sum_{u=1}^{p}(\sigma_{uv}s^{u1})] + \sum_{v=1}^{p}[s^{1v}\sum_{u=1}^{p}(\sigma_{uv}s^{ui})]\right\}$$

$$\left.-s^{ji}\left\{m^2\sum_{v=1}^{p}[s^{v1}(\sum_{u=1}^{p}s^{1u}\sigma_{uv})]\right\}^{-\frac{1}{2}}\right) - \left(\frac{1}{2}ms^{11}\left\{\sum_{v=1}^{p}[s^{v1}(\sum_{u=1}^{p}s^{1u}\sigma_{uv})]\right\}^{-\frac{1}{2}}\right.$$

$$\times\left[-\frac{1}{2}s^{11}\times\left\{\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}}\times\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{ju}\sigma_{uv})]\right]$$

$$\left.\left.\times\left[-\frac{1}{2}s^{11}\times\left\{\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}}\times\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{iu}\sigma_{uv})]\right]\right)\right\} \qquad (A1.26)$$

Now we need to find $\dfrac{\partial^2\Phi}{\partial\overline{x}_{2i}\partial\overline{x}_{2j}}$:

$$\frac{\partial^2\Phi}{\partial\overline{x}_{1i}\partial\overline{x}_{1j}} = -\phi(-A)\left[\frac{\partial^2 A}{\partial\overline{x}_{2i}\partial\overline{x}_{2j}} - A\frac{\partial A}{\partial\overline{x}_{2i}}\frac{\partial A}{\partial\overline{x}_{2j}}\right] \qquad (A1.27)$$

$$A = [(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)_1^T \mathbf{S}^{-1}\boldsymbol{\Sigma}\mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)]^{-\frac{1}{2}}[\boldsymbol{\mu}_1 - \frac{1}{2}(\overline{\mathbf{x}}_1 + \overline{\mathbf{x}}_2)]^T \mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)$$

$$\begin{aligned}\frac{\partial A}{\partial\overline{\mathbf{x}}_2} = {} & [(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)_1^T \mathbf{S}^{-1}\Sigma\mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)]^{-\frac{3}{2}}\mathbf{S}^{-1}\Sigma\mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) \\ & \times[\boldsymbol{\mu}_1 - \frac{1}{2}(\overline{\mathbf{x}}_1 + \overline{\mathbf{x}}_2)]^T \mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) \\ & + [(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)_1^T \mathbf{S}^{-1}\boldsymbol{\Sigma}\mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)]^{-\frac{1}{2}}[-\mathbf{S}^{-1}\boldsymbol{\mu}_1 + \mathbf{S}^{-1}\overline{\mathbf{x}}_2]\end{aligned} \qquad (A1.28)$$

where

$$(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)_1^T \mathbf{S}^{-1}\boldsymbol{\Sigma}\mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = m^2\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})]$$

$$\mathbf{S}^{-1}\boldsymbol{\Sigma}\mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)[\boldsymbol{\mu}_1 - \frac{1}{2}(\overline{\mathbf{x}}_1 + \overline{\mathbf{x}}_2)]^T \mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)$$

$$= \frac{1}{2}m^3 s^{11}\begin{bmatrix}\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})] \\ \vdots \\ \sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{pu}\sigma_{uv})]\end{bmatrix}$$

$$\begin{aligned}\therefore\quad \frac{\partial A}{\partial\overline{\mathbf{x}}_2} = {} & \frac{1}{2}s^{11}\left\{\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\right\}^{-\frac{3}{2}}\begin{bmatrix}\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})] \\ \vdots \\ \sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{pu}\sigma_{uv})]\end{bmatrix} \\ & - \left\{\sum_{v=1}^{p}[s^{v1}\sum_{u=1}^{p}(s^{1u}\sigma_{uv})]\right\}^{-\frac{1}{2}}\begin{bmatrix}s^{11} \\ \vdots \\ s^{p1}\end{bmatrix}\end{aligned}$$

$$\therefore \quad \frac{\partial A}{\partial \bar{x}_{2j}} = \frac{1}{2} s^{11} \left\{ \sum_{v=1}^{p} [s^{v1} \sum_{u=1}^{p} (s^{1u} \sigma_{uv})] \right\}^{-\frac{3}{2}} \sum_{v=1}^{p} [s^{v1} \sum_{u=1}^{p} (s^{ju} \sigma_{uv})]$$

$$- \left\{ \sum_{v=1}^{p} [s^{v1} \sum_{u=1}^{p} (s^{1u} \sigma_{uv})] \right\}^{-\frac{1}{2}} s^{j1} \qquad (A1.28a)$$

and

$$\frac{\partial A}{\partial \bar{x}_{2i}} = \frac{1}{2} s^{11} \left\{ \sum_{v=1}^{p} [s^{v1} \sum_{u=1}^{p} (s^{1u} \sigma_{uv})] \right\}^{-\frac{3}{2}} \sum_{v=1}^{p} [s^{v1} \sum_{u=1}^{p} (s^{iu} \sigma_{uv})]$$

$$- \left\{ \sum_{v=1}^{p} [s^{v1} \sum_{u=1}^{p} (s^{1u} \sigma_{uv})] \right\}^{-\frac{1}{2}} s^{i1} \qquad (A1.28b)$$

Aside:

$$(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)_1^T \mathbf{S}^{-1} \boldsymbol{\Sigma} \mathbf{S}^{-1} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) = C1$$

$$= (m - \bar{x}_{21}) \left\{ (m - \bar{x}_{21}) \sum_{v=1}^{p} [s^{v1} \sum_{u=1}^{p} (s^{1u} \sigma_{uv})] - \sum_{w=2}^{p} \{ \bar{x}_{2w} \sum_{v=1}^{p} [ s^{v1} ( \sum_{u=1}^{p} s^{wu} \sigma_{uv} ) ] \} \right\}$$

$$- \sum_{y=2}^{p} \left\{ \bar{x}_{2y} \left[ (m - \bar{x}_{21}) \sum_{v=1}^{p} [s^{vy} \sum_{u=1}^{p} (s^{1u} \sigma_{uv})] \right] - \sum_{w=2}^{p} \{ \bar{x}_{2w} \sum_{v=1}^{p} [ s^{v1} ( \sum_{u=1}^{p} s^{wu} \sigma_{uv} ) ] \} \right\}$$

$$\mathbf{S}^{-1} \boldsymbol{\Sigma} \mathbf{S}^{-1} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) [\boldsymbol{\mu}_1 - \frac{1}{2} (\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2)]^T \mathbf{S}^{-1} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)$$

$$= \frac{1}{2} \left\{ (m - \bar{x}_{21}) [(m - \bar{x}_{21}) \sigma_{11} - \sum_{u=2}^{p} (\bar{x}_{2u} \sigma_{u1})] \right.$$

$$\left. - \sum_{v=2}^{p} \bar{x}_{2v} [(m - \bar{x}_{21}) \sigma_{1v} - \sum_{u=2}^{p} (\bar{x}_{2u} \sigma_{uv})] \right\}$$

$$\frac{\partial A}{\partial \bar{x}_{2j}} = \{C1\}^{-\frac{3}{2}} \left\{ (m - \bar{x}_{21}) \sum_{v=1}^{p} [s^{v1} \sum_{u=1}^{p} (s^{ju} \sigma_{uv})] - \sum_{w=2}^{p} \{ \bar{x}_{2w} \sum_{v=1}^{p} [s^{vw} \sum_{u=1}^{p} (s^{ju} \sigma_{uv})] \} \right\}$$

$$\times \frac{1}{2} \left\{ (m - \bar{x}_{21}) [(m - \bar{x}_{21}) \sigma_{11} - \sum_{u=2}^{p} (\bar{x}_{2u} \sigma_{u1})] \right.$$

$$\left. - \sum_{v=2}^{p} \bar{x}_{2v} [(m - \bar{x}_{21}) \sigma_{1v} - \sum_{u=2}^{p} (\bar{x}_{2u} \sigma_{uv})] \right\}$$

$$+ \{C1\}^{-\frac{1}{2}} \left( -m s^{j1} + \sum_{u=1}^{p} (s^{ju} \bar{x}_{2u}) \right)$$

$$= a \quad + \quad b = (a1 \times a2 \times a3) \quad + \quad b$$

$$\therefore \quad \frac{\partial^2 A}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}} = \frac{\partial a}{\partial \overline{x}_{2i}} + \frac{\partial b}{\partial \overline{x}_{2i}} \qquad \textit{(A1.29)}$$

(1) Find $\frac{\partial a}{\partial \overline{x}_{2i}}$ :

$$\frac{\partial a}{\partial \overline{x}_{2i}} = (a1)(a2)\frac{\partial (a3)}{\partial \overline{x}_{2i}} \quad + \quad (a1)(a3)\frac{\partial (a2)}{\partial \overline{x}_{2i}} \quad + \quad (a2)(a3)\frac{\partial (a1)}{\partial \overline{x}_{2i}}$$

$$a1 = \{C1\}^{-\frac{3}{2}}$$

$$\frac{\partial (a1)}{\partial \overline{x}_{1i}} = -\frac{3}{2}\{C1\}^{-\frac{5}{2}} \times \frac{\partial (C1)}{\partial \overline{x}_{1i}} = 3m \sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{iu}\sigma_{uv})]$$

$$a2 = (m - \overline{x}_{21}) \sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})] \quad - \quad \sum_{w=2}^{p}\{\overline{x}_{2w} \sum_{v=1}^{p}[s^{vw} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})]\}$$

$$\frac{\partial (a2)}{\partial \overline{x}_{2i}} = \begin{cases} -\sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})] & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \quad - \quad \sum_{v=1}^{p}[s^{vi} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})]$$

$$= \begin{cases} -\sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})] \quad - \quad \sum_{v=1}^{p}[s^{vi} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})] & \text{if} \quad i = j \\ -\sum_{v=1}^{p}[s^{vi} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})] & \text{if} \quad i \neq j \end{cases}$$

$$a3 = \frac{1}{2}\Big\{(m - \overline{x}_{21})[(m - \overline{x}_{21})\sigma_{11} - \sum_{u=2}^{p}(\overline{x}_{2u}\sigma_{u1})]$$
$$- \sum_{v=2}^{p}\overline{x}_{2v}[(m - \overline{x}_{21})\sigma_{1v} - \sum_{u=2}^{p}(\overline{x}_{2u}\sigma_{uv})]\Big\}$$

$$\frac{\partial (a3)}{\partial \overline{x}_{1i}} = \frac{1}{2}\begin{cases} -2(m - \overline{x}_{21})\sigma_{11} & \text{if} \quad i = 1 \\ -\sigma_{i1} & \text{if} \quad i \neq 1 \end{cases}$$
$$= \frac{1}{2}\begin{cases} -2m\sigma_{11} & \text{if} \quad i = 1 \\ -\sigma_{i1} & \text{if} \quad i \neq 1 \end{cases}$$

$$\therefore \frac{\partial a}{\partial \overline{x}_{2i}} = \frac{m}{2}\{C1\}^{-\frac{3}{2}} \sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})] \times \begin{cases} -2m\sigma_{11} & \text{if } i=1 \\ -\sigma_{i1} & \text{if } i \neq 1 \end{cases}$$

$$-\frac{m^2}{2}\{C1\}^{-\frac{3}{2}} \begin{cases} \sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})] + \sum_{v=1}^{p}[s^{vi} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})] & \text{if } i=j \\ \sum_{v=1}^{p}[s^{vi} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})] & \text{if } i \neq j \end{cases}$$

$$+\frac{3}{2}m^4 \sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})] \times \sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{iu}\sigma_{uv})]$$

*expression (A1.29a)*

where $C1 = m^2 \sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{1u}\sigma_{uv})]$

(2) Find $\frac{\partial b}{\partial \overline{x}_{2i}}$ :

$$b = \{C1\}^{-\frac{1}{2}}\left(-ms^{j1} + \sum_{u=1}^{p}(s^{ju}\overline{x}_{2u})\right)$$

$$\frac{\partial b}{\partial \overline{x}_{2i}} = s^{ji}\{C1\}^{-\frac{1}{2}} - \frac{1}{2}\{C1\}^{-\frac{3}{2}} \times (-2m) \times \sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{iu}\sigma_{uv})] \times (-ms^{j1})$$

$$= s^{ji}\{C1\}^{-\frac{1}{2}} - ms^{j1}\{C1\}^{-\frac{3}{2}} \sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{iu}\sigma_{uv})] \qquad (A1.29b)$$

Now substitute *(A1.29a)* and *(A1.29b)* into *(A1.29)*,

$$\therefore \frac{\partial^2 A}{\partial \overline{x}_{2i}\partial \overline{x}_{2j}} = \frac{\partial a}{\partial \overline{x}_{2i}} + \frac{\partial b}{\partial \overline{x}_{2i}}$$

$$= \frac{m}{2}\{C1\}^{-\frac{3}{2}} \sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})] \times \begin{cases} -2m\sigma_{11} & \text{if } i=1 \\ -\sigma_{i1} & \text{if } i \neq 1 \end{cases}$$

$$-\frac{m^2}{2}\{C1\}^{-\frac{3}{2}} \begin{cases} \sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})] + \sum_{v=1}^{p}[s^{vi} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})] & \text{if } i=j \\ \sum_{v=1}^{p}[s^{vi} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})] & \text{if } i \neq j \end{cases}$$

$$+\frac{3}{2}m^4 \sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{ju}\sigma_{uv})] \times \sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{iu}\sigma_{uv})]$$

$$+ s^{ji}\{C1\}^{-\frac{1}{2}} - ms^{j1}\{C1\}^{-\frac{3}{2}} \sum_{v=1}^{p}[s^{v1} \sum_{u=1}^{p}(s^{iu}\sigma_{uv})]$$

*expression (A1.30)*

Therefore substitute *(A1.28a)* , *(A1.28b)* and *(A1.30)* into *(A1.27)*

gives $\dfrac{\partial^2 \Phi}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}}$

$$
\begin{aligned}
= -\phi\Bigg( & -\frac{1}{2} m s^{11} \left\{ \sum_{v=1}^{p} [ s^{v1} ( \sum_{u=1}^{p} s^{1u} \sigma_{uv} ) ] \right\}^{-\frac{1}{2}} \Bigg) \\
& \times \Bigg\{ \frac{m}{2} \{C1\}^{-\frac{3}{2}} \sum_{v=1}^{p} [ s^{v1} \sum_{u=1}^{p} ( s^{ju} \sigma_{uv} ) ] \times \begin{cases} -2m\sigma_{11} & \text{if } \; i = 1 \\ -\sigma_{i1} & \text{if } \; i \neq 1 \end{cases} \\
& \quad -\frac{m^2}{2} \{C1\}^{-\frac{3}{2}} \begin{cases} \sum_{v=1}^{p} [ s^{v1} \sum_{u=1}^{p} ( s^{ju} \sigma_{uv} ) ] + \sum_{v=1}^{p} [ s^{vi} \sum_{u=1}^{p} ( s^{ju} \sigma_{uv} ) ] & \text{if } \; i = j \\ \sum_{v=1}^{p} [ s^{vi} \sum_{u=1}^{p} ( s^{ju} \sigma_{uv} ) ] & \text{if } \; i \neq j \end{cases} \\
& \quad + \frac{3}{2} m^4 \sum_{v=1}^{p} [ s^{v1} \sum_{u=1}^{p} ( s^{ju} \sigma_{uv} ) ] \; \times \; \sum_{v=1}^{p} [ s^{v1} \sum_{u=1}^{p} ( s^{iu} \sigma_{uv} ) ] \\
& \quad + s^{ji} \{C1\}^{-\frac{1}{2}} \; - \; m s^{j1} \{C1\}^{-\frac{3}{2}} \sum_{v=1}^{p} [ s^{v1} \sum_{u=1}^{p} ( s^{iu} \sigma_{uv} ) ] \\
& \quad - \Bigg( \frac{1}{2} m s^{11} \left\{ \sum_{v=1}^{p} [ s^{v1} ( \sum_{u=1}^{p} s^{1u} \sigma_{uv} ) ] \right\}^{-\frac{1}{2}} \\
& \quad \times \left[ \frac{1}{2} s^{11} \times \left\{ \sum_{v=1}^{p} [ s^{v1} \sum_{u=1}^{p} ( s^{1u} \sigma_{uv} ) ] \right\}^{-\frac{3}{2}} \times \sum_{v=1}^{p} [ s^{v1} \sum_{u=1}^{p} ( s^{ju} \sigma_{uv} ) ] \right] \\
& \quad \times \left[ \frac{1}{2} s^{11} \times \left\{ \sum_{v=1}^{p} [ s^{v1} \sum_{u=1}^{p} ( s^{1u} \sigma_{uv} ) ] \right\}^{-\frac{3}{2}} \times \sum_{v=1}^{p} [ s^{v1} \sum_{u=1}^{p} ( s^{iu} \sigma_{uv} ) ] \right] \Bigg) \Bigg\}
\end{aligned}
$$

*expression (A1.31)*

Now need to find $\dfrac{\partial^2 \Phi}{\partial s_{kl} \partial s_{ij}}$ .

Consider case A1 where the conditions are :

$$\boldsymbol{\mu}_1 = (m, 0, \ldots\ldots\ldots, 0)^T$$

$$\boldsymbol{\mu}_2 = (0, \ldots\ldots\ldots\ldots, 0)^T$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho & \rho & \cdots\cdots & \rho \\ \rho & 1 & \rho & \cdots\cdots & \rho \\ \vdots & & & & \vdots \\ \rho & \cdots\cdots\cdots & & \rho & 1 \end{bmatrix}$$

we have $p_{21}^{(A)} = \Phi[-D]$

where $D = \dfrac{m}{2} s^{11} [\sum_{v=1}^{p} s^{v1} (s^{1v} + \rho \sum_{u \neq v=1}^{p} s^{1u})]^{-1/2}$

Now need to find $\dfrac{\partial^2 \Phi}{\partial s_{kl} \partial s_{ij}}$ by considering the partial differentials :

$$\frac{\partial^2 \Phi}{\partial s_{kl} \partial s_{ij}} = \frac{\partial}{\partial s_{kl}} [\frac{\partial \Phi}{\partial s_{ij}}]$$

$$= \frac{\partial}{\partial s_{kl}} \left[ -\frac{m}{2} \phi[-D] \frac{\partial}{\partial s_{ij}} \left( s^{11} [\sum_{v=1}^{p} \{s^{v1} (s^{1v} + \rho \sum_{u \neq v=1}^{p} s^{1u})\}]^{-1/2} \right) \right]$$

$$= -\frac{m}{2}\left[ \phi[-D]\frac{\partial^2}{\partial s_{kl}\,\partial s_{ij}}\left( s^{11}[\sum_{v=1}^{p}\{s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})\}]^{-1/2}\right) + \frac{\partial}{\partial s_{ij}}\left( s^{11}[\sum_{v=1}^{p}\{s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})\}]^{-1/2}\right)\frac{\partial}{\partial s_{kl}}(\phi[-D])\right]$$

Let $\quad A = \frac{\partial^2}{\partial s_{kl}\,\partial s_{ij}}\left( s^{11}[\sum_{v=1}^{p}\{s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})\}]^{-1/2}\right)$

$$B = \frac{\partial}{\partial s_{ij}}\left( s^{11}[\sum_{v=1}^{p}\{s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})\}]^{-1/2}\right)$$

$$C = \frac{\partial}{\partial s_{kl}}(\phi[-D])$$

Now find B :

$$B = s^{11}\frac{\partial}{\partial s_{ij}}\left( [\sum_{v=1}^{p}\{s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})\}]^{-1/2}\right) + [\sum_{v=1}^{p}\{s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})\}]^{-1/2}\frac{\partial s^{11}}{\partial s_{ij}}$$

Let $\quad B_2 = \frac{\partial s^{11}}{\partial s_{ij}} = w_0\,(s^{1i}s^{j1} + s^{1j}s^{i1}) = 2w_0\,s^{1i}s^{j1}$

where $\quad w_0 = \begin{cases} -0.5 & \text{if } i = j \\ -1 & \text{if } i \neq j \end{cases}$

$$B_1 = \frac{\partial}{\partial s_{ij}}\left( [\sum_{v=1}^{p}\{s^{v1}(s^{1v} + \rho \sum_{u\neq v=1}^{p} s^{1u})\}]^{-1/2} \right)$$

$$= -\frac{1}{2}[\sum_{v=1}^{p}\{s^{v1}(s^{1v} + \rho \sum_{u\neq v=1}^{p} s^{1u})\}]^{-3/2}$$

$$\times \frac{\partial}{\partial s_{ij}}[\sum_{v=1}^{p}\{s^{v1}(s^{1v} + \rho \sum_{u\neq v=1}^{p} s^{1u})\}]$$

Let $B_{11} = \frac{\partial}{\partial s_{ij}}[\sum_{v=1}^{p}\{s^{v1}(s^{1v} + \rho \sum_{u\neq v=1}^{p} s^{1u})\}]$

$$= \sum_{v=1}^{p} \frac{\partial}{\partial s_{ij}}\{s^{v1}(s^{1v} + \rho \sum_{u\neq v=1}^{p} s^{1u})\}$$

$$= \sum_{v=1}^{p} \{s^{v1} \frac{\partial}{\partial s_{ij}}(s^{1v} + \rho \sum_{u\neq v=1}^{p} s^{1u}) + (s^{1v} + \rho \sum_{u\neq v=1}^{p} s^{1u}) \frac{\partial s^{v1}}{\partial s_{ij}}\}$$

$$= \sum_{v=1}^{p} \{s^{v1} (\frac{\partial s^{1v}}{\partial s_{ij}} + \rho \sum_{u\neq v=1}^{p} \frac{\partial s^{1u}}{\partial s_{ij}})$$

$$+ (s^{1v} + \rho \sum_{u\neq v=1}^{p} s^{1u}) w_0 (s^{vi}s^{j1} + s^{vj}s^{i1})\}$$

$$= w_0 \sum_{v=1}^{p} \{s^{v1}[(s^{1i}s^{jv} + s^{1j}s^{iv}) + \rho \sum_{u\neq v=1}^{p} (s^{1i}s^{ju} + s^{1j}s^{iu})]$$

$$+ (s^{1v} + \rho \sum_{u\neq v=1}^{p} s^{1u})(s^{vi}s^{j1} + s^{vj}s^{i1})\}$$

$$\therefore B_1 = -\frac{1}{2}[\sum_{v=1}^{p}\{s^{v1}(s^{1v} + \rho \sum_{u\neq v=1}^{p} s^{1u})\}]^{-3/2} w_0 \sum_{v=1}^{p} \{s^{v1}[(s^{1i}s^{jv} + s^{1j}s^{iv})$$

$$+ \rho \sum_{u\neq v=1}^{p} (s^{1i}s^{ju} + s^{1j}s^{iu})] + (s^{1v} + \rho \sum_{u\neq v=1}^{p} s^{1u})(s^{vi}s^{j1} + s^{vj}s^{i1})\}$$

$$\therefore B=\left[-\frac{1}{2}s^{11}\ w_0\,[\sum_{v=1}^{p}\{s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})\}]^{-3/2}\sum_{v=1}^{p}\{s^{v1}\times[(s^{1i}s^{jv}+s^{1j}s^{iv})\right.$$

$$\left.+\rho\sum_{u\neq v=1}^{p}(s^{1i}s^{ju}+s^{1j}s^{iu})]+(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})(s^{vi}s^{j1}+s^{vj}s^{i1})\}\right]$$

$$+\left[[\sum_{v=1}^{p}\{s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})\}]^{-1/2}\,2w_0\,s^{1i}s^{j1}\right]$$

$$=w_0\,[\sum_{v=1}^{p}\{s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})\}]^{-3/2}\left\{\left[-\frac{s^{11}}{2}\sum_{v=1}^{p}\{s^{v1}[(s^{1i}s^{jv}+s^{1j}s^{iv})\right.\right.$$

$$\left.+\rho\sum_{u\neq v=1}^{p}(s^{1i}s^{ju}+s^{1j}s^{iu})]+(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})(s^{vi}s^{j1}+s^{vj}s^{i1})\}\right]$$

$$\left.+\left[(s^{1i}s^{j1})\times\sum_{v=1}^{p}\{s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})\}\right]\right\} \qquad (A1.32)$$

Let $B=w_0\,(F_1)^{-3/2}\,\{(-\frac{s^{11}}{2}F_2)+(F_3\,F_1)\}$

where $F_1=\sum_{v=1}^{p}\{s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})\}$

$$F_2=\sum_{v=1}^{p}\{s^{v1}[F_{21}+F_{22}]+[F_{23}\,F_{24}]\}$$

$$F_{21}=s^{1i}s^{jv}+s^{1j}s^{iv}$$

$$F_{22}=\rho\sum_{u\neq v=1}^{p}(s^{1i}s^{ju}+s^{1j}s^{iu})$$

$$F_{23}=s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u}$$

$$F_{24} = s^{vi}s^{jl} + s^{vj}s^{il}$$

$$F_3 = s^{li}s^{jl}$$

Now find A :

$$A = \frac{\partial^2}{\partial s_{kl}\, \partial s_{ij}} \left( s^{11} [ \sum_{v=1}^{p} \{ s^{vl} ( s^{lv} + \rho \sum_{u \neq v=1}^{p} s^{lu} ) \} ]^{-1/2} \right)$$

$$= \frac{\partial}{\partial s_{kl}} \{B\}$$

$$= \frac{\partial}{\partial s_{kl}} \left\{ w_0 (F_1)^{-3/2} [ ( -\frac{s^{11}}{2} F_2 ) + ( F_3 F_1 ) ] \right\}$$

$$= w_0 \left\{ \left( (F_1)^{-3/2} \frac{\partial}{\partial s_{kl}} [ ( -\frac{s^{11}}{2} F_2 ) + ( F_3 F_1 ) ] \right) + \left( [ ( -\frac{s^{11}}{2} F_2 ) + ( F_3 F_1 ) ] \frac{\partial (F1)^{-3/2}}{\partial s_{kl}} \right) \right\}$$

where

$$\frac{\partial}{\partial s_{kl}} [ ( -\frac{s^{11}}{2} F_2 ) + ( F_3 F_1 ) ]$$

$$= ( -\frac{s^{11}}{2} \frac{\partial F_2}{\partial s_{kl}} ) + ( -\frac{1}{2} F_2 \frac{\partial s^{11}}{\partial s_{kl}} ) ( F_3 \frac{\partial F_1}{\partial s_{kl}} ) + ( F_1 \frac{\partial F_3}{\partial s_{kl}} )$$

where

$$\frac{\partial s^{11}}{\partial s_{kl}} = w_1 ( s^{1k} s^{l1} + s^{1l} s^{k1} ) = 2 w_1 \, s^{1k} s^{l1}$$

where $w_1 = \begin{cases} -0.5 & \text{if} \quad k = l \\ -1 & \text{if} \quad k \neq l \end{cases}$

$$\frac{\partial s^{vl}}{\partial s_{kl}} = w_1 (s^{vk} s^{ll} + s^{vl} s^{kl})$$

$$\frac{\partial F_1}{\partial s_{kl}} = \frac{\partial}{\partial s_{kl}} \{ \sum_{v=1}^{p} [s^{vl} (s^{lv} + \rho \sum_{u \neq v=1}^{p} s^{lu})] \}$$

$$= w_1 \sum_{v=1}^{p} \{ [s^{vl} ((s^{lk} s^{lv} + s^{ll} s^{kv}) + \rho \sum_{u \neq v=1}^{p} (s^{lk} s^{lu} + s^{ll} s^{ku}))]$$

$$+ [(s^{lv} + \rho \sum_{u \neq v=1}^{p} s^{lu})(s^{vk} s^{ll} + s^{vl} s^{kl})] \}$$

$$\frac{\partial F_{21}}{\partial s_{kl}} = \frac{\partial}{\partial s_{kl}} [s^{li} s^{jv} + s^{lj} s^{iv}]$$

$$= w_1 \{ s^{li} (s^{jk} s^{lv} + s^{jl} s^{kv}) + s^{jv} (s^{lk} s^{li} + s^{ll} s^{ki})$$

$$+ s^{lj} (s^{ik} s^{lv} + s^{il} s^{kv}) + s^{iv} (s^{lk} s^{lj} + s^{ll} s^{kj}) \}$$

$$\frac{\partial F_{22}}{\partial s_{kl}} = \frac{\partial}{\partial s_{kl}} [\rho \sum_{u \neq v=1}^{p} (s^{li} s^{ju} + s^{lj} s^{iu})]$$

$$= \rho w_1 \sum_{u \neq v=1}^{p} \{ s^{li} (s^{lk} s^{li} + s^{ll} s^{ki}) + s^{ju} (s^{lk} s^{li} + s^{ll} s^{ki})$$

$$+ s^{lj} (s^{ik} s^{lu} + s^{il} s^{ku}) + s^{iu} (s^{lk} s^{lj} + s^{ll} s^{kj}) \}$$

$$\frac{\partial F_{23}}{\partial s_{kl}} = \frac{\partial}{\partial s_{kl}}[s^{1v} + \rho \sum_{u \neq v=1}^{p} s^{1u}]$$

$$= w_1 \{(s^{1k}s^{lv} + s^{11}s^{kv}) + \rho \sum_{u \neq v=1}^{p}(s^{1k}s^{lu} + s^{11}s^{ku})\}$$

$$\frac{\partial F_{24}}{\partial s_{kl}} = \frac{\partial}{\partial s_{kl}}[s^{vi}s^{j1} + s^{vj}s^{i1}]$$

$$= w_1 \{s^{vi}(s^{jk}s^{11} + s^{jl}s^{k1}) + s^{jl}(s^{vk}s^{li} + s^{vl}s^{ki})$$
$$+ s^{vj}(s^{ik}s^{11} + s^{il}s^{k1}) + s^{i1}(s^{vk}s^{lj} + s^{vl}s^{kj})\}$$

$$\frac{\partial F_2}{\partial s_{kl}} = \frac{\partial}{\partial s_{kl}}\{\sum_{v=1}^{p}\{[s^{v1}(F_{21}\, F_{22})] + [F_{23}\, F_{24}]\}\}$$

$$= \sum_{v=1}^{p}\{[s^{v1}\frac{\partial}{\partial s_{kl}}(F_{21}\, F_{22})] + [(F_{21}\, F_{22})\frac{\partial s^{v1}}{\partial s_{kl}}] + [F_{23}\frac{\partial F_{24}}{\partial s_{kl}}] + [F_{24}\frac{\partial F_{23}}{\partial s_{kl}}]\}$$

where

$$\frac{\partial}{\partial s_{kl}}[F_{21}\, F_{22}] = (F_{21}\frac{\partial F_{22}}{\partial s_{kl}}) + (F_{22}\frac{\partial F_{21}}{\partial s_{kl}})$$

$$= \Big[(s^{1i}s^{jv} + s^{1j}s^{iv})\rho\, w_1 \sum_{u \neq v=1}^{p}\{s^{1i}(s^{1k}s^{li} + s^{11}s^{ki})$$
$$+ s^{ju}(s^{1k}s^{li} + s^{11}s^{ki}) + s^{1j}(s^{ik}s^{lu} + s^{il}s^{ku}) + s^{iu}(s^{1k}s^{lj} + s^{11}s^{kj})\}\Big]$$
$$+ \Big[\rho \sum_{u \neq v=1}^{p}(s^{1i}s^{ju} + s^{1j}s^{iu})\, w_1 \{s^{1i}(s^{jk}s^{lv} + s^{jl}s^{kv})$$
$$+ s^{jv}(s^{1k}s^{li} + s^{11}s^{ki}) + s^{1j}(s^{ik}s^{lv} + s^{il}s^{kv}) + s^{iv}(s^{1k}s^{lj} + s^{11}s^{kj})\}\Big]$$

$$= w_1 \rho \Big\{ \Big[ (s^{1i} s^{jv} + s^{1j} s^{iv}) \times \sum_{u \neq v = 1}^{p} \{ s^{1i} (s^{1k} s^{li} + s^{11} s^{ki})$$

$$+ s^{ju} (s^{1k} s^{li} + s^{11} s^{ki}) + s^{1j} (s^{ik} s^{lu} + s^{il} s^{ku}) + s^{iu} (s^{1k} s^{lj} + s^{11} s^{kj}) \} \Big]$$

$$+ \Big[ \sum_{u \neq v = 1}^{p} (s^{1i} s^{ju} + s^{1j} s^{iu}) \times \{ s^{1i} (s^{jk} s^{lv} + s^{jl} s^{kv})$$

$$+ s^{jv} (s^{1k} s^{li} + s^{11} s^{ki}) + s^{1j} (s^{ik} s^{lv} + s^{il} s^{kv}) + s^{iv} (s^{1k} s^{lj} + s^{11} s^{kj}) \} \Big]$$

$$\therefore \frac{\partial F_2}{\partial s_{kl}} = w_1 \sum_{v=1}^{p} \Big\{ \Big[ s^{v1} \rho \{ [ (s^{1i} s^{jv} + s^{1j} s^{iv}) \sum_{u \neq v = 1}^{p} \{ s^{1i} (s^{1k} s^{li} + s^{11} s^{ki})$$

$$+ s^{ju} (s^{1k} s^{li} + s^{11} s^{ki}) + s^{1j} (s^{ik} s^{lu} + s^{il} s^{ku}) + s^{iu} (s^{1k} s^{lj} + s^{11} s^{kj}) \} ]$$

$$+ [ \{ s^{1i} (s^{jk} s^{lv} + s^{jl} s^{kv}) + s^{jv} (s^{1k} s^{li} + s^{11} s^{ki})$$

$$+ s^{1j} (s^{ik} s^{lv} + s^{il} s^{kv}) + s^{iv} (s^{1k} s^{lj} + s^{11} s^{kj}) \} \times \sum_{u \neq v = 1}^{p} (s^{1i} s^{ju} + s^{1j} s^{iu}) ] \} \Big]$$

$$+ [ (s^{1i} s^{jv} + s^{1j} s^{iv}) \times (s^{vk} s^{11} + s^{vl} s^{k1}) \times \rho \sum_{u \neq v = 1}^{p} (s^{1i} s^{ju} + s^{1j} s^{iu}) ]$$

$$+ [ (s^{1v} + \rho \sum_{u \neq v = 1}^{p} s^{1u}) \times \{ s^{vi} (s^{jk} s^{11} + s^{jl} s^{k1}) + s^{jl} (s^{vk} s^{li} + s^{vl} s^{ki})$$

$$+ s^{vj} (s^{ik} s^{11} + s^{il} s^{k1}) + s^{il} (s^{vk} s^{lj} + s^{vl} s^{kj}) \} ]$$

$$+ [ (s^{vi} s^{j1} + s^{vj} s^{i1}) \times \{ (s^{1k} s^{lv} + s^{11} s^{kv}) + \rho \sum_{u \neq v = 1}^{p} (s^{1k} s^{lu} + s^{11} s^{ku}) \} ] \Big\}$$

$$\frac{\partial F_3}{\partial s_{kl}} = \frac{\partial}{\partial s_{kl}} [ s^{1i} s^{j1} ] = w_1 \{ s^{1i} (s^{jk} s^{11} + s^{jl} s^{k1}) + s^{jl} (s^{1k} s^{li} + s^{11} s^{ki}) \}$$

$$\therefore \frac{\partial}{\partial s_{kl}}[(-\frac{s^{11}}{2}F_2)+(F_3\,F_1)]$$

$$=\Bigg[-\frac{s^{11}}{2}w_1\sum_{v=1}^{p}\Big\{\Big[s^{v1}\rho\{[(s^{1i}s^{jv}+s^{1j}s^{iv})\sum_{u\neq v=1}^{p}\{s^{1i}(s^{1k}s^{li}+s^{11}s^{ki})$$

$$+s^{ju}(s^{1k}s^{li}+s^{11}s^{ki})+s^{1j}(s^{ik}s^{lu}+s^{il}s^{ku})+s^{iu}(s^{1k}s^{lj}+s^{11}s^{kj})\}]$$

$$+[\{s^{1i}(s^{jk}s^{lv}+s^{jl}s^{kv})+s^{jv}(s^{1k}s^{li}+s^{11}s^{ki})$$

$$+s^{1j}(s^{ik}s^{lv}+s^{il}s^{kv})+s^{iv}(s^{1k}s^{lj}+s^{11}s^{kj})\}\sum_{u\neq v=1}^{p}(s^{1i}s^{ju}+s^{1j}s^{iu})]\}]$$

$$+[(s^{1i}s^{jv}+s^{1j}s^{iv})(s^{vk}s^{11}+s^{vl}s^{k1})\rho\sum_{u\neq v=1}^{p}(s^{1i}s^{ju}+s^{1j}s^{iu})]$$

$$+[(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})\{s^{vi}(s^{jk}s^{11}+s^{jl}s^{k1})+s^{jl}(s^{vk}s^{li}+s^{vl}s^{ki})$$

$$+s^{vj}(s^{ik}s^{11}+s^{il}s^{k1})+s^{il}(s^{vk}s^{lj}+s^{vl}s^{kj})\}]$$

$$+[(s^{vi}s^{jl}+s^{vj}s^{il})\times\{(s^{1k}s^{lv}+s^{11}s^{kv})+\rho\sum_{u\neq v=1}^{p}(s^{1k}s^{lu}+s^{11}s^{ku})\}]\Big\}\Bigg]$$

$$+\Bigg[-s^{1k}s^{11}\sum_{u\neq v=1}^{p}\{[s^{v1}((s^{1i}s^{jv}+s^{1j}s^{iv})+\rho\sum_{u\neq v=1}^{p}(s^{1i}s^{ju}+s^{1j}s^{iu}))]$$

$$+[(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})(s^{vi}s^{jl}+s^{vj}s^{il})]\}\Bigg]$$

$$+\Bigg[s^{1i}s^{jl}\sum_{u\neq v=1}^{p}\{[s^{v1}((s^{1k}s^{lv}+s^{11}s^{kv})+\rho\sum_{u\neq v=1}^{p}(s^{1k}s^{lu}+s^{11}s^{ku}))]$$

$$+[(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})(s^{vk}s^{11}+s^{vl}s^{k1})]\}\Bigg]$$

$$+\Bigg[\{s^{1i}(s^{jk}s^{11}+s^{jl}s^{k1})+s^{jl}(s^{1k}s^{li}+s^{11}s^{ki})\}$$

$$\times\sum_{v=1}^{p}[s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})]\Bigg]\Bigg)$$

$$
\begin{aligned}
\therefore A = w_0 \Bigg\{ \Bigg( & [\sum_{v=1}^{p} \{ s^{v1} ( s^{1v} + \rho \sum_{u \neq v=1}^{p} s^{1u} ) \} ]^{-3/2} \, w_1 \Bigg[ - \frac{s^{11}}{2} \\
& \sum_{v=1}^{p} \Big\{ \Big[ s^{v1} \rho \{ [ ( s^{1i} s^{jv} + s^{1j} s^{iv} ) \sum_{u \neq v=1}^{p} \{ s^{1i} ( s^{1k} s^{li} + s^{11} s^{ki} ) \\
& + s^{ju} ( s^{1k} s^{li} + s^{11} s^{ki} ) + s^{1j} ( s^{ik} s^{lu} + s^{il} s^{ku} ) + s^{iu} ( s^{1k} s^{lj} + s^{11} s^{kj} ) \} ] \\
& + [ \{ s^{li} ( s^{jk} s^{lv} + s^{jl} s^{kv} ) + s^{jv} ( s^{1k} s^{li} + s^{11} s^{ki} ) + s^{1j} ( s^{ik} s^{lv} + s^{il} s^{kv} ) \\
& + s^{iv} ( s^{1k} s^{lj} + s^{11} s^{kj} ) \} \sum_{u \neq v=1}^{p} ( s^{1i} s^{ju} + s^{1j} s^{iu} ) ] \} \Big] \\
& + \Big[ ( s^{1i} s^{jv} + s^{1j} s^{iv} ) ( s^{vk} s^{11} + s^{vl} s^{k1} ) \rho \sum_{u \neq v=1}^{p} ( s^{1i} s^{ju} + s^{1j} s^{iu} ) \Big] \\
& + \Big[ ( s^{1v} + \rho \sum_{u \neq v=1}^{p} s^{1u} ) \{ s^{vi} ( s^{jk} s^{11} + s^{jl} s^{k1} ) + s^{jl} ( s^{vk} s^{li} + s^{vl} s^{ki} ) \\
& \qquad + s^{vj} ( s^{ik} s^{11} + s^{il} s^{k1} ) + s^{il} ( s^{vk} s^{lj} + s^{vl} s^{kj} ) \} \Big] \\
& + \Big[ ( s^{vi} s^{jl} + s^{vj} s^{il} ) \{ ( s^{1k} s^{lv} + s^{11} s^{kv} ) + \rho \sum_{u \neq v=1}^{p} ( s^{1k} s^{lu} + s^{11} s^{ku} ) \} \Big] \Big\} \Bigg] \\
& + \Bigg[ - s^{1k} s^{11} \sum_{u \neq v=1}^{p} \{ [ s^{v1} \, ( ( s^{1i} s^{jv} + s^{1j} s^{iv} ) + \rho \sum_{u \neq v=1}^{p} ( s^{1i} s^{ju} + s^{1j} s^{iu} ) ) ] \\
& \qquad + [ ( s^{1v} + \rho \sum_{u \neq v=1}^{p} s^{1u} ) ( s^{vi} s^{jl} + s^{vj} s^{il} ) ] \} \Bigg] \\
& + \Bigg[ s^{1i} s^{jl} \sum_{u \neq v=1}^{p} \{ [ s^{v1} \, ( ( s^{1k} s^{lv} + s^{11} s^{kv} ) + \rho \sum_{u \neq v=1}^{p} ( s^{1k} s^{lu} + s^{11} s^{ku} ) ) ] \\
& \qquad + [ ( s^{1v} + \rho \sum_{u \neq v=1}^{p} s^{1u} ) ( s^{vk} s^{11} + s^{vl} s^{k1} ) ] \} \Bigg] \\
& + \Bigg[ \{ s^{1i} ( s^{jk} s^{11} + s^{jl} s^{k1} ) + s^{jl} ( s^{1k} s^{li} + s^{11} s^{ki} ) \} \\
& \qquad \times \sum_{v=1}^{p} [ s^{v1} ( s^{1v} + \rho \sum_{u \neq v=1}^{p} s^{1u} ) ] \Bigg] \Bigg)
\end{aligned}
$$

$$+\Bigg(\{[-\frac{s^{11}}{2}\sum_{v=1}^{p}\{[s^{v1}((s^{1i}s^{jv}+s^{1j}s^{iv})+\rho\sum_{v=1}^{p}(s^{1i}s^{jv}+s^{1j}s^{iv}))]$$

$$+[(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})(s^{vi}s^{j1}+s^{vj}s^{i1})]\}]$$

$$+\Big[s^{1i}s^{j1}\sum_{v=1}^{p}[s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})]\Big]\Big\}\times(\sum_{v=1}^{p}[s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})])^{-5/2}$$

$$\times-\frac{3}{2}w_1\sum_{v=1}^{p}\{[s^{v1}((s^{1k}s^{lv}+s^{11}s^{kv})+\rho\sum_{v=1}^{p}(s^{1k}s^{lu}+s^{11}s^{ku}))]$$

$$+[(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})(s^{vk}s^{l1}+s^{vl}s^{k1})]\}\Bigg)\Bigg\} \qquad (A1.33)$$

Now need to find C :

$$C=\frac{\partial}{\partial s_{kl}}\left[\phi[-\frac{m}{2}s^{11}[\sum_{v=1}^{p}s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})]^{-1/2}]\right]=\frac{\partial}{\partial s_{kl}}[\phi[-D]]$$

$$=-D\phi[-D]\frac{\partial D}{\partial s_{kl}}$$

$$=-\frac{m}{2}s^{11}[\sum_{v=1}^{p}s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})]^{-1/2}\phi[-D]\frac{m}{2}$$

$$\times\frac{\partial}{\partial s_{kl}}\{s^{11}[\sum_{v=1}^{p}s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})]^{-1/2}\}$$

$$=-\frac{m^2}{4}s^{11}[\sum_{v=1}^{p}s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})]^{-1/2}\phi[-D]w_1$$

$$\times[\sum_{v=1}^{p}s^{v1}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})]^{-3/2}\Bigg\{[-\frac{s^{11}}{2}[\sum_{v=1}^{p}\{[s^{v1}+((s^{1k}s^{lv}+s^{11}s^{kv})$$

$$+\rho\sum_{u\neq v=1}^{p}(s^{1k}s^{lu}+s^{11}s^{ku}))]+[(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})(s^{vk}s^{11}+s^{vl}s^{kl})\}]$$

$$+[(s^{1k}s^{lv})\sum_{v=1}^{p}\{s^{vl}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})\}]\Bigg\}$$

$$=-\frac{m^2}{4}s^{11}\phi[-D]w_1[\sum_{v=1}^{p}s^{vl}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})]^{-2}\Bigg\{[-\frac{s^{11}}{2}[\sum_{v=1}^{p}\{[s^{vl}$$

$$+((s^{1k}s^{lv}+s^{11}s^{kv})+\rho\sum_{u\neq v=1}^{p}(s^{1k}s^{lu}+s^{11}s^{ku}))]+[(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})$$

$$\times(s^{vk}s^{11}+s^{vl}s^{kl})\}]+[(s^{1k}s^{lv})\sum_{v=1}^{p}\{s^{vl}(s^{1v}+\rho\sum_{u\neq v=1}^{p}s^{1u})\}]\Bigg\} \qquad (A1.34)$$

$$\therefore\frac{\partial^2\Phi}{\partial s_{kl}\partial s_{ij}}=-\frac{m}{2}[\phi(-D)\ A+B\ C] \qquad (A1.35)$$

where A, B and C are as defined in expressions *(A1.32)* , *(A1.33)* and *(A1.34)* respectively.

$$\text{Let}\quad A_{klij}=\frac{\partial^2\Phi}{\partial s_{kl}\partial s_{ij}}\ . \qquad (A1.36)$$

Now need to find $\frac{\partial^2 \Phi}{\partial s_{kl} \partial s_{ij}}$ .

Consider case A1 where the conditions are :

$$\mu_1 = (m, 0, \ldots \ldots \ldots, 0)^T$$

$$\mu_2 = (0, \ldots \ldots \ldots \ldots, 0)^T$$

$$\Sigma = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{p-1} \\ \rho & 1 & \rho & \cdots & \rho^{p-2} \\ \vdots & & & & \vdots \\ \rho^{p-1} & \ldots\ldots & & \rho & 1 \end{bmatrix}$$

we have $p_{21}^{(A)} = \Phi[-D]$

where $D = \frac{m}{2} s^{11} [\sum_{u=1}^{p} s^{u1} (\sum_{v=1}^{p} s^{1v} \rho^{|v-u|})]^{-1/2}$

Now need to find $\frac{\partial^2 \Phi}{\partial s_{kl}\, \partial s_{ij}}$ by considering the partial differentials :

$$\frac{\partial^2 \Phi}{\partial s_{kl} \partial s_{ij}} = \frac{\partial}{\partial s_{kl}} [\frac{\partial \Phi}{\partial s_{ij}}]$$

$$= \frac{\partial}{\partial s_{kl}} \left[ -\frac{m}{2} \phi[-D] \frac{\partial}{\partial s_{ij}} \left( s^{11} [ \sum_{u=1}^{p} s^{u1} ( \sum_{v=1}^{p} s^{1v} \rho^{|v-u|} )]^{-1/2} \right) \right]$$

$$= -\frac{m}{2} \left[ \phi[-D] \frac{\partial^2}{\partial s_{kl} \, \partial s_{ij}} \left( s^{11} [ \sum_{u=1}^{p} s^{u1} ( \sum_{v=1}^{p} s^{1v} \rho^{|v-u|} )]^{-1/2} \right) \right.$$

$$\left. + \frac{\partial}{\partial s_{ij}} \left( s^{11} [ \sum_{u=1}^{p} s^{u1} ( \sum_{v=1}^{p} s^{1v} \rho^{|v-u|} )]^{-1/2} \right) \frac{\partial}{\partial s_{kl}} (\phi[-D]) \right]$$

Let $A = \frac{\partial^2}{\partial s_{kl} \, \partial s_{ij}} \left( s^{11} [ \sum_{u=1}^{p} s^{u1} ( \sum_{v=1}^{p} s^{1v} \rho^{|v-u|} )]^{-1/2} \right)$

$$B = \frac{\partial}{\partial s_{ij}} \left( s^{11} [ \sum_{u=1}^{p} s^{u1} ( \sum_{v=1}^{p} s^{1v} \rho^{|v-u|} )]^{-1/2} \right)$$

$$C = \frac{\partial}{\partial s_{kl}} (\phi[-D])$$

Now find B :

$$B = s^{11} \frac{\partial}{\partial s_{ij}} \left( [ \sum_{u=1}^{p} s^{u1} ( \sum_{v=1}^{p} s^{1v} \rho^{|v-u|} )]^{-1/2} \right) + [ \sum_{u=1}^{p} s^{u1} ( \sum_{v=1}^{p} s^{1v} \rho^{|v-u|} )]^{-1/2} \frac{\partial s^{11}}{\partial s_{ij}}$$

Let $B_2 = \frac{\partial s^{11}}{\partial s_{ij}} = w_0 (s^{1i}s^{j1} + s^{1j}s^{i1}) = 2w_0 s^{1i}s^{j1}$

$$\text{where} \qquad w_0 = \begin{cases} -0.5 & \text{if } i = j \\ -1 & \text{if } i \neq j \end{cases}$$

$$B_1 = \frac{\partial}{\partial s_{ij}}\left( [\sum_{u=1}^{p} s^{u1} (\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})]^{-1/2} \right)$$
$$= -\frac{1}{2}[\sum_{u=1}^{p} s^{u1} (\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})]^{-3/2} \frac{\partial}{\partial s_{ij}}[\sum_{u=1}^{p} s^{u1} (\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})]$$

Let $B_{11} = \frac{\partial}{\partial s_{ij}}[\sum_{u=1}^{p} s^{u1} (\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})]$

$$= \sum_{u=1}^{p} \frac{\partial}{\partial s_{ij}}\{s^{u1} (\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})\}$$
$$= \sum_{u=1}^{p} \{s^{u1} \frac{\partial}{\partial s_{ij}}( \sum_{v=1}^{p} s^{1v}\rho^{|v-u|}) + (\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})\frac{\partial s^{u1}}{\partial s_{ij}}\}$$
$$= w_0 \sum_{u=1}^{p} [s^{u1} \sum_{v=1}^{p}\{(s^{1i}s^{jv} + s^{1j}s^{iv})\rho^{|u-v|}\}$$
$$+(\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})(s^{ui}s^{j1} + s^{uj}s^{i1})]$$

$$\therefore B_1 = -\frac{1}{2}[\sum_{u=1}^{p} s^{u1} (\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})]^{-3/2} w_0 \sum_{u=1}^{p} [s^{u1} \sum_{v=1}^{p}\{(s^{1i}s^{jv} + s^{1j}s^{iv})\rho^{|u-v|}\}$$
$$+ (\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})(s^{ui}s^{j1} + s^{uj}s^{i1})]$$

$$\therefore B = w_0 [\sum_{u=1}^{p} s^{u1} (\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})]^{-3/2}\left[ \left\{ -\frac{s^{11}}{2}\sum_{u=1}^{p} \{s^{u1} \sum_{v=1}^{p}\{(s^{1i}s^{jv} + s^{1j}s^{iv})\rho^{|v-u|}\}\right.\right.$$
$$\left.\left. +(\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})(s^{ui}s^{j1} + s^{uj}s^{i1})\}\right\} + \left\{ 2(s^{1i}s^{j1})[\sum_{u=1}^{p} s^{u1} (\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})]\right\}\right]$$

*expression (A1.37)*

Let $B = w_0 (E_1)^{-3/2} \{(-\frac{s^{11}}{2} E_2) + (2E_3 E_1)\}$

where $E_1 = [\sum_{u=1}^{p} s^{u1} (\sum_{v=1}^{p} s^{1v} \rho^{|v-u|})]$

$$E_2 = \sum_{u=1}^{p} \{[s^{u1} E_{21}] + [E_{22} E_{23}]\}$$

$$E_{21} = \sum_{v=1}^{p} \{(s^{1i}s^{jv} + s^{1j}s^{iv})\rho^{|v-u|}\}$$

$$E_{22} = \sum_{u=1}^{p} (s^{1v}\rho^{|v-u|})$$

$$E_{23} = s^{ui}s^{jl} + s^{uj}s^{il}$$

$$E_3 = s^{1i}s^{jl}$$

Now find A :

$$A = \frac{\partial^2}{\partial s_{kl}\,\partial s_{ij}}\left( s^{11} [\sum_{u=1}^{p} s^{u1} (\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})]^{-1/2} \right) = \frac{\partial}{\partial s_{kl}}\{B\}$$

$$= \frac{\partial}{\partial s_{kl}}\left\{ w_0 (E_1)^{-3/2} [(-\frac{s^{11}}{2} E_2) + (2E_3 E_1)] \right\}$$

$$= w_0 \left\{ \left( (E_1)^{-3/2} \frac{\partial}{\partial s_{kl}} [(-\frac{s^{11}}{2} E_2) + (2E_3 E_1)] \right) + \left( [(-\frac{s^{11}}{2} E_2) + (2E_3 E_1)] \frac{\partial (E_1)^{-3/2}}{\partial s_{kl}} \right) \right\}$$

where

$$\frac{\partial}{\partial s_{kl}}[(-\frac{s^{11}}{2}E_2)+(2E_3\,E_1)]$$

$$=(-\frac{s^{11}}{2}\frac{\partial E_2}{\partial s_{kl}})+(-\frac{E_2}{2}\frac{\partial s^{11}}{\partial s_{kl}})+(2E_1\frac{\partial E_3}{\partial s_{kl}})+(2E_3\frac{\partial E_1}{\partial s_{kl}})$$

where

$$\frac{\partial s^{11}}{\partial s_{kl}}=w_1\,(s^{1k}s^{l1}+s^{1l}s^{k1})=2w_1\,s^{1k}s^{l1}$$

$$\text{where}\quad w_1=\begin{cases}-0.5 & \text{if}\quad k=l\\ -1 & \text{if}\quad k\neq l\end{cases}$$

$$\frac{\partial s^{u1}}{\partial s_{kl}}=w_1\,(s^{uk}s^{l1}+s^{ul}s^{k1})$$

$$\frac{\partial E_1}{\partial s_{kl}}=\frac{\partial}{\partial s_{kl}}[\sum_{u=1}^{p}s^{u1}(\sum_{v=1}^{p}s^{1v}\rho^{|v-u|})]$$

$$=\sum_{u=1}^{p}[\frac{\partial}{\partial s_{kl}}s^{u1}(\sum_{v=1}^{p}s^{1v}\rho^{|v-u|})]$$

$$=\sum_{u=1}^{p}[s^{u1}\frac{\partial}{\partial s_{kl}}(\sum_{v=1}^{p}s^{1v}\rho^{|v-u|})+(\sum_{v=1}^{p}s^{1v}\rho^{|v-u|})\frac{\partial s^{u1}}{\partial s_{kl}}]$$

$$=w_1\sum_{u=1}^{p}[s^{u1}\sum_{v=1}^{p}\{(s^{1k}s^{lv}+s^{1l}s^{kv})\rho^{|v-u|}\}$$

$$+(\sum_{v=1}^{p}s^{1v}\rho^{|v-u|})(s^{uk}s^{l1}+s^{ul}s^{k1})]$$

$$\frac{\partial E_{21}}{\partial s_{kl}}=\frac{\partial}{\partial s_{kl}}[\sum_{v=1}^{p}(s^{1i}s^{jv}+s^{1j}s^{iv})\rho^{|v-u|}]$$

$$=w_1\sum_{v=1}^{p}[\rho^{|v-u|}\{s^{1i}(s^{jk}s^{lv}+s^{jl}s^{kv})+s^{jv}(s^{1k}s^{li}+s^{1l}s^{ki})$$

$$+s^{1j}(s^{ik}s^{lv}+s^{il}s^{kv})+s^{iv}(s^{1k}s^{lj}+s^{1l}s^{kj})\}]$$

$$\frac{\partial E_{22}}{\partial s_{kl}} = \frac{\partial}{\partial s_{kl}}[\sum_{u=1}^{p}(s^{1v}\rho^{|v-u|})]$$
$$= \sum_{u=1}^{p}\frac{\partial}{\partial s_{kl}}(s^{1v}\rho^{|v-u|})$$
$$= w_1\sum_{u=1}^{p}\{(\rho^{|v-u|})(s^{1k}s^{1v}+s^{11}s^{kv})\}$$

$$\frac{\partial E_{23}}{\partial s_{kl}} = \frac{\partial}{\partial s_{kl}}[s^{ui}s^{jl}+s^{uj}s^{il}]$$
$$= w_1\{s^{ui}(s^{jk}s^{l1}+s^{jl}s^{k1})+s^{jl}(s^{uk}s^{li}+s^{ul}s^{ki})$$
$$+s^{uj}(s^{ik}s^{l1}+s^{il}s^{k1})+s^{i1}(s^{uk}s^{lj}+s^{ul}s^{kj})\}$$

$$\frac{\partial E_3}{\partial s_{kl}} = \frac{\partial}{\partial s_{kl}}[s^{1i}s^{jl}] = w_1\{s^{1i}(s^{jk}s^{l1}+s^{jl}s^{k1})+s^{jl}(s^{1k}s^{li}+s^{11}s^{ki})\}$$

$$\frac{\partial E_2}{\partial s_{kl}} = \frac{\partial}{\partial s_{kl}}\{\sum_{u=1}^{p}[(s^{ul}E_{21})+(E_{22}E_{23})]\}$$
$$= \sum_{u=1}^{p}[(s^{ul}\frac{\partial E_{21}}{\partial s_{kl}})+(E_{21}\frac{\partial s^{ul}}{\partial s_{kl}})+(E_{22}\frac{\partial E_{23}}{\partial s_{kl}})+(E_{23}\frac{\partial E_{22}}{\partial s_{kl}})]$$
$$= \sum_{u=1}^{p}\Big[\Big\{s^{ul}w_1\sum_{v=1}^{p}[\rho^{|v-u|}\{s^{1i}(s^{jk}s^{lv}+s^{jl}s^{kv})+s^{jv}(s^{1k}s^{li}+s^{11}s^{ki})$$
$$+s^{1j}(s^{ik}s^{lv}+s^{il}s^{kv})+s^{iv}(s^{1k}s^{lj}+s^{11}s^{kj})\}]\Big\}$$
$$+\Big\{\sum_{v=1}^{p}\{(s^{1i}s^{jv}+s^{1j}s^{iv})\rho^{|v-u|}\}w_1(s^{uk}s^{l1}+s^{ul}s^{k1})\Big\}$$
$$+\Big\{\sum_{u=1}^{p}(s^{1v}\rho^{|v-u|})w_1\{s^{ui}(s^{jk}s^{l1}+s^{jl}s^{k1})+s^{jl}(s^{uk}s^{li}+s^{ul}s^{ki})$$
$$+s^{uj}(s^{ik}s^{l1}+s^{il}s^{k1})+s^{i1}(s^{uk}s^{lj}+s^{ul}s^{kj})\}\Big\}$$
$$+\Big\{(s^{ui}s^{jl}+s^{uj}s^{il})w_1\sum_{u=1}^{p}\{(\rho^{|v-u|})(s^{1k}s^{lv}+s^{11}s^{kv})\}\Big\}\Big]$$

$$\therefore \frac{\partial}{\partial s_{kl}}[(-\frac{s^{11}}{2}E_2)+(2E_3\,E_1)]$$

$$=\Bigg[-\frac{s^{11}}{2}w_1\sum_{u=1}^{p}\Big[\Big\{s^{u1}\sum_{v=1}^{p}[\rho^{|v-u|}\{s^{1i}(s^{jk}s^{lv}+s^{jl}s^{kv})+s^{jv}(s^{1k}s^{li}+s^{11}s^{ki})$$

$$+s^{1j}(s^{ik}s^{lv}+s^{il}s^{kv})+s^{iv}(s^{1k}s^{lj}+s^{11}s^{kj})\}]\Big\}+\Big\{\sum_{v=1}^{p}\{(s^{1i}s^{jv}+s^{1j}s^{iv})\rho^{|v-u|}\}$$

$$\times(s^{uk}s^{11}+s^{ul}s^{k1})\Big\}+\Big\{\sum_{u=1}^{p}(s^{1v}\rho^{|v-u|})\{s^{ui}(s^{jk}s^{11}+s^{jl}s^{k1})$$

$$+s^{j1}(s^{uk}s^{li}+s^{ul}s^{ki})+s^{uj}(s^{ik}s^{11}+s^{il}s^{k1})+s^{i1}(s^{uk}s^{lj}+s^{ul}s^{kj})\}\Big\}$$

$$+\Big\{(s^{ui}s^{j1}+s^{uj}s^{i1})\times\sum_{u=1}^{p}\{(\rho^{|v-u|})\times(s^{1k}s^{lv}+s^{11}s^{kv})\}\Big\}\Big]\Bigg]$$

$$-\Bigg[\frac{1}{2}\sum_{u=1}^{p}\{[s^{u1}\sum_{v=1}^{p}\{(s^{1i}s^{jv}+s^{1j}s^{iv})\rho^{|v-u|}\}]$$

$$+[\sum_{u=1}^{p}(s^{1v}\rho^{|v-u|})(s^{ui}s^{j1}+s^{uj}s^{i1})]\}2w_1\,s^{1k}s^{11}\Bigg]$$

$$+\Big[2[\sum_{u=1}^{p}s^{u1}(\sum_{v=1}^{p}s^{1v}\rho^{|v-u|})]w_1\{s^{1i}(s^{jk}s^{11}+s^{jl}s^{k1})$$

$$+s^{j1}(s^{1k}s^{li}+s^{11}s^{ki})\}\Big]$$

$$+\Big[2s^{1i}s^{j1}\,w_1\sum_{u=1}^{p}[s^{u1}\sum_{v=1}^{p}\{(s^{1k}s^{lv}+s^{11}s^{kv})\rho^{|v-u|}\}$$

$$+(\sum_{v=1}^{p}s^{1v}\rho^{|v-u|})(s^{uk}s^{11}+s^{ul}s^{k1})]\Big]$$

$$\therefore A = w_0\Bigg[\Bigg\{[\sum_{u=1}^{p} s^{u1}(\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})]^{-3/2}\Bigg\{\Bigg[-\frac{s^{11}}{2}w_1\sum_{u=1}^{p}\Big[\Big\{s^{u1}\sum_{v=1}^{p}[\rho^{|v-u|}$$

$$\times\{s^{1i}(s^{jk}s^{1v}+s^{j1}s^{kv})+s^{jv}(s^{1k}s^{1i}+s^{11}s^{ki})+s^{1j}(s^{ik}s^{1v}+s^{i1}s^{kv})$$

$$+s^{iv}(s^{1k}s^{1j}+s^{11}s^{kj})\}]\Big\}+\{\sum_{v=1}^{p}\{(s^{1i}s^{jv}+s^{1j}s^{iv})\rho^{|v-u|}\}$$

$$\times(s^{uk}s^{11}+s^{u1}s^{k1})\Big\}+\{\sum_{u=1}^{p}(s^{1v}\rho^{|v-u|})\{s^{ui}(s^{jk}s^{11}+s^{j1}s^{k1})$$

$$+s^{j1}(s^{uk}s^{1i}+s^{u1}s^{ki})+s^{uj}(s^{ik}s^{11}+s^{i1}s^{k1})+s^{i1}(s^{uk}s^{1j}+s^{u1}s^{kj})\}\Big\}$$

$$+\Big\{(s^{ui}s^{j1}+s^{uj}s^{i1})\times\sum_{u=1}^{p}\{(\rho^{|v-u|})\times(s^{1k}s^{1v}+s^{11}s^{kv})\Big\}\Big]\Big]$$

$$-\Bigg[\frac{1}{2}\sum_{u=1}^{p}\{[s^{u1}\sum_{v=1}^{p}\{(s^{1i}s^{jv}+s^{1j}s^{iv})\rho^{|v-u|}\}]$$

$$+[\sum_{u=1}^{p}(s^{1v}\rho^{|v-u|})(s^{ui}s^{j1}+s^{uj}s^{i1})]\}2w_1 s^{1k}s^{11}\Bigg]$$

$$+\Big[2[\sum_{u=1}^{p}s^{u1}(\sum_{v=1}^{p}s^{1v}\rho^{|v-u|})]w_1\{s^{1i}(s^{jk}s^{11}+s^{j1}s^{k1})$$

$$+s^{j1}(s^{1k}s^{1i}+s^{11}s^{ki})\}\Big]$$

$$+\Big[2s^{1i}s^{j1}w_1\sum_{u=1}^{p}[s^{u1}\sum_{v=1}^{p}\{(s^{1k}s^{1v}+s^{11}s^{kv})\rho^{|v-u|}\}$$

$$+(\sum_{v=1}^{p}s^{1v}\rho^{|v-u|})(s^{uk}s^{11}+s^{u1}s^{k1})]\Big]\Bigg\}\Bigg\}$$

$$+\Bigg\{\Bigg[-\frac{s^{11}}{2}\sum_{u=1}^{p}\{[s^{u1}\sum_{v=1}^{p}\{(s^{1i}s^{jv}+s^{1j}s^{iv})\rho^{|v-u|}\}]+[\sum_{u=1}^{p}(s^{1v}\rho^{|v-u|})$$

$$(s^{ui}s^{j1}+s^{uj}s^{i1})]\}+\{2[\sum_{u=1}^{p}s^{u1}(\sum_{v=1}^{p}s^{1v}\rho^{|v-u|})]\times(s^{1i}s^{j1})\}\Bigg]\times-\frac{3}{2}$$

$$\times[\sum_{u=1}^{p}s^{u1}(\sum_{v=1}^{p}s^{1v}\rho^{|v-u|})]^{-5/2}w_1\sum_{u=1}^{p}[s^{u1}\sum_{v=1}^{p}\{(s^{1k}s^{1v}+s^{11}s^{kv})\rho^{|v-u|}\}$$

$$+(\sum_{v=1}^{p}s^{1v}\rho^{|v-u|})(s^{uk}s^{11}+s^{u1}s^{k1})]\Bigg\}\Bigg] \qquad (A1.38)$$

Now need to find C :

$$C = \frac{\partial}{\partial s_{kl}}\left[ \phi[-\frac{m}{2}s^{11}[\sum_{u=1}^{p} s^{ul}(\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})]^{-1/2}]\right] = \frac{\partial}{\partial s_{kl}}[\phi[-D]]$$

$$= -D\phi[-D]\frac{\partial D}{\partial s_{kl}}$$

$$= -\frac{m}{2}s^{11}[\sum_{u=1}^{p} s^{ul}(\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})]^{-1/2}\phi[-D]\frac{m}{2}$$

$$\times\frac{\partial}{\partial s_{kl}}\{s^{11}[\sum_{u=1}^{p} s^{ul}(\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})]^{-1/2}\}$$

$$= -\frac{m^2}{4}s^{11}[\sum_{u=1}^{p} s^{ul}(\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})]^{-1/2}\phi[-D]w_1[\sum_{u=1}^{p} s^{ul}(\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})]^{-3/2}$$

$$\left\{\{-\frac{s^{11}}{2}\sum_{u=1}^{p}\{[s^{ul}\sum_{u\neq v=1}^{p}\{(s^{1k}s^{lv}+s^{1l}s^{kv})\rho^{|v-u|}\}+(\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})\right.$$

$$\left.\times(s^{uk}s^{1l}+s^{ul}s^{k1})]\}+\{2[\sum_{u=1}^{p} s^{ul}(\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})]\ (s^{1k}s^{1l})\}\right\}$$

$$= -\frac{m^2}{4}s^{11}\phi[-D]w_1[\sum_{u=1}^{p} s^{ul}(\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})]^{-2}\times\left\{\{-\frac{s^{11}}{2}\sum_{u=1}^{p}[s^{ul}\right.$$

$$\times\sum_{u\neq v=1}^{p}\{(s^{1k}s^{lv}+s^{1l}s^{kv})\rho^{|v-u|}\}+(\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})(s^{uk}s^{1l}+s^{ul}s^{k1})]\}$$

$$\left.+\{2[\sum_{u=1}^{p} s^{ul}(\sum_{v=1}^{p} s^{1v}\rho^{|v-u|})](s^{1k}s^{1l})\}\right\} \qquad (A1.39)$$

$$\therefore \frac{\partial^2\Phi}{\partial s_{kl}\partial s_{ij}} = -\frac{m}{2}[\phi(-D)\ A + BC] \qquad (A1.40)$$

where A, B and C are as defined in expressions *(A1.37)*, *(A1.38)* and *(A1.39)* respectively.

$$\text{Let}\quad A_{klij} = \frac{\partial^2\Phi}{\partial s_{kl}\partial s_{ij}}\ . \qquad (A1.41)$$

# *Appendix A2.1*

# *ASYMPTOTIC EXPANSION FOR EUCLIDEAN DISTANCE CLASSIFIER*

## *Plug-in error rate*

We have $p_{12}^{(P)} = \Phi[-\frac{1}{2}\frac{(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)}{[(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T \mathbf{S}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)]^{1/2}}] = \Phi[-\frac{A}{2}].$

Using Taylor series expansion (to second order approximation),we obtain:-

$$
\begin{aligned}
\Phi(\overline{\mathbf{x}}_1, \overline{\mathbf{x}}_2, \mathbf{S}) = {} & \Phi(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\Sigma}) + \sum_{j=1}^{p}\frac{\partial\Phi}{\partial\overline{x}_{1j}}(\overline{x}_{1j} - \mu_{1j}) + \sum_{j=1}^{p}\frac{\partial\Phi}{\partial\overline{x}_{2j}}(\overline{x}_{2j} - \mu_{2j}) \\
& + \sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial\Phi}{\partial s_{ij}}(s_{ij} - \Sigma_{ij}) \\
& + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{1i}\partial\overline{x}_{1j}}(\overline{x}_{1i} - \mu_{1i})(\overline{x}_{1j} - \mu_{1j}) \\
& + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{2i}\partial\overline{x}_{2j}}(\overline{x}_{2i} - \mu_{2i})(\overline{x}_{2j} - \mu_{2j}) \\
& + \frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{p}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial s_{kl}\partial s_{ij}}(s_{kl} - \Sigma_{kl})(s_{ij} - \Sigma_{ij}) \\
& + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{1i}\partial\overline{x}_{2j}}(\overline{x}_{1i} - \mu_{1i})(\overline{x}_{2j} - \mu_{2j}) \\
& + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{1i}\partial s_{ij}}(\overline{x}_{1i} - \mu_{1i})(s_{ij} - \Sigma_{ij}) \\
& + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{2i}\partial s_{ij}}(\overline{x}_{2i} - \mu_{2i})(s_{ij} - \Sigma_{ij})
\end{aligned}
$$

where $\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{2j}}=0,\ \sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial s_{ij}}=0$ and $\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial s_{ij}}=0$ .

Taking expectations gives:-

$$
\begin{aligned}
E[\Phi(\bar{\mathbf{x}}_1,\bar{\mathbf{x}}_2,\mathbf{S})] = \Phi(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\boldsymbol{\Sigma}) &+ \sum_{j=1}^{p}\frac{\partial\Phi}{\partial\bar{x}_{1j}}E(\bar{x}_{1j}-\mu_{1j}) \\
&+\sum_{j=1}^{p}\frac{\partial\Phi}{\partial\bar{x}_{2j}}E(\bar{x}_{2j}-\mu_{2j})+\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial\Phi}{\partial s_{ij}}E(s_{ij}-\Sigma_{ij}) \\
&+\frac{1}{2}\sum_{i=1}^{p}\ \sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\ \partial\bar{x}_{1j}}\operatorname{cov}(\bar{x}_{1,ij}) \\
&+\frac{1}{2}\sum_{i=1}^{p}\ \sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\ \partial\bar{x}_{2j}}\operatorname{cov}(\bar{x}_{2,ij}) \\
&+\frac{1}{2}\sum_{k=1}^{p}\ \sum_{l=1}^{p}\ \sum_{i=1}^{p}\ \sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial s_{kl}\ \partial s_{ij}}\operatorname{cov}(s_{kl},s_{ij})
\end{aligned}
$$

where $\sum_{j=1}^{p}\frac{\partial\Phi}{\partial\bar{x}_{1j}}E(\bar{x}_{1j}-\mu_{1j})=\sum_{j=1}^{p}\frac{\partial\Phi}{\partial\bar{x}_{2j}}E(\bar{x}_{2j}-\mu_{2j})=0$

and $\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial\Phi}{\partial s_{ij}}E(s_{ij}-\Sigma_{ij})=0.$

$$
\begin{aligned}
\therefore E[\Phi(\bar{\mathbf{x}}_1,\bar{\mathbf{x}}_2,\mathbf{S})] = \Phi(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\boldsymbol{\Sigma}) &+\frac{1}{2}\sum_{i=1}^{p}\ \sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\ \partial\bar{x}_{1j}}\operatorname{cov}(\bar{x}_{1,ij}) \\
&+\frac{1}{2}\sum_{i=1}^{p}\ \sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\ \partial\bar{x}_{2j}}\operatorname{cov}(\bar{x}_{2,ij}) \\
&+\frac{1}{2}\sum_{k=1}^{p}\ \sum_{l=1}^{p}\ \sum_{i=1}^{p}\ \sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial s_{kl}\ \partial s_{ij}}\operatorname{cov}(s_{kl},s_{ij})
\end{aligned}
$$

So now we want to find $\frac{\partial \Phi}{\partial \overline{x}_{1j}}, \frac{\partial \Phi}{\partial \overline{x}_{2j}}, \frac{\partial^2 \Phi}{\partial \overline{x}_{1j}\,\partial \overline{x}_{1i}}, \frac{\partial^2 \Phi}{\partial \overline{x}_{2j}\,\partial \overline{x}_{2i}}, \frac{\partial^2 \Phi}{\partial s_{kl}\,\partial s_{ij}}$ .

Consider cases 1A and 1B where the conditions are as follow (respectively) :

$$\boldsymbol{\mu}_1 = (m, 0, \dots\dots, 0)^T \qquad\qquad \boldsymbol{\mu}_1 = (m, 0, \dots\dots, 0)^T$$

$$\boldsymbol{\mu}_2 = (0, \dots\dots\dots, 0)^T \qquad \text{and} \qquad \boldsymbol{\mu}_2 = (0, \dots\dots\dots, 0)^T$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho & \rho & \dots & \rho \\ \rho & 1 & \rho & \dots & \rho \\ \vdots & & & & \vdots \\ \rho & \dots & \dots & \rho & 1 \end{bmatrix} \qquad\qquad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^{p-1} \\ \rho & 1 & \rho & \dots & \rho^{p-2} \\ \vdots & & & & \vdots \\ \rho^{p-1} & \dots & \dots & \rho & 1 \end{bmatrix}$$

Need to find $\frac{\partial^2 \Phi}{\partial \overline{x}_{1i}\partial \overline{x}_{1j}}$ :

$$\frac{\partial^2 \Phi}{\partial \overline{x}_{1i}\partial \overline{x}_{1j}} = \frac{\partial}{\partial \overline{x}_{1i}}\left[-\frac{1}{2}\phi(-A)\frac{\partial A}{\partial \overline{x}_{1j}}\right] = -\frac{1}{2}\left[\phi(-A)\frac{\partial^2 A}{\partial \overline{x}_{1i}\partial \overline{x}_{1j}} + \frac{\partial \phi}{\partial \overline{x}_{1i}}\frac{\partial A}{\partial \overline{x}_{1j}}\right]$$

where $$\frac{\partial \phi}{\partial \overline{x}_{1i}} = -\frac{A}{4}\phi(-\frac{A}{2})\frac{\partial A}{\partial \overline{x}_{1i}}$$

$$\therefore \quad \frac{\partial^2 \Phi}{\partial \overline{x}_{1i}\partial \overline{x}_{1j}} = -\frac{1}{2}\left[\phi(-\frac{A}{2})\frac{\partial^2 A}{\partial \overline{x}_{1i}\partial \overline{x}_{1j}} - \frac{A}{4}\phi(-\frac{A}{2})\frac{\partial A}{\partial \overline{x}_{1i}}\frac{\partial A}{\partial \overline{x}_{1j}}\right]$$

$$= -\frac{1}{2}\phi(-\frac{A}{2})\left[\frac{\partial^2 A}{\partial \overline{x}_{1i}\partial \overline{x}_{1j}} - \frac{A}{4}\frac{\partial A}{\partial \overline{x}_{1i}}\frac{\partial A}{\partial \overline{x}_{1j}}\right] \qquad \textit{(A2.1)}$$

Need to find $\frac{\partial A}{\partial \overline{x}_{1i}}, \frac{\partial A}{\partial \overline{x}_{1j}}$ and $\frac{\partial^2 A}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}}$ :

Under cases 1A and 1B ,

let $\quad A = (\overline{\mathbf{x}}_1^T \mathbf{S} \overline{\mathbf{x}}_1)^{-\frac{1}{2}} \overline{\mathbf{x}}_1^T \overline{\mathbf{x}}_1$

$$\frac{\partial A}{\partial \overline{\mathbf{x}}_1} = -\frac{1}{2}(\overline{\mathbf{x}}_1^T \mathbf{S} \overline{\mathbf{x}}_1)^{-\frac{3}{2}} 2\mathbf{S}\overline{\mathbf{x}}_1 \overline{\mathbf{x}}_1^T \overline{\mathbf{x}}_1 + (\overline{\mathbf{x}}_1^T \mathbf{S} \overline{\mathbf{x}}_1)^{-\frac{1}{2}} 2\overline{\mathbf{x}}_1$$

$$= -(\overline{\mathbf{x}}_1^T \mathbf{S} \overline{\mathbf{x}}_1)^{-\frac{3}{2}} \mathbf{S}\overline{\mathbf{x}}_1 \overline{\mathbf{x}}_1^T \overline{\mathbf{x}}_1 + 2(\overline{\mathbf{x}}_1^T \mathbf{S} \overline{\mathbf{x}}_1)^{-\frac{1}{2}} \overline{\mathbf{x}}_1 \qquad (A2.2)$$

where $\quad \overline{\mathbf{x}}_1^T \mathbf{S} \overline{\mathbf{x}}_1 = m^2 s_{11}$

$$\mathbf{S}\overline{\mathbf{x}}_1 \overline{\mathbf{x}}_1^T \overline{\mathbf{x}}_1 = m^3 \begin{bmatrix} s_{11} \\ \vdots \\ s_{p1} \end{bmatrix}$$

$$\overline{\mathbf{x}}_1^T \overline{\mathbf{x}}_1 = m^2$$

$$A = \frac{1}{2} m (s_{11})^{-\frac{1}{2}} = \frac{1}{2} m$$

$$\therefore \quad \frac{\partial A}{\partial \overline{\mathbf{x}}_1} = -\left(m^2 s_{11}\right)^{-\frac{3}{2}} \quad m^3 \begin{bmatrix} s_{11} \\ \vdots \\ s_{p1} \end{bmatrix} \quad + \quad 2\left(m^2 s_{11}\right)^{-\frac{1}{2}} \begin{bmatrix} m \\ 0 \\ \vdots \\ 0 \end{bmatrix} = -\begin{bmatrix} -1 \\ s_{21} \\ \vdots \\ s_{p1} \end{bmatrix}$$

$$\therefore \quad \frac{\partial A}{\partial \overline{x}_{1j}} = \begin{cases} 1 & \text{if} \quad j = 1 \\ -s_{j1} & \text{if} \quad j \neq 1 \end{cases} \qquad \text{and} \qquad \frac{\partial A}{\partial \overline{x}_{1i}} = \begin{cases} 1 & \text{if} \quad i = 1 \\ -s_{i1} & \text{if} \quad i \neq 1 \end{cases}$$

Aside:

$$\overline{\mathbf{x}}_1^T \mathbf{S} \overline{\mathbf{x}}_1 = \sum_{v=1}^{p} [\overline{x}_{1v} (\sum_{u=1}^{p} \overline{x}_{1u} s_{uv})]$$

$$\mathbf{S}\overline{\mathbf{x}}_1 \overline{\mathbf{x}}_1^T \overline{\mathbf{x}}_1 = \begin{bmatrix} \sum_{u=1}^{p} \overline{x}_{1u} s_{1u} \\ \vdots \\ \sum_{u=1}^{p} \overline{x}_{1u} s_{pu} \end{bmatrix} \times \sum_{u=1}^{p} (\overline{x}_{1u})^2$$

Using the above expressions rewrite *(A2.2)* as:

$$\frac{\partial A}{\partial \overline{\mathbf{x}}_1} = -\left\{ \sum_{v=1}^{p} [\overline{x}_{1v} (\sum_{u=1}^{p} \overline{x}_{1u} s_{uv})] \right\}^{-\frac{3}{2}} \begin{bmatrix} \sum_{u=1}^{p} \overline{x}_{1u} s_{1u} \\ \vdots \\ \sum_{u=1}^{p} \overline{x}_{1u} s_{pu} \end{bmatrix} \times \sum_{u=1}^{p} (\overline{x}_{1u})^2$$
$$+2 \left\{ \sum_{v=1}^{p} [\overline{x}_{1v} (\sum_{u=1}^{p} \overline{x}_{1u} s_{uv})] \right\}^{-\frac{1}{2}} \begin{bmatrix} \overline{x}_{11} \\ \vdots \\ \overline{x}_{1p} \end{bmatrix}$$

$$\therefore \quad \frac{\partial A}{\partial \overline{x}_{1j}} = -\left\{ \sum_{v=1}^{p} [\overline{x}_{1v} (\sum_{u=1}^{p} \overline{x}_{1u} s_{uv})] \right\}^{-\frac{3}{2}} \times \sum_{u=1}^{p} \overline{x}_{1u} s_{ju} \times \sum_{u=1}^{p} (\overline{x}_{1u})^2$$
$$+ \left\{ \sum_{v=1}^{p} [\overline{x}_{1v} (\sum_{u=1}^{p} \overline{x}_{1u} s_{uv})] \right\}^{-\frac{1}{2}} \overline{x}_{1j}$$

$$\therefore \quad \frac{\partial^2 A}{\partial \overline{x}_{1i}\partial \overline{x}_{1j}} = -\frac{\partial}{\partial \overline{x}_{1i}}\left[\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{3}{2}}(\sum_{u=1}^{p}\overline{x}_{1u}s_{ju})\sum_{u=1}^{p}(\overline{x}_{1u})^2\right] + 2\frac{\partial}{\partial \overline{x}_{1i}}\left[\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{1}{2}}\quad \overline{x}_{1j}\right]$$

where $\frac{\partial}{\partial \overline{x}_{1i}}\left[\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{3}{2}}(\sum_{u=1}^{p}\overline{x}_{1u}s_{ju})\sum_{u=1}^{p}(\overline{x}_{1u})^2\right]$

$$= \left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{3}{2}} \times \frac{\partial}{\partial \overline{x}_{1i}}\left[(\sum_{u=1}^{p}\overline{x}_{1u}s_{ju})\sum_{u=1}^{p}(\overline{x}_{1u})^2\right] + \frac{\partial}{\partial \overline{x}_{1i}}\left[\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{3}{2}}\right]\times(\sum_{u=1}^{p}\overline{x}_{1u}s_{ju})\sum_{u=1}^{p}(\overline{x}_{1u})^2$$

where $\frac{\partial}{\partial \overline{x}_{1i}}\left[(\sum_{u=1}^{p}\overline{x}_{1u}s_{ju})\sum_{u=1}^{p}(\overline{x}_{1u})^2\right]$

$$= (\sum_{u=1}^{p}\overline{x}_{1u}s_{ju})\frac{\partial}{\partial \overline{x}_{1i}}\left[\sum_{u=1}^{p}(\overline{x}_{1u})^2\right] + \frac{\partial}{\partial \overline{x}_{1i}}\left[(\sum_{u=1}^{p}\overline{x}_{1u}s_{ju})\right]\times\sum_{u=1}^{p}(\overline{x}_{1u})^2$$

$$= 2\overline{x}_{1i}(\sum_{u=1}^{p}\overline{x}_{1u}s_{ju}) + s_{ji}\sum_{u=1}^{p}(\overline{x}_{1u})^2$$

and $\frac{\partial}{\partial \overline{x}_{1i}}\left[\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{3}{2}}\right]$

$$= -\frac{3}{2}\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{5}{2}}\quad \frac{\partial}{\partial \overline{x}_{1i}}\left[\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right]$$

$$= -\frac{3}{2}\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{5}{2}} \times 2(\sum_{u=1}^{p}\overline{x}_{1u}s_{ui})$$

$$\therefore \quad \frac{\partial}{\partial \overline{x}_{1i}}\left[\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{3}{2}}(\sum_{u=1}^{p}\overline{x}_{1u}s_{ju})\sum_{u=1}^{p}(\overline{x}_{1u})^2\right]$$

$$=\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{3}{2}}\times\left\{2\overline{x}_{1i}(\sum_{u=1}^{p}\overline{x}_{1u}s_{ju}) \quad + \quad s_{ji}\sum_{u=1}^{p}(\overline{x}_{1u})^2\right\}$$

$$-\frac{3}{2}\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{5}{2}}\times\ 2(\sum_{u=1}^{p}\overline{x}_{1u}s_{ui})(\sum_{u=1}^{p}\overline{x}_{1u}s_{ju})\sum_{u=1}^{p}(\overline{x}_{1u})^2$$

$$=\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{5}{2}}\left[\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\{2\overline{x}_{1i}\sum_{u=1}^{p}\overline{x}_{1u}s_{ju}\right.$$

$$\left.+s_{ji}\sum_{u=1}^{p}(\overline{x}_{1u})^2\}-3(\sum_{u=1}^{p}\overline{x}_{1u}s_{ui})(\sum_{u=1}^{p}\overline{x}_{1u}s_{ju})\sum_{u=1}^{p}(\overline{x}_{1u})^2\right]$$

and $$\frac{\partial}{\partial \overline{x}_{1i}}\left[\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{1}{2}}\quad \overline{x}_{1j}\right]$$

$$=-\frac{1}{2}\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{3}{2}}\times 2\sum_{u=1}^{p}\overline{x}_{1u}s_{ui}\times\overline{x}_{1j}$$

$$+\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{1}{2}}\times\begin{cases}1 & \text{if} \quad i=j\\ 0 & \text{if} \quad i\neq j\end{cases}$$

$$=\begin{cases}-\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{3}{2}}\overline{x}_{1j}\sum_{u=1}^{p}\overline{x}_{1u}s_{ui} & \\ \qquad +\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{1}{2}} & \text{if} \quad i=j\\ -\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{3}{2}}\overline{x}_{1j}\sum_{u=1}^{p}\overline{x}_{1u}s_{ui} & \text{if} \quad i\neq j\end{cases}$$

$$=\begin{cases}\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{3}{2}}\left[\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right. & \\ \qquad \left.-\overline{x}_{1j}\sum_{u=1}^{p}\overline{x}_{1u}s_{ui}\right] & \text{if} \quad i=j\\ -\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{3}{2}}\overline{x}_{1j}\sum_{u=1}^{p}\overline{x}_{1u}s_{ui} & \text{if} \quad i\neq j\end{cases}$$

Under cases 1A and 1B :

$$\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]=m^2s_{11}$$

$$2\overline{x}_{1i}(\sum_{u=1}^{p}\overline{x}_{1u}s_{ju}) \quad + \quad s_{ji}\sum_{u=1}^{p}(\overline{x}_{1u})^2=\begin{cases}3m^2s_{j1} & \text{if} \quad i=1\\ m^2s_{ji} & \text{if} \quad i\neq 1\end{cases}$$

$$3(\sum_{u=1}^{p}\overline{x}_{1u}s_{ui})(\sum_{u=1}^{p}\overline{x}_{1u}s_{ju})\sum_{u=1}^{p}(\overline{x}_{1u})^2=3m^4s_{1i}s_{j1}$$

$$\overline{x}_{1j}\sum_{u=1}^{p}\overline{x}_{1u}s_{ui}=\begin{cases}m^2s_{1i} & \text{if} \quad j=1\\ 0 & \text{if} \quad j\neq 1\end{cases}$$

$$\therefore \quad \frac{\partial^2 A}{\partial\overline{x}_{1i}\partial\overline{x}_{1j}}$$

$$\begin{aligned}=&-\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{5}{2}}\left[\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\{2\overline{x}_{1i}\sum_{u=1}^{p}\overline{x}_{1u}s_{ju})\right.\\ &\left.+s_{ji}\sum_{u=1}^{p}(\overline{x}_{1u})^2\}-3(\sum_{u=1}^{p}\overline{x}_{1u}s_{ui})(\sum_{u=1}^{p}\overline{x}_{1u}s_{ju})\sum_{u=1}^{p}(\overline{x}_{1u})^2\right]\\ &+2\begin{cases}\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{3}{2}}\left[\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]-\overline{x}_{1j}\sum_{u=1}^{p}\overline{x}_{1u}s_{ui}\right] & \text{if} \quad i=j\\ -\left\{\sum_{v=1}^{p}[\overline{x}_{1v}(\sum_{u=1}^{p}\overline{x}_{1u}s_{uv})]\right\}^{-\frac{3}{2}}\overline{x}_{1j}\sum_{u=1}^{p}\overline{x}_{1u}s_{ui} & \text{if} \quad i\neq j\end{cases}\end{aligned}$$

$$\begin{aligned}=&-\{m^2s_{11}\}^{-\frac{5}{2}}\left[m^2s_{11}\begin{cases}3m^2s_{j1} & \text{if} \quad i=1\\ m^2s_{ji} & \text{if} \quad i\neq 1\end{cases}-3m^4s_{1i}s_{j1}\right]\\ &-2\{m^2s_{11}\}^{-\frac{3}{2}}\begin{cases}\begin{cases}m^2s_{1i} & \text{if} \quad j=1\\ 0 & \text{if} \quad j\neq 1\end{cases}-m^2s_{11} & \text{if} \quad i=j\\ \begin{cases}m^2s_{1i} & \text{if} \quad j=1\\ 0 & \text{if} \quad j\neq 1\end{cases} & \text{if} \quad i\neq j\end{cases}\end{aligned}$$

$$= -\frac{1}{m}\left[\begin{cases} 3s_{j1} - 3s_{1i}s_{j1} & \text{if} \quad i = 1 \\ s_{ji} - 3s_{1i}s_{j1} & \text{if} \quad i \neq 1 \end{cases}\right] + \frac{2}{m}\begin{cases} \begin{cases} 1 - s_{1i} & \text{if} \quad j = 1, i = j \\ 1 & \text{if} \quad j \neq 1, i = j \end{cases} \\ \begin{cases} -s_{1i} & \text{if} \quad j = 1, i \neq j \\ 0 & \text{if} \quad j \neq 1, i \neq j \end{cases} \end{cases}$$

$$= -\frac{1}{m}\left[\begin{cases} 3s_{j1} - 3s_{1i}s_{j1} & \text{if} \quad i = 1 \\ s_{ji} - 3s_{1i}s_{j1} & \text{if} \quad i \neq 1 \end{cases}\right] + \frac{2}{m}\begin{cases} 1 - s_{1i} & \text{if} \quad j = 1, i = j \\ 1 & \text{if} \quad j \neq 1, i = j \\ -s_{1i} & \text{if} \quad j = 1, i \neq j \\ 0 & \text{if} \quad j \neq 1, i \neq j \end{cases}$$

$$\therefore \quad \frac{\partial^2 A}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}} = \frac{1}{m}\begin{cases} -3(s_{j1} - s_{1i}s_{j1}) + 2(1 - s_{1i}) & \text{if} \quad i = 1, j = 1, i = j \\ -3(s_{j1} - s_{1i}s_{j1}) & \text{if} \quad i = 1, j \neq 1, i \neq j \\ -(s_{ji} - 3s_{1i}s_{j1}) - 2s_{1i} & \text{if} \quad i \neq 1, j = 1, i \neq j \\ -(s_{ji} - 3s_{1i}s_{j1}) + 2 & \text{if} \quad i \neq 1, j \neq 1, i = j \\ -(s_{ji} - 3s_{1i}s_{j1}) & \text{if} \quad i \neq 1, j \neq 1, i \neq j \end{cases}$$

*expression (A2.3)*

$$\frac{A}{4}\frac{\partial A}{\partial \bar{x}_{1i}}\frac{\partial A}{\partial \bar{x}_{1j}} = \frac{1}{4} m \{s_{11}\}^{-\frac{1}{2}} \begin{cases} 1 & \text{if} \quad j = 1 \\ -s_{j1} & \text{if} \quad j \neq 1 \end{cases} \times \begin{cases} 1 & \text{if} \quad i = 1 \\ -s_{i1} & \text{if} \quad i \neq 1 \end{cases}$$

$$= \frac{m}{4}\begin{cases} 1 & \text{if} \quad i = j = 1 \\ -s_{j1} & \text{if} \quad i = 1, j \neq 1 \\ -s_{i1} & \text{if} \quad i \neq 1, j = 1 \\ s_{j1}s_{i1} & \text{if} \quad i \neq 1, j \neq 1 \end{cases} \qquad (A2.4)$$

Substitute *(A2.3)* and *(A2.4)* into *(A2.1)* gives $\dfrac{\partial^2 \Phi}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}}$

$$= -\frac{1}{2}\phi\left(-\frac{m}{2}\right)\begin{cases} -\dfrac{1}{m}\left[3(s_{j1} - s_{1i}s_{j1}) - 2(1 - s_{1i})\right] - \dfrac{m}{4} & \text{if} \quad i = 1, j = 1, i = j \\ -\dfrac{3}{m}(s_{j1} - s_{1i}s_{j1}) + \dfrac{m}{4}s_{j1} & \text{if} \quad i = 1, j \neq 1, i \neq j \\ -\dfrac{1}{m}\left[s_{ji} - 3s_{1i}s_{j1} + 2s_{1i}\right] + \dfrac{m}{4}s_{i1} & \text{if} \quad i \neq 1, j = 1, i \neq j \\ -\dfrac{1}{m}\left[s_{ji} - 3s_{1i}s_{j1} - 2\right] - \dfrac{m}{4}s_{j1}s_{i1} & \text{if} \quad i \neq 1, j \neq 1, i = j \\ -\dfrac{1}{m}(s_{ji} - 3s_{1i}s_{j1}) - \dfrac{m}{4}s_{j1}s_{i1} & \text{if} \quad i \neq 1, j \neq 1, i \neq j \end{cases}$$

*expression (A2.5)*

Now need to find $\dfrac{\partial^2 \Phi}{\partial \bar{x}_{2i} \partial \bar{x}_{2j}}$ :

$$\frac{\partial^2 \Phi}{\partial \bar{x}_{2i} \partial \bar{x}_{2j}} = -\frac{1}{2}\phi(-\frac{A}{2})\left[\frac{\partial^2 A}{\partial \bar{x}_{2i} \partial \bar{x}_{2j}} - \frac{A}{4}\frac{\partial A}{\partial \bar{x}_{2i}}\frac{\partial A}{\partial \bar{x}_{2j}}\right] \qquad (A2.6)$$

Need to find $\dfrac{\partial A}{\partial \bar{x}_{2j}}$ , $\dfrac{\partial A}{\partial \bar{x}_{2i}}$ , $\dfrac{\partial^2 A}{\partial \bar{x}_{2i} \partial \bar{x}_{2j}}$ :

$$\frac{\partial A}{\partial \bar{\mathbf{x}}_2} = \left[(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T \mathbf{S}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)\right]^{-\frac{3}{2}} \mathbf{S}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)$$
$$-2\left[(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T \mathbf{S}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)\right]^{-\frac{1}{2}} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) \qquad (A2.7)$$

where $$(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T \mathbf{S}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = m^2 s_{11} = m^2$$

$$\mathbf{S}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = m^3 \begin{bmatrix} s_{11} \\ \vdots \\ s_{p1} \end{bmatrix}$$

$$\therefore \quad \frac{\partial A}{\partial \overline{\mathbf{x}}_2} = \left\{m^2 s_{11}\right\}^{-\frac{3}{2}} m^3 \begin{bmatrix} s_{11} \\ \vdots \\ s_{p1} \end{bmatrix} - 2\left\{m^2 s_{11}\right\}^{-\frac{1}{2}} \begin{bmatrix} m \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ s_{21} \\ \vdots \\ s_{p1} \end{bmatrix}$$

$$\therefore \frac{\partial A}{\partial \overline{x}_{2j}} = \begin{cases} -1 & \text{if } j = 1 \\ s_{j1} & \text{if } j \neq 1 \end{cases} \quad \text{and} \quad \frac{\partial A}{\partial \overline{x}_{2i}} = \begin{cases} -1 & \text{if } i = 1 \\ s_{i1} & \text{if } i \neq 1 \end{cases}$$

Aside :

$$(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = (m - \overline{x}_{21})^2 + \sum_{u=2}^{p}(\overline{x}_{2u})^2$$

$$\begin{aligned}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T \mathbf{S}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) &= (m - \overline{x}_{21})\left[(m - \overline{x}_{21})s_{11} - \sum_{u=2}^{p}\overline{x}_{2u} s_{u1}\right] \\ &\quad - \sum_{v=2}^{p}\overline{x}_{2v}\left[(m - \overline{x}_{21})s_{1v} - \sum_{u=2}^{p}\overline{x}_{2u} s_{uv}\right] \\ &= C1\end{aligned}$$

$$\mathbf{S}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)$$

$$= \begin{bmatrix} (m - \overline{x}_{21})s_{11} - \sum_{u=2}^{p}\overline{x}_{2u}s_{1u} \\ \vdots \\ (m - \overline{x}_{21})s_{p1} - \sum_{u=2}^{p}\overline{x}_{2u}s_{pu} \end{bmatrix} \times \left\{(m - \overline{x}_{21})^2 + \sum_{u=2}^{p}(\overline{x}_{2u})^2\right\}$$

$(= \mathbf{S}\mathbf{y}\mathbf{y}^T\mathbf{y})$

Using the above expressions rewrite *(A2.7)* as :

$$\frac{\partial A}{\partial \overline{\mathbf{x}}_2} = \{C1\}^{-\frac{3}{2}} \begin{bmatrix} (m - \overline{x}_{21})s_{11} - \sum_{u=2}^{p} \overline{x}_{2u} s_{1u} \\ \vdots \\ (m - \overline{x}_{21})s_{p1} - \sum_{u=2}^{p} \overline{x}_{2u} s_{pu} \end{bmatrix} \times \left\{ (m - \overline{x}_{21})^2 + \sum_{u=2}^{p} (\overline{x}_{2u})^2 \right\}$$

$$- 2 \; \{C1\}^{-\frac{1}{2}} \begin{bmatrix} m + \overline{x}_{21} \\ - \overline{x}_{22} \\ \vdots \\ - \overline{x}_{2p} \end{bmatrix}$$

$$\therefore \frac{\partial A}{\partial \overline{x}_{2j}} = \{C1\}^{-\frac{3}{2}} \left\{ (m - \overline{x}_{21})s_{j1} - \sum_{u=2}^{p} \overline{x}_{2u} s_{ju} \right\} \left\{ (m - \overline{x}_{21})^2 + \sum_{u=2}^{p} (\overline{x}_{2u})^2 \right\}$$

$$- 2 \{C1\}^{-\frac{1}{2}} \begin{cases} m - \overline{x}_{21} & \text{if} \quad j = 1 \\ -\overline{x}_{2j} & \text{if} \quad j \neq 1 \end{cases}$$

$$= a + b$$

$$\therefore \quad \frac{\partial^2 A}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}} = \frac{\partial}{\partial \overline{x}_{2i}} \left[ \frac{\partial A}{\partial \overline{x}_{2j}} \right] = \frac{\partial a}{\partial \overline{x}_{2i}} + \frac{\partial b}{\partial \overline{x}_{2i}} \qquad (A2.8)$$

(1) Find $\frac{\partial a}{\partial \overline{x}_{2i}}$ :

$$a = \{C1\}^{-\frac{3}{2}} \times \left( \mathbf{S y y}^{\mathrm{T}} \mathbf{y} \right)_j$$

$$\therefore \quad \frac{\partial a}{\partial \bar{x}_{2i}} = \{C1\}^{-\frac{3}{2}} \frac{\partial}{\partial \bar{x}_{2i}}\left[\left(\mathbf{Syy}^T\mathbf{y}\right)_j\right] + \frac{\partial}{\partial \bar{x}_{2i}}\left[\{C1\}^{-\frac{3}{2}}\right] \times \left(\mathbf{Syy}^T\mathbf{y}\right)_j$$

where
$$\frac{\partial}{\partial \bar{x}_{2i}}\left[\left(\mathbf{Syy}^T\mathbf{y}\right)_j\right] = \begin{cases} -3s_{j1}m^2 & \text{if} \quad i=1 \\ -s_{ji}m^2 & \text{if} \quad i\neq 1 \end{cases}$$

and
$$\frac{\partial}{\partial \bar{x}_{2i}}\left[\{C1\}^{-\frac{3}{2}}\right] = -\frac{3}{2}\{C1\}^{-\frac{5}{2}} \times -2ms_{1i} = 3ms_{1i}\{C1\}^{-\frac{5}{2}}$$

$$\therefore \frac{\partial a}{\partial \bar{x}_{2i}} = \{C1\}^{-\frac{3}{2}} \begin{cases} -3s_{j1}m^2 & \text{if} \quad i=1 \\ -s_{ji}m^2 & \text{if} \quad i\neq 1 \end{cases} + \quad 3ms_{1i}\{C1\}^{-\frac{5}{2}} \times (m^3 s_{j1})$$

$$= \frac{1}{m}\begin{cases} -3s_{j1} + 3s_{1i}s_{j1} & \text{if} \quad i=1 \\ -s_{ji} + 3s_{1i}s_{j1} & \text{if} \quad i\neq 1 \end{cases} \qquad \textit{(A2.8a)}$$

(2) Find $\frac{\partial b}{\partial \bar{x}_{2i}}$ :

$$b = -2\{C1\}^{-\frac{1}{2}} \begin{cases} m - \bar{x}_{21} & \text{if} \quad j=1 \\ -\bar{x}_{2j} & \text{if} \quad j\neq 1 \end{cases}$$

$$\frac{\partial b}{\partial \bar{x}_{2i}} = -2\{C1\}^{-\frac{1}{2}} \quad \times \quad \begin{cases} -1 & \text{if} \quad i=j \\ 0 & \text{if} \quad i\neq j \end{cases}$$
$$+ \quad \{C1\}^{-\frac{3}{2}} \frac{\partial (C1)}{\partial \bar{x}_{2i}} \quad \times \quad \begin{cases} m - \bar{x}_{21} & \text{if} \quad j=1 \\ -\bar{x}_{2j} & \text{if} \quad j\neq 1 \end{cases}$$

$$= \begin{cases} 2\{C1\}^{-\frac{1}{2}} & \text{if} \quad i=j \\ 0 & \text{if} \quad i\neq j \end{cases} - 2\{C1\}^{-\frac{3}{2}} ms_{1i} \begin{cases} m - \bar{x}_{21} & \text{if} \quad j=1 \\ -\bar{x}_{2j} & \text{if} \quad j\neq 1 \end{cases}$$

$$= \begin{cases} 2\left\{m^2 s_{11}\right\}^{-\frac{1}{2}} & \text{if} \quad i=j \\ 0 & \text{if} \quad i \neq j \end{cases} - 2\left\{m^2 s_{11}\right\}^{-\frac{3}{2}} m s_{1i} \begin{cases} m & \text{if} \quad j=1 \\ 0 & \text{if} \quad j \neq 1 \end{cases}$$

$$= \frac{1}{m}\begin{cases} 2 & \text{if} \quad i=j \\ 0 & \text{if} \quad i \neq j \end{cases} - \frac{2}{m} s_{1i} \begin{cases} 1 & \text{if} \quad j=1 \\ 0 & \text{if} \quad j \neq 1 \end{cases}$$

$$= \frac{2}{m}\begin{cases} 1 - s_{1i} & \text{if} \quad i=j=1 \\ 1 & \text{if} \quad i=j,\, j\neq 1 \\ -s_{1i} & \text{if} \quad i \neq j,\, j=1 \\ 0 & \text{if} \quad i\neq j,\, j \neq 1 \end{cases} \qquad (A2.8b)$$

Therefore substitute *(A2.8a)* and *(A2.8b)* into *(A2.8)* gives

$$\frac{\partial^2 A}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}} = \frac{1}{m}\begin{cases} -3s_{j1} + 3s_{1i}s_{j1} & \text{if} \quad i=1 \\ -s_{ji} + 3s_{1i}s_{j1} & \text{if} \quad i \neq 1 \end{cases} + \frac{2}{m}\begin{cases} 1 - s_{1i} & \text{if} \quad i=j=1 \\ 1 & \text{if} \quad i=j,\, j\neq 1 \\ -s_{1i} & \text{if} \quad i \neq j,\, j=1 \\ 0 & \text{if} \quad i\neq j,\, j \neq 1 \end{cases}$$

$$= \frac{1}{m}\begin{cases} -3s_{j1} + 3s_{1i}s_{j1} + 2(1 - s_{1i}) & \text{if} \quad i=j=1 \\ -3s_{j1} + 3s_{1i}s_{j1} & \text{if} \quad i=1,\, j\neq 1,\, i\neq j \\ -s_{ji} + 3s_{1i}s_{j1} + 2 & \text{if} \quad i\neq 1,\, i=j,\, j\neq 1 \\ -s_{ji} + 3s_{1i}s_{j1} - 2s_{1i} & \text{if} \quad i\neq 1,\, i\neq j,\, j=1 \\ -s_{ji} + 3s_{1i}s_{j1} & \text{if} \quad i\neq 1,\, j\neq 1,\, i\neq j \end{cases}$$

*expression (A2.9)*

$$-\frac{A}{4}\frac{\partial A}{\partial \overline{x}_{2i}}\frac{\partial A}{\partial \overline{x}_{2j}} = -\frac{m}{4}\begin{cases} 1 & \text{if} \quad i=j=1 \\ -s_{j1} & \text{if} \quad i=j,\, j\neq 1 \\ -s_{i1} & \text{if} \quad i\neq j,\, j=1 \\ s_{i1}s_{j1} & \text{if} \quad i\neq j,\, j\neq 1 \end{cases} \qquad (A2.10)$$

Therefore subsitute *(A2.9)* and *(A2.10)* into *(A2.6)* gives $\dfrac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}$

$$
= -\frac{1}{2}\phi\left(-\frac{m}{2}\right)\begin{cases}
\frac{1}{m}\left[3s_{1i}s_{j1} - 3s_{j1} + 2(1-s_{1i})\right] - \frac{m}{4} & \text{if} \quad i=j=1 \\
\frac{3}{m}\left[s_{1i}s_{j1} - s_{j1}\right] + \frac{m}{4}s_{j1} & \text{if} \quad i=1,\, j\neq 1,\, i\neq j \\
\frac{1}{m}\left[3s_{1i}s_{j1} - s_{ji} + 2\right] - \frac{m}{4}s_{i1}s_{j1} & \text{if} \quad i\neq 1,\, i=j,\, j\neq 1 \\
\frac{1}{m}\left[3s_{1i}s_{j1} - s_{ji} - 2s_{1i}\right] + \frac{m}{4}s_{i1} & \text{if} \quad i\neq 1,\, i\neq j,\, j=1 \\
\frac{1}{m}\left[3s_{1i}s_{j1} - s_{ji}\right] - \frac{m}{4}s_{i1}s_{j1} & \text{if} \quad i\neq 1,\, j\neq 1,\, i\neq j
\end{cases}
$$

*expression (A2.11)*

$$
\therefore E[\Phi(\bar{x}_1,\bar{x}_2)] = \Phi(\mu_1,\mu_2) + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\,\partial\bar{x}_{1j}}\operatorname{cov}(\bar{x}_{1,ij}) + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\,\partial\bar{x}_{2j}}\operatorname{cov}(\bar{x}_{2,ij})
$$

$$
= \Phi(\mu_1,\mu_2) + \frac{1}{2n_1}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\,\partial\bar{x}_{1j}}\times\sigma_{ij} + \frac{1}{2n_2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\,\partial\bar{x}_{2j}}\times\sigma_{ij}
$$

where $\dfrac{\partial^2\Phi}{\partial\bar{x}_{1i}\,\partial\bar{x}_{1j}}$ and $\dfrac{\partial^2\Phi}{\partial\bar{x}_{2i}\,\partial\bar{x}_{2j}}$ are as in *(A2.5)* and *(A2.11)* respectively.

Now need to find $\dfrac{\partial^2 \Phi}{\partial s_{kl}\, \partial s_{ij}}$:

$$A = \frac{m}{\sqrt{s_{11}}} = m(s_{11})^{-1/2}$$

note that $\dfrac{\partial^2 s_{11}}{\partial s_{kl} \partial s_{ij}} = 0 \qquad \forall\ k, l, i, j = 1, \ldots, p$

$$\frac{\partial^2 \Phi}{\partial s_{kl}\, \partial s_{ij}} = \frac{\partial}{\partial s_{kl}}\left[\frac{\partial \Phi}{\partial s_{ij}}\right]$$

$$= \frac{\partial}{\partial s_{kl}}\left[-\frac{1}{2} m \frac{\partial}{\partial s_{ij}}\left[\{s_{11}\}^{-\frac{1}{2}}\right]\phi\left(-\frac{m}{2}\{s_{11}\}^{-\frac{1}{2}}\right)\right]$$

$$= -\frac{m}{2}\left[\phi\left(-\frac{m}{2}\{s_{11}\}^{-\frac{1}{2}}\right)\frac{\partial^2}{\partial s_{kl} \partial s_{ij}}\left[\{s_{11}\}^{-\frac{1}{2}}\right] + \frac{\partial}{\partial s_{ij}}\left[\{s_{11}\}^{-\frac{1}{2}}\right]\frac{\partial}{\partial s_{kl}}\left[\phi\left(-\frac{m}{2}\{s_{11}\}^{-\frac{1}{2}}\right)\right]\right] \qquad (A212)$$

where $$\frac{\partial}{\partial s_{ij}}\left[\{s_{11}\}^{-\frac{1}{2}}\right] = -\frac{1}{2}\{s_{11}\}^{-\frac{3}{2}} \frac{\partial (s_{11})}{\partial s_{ij}} = \begin{cases} -\frac{1}{2} & \text{if} \quad i = j = 1 \\ 0 & \text{otherwise} \end{cases}$$

*expression (A2.12a)*

For both cases of $\mathbf{\Sigma}$, $s_{11} = 1$

$$\frac{\partial s_{11}}{\partial s_{ij}} = \begin{cases} 1 & \text{if} \quad i = j = 1 \\ 0 & \text{otherwise} \end{cases}$$

and $$\frac{\partial s_{11}}{\partial s_{kl}} = \begin{cases} 1 & \text{if} \quad k = l = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial}{\partial s_{kl}}\left[\phi\left(-\frac{m}{2}\{s_{11}\}^{-\frac{1}{2}}\right)\right] = -\frac{1}{2}\frac{m^2}{4}\frac{\partial}{\partial s_{kl}}\left[\{s_{11}\}^{-1}\right]\phi\left(-\frac{m}{2}\{s_{11}\}^{-\frac{1}{2}}\right)$$

$$= \frac{m^2}{8}\{s_{11}\}^{-2}\frac{\partial(s_{11})}{\partial s_{kl}}\phi\left(-\frac{m}{2}\{s_{11}\}^{-\frac{1}{2}}\right)$$

$$= \begin{cases} \frac{m^2}{8}\phi\left(-\frac{m}{2}\{s_{11}\}^{-\frac{1}{2}}\right) & \text{if} \quad k = l = 1 \\ 0 & \text{otherwise} \end{cases}$$

*expression (A2.12b)*

$$\frac{\partial^2}{\partial s_{kl}\partial s_{ij}}\left[\{s_{11}\}^{-\frac{1}{2}}\right] = \frac{\partial}{\partial s_{kl}}\left[\frac{\partial}{\partial s_{ij}}\{s_{11}\}^{-\frac{1}{2}}\right] = \frac{\partial}{\partial s_{kl}}\left[-\frac{1}{2}\{s_{11}\}^{-\frac{3}{2}}\frac{\partial(s_{11})}{\partial s_{ij}}\right]$$

$$= -\frac{1}{2}\left[\{s_{11}\}^{-\frac{3}{2}}\frac{\partial^2(s_{11})}{\partial s_{kl}\partial s_{ij}} + \frac{\partial}{\partial s_{kl}}\{s_{11}\}^{-\frac{3}{2}}\frac{\partial(s_{11})}{\partial s_{ij}}\right]$$

$$= \frac{3}{4}\{s_{11}\}^{-\frac{5}{2}}\frac{\partial(s_{11})}{\partial s_{kl}}\frac{\partial(s_{11})}{\partial s_{ij}} \qquad (A2.12c)$$

Now substitute *(A2.12a)* , *(A2.12b)* and *(A2.12c)* into *(A2.12)* ,

$$\therefore \frac{\partial^2\Phi}{\partial s_{kl}\partial s_{ij}} = -\frac{m}{2}\left[\phi\left(-\frac{m}{2}\{s_{11}\}^{-\frac{1}{2}}\right)\frac{3}{4}\{s_{11}\}^{-\frac{5}{2}}\frac{\partial(s_{11})}{\partial s_{kl}}\frac{\partial(s_{11})}{\partial s_{ij}} \right.$$
$$\left. -\frac{1}{2}\{s_{11}\}^{-\frac{7}{2}}\frac{\partial(s_{11})}{\partial s_{ij}}\frac{m^2}{8}\frac{\partial(s_{11})}{\partial s_{kl}}\phi\left(-\frac{m}{2}\{s_{11}\}^{-\frac{1}{2}}\right)\right]$$

$$= -\frac{m}{8}\phi\left(-\frac{m}{2}\{s_{11}\}^{-\frac{1}{2}}\right)\frac{\partial(s_{11})}{\partial s_{ij}}\frac{\partial(s_{11})}{\partial s_{kl}}\left[3-\frac{m^2}{4}\right]$$

$$= -\frac{m}{8}\phi\left(-\frac{m}{2}\{s_{11}\}^{-\frac{1}{2}}\right)\left[3-\frac{m^2}{4}\right] \qquad (A2.13)$$

Consider cases 2A and 2B where the conditions are as follow (respectively) :

$\boldsymbol{\mu}_1 = (m,\ldots\ldots\ldots,m)^T$ $\qquad\qquad$ $\boldsymbol{\mu}_1 = (m,\ldots\ldots\ldots,m)^T$

$\boldsymbol{\mu}_2 = (0,\ldots\ldots\ldots,0)^T$ $\qquad$ and $\qquad$ $\boldsymbol{\mu}_2 = (0,\ldots\ldots\ldots,0)^T$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \vdots & & & & \vdots \\ \rho & \cdots\cdots\cdots & & \rho & 1 \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{p-1} \\ \rho & 1 & \rho & \cdots & \rho^{p-2} \\ \vdots & & & & \vdots \\ \rho^{p-1} & \cdots\cdots & & \rho & 1 \end{bmatrix}$$

Need to find $\dfrac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}$ :

$$\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}} = \ = -\frac{1}{2}\phi(-\frac{A}{2})\left[\frac{\partial^2 A}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}} - \frac{A}{4}\frac{\partial A}{\partial\bar{x}_{1i}}\frac{\partial A}{\partial\bar{x}_{1j}}\right] \qquad (A1.14)$$

As in cases 1A and 1B :

$$\frac{\partial A}{\partial\bar{\mathbf{x}}_1} = -(\bar{\mathbf{x}}_1^T\mathbf{S}\bar{\mathbf{x}}_1)^{-\frac{3}{2}}\mathbf{S}\bar{\mathbf{x}}_1\bar{\mathbf{x}}_1^T\bar{\mathbf{x}}_1 + 2(\bar{\mathbf{x}}_1^T\mathbf{S}\bar{\mathbf{x}}_1)^{-\frac{1}{2}}\bar{\mathbf{x}}_1$$

where $\overline{\mathbf{x}}_1^T \mathbf{S} \overline{\mathbf{x}}_1 = m^2 \sum_{v=1}^{p} \sum_{u=1}^{p} s_{uv}$

$$\mathbf{S}\overline{\mathbf{x}}_1 \overline{\mathbf{x}}_1^T \overline{\mathbf{x}}_1 = m^3 \begin{bmatrix} \sum_{u=1}^{p} s_{1u} \\ \vdots \\ \sum_{u=1}^{p} s_{pu} \end{bmatrix}$$

$$\therefore \quad \frac{\partial A}{\partial \overline{\mathbf{x}}_1} = \left(\sum_{v=1}^{p} \sum_{u=1}^{p} \sigma_{uv}\right)^{-\frac{3}{2}} \left\{ - \begin{bmatrix} \sum_{u=1}^{p} s_{1u} \\ \vdots \\ \sum_{u=1}^{p} s_{pu} \end{bmatrix} + 2 \sum_{v=1}^{p} \sum_{u=1}^{p} s_{uv} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

$$\therefore \quad \frac{\partial A}{\partial \overline{x}_{1j}} = \left(\sum_{v=1}^{p} \sum_{u=1}^{p} s_{uv}\right)^{-\frac{3}{2}} \left\{ - \sum_{u=1}^{p} s_{ju} + 2 \sum_{v=1}^{p} \sum_{u=1}^{p} s_{uv} \right\}$$

$$\text{and} \quad \frac{\partial A}{\partial \overline{x}_{1i}} = \left(\sum_{v=1}^{p} \sum_{u=1}^{p} s_{uv}\right)^{-\frac{3}{2}} \left\{ - \sum_{u=1}^{p} s_{iu} + 2 \sum_{v=1}^{p} \sum_{u=1}^{p} s_{uv} \right\}$$

As in cases 1A and 1B :

$$\frac{\partial^2 A}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}} = -\left\{ \sum_{v=1}^{p} [\overline{x}_{1v} (\sum_{u=1}^{p} \overline{x}_{1u} s_{uv})] \right\}^{-\frac{5}{2}} \left[ \sum_{v=1}^{p} [\overline{x}_{1v} (\sum_{u=1}^{p} \overline{x}_{1u} s_{uv})] \{ 2\overline{x}_{1i} \sum_{u=1}^{p} \overline{x}_{1u} s_{ju} ) \right.$$

$$\left. + s_{ji} \sum_{u=1}^{p} (\overline{x}_{1u})^2 \} - 3 (\sum_{u=1}^{p} \overline{x}_{1u} s_{ui}) (\sum_{u=1}^{p} \overline{x}_{1u} s_{ju}) \sum_{u=1}^{p} (\overline{x}_{1u})^2 \right]$$

$$+ 2 \begin{cases} \left\{ \sum_{v=1}^{p} [\overline{x}_{1v} (\sum_{u=1}^{p} \overline{x}_{1u} s_{uv})] \right\}^{-\frac{3}{2}} \left[ \sum_{v=1}^{p} [\overline{x}_{1v} (\sum_{u=1}^{p} \overline{x}_{1u} s_{uv})] - \overline{x}_{1j} \sum_{u=1}^{p} \overline{x}_{1u} s_{ui} \right] & \text{if } i = j \\ -\left\{ \sum_{v=1}^{p} [\overline{x}_{1v} (\sum_{u=1}^{p} \overline{x}_{1u} s_{uv})] \right\}^{-\frac{3}{2}} \overline{x}_{1j} \sum_{u=1}^{p} \overline{x}_{1u} s_{ui} & \text{if } i \neq j \end{cases}$$

Under cases 2A and 2B :

$$\sum_{v=1}^{p}[\bar{x}_{1v}(\sum_{u=1}^{p}\bar{x}_{1u}\sigma_{uv})] = m^2 \sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}$$

$$2\bar{x}_{1i}\sum_{u=1}^{p}\bar{x}_{1u}s_{ju} + s_{ji}\sum_{u=1}^{p}(\bar{x}_{1u})^2 = m^2\left[2\sum_{u=1}^{p}s_{ju} + ps_{ji}\right]$$

$$3(\sum_{u=1}^{p}\bar{x}_{1u}s_{ui})(\sum_{u=1}^{p}\bar{x}_{1u}s_{ju})\sum_{u=1}^{p}(\bar{x}_{1u})^2 = 3m^4 p\sum_{u=1}^{p}s_{ui}\sum_{u=1}^{p}s_{ju}$$

$$\bar{x}_{1j}\sum_{u=1}^{p}\bar{x}_{1u}s_{ui} = m^2\sum_{u=1}^{p}s_{ui}$$

$$\therefore \frac{\partial^2 A}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}} = -\frac{1}{m}\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\}^{-\frac{5}{2}}\left[\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\left(2\sum_{u=1}^{p}s_{ju} + ps_{ij}\right) - 3p\sum_{u=1}^{p}s_{ui}\sum_{u=1}^{p}s_{ju}\right]$$
$$+\frac{2}{m}\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\}^{-\frac{3}{2}}\begin{cases}-\sum_{u=1}^{p}s_{ui} + \sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv} & \text{if } i = j\\ -\sum_{u=1}^{p}s_{ui} & \text{if } i \neq j\end{cases}$$

$$= \frac{1}{m}\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\}^{-\frac{5}{2}}\begin{cases}-\left[\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\left(2\sum_{u=1}^{p}s_{ju} + ps_{ji}\right) - 3p\sum_{u=1}^{p}s_{ui}\sum_{u=1}^{p}s_{ju}\right] & \\ +2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\left(-\sum_{u=1}^{p}s_{ui} + \sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right) & \text{if } i = j\\ -\left[\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\left(2\sum_{u=1}^{p}s_{ju} + ps_{ji}\right) - 3p\sum_{u=1}^{p}s_{ui}\sum_{u=1}^{p}s_{ju}\right] & \\ -2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\sum_{u=1}^{p}s_{ui} & \text{if } i \neq j\end{cases}$$

*expression (A2.15)*

$$\frac{A}{4}\frac{\partial A}{\partial\bar{x}_{1i}}\frac{\partial A}{\partial\bar{x}_{1j}} = \frac{1}{4}mp\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\}^{-\frac{7}{2}} \times\left\{-\sum_{u=1}^{p}s_{ju} + 2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\}$$
$$\times\left\{-\sum_{u=1}^{p}s_{iu} + 2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\} \qquad (A2.16)$$

Substitute (A2.13) and (A2.14) into (A1.12) gives $\frac{\partial^2 \Phi}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}}$

$$= -\frac{1}{2}\phi\left(-\frac{1}{2}mp(\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv})^{-\frac{1}{2}}\right) \times \left\{\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\}^{-\frac{7}{2}}$$

$$\times \begin{cases} \frac{1}{m}\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\left\{\left[-\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\left(2\sum_{u=1}^{p}s_{ju}+ps_{ji}\right)+3p\sum_{u=1}^{p}s_{ui}\sum_{u=1}^{p}s_{ju}\right] \right. \\ \left. +2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\left(-\sum_{u=1}^{p}s_{ui}+\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right)\right\} \\ -\frac{1}{4}mp\left\{-\sum_{u=1}^{p}s_{ju}+2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\}\left\{-\sum_{u=1}^{p}s_{iu}+2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\} & \text{if } i=j \\ \frac{1}{m}\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\left\{\left[-\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\left(2\sum_{u=1}^{p}s_{ju}+ps_{ji}\right)+3p\sum_{u=1}^{p}s_{ui}\sum_{u=1}^{p}s_{ju}\right] \right. \\ \left. -2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\sum_{u=1}^{p}s_{ui} \right. \\ -\frac{1}{4}mp\left\{-\sum_{u=1}^{p}s_{ju}+2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\}\left\{-\sum_{u=1}^{p}s_{iu}+2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\} & \text{if } i\neq j \end{cases}$$

*expression (A2.17)*

Now we need to find $\frac{\partial^2 \Phi}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}}$:

$$\frac{\partial^2 \Phi}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}} = -\frac{1}{2}\phi(-\frac{A}{2})\left[\frac{\partial^2 A}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}} - \frac{A}{4}\frac{\partial A}{\partial \overline{x}_{2i}}\frac{\partial A}{\partial \overline{x}_{2j}}\right] \quad (A2.18)$$

As in cases 1A and 1B :

$$\frac{\partial A}{\partial \overline{\mathbf{x}}_2} = \left[(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T \mathbf{S}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)\right]^{-\frac{3}{2}} \mathbf{S}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)$$
$$-2\left[(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T \mathbf{S}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)\right]^{-\frac{1}{2}}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) \quad (A2.19)$$

where $$(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T \mathbf{S}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = m^2 \sum_{v=1}^{p} \sum_{u=1}^{p} s_{uv}$$

$$\mathbf{S}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T (\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = m^3 \begin{bmatrix} \sum_{u=1}^{p} s_{1u} \\ \vdots \\ \sum_{u=1}^{p} s_{pu} \end{bmatrix}$$

$$\therefore \quad \frac{\partial A}{\partial \overline{\mathbf{x}}_2} = \left\{ \sum_{v=1}^{p} \sum_{u=1}^{p} s_{uv} \right\}^{-\frac{3}{2}} \left\{ \begin{bmatrix} \sum_{u=1}^{p} s_{1u} \\ \vdots \\ \sum_{u=1}^{p} s_{pu} \end{bmatrix} - 2 \left\{ \sum_{v=1}^{p} \sum_{u=1}^{p} s_{uv} \right\} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

$$\therefore \quad \frac{\partial A}{\partial \overline{x}_{2j}} = \left\{ \sum_{v=1}^{p} \sum_{u=1}^{p} s_{uv} \right\}^{-\frac{3}{2}} \left\{ (\sum_{u=1}^{p} s_{ju}) - 2 \sum_{v=1}^{p} \sum_{u=1}^{p} s_{uv} \right\} \qquad (A2.19a)$$

and $$\frac{\partial A}{\partial \overline{x}_{2i}} = \left\{ \sum_{v=1}^{p} \sum_{u=1}^{p} s_{uv} \right\}^{-\frac{3}{2}} \left\{ (\sum_{u=1}^{p} s_{iu}) - 2 \sum_{v=1}^{p} \sum_{u=1}^{p} s_{uv} \right\} \qquad (A2.19b)$$

Aside:

$$(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T \mathbf{S}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = \sum_{v=1}^{p} \left[ (m - \overline{x}_{2v}) \sum_{u=1}^{p} (m - \overline{x}_{2u}) s_{uv} \right] = C1$$

$$\mathbf{S}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T (\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = \begin{bmatrix} \sum_{u=1}^{p} (m - \overline{x}_{2u}) s_{1u} \\ \vdots \\ \sum_{u=1}^{p} (m - \overline{x}_{2u}) s_{pu} \end{bmatrix} \times \sum_{u=1}^{p} (m - \overline{x}_{2u})^2$$

$$(= \mathbf{S}\mathbf{y}\mathbf{y}^T \mathbf{y})$$

Using the above expressions rewrite *(A2.19)* as :

$$\frac{\partial A}{\partial \mathbf{\bar{x}}_2} = \{C1\}^{-\frac{3}{2}} \begin{bmatrix} \sum_{u=1}^{p}(m - \bar{x}_{2u})s_{1u} \\ \vdots \\ \sum_{u=1}^{p}(m - \bar{x}_{2u})s_{pu} \end{bmatrix} \times \sum_{u=1}^{p}(m - \bar{x}_{2u})^2 - 2\{C1\}^{-\frac{1}{2}} \begin{bmatrix} m - \bar{x}_{21} \\ \vdots \\ m - \bar{x}_{2p} \end{bmatrix}$$

$$\frac{\partial A}{\partial \bar{x}_{2j}} = \{C1\}^{-\frac{3}{2}} \sum_{u=1}^{p}(m - \bar{x}_{2u})s_{ju} \times \sum_{u=1}^{p}(m - \bar{x}_{2u})^2 - 2\{C1\}^{-\frac{1}{2}}(m - \bar{x}_{2j})$$

$$= a + b$$

$$\frac{\partial^2 A}{\partial \bar{x}_{2i} \partial \bar{x}_{2j}} = \frac{\partial}{\partial \bar{x}_{2i}} \left[ \frac{\partial A}{\partial \bar{x}_{2j}} \right] = \frac{\partial a}{\partial \bar{x}_{2i}} + \frac{\partial b}{\partial \bar{x}_{2i}} \qquad (A2.20)$$

(1) find $\frac{\partial a}{\partial \bar{x}_{2i}}$ :

$$a = \{C1\}^{-\frac{3}{2}} \sum_{u=1}^{p}(m - \bar{x}_{2u})s_{ju} \times \sum_{u=1}^{p}(m - \bar{x}_{2u})^2$$

$$\frac{\partial a}{\partial \bar{x}_{2i}} = \{C1\}^{-\frac{3}{2}} \times \frac{\partial}{\partial \bar{x}_{2i}} \left[ \sum_{u=1}^{p}(m - \bar{x}_{2u})s_{ju} \times \sum_{u=1}^{p}(m - \bar{x}_{2u})^2 \right]$$
$$+ \frac{1}{2} \frac{\partial}{\partial \bar{x}_{2i}} \left[ \{C1\}^{-\frac{3}{2}} \right] \times \sum_{u=1}^{p}(m - \bar{x}_{2u})s_{ju} \times \sum_{u=1}^{p}(m - \bar{x}_{2u})^2$$

where

$$\frac{\partial}{\partial \overline{x}_{2i}}\left[\sum_{u=1}^{p}(m-\overline{x}_{2u})s_{ju} \times \sum_{u=1}^{p}(m-\overline{x}_{2u})^2\right]$$
$$=\sum_{u=1}^{p}(m-\overline{x}_{2u})s_{ju} \times 2(m-\overline{x}_{2i})(-1) - s_{ji}\sum_{u=1}^{p}(m-\overline{x}_{2u})^2$$
$$=-2m^2\sum_{u=1}^{p}s_{ju} - m^2 s_{ji}p$$
$$=-m^2\left(2\sum_{u=1}^{p}s_{ju} + s_{ji}p\right)$$

and

$$\frac{\partial}{\partial \overline{x}_{2i}}\left[\{C1\}^{-\frac{3}{2}}\right] = -\frac{3}{2}\frac{\partial(C1)}{\partial \overline{x}_{2i}}\{C1\}^{-\frac{5}{2}}$$
$$= -\frac{3}{2}\{C1\}^{-\frac{5}{2}} \times -2 \times \sum_{u=1}^{p}(m-\overline{x}_{2u})s_{ui}$$
$$=3\{C1\}^{-\frac{5}{2}} \times \sum_{u=1}^{p}(m-\overline{x}_{2u})s_{ui}$$

$$\therefore \quad \frac{\partial a}{\partial \overline{x}_{2i}} = \{C1\}^{-\frac{3}{2}} \times -m^2(2\sum_{u=1}^{p}s_{ju} + s_{ji}p)$$
$$+3\{C1\}^{-\frac{5}{2}}\sum_{u=1}^{p}(m-\overline{x}_{2u})s_{ui} \times m\sum_{u=1}^{p}s_{ju} \times m^2 p$$
$$=-\left\{m^2\sum_{v=1}^{p}\sum_{u=1}^{p}\sigma_{uv}\right\}^{-\frac{3}{2}} \times m^2(2\sum_{u=1}^{p}s_{ju} + s_{ji}p)$$
$$+3\left\{m^2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\}^{-\frac{5}{2}} \times m\sum_{u=1}^{p}s_{ui} \times m\sum_{u=1}^{p}s_{ju} \times m^2 p$$
$$=-\frac{1}{m}\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\}^{-\frac{3}{2}} \times (2\sum_{u=1}^{p}s_{ju} + s_{ji}p)$$
$$+\frac{3p}{m}\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\}^{-\frac{5}{2}}\sum_{u=1}^{p}s_{ui}\sum_{u=1}^{p}s_{ju}$$
$$=\frac{1}{m}\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\}^{-\frac{5}{2}}\left[3p\sum_{u=1}^{p}s_{ui}\sum_{u=1}^{p}s_{ju} - (\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv})(2\sum_{u=1}^{p}s_{ju} + s_{ji}p)\right]$$

*expression (A2.20a)*

(2) find $\dfrac{\partial b}{\partial \overline{x}_{2i}}$ :

$$b = -2\{C1\}^{-\frac{1}{2}}(m - \overline{x}_{2j})$$

$$\frac{\partial b}{\partial \overline{x}_{2i}} = -2\{C1\}^{-\frac{1}{2}}\begin{cases} -1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} + \{C1\}^{-\frac{3}{2}}\frac{\partial (C1)}{\partial \overline{x}_{2i}}(m - \overline{x}_{2j})$$

$$= 2\{C1\}^{-\frac{1}{2}}\begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} - 2\{C1\}^{-\frac{3}{2}}(m - \overline{x}_{2j})\sum_{u=1}^{p}(m - \overline{x}_{2u})s_{ui}$$

$$= 2\left\{m^2 \sum_{v=1}^{p}\sum_{u=1}^{p} s_{uv}\right\}^{-\frac{1}{2}}\begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} - 2\left\{m^2 \sum_{v=1}^{p}\sum_{u=1}^{p} \sigma_{uv}\right\}^{-\frac{3}{2}} m^2 \sum_{u=1}^{p} s_{ui}$$

$$= \frac{2}{m}\left\{\sum_{v=1}^{p}\sum_{u=1}^{p} s_{uv}\right\}^{-\frac{3}{2}}\begin{cases} \sum_{v=1}^{p}\sum_{u=1}^{p} s_{uv} - \sum_{u=1}^{p} s_{ui} & \text{if} \quad i = j \\ -\sum_{u=1}^{p} s_{ui} & \text{if} \quad i \neq j \end{cases} \qquad (A2.20b)$$

Therefore subsitute *(A2.20a)* and *(A2.20b)* into *(A2.20)* gives

$$\frac{\partial^2 A}{\partial \overline{x}_{2i}\partial \overline{x}_{2j}} = \frac{1}{m}\left\{\sum_{v=1}^{p}\sum_{u=1}^{p} s_{uv}\right\}^{-\frac{5}{2}}\left[3p\sum_{u=1}^{p} s_{ui}\sum_{u=1}^{p} s_{ju} - (\sum_{v=1}^{p}\sum_{u=1}^{p} s_{uv})(2\sum_{u=1}^{p} s_{ju} + s_{ji}p)\right]$$
$$+ \frac{2}{m}\left\{\sum_{v=1}^{p}\sum_{u=1}^{p} s_{uv}\right\}^{-\frac{3}{2}}\begin{cases} \sum_{v=1}^{p}\sum_{u=1}^{p} s_{uv} - \sum_{u=1}^{p} s_{ui} & \text{if} \quad i = j \\ -\sum_{u=1}^{p} s_{ui} & \text{if} \quad i \neq j \end{cases}$$

$$
=\frac{1}{m}\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\}^{-\frac{5}{2}}\times\begin{cases}3p\sum_{u=1}^{p}s_{ui}\sum_{u=1}^{p}s_{ju}-(\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv})(2\sum_{u=1}^{p}s_{ju}+s_{ji}p) \\ +2(\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv})(\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}-\sum_{u=1}^{p}s_{ui}) & \text{if} \quad i=j \\ 3p\sum_{u=1}^{p}s_{ui}\sum_{u=1}^{p}s_{ju}-(\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv})(2\sum_{u=1}^{p}s_{ju}+s_{ji}p) \\ -2(\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv})\sum_{u=1}^{p}s_{ui} & \text{if} \quad i\neq j\end{cases}
$$

*expression (A2.21)*

Therefore substitute *(A2.19a)* , *(A2.19b)* and *(A2.21)* into *(A2.18)*

gives $\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}$

$$
=-\frac{1}{2}\phi\left(-\frac{1}{2}mp(\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv})^{-\frac{1}{2}}\right)\times\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\right\}^{-\frac{7}{2}}
$$

$$
\times\begin{cases}\frac{1}{m}\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\Big[3p\sum_{u=1}^{p}s_{ui}\sum_{u=1}^{p}s_{ju}-(\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv})(2\sum_{u=1}^{p}s_{ju}+s_{ji}p) \\ +2(\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv})(\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}-\sum_{u=1}^{p}s_{ui})\Big] \\ -\frac{mp}{4}(\sum_{u=1}^{p}s_{ju}-2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv})(\sum_{u=1}^{p}s_{iu}-2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}) & \text{if} \quad i=j \\ \frac{1}{m}\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}\Big[3p\sum_{u=1}^{p}s_{ui}\sum_{u=1}^{p}s_{ju}-(\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv})(2\sum_{u=1}^{p}s_{ju}+s_{ji}p) \\ -2(\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv})\sum_{u=1}^{p}s_{ui}\Big] \\ -\frac{mp}{4}(\sum_{u=1}^{p}s_{ju}-2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv})(\sum_{u=1}^{p}s_{iu}-2\sum_{v=1}^{p}\sum_{u=1}^{p}s_{uv}) & \text{if} \quad i\neq j\end{cases}
$$

*expression (A2.22)*

$$\therefore E[\Phi(\bar{x}_1, \bar{x}_2)] = \Phi(\mu_1, \mu_2) + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\,\partial\bar{x}_{1j}}\operatorname{cov}(\bar{x}_{1,ij})$$

$$+\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\,\partial\bar{x}_{2j}}\operatorname{cov}(\bar{x}_{2,ij})$$

$$= \Phi(\mu_1, \mu_2) + \frac{1}{2n_1}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\,\partial\bar{x}_{1j}}\times\sigma_{ij}$$

$$+\frac{1}{2n_2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\,\partial\bar{x}_{2j}}\times\sigma_{ij}$$

where $\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\,\partial\bar{x}_{1j}}$ and $\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\,\partial\bar{x}_{2j}}$ are as in *(A2.17)* and *(A2.22)* respectively.

Now need to find $\frac{\partial^2\Phi}{\partial s_{kl}\,\partial s_{ij}}$:

$$A = \frac{mp}{[\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}]^{1/2}} = mp[\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}]^{-1/2}$$

$$\frac{\partial^2\Phi}{\partial s_{kl}\,\partial s_{ij}} = \frac{\partial}{\partial s_{kl}}\left[\frac{\partial\Phi}{\partial s_{ij}}\right]$$

$$= -\frac{mp}{2}\frac{\partial}{\partial s_{kl}}\left[\phi\left(-\frac{mp}{2}[\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}]^{-1/2}\right)\frac{\partial}{\partial s_{ij}}\left[(\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv})^{-1/2}\right]\right]$$

$$= -\frac{mp}{2}\left[ \frac{\partial^2}{\partial s_{kl}\partial s_{ij}}\left[ [\sum_{u=1}^{p}\sum_{v=1}^{p} s_{uv}]^{-1/2} \right] \times \phi\left( -\frac{mp}{2}[\sum_{u=1}^{p}\sum_{v=1}^{p} s_{uv}]^{-1/2} \right) + \frac{\partial}{\partial s_{ij}}\left[ (\sum_{u=1}^{p}\sum_{v=1}^{p} s_{uv})^{-1/2} \right] \times \frac{\partial}{\partial s_{kl}}\left[ \phi\left( -\frac{mp}{2}[\sum_{u=1}^{p}\sum_{v=1}^{p} s_{uv}]^{-1/2} \right) \right] \right]$$

*expression (A2.23)*

where

$$\frac{\partial}{\partial s_{ij}}\left[ (\sum_{u=1}^{p}\sum_{v=1}^{p} s_{uv})^{-1/2} \right] = -\frac{1}{2}\left\{ \sum_{v=1}^{p}\sum_{u=1}^{p} s_{uv} \right\}^{-\frac{3}{2}} \frac{\partial}{\partial s_{ij}}\left[ \sum_{u=1}^{p}\sum_{v=1}^{p} s_{uv} \right] \qquad (A2.23a)$$

$$\frac{\partial}{\partial s_{kl}}\left[ \phi\left( -\frac{mp}{2}[\sum_{u=1}^{p}\sum_{v=1}^{p} s_{uv}]^{-1/2} \right) \right]$$

$$= -\frac{1}{2}\frac{m^2p^2}{4}\frac{\partial}{\partial s_{kl}}\left[ \left\{ \sum_{u=1}^{p}\sum_{v=1}^{p} s_{uv} \right\}^{-1} \right] \times \phi\left( -\frac{mp}{2}[\sum_{u=1}^{p}\sum_{v=1}^{p} s_{uv}]^{-1/2} \right)$$

$$= \frac{m^2p^2}{8}\left\{ \sum_{u=1}^{p}\sum_{v=1}^{p} s_{uv} \right\}^{-2} \frac{\partial}{\partial s_{kl}}\left[ \sum_{u=1}^{p}\sum_{v=1}^{p} s_{uv} \right] \times \phi\left( -\frac{mp}{2}[\sum_{u=1}^{p}\sum_{v=1}^{p} s_{uv}]^{-1/2} \right)$$

*expression (A2.23b)*

$$\frac{\partial^2}{\partial s_{kl}\partial s_{ij}}\left[ \left\{ \sum_{u=1}^{p}\sum_{v=1}^{p} s_{uv} \right\}^{-\frac{1}{2}} \right] = \frac{\partial}{\partial s_{kl}}\left[ \frac{\partial}{\partial s_{ij}}\left[ \left\{ \sum_{u=1}^{p}\sum_{v=1}^{p} s_{uv} \right\}^{-\frac{1}{2}} \right] \right]$$

$$= \frac{\partial}{\partial s_{kl}}\left[ -\frac{1}{2}\left\{ \sum_{u=1}^{p}\sum_{v=1}^{p} s_{uv} \right\}^{-\frac{3}{2}} \frac{\partial}{\partial s_{ij}}\left[ \sum_{u=1}^{p}\sum_{v=1}^{p} s_{uv} \right] \right]$$

$$= -\frac{1}{2}\left[\left\{\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}\right\}^{-\frac{3}{2}}\frac{\partial^2}{\partial s_{kl}\partial s_{ij}}\left[\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}\right] + \frac{\partial}{\partial s_{kl}}\left\{\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}\right\}^{-\frac{3}{2}}\frac{\partial}{\partial s_{ij}}\left[\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}\right]\right]$$

$$= \frac{3}{4}\left\{\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}\right\}^{-\frac{5}{2}}\frac{\partial}{\partial s_{kl}}\left[\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}\right]\frac{\partial}{\partial s_{ij}}\left[\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}\right]$$

*expression (A2.23c)*

since $\frac{\partial^2}{\partial s_{kl}\partial s_{ij}}\left[\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}\right] = 0 \quad \forall\ k,l,i,j = 1,\ldots,p$

Now substitute *(A2.23a)* , *(A2.23b)* and *(A2.23c)* into *(A2.23)*,

$$\therefore \frac{\partial^2\Phi}{\partial s_{kl}\partial s_{ij}} = -\frac{mp}{2}\phi\left(-\frac{mp}{2}\left[\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}\right]^{-1/2}\right)\left\{\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}\right\}^{-\frac{7}{2}} \times\frac{\partial}{\partial s_{kl}}\left[\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}\right]\frac{\partial}{\partial s_{ij}}\left[\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}\right]\left[\frac{3}{4}\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv} - \frac{1}{16}m^2p^2\right]$$

where $\frac{\partial}{\partial s_{ij}}\left[\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}\right] = 1 \qquad \forall\ i,j = 1,\ldots,p$

$\frac{\partial}{\partial s_{kl}}\left[\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}\right] = 1 \qquad \forall\ k,l = 1,\ldots,p$

$$\therefore \frac{\partial^2\Phi}{\partial s_{kl}\partial s_{ij}} = -\frac{mp}{8}\phi\left(-\frac{mp}{2}\left[\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}\right]^{-1/2}\right)\left\{\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv}\right\}^{-\frac{7}{2}} \times\left[3\sum_{u=1}^{p}\sum_{v=1}^{p}s_{uv} - \frac{1}{4}m^2p^2\right] \qquad (A2.24)$$

# *Appendix A2.2*

## *ASYPMTOTIC EXPANSION FOR LINEAR DISCRIMINANT FUNCTION*

### *Plug-in error rate*

We have $p_{12}^{(P)} = \Phi\left( -\frac{1}{2}[(\overline{x}_1 - \overline{x}_2)^T S^{-1} (\overline{x}_1 - \overline{x}_2)]^{\frac{1}{2}} \right) = \Phi\left( -\frac{D}{2} \right)$.

Using Taylor series expansion (to second order approximation),we obtain:-

$$
\begin{aligned}
\Phi(\overline{x}_1, \overline{x}_2, S) = {} & \Phi(\mu_1, \mu_2, \Sigma) + \sum_{j=1}^{p} \frac{\partial \Phi}{\partial \overline{x}_{1j}} (\overline{x}_{1j} - \mu_{1j}) + \sum_{j=1}^{p} \frac{\partial \Phi}{\partial \overline{x}_{2j}} (\overline{x}_{2j} - \mu_{2j}) \\
& + \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial \Phi}{\partial s_{ij}} (s_{ij} - \Sigma_{ij}) \\
& + \frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}} (\overline{x}_{1i} - \mu_{1i})(\overline{x}_{1j} - \mu_{1j}) \\
& + \frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}} (\overline{x}_{2i} - \mu_{2i})(\overline{x}_{2j} - \mu_{2j}) \\
& + \frac{1}{2} \sum_{k=1}^{p} \sum_{l=1}^{p} \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial s_{kl} \partial s_{ij}} (s_{kl} - \Sigma_{kl})(s_{ij} - \Sigma_{ij}) \\
& + \frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{1i} \partial \overline{x}_{2j}} (\overline{x}_{1i} - \mu_{1i})(\overline{x}_{2j} - \mu_{2j}) \\
& + \frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{1i} \partial s_{ij}} (\overline{x}_{1i} - \mu_{1i})(s_{ij} - \Sigma_{ij}) \\
& + \frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 \Phi}{\partial \overline{x}_{2i} \partial s_{ij}} (\overline{x}_{2i} - \mu_{2i})(s_{ij} - \Sigma_{ij})
\end{aligned}
$$

where $\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{1i}\partial\overline{x}_{2j}}=0$, $\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{1i}\partial s_{ij}}=0$ and $\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{2i}\partial s_{ij}}=0$.

Taking expectations gives:-

$$\begin{aligned}E[\Phi(\overline{\mathbf{x}}_1,\overline{\mathbf{x}}_2,\mathrm{S})]=\Phi(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\boldsymbol{\Sigma})&+\sum_{j=1}^{p}\frac{\partial\Phi}{\partial\overline{x}_{1j}}E(\overline{x}_{1j}-\mu_{1j})\\&+\sum_{j=1}^{p}\frac{\partial\Phi}{\partial\overline{x}_{2j}}E(\overline{x}_{2j}-\mu_{2j})+\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial\Phi}{\partial s_{ij}}E(s_{ij}-\Sigma_{ij})\\&+\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{1i}\,\partial\overline{x}_{1j}}\operatorname{cov}(\overline{x}_{1,ij})\\&+\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{2i}\,\partial\overline{x}_{2j}}\operatorname{cov}(\overline{x}_{2,ij})\\&+\frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{p}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial s_{kl}\,\partial s_{ij}}\operatorname{cov}(s_{kl},s_{ij})\end{aligned}$$

where $\sum_{j=1}^{p}\frac{\partial\Phi}{\partial\overline{x}_{1j}}E(\overline{x}_{1j}-\mu_{1j})=\sum_{j=1}^{p}\frac{\partial\Phi}{\partial\overline{x}_{2j}}E(\overline{x}_{2j}-\mu_{2j})=0$

and $\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial\Phi}{\partial s_{ij}}E(s_{ij}-\Sigma_{ij})=0$.

$$\begin{aligned}\therefore E[\Phi(\overline{\mathbf{x}}_1,\overline{\mathbf{x}}_2,\mathrm{S})]=\Phi(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\boldsymbol{\Sigma})&+\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{1i}\,\partial\overline{x}_{1j}}\operatorname{cov}(\overline{x}_{1,ij})\\&+\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial\overline{x}_{2i}\,\partial\overline{x}_{2j}}\operatorname{cov}(\overline{x}_{2,ij})\\&+\frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{p}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2\Phi}{\partial s_{kl}\,\partial s_{ij}}\operatorname{cov}(s_{kl},s_{ij})\end{aligned}$$

So now we want to find $\frac{\partial \Phi}{\partial \bar{x}_{1j}}, \frac{\partial \Phi}{\partial \bar{x}_{2j}}, \frac{\partial^2 \Phi}{\partial \bar{x}_{1j}\, \partial \bar{x}_{1i}}, \frac{\partial^2 \Phi}{\partial \bar{x}_{2j}\, \partial \bar{x}_{2i}}, \frac{\partial^2 \Phi}{\partial s_{kl}\, \partial s_{ij}}$ .

Consider cases 1A and 1B where the conditions are as follow (respectively) :

$$\boldsymbol{\mu}_1 = (m, 0, \dots \dots, 0)^T \qquad\qquad \boldsymbol{\mu}_1 = (m, 0, \dots \dots, 0)^T$$

$$\boldsymbol{\mu}_2 = (0, \dots \dots \dots, 0)^T \qquad \text{and} \qquad \boldsymbol{\mu}_2 = (0, \dots \dots \dots, 0)^T$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho & \rho & \dots & \rho \\ \rho & 1 & \rho & \dots & \rho \\ \vdots & & & & \vdots \\ \rho & \dots & \dots & \rho & 1 \end{bmatrix} \qquad\qquad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^{p-1} \\ \rho & 1 & \rho & \dots & \rho^{p-2} \\ \vdots & & & & \vdots \\ \rho^{p-1} & \dots & \dots & \rho & 1 \end{bmatrix}$$

Need to find $\frac{\partial^2 \Phi}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}}$ :

$$\frac{\partial^2 \Phi}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}} = \frac{\partial}{\partial \bar{x}_{1i}} \left[ -\frac{1}{2} \phi(-\frac{D}{2}) \frac{\partial D}{\partial \bar{x}_{1j}} \right] = -\frac{1}{2} \left[ \phi(-\frac{D}{2}) \frac{\partial^2 D}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}} + \frac{\partial \phi}{\partial \bar{x}_{1i}} \frac{\partial D}{\partial \bar{x}_{1j}} \right]$$

where $$\frac{\partial \phi}{\partial \bar{x}_{1i}} = -\frac{D}{4} \phi(-\frac{D}{2}) \frac{\partial D}{\partial \bar{x}_{1i}}$$

$$\therefore \quad \frac{\partial^2 \Phi}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}} = -\frac{1}{2}\left[ \phi(-\frac{D}{2}) \frac{\partial^2 D}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}} - \frac{D}{4}\phi(-\frac{D}{2}) \frac{\partial D}{\partial \overline{x}_{1i}} \frac{\partial D}{\partial \overline{x}_{1j}} \right]$$

$$= -\frac{1}{2}\phi(-\frac{D}{2}) \left[ \frac{\partial^2 D}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}} - \frac{D}{4} \frac{\partial D}{\partial \overline{x}_{1i}} \frac{\partial D}{\partial \overline{x}_{1j}} \right] \qquad (A2.25)$$

Need to find $\frac{\partial A}{\partial \overline{x}_{1i}}, \frac{\partial A}{\partial \overline{x}_{1j}}$ and $\frac{\partial^2 A}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}}$ :

Under cases 1A and 1B ,

let $\quad D = (\overline{\mathbf{x}}_1^T S^{-1} \overline{\mathbf{x}}_1)^{\frac{1}{2}}$

$$\frac{\partial D}{\partial \overline{\mathbf{x}}_1} = (\overline{\mathbf{x}}_1^T S^{-1} \overline{\mathbf{x}}_1)^{-\frac{1}{2}} S^{-1} \overline{\mathbf{x}}_1 \qquad (A2.26)$$

where $\quad \overline{\mathbf{x}}_1^T S^{-1} \overline{\mathbf{x}}_1 = m^2 s^{11}$

$$S^{-1}\overline{\mathbf{x}}_1 = m \begin{bmatrix} s^{11} \\ \vdots \\ s^{p1} \end{bmatrix}$$

$$\overline{\mathbf{x}}_1^T \overline{\mathbf{x}}_1 = m^2$$

$$D = \frac{1}{2} m \left(s^{11}\right)^{\frac{1}{2}}$$

$$\therefore \quad \frac{\partial D}{\partial \overline{\mathbf{x}}_1} = \left(m^2 s^{11}\right)^{-\frac{1}{2}} \quad m\begin{bmatrix} s^{11} \\ \vdots \\ s^{p1} \end{bmatrix} = \left(s^{11}\right)^{-\frac{1}{2}} \quad \begin{bmatrix} s^{11} \\ \vdots \\ s^{p1} \end{bmatrix}$$

$$\therefore \quad \frac{\partial D}{\partial \overline{x}_{1j}} = \left(s^{11}\right)^{-\frac{1}{2}} \times s^{j1} \qquad \text{and} \qquad \frac{\partial D}{\partial \overline{x}_{1i}} = \left(s^{11}\right)^{-\frac{1}{2}} \times s^{i1}$$

Aside:

$$\overline{\mathbf{x}}_1^T \mathbf{S}^{-1} \overline{\mathbf{x}}_1 = \sum_{v=1}^{p} [\overline{x}_{1v} ( \sum_{u=1}^{p} \overline{x}_{1u} s^{uv} )]$$

$$\mathbf{S}^{-1}\overline{\mathbf{x}}_1 = \begin{bmatrix} \sum_{u=1}^{p} \overline{x}_{1u} s^{1u} \\ \vdots \\ \sum_{u=1}^{p} \overline{x}_{1u} s^{pu} \end{bmatrix}$$

Using the expressions above rewrite *(A2.26)* as:

$$\frac{\partial D}{\partial \overline{\mathbf{x}}_1} = \left\{ \sum_{v=1}^{p} [\overline{x}_{1v} ( \sum_{u=1}^{p} \overline{x}_{1u} s^{uv} )] \right\}^{-\frac{1}{2}} \quad \begin{bmatrix} \sum_{u=1}^{p} \overline{x}_{1u} s^{1u} \\ \vdots \\ \sum_{u=1}^{p} \overline{x}_{1u} s^{pu} \end{bmatrix}$$

$$\therefore \quad \frac{\partial D}{\partial \overline{x}_{1j}} = \left\{ \sum_{v=1}^{p} [\overline{x}_{1v} ( \sum_{u=1}^{p} \overline{x}_{1u} s^{uv} )] \right\}^{-\frac{1}{2}} \times \sum_{u=1}^{p} \overline{x}_{1u} s^{ju}$$

$$\therefore \quad \frac{\partial^2 D}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}} = \frac{\partial}{\partial \overline{x}_{1i}} \left[ \left\{ \sum_{v=1}^{p} [\overline{x}_{1v} ( \sum_{u=1}^{p} \overline{x}_{1u} s^{uv} )] \right\}^{-\frac{1}{2}} \quad \sum_{u=1}^{p} \overline{x}_{1u} s^{ju} \right]$$

$$= \left\{ \sum_{v=1}^{p} [\overline{x}_{1v} ( \sum_{u=1}^{p} \overline{x}_{1u} s^{uv} )] \right\}^{-\frac{1}{2}} \frac{\partial}{\partial \overline{x}_{1i}} \left[ \sum_{u=1}^{p} \overline{x}_{1u} s^{ju} \right]$$

$$+ \frac{\partial}{\partial \overline{x}_{1i}} \left[ \left\{ \sum_{v=1}^{p} [\overline{x}_{1v} ( \sum_{u=1}^{p} \overline{x}_{1u} s^{uv} )] \right\}^{-\frac{1}{2}} \right] \times \quad \sum_{u=1}^{p} \overline{x}_{1u} s^{ju}$$

where $\quad \frac{\partial}{\partial \overline{x}_{1i}} \left[ \sum_{u=1}^{p} \overline{x}_{1u} s^{ju} \right] = s^{ji}$

$$\frac{\partial}{\partial \overline{x}_{1i}} \left[ \left\{ \sum_{v=1}^{p} [\overline{x}_{1v} ( \sum_{u=1}^{p} \overline{x}_{1u} s^{uv} )] \right\}^{-\frac{1}{2}} \right]$$

$$= -\frac{1}{2} \left\{ \sum_{v=1}^{p} [\overline{x}_{1v} ( \sum_{u=1}^{p} \overline{x}_{1u} s^{uv} )] \right\}^{-\frac{3}{2}} \times \frac{\partial}{\partial \overline{x}_{1i}} \left[ \sum_{v=1}^{p} [\overline{x}_{1v} ( \sum_{u=1}^{p} \overline{x}_{1u} s^{uv} )] \right]$$

$$= -\left\{ \sum_{v=1}^{p} [\overline{x}_{1v} ( \sum_{u=1}^{p} \overline{x}_{1u} s^{uv} )] \right\}^{-\frac{3}{2}} \times \sum_{u=1}^{p} \overline{x}_{1u} s^{ui}$$

$$\therefore \quad \frac{\partial^2 D}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}} = \left\{ \sum_{v=1}^{p} [\overline{x}_{1v} ( \sum_{u=1}^{p} \overline{x}_{1u} s^{uv} )] \right\}^{-\frac{1}{2}} \times s^{ji}$$

$$- \left\{ \sum_{v=1}^{p} [\overline{x}_{1v} ( \sum_{u=1}^{p} \overline{x}_{1u} s^{uv} )] \right\}^{-\frac{3}{2}} \times \sum_{u=1}^{p} \overline{x}_{1u} s^{ui} \times \sum_{u=1}^{p} \overline{x}_{1u} s^{ju}$$

$$= \left( m^2 s^{11} \right)^{-\frac{1}{2}} s^{ji} - \left( m^2 s^{11} \right)^{-\frac{3}{2}} \times m s^{1i} \times m s^{j1}$$

$$= \left( m^2 s^{11} \right)^{-\frac{3}{2}} \left[ m^2 s^{11} s^{ji} - m^2 s^{1i} s^{j1} \right]$$

$$= \frac{1}{m} \left( s^{11} \right)^{-\frac{3}{2}} \left[ s^{11} s^{ji} - s^{1i} s^{j1} \right] \qquad (A2.27)$$

$$\therefore \quad \frac{\partial^2 \Phi}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}} = -\frac{1}{2}\phi\left(-\frac{1}{2}m\left(s^{11}\right)^{\frac{1}{2}}\right)\left[\frac{1}{m}\left(s^{11}\right)^{-\frac{3}{2}}\left[s^{11}s^{ji} - s^{1i}s^{j1}\right] - \frac{m}{4}\left(s^{11}\right)^{-\frac{1}{2}} s^{j1}s^{i1}\right]$$

$$= -\frac{1}{2}\phi\left(-\frac{1}{2}m\left(s^{11}\right)^{\frac{1}{2}}\right)\left(s^{11}\right)^{-\frac{3}{2}}\left[\frac{1}{m}\left[s^{11}s^{ji} - s^{1i}s^{j1}\right] - \frac{m}{4}s^{11}s^{j1}s^{i1}\right]$$

*expression (A2.28)*

Need to find $\frac{\partial^2 \Phi}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}}$ :

$$\frac{\partial^2 \Phi}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}} = -\frac{1}{2}\left[\phi(-\frac{D}{2})\frac{\partial^2 D}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}} - \frac{D}{4}\phi(-\frac{D}{2})\frac{\partial D}{\partial \overline{x}_{2i}}\frac{\partial D}{\partial \overline{x}_{2j}}\right]$$

$$= -\frac{1}{2}\phi(-\frac{D}{2})\left[\frac{\partial^2 D}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}} - \frac{D}{4}\frac{\partial D}{\partial \overline{x}_{2i}}\frac{\partial D}{\partial \overline{x}_{2j}}\right] \quad (A2.29)$$

$$\frac{\partial D}{\partial \overline{\mathbf{x}}_2} = -\left[(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T \mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)\right]^{-\frac{1}{2}} \mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) \quad (A2.30)$$

where $(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T \mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = m^2 s^{11}$

$$\mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = m\begin{bmatrix} s^{11} \\ \vdots \\ s^{p1} \end{bmatrix}$$

$$\overline{\mathbf{x}}_1^T \overline{\mathbf{x}}_1 = m^2$$

$$D = \frac{1}{2}m\left(s^{11}\right)^{\frac{1}{2}}$$

$$\therefore \quad \frac{\partial D}{\partial \overline{\mathbf{x}}_2} = -\left(m^2 s^{11}\right)^{-\frac{1}{2}} \quad m \begin{bmatrix} s^{11} \\ \vdots \\ s^{p1} \end{bmatrix} = -\left(s^{11}\right)^{-\frac{1}{2}} \begin{bmatrix} s^{11} \\ \vdots \\ s^{p1} \end{bmatrix}$$

$$\therefore \quad \frac{\partial D}{\partial \overline{x}_{2j}} = -\left(s^{11}\right)^{-\frac{1}{2}} \times s^{j1} \qquad \text{and} \qquad \frac{\partial D}{\partial \overline{x}_{2i}} = -\left(s^{11}\right)^{-\frac{1}{2}} \times s^{i1} \qquad \textit{(A2.30a)}$$

Aside:

$$(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T \mathbf{S}^{-1} (\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = (m - \overline{x}_{21})\left[(m - \overline{x}_{21}) s^{11} - (\sum_{u=2}^{p} \overline{x}_{2u} s^{u1})\right]$$
$$- \sum_{v=2}^{p} \overline{x}_{2v} \left[(m - \overline{x}_{21}) s^{1v} - (\sum_{u=2}^{p} \overline{x}_{2u} s^{uv})\right]$$
$$= C1$$

$$\mathbf{S}^{-1} (\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = \begin{bmatrix} (m - \overline{x}_{21}) s^{11} - \sum_{u=2}^{p} \overline{x}_{2u} s^{1u} \\ \vdots \\ (m - \overline{x}_{21}) s^{p1} - \sum_{u=2}^{p} \overline{x}_{2u} s^{pu} \end{bmatrix}$$

Using the expressions above rewrite *(A2.30)* as:

$$\frac{\partial D}{\partial \overline{\mathbf{x}}_2} = -\left\{ \sum_{v=1}^{p} [\overline{x}_{1v} (\sum_{u=1}^{p} \overline{x}_{1u} s^{uv})] \right\}^{-\frac{1}{2}} \begin{bmatrix} (m - \overline{x}_{21}) s^{11} - \sum_{u=2}^{p} \overline{x}_{2u} s^{1u} \\ \vdots \\ (m - \overline{x}_{21}) s^{p1} - \sum_{u=2}^{p} \overline{x}_{2u} s^{pu} \end{bmatrix}$$

$$\therefore \quad \frac{\partial D}{\partial \overline{x}_{2j}} = -\left\{ \sum_{v=1}^{p} [\overline{x}_{1v} ( \sum_{u=1}^{p} \overline{x}_{1u} s^{uv} )] \right\}^{-\frac{1}{2}} \left\{ (m - \overline{x}_{21}) s^{j1} - \sum_{u=2}^{p} \overline{x}_{2u} s^{ju} \right\}$$

$$\frac{\partial^2 D}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}} = \frac{\partial}{\partial \overline{x}_{2i}} \left[ -\left\{ \sum_{v=1}^{p} [\overline{x}_{1v} ( \sum_{u=1}^{p} \overline{x}_{1u} s^{uv} )] \right\}^{-\frac{1}{2}} \left\{ (m - \overline{x}_{21}) s^{j1} - \sum_{u=2}^{p} \overline{x}_{2u} s^{ju} \right\} \right]$$

$$= -\{C1\}^{-\frac{1}{2}} \frac{\partial}{\partial \overline{x}_{1i}} \left[ (m - \overline{x}_{21}) s^{j1} - \sum_{u=2}^{p} \overline{x}_{2u} s^{ju} \right]$$
$$+ \frac{1}{2} \{C1\}^{-\frac{3}{2}} \frac{\partial}{\partial \overline{x}_{1i}} [C1] \quad \times \quad \left\{ (m - \overline{x}_{21}) s^{j1} - \sum_{u=2}^{p} \overline{x}_{2u} s^{ju} \right\}$$

where

$$\frac{\partial}{\partial \overline{x}_{2i}} \left[ (m - \overline{x}_{21}) s^{j1} - \sum_{u=2}^{p} \overline{x}_{2u} s^{ju} \right] = -s^{ji}$$

$$\frac{\partial}{\partial \overline{x}_{2i}} [C1] = -2 m s^{1i}$$

$$\therefore \frac{\partial^2 D}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}} = \{C1\}^{-\frac{1}{2}} \; s^{ji} - \{C1\}^{-\frac{3}{2}} \times m\, s^{1i} \times \left\{ (m - \overline{x}_{21}) s^{j1} - \sum_{u=2}^{p} \overline{x}_{2u} s^{ju} \right\}$$

$$= \left( m^2 s^{11} \right)^{-\frac{1}{2}} s^{ji} - \left( m^2 s^{11} \right)^{-\frac{3}{2}} \times m s^{1i} \times m s^{j1}$$

$$= \left( m^2 s^{11} \right)^{-\frac{3}{2}} \left[ m^2 s^{11} s^{ji} - m^2 s^{1i} s^{j1} \right]$$

$$= \frac{1}{m} \left( s^{11} \right)^{-\frac{3}{2}} \left[ s^{11} s^{ji} - s^{1i} s^{j1} \right] \qquad (A2.31)$$

Now subsitute *(A2.30a)* and *(A2.31)* into *(A2.29)*,

$$\therefore \quad \frac{\partial^2 \Phi}{\partial \bar{x}_{2i} \partial \bar{x}_{2j}} = -\frac{1}{2}\phi\left(-\frac{1}{2}m\left(s^{11}\right)^{\frac{1}{2}}\right)\left[\frac{1}{m}\left(s^{11}\right)^{-\frac{3}{2}}\left[s^{11}s^{ji} - s^{1i}s^{j1}\right] - \frac{m}{4}\left(s^{11}\right)^{-\frac{1}{2}} s^{j1}s^{i1}\right]$$

$$= -\frac{1}{2}\phi\left(-\frac{1}{2}m\left(s^{11}\right)^{\frac{1}{2}}\right)\left(s^{11}\right)^{-\frac{3}{2}}\left[\frac{1}{m}\left[s^{11}s^{ji} - s^{1i}s^{j1}\right] - \frac{m}{4}s^{11}s^{j1}s^{i1}\right]$$

*expression (A2.32)*

Note that $\dfrac{\partial^2 \Phi}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}} = \dfrac{\partial^2 \Phi}{\partial \bar{x}_{2i} \partial \bar{x}_{2j}}$ .

Now need to find $\dfrac{\partial^2 \Phi}{\partial s_{kl}\, \partial s_{ij}}$:

$$D = m\left\{s^{11}\right\}^{\frac{1}{2}}$$

$$\frac{\partial^2 \Phi}{\partial s_{kl} \partial s_{ij}} = \frac{\partial}{\partial s_{kl}}\left[\frac{\partial \Phi}{\partial s_{ij}}\right]$$

$$= \frac{\partial}{\partial s_{kl}}\left[-\frac{m}{2}\phi\left(-\frac{m}{2}\left\{s^{11}\right\}^{\frac{1}{2}}\right)\frac{\partial}{\partial s_{ij}}\left[\left\{s^{11}\right\}^{\frac{1}{2}}\right]\right]$$

$$= -\frac{m}{2}\left[\phi\left(-\frac{m}{2}\left\{s^{11}\right\}^{\frac{1}{2}}\right)\frac{\partial^2}{\partial s_{kl}\partial s_{ij}}\left[\left\{s^{11}\right\}^{\frac{1}{2}}\right] + \frac{\partial}{\partial s_{ij}}\left[\left\{s^{11}\right\}^{\frac{1}{2}}\right]\frac{\partial}{\partial s_{kl}}\left[\phi\left(-\frac{m}{2}\left\{s^{11}\right\}^{\frac{1}{2}}\right)\right]\right] \qquad (A2.33)$$

where

(1) $\dfrac{\partial}{\partial s_{ij}}\left[\left\{s^{11}\right\}^{\frac{1}{2}}\right] = \dfrac{1}{2}\left\{s^{11}\right\}^{-\frac{1}{2}}\dfrac{\partial (s^{11})}{\partial s_{ij}}$

$$= \frac{1}{2}\left\{s^{11}\right\}^{-\frac{1}{2}} \times w_o \times (s^{1i}s^{jl} + s^{1j}s^{il})$$

$$= \left\{s^{11}\right\}^{-\frac{1}{2}} \times w_o \times s^{1i}s^{jl} \qquad \textit{(A2.33a)}$$

where $w_o = \begin{cases} -\frac{1}{2} & \text{if} \quad i = j \\ -1 & \text{if} \quad i \neq j \end{cases}$

(2) $$\frac{\partial}{\partial s_{kl}}\left[\phi\left(-\frac{m}{2}\left\{s^{11}\right\}^{\frac{1}{2}}\right)\right] = -\frac{m^2}{8}\frac{\partial(s^{11})}{\partial s_{kl}}\phi\left(-\frac{m}{2}\left\{s^{11}\right\}^{\frac{1}{2}}\right)$$

$$= -\frac{m^2}{8} w_1 (s^{1k}s^{l1} + s^{l1}s^{kl})\phi\left(-\frac{m}{2}\left\{s^{11}\right\}^{\frac{1}{2}}\right)$$

$$= -\frac{m^2}{4} \times w_1 \times s^{1k}s^{l1}\phi\left(-\frac{m}{2}\left\{s^{11}\right\}^{\frac{1}{2}}\right)$$

*expression (A2.33b)*

where $w_1 = \begin{cases} -\frac{1}{2} & \text{if} \quad k = l \\ -1 & \text{if} \quad k \neq l \end{cases}$

(3) $$\frac{\partial^2}{\partial s_{kl}\partial s_{ij}}\left[\left\{s^{11}\right\}^{\frac{1}{2}}\right] = \frac{\partial}{\partial s_{kl}}\left[\frac{\partial}{\partial s_{ij}}\left\{s^{11}\right\}^{\frac{1}{2}}\right]$$

$$= \frac{1}{2}\left[\left\{s^{11}\right\}^{-\frac{1}{2}}\frac{\partial^2(s^{11})}{\partial s_{kl}\partial s_{ij}} + \frac{\partial}{\partial s_{kl}}\left[\left\{s^{11}\right\}^{-\frac{1}{2}}\right]\frac{\partial(s^{11})}{\partial s_{ij}}\right]$$

$$= \frac{1}{2}\left[\left\{s^{11}\right\}^{-\frac{1}{2}}\frac{\partial^2(s^{11})}{\partial s_{kl}\partial s_{ij}} - \frac{1}{2}\left\{s^{11}\right\}^{-\frac{3}{2}}\frac{\partial(s^{11})}{\partial s_{kl}}\frac{\partial(s^{11})}{\partial s_{ij}}\right]$$

where

$$\frac{\partial(s^{11})}{\partial s_{ij}} = w_0 (s^{1i}s^{jl} + s^{1j}s^{il}) = 2w_0 \times s^{1i}s^{jl}$$

$$\frac{\partial(s^{11})}{\partial s_{kl}} = 2w_1 \times s^{1k}s^{l1}$$

$$\frac{\partial^2(s^{11})}{\partial s_{kl}\partial s_{ij}} = \frac{\partial}{\partial s_{kl}}\left[2w_0 \times s^{1i}s^{j1}\right] = 2w_0 \frac{\partial}{\partial s_{kl}}\left[s^{1i}s^{j1}\right]$$

$$= 2w_0 w_1\left\{s^{1i}(s^{jk}s^{l1} + s^{j1}s^{k1}) + s^{j1}(s^{1k}s^{li} + s^{11}s^{ki})\right\}$$

$$\frac{\partial^2}{\partial s_{kl}\partial s_{ij}}\left[\left\{s^{11}\right\}^{\frac{1}{2}}\right] = \frac{1}{2}\left[2\left\{s^{11}\right\}^{-\frac{1}{2}} w_0 w_1\left\{s^{1i}(s^{jk}s^{l1} + s^{j1}s^{k1}) + s^{j1}(s^{1k}s^{li} + s^{11}s^{ki})\right\} - \frac{1}{2}\left\{s^{11}\right\}^{-\frac{3}{2}} \times 2w_0 s^{1i}s^{j1} \times 2w_1 s^{1k}s^{l1}\right]$$

$$= w_0 w_1\left\{s^{11}\right\}^{-\frac{3}{2}}\left[s^{11}\left\{s^{1i}(s^{jk}s^{l1} + s^{j1}s^{k1}) + s^{j1}(s^{1k}s^{li} + s^{11}s^{ki})\right\} - s^{1i}s^{j1} \times s^{1k}s^{l1}\right] \qquad (A2.33c)$$

Substitute *(A2.33a)*, *(A2.33b)* and *(A2.33c)* into *(A2.33)* gives

$$\frac{\partial^2\Phi}{\partial s_{kl}\partial s_{ij}} = -\frac{m}{2} w_0 w_1\left\{s^{11}\right\}^{-\frac{3}{2}}\phi\left(-\frac{m}{2}\left\{s^{11}\right\}^{-\frac{1}{2}}\right) \times\left[s^{11}\left\{s^{1i}(s^{jk}s^{l1} + s^{j1}s^{k1}) + s^{j1}(s^{1k}s^{li} + s^{11}s^{ki})\right\} - s^{1i}s^{j1} \times s^{1k}s^{l1} - \frac{m^2}{4} s^{11}s^{1i}s^{j1} \times s^{1k}s^{l1}\right]$$

$$= -\frac{m}{2} w_0 w_1\left\{s^{11}\right\}^{-\frac{3}{2}}\phi\left(-\frac{m}{2}\left\{s^{11}\right\}^{-\frac{1}{2}}\right) \times\left[s^{11}\left\{s^{1i}(s^{jk}s^{l1} + s^{j1}s^{k1}) + s^{j1}(s^{1k}s^{li} + s^{11}s^{ki})\right\} - (1 + \frac{m^2}{4})s^{1i}s^{j1} \times s^{1k}s^{l1}\right] \qquad (A2.34)$$

Consider cases 2A and 2B where the conditions are as follow (respectively) :

$$\boldsymbol{\mu}_1 = (m,\ldots\ldots\ldots,m)^T \qquad\qquad \boldsymbol{\mu}_1 = (m,\ldots\ldots\ldots,m)^T$$

$$\boldsymbol{\mu}_2 = (0,\ldots\ldots\ldots,0)^T \qquad \text{and} \qquad \boldsymbol{\mu}_2 = (0,\ldots\ldots\ldots,0)^T$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \vdots & & & & \vdots \\ \rho & \cdots\cdots\cdots & & \rho & 1 \end{bmatrix} \qquad\qquad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{p-1} \\ \rho & 1 & \rho & \cdots & \rho^{p-2} \\ \vdots & & & & \vdots \\ \rho^{p-1} & \cdots\cdots & & \rho & 1 \end{bmatrix}$$

Need to find $\dfrac{\partial^2\Phi}{\partial\overline{x}_{1i}\partial\overline{x}_{1j}}$ :

$$\frac{\partial^2\Phi}{\partial\overline{x}_{1i}\partial\overline{x}_{1j}} = -\frac{1}{2}\phi\left(-\frac{D}{2}\right)\left[\frac{\partial^2 D}{\partial\overline{x}_{1i}\partial\overline{x}_{1j}} - \frac{D}{4}\frac{\partial D}{\partial\overline{x}_{1i}}\frac{\partial D}{\partial\overline{x}_{1j}}\right] \tag{A2.35}$$

$$\frac{\partial D}{\partial\overline{\mathbf{x}}_1} = (\overline{\mathbf{x}}_1^T\mathbf{S}^{-1}\overline{\mathbf{x}}_1)^{-\frac{1}{2}}\mathbf{S}^{-1}\overline{\mathbf{x}}_1$$

where $\overline{\mathbf{x}}_1^T\mathbf{S}^{-1}\overline{\mathbf{x}}_1 = m^2\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv}$

$$\mathbf{S}^{-1}\overline{\mathbf{x}}_1 = m\begin{bmatrix}\sum_{u=1}^{p}s^{1u} \\ \vdots \\ \sum_{u=1}^{p}s^{pu}\end{bmatrix}$$

$$\overline{\mathbf{x}}_1^T \overline{\mathbf{x}}_1 = m^2 p$$

$$D = \frac{1}{2} m \left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{\frac{1}{2}}$$

$$\therefore \quad \frac{\partial D}{\partial \overline{\mathbf{x}}_1} = \left( m^2 \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{-\frac{1}{2}} \quad m \begin{bmatrix} \sum_{u=1}^{p} s^{1u} \\ \vdots \\ \sum_{u=1}^{p} s^{pu} \end{bmatrix} = \left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{-\frac{1}{2}} \quad \begin{bmatrix} \sum_{u=1}^{p} s^{1u} \\ \vdots \\ \sum_{u=1}^{p} s^{pu} \end{bmatrix}$$

$$\therefore \quad \frac{\partial D}{\partial \overline{x}_{1j}} = \left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{-\frac{1}{2}} \times \sum_{u=1}^{p} s^{j1} \qquad \textit{(A2.36a)}$$

and

$$\frac{\partial D}{\partial \overline{x}_{1i}} = \left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{-\frac{1}{2}} \times \sum_{u=1}^{p} s^{i1} \qquad \textit{(A2.36b)}$$

Aside:

$$\overline{\mathbf{x}}_1^T \mathbf{S}^{-1} \overline{\mathbf{x}}_1 = \sum_{v=1}^{p} [\overline{x}_{1v} ( \sum_{u=1}^{p} \overline{x}_{1u} s^{uv} )]$$

$$\mathbf{S}^{-1} \overline{\mathbf{x}}_1 = \begin{bmatrix} \sum_{u=1}^{p} \overline{x}_{1u} s^{1u} \\ \vdots \\ \sum_{u=1}^{p} \overline{x}_{1u} s^{pu} \end{bmatrix}$$

As in cases 1A and 1B and with the expressions above we have

$$\frac{\partial^2 D}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}} = \left\{ \sum_{v=1}^{p} [\overline{x}_{1v} ( \sum_{u=1}^{p} \overline{x}_{1u} s^{uv} )] \right\}^{-\frac{1}{2}} \times s^{ji}$$

$$- \left\{ \sum_{v=1}^{p} [\overline{x}_{1v} ( \sum_{u=1}^{p} \overline{x}_{1u} s^{uv} )] \right\}^{-\frac{3}{2}} \times \sum_{u=1}^{p} \overline{x}_{1u} s^{ui} \times \sum_{u=1}^{p} \overline{x}_{1u} s^{ju}$$

$$= \left( m^2 \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{-\frac{1}{2}} s^{ji} - \left( m^2 \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{-\frac{3}{2}} \times m \sum_{u=1}^{p} s^{ui} \times m \sum_{u=1}^{p} s^{ju}$$

$$= \frac{1}{m} \left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{-\frac{3}{2}} \left[ s^{ji} \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} - \sum_{u=1}^{p} s^{ui} \sum_{u=1}^{p} s^{ju} \right] \qquad (A2.37)$$

Substitute *(A2.36a)* , *(A2.36b)* and *(A2.37)* into *(A2.35)* gives $\frac{\partial^2 \Phi}{\partial \overline{x}_{1i} \partial \overline{x}_{1j}}$

$$= -\frac{1}{2} \phi \left( -\frac{1}{2} m \left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{\frac{1}{2}} \right) \left[ \frac{1}{m} \left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{-\frac{3}{2}} \left[ s^{ji} \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} - \sum_{u=1}^{p} s^{ui} \sum_{u=1}^{p} s^{ju} \right] \right.$$

$$\left. - \frac{m}{4} \left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{-\frac{1}{2}} \sum_{u=1}^{p} s^{iu} \sum_{u=1}^{p} s^{ju} \right]$$

$$= -\frac{1}{2} \phi \left( -\frac{1}{2} m \left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{\frac{1}{2}} \right) \left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{-\frac{3}{2}} \left[ \frac{1}{m} \left[ s^{ji} \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} - \sum_{u=1}^{p} s^{ui} \sum_{u=1}^{p} s^{ju} \right] \right.$$

$$\left. - \frac{m}{4} \left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right) \sum_{u=1}^{p} s^{iu} \sum_{u=1}^{p} s^{ju} \right] \qquad (A2.38)$$

Need to find $\frac{\partial^2 \Phi}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}}$ :

$$\frac{\partial^2 \Phi}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}} = -\frac{1}{2}\phi(-\frac{D}{2})\left[\frac{\partial^2 D}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}} - \frac{D}{4}\frac{\partial D}{\partial \overline{x}_{2i}}\frac{\partial D}{\partial \overline{x}_{2j}}\right] \quad (A2.39)$$

$$\frac{\partial D}{\partial \overline{\mathbf{x}}_2} = -\left[(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T \mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)\right]^{-\frac{1}{2}} \mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) \quad (A2.40)$$

where $(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T \mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = m^2 \sum_{v=1}^{p}\sum_{u=1}^{p} s^{uv}$

$$\mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = m\begin{bmatrix} \sum_{u=1}^{p} s^{1u} \\ \vdots \\ \sum_{u=1}^{p} s^{pu} \end{bmatrix}$$

$$\therefore \quad \frac{\partial D}{\partial \overline{\mathbf{x}}_2} = -\left(m^2 \sum_{v=1}^{p}\sum_{u=1}^{p} s^{uv}\right)^{-\frac{1}{2}} \times m \begin{bmatrix} \sum_{u=1}^{p} s^{1u} \\ \vdots \\ \sum_{u=1}^{p} s^{pu} \end{bmatrix} = -\left(\sum_{v=1}^{p}\sum_{u=1}^{p} s^{uv}\right)^{-\frac{1}{2}} \begin{bmatrix} \sum_{u=1}^{p} s^{1u} \\ \vdots \\ \sum_{u=1}^{p} s^{pu} \end{bmatrix}$$

$$\therefore \frac{\partial D}{\partial \overline{x}_{2j}} = -\left(\sum_{v=1}^{p}\sum_{u=1}^{p} s^{uv}\right)^{-\frac{1}{2}} \sum_{u=1}^{p} s^{ju} \text{ and } \qquad \frac{\partial D}{\partial \overline{x}_{2i}} = -\left(\sum_{v=1}^{p}\sum_{u=1}^{p} s^{uv}\right)^{-\frac{1}{2}} \sum_{u=1}^{p} s^{iu} \quad (A2.40a)$$

Aside:

$$(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)^T \mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = \sum_{v=1}^{p}\left[(m - \overline{x}_{2v})\sum_{u=2}^{p}(m - \overline{x}_{2u})s^{uv})\right] = C1$$

$$\mathbf{S}^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = \begin{bmatrix} \sum_{u=1}^{p} s^{1u}(m - \overline{x}_{2u}) \\ \vdots \\ \sum_{u=1}^{p} s^{pu}(m - \overline{x}_{2u}) \end{bmatrix}$$

Using the above expressions rewrite *(A2.40)* as:

$$\frac{\partial D}{\partial \overline{\mathbf{x}}_2} = -\left\{ \sum_{v=1}^{p} [\overline{x}_{1v}(\sum_{u=1}^{p} \overline{x}_{1u} s^{uv})] \right\}^{-\frac{1}{2}} \begin{bmatrix} \sum_{u=1}^{p} s^{1u}(m - \overline{x}_{2u}) \\ \vdots \\ \sum_{u=1}^{p} s^{pu}(m - \overline{x}_{2u}) \end{bmatrix}$$

$$\therefore \quad \frac{\partial D}{\partial \overline{x}_{2j}} = -\left\{ \sum_{v=1}^{p} [\overline{x}_{1v}(\sum_{u=1}^{p} \overline{x}_{1u} s^{uv})] \right\}^{-\frac{1}{2}} \left\{ \sum_{u=1}^{p} s^{ju}(m - \overline{x}_{2u}) \right\}$$

$$\frac{\partial^2 D}{\partial \overline{x}_{2i} \partial \overline{x}_{2j}} = \frac{\partial}{\partial \overline{x}_{2i}} \left[ -\left\{ \sum_{v=1}^{p} [\overline{x}_{1v}(\sum_{u=1}^{p} \overline{x}_{1u} s^{uv})] \right\}^{-\frac{1}{2}} \left\{ \sum_{u=1}^{p} s^{ju}(m - \overline{x}_{2u}) \right\} \right]$$

$$\begin{aligned} &= -\{C1\}^{-\frac{1}{2}} \frac{\partial}{\partial \overline{x}_{2i}} \left[ \sum_{u=1}^{p} s^{ju}(m - \overline{x}_{2u}) \right] \\ &\quad + \frac{1}{2}\{C1\}^{-\frac{3}{2}} \frac{\partial}{\partial \overline{x}_{2i}}[C1] \times \left\{ \sum_{u=1}^{p} s^{ju}(m - \overline{x}_{2u}) \right\} \\ &= \{C1\}^{-\frac{1}{2}} s^{ji} - \{C1\}^{-\frac{3}{2}} \times \sum_{u=1}^{p}(m - \overline{x}_{2u}) s^{ui} \times \left\{ \sum_{u=1}^{p} s^{ju}(m - \overline{x}_{2u}) \right\} \\ &= \frac{1}{m}\left\{ \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right\}^{-\frac{1}{2}} s^{ji} - \frac{1}{m}\left\{ \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right\}^{-\frac{3}{2}} \times \sum_{u=1}^{p} s^{ui} \times \sum_{u=1}^{p} s^{ju} \\ &= \frac{1}{m}\left\{ \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right\}^{-\frac{3}{2}} \left[ s^{ji} \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} - \sum_{u=1}^{p} s^{ui} \times \sum_{u=1}^{p} s^{ju} \right] \end{aligned} \qquad (A2.41)$$

Substitute *(A2.40a)* , *(A2.40b)* and *(A2.41)* gives $\dfrac{\partial^2\Phi}{\partial\overline{x}_{2i}\partial\overline{x}_{2j}}$

$$= -\frac{1}{2}\phi\left(-\frac{1}{2}m\left(\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv}\right)^{\frac{1}{2}}\right)\left[\frac{1}{m}\left(\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv}\right)^{-\frac{3}{2}}\left[s^{ji}\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv} - \sum_{u=1}^{p}s^{ui}\sum_{u=1}^{p}s^{ju}\right]\right.$$

$$\left. - \frac{m}{4}\left(\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv}\right)^{-\frac{1}{2}}\sum_{u=1}^{p}s^{iu}\sum_{u=1}^{p}s^{ju}\right]$$

$$= -\frac{1}{2}\phi\left(-\frac{1}{2}m\left(\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv}\right)^{\frac{1}{2}}\right)\left(\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv}\right)^{-\frac{3}{2}}\left[\frac{1}{m}\left[s^{ji}\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv} - \sum_{u=1}^{p}s^{ui}\sum_{u=1}^{p}s^{ju}\right]\right.$$

$$\left. - \frac{m}{4}\left(\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv}\right)\sum_{u=1}^{p}s^{iu}\sum_{u=1}^{p}s^{ju}\right] \qquad (A2.42)$$

Note that $\dfrac{\partial^2\Phi}{\partial\overline{x}_{1i}\partial\overline{x}_{1j}} = \dfrac{\partial^2\Phi}{\partial\overline{x}_{2i}\partial\overline{x}_{2j}}$ .

Now need to find $\dfrac{\partial^2\Phi}{\partial s_{kl}\,\partial s_{ij}}$:

$$D = m[\sum_{u=1}^{p}\sum_{v=1}^{p}s^{uv}]^{1/2}$$

$$\frac{\partial^2\Phi}{\partial s_{kl}\,\partial s_{ij}} = \frac{\partial}{\partial s_{kl}}\left[\frac{\partial\Phi}{\partial s_{ij}}\right]$$

$$= \frac{\partial}{\partial s_{kl}}\left[-\frac{m}{2}\frac{\partial}{\partial s_{ij}}\left(\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv}\right)^{\frac{1}{2}} \times \phi\left(-\frac{m}{2}\left(\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv}\right)^{\frac{1}{2}}\right)\right]$$

$$= -\frac{m}{2}\left[ \phi\left( -\frac{m}{2}\left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{\frac{1}{2}} \right) \frac{\partial^2}{\partial s_{kl}\, \partial s_{ij}} \left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{\frac{1}{2}} \right.$$

$$\left. + \frac{\partial}{\partial s_{ij}} \left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{\frac{1}{2}} \times \frac{\partial}{\partial s_{kl}} \phi\left( -\frac{m}{2}\left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{\frac{1}{2}} \right) \right] \qquad \textit{(A2.43)}$$

where

(1) $$\frac{\partial}{\partial s_{ij}} \left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{\frac{1}{2}} = \frac{1}{2}\left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{-\frac{1}{2}} \frac{\partial}{\partial s_{ij}} \left[ \sum_{u=1}^{p} \sum_{v=1}^{p} s^{uv} \right]$$

$$= \frac{1}{2}\left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{-\frac{1}{2}} \times w_0 \times \sum_{v=1}^{p} \sum_{u=1}^{p} (s^{ui} s^{jv} + s^{uj} s^{iv})$$

*expression (A2.44)*

(2) $$\frac{\partial}{\partial s_{kl}} \phi\left( -\frac{m}{2}\left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{\frac{1}{2}} \right)$$

$$= -\frac{m^2}{8} \frac{\partial}{\partial s_{kl}} \left[ \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right] \times \phi\left( -\frac{m}{2}\left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{\frac{1}{2}} \right)$$

$$= -\frac{m^2}{8} \phi\left( -\frac{m}{2}\left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{\frac{1}{2}} \right) \times w_1 \times \sum_{v=1}^{p} \sum_{u=1}^{p} (s^{uk} s^{lv} + s^{ul} s^{kv}) \qquad \textit{(A2.45)}$$

(3) $$\frac{\partial^2}{\partial s_{kl}\, \partial s_{ij}} \left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{\frac{1}{2}}$$

$$= \frac{\partial}{\partial s_{kl}} \left[ \frac{\partial}{\partial s_{ij}} \left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{\frac{1}{2}} \right]$$

$$= \frac{\partial}{\partial s_{kl}} \left[ \frac{1}{2}\left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right)^{-\frac{1}{2}} \frac{\partial}{\partial s_{ij}} \left( \sum_{v=1}^{p} \sum_{u=1}^{p} s^{uv} \right) \right]$$

$$= \frac{1}{2}\left[\left(\sum_{v=1}^{p}\sum_{u=1}^{p} s^{uv}\right)^{-\frac{1}{2}} \frac{\partial^2}{\partial s_{kl}\,\partial s_{ij}}\left[\sum_{v=1}^{p}\sum_{u=1}^{p} s^{uv}\right] + \frac{\partial}{\partial s_{kl}}\left(\sum_{v=1}^{p}\sum_{u=1}^{p} s^{uv}\right)^{-\frac{1}{2}} \frac{\partial}{\partial s_{ij}}\left(\sum_{v=1}^{p}\sum_{u=1}^{p} s^{uv}\right)\right] \tag{A2.46}$$

where

$$\frac{\partial}{\partial s_{ij}}\left(\sum_{u=1}^{p}\sum_{v=1}^{p} s^{uv}\right) = w_0 \sum_{u=1}^{p}\sum_{v=1}^{p}\left(s^{ui}s^{jv} + s^{uj}s^{iv}\right) \tag{A2.46a}$$

$$w_0 = \begin{cases} -0.5 & \text{if } i = j \\ -1 & \text{if } i \neq j \end{cases}$$

$$\frac{\partial}{\partial s_{kl}}\left(\sum_{u=1}^{p}\sum_{v=1}^{p} s^{uv}\right) = w_1 \sum_{u=1}^{p}\sum_{v=1}^{p}\left(s^{uk}s^{lv} + s^{ul}s^{kv}\right)$$

$$w_1 = \begin{cases} -0.5 & \text{if } k = l \\ -1 & \text{if } k \neq l \end{cases}$$

$$\begin{aligned} &\frac{\partial^2}{\partial s_{kl}\,\partial s_{ij}}\left[\sum_{u=1}^{p}\sum_{v=1}^{p} s^{uv}\right] \\ &= \frac{\partial}{\partial s_{kl}}\left[\frac{\partial}{\partial s_{ij}}\left[\sum_{u=1}^{p}\sum_{v=1}^{p} s^{uv}\right]\right] \\ &= \frac{\partial}{\partial s_{kl}}\left[w_0 \sum_{u=1}^{p}\sum_{v=1}^{p}\left(s^{ui}s^{jv} + s^{uj}s^{iv}\right)\right] \\ &= w_0 w_1 \sum_{u=1}^{p}\sum_{v=1}^{p}\Big[s^{ui}\left(s^{jk}s^{lv} + s^{jl}s^{kv}\right) + s^{jv}\left(s^{uk}s^{li} + s^{ul}s^{ki}\right) \\ &\qquad + s^{uj}\left(s^{ik}s^{lv} + s^{il}s^{kv}\right) + s^{iv}\left(s^{uk}s^{lj} + s^{ul}s^{kj}\right)\Big] \end{aligned} \tag{A2.46b}$$

$$\frac{\partial}{\partial s_{kl}}\left(\sum_{u=1}^{p}\sum_{v=1}^{p} s^{uv}\right)^{-\frac{1}{2}} = -\frac{w_1}{2}\left(\sum_{u=1}^{p}\sum_{v=1}^{p} s^{uv}\right)^{-\frac{3}{2}} \times \sum_{v=1}^{p}\sum_{u=1}^{p}\left(s^{uk}s^{lv} + s^{ul}s^{kv}\right) \tag{A2.46c}$$

Therefore substitute *(A2.46a)* , *(A2.46b)* and *(A2.46c)* into *(A2.46)* gives

$$\frac{\partial^2}{\partial s_{kl}\partial s_{ij}}\left(\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv}\right)^{\frac{1}{2}}$$

$$=\frac{1}{2}w_0\,w_1\,(\sum_{u=1}^{p}\sum_{v=1}^{p}s^{uv})^{-3/2}\times\Bigg[(\sum_{u=1}^{p}\sum_{v=1}^{p}s^{uv})\sum_{u=1}^{p}\sum_{v=1}^{p}[s^{ui}(s^{jk}s^{lv}+s^{jl}s^{kv})$$
$$+s^{jv}(s^{uk}s^{li}+s^{ul}s^{ki})+s^{uj}(s^{ik}s^{lv}+s^{il}s^{kv})+s^{iv}(s^{uk}s^{lj}+s^{ul}s^{kj})]$$
$$-(\frac{1}{2}\sum_{u=1}^{p}\sum_{v=1}^{p}(s^{ui}s^{jv}+s^{uj}s^{iv})\sum_{u=1}^{p}\sum_{v=1}^{p}(s^{uk}s^{lv}+s^{ul}s^{kv}))\Bigg] \qquad (A2.47)$$

Substitute *(A2.44)* , *(A2.45)* and *(A2.47)* into *(A2.43)* gives $\frac{\partial^2\Phi}{\partial s_{kl}\partial s_{ij}}$

$$=-\frac{m}{4}w_0w_1\left\{\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv}\right\}^{-\frac{3}{2}}\phi\left(-\frac{m}{2}\left(\sum_{v=1}^{p}\sum_{u=1}^{p}s^{uv}\right)^{\frac{1}{2}}\right)$$
$$\times\Bigg[\Bigg(\{(\sum_{u=1}^{p}\sum_{v=1}^{p}s^{uv})\sum_{u=1}^{p}\sum_{v=1}^{p}[s^{ui}(s^{jk}s^{lv}+s^{jl}s^{kv})+s^{jv}(s^{uk}s^{li}+s^{ul}s^{ki})$$
$$+s^{uj}(s^{ik}s^{lv}+s^{il}s^{kv})+s^{iv}(s^{uk}s^{lj}+s^{ul}s^{kj})]\}$$
$$-\{\frac{1}{2}\left(1+\frac{m^2}{8}\sum_{u=1}^{p}\sum_{v=1}^{p}s^{uv}\right)\times\sum_{u=1}^{p}\sum_{v=1}^{p}(s^{ui}s^{jv}+s^{uj}s^{iv})\times\sum_{u=1}^{p}\sum_{v=1}^{p}(s^{uk}s^{lv}+s^{ul}s^{kv})\}\Bigg)\Bigg]$$

*expression (A2.48)*

# *Appendix A3*

# *COMPUTER PROGRAM*

## *Euclidean distance classifier*

## ***Actual error rate***

---

Function to find $y = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$

---

```
function y=cf(x)

y=exp(-0.5*(x.^2))/sqrt(2.*pi);
```

---

Function to find $cdf = \Phi(x)$

---

```
function cdf=cdff(x)

if x > 0
        cdf=0.5+quad('cf',0,x);
        end
else if x < 0
        temp1=abs(x);
        cdf=0.5-quad('cf',0,temp1);
        end
else if x==0
        cdf=quad('cf',0,x);
        end;
```

---

Function to find $p = \phi(x)$

---

```
function p=pdff(x)

temp=-1*(x.^2)/2;
p=exp(temp)/sqrt(2*pi);
```

---

Main program to find (for cases 1A and 1B)

$$\text{rate} = \Phi(\cdot) + \frac{1}{2n_1}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}\sigma_{ij} + \frac{1}{2n_2}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}\sigma_{ij}$$

$$= \text{cdf} + \frac{1}{2}(\text{ndx1}) + \frac{1}{2}(\text{ndx2})$$

---

```
function rate=eratef(x1,x2,s,p,n1,n2)

m=x1(1,1);
cdf=cdff(-0.5*m);
tx1=0.5*ndx1(x1,s,p,n1)
tx2=0.5*ndx2(x1,s,p,n2)
rate=cdf+tx1+tx2;
```

---

Function to find (for cases 1A and 1B) $\text{ndx1} = \frac{1}{n_1}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}\sigma_{ij}$

---

```
function d1=ndx1(x1,s,p,n1);

m=x1(1,1);
pdf=pdff(-0.5*m);
c1=1/(2*m);
d1=0;
```

```
for i=1:p
        for j=1:p
                temp1=s(1,i)*s(j,1);
                if i==j
                        temp2=s(j,i)-(3*temp1)+2;
                else
                        temp2=s(j,i)-(3*temp1);end;
                d=(-1*c1*temp2)-(temp1*m/8);
                temp=-1*pdf*d*s(i,j)/n1;
                d1=d1+temp;
        end;
end;
```

---

Function to find (for cases 1A and 1B) $ndx2 = \frac{1}{n_2}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}\sigma_{ij}$

---

```
function d2=ndx2(x1,s,p,n2);

m=x1(1,1);
pdf=pdff(-0.5*m);
c1=1/m;
c2=m/8;
d2=0;
for i=1:p
        for j=1:p
                temp1=3*s(1,i)*s(j,1);
                if (i==1) & (j==1)
                  temp2=((-3*s(j,1))+temp1)/2;
                  d=(c1*(temp2+1-s(1,i)))-c2;
                elseif (i==1) & (j~=1) & (i~=j)
                  temp2=((-3*s(j,1))+temp1)/2;
                  d=(c1*temp2)+(c2*s(j,1));
                elseif (i~=1) & (j==1) & (i~=j)
                  temp2=((-1*s(j,i))+temp1)/2;
                  d=(c1*(temp2-s(1,i)))+(c2*s(i,1));
                elseif (i~=1) & (j~=1) & (i==j)
                  temp2=((-1*s(j,i))+temp1)/2;
                  d=(c1*(temp2+1))-(c2*s(1,i)*s(j,1));
             else
                  temp2=((-1*s(j,i))+temp1)/2;
                  d=(c1*temp2)-(s(1,i)*s(j,1)*c2);
                end;
                temp=-1*pdf*d*s(i,j)/n2;
                d2=d2+temp;
        end;
end;
```

---

Main program to find (for cases 2A and 2B)

$$\text{rate} = \Phi(\cdot) + \frac{1}{2n_1}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}\sigma_{ij} + \frac{1}{2n_2}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}\sigma_{ij}$$

$$= \text{cdf} + \frac{1}{2}(\text{edx1}) + \frac{1}{2}(\text{edx2})$$

---

```
function rate=eratef2(x1,x2,s,p,n1,n2)

m=x1(1,1);
ssum=sum(sum(s));
a=0.5*m*p*(ssum^(-0.5));
cdf=cdff(-1*a);
tx1=0.5*edx1(x1,s,p,n1)
tx2=0.5*edx2(x1,s,p,n2)
rate=cdf+tx1+tx2;
```

---

Function to find (for cases 2A and 2B) $\text{edx1} = \frac{1}{n_1}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}\sigma_{ij}$

---

```
function d1=edx1(x1,s,p,n1);

m=x1(1,1);
ssum=sum(sum(s));
a=m*p*(ssum^(-0.5));
pdf=pdff(-0.5*a);
prod1=ssum^(-3.5);
tsum=sum(s);
c1=ssum/(2*m);
d1=0;
for i=1:p
        for j=1:p
        temp1=m*p*tsum(1,j)*tsum(1,i)/8;
        temp2=(ssum*s(j,i))-(3*tsum(1,i)*tsum(1,j));
        if (i==j)
                temp4=(p*temp2)+(2*(ssum^2));
                d=(c1*temp4)+temp1;
        else
                d=(c1*p*temp2)+temp1;end;
        temp=pdf*prod1*d*s(i,j)/n1;
        d1=d1+temp;
        end;
end;
```

---

Function to find (for cases 2A and 2B) $\mathrm{edx2} = \frac{1}{n_2}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}\sigma_{ij}$

---

```
function d2=edx2(x1,s,p,n2);

m=x1(1,1);
ssum=sum(sum(s));
a=m*p*(ssum^(-0.5));
pdf=pdff(-0.5*a);
tsum=sum(s);
prod1=ssum^(-3.5);
c1=ssum/m;
d2=0;
for i=1:p
        temp4=(0.5*tsum(1,i))-ssum;
        for j=1:p
                temp5=(0.5*tsum(1,j))-ssum;
                temp1=-1*ssum*((2*tsum(1,j))+(p*s(j,i)));
                temp2=3*p*tsum(1,i)*tsum(1,j);
                temp3=m*p*temp4*temp5;
                if (i==j)
                  temp6=2*ssum*(ssum-tsum(1,i));
                else
                  temp6=-2*ssum*tsum(1,i);
                end;
                d=(c1*(temp1+temp2+temp6))-temp3;
                temp=-0.5*pdf*prod1*d*s(i,j)/n2;
                d2=d2+temp;
        end;
end;
```

# *Linear discriminant function*

# *Actual error rate*

---

Function to find $y = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$

---

```
function y=cf(x)

y=exp(-0.5*(x.^2))/sqrt(2.*pi);
```

---

Function to find $cdf = \Phi(x)$

---

```
function cdf=cdff(x)

if x > 0
        cdf=0.5+quad('cf',0,x);
        end
else if x < 0
        temp1=abs(x);
        cdf=0.5-quad('cf',0,temp1);
        end
else if x==0
        cdf=quad('cf',0,x);
        end;
```

---

Function to find $p = o(x)$

---

```
function p=pdff(x)

temp=-1*(x.^2)/2;
p=exp(temp)/sqrt(2*pi);
```

---

Function to find (for cases 1A and 1B) $ndx1 = \frac{1}{n_1}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}\sigma_{ij}$

---

```
function d1=ndx1(x1,s,p,n1)
m=x1(1,1);
si=s^(-1);
cs=si*s*si;
c1=x1'*si*s*si*x1;
c2=c1/(m^2);
c3=x1'*si*x1;
a=0.5*c3*(c1^(-0.5));
pdf=pdff(-1*a);
d1=0;
for i=1:p
        d1ai=-0.5*si(1,1)*cs(i,1)*(c2^(-1.5));
        temp4=cs(1,i)+cs(i,1);
        for j=1:p
                d1aj=-0.5*si(1,1)*cs(j,1)*(c2^(-1.5));
                temp1=-0.5*si(1,1)*(c2^(-1.5))*cs(j,i)/m;
                temp2=si(1,i)*(c2^(-1.5))*cs(j,1)/m;
                temp3=3*si(1,1)*(c2^(-2.5))*cs(j,1)/(m*4);
                da=temp1+temp2+(temp3*temp4);
                db=-1*si(j,i)*(c1^(-0.5));
                d=da+db-(a*d1ai*d1aj);
                temp=-1*pdf*d*s(i,j)/n1;
                d1=d1+temp;
        end;
end;
```

---

Function to find (for cases 1A and 1B) $ndx2 = \frac{1}{n_2}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}\sigma_{ij}$

---

```
function d2=ndx2(x1,s,p,n2)
m=x1(1,1);
si=s^(-1);
cs=si*s*si;
c1=x1'*si*s*si*x1;
c2=c1/(m^2);
c3=x1'*si*x1;
a=0.5*c3*(c1^(-0.5));
pdf=pdff(-1*a);
d2=0;
```

```
for i=1:p
        if i==1
                temp1=-2*m;
        else
                temp1=-1*s(i,1);end;
        d1ai=(0.5*si(1,1)*cs(i,1)*(c2^(-1.5)))-((c2^(-0.5))*si(i,1));
        for j=1:p
                d1aj=(0.5*si(1,1)*cs(j,1)*(c2^(-1.5)))-((c2^(-0.5))*si(j,1));
                if i==1
                        temp2=cs(j,1)+cs(j,i);
                else
                        temp2=cs(j,i);end;
                temp3=0.5*m*(c1^(-1.5))*cs(j,1)*temp1;
                temp4=-0.5*m^2*(c1^(-1.5))*temp2;
                temp5=3*m^4*cs(j,1)*cs(i,1)/2;
                da=temp3+temp4+temp5;
                temp6=c1^(-0.5)*si(j,i);
                temp7=(m^2)*si(j,1)*(c1^(-1.5))*cs(i,1);
                db=temp6-temp7;
                d=da+db-(a*d1ai*d1aj);
                temp=-1*pdf*d*s(i,j)/n2;
                d2=d2+temp;
        end;
end;
```

---

Function to find D in case 1A (used in the main program).

---

```
function d1=getd1(xbs,s,m)
si=s^(-1);
temp1=m/2;
temp2=temp1*si(1,1);
d1=temp2*(xbs^(-0.5));
```

---

The following 4 functions are for case 1A (used in the function diff2s).

---

```
function ssum=ssum1a(s,v,p)

si=s^(-1);
sum1=0;
for u=1:p
        if u~=v
        sum1=sum1+si(1,u);
        end;
end;
ssum=sum1;
```

```
function bsum=bigsum(s,p,rho);
si=s^(-1);
bsum=0;
for v=1:p
        ssum=ssum1a(s,v,p);
        temp1=rho*ssum;
        temp2=temp1+si(1,v);
        temp3=si(v,1)*temp2;
        bsum=bsum+temp3;
end;

function f=getf(s,p,i,j,k,l)

si=s^(-1);
rho=s(1,2);
temp12=(si(1,i)*si(j,k))+(si(1,j)*si(i,k));
temp13=(si(1,k)*si(l,i))+(si(1,l)*si(k,i));
temp14=(si(1,i)*si(j,l))+(si(1,j)*si(i,l));
temp15=(si(1,k)*si(l,j))+(si(1,l)*si(k,j));
f=0;
for v=1:p
        sumt1=0;
        sumt3=0;
        sumt4=0;
        sumt5=0;
        for u=1:p
          if u~=v
                temp6=si(l,u)*temp12;
                temp7=si(j,u)*temp13;
                temp8=si(k,u)*temp14;
                temp9=si(i,u)*temp15;
                sumt3=sumt3+temp6+temp7+temp8+temp9;
                sumt1=sumt1+(si(1,i)*si(j,u))+(si(1,j)*si(i,u));
                sumt4=sumt4+(si(1,k)*si(l,u))+(si(1,l)*si(k,u));
                sumt5=sumt5+si(1,u);
          end;
        end;
        temp1=(si(1,i)*si(j,v))+(si(1,j)*si(i,v));
        temp2=si(l,v)*temp12;
        temp3=si(j,v)*temp13;
        temp4=si(k,v)*temp14;
        temp5=si(i,v)*temp15;
        sumt2=temp2+temp3+temp4+temp5;
        sum1=si(v,1)*rho*((temp1*sumt3)+(sumt2*sumt1));
        temp10=(si(v,k)*si(l,1))+(si(v,l)*si(k,1));
        sum2=temp10+temp1*rho*sumt1;
        temp11=si(1,v)+(rho*sumt5);
        sum3=temp11*sumt2;
        sum4=temp1*(temp10+(rho*sumt4));
        f=f+sum1+sum2+sum3+sum4;
end;
```

```
function f26=getf26(s,p,i,j)

si=s^(-1);
rho=s(1,2);
f26=0;
for v=1:p
        temp1=(si(1,i)*si(j,v))+(si(1,j)*si(i,v));
        temp3=(si(v,i)*si(j,1))+(si(v,j)*si(i,1));
        sumt1=0;
        sumt2=0;
        for u=1:p
                if u~=v
                        sumt1=sumt1+(si(1,i)*si(j,u))+(si(1,j)*si(i,u));
                        sumt2=sumt2+si(1,u);
                end;
        end;
        temp2=temp1+(rho*sumt1);
        sum1=si(v,1)*temp2;
        temp4=si(1,v)+(rho*sumt2);
        sum2=temp4*temp3;
        f26=f26+sum1+sum2;
end;
```

---

Function to find (for case 1A) $\text{nds} = \frac{n_1 + n_2}{(n_1 + n_2 - 2)^2} \sum_{k,l,i,j=1}^{p} \frac{\partial^2 \Phi}{\partial s_{kl} \partial \bar{x}_{ij}} (\sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk})$

---

```
function d=diff2s(x1,s,p,n1,n2)

si=s^(-1);
m=x1(1,1);
rho=s(1,2);
f1=0;
for v=1:p
        sum1=0;
        for u=1:p
                if u~=v
                        sum1=sum1+si(1,u);
                end;
        temp1=sum1*rho;
        f1=f1+(temp1+si(1,v))*si(v,1);
        end;
end;
temp=-0.5*m*si(1,1)*f1^(-0.5);
pdf=pdff(temp);
temp1=-0.5*pdf*f1^(-3.5);
temp2=f1^2;
temp3=-0.5*si(1,1);
temp4=(n1+n2)/((n1+n2-2)^2);
sum3=0;
```

```
sum4=0;
for k=1:p
   for l=1:p
    if k<=l
        if k==l
          w1=-0.5;
        else
          w1=-1;
        end;
        f4=si(1,k)*si(l,1);
        f6=getf26(s,p,k,l);
    end;
        for i=1:p
           for j=1:p
                if (i<=j) & (k<=l)
                        if i==j
                                w0=-0.5;
                        else
                                w0=-1;
                        end;
                f=getf(s,p,i,j,k,l);
                f2=getf26(s,p,i,j);
                f3=si(1,i)*si(j,1);
                temp9=f1*f3;
                s1=si(1,i)*((si(j,k)*si(l,1))+(si(j,l)*si(k,1)));
                s2=si(j,1)*((si(1,k)*si(l,i))+(si(1,l)*si(k,i)));
                f5=s1+s2;
                sum11=temp2*((temp3*f)-(f4*f2)+(f3*f6)+(f5*f1));
                sum12=-1.5*f1*f6*((temp3*f2)+temp9);
                sum1=sum11+sum12;
                sum21=(temp3*f2)+temp9;
                sum22=(temp3*f6)+(f4*f1);
                sum2=-0.25*(m^2)*si(1,1)*sum21*sum22;
                ds=temp1*w0*w1*(sum1+sum2);
                temp5=(s(i,k)*s(j,l))+(s(i,l)*s(j,k));
                cv=temp4*temp5;
                if i==j
                        temp6=ds*cv;
                else
                        temp6=2*ds*cv;
                end;
                if k==l
                        sum3=sum3+temp6;
                end;
                sum4=sum4+temp6;
                end;
                end;
        end;
        end;
        end;
sum5=sum4-sum3;
d=(2*sum5)+sum3;
```

---

Main program to find (for case 1A)

$$\text{rate} = \Phi(\cdot) + \frac{1}{2n_1}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}\sigma_{ij} + \frac{1}{2n_2}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}\sigma_{ij} + \frac{n_1+n_2}{2(n_1+n_2-2)^2}\sum_{k,l,i,j=1}^{p}\frac{\partial^2\Phi}{\partial s_{kl}\partial\bar{x}_{ij}}(\sigma_{ik}\sigma_{jl}+\sigma_{il}\sigma_{jk})$$

$$= \text{cdf} + \frac{1}{2}(\text{ndx1}) + \frac{1}{2}(\text{ndx2}) + \frac{1}{2}(\text{nds})$$

---

```
function rate=eratef1(x1,x2,s,p,n1,n2)
rho=s(1,2);
m=x1(1,1);
xbs=bigsum1a(s,p,rho);
d=getd1a(xbs,s,m)
xx=-1*d;
cdf=cdff(xx)
pdf=pdff(xx)
tx1=0.5*ndx1(x1,s,p,n1);
tx2=0.5*ndx2(x1,s,p,n2);
ts=0.5*diff2s(x1,s,p,n1,n2)
rate=cdf+tx1+tx2+ts;
```

---

Function to find D in case 1B (used in the main program).

---

```
function d1b=getd1b(xbs,m,s)
si=s^(-1);
temp1=0.5*m*si(1,1);
temp3=xbs^(-0.5);
d1b=temp1*temp3;
```

---

The following 4 functions are for case 1B (used in function diff2s2).

---

```
function ssum=ssum1b(p,u,s,rho)
si=s^(-1);
ssum=0;
for v=1:p
        power=abs(v-u);
        temp1=rho^(power);
        temp2=si(1,v)*temp1;
        ssum=ssum+temp2;
end;
```

```
function bsum=bigsum1(s,p,rho)

si=s^(-1);
bsum=0;
for u=1:p
        ssum=ssum1b(p,u,s,rho);
        temp=si(u,1)*ssum;
        bsum=bsum+temp;
end;


function e=gete(x1,s,p,i,j,k,l);

si=s^(-1);
rho=s(1,2);
temp30=si(l,i);
temp31=(si(l,k)*si(l,i))+(si(l,l)*si(k,i));
temp32=si(l,j);
temp33=(si(l,k)*si(l,j))+(si(l,l)*si(k,j));
temp22=(si(j,k)*si(l,1))+(si(j,l)*si(k,1));
temp23=(si(i,k)*si(l,1))+(si(i,l)*si(k,1));
temp24=(si(j,1)*si(l,i))+(si(i,1)*si(l,j));
temp25=(si(j,1)*si(k,i))+(si(i,1)*si(k,j));
e=0;
for u=1:p
   sumt1=0;
   for v=1:p
        temp1=temp30*((si(j,k)*si(l,v))+(si(j,l)*si(k,v)));
        temp2=si(j,v)*temp31;
        temp3=temp32*((si(i,k)*si(l,v))+(si(i,l)*si(k,v)));
        temp4=si(i,v)*temp33;
        temp5=abs(v-u);
        temp6=rho^(temp5);
        temp7=temp6*(temp1+temp2+temp3+temp4);
        sumt1=sumt1+temp7;
   end;
   sum1=si(u,1)*sumt1;
   temp8=(si(u,k)*si(l,1))+(si(u,l)*si(k,1));
   sumt2=0;
   sumt3=0;
   sumt4=0;
   for v=1:p
        temp9=(si(1,i)*si(j,v))+(si(1,j)*si(i,v));
        temp10=abs(v-u);
        temp11=rho^(temp10);
        sumt2=sumt2+(temp9*temp11);
        sumt3=sumt3+(si(1,v)*temp11);
        temp20=(si(1,k)*si(l,v))+(si(1,l)*si(k,v));
        sumt4=sumt4+(temp20*temp11);
   end;
   sum2=temp8*sumt2;
   temp13=si(u,i)*temp22;
   temp14=si(u,j)*temp23;
   temp15=si(u,k)*temp24;
```

```
   temp16=si(u,1)*temp25;
   temp17=temp13+temp14+temp15+temp16;
   sum3=temp17*sumt3;
   temp19=(si(u,i)*si(j,1))+(si(u,j)*si(i,1));
   sum4=temp19*sumt4;
   e=e+sum1+sum2+sum3+sum4;
end;
```

```
function e24=gete24(s,p,i,j)
si=s^(-1);
rho=s(1,2);
e24=0;
for u=1:p
        sumt1=0;
        sumt2=0;
        for v=1:p
                temp1=(si(1,i)*si(j,v))+(si(1,j)*si(i,v));
                temp2=rho^(abs(v-u));
                sumt1=sumt1+(temp1*temp2);
                sumt2=sumt2+(temp2*si(1,v));
        end;
        sum1=si(u,1)*sumt1;
        temp3=(si(u,i)*si(j,1))+(si(u,j)*si(i,1));
        sum2=temp3*sumt2;
        e24=e24+sum1+sum2;
end;
```

---

Function to find (for case 1B) $nds = \frac{n_1+n_2}{(n_1+n_2-2)^2}\sum_{k,l,i,j=1}^{p}\frac{\partial^2\Phi}{\partial s_{kl}\partial\bar{x}_{ij}}(\sigma_{ik}\sigma_{jl}+\sigma_{il}\sigma_{jk})$

---

```
function d=diff2s2(x1,s,p,n1,n2)
si=s^(-1);
m=x1(1,1);
rho=s(1,2);
e1=0;
sum1=0;
for u=1:p
        for v=1:p
                temp1=abs(v-u);
                temp2=si(1,v)*(rho^(temp1));
                sum1=sum1+temp2;
        end;
        e1=e1+(sum1*si(u,1));
end;
temp=-0.5*m*si(1,1)*e1^(-0.5);
pdf=pdff(temp);
temp1=-0.5*m*pdf*e1^(-3.5);
temp2=e1^2;
```

```
temp3=-0.5*si(1,1);
temp5=(n1+n2)/((n1+n2-2)^2);
sum3=0;
sum4=0;
for k=1:p
   for l=1:p
    if k <= l
        if k==l
          w1=-0.5;
        else
          w1=-1;end;
        e5=si(1,k)*si(l,1);
        e4=gete24(s,p,k,l);
     end;
          for i=1:p
             for j=1:p
                if ( i<=j ) & ( k<=l )
                if i==j
                  w0=-0.5;
                else
                  w0=-1;
                end;
                e=gete(x1,s,p,i,j,k,l);
                e2=gete24(s,p,i,j);
                s1=si(1,i)*((si(j,k)*si(l,1))+(si(j,l)*si(k,1)));
                s2=si(j,1)*((si(1,k)*si(l,i))+(si(1,l)*si(k,i)));
                e6=s1+s2;
                e3=si(1,i)*si(j,1);
                e7=si(1,i)*si(j,1);
                sum11=temp2*((2*e1*e6)+(2*e3*e4)+(temp3*e)-(e2*e5));
                sum12=-1.5*e1*e4*((temp3*e2)+(2*e1*e7));
                sum1=sum11+sum12;
                sum21=(temp3*e2)+(2*e3*e1);
                sum22=(temp3*e4)+(2*e5*e1);
                sum2=-0.25*(m^2)*si(1,1)*sum21*sum22;
                ds=temp1*w0*w1*(sum1+sum2);
                temp4=(s(i,k)*s(j,l))+(s(i,l)*s(j,k));
                cv=temp4*temp5;
                if i==j
                  temp6=ds*cv;
                else
                  temp6=2*ds*cv;end;
                if k==l
                  sum3=sum3+temp6;
                end;
                sum4=sum4+temp6;
                end;
        end;
    end;
  end;
end;
sum5=sum4-sum3;
d=(2*sum5)+sum3;
```

---

Main program to find (for case 1B)

$$\text{rate} = \Phi(\cdot) + \frac{1}{2n_1}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}\sigma_{ij} + \frac{1}{2n_2}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}\sigma_{ij} + \frac{n_1+n_2}{2(n_1+n_2-2)^2}\sum_{k,l,i,j=1}^{p}\frac{\partial^2\Phi}{\partial s_{kl}\partial\bar{x}_{ij}}(\sigma_{ik}\sigma_{jl}+\sigma_{il}\sigma_{jk})$$

$$= \text{cdf} + \frac{1}{2}(\text{ndx1}) + \frac{1}{2}(\text{ndx2}) + \frac{1}{2}(\text{nds})$$

---

```
function rate=eratef2(x1,x2,s,p,n1,n2)
rho=s(1,2);
m=x1(1,1);
xbs=bigsum1b(s,p,rho);
d=getd1b(xbs,m,s)
xd=-1*d;
cdf=cdff(xd)
pdf=pdff(xd)
tx1=0.5*ndx1(x1,x2,s);
tx2=0.5*ndx2(x1,x2,s);
ts=0.5*diff2s2(x1,s,p,n1,n2)
rate=cdf+tx1+tx2+ts;
```

# *Euclidean distance classifier*

## ***Plug-in error rate***

---

Function to find $y = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)$

---

```
function y=cf(x)

y=exp(-0.5*(x.^2))/sqrt(2.*pi);
```

---

Function to find $cdf = \Phi(x)$

---

```
function cdf=cdff(x)

if x > 0
        cdf=0.5+quad('cf',0,x);
        end
else if x < 0
        temp1=abs(x);
        cdf=0.5-quad('cf',0,temp1);
        end
else if x==0
        cdf=quad('cf',0,x);
        end;
```

---

Function to find $p = \phi(x)$

---

```
function p=pdff(x)

temp=-1*(x.^2)/2;
p=exp(temp)/sqrt(2*pi);
```

---

Main program to find (for cases 1A and 1B)

$$\text{rate} = \Phi(\cdot) + \frac{1}{2n_1}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}\sigma_{ij} + \frac{1}{2n_2}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}\sigma_{ij} + \frac{n_1+n_2}{2(n_1+n_2-2)^2}\sum_{k,l,i,j=1}^{p}\frac{\partial^2\Phi}{\partial s_{kl}\partial\bar{x}_{ij}}(\sigma_{ik}\sigma_{jl}+\sigma_{il}\sigma_{jk})$$

$$= \text{cdf} + \frac{1}{2}(\text{ndx1}) + \frac{1}{2}(\text{ndx2}) + \frac{1}{2}(\text{nds})$$

---

```
function rate=eratef(x1,x2,s,p,n1,n2)

m=x1(1,1);
temp1=-0.5*m;
cdf=cdff(temp1)
tx1=0.5*dx1(x1,s,p,n1)
tx2=0.5*dx2(x1,s,p,n2)
ts=0.5*nds(x1,s,n1,n2)
rate=cdf+tx1+tx2+ts;
```

---

Function to find (for cases 1A and 1B) $\text{ndx1} = \frac{1}{n_1}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}\sigma_{ij}$

---

```
function d1=dx1(x1,s,p,n1);

m=x1(1,1);
a=m;
pdf=pdff(-0.5*a);
d1=0;
c2=-1/m;
for i=1:p
        if i==1
                d1ai=1;
        else d1ai=-1*s(i,1); end;
        for j=1:p
                temp1=(3*s(j,1))-(3*s(1,i)*s(j,1));
                temp2=s(j,i)-(3*s(1,i)*s(j,1));
                temp3=2*((-1*s(1,i))+1);
                if (i==1) & (j==1)
                        d2a=c2*(temp1-temp3);
                elseif (i==1) & (j~=1)
                        d2a=c2*temp1;
                elseif (i~=1) & (j==1)
                        d2a=c2*(temp2+(2*s(1,i)));
                elseif (i~=1) & (j~=1) & (i==j)
```

```
                    d2a=c2*(temp2-2);
            else
                    d2a=c2*temp2;end;
            if j==1
                    d1aj=1;
            else d1aj=-1*s(j,1);end;
            temp6=0.25*a*d1ai*d1aj;
            dp=-0.5*pdf*(d2a-temp6)*s(i,j)/n1;
            d1=d1+dp;
        end;
end;
```

---

Function to find (for cases 1A and 1B) $ndx2 = \frac{1}{n_2}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}\sigma_{ij}$

---

```
function d2=dx2(x1,s,p,n2)

m=x1(1,1);
pdf=pdff(-0.5*m);
c1=1/m;
d2=0;
for i=1:p
 if i==1
  d1ai=-1;
 else
  d1ai=s(i,1); end;
  for j=1:p
        temp1=3*s(1,i)*s(j,1);
        temp2=2*(1-s(1,i));
        if (i==1) & (j==1)
                d2a=c1*((-3*s(j,1))+temp1+temp2);
        elseif (i==1) & (j~=1) & (i~=j);
                d2a=c1*((-3*s(j,1))+temp1);
        elseif (i~=1) & (j==1) &(i~=j)
                d2a=c1*((-1*s(j,i))+temp1-(2*s(1,i)));
        elseif (i~=1) & (j~=1) & (i==j)
                d2a=c1*((-1*s(j,i))+temp1+2);
        else
                d2a=c1*((-1*s(j,1))+temp1);end;
        if j==1
                d1aj=-1;
        else
                d1aj=s(j,1);end;
        temp2=0.25*m*d1ai*d1aj;
        dp=-0.5*pdf*(d2a-temp2)*s(i,j)/n2;
        d2=d2+dp;
        end;
end;
```

---

Function to find (for cases 1A and 1B) $\text{nds} = \frac{n_1+n_2}{(n_1+n_2-2)^2} \sum_{k,l,i,j=1}^{p} \frac{\partial^2 \Phi}{\partial s_{kl} \partial \bar{x}_{ij}} (\sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk})$

---

```
function ds=nds(x1,s,n1,n2);

m=x1(1,1);
pdf=pdff(-0.5*m);
temp1=3-((m^2)/4);
temp2=2*(n1+n2)/((n1+n2-2)^2);
ds=-1*pdf*m*temp1*temp2/8;
```

---

Main program to find (for cases 2A and 2B)

$$\text{rate} = \Phi(\cdot) + \frac{1}{2n_1}\sum_{i,j=1}^{p} \frac{\partial^2 \Phi}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}} \sigma_{ij} + \frac{1}{2n_2}\sum_{i,j=1}^{p} \frac{\partial^2 \Phi}{\partial \bar{x}_{2i} \partial \bar{x}_{2j}} \sigma_{ij} + \frac{n_1+n_2}{2(n_1+n_2-2)^2} \sum_{k,l,i,j=1}^{p} \frac{\partial^2 \Phi}{\partial s_{kl} \partial \bar{x}_{ij}} (\sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk})$$

$$= \text{cdf} + \frac{1}{2}(\text{edx1}) + \frac{1}{2}(\text{edx2}) + \frac{1}{2}(\text{eds})$$

---

```
function rate=eratef(x1,x2,s,p,n1,n2)

m=x1(1,1);
ssum=sum(sum(s));
a=m*p*(ssum^(-0.5));
temp1=-0.5*a;
cdf=cdff(temp1);
tx1=0.5*edx1(x1,s,p,n1)
tx2=0.5*edx2(x1,s,p,n2)
ts=0.5*eds(x1,s,p,n1,n2)
rate=cdf+tx1+tx2+ts;
```

---

Function to find (for cases 2A and 2B) $\text{edx1} = \frac{1}{n_1}\sum_{i,j=1}^{p} \frac{\partial^2 \Phi}{\partial \bar{x}_{1i} \partial \bar{x}_{1j}} \sigma_{ij}$

---

```
function d1=edx1(x1,s,p,n1);

m=x1(1,1);
ssum=sum(sum(s));
a=m*p*(ssum^(-0.5));
pdf=pdff(-0.5*a);
prod1=ssum^(-3.5);
tsum=sum(s);
```

```
d1=0;
c1=ssum/m;
for i=1:p
        temp4=(2*ssum)-tsum(1,i);
        for j=1:p
          temp1=-1*ssum*((2*tsum(1,j))+(p*s(j,i)));
          temp2=3*p*tsum(1,i)*tsum(1,j);
          temp5=(2*ssum)-tsum(1,j);
          temp3=m*p*temp4*temp5/4;
         if i==j
                temp6=2*ssum*(ssum-tsum(1,i));
         else
                temp6=-2*ssum*tsum(1,i);end;
         d=c1*(temp1+temp2+temp6)-temp3;
          temp=-0.5*pdf*prod1*d*s(i,j)/n1;
          d1=d1+temp;
        end;
end;
```

---

Function to find (for cases 2A and 2B) $edx2 = \frac{1}{n_2}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}\sigma_{ij}$

---

```
function d2=edx2(x1,s,p,n2);

m=x1(1,1);
ssum=sum(sum(s));
a=m*p*(ssum^(-0.5));
pdf=pdff(-0.5*a);
prod1=ssum^(-3.5);
tsum=sum(s);
c1=ssum/m;
d2=0;
for i=1:p
        temp1=tsum(1,i)-(2*ssum);
        for j=1:p
                temp2=tsum(1,j)-(2*ssum);
                temp3=m*p*temp1*temp2/4;
                temp4=3*p*tsum(1,j)*tsum(1,i);
                temp6=-1*ssum*((2*tsum(1,j))+(s(j,i)*p));
                if i==j
                  temp5=2*ssum*(ssum-tsum(1,i));
                else
                  temp5=-2*ssum*tsum(1,i);end;
                d=c1*(temp6+temp4+temp5)-temp3;
                temp=-0.5*pdf*prod1*d*s(i,j)/n2;
                d2=d2+temp;
        end;
end;
```

---

Function to find (for cases 2A and 2B) $nds = \frac{n_1 + n_2}{(n_1 + n_2 - 2)^2} \sum_{k,l,i,j=1}^{p} \frac{\partial^2 \Phi}{\partial s_{kl} \partial \bar{x}_{ij}} (\sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk})$

---

```
function ds=eds(x1,s,p,n1,n2);

m=x1(1,1);
ssum=sum(sum(s));
a=m*p*(ssum^(-0.5));
pdf=pdff(-0.5*a);
prod1=ssum^(-3.5);
c1=(n1+n2)/((n1+n2-2)^2);
prod2=(3*ssum)-(((m*p)^2)/4);
d=-1*m*p*prod1*prod2*pdf/8;
ds=0;
for k=1:p
  for l=1:p
   for i=1:p
       for j=1:p
       prod3=((s(i,k)*s(j,l))+(s(i,l)*s(j,k)))*c1;
       ds=ds+(d*prod3);
       end;
   end;
  end;
end;
```

# *Linear discriminant function*

# *Plug-in error rate*

---

Function to find $y = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} x^2\right)$

---

```
function y=cf(x)

y=exp(-0.5*(x.^2))/sqrt(2.*pi);
```

---

Function to find $\mathrm{cdf} = \Phi(x)$

---

```
function cdf=cdff(x)

if x > 0
        cdf=0.5+quad('cf',0,x);
        end
else if x < 0
        temp1=abs(x);
        cdf=0.5-quad('cf',0,temp1);
        end
else if x==0
        cdf=quad('cf',0,x);
        end;
```

---

Function to find $p = \phi(x)$

---

```
function p=pdff(x)

temp=-1*(x.^2)/2;
p=exp(temp)/sqrt(2*pi);
```

---

Main program to find (for cases 1A and 1B)

$$\text{rate} = \Phi(\cdot) + \frac{1}{2n_1}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}\sigma_{ij} + \frac{1}{2n_2}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}\sigma_{ij} + \frac{n_1+n_2}{2(n_1+n_2-2)^2}\sum_{k,1,i,j=1}^{p}\frac{\partial^2\Phi}{\partial s_{kl}\partial\bar{x}_{ij}}(\sigma_{ik}\sigma_{jl}+\sigma_{il}\sigma_{jk})$$

$$= \text{cdf} + \frac{1}{2}(\text{ndx1}) + \frac{1}{2}(\text{ndx2}) + \frac{1}{2}(\text{nds})$$

$$= \text{cdf} + (\text{ndx1}) + \frac{1}{2}(\text{nds})$$

since ndx1=ndx2.

---

```
function rate=eratef1(x1,x2,s,p,n1,n2);

m=x1(1,1);
si=s^(-1);
a=m*(si(1,1)^(0.5));
cdf=cdff(-0.5*a)
tx1=0.5*ndx1(x1,s,p,n1)
ts=0.5*nds(x1,s,p,n1,n2)
rate=cdf+(2*tx1)+ts;
```

---

Function to find (for cases 1A and 1B) $\text{ndx1} = \frac{1}{n_1}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}\sigma_{ij}$

---

```
function d1=ndx1(x1,s,p,n1);

m=x1(1,1);
si=s^(-1);
a=m*si(1,1)^(0.5);
pdf=pdff(-0.5*a);
d1=0;
for i=1:p
        for j=1:p
                temp1=(si(1,1)*si(j,i))-(si(1,i)*si(j,1));
                temp2=si(1,1)*si(j,1)*si(i,1)/4;
                prod=(temp1/m)-(m*temp2);
                dp=-0.5*pdf*(si(1,1)^(-1.5))*prod*s(i,j)/n1;
                d1=d1+dp;
        end;
end;
```

---

Function to find (for cases 1A and 1B) $nds = \frac{n_1 + n_2}{(n_1 + n_2 - 2)^2} \sum_{k,l,i,j=1}^{p} \frac{\partial^2 \Phi}{\partial s_{kl} \partial \bar{x}_{ij}} (\sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk})$

---

```
function eee=nds(x1,s,p,n1,n2)

m=x1(1,1);
si=s^(-1);
temp=-0.5*m*sqrt(si(1,1));
pdf=pdff(temp);
temp12=si(1,1)^(-1.5);
temp10=(n1+n2)/((n1+n2-2)^2);
eee=0;
for k=1:p
        for l=1:p
                if k==l
                w1=-0.5;
                else
                w1=-1;end;
                for i=1:p
                        for j=1:p
                        if (i <= j) & (k <= l)
                                if i==j
                                w0=-0.5;
                                else
                                w0=-1;end;
                        temp1=si(1,i)*((si(j,k)*si(l,1))+(si(j,l)*si(k,1)));
                        temp2=si(j,1)*((si(1,k)*si(l,i))+(si(1,l)*si(k,i)));
                        temp5=si(1,1)*(temp1+temp2);
                        temp6=si(1,i)*si(j,1)*si(1,k)*si(l,1);
                        temp8=(1+((m^2)*si(1,1)/4))*temp6;
                        prod1=temp5-temp8;
                        ds=-0.5*m*w0*w1*pdf*temp12*prod1;
                        temp9=(s(i,k)*s(j,l))+(s(i,l)*s(j,k));
                        cv=temp9*temp10;
                        if (i == j) & (k==l)
                        temp11=ds*cv;
                        elseif (i==j) & (k~=l)
                        temp11=2*ds*cv;
                        elseif (i~=j) & (k==l)
                        temp11=2*ds*cv;
                        else
                        temp11=4*ds*cv;end;
                        eee=eee+temp11;
                        end;
                        end;
                end;
        end;
end;
```

---

Main program to find (for cases 2A and 2B)

$$\text{rate} = \Phi(\cdot) + \frac{1}{2n_1}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}\sigma_{ij} + \frac{1}{2n_2}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{2i}\partial\bar{x}_{2j}}\sigma_{ij} + \frac{n_1+n_2}{2(n_1+n_2-2)^2}\sum_{k,l,i,j=1}^{p}\frac{\partial^2\Phi}{\partial s_{kl}\partial\bar{x}_{ij}}(\sigma_{ik}\sigma_{jl}+\sigma_{il}\sigma_{jk})$$

$$= \text{cdf} + \frac{1}{2}(\text{edx1}) + \frac{1}{2}(\text{edx2}) + \frac{1}{2}(\text{eds})$$

$$= \text{cdf} + (\text{edx1}) + \frac{1}{2}(\text{eds})$$

since edx1=edx2.

---

```
function rate=eratef2(x1,x2,s,p,n1,n2)

m=x1(1,1);
si=s^(-1);
sum1=sum(sum(si));
temp=m*(sum1^(0.5));
cdf=cdff(-0.5*temp);
tx1=edx1(x1,s,p,n1)
ts=0.5*eds(x1,s,p,n1,n2)
rate=cdf+tx1+ts;
```

---

Function to find (for cases 2A and 2B) $\text{edx1} = \frac{1}{n_1}\sum_{i,j=1}^{p}\frac{\partial^2\Phi}{\partial\bar{x}_{1i}\partial\bar{x}_{1j}}\sigma_{ij}$

---

```
function d1=edx1(x1,s,p,n1);

m=x1(1,1);
si=s^(-1);
ssum=sum(sum(si));
tsum=sum(si);
a=m*(ssum^(-0.5));
pdf=pdff(-0.5*a);
prod1=ssum^(-1.5);
d1=0;
for i=1:p
        for j=1:p
                temp1=tsum(1,i)*tsum(1,j);
                sum1=((si(j,i)*ssum)-temp1)/m;
                sum2=temp1*m*ssum/4;
                temp=-0.5*pdf*prod1*(sum1-sum2)*s(i,j)/n1;
                d1=d1+temp;
        end;
end;
```

---

Function to find (for cases 2A and 2B) $nds = \frac{n_1+n_2}{(n_1+n_2-2)^2}\sum_{k,l,i,j=1}^{p}\frac{\partial^2\Phi}{\partial s_{kl}\partial \bar{x}_{ij}}(\sigma_{ik}\sigma_{jl}+\sigma_{il}\sigma_{jk})$

---

```
function ddd=eds(x1,s,p,n1,n2)

m=x1(1,1);
si=s^(-1);
sum1=sum(sum(si));
prod1=sum1^(-1.5);
temp1=-0.5*m*sqrt(sum1);
pdf=pdff(temp1);
temp12=(n1+n2)/((n1+n2-2)^2);
ddd=0;
for k=1:p
        for l=1:p
                if k==l
                w1=-0.5;
                else
                w1=-1;end;
                sum4=0;
                for u=1:p
                for v=1:p
                temp7=(si(u,k)*si(l,v))+(si(u,l)*si(k,v));
                sum4=sum4+temp7;
                end;
                end;
                for i=1:p
                        for j=1:p
                        if (i <= j) & (k <= l)
                                if i==j
                                w0=-0.5;
                                else
                                w0=-1;end;
                                sum2=0;
                                sum3=0;
                                for u=1:p
                                for v=1:p
                        temp2=si(u,i)*((si(j,k)*si(l,v))+(si(j,l)*si(k,v)));
                        temp3=si(j,v)*((si(u,k)*si(l,i))+(si(u,l)*si(k,i)));
                        temp4=si(u,j)*((si(i,k)*si(l,v))+(si(i,l)*si(k,v)));
                        temp25=si(i,v)*((si(u,k)*si(l,j))+(si(u,l)*si(k,j)));
                        sum2=sum2+temp2+temp3+temp4+temp25;
                        temp6=(si(u,i)*si(j,v))+(si(u,j)*si(i,v));
                        sum3=sum3+temp6;
                                end;
                                end;


                                temp5=sum1*sum2;
```

```
                    temp8=-0.5*(1+((m^2)*sum1/4))*sum3*sum4;
                    prod2=temp5+temp8;
                    ds=-0.25*m*w0*w1*pdf*prod1*prod2;
                    temp11=(s(i,k)*s(j,l))+(s(i,l)*s(j,k));
                    cv=temp11*temp12;
                    if (i==j) & (k==l)
                    temp15=ds*cv;
                    elseif (i==j) & (k~=l)
                    temp15=2*ds*cv;
                    elseif (i~=j) & (k==l)
                    temp15=2*ds*cv;
                    else
                    temp15=4*ds*cv;end;
                    ddd=ddd+temp15;
               end;
               end;
          end;
     end;
end;
```

# *Appendix A4*

## *COMPUTATIONAL RESULTS*

| ρ | $\Delta^2$ | p=4 | | p=8 | | p=12 | | p=16 | |
|---|---|---|---|---|---|---|---|---|---|
| | | $p_A^{(E)}$ | *true* | $p_A^{(E)}$ | *true* | $p_A^{(E)}$ | *true* | $p_A^{(E)}$ | *true* |
| -0.06 | 0.5 | 0.3792 | *0.3626* | 0.4034 | *0.3645* | 0.4325 | *0.3686* | 0.4912 | *0.3868* |
| | 1.0 | 0.3211 | *0.3096* | 0.3384 | *0.3120* | 0.3607 | *0.3175* | 0.4133 | *0.3420* |
| | 1.5 | 0.2805 | *0.2714* | 0.2948 | *0.2742* | 0.3142 | *0.2805* | 0.3652 | *0.3091* |
| | 2.0 | 0.2488 | *0.2411* | 0.2612 | *0.2441* | 0.2789 | *0.2509* | 0.3293 | *0.2825* |
| | 2.5 | 0.2227 | *0.2160* | 0.2338 | *0.2192* | 0.2503 | *0.2265* | 0.3004 | *0.2599* |
| 0.00 | 0.5 | 0.3784 | *0.3618* | 0.3996 | *0.3618* | 0.4208 | *0.3618* | 0.4420 | *0.3618* |
| | 1.0 | 0.3200 | *0.3086* | 0.3341 | *0.3085* | 0.3481 | *0.3085* | 0.3622 | *0.3085* |
| | 1.5 | 0.2793 | *0.2702* | 0.2901 | *0.2702* | 0.3009 | *0.2702* | 0.3117 | *0.2702* |
| | 2.0 | 0.2474 | *0.2398* | 0.2562 | *0.2398* | 0.2650 | *0.2398* | 0.2728 | *0.2398* |
| | 2.5 | 0.2213 | *0.2146* | 0.2287 | *0.2146* | 0.2361 | *0.2146* | 0.2434 | *0.2146* |
| 0.01 | 0.5 | 0.3784 | *0.3618* | 0.3997 | *0.3619* | 0.4209 | *0.3619* | 0.4422 | *0.3619* |
| | 1.0 | 0.3200 | *0.3086* | 0.3341 | *0.3086* | 0.3483 | *0.3086* | 0.3624 | *0.3087* |
| | 1.5 | 0.2793 | *0.2702* | 0.2901 | *0.2702* | 0.3010 | *0.2703* | 0.3119 | *0.2703* |
| | 2.0 | 0.2475 | *0.2398* | 0.2563 | *0.2398* | 0.2652 | *0.2399* | 0.2740 | *0.2399* |
| | 2.5 | 0.2213 | *0.2146* | 0.2288 | *0.2147* | 0.2362 | *0.2147* | 0.2436 | *0.2147* |
| 0.2 | 0.5 | 0.3823 | *0.3677* | 0.4075 | *0.3706* | 0.4346 | *0.3720* | 0.4644 | *0.3728* |
| | 1.0 | 0.3266 | *0.3163* | 0.3456 | *0.3202* | 0.3650 | *0.3221* | 0.3860 | *0.3232* |
| | 1.5 | 0.2874 | *0.2791* | 0.3038 | *0.2836* | 0.3198 | *0.2858* | 0.3368 | *0.2871* |
| | 2.0 | 0.2566 | *0.2495* | 0.2714 | *0.2544* | 0.2853 | *0.2568* | 0.2999 | *0.2582* |
| | 2.5 | 0.2311 | *0.2248* | 0.2448 | *0.2301* | 0.2573 | *0.2326* | 0.2702 | *0.2341* |
| 0.4 | 0.5 | 0.3847 | *0.3810* | 0.3891 | *0.3861* | 0.3844 | *0.3880* | 0.3728 | *0.3890* |
| | 1.0 | 0.3375 | *0.3343* | 0.3450 | *0.3411* | 0.3449 | *0.3437* | 0.3402 | *0.3450* |
| | 1.5 | 0.3031 | *0.3000* | 0.3125 | *0.3080* | 0.3148 | *0.3110* | 0.3137 | *0.3126* |
| | 2.0 | 0.2754 | *0.2724* | 0.2861 | *0.2813* | 0.2901 | *0.2846* | 0.2913 | *0.2863* |
| | 2.5 | 0.2522 | *0.2492* | 0.2639 | *0.2587* | 0.2690 | *0.2623* | 0.2719 | *0.2641* |
| 0.65 | 0.5 | 0.3611 | *0.4064* | 0.2032 | *0.4119* | ** | *0.4137* | | |
| | 1.0 | 0.3389 | *0.3688* | 0.2371 | *0.3765* | ** | *0.3789* | | |
| | 1.5 | 0.3181 | *0.3408* | 0.2426 | *0.3499* | 0.1007 | *0.3528* | | |
| | 2.0 | 0.2995 | *0.3174* | 0.2405 | *0.3281* | 0.1252 | *0.3314* | | |
| | 2.5 | 0.2830 | *0.2982* | 0.2357 | *0.3093* | 0.1390 | *0.3129* | | |

**Table 9 :** The expected actual error rate of the EDC under the case of non - equivalence with $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ ( i.e. case A1 ).

| ρ | $\Delta^2$ | p=4 | | p=8 | | p=12 | |
|---|---|---|---|---|---|---|---|
| | | $p_A^{(L)}$ | *true* | $p_A^{(L)}$ | *true* | $p_A^{(L)}$ | *true* |
| -0.06 | 0.5 | 0.3783 | *0.3618* | 0.3939 | *0.3618* | | |
| | 1.0 | 0.3145 | *0.3085* | 0.3232 | *0.3085* | | |
| | 1.5 | 0.2673 | *0.2701* | 0.2730 | *0.2701* | | |
| | 2.0 | 0.2346 | *0.2398* | 0.2317 | *0.2398* | | |
| | 2.5 | 0.1926 | *0.2146* | 0.1955 | *0.2146* | | |
| 0.00 | 0.5 | 0.3857 | *0.3618* | 0.3954 | *0.3618* | 0.4133 | *0.3618* |
| | 1.0 | 0.3148 | *0.3085* | 0.3242 | *0.3085* | | |
| | 1.5 | 0.2675 | *0.2702* | 0.2737 | *0.2702* | | |
| | 2.0 | 0.2277 | *0.2398* | 0.2319 | *0.2398* | | |
| | 2.5 | 0.1925 | *0.2146* | 0.1953 | *0.2146* | 0.1994 | *0.2146* |
| 0.01 | 0.5 | 0.3787 | *0.3618* | 0.3953 | *0.3618* | | |
| | 1.0 | 0.3148 | *0.3085* | 0.3242 | *0.3085* | | |
| | 1.5 | 0.2674 | *0.2701* | 0.2737 | *0.2702* | | |
| | 2.0 | 0.2277 | *0.2398* | 0.2319 | *0.2398* | | |
| | 2.5 | 0.1926 | *0.2146* | 0.1953 | *0.2146* | | |
| 0.2 | 0.5 | 0.3766 | *0.3618* | 0.3899 | *0.3618* | 0.4477 | *0.3618* |
| | 1.0 | 0.3135 | *0.3085* | 0.3206 | *0.3085* | | |
| | 1.5 | 0.2669 | *0.2701* | 0.2909 | *0.2701* | | |
| | 2.0 | 0.2280 | *0.2398* | 0.2308 | *0.2397* | | |
| | 2.5 | 0.1937 | *0.2146* | 0.1956 | *0.2146* | 0.1984 | *0.2146* |
| 0.4 | 0.5 | 0.3685 | *0.3618* | 0.3673 | *0.3618* | 0.4359 | *0.3618* |
| | 1.0 | 0.3073 | *0.3085* | 0.3008 | *0.3085* | | |
| | 1.5 | 0.2623 | *0.2701* | 0.2538 | *0.2702* | | |
| | 2.0 | 0.2252 | *0.2398* | 0.2155 | *0.2398* | | |
| | 2.5 | 0.1927 | *0.2416* | 0.1826 | *0.2416* | 0.1699 | *0.2146* |

**Table 10 :** The expected actual error rate of the LDF under the case of "non-equivalence " with $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ ( i.e. case A1 ).

| ρ | $\Delta^2$ | p=4 | | p=8 | | p=12 | | p=16 | |
|---|---|---|---|---|---|---|---|---|---|
| | | $p_A^{(E)}$ | *true* | $p_A^{(E)}$ | *true* | $p_A^{(E)}$ | *true* | $p_A^{(E)}$ | *true* |
| 0.00 | 0.5 | 0.3784 | *0.3618* | 0.3996 | *0.3618* | 0.4208 | *0.3618* | 0.4420 | *0.3618* |
| | 1.0 | 0.3200 | *0.3085* | 0.3341 | *0.3085* | 0.3481 | *0.3085* | 0.3622 | *0.3085* |
| | 1.5 | 0.2793 | *0.2702* | 0.2901 | *0.2702* | 0.3009 | *0.2702* | 0.3117 | *0.2702* |
| | 2.0 | 0.2474 | *0.2398* | 0.2562 | *0.2398* | 0.2650 | *0.2398* | 0.2738 | *0.2398* |
| | 2.5 | 0.2213 | *0.2146* | 0.2287 | *0.2146* | 0.2361 | *0.2146* | 0.2434 | *0.2146* |
| 0.01 | 0.5 | 0.3784 | *0.3618* | 0.3996 | *0.3618* | 0.4208 | *0.3618* | 0.4420 | *0.3618* |
| | 1.0 | 0.3200 | *0.3086* | 0.3341 | *0.3086* | 0.3482 | *0.3086* | 0.3623 | *0.3086* |
| | 1.5 | 0.2793 | *0.2702* | 0.2901 | *0.2702* | 0.3009 | *0.2702* | 0.3117 | *0.2702* |
| | 2.0 | 0.2475 | *0.2398* | 0.2562 | *0.2398* | 0.2650 | *0.2398* | 0.2738 | *0.2398* |
| | 2.5 | 0.2213 | *0.2146* | 0.2287 | *0.2146* | 0.2361 | *0.2146* | 0.2435 | *0.2146* |
| 0.2 | 0.5 | 0.3812 | *0.3645* | 0.4047 | *0.3645* | 0.4282 | *0.3645* | 0.4517 | *0.3645* |
| | 1.0 | 0.3236 | *0.3121* | 0.3393 | *0.3121* | 0.3549 | *0.3121* | 0.3706 | *0.3121* |
| | 1.5 | 0.2835 | *0.2743* | 0.2955 | *0.2743* | 0.3075 | *0.2743* | 0.3196 | *0.2743* |
| | 2.0 | 0.2520 | *0.2442* | 0.2618 | *0.2442* | 0.2717 | *0.2442* | 0.2815 | *0.2442* |
| | 2.5 | 0.2261 | *0.2193* | 0.2344 | *0.2193* | 0.2426 | *0.2193* | 0.2509 | *0.2193* |
| 0.4 | 0.5 | 0.3886 | *0.3730* | 0.4207 | *0.3730* | 0.4530 | *0.3730* | 0.4852 | *0.3730* |
| | 1.0 | 0.3344 | *0.3234* | 0.3559 | *0.3234* | 0.3776 | *0.3234* | 0.3992 | *0.3234* |
| | 1.5 | 0.2962 | *0.2873* | 0.3129 | *0.2873* | 0.3297 | *0.2873* | 0.3465 | *0.2873* |
| | 2.0 | 0.2661 | *0.2585* | 0.2799 | *0.2585* | 0.2936 | *0.2585* | 0.3074 | *0.2585* |
| | 2.5 | 0.2411 | *0.2344* | 0.2528 | *0.2344* | 0.2645 | *0.2344* | 0.2762 | *0.2344* |
| 0.65 | 0.5 | 0.3931 | *0.3941* | 0.4523 | *0.3941* | 0.5222 | *0.3941* | 0.5222 | *0.5928* |
| | 1.0 | 0.3521 | *0.3520* | 0.3927 | *0.3520* | 0.4404 | *0.3520* | 0.4404 | *0.4520* |
| | 1.5 | 0.3216 | *0.3208* | 0.3537 | *0.3208* | 0.3912 | *0.3208* | 0.3912 | *0.4291* |
| | 2.0 | 0.2967 | *0.2955* | 0.3236 | *0.2955* | 0.3550 | *0.2955* | 0.3550 | *0.3866* |
| | 2.5 | 0.2755 | *0.2740* | 0.2988 | *0.2740* | 0.3258 | *0.2740* | 0.3258 | *0.3531* |

**Table 11 :** The expected actual error rate of the EDC under the case of " non-equivalence " when $\mathbf{\Sigma} = AR(1)$ (with positive ρ in case A2).

| ρ | $\Delta^2$ | p=4 | | p=8 | | p=12 | | p=16 | |
|---|---|---|---|---|---|---|---|---|---|
| | | $p_A^{(E)}$ | *true* | $p_A^{(E)}$ | *true* | $p_A^{(E)}$ | *true* | $p_A^{(E)}$ | *true* |
| -0.65 | 0.5 | 0.3931 | *0.3941* | 0.4523 | *0.3941* | 0.5222 | *0.3941* | 0.5222 | *0.5928* |
| | 1.0 | 0.3521 | *0.3520* | 0.3927 | *0.3520* | 0.4404 | *0.3520* | 0.4404 | *0.4520* |
| | 1.5 | 0.3216 | *0.3208* | 0.3537 | *0.3208* | 0.3912 | *0.3208* | 0.3912 | *0.4291* |
| | 2.0 | 0.2967 | *0.2955* | 0.3236 | *0.2955* | 0.3550 | *0.2955* | 0.3550 | *0.3866* |
| | 2.5 | 0.2755 | *0.2740* | 0.2988 | *0.2740* | 0.3258 | *0.2740* | 0.3258 | *0.3531* |
| -0.4 | 0.5 | 0.3886 | *0.3730* | 0.4207 | *0.3730* | 0.4530 | *0.3730* | 0.4852 | *0.3730* |
| | 1.0 | 0.3344 | *0.3234* | 0.3559 | *0.3234* | 0.3776 | *0.3234* | 0.3992 | *0.3234* |
| | 1.5 | 0.2962 | *0.2873* | 0.3129 | *0.2873* | 0.3297 | *0.2873* | 0.3465 | *0.2873* |
| | 2.0 | 0.2661 | *0.2585* | 0.2799 | *0.2585* | 0.2936 | *0.2585* | 0.3074 | *0.2585* |
| | 2.5 | 0.2441 | *0.2344* | 0.2528 | *0.2344* | 0.2645 | *0.2344* | 0.2762 | *0.2344* |
| -0.2 | 0.5 | 0.3812 | *0.3645* | 0.4047 | *0.3645* | 0.4282 | *0.3645* | 0.4517 | *0.3645* |
| | 1.0 | 0.3236 | *0.3121* | 0.3393 | *0.3121* | 0.3549 | *0.3121* | 0.3706 | *0.3121* |
| | 1.5 | 0.2835 | *0.2743* | 0.2955 | *0.2743* | 0.3075 | *0.2743* | 0.3196 | *0.2743* |
| | 2.0 | 0.2520 | *0.2442* | 0.2618 | *0.2442* | 0.2717 | *0.2442* | 0.2815 | *0.2442* |
| | 2.5 | 0.2661 | *0.2193* | 0.2344 | *0.2193* | 0.2426 | *0.2193* | 0.2509 | *0.2193* |
| -0.06 | 0.5 | 0.3787 | *0.3621* | 0.4001 | *0.3621* | 0.4215 | *0.3621* | 0.4429 | *0.3621* |
| | 1.0 | 0.3203 | *0.3089* | 0.3345 | *0.3089* | 0.3487 | *0.3089* | 0.3630 | *0.3089* |
| | 1.5 | 0.2796 | *0.2705* | 0.2906 | *0.2705* | 0.3015 | *0.2705* | 0.3124 | *0.2705* |
| | 2.0 | 0.2478 | *0.2401* | 0.2567 | *0.2401* | 0.2656 | *0.2401* | 0.2745 | *0.2401* |
| | 2.5 | 0.2217 | *0.2150* | 0.2292 | *0.2150* | 0.2366 | *0.2150* | 0.2441 | *0.2150* |

**Table 11a:** The expected actual error rate of the EDC under the case of "non-equivalence" when $\Sigma = AR(1)$ (with negative ρ in case A2).

| ρ | $\Delta^2$ | p=4 | | p=8 | | p=12 | |
|---|---|---|---|---|---|---|---|
| | | $p_A^{(L)}$ | *true* | $p_A^{(L)}$ | *true* | $p_A^{(L)}$ | *true* |
| -0.06 | 0.5 | 0.3889 | *0.3618* | 0.4191 | *0.3618* | | |
| | 1.0 | 0.3272 | *0.3085* | 0.3531 | *0.3085* | | |
| | 1.5 | 0.2812 | *0.2702* | 0.3057 | *0.2702* | | |
| | 2.0 | 0.2423 | *0.2397* | 0.2660 | *0.2397* | | |
| | 2.5 | 0.2077 | *0.2146* | 0.2309 | *0.2146* | | |
| 0.00 | 0.5 | 0.3906 | *0.3618* | 0.4228 | *0.3618* | 0.4550 | *0.3618* |
| | 1.0 | 0.3292 | *0.3085* | 0.3579 | *0.3085* | | |
| | 1.5 | 0.2835 | *0.2702* | 0.3112 | *0.2702* | | |
| | 2.0 | 0.2447 | *0.2398* | 0.2719 | *0.2398* | | |
| | 2.5 | 0.2102 | *0.2146* | 0.2368 | *0.2146* | 0.2634 | *0.2146* |
| 0.01 | 0.5 | 0.3908 | *0.3618* | 0.4234 | *0.3618* | | |
| | 1.0 | 0.3295 | *0.3085* | 0.3588 | *0.3085* | | |
| | 1.5 | 0.2835 | *0.2701* | 0.3121 | *0.2701* | | |
| | 2.0 | 0.2451 | *0.2398* | 0.2729 | *0.2398* | | |
| | 2.5 | 0.2106 | *0.2146* | 0.2379 | *0.2146* | | |
| 0.2 | 0.5 | 0.3969 | *0.3618* | 0.4389 | *0.3618* | 0.4844 | *0.3618* |
| | 1.0 | 0.3380 | *0.3085* | 0.3796 | *0.3085* | | |
| | 1.5 | 0.2936 | *0.2701* | 0.3357 | *0.2701* | | |
| | 2.0 | 0.2558 | *0.2398* | 0.2981 | *0.2398* | | |
| | 2.5 | 0.2218 | *0.2146* | 0.2638 | *0.2146* | 0.3059 | *0.2146* |
| 0.4 | 0.5 | 0.4064 | *0.3618* | 0.4669 | *0.3618* | 0.5169 | *0.3618* |
| | 1.0 | 0.3517 | *0.3085* | 0.4167 | *0.3085* | | |
| | 1.5 | 0.3096 | *0.2701* | 0.3772 | *0.2701* | | |
| | 2.0 | 0.2733 | *0.2398* | 0.3414 | *0.2398* | | |
| | 2.5 | 0.2402 | *0.2146* | 0.3075 | *0.2146* | 0.3747 | *0.2146* |

**Table 12 :** The expected actual error rate of the LDF under the case of " non - equivalence " with $\Sigma = AR(1)$ ( i.e. case A2 ).

| ρ | $\Delta^2$ | p=4 | | p=8 | | p=12 | | p=16 | |
|---|---|---|---|---|---|---|---|---|---|
| | | $p_A^{(E)}$ | *true* | $p_A^{(E)}$ | *true* | $p_A^{(E)}$ | *true* | $p_A^{(E)}$ | *true* |
| -0.06 | 0.5 | 0.3894 | *0.3618* | 0.4869 | *0.3618* | 0.9299 | *0.3619* | ** | *0.3618* |
| | 1.0 | 0.3275 | *0.3085* | 0.3924 | *0.3085* | 0.6867 | *0.3086* | ** | *0.3084* |
| | 1.5 | 0.2853 | *0.2702* | 0.3350 | *0.2701* | 0.5606 | *0.2701* | ** | *0.2702* |
| | 2.0 | 0.2525 | *0.2398* | 0.2930 | *0.2397* | 0.4768 | *0.2398* | ** | *0.2398* |
| | 2.5 | 0.2257 | *0.2146* | 0.2598 | *0.2146* | 0.4142 | *0.2146* | ** | *0.2146* |
| 0.00 | 0.5 | 0.3788 | *0.3618* | 0.4001 | *0.3618* | 0.4214 | *0.3619* | 0.4426 | *0.3618* |
| | 1.0 | 0.3205 | *0.3085* | 0.3347 | *0.3085* | 0.3489 | *0.3085* | 0.3630 | *0.3085* |
| | 1.5 | 0.2798 | *0.2701* | 0.2908 | *0.2702* | 0.3017 | *0.2701* | 0.3125 | *0.2701* |
| | 2.0 | 0.2481 | *0.2398* | 0.2571 | *0.2397* | 0.2660 | *0.2398* | 0.2747 | *0.2397* |
| | 2.5 | 0.2219 | *0.2146* | 0.2296 | *0.2146* | 0.2370 | *0.2146* | 0.2444 | *0.2146* |
| 0.01 | 0.5 | 0.3776 | *0.3618* | 0.3948 | *0.3618* | 0.4094 | *0.3618* | 0.4220 | *0.3618* |
| | 1.0 | 0.3197 | *0.3086* | 0.3312 | *0.3085* | 0.3410 | *0.3086* | 0.3493 | *0.3085* |
| | 1.5 | 0.2792 | *0.2701* | 0.2881 | *0.2702* | 0.2956 | *0.2701* | 0.3021 | *0.2702* |
| | 2.0 | 0.2476 | *0.2398* | 0.2549 | *0.2398* | 0.2610 | *0.2398* | 0.2663 | *0.2398* |
| | 2.5 | 0.2215 | *0.2146* | 0.2277 | *0.2146* | 0.2329 | *0.2146* | 0.2373 | *0.2146* |
| 0.2 | 0.5 | 0.3669 | *0.3618* | 0.3671 | *0.3618* | 0.3667 | *0.3619* | 0.3662 | *0.3618* |
| | 1.0 | 0.3125 | *0.3085* | 0.3128 | *0.3085* | 0.3126 | *0.3085* | 0.3123 | *0.3085* |
| | 1.5 | 0.2738 | *0.2701* | 0.2740 | *0.2702* | 0.2739 | *0.2701* | 0.2737 | *0.2701* |
| | 2.0 | 0.2431 | *0.2398* | 0.2434 | *0.2397* | 0.2433 | *0.2397* | 0.2431 | *0.2398* |
| | 2.5 | 0.2178 | *0.2146* | 0.2181 | *0.2146* | 0.2180 | *0.2146* | 0.2179 | *0.2146* |
| 0.4 | 0.5 | 0.3641 | *0.3618* | 0.3639 | *0.3618* | 0.3638 | *0.3618* | 0.3673 | *0.3618* |
| | 1.0 | 0.3107 | *0.3085* | 0.3107 | *0.3085* | 0.3106 | *0.3085* | 0.3106 | *0.3085* |
| | 1.5 | 0.2723 | *0.2701* | 0.2724 | *0.2701* | 0.2724 | *0.2701* | 0.2723 | *0.2701* |
| | 2.0 | 0.2420 | *0.2398* | 0.2421 | *0.2397* | 0.2421 | *0.2397* | 0.2421 | *0.2398* |
| | 2.5 | 0.2168 | *0.2146* | 0.2170 | *0.2146* | 0.2170 | *0.2146* | 0.2170 | *0.2146* |
| 0.65 | 0.5 | 0.3631 | *0.3618* | 0.3631 | *0.3618* | 0.3632 | *0.3618* | | |
| | 1.0 | 0.3101 | *0.3085* | 0.3102 | *0.3085* | 0.3102 | *0.3085* | | |
| | 1.5 | 0.2718 | *0.2701* | 0.2720 | *0.2701* | 0.2721 | *0.2701* | | |
| | 2.0 | 0.2416 | *0.2398* | 0.2418 | *0.2397* | 0.2418 | *0.2397* | | |
| | 2.5 | 0.2165 | *0.2146* | 0.2167 | *0.2146* | 0.2168 | *0.2146* | | |

**Table 13 :** The expected actual error rate of the EDC under the case of " equivalence " with $\Sigma = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ ( i.e. case A3 ).

| ρ | $\Delta^2$ | p=4 | | p=8 | | p=12 | | p=16 | |
|---|---|---|---|---|---|---|---|---|---|
| | | $p_A^{(E)}$ | *true* | $p_A^{(E)}$ | *true* | $p_A^{(E)}$ | *true* | $p_A^{(E)}$ | *true* |
| 0.00 | 0.5 | 0.3788 | *0.3618* | 0.4001 | *0.3618* | 0.4214 | *0.3619* | 0.4426 | *0.3618* |
| | 1.0 | 0.3205 | *0.3085* | 0.3347 | *0.3085* | 0.3489 | *0.3085* | 0.3630 | *0.3085* |
| | 1.5 | 0.2798 | *0.2701* | 0.2908 | *0.2702* | 0.3017 | *0.2701* | 0.3125 | *0.2701* |
| | 2.0 | 0.2481 | *0.2398* | 0.2571 | *0.2397* | 0.2660 | *0.2398* | 0.2747 | *0.2397* |
| | 2.5 | 0.2219 | *0.2146* | 0.2296 | *0.2146* | 0.2370 | *0.2146* | 0.2444 | *0.2146* |
| 0.01 | 0.5 | 0.3782 | *0.3618* | 0.3987 | *0.3618* | 0.4191 | *0.3618* | 0.4395 | *0.3619* |
| | 1.0 | 0.3200 | *0.3085* | 0.3337 | *0.3085* | 0.3474 | *0.3086* | 0.3609 | *0.3085* |
| | 1.5 | 0.2795 | *0.3701* | 0.2901 | *0.2702* | 0.3006 | *0.2702* | 0.3109 | *0.2701* |
| | 2.0 | 0.2478 | *0.2398* | 0.2565 | *0.2397* | 0.2650 | *0.2398* | 0.2735 | *0.2397* |
| | 2.5 | 0.2217 | *0.2146* | 0.2291 | *0.2146* | 0.2363 | *0.2146* | 0.2434 | *0.2146* |
| 0.2 | 0.5 | 0.3704 | *0.3624* | 0.3806 | *0.3623* | 0.3907 | *0.3622* | 0.4009 | *0.3621* |
| | 1.0 | 0.3153 | *0.3092* | 0.3221 | *0.3092* | 0.3288 | *0.3090* | 0.3355 | *0.3089* |
| | 1.5 | 0.2761 | *0.2710* | 0.2814 | *0.2709* | 0.2865 | *0.2707* | 0.2916 | *0.2706* |
| | 2.0 | 0.2453 | *0.2406* | 0.2496 | *0.2405* | 0.2537 | *0.2404* | 0.2579 | *0.2402* |
| | 2.5 | 0.2198 | *0.2155* | 0.2234 | *0.2154* | 0.2269 | *0.2152* | 0.2304 | *0.2151* |
| 0.4 | 0.5 | 0.3673 | *0.3634* | 0.3725 | *0.3636* | 0.3776 | *0.3633* | 0.3827 | *0.3631* |
| | 1.0 | 0.3139 | *0.3106* | 0.3176 | *0.3109* | 0.3208 | *0.3105* | 0.3241 | *0.3102* |
| | 1.5 | 0.2756 | *0.2725* | 0.2786 | *0.2729* | 0.2810 | *0.2725* | 0.2833 | *0.2720* |
| | 2.0 | 0.2452 | *0.2424* | 0.2479 | *0.2428* | 0.2497 | *0.2422* | 0.2515 | *0.2418* |
| | 2.5 | 0.2201 | *0.2173* | 0.2225 | *0.2178* | 0.2239 | *0.2172* | 0.2254 | *0.2168* |

**Table 14 :** The expected actual error rate of the EDC under the case of "equivalence" when $\Sigma = AR(1)$ (with positive ρ in case A4 ).

| ρ | $\Delta^2$ | p=4 | | p=8 | | p=12 | | p=16 | |
|---|---|---|---|---|---|---|---|---|---|
| | | $p_A^{(E)}$ | *true* | $p_A^{(E)}$ | *true* | $p_A^{(E)}$ | *true* | $p_A^{(E)}$ | *true* |
| -0.65 | 0.5 | 0.7889 | *0.3721* | ** | *0.3706* | ** | *0.3685* | | |
| | 1.0 | 0.6022 | *0.3222* | ** | *0.3203* | ** | *0.3175* | | |
| | 1.5 | 0.5033 | *0.2860* | 0.9368 | *0.2838* | ** | *0.2804* | | |
| | 2.0 | 0.4359 | *0.2570* | 0.7905 | *0.2546* | ** | *0.2509* | | |
| | 2.5 | 0.3848 | *0.2328* | 0.6843 | *0.2302* | ** | *0.2264* | | |
| -0.4 | 0.5 | 0.4549 | *0.3654* | 0.6034 | *0.3644* | 0.7601 | *0.3637* | | |
| | 1.0 | 0.3736 | *0.3133* | 0.4717 | *0.3119* | 0.5756 | *0.3110* | | |
| | 1.5 | 0.3226 | *0.2756* | 0.3974 | *0.2740* | 0.4767 | *0.2729* | | |
| | 2.0 | 0.2844 | *0.2457* | 0.3451 | *0.2439* | 0.4093 | *0.2427* | | |
| | 2.5 | 0.2539 | *0.2209* | 0.3046 | *0.2190* | 0.3584 | *0.2178* | | |
| -0.2 | 0.5 | 0.3994 | *0.3626* | 0.4505 | *0.3624* | 0.5021 | *0.3623* | | |
| | 1.0 | 0.3347 | *0.3096* | 0.3686 | *0.3092* | 0.4028 | *0.3091* | | |
| | 1.5 | 0.2912 | *0.2714* | 0.3171 | *0.2710* | 0.3432 | *0.2707* | | |
| | 2.0 | 0.2576 | *0.2411* | 0.2787 | *0.2407* | 0.2999 | *0.2404* | | |
| | 2.5 | 0.2302 | *0.2160* | 0.2479 | *0.2155* | 0.2657 | *0.2153* | | |
| -0.06 | 0.5 | 0.3831 | *0.3619* | 0.4104 | *0.3619* | 0.4376 | *0.3619* | 0.4647 | *0.3618* |
| | 1.0 | 0.3234 | *0.3086* | 0.3416 | *0.3086* | 0.3597 | *0.3086* | 0.3777 | *0.3085* |
| | 1.5 | 0.2821 | *0.2702* | 0.2961 | *0.2702* | 0.3100 | *0.2702* | 0.3238 | *0.2702* |
| | 2.0 | 0.2500 | *0.2399* | 0.2614 | *0.2398* | 0.2727 | *0.2398* | 0.2840 | *0.2398* |
| | 2.5 | 0.2236 | *0.2147* | 0.2332 | *0.2147* | 0.2427 | *0.2146* | 0.2522 | *0.2146* |

**Table 14a:** The expected actual error rate of the EDC under the case of "equivalence" when $\Sigma = AR(1)$ (with negative ρ in case A4 ).

| ρ | $\Delta^2$ | p=4 | | p=8 | | p=12 | | p=16 | |
|---|---|---|---|---|---|---|---|---|---|
| | | $p_p^{(E)}$ | $p_A^{(E)}$ | $p_p^{(E)}$ | $p_A^{(E)}$ | $p_p^{(E)}$ | $p_A^{(E)}$ | $p_p^{(E)}$ | $p_A^{(E)}$ |
| -0.06 | 0.5 | 0.3463 | *0.3792* | 0.3267 | *0.4034* | 0.3084 | *0.4325* | 0.2915 | *0.4912* |
| | 1.0 | 0.2986 | *0.3211* | 0.2867 | *0.3384* | 0.2771 | *0.3607* | 0.2772 | *0.4133* |
| | 1.5 | 0.2628 | *0.2805* | 0.2546 | *0.2948* | 0.2492 | *0.3142* | 0.2582 | *0.3652* |
| | 2.0 | 0.2341 | *0.2488* | 0.2281 | *0.2612* | 0.2254 | *0.2789* | 0.2401 | *0.3293* |
| | 2.5 | 0.2101 | *0.2227* | 0.2057 | *0.2338* | 0.2048 | *0.2503* | 0.2235 | *0.3004* |
| 0.00 | 0.5 | 0.3456 | *0.3784* | 0.3244 | *0.3996* | 0.3032 | *0.4208* | 0.2820 | *0.4420* |
| | 1.0 | 0.2976 | *0.3200* | 0.2835 | *0.3341* | 0.2694 | *0.3481* | 0.2553 | *0.3622* |
| | 1.5 | 0.2617 | *0.2793* | 0.2509 | *0.2901* | 0.2401 | *0.3009* | 0.2293 | *0.3117* |
| | 2.0 | 0.2328 | *0.2474* | 0.2240 | *0.2562* | 0.2153 | *0.2650* | 0.2065 | *0.2728* |
| | 2.5 | 0.2088 | *0.2213* | 0.2014 | *0.2287* | 0.1940 | *0.2361* | 0.1866 | *0.2434* |
| 0.01 | 0.5 | 0.3456 | *0.3784* | 0.3244 | *0.3997* | 0.3033 | *0.4209* | 0.2821 | *0.4422* |
| | 1.0 | 0.2976 | *0.3200* | 0.2836 | *0.3341* | 0.2695 | *0.3483* | 0.2555 | *0.3624* |
| | 1.5 | 0.2617 | *.2793* | 0.2509 | *0.2901* | 0.2402 | *0.3010* | 0.2294 | *0.3119* |
| | 2.0 | 0.2329 | *0.2475* | 0.2241 | *0.2563* | 0.2154 | *0.2652* | 0.2066 | *0.2740* |
| | 2.5 | 0.2088 | *0.2213* | 0.2015 | *0.2288* | 0.1941 | *0.2362* | 0.1868 | *0.2436* |
| 0.2 | 0.5 | 0.3491 | *0.3823* | 0.3293 | *0.4075* | 0.3108 | *0.4346* | 0.2948 | *0.4644* |
| | 1.0 | 0.3037 | *0.3266* | 0.2924 | *0.3456* | 0.2811 | *0.3650* | 0.2711 | *0.3860* |
| | 1.5 | 0.2693 | *0.2874* | 0.2621 | *0.3038* | 0.2541 | *0.3198* | 0.2470 | *0.3368* |
| | 2.0 | 0.2414 | *0.2566* | 0.2367 | *0.2714* | 0.2309 | *0.2853* | 0.2255 | *0.2999* |
| | 2.5 | 0.2180 | *0.2311* | 0.2151 | *0.2448* | 0.2107 | *0.2573* | 0.2066 | *0.2702* |
| 0.4 | 0.5 | 0.3515 | *0.3847* | 0.3093 | *0.3891* | 0.2573 | *0.3844* | 0.1981 | *0.3728* |
| | 1.0 | 0.3141 | *0.3375* | 0.2895 | *0.3450* | 0.2567 | *0.3449* | 0.2192 | *0.3402* |
| | 1.5 | 0.2842 | *0.3031* | 0.2680 | *0.3125* | 0.2443 | *0.3148* | 0.2170 | *0.3137* |
| | 2.0 | 0.2592 | *0.2754* | 0.2483 | *0.2861* | 0.2303 | *0.2901* | 0.2093 | *0.2913* |
| | 2.5 | 0.2379 | *0.2522* | 0.2307 | *0.2639* | 0.2167 | *0.2690* | 0.2002 | *0.2719* |
| 0.65 | 0.5 | 0.3309 | *0.3611* | 0.1293 | *0.2032* | ** | ** | | |
| | 1.0 | 0.3169 | *0.3389* | 0.1839 | *0.2371* | ** | ** | | |
| | 1.5 | 0.2996 | *0.3181* | 0.1985 | *0.2426* | ** | *0.1007* | | |
| | 2.0 | 0.2831 | *0.2995* | 0.2018 | *0.2405* | 0.0639 | *0.1252* | | |
| | 2.5 | 0.2680 | *0.2830* | 0.2006 | *0.2357* | 0.0836 | *0.1390* | | |

**Table 15** : The expected plug-in and the expected actual error rates of the EDC under the case of " non - equivalence " with $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ ( i.e. case P1, n1=n2=50).

| ρ | $\Delta^2$ | p=4 | | p=8 | | p=12 | | p=16 | |
|---|---|---|---|---|---|---|---|---|---|
| | | $p_P^{(E)}$ | $p_A^{(E)}$ | $p_P^{(E)}$ | $p_A^{(E)}$ | $p_P^{(E)}$ | $p_A^{(E)}$ | $p_P^{(E)}$ | $p_A^{(E)}$ |
| -0.06 | 0.5 | 0.3545 | *0.3626* | 0.3456 | *0.3645* | 0.3385 | *0.3686* | 0.3392 | *0.3868* |
| | 1.0 | 0.3041 | *0.3096* | 0.2994 | *0.3120* | 0.2973 | *0.3175* | 0.3096 | *0.3420* |
| | 1.5 | 0.2671 | *0.2714* | 0.2644 | *0.2742* | 0.2649 | *0.2805* | 0.2836 | *0.3091* |
| | 2.0 | 0.2376 | *0.2411* | 0.2362 | *0.2441* | 0.2381 | *0.2509* | 0.2613 | *0.2825* |
| | 2.5 | 0.2131 | *0.2160* | 0.2125 | *0.2192* | 0.2157 | *0.2265* | 0.2418 | *0.2599* |
| 0.00 | 0.5 | 0.3537 | *0.3618* | 0.3431 | *0.3618* | 0.3325 | *0.3618* | 0.3219 | *0.3618* |
| | 1.0 | 0.3031 | *0.3085* | 0.2960 | *0.3085* | 0.2890 | *0.3085* | 0.2820 | *0.3085* |
| | 1.5 | 0.2659 | *0.2702* | 0.2605 | *0.2702* | 0.2551 | *0.2702* | 0.2497 | *0.2702* |
| | 2.0 | 0.2363 | *0.2398* | 0.2319 | *0.2398* | 0.2275 | *0.2398* | 0.2231 | *0.2398* |
| | 2.5 | 0.2117 | *0.2146* | 0.2080 | *0.2146* | 0.2043 | *0.2146* | 0.2006 | *0.2146* |
| 0.01 | 0.5 | 0.3537 | *0.3618* | 0.3432 | *0.3619* | 0.3326 | *0.3619* | 0.3220 | *0.3619* |
| | 1.0 | 0.3031 | *0.3086* | 0.2961 | *0.3086* | 0.2891 | *0.3086* | 0.2821 | *0.3087* |
| | 1.5 | 0.2659 | *0.2702* | 0.2606 | *0.2702* | 0.2552 | *0.2703* | 0.2499 | *0.2703* |
| | 2.0 | 0.2363 | *0.2398* | 0.2320 | *0.2398* | 0.2275 | *0.2398* | 0.2233 | *0.2398* |
| | 2.5 | 0.2117 | *0.2146* | 0.2081 | *0.2147* | 0.2043 | *0.2146* | 0.2008 | *0.2147* |
| 0.2 | 0.5 | 0.3584 | *0.3677* | 0.3500 | *0.3706* | 0.3414 | *0.3720* | 0.3338 | *0.3728* |
| | 1.0 | 0.3100 | *0.3163* | 0.3063 | *0.3202* | 0.3016 | *0.3221* | 0.2971 | *0.3232* |
| | 1.5 | 0.2742 | *0.2791* | 0.2729 | *0.2836* | 0.2700 | *0.2858* | 0.2670 | *0.2871* |
| | 2.0 | 0.2455 | *0.2495* | 0.2456 | *0.2544* | 0.2439 | *0.2568* | 0.2419 | *0.2582* |
| | 2.5 | 0.2214 | *0.2248* | 0.2226 | *0.2301* | 0.2217 | *0.2326* | 0.2203 | *0.2341* |
| 0.4 | 0.5 | 0.3663 | *0.3810* | 0.3477 | *0.3861* | 0.3226 | *0.3880* | 0.2935 | *0.3890* |
| | 1.0 | 0.3242 | *0.3343* | 0.3153 | *0.3411* | 0.3002 | *0.3437* | 0.2821 | *0.3450* |
| | 1.5 | 0.2921 | *0.3000* | 0.2880 | *0.3080* | 0.2777 | *0.3110* | 0.2648 | *0.3126* |
| | 2.0 | 0.2658 | *0.2724* | 0.2648 | *0.2813* | 0.2575 | *0.2846* | 0.2479 | *0.2863* |
| | 2.5 | 0.2436 | *0.2492* | 0.2447 | *0.2587* | 0.2395 | *0.2623* | 0.2322 | *0.2641* |

**Table 15a** : The expected plug-in and the expected actual error rates of the EDC under the case of " non - equivalence " with $\Sigma = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ ( i.e. case P1, n1=n2=100).

| ρ | $\Delta^2$ | p=4 | | | p=8 | | | p=12 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $p_p^{(L)}$ | $p_A^{(L)}$ | *true* | $p_p^{(L)}$ | $p_A^{(L)}$ | *true* | $p_p^{(L)}$ | $p_A^{(L)}$ | *true* |
| -0.06 | 0.5 | 0.3373 | *0.3783* | *0.3618* | 0.3049 | *0.3939* | *0.3618* | 0.2722 | | *0.3618* |
| | 1.0 | 0.2865 | *0.3145* | *0.3085* | 0.2576 | *0.3232* | *0.3085* | 0.2283 | | *0.3085* |
| | 1.5 | 0.2490 | *0.2673* | *0.2701* | 0.2211 | *0.2730* | *0.2701* | 0.1927 | | *0.2701* |
| | 2.0 | 0.2190 | *0.2346* | *0.2398* | 0.1917 | *0.2317* | *0.2397* | 0.1637 | | *0.2398* |
| | 2.5 | 0.1943 | *0.1926* | *0.2146* | 0.1675 | *0.1955* | *0.2146* | 0.1401 | | *0.2146* |
| 0.00 | 0.5 | 0.3373 | *0.3857* | *0.3618* | 0.3051 | *0.3954* | *0.3618* | 0.2729 | *0.4133* | *0.3618* |
| | 1.0 | 0.2866 | *0.3148* | *0.3085* | 0.2579 | *0.3242* | *0.3085* | 0.2291 | | *0.3085* |
| | 1.5 | 0.2490 | *0.2675* | *0.2702* | 0.2214 | *0.2737* | *0.2702* | 0.1937 | | *0.2701* |
| | 2.0 | 0.2191 | *0.2277* | *0.2398* | 0.1920 | *0.2319* | *0.2398* | 0.1649 | | *0.2398* |
| | 2.5 | 0.1944 | *0.1925* | *0.2146* | 0.1678 | *0.1953* | *0.2146* | 0.1412 | *0.1994* | *0.2146* |
| 0.01 | 0.5 | 0.3373 | *0.3787* | *0.3618* | 0.3051 | *0.3953* | *0.3618* | 0.2728 | | *0.3618* |
| | 1.0 | 0.2866 | *0.3148* | *0.3085* | 0.2578 | *0.3242* | *0.3085* | 0.2291 | | *0.3085* |
| | 1.5 | 0.2490 | *0.2674* | *0.2701* | 0.2214 | *0.2737* | *0.2702* | 0.1937 | | *0.2701* |
| | 2.0 | 0.2191 | *0.2277* | *0.2398* | 0.1920 | *0.2319* | *0.2398* | 0.1649 | | *0.2398* |
| | 2.5 | 0.1944 | *0.1926* | *0.2140* | 0.1678 | *0.1953* | *0.2146* | 0.1412 | | *0.2146* |
| 0.2 | 0.5 | 0.3370 | *0.3766* | *0.3618* | 0.3047 | *0.3899* | *0.3618* | 0.2724 | *0.4477* | *0.3618* |
| | 1.0 | 0.2862 | *0.3135* | *0.3085* | 0.2573 | *0.3206* | *0.3085* | 0.2285 | | *0.3085* |
| | 1.5 | 0.2486 | *0.2669* | *0.2701* | 0.2207 | *0.2909* | *0.2701* | 0.1930 | | *0.2701* |
| | 2.0 | 0.2187 | *0.2280* | *0.2398* | 0.1914 | *0.2308* | *0.2397* | 0.1642 | | *0.2398* |
| | 2.5 | 0.1939 | *0.1937* | *0.2146* | 0.1671 | *0.1956* | *0.2146* | 0.1405 | *0.1984* | *0.2146* |
| 0.4 | 0.5 | 0.3360 | *0.3685* | *0.3618* | 0.3036 | *0.3673* | *0.3618* | 0.2172 | *0.4359* | *0.3618* |
| | 1.0 | 0.2850 | *0.3073* | *0.3085* | 0.2559 | *0.3008* | *0.3085* | 0.2270 | | *0.3085* |
| | 1.5 | 0.2472 | *0.2623* | *0.2701* | 0.2192 | *0.2538* | *0.2702* | 0.1914 | | *0.2701* |
| | 2.0 | 0.2172 | *0.2252* | *0.2397* | 0.1897 | *0.2155* | *0.2397* | 0.1625 | | *0.2398* |
| | 2.5 | 0.1924 | *0.1927* | *0.2146* | 0.1655 | *0.1826* | *0.2146* | 0.1388 | *0.1699* | *0.2146* |
| 0.65 | 0.5 | 0.3312 | | *0.3618* | 0.2984 | | *0.3618* | 0.2660 | | *0.3618* |
| | 1.0 | 0.2788 | | *0.3085* | 0.2493 | | *0.3085* | 0.2204 | | *0.3085* |
| | 1.5 | 0.2404 | | *0.2701* | 0.2120 | | *0.2702* | 0.1841 | | *0.2701* |
| | 2.0 | 0.2101 | | *0.2398* | 0.1823 | | *0.2398* | 0.1550 | | *0.2398* |
| | 2.5 | 0.1853 | | *0.2146* | 0.1580 | | *0.2146* | 0.1312 | | *0.2146* |

**Table 16 :** The expected plug-in and expected actual error rates of the LDF under the case of "non-equivalence" with $\Sigma = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ (i.e. case P1).

| ρ | $\Delta^2$ | p=4 | | p=8 | | p=12 | | p=16 | |
|---|---|---|---|---|---|---|---|---|---|
| | | $p_p^{(E)}$ | $p_A^{(E)}$ | $p_p^{(E)}$ | $p_A^{(E)}$ | $p_p^{(E)}$ | $p_A^{(E)}$ | $p_p^{(E)}$ | $p_A^{(E)}$ |
| 0.00 | 0.5 | 0.3456 | *0.3784* | 0.3244 | *0.3996* | 0.3032 | *0.4208* | 0.2820 | *0.4420* |
| | 1.0 | 0.2976 | *0.3200* | 0.2835 | *0.3341* | 0.2694 | *0.3481* | 0.2553 | *0.3622* |
| | 1.5 | 0.2617 | *0.2793* | 0.2509 | *0.2901* | 0.2401 | *0.3009* | 0.2293 | *0.3117* |
| | 2.0 | 0.2328 | *0.2474* | 0.2240 | *0.2562* | 0.2153 | *0.2650* | 0.2065 | *0.2738* |
| | 2.5 | 0.2088 | *0.2213* | 0.2014 | *0.2287* | 0.1940 | *0.2361* | 0.1866 | *0.2434* |
| 0.01 | 0.5 | 0.3456 | *0.3784* | 0.3244 | *0.3996* | 0.3032 | *0.4208* | 0.2820 | *0.4420* |
| | 1.0 | 0.2976 | *0.3200* | 0.2835 | *0.3341* | 0.2694 | *0.3482* | 0.2554 | *0.3623* |
| | 1.5 | 0.2617 | *0.2793* | 0.2509 | *0.2901* | 0.2401 | *0.3009* | 0.2293 | *0.3117* |
| | 2.0 | 0.2328 | *0.2475* | 0.2241 | *0.2562* | 0.2153 | *0.2650* | 0.2065 | *0.2738* |
| | 2.5 | 0.2088 | *0.2213* | 0.2014 | *0.2287* | 0.1940 | *0.2361* | 0.1866 | *0.2435* |
| 0.2 | 0.5 | 0.3478 | *0.3812* | 0.3270 | *0.4047* | 0.3062 | *0.4282* | 0.2854 | *0.4517* |
| | 1.0 | 0.3008 | *0.3236* | 0.2870 | *0.3393* | 0.2731 | *0.3549* | 0.2593 | *0.3706* |
| | 1.5 | 0.2655 | *0.2835* | 0.2548 | *0.2955* | 0.2442 | *0.3075* | 0.2335 | *0.3196* |
| | 2.0 | 0.2370 | *0.2520* | 0.2283 | *0.2618* | 0.2196 | *0.2717* | 0.2110 | *0.2815* |
| | 2.5 | 0.2132 | *0.2261* | 0.2059 | *0.2344* | 0.1986 | *0.2426* | 0.1913 | *0.2509* |
| 0.4 | 0.5 | 0.3537 | *0.3886* | 0.3352 | *0.4207* | 0.3163 | *0.4530* | 0.2974 | *0.4852* |
| | 1.0 | 0.3102 | *0.3344* | 0.2978 | *0.3559* | 0.2852 | *0.3776* | 0.2725 | *0.3992* |
| | 1.5 | 0.2770 | *0.2962* | 0.2674 | *0.3129* | 0.2576 | *0.3297* | 0.2477 | *0.3465* |
| | 2.0 | 0.2499 | *0.2661* | 0.2420 | *0.2799* | 0.2340 | *0.2936* | 0.2259 | *0.3074* |
| | 2.5 | 0.2271 | *0.2411* | 0.2204 | *0.2528* | 0.2135 | *0.2645* | 0.2067 | *0.2762* |
| 0.65 | 0.5 | 0.3565 | *0.3931* | 0.3471 | *0.4523* | 0.3414 | *0.5222* | 0.3344 | *0.5928* |
| | 1.0 | 0.3262 | *0.3521* | 0.3199 | *0.3927* | 0.3161 | *0.4404* | 0.3112 | *0.4520* |
| | 1.5 | 0.3004 | *0.3216* | 0.2956 | *0.3537* | 0.2925 | *0.3912* | 0.2887 | *0.4291* |
| | 2.0 | 0.2784 | *0.2967* | 0.2744 | *0.3236* | 0.2719 | *0.3550* | 0.2687 | *0.3866* |
| | 2.5 | 0.2591 | *0.2755* | 0.2558 | *0.2988* | 0.2536 | *0.3258* | 0.2509 | *0.3531* |

**Table 17** : The expected plug-in and the expected actual error rates of the EDC under the case of "non-equivalence" when $\Sigma = AR(1)$ ( with positive ρ in case P2 ).

| $\rho$ | $\Delta^2$ | p=4 | | p=8 | | p=12 | | p=16 | |
|---|---|---|---|---|---|---|---|---|---|
| | | $p_p^{(E)}$ | $p_A^{(E)}$ | $p_p^{(E)}$ | $p_A^{(E)}$ | $p_p^{(E)}$ | $p_A^{(E)}$ | $p_p^{(E)}$ | $p_A^{(E)}$ |
| -0.65 | 0.5 | 0.3498 | *0.3931* | 0.3325 | *0.4523* | 0.3245 | *0.5222* | 0.3169 | *0.5920* |
| | 1.0 | 0.3216 | *0.3521* | 0.3099 | *0.3927* | 0.3045 | *0.4404* | 0.2993 | *0.4520* |
| | 1.5 | 0.2968 | *0.3216* | 0.2877 | *0.3537* | 0.2834 | *0.3912* | 0.2793 | *0.4291* |
| | 2.0 | 0.2754 | *0.2967* | 0.2678 | *0.3236* | 0.2643 | *0.3550* | 0.2608 | *0.3866* |
| | 2.5 | 0.2565 | *0.2755* | 0.2501 | *0.2988* | 0.2470 | *0.3258* | 0.2441 | *0.3531* |
| -0.4 | 0.5 | 0.3525 | *0.3886* | 0.3336 | *0.4207* | 0.3147 | *0.4530* | 0.2958 | *0.4852* |
| | 1.0 | 0.3095 | *0.3344* | 0.2968 | *0.3559* | 0.2841 | *0.3776* | 0.2714 | *0.3992* |
| | 1.5 | 0.2764 | *0.2962* | 0.2666 | *0.3129* | 0.2567 | *0.3297* | 0.2469 | *0.3465* |
| | 2.0 | 0.2494 | *0.2661* | 0.2414 | *0.2799* | 0.2333 | *0.2936* | 0.2252 | *0.3074* |
| | 2.5 | 0.2266 | *0.2441* | 0.2198 | *0.2528* | 0.2129 | *0.2645* | 0.2061 | *0.2762* |
| -0.2 | 0.5 | 0.3477 | *0.3812* | 0.3269 | *0.4047* | 0.3061 | *0.4282* | 0.2853 | *0.4517* |
| | 1.0 | 0.3007 | *0.3236* | 0.2869 | *0.3393* | 0.2730 | *0.3549* | 0.2592 | *0.3706* |
| | 1.5 | 0.2654 | *0.2835* | 0.2547 | *0.2955* | 0.2441 | *0.3075* | 0.2335 | *0.3196* |
| | 2.0 | 0.2369 | *0.2520* | 0.2283 | *0.2618* | 0.2196 | *0.2717* | 0.2109 | *0.2815* |
| | 2.5 | 0.2131 | *0.2661* | 0.2058 | *0.2344* | 0.1985 | *0.2426* | 0.1912 | *0.2509* |
| -0.06 | 0.5 | 0.3458 | *0.3787* | 0.3246 | *0.4001* | 0.3035 | *0.4215* | 0.2823 | *0.4429* |
| | 1.0 | 0.2979 | *0.3203* | 0.2838 | *0.3345* | 0.2698 | *0.3487* | 0.2557 | *0.3630* |
| | 1.5 | 0.2620 | *0.2796* | 0.2512 | *0.2906* | 0.2404 | *0.3015* | 0.2296 | *0.3124* |
| | 2.0 | 0.2332 | *0.2478* | 0.2244 | *0.2567* | 0.2156 | *0.2656* | 0.2069 | *0.2745* |
| | 2.5 | 0.2092 | *0.2217* | 0.2018 | *0.2292* | 0.1944 | *0.2366* | 0.1870 | *0.2441* |

**Table 17a :** The expected plug-in and the expected actual error rates of the EDC under the case of "non -equivalence" when $\Sigma = AR(1)$ ( with negative $\rho$ in case P2 ).

| ρ | $\Delta^2$ | p=4 | | | p=8 | | | p=12 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $p_p^{(L)}$ | $p_A^{(L)}$ | *true* | $p_p^{(L)}$ | $p_A^{(L)}$ | *true* | $p_p^{(L)}$ | $p_A^{(L)}$ | *true* |
| 0.00 | 0.5 | 0.3373 | *0.3906* | *0.3618* | 0.3051 | *0.4228* | *0.3618* | 0.2729 | *0.4550* | *0.3618* |
| | 1.0 | 0.2866 | *0.3292* | *0.3085* | 0.2579 | *0.3579* | *0.3085* | 0.2291 | | *0.3085* |
| | 1.5 | 0.2490 | *0.2835* | *0.2702* | 0.2214 | *0.3112* | *0.2702* | 0.1937 | | *0.2702* |
| | 2.0 | 0.2191 | *0.2447* | *0.2398* | 0.1920 | *0.2719* | *0.2398* | 0.1649 | | *0.2398* |
| | 2.5 | 0.1944 | *0.2102* | *0.2146* | 0.1678 | *0.2368* | *0.2146* | 0.1412 | *0.2634* | *0.2146* |
| 0.01 | 0.5 | 0.3373 | *0.3908* | *0.3618* | 0.3051 | *0.4234* | *0.3618* | 0.2729 | | *0.3618* |
| | 1.0 | 0.2866 | *0.3295* | *0.3085* | 0.2579 | *0.2588* | *0.3085* | 0.2291 | | *0.3085* |
| | 1.5 | 0.2490 | *0.2838* | *0.2701* | 0.2213 | *0.3121* | *0.2701* | 0.1937 | | *0.2701* |
| | 2.0 | 0.2191 | *0.2451* | *0.2398* | 0.1920 | *0.2729* | *0.2398* | 0.1650 | | *0.2398* |
| | 2.5 | 0.1944 | *0.2106* | *0.2146* | 0.1678 | *0.2379* | *0.2146* | 0.1413 | | *0.2146* |
| 0.2 | 0.5 | 0.3372 | *0.3969* | *0.3618* | 0.3049 | *0.4389* | *0.3618* | 0.2727 | *0.4844* | *0.3618* |
| | 1.0 | 0.2864 | *0.3380* | *0.3085* | 0.2577 | *0.3796* | *0.3085* | 0.2289 | | *0.3085* |
| | 1.5 | 0.2488 | *0.2936* | *0.2701* | 0.2211 | *0.3357* | *0.2701* | 0.1935 | | *0.2701* |
| | 2.0 | 0.2189 | *0.2558* | *0.2398* | 0.1918 | *0.2981* | *0.2398* | 0.1647 | | *0.2398* |
| | 2.5 | 0.1941 | *0.2218* | *0.2146* | 0.1675 | *0.2638* | *0.2146* | 0.1409 | *0.3059* | *0.2146* |
| 0.4 | 0.5 | 0.3365 | *0.4064* | *0.3618* | 0.3043 | *0.4669* | *0.3618* | 0.2720 | *0.5169* | *0.3618* |
| | 1.0 | 0.2855 | *0.3517* | *0.3085* | 0.2568 | *0.4167* | *0.3085* | 0.2280 | | *0.3085* |
| | 1.5 | 0.2478 | *0.3096* | *0.2701* | 0.2201 | *0.3772* | *0.2701* | 0.1925 | | *0.2701* |
| | 2.0 | 0.2178 | *0.2733* | *0.2398* | 0.1908 | *0.3414* | *0.2398* | 0.1637 | | *0.2398* |
| | 2.5 | 0.1931 | *0.2402* | *0.2146* | 0.1665 | *0.3075* | *0.2146* | 0.1399 | *0.3747* | *0.2146* |
| 0.65 | 0.5 | 0.3320 | | *0.3618* | 0.2998 | | *0.3618* | 0.2675 | | *0.3618* |
| | 1.0 | 0.2797 | | *0.3085* | 0.2510 | | *0.3085* | 0.2222 | | *0.3085* |
| | 1.5 | 0.2414 | | *0.2702* | 0.2137 | | *0.2702* | 0.1860 | | *0.2702* |
| | 2.0 | 0.2111 | | *0.2398* | 0.1840 | | *0.2398* | 0.1569 | | *0.2398* |
| | 2.5 | 0.1862 | | *0.2146* | 0.1596 | | *0.2146* | 0.1330 | | *0.2146* |

**Table 18 :** The expected plug-in and expected actual error rates of the LDF under the case of " non - equivalence " when $\Sigma = AR(1)$ ( with positive ρ in case P2 ).

| ρ | $\Delta^2$ | p=4 | | | p=8 | | | p=12 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $p_p^{(L)}$ | $p_A^{(L)}$ | *true* | $p_p^{(L)}$ | $p_A^{(L)}$ | *true* | $p_p^{(L)}$ | $p_A^{(L)}$ | *true* |
| -0.65 | 0.5 | 0.3320 | | *0.3618* | 0.2998 | | *0.3618* | 0.2675 | | *0.3618* |
| | 1.0 | 0.2797 | | *0.3085* | 0.2510 | | *0.3085* | 0.2222 | | *0.3085* |
| | 1.5 | 0.2414 | | *0.2702* | 0.2137 | | *0.2702* | 0.1860 | | *0.2702* |
| | 2.0 | 0.2111 | | *0.2398* | 0.1840 | | *0.2398* | 0.1569 | | *0.2398* |
| | 2.5 | 0.1862 | | *0.2146* | 0.1596 | | *0.2146* | 0.1330 | | *0.2146* |
| -0.4 | 0.5 | 0.3365 | | *0.3618* | 0.3043 | | *0.3618* | 0.2720 | | *0.3618* |
| | 1.0 | 0.2855 | | *0.3085* | 0.2568 | | *0.3085* | 0.2280 | | *0.3085* |
| | 1.5 | 0.2478 | | *0.2701* | 0.2201 | | *0.2701* | 0.1925 | | *0.2701* |
| | 2.0 | 0.2178 | | *0.2398* | 0.1908 | | *0.2398* | 0.1637 | | *0.2398* |
| | 2.5 | 0.1931 | | *0.2146* | 0.1665 | | *0.2146* | 0.1399 | | *0.2146* |
| -0.2 | 0.5 | 0.3372 | | *0.3618* | 0.3049 | | *0.3618* | 0.2727 | | *0.3618* |
| | 1.0 | 0.2864 | | *0.3085* | 0.2577 | | *0.3085* | 0.2289 | | *0.3085* |
| | 1.5 | 0.2488 | | *0.2701* | 0.2211 | | *0.2701* | 0.1935 | | *0.2701* |
| | 2.0 | 0.2189 | | *0.2398* | 0.1918 | | *0.2398* | 0.1647 | | *0.2398* |
| | 2.5 | 0.1941 | | *0.2146* | 0.1675 | | *0.2146* | 0.1409 | | *0.2146* |
| -0.06 | 0.5 | 0.3373 | *0.3889* | *0.3618* | 0.3051 | *0.4191* | *0.3618* | 0.2728 | | *0.3618* |
| | 1.0 | 0.2866 | *0.3272* | *0.3085* | 0.2578 | *0.3531* | *0.3085* | 0.2291 | | *0.3085* |
| | 1.5 | 0.2490 | *0.2812* | *0.2702* | 0.2213 | *0.3057* | *0.2702* | 0.1937 | | *0.2702* |
| | 2.0 | 0.2141 | *0.2423* | *0.2397* | 0.1920 | *0.2660* | *0.2397* | 0.1649 | | *0.2397* |
| | 2.5 | 0.1944 | *0.2077* | *0.2146* | 0.1677 | *0.2309* | *0.2146* | 0.1411 | | *0.2146* |

**Table 18a:** The expected plug-in and expected actual error rates of the LDF under the case of " non - equivalence " when $\Sigma = AR(1)$ (with negative ρ in case P2 ).

| ρ | $\Delta^2$ | p=4 | | p=8 | | p=12 | | p=16 | |
|---|---|---|---|---|---|---|---|---|---|
| | | $p_p^{(E)}$ | $p_A^{(E)}$ | $p_p^{(E)}$ | $p_A^{(E)}$ | $p_p^{(E)}$ | $p_A^{(E)}$ | $p_p^{(E)}$ | $p_A^{(E)}$ |
| -0.06 | 0.5 | 0.3483 | *0.3756* | 0.3515 | *0.4244* | 0.5665 | *0.6459* | ** | ** |
| | 1.0 | 0.3003 | *0.3180* | 0.3026 | *0.3504* | 0.4455 | *0.4976* | | |
| | 1.5 | 0.2645 | *0.2777* | 0.2663 | *0.3026* | 0.3759 | *0.4154* | | |
| | 2.0 | 0.2357 | *0.2461* | 0.2373 | *0.2664* | 0.3266 | *0.3583* | | |
| | 2.5 | 0.2116 | *0.2201* | 0.2132 | *0.2372* | 0.2882 | *0.3144* | | |
| 0.00 | 0.5 | 0.3470 | *0.3703* | 0.3261 | *0.3810* | 0.3050 | *0.3916* | 0.2838 | *0.4022* |
| | 1.0 | 0.2994 | *0.3145* | 0.2857 | *0.3216* | 0.2718 | *0.3287* | | |
| | 1.5 | 0.2638 | *0.2750* | 0.2534 | *0.2805* | 0.2428 | *0.2859* | | |
| | 2.0 | 0.2351 | *0.2439* | 0.2268 | *0.2484* | 0.2182 | *0.2529* | | |
| | 2.5 | 0.2112 | *0.2183* | 0.2043 | *0.2221* | 0.1971 | *0.2258* | | |
| 0.01 | 0.5 | 0.3470 | *0.3697* | 0.3263 | *0.3783* | 0.3056 | *0.3856* | 0.2854 | *0..3919* |
| | 1.0 | 0.2994 | *0.3141* | 0.2859 | *0.3199* | 0.2723 | *0.3248* | | |
| | 1.5 | 0.2638 | *0.2747* | 0.2535 | *0.2791* | 0.2431 | *0.2829* | | |
| | 2.0 | 0.2351 | *0.2437* | 0.2269 | *0.2473* | 0.2185 | *0.2504* | | |
| | 2.5 | 0.2112 | *0.2181* | 0.2044 | *0.2211* | 0.1973 | *0.2237* | | |
| 0.2 | 0.5 | 0.3510 | *0.3643* | 0.3426 | *0.3645* | 0.3378 | *0.3643* | 0.3347 | *0.3640* |
| | 1.0 | 0.3020 | *0.3105* | 0.2967 | *0.3107* | 0.2936 | *0.3106* | | |
| | 1.5 | 0.2658 | *0.2719* | 0.2618 | *0.2721* | 0.2595 | *0.2720* | | |
| | 2.0 | 0.2367 | *0.2414* | 0.2336 | *0.2416* | 0.2318 | *0.2415* | | |
| | 2.5 | 0.2126 | *0.2162* | 0.2101 | *0.2163* | 0.2085 | *0.2163* | | |
| 0.4 | 0.5 | 0.3554 | *0.3629* | 0.3524 | *0.3629* | 0.3511 | *0.3628* | 0.3503 | *0.3628* |
| | 1.0 | 0.3050 | *0.3096* | 0.3032 | *0.3096* | 0.3024 | *0.3096* | | |
| | 1.5 | 0.2680 | *0.2712* | 0.2668 | *0.2713* | 0.2662 | *0.2713* | | |
| | 2.0 | 0.2386 | *0.2409* | 0.2377 | *0.2409* | 0.2373 | *0.2409* | | |
| | 2.5 | 0.2141 | *0.2157* | 0.2135 | *0.2158* | 0.2131 | *0.2158* | | |
| 0.65 | 0.5 | 0.3593 | *0.3631* | 0.3586 | *0.3631* | 0.3584 | *0.3632* | | |
| | 1.0 | 0.3076 | *0.3101* | 0.3074 | *0.3102* | 0.3073 | *0.3102* | | |
| | 1.5 | 0.2701 | *0.2718* | 0.2700 | *0.2720* | 0.2700 | *0.2721* | | |
| | 2.0 | 0.2402 | *0.2416* | 0.2403 | *0.2418* | 0.2403 | *0.2418* | | |
| | 2.5 | 0.2155 | *0.2165* | 0.2156 | *0.2167* | 0.2157 | *0.2168* | | |

**Table 19** : The expected plug-in and the expected actual error rate of the EDC under the case of " equivalence " with $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ ( i.e. case P3 ).

| ρ | $\Delta^2$ | p=4 | | p=8 | | p=12 | | p=16 | |
|---|---|---|---|---|---|---|---|---|---|
| | | $p_p^{(E)}$ | $p_A^{(E)}$ | $p_p^{(E)}$ | $p_A^{(E)}$ | $p_p^{(E)}$ | $p_A^{(E)}$ | $p_p^{(E)}$ | $p_A^{(E)}$ |
| -0.06 | 0.5 | 0.3473 | *0.3831* | 0.3269 | *0.4104* | 0.3064 | *0.4376* | 0.4647 | *0.3618* |
| | 1.0 | 0.2997 | *0.3234* | 0.2863 | *0.3416* | 0.2727 | *0.3597* | 0.3777 | *0.3085* |
| | 1.5 | 0.2640 | *0.2821* | 0.2539 | *0.2961* | 0.2435 | *0.3100* | 0.3238 | *0.2702* |
| | 2.0 | 0.2353 | *0.2500* | 0.2272 | *0.2614* | 0.2188 | *0.2727* | 0.2840 | *0.2398* |
| | 2.5 | 0.2114 | *0.2236* | 0.2047 | *0.2332* | 0.1976 | *0.2427* | 0.2522 | *0.2146* |
| 0.00 | 0.5 | 0.3470 | *0.3788* | 0.3261 | *0.4001* | 0.3050 | *0.4214* | 0.2838 | *0.4426* |
| | 1.0 | 0.2994 | *0.3205* | 0.2857 | *0.3347* | 0.2718 | *0.3489* | 0.2578 | *0.3630* |
| | 1.5 | 0.2638 | *0.2798* | 0.2534 | *0.2908* | 0.2428 | *0.3017* | | |
| | 2.0 | 0.2351 | *0.2481* | 0.2268 | *0.2571* | 0.2182 | *0.2660* | | |
| | 2.5 | 0.2112 | *0.2219* | 0.2043 | *0.2296* | 0.1971 | *0.2370* | | |
| 0.01 | 0.5 | 0.3470 | *0.3782* | 0.3261 | *0.3987* | 0.3050 | *0.4191* | 0.2838 | *0.4395* |
| | 1.0 | 0.2994 | *0.3200* | 0.2857 | *0.3337* | 0.2718 | *0.3474* | | |
| | 1.5 | 0.2638 | *0.2795* | 0.2534 | *0.2901* | 0.2428 | *0.3006* | | |
| | 2.0 | 0.2351 | *0.2478* | 0.2268 | *0.2565* | 0.2182 | *0.2650* | | |
| | 2.5 | 0.2112 | *0.2217* | 0.2043 | *0.2291* | 0.1971 | *0.2363* | | |
| 0.2 | 0.5 | 0.3495 | *0.3704* | 0.3316 | *0.3806* | 0.3135 | *0.3907* | 0.2954 | *0.4009* |
| | 1.0 | 0.3014 | *0.3153* | 0.2897 | *0.3221* | 0.2777 | *0.3288* | | |
| | 1.5 | 0.2655 | *0.2761* | 0.2567 | *0.2814* | 0.2474 | *0.2865* | | |
| | 2.0 | 0.2368 | *0.2453* | 0.2296 | *0.2496* | 0.2221 | *0.2537* | | |
| | 2.5 | 0.2127 | *0.2198* | 0.2068 | *0.2234* | 0.2006 | *0.2269* | | |
| 0.4 | 0.5 | 0.3541 | *0.3673* | 0.3417 | *0.3725* | 0.3287 | *0.3776* | 0.3157 | *0.3827* |
| | 1.0 | 0.3052 | *0.3139* | 0.2973 | *0.3176* | 0.2885 | *0.3208* | | |
| | 1.5 | 0.2690 | *0.2756* | 0.2632 | *0.2786* | 0.2563 | *0.2810* | | |
| | 2.0 | 0.2400 | *0.2452* | 0.2355 | *0.2479* | 0.2298 | *0.2497* | | |
| | 2.5 | 0.2158 | *0.2201* | 0.2122 | *0.2225* | 0.2074 | *0.2239* | | |
| 0.65 | 0.5 | 0.3597 | *0.3659* | 0.3549 | *0.3692* | 0.3482 | *0.3712* | | |
| | 1.0 | 0.3095 | *0.3135* | 0.3079 | *0.3170* | 0.3032 | *0.3183* | | |
| | 1.5 | 0.2727 | *0.2757* | 0.2723 | *0.2793* | 0.2689 | *0.2804* | | |
| | 2.0 | 0.2433 | *0.2457* | 0.2439 | *0.2495* | 0.2412 | *0.2503* | | |
| | 2.5 | 0.2189 | *0.2208* | 0.2201 | *0.2246* | 0.2178 | *0.2254* | | |

**Table 20** : The expected plug-in and the expected actual error rates of the EDC under the case of " equivalence " with $\Sigma = AR(1)$ ( i.e. case P4 ).

| ρ | $\Delta^2$ | p=4 | p=8 |
|---|---|---|---|
| | | $P_p^{(L)}$ | $P_p^{(L)}$ |
| -0.06 | 0.5 | 0.3370 | 0.3057 |
| | 1.0 | 0.2863 | 0.2549 |
| | 1.5 | 0.2489 | 0.2180 |
| | 2.0 | 0.2191 | 0.1886 |
| | 2.5 | 0.1946 | 0.1643 |
| 0.00 | 0.5 | 0.3373 | 0.3037 |
| | 1.0 | 0.2867 | 0.2562 |
| | 1.5 | 0.2493 | 0.2197 |
| | 2.0 | 0.2196 | 0.1902 |
| | 2.5 | 0.1951 | 0.1661 |
| 0.01 | 0.5 | 0.3373 | 0.3039 |
| | 1.0 | 0.2867 | 0.2563 |
| | 1.5 | 0.2493 | 0.2198 |
| | 2.0 | 0.2197 | 0.1904 |
| | 2.5 | 0.1951 | 0.1662 |

| ρ | $\Delta^2$ | p=4 | p=8 |
|---|---|---|---|
| | | $P_p^{(L)}$ | $P_p^{(L)}$ |
| 0.2 | 0.5 | 0.3378 | 0.3046 |
| | 1.0 | 0.2873 | 0.2574 |
| | 1.5 | 0.2500 | 0.2211 |
| | 2.0 | 0.2204 | 0.1917 |
| | 2.5 | 0.1959 | 0.1677 |
| 0.4 | 0.5 | 0.3381 | 0.3052 |
| | 1.0 | 0.2877 | 0.2580 |
| | 1.5 | 0.2505 | 0.2217 |
| | 2.0 | 0.2209 | 0.1926 |
| | 2.5 | 0.1964 | 0.1686 |
| 0.65 | 0.5 | 0.3384 | 0.3059 |
| | 1.0 | 0.2882 | 0.2590 |
| | | | |
| | 2.0 | 0.2214 | 0.1939 |
| | 2.5 | 0.1970 | 0.1699 |

**Table 21** : The expected plug-in error rate of the LDF under the case of " equivalence " with $\Sigma = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ (i.e. case P3).

| $\rho$ | $\Delta^2$ | p=4 | p=8 |
|---|---|---|---|
| | | $p_p^{(L)}$ | $p_p^{(L)}$ |
| -0.65 | 0.5 | 0.3250 | 0.2875 |
| | 1.0 | 0.2790 | 0.2350 |
| | 1.5 | | |
| | 2.0 | 0.2011 | 0.1646 |
| | 2.5 | 0.1762 | 0.1394 |
| -0.4 | 0.5 | 0.3350 | 0.3009 |
| | 1.0 | 0.2837 | 0.2524 |
| | 1.5 | | |
| | 2.0 | 0.2161 | 0.1856 |
| | 2.5 | 0.1915 | 0.1612 |
| -0.2 | 0.5 | 0.3362 | 0.3031 |
| | 1.0 | 0.2859 | 0.2552 |
| | 1.5 | | |
| | 2.0 | 0.2186 | 0.1891 |
| | 2.5 | 0.1941 | 0.1648 |
| -0.06 | 0.5 | 0.3371 | 0.3036 |
| | | 0.2865 | 0.2560 |
| | | 0.2491 | 0.2194 |
| | | 0.2194 | 0.1900 |
| | | 0.1949 | 0.1658 |

| $\rho$ | $\Delta^2$ | p=4 | p=8 |
|---|---|---|---|
| | | $p_p^{(L)}$ | $p_p^{(L)}$ |
| 0.00 | 0.5 | 0.3373 | 0.3037 |
| | 1.0 | 0.2867 | 0.2562 |
| | 1.5 | 0.2493 | 0.2197 |
| | 2.0 | 0.2196 | 0.1902 |
| | 2.5 | 0.1951 | 0.1661 |
| 0.01 | 0.5 | 0.3373 | 0.3038 |
| | 1.0 | 0.2867 | 0.2562 |
| | 1.5 | 0.2493 | 0.2197 |
| | 2.0 | 0.2196 | 0.1902 |
| | 2.5 | 0.1951 | 0.1653 |
| 0.2 | 0.5 | 0.3376 | 0.3042 |
| | | 0.2871 | 0.2567 |
| | | 0.2497 | 0.2201 |
| | | 0.2200 | 0.1908 |
| | | 0.1955 | 0.1667 |
| 0.4 | 0.5 | 0.3378 | 0.3044 |
| | | 0.2873 | 0.2570 |
| | | 0.2500 | 0.2205 |
| | | 0.2203 | 0.1913 |
| | | 0.1959 | 0.1671 |
| 0.65 | 0.5 | 0.3378 | 0.3046 |
| | 1.0 | 0.2874 | 0.2573 |
| | 2.0 | 0.2204 | 0.1917 |
| | 2.5 | 0.1959 | 0.1676 |

**Table 22** : The expected plug-in error rate of the LDF under the case of " equivalence " with $\Sigma = AR(1)$ (i.e. case P4).

| | | $\Sigma = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ | | $\Sigma = AR(1)$ | |
|---|---|---|---|---|---|
| ρ | $\Delta^2$ | p=12 | p=16 | p=12 | p=16 |
| | | $p_p^{(L)}$ | $p_p^{(L)}$ | $p_p^{(L)}$ | $p_p^{(L)}$ |
| -0.06 | 0.5 | 0.2722 | 0.2328 | 0.2728 | 0.2406 |
| | 1.0 | 0.2283 | 0.1901 | 0.2291 | 0.2003 |
| | 1.5 | 0.1927 | 0.1544 | 0.1937 | 0.1660 |
| | 2.0 | 0.1637 | 0.1254 | 0.1649 | 0.1378 |
| | 2.5 | 0.1401 | 0.1016 | 0.1411 | 0.1145 |
| 0.00 | 0.5 | 0.2729 | 0.2406 | 0.2729 | 0.2406 |
| | 1.0 | 0.2291 | 0.2004 | 0.2291 | 0.2004 |
| | 1.5 | 0.1937 | 0.1660 | 0.1937 | 0.1660 |
| | 2.0 | 0.1649 | 0.1378 | 0.1649 | 0.1378 |
| | 2.5 | 0.1412 | 0.1146 | 0.1412 | 0.1146 |
| 0.01 | 0.5 | 0.2728 | 0.2406 | 0.2729 | 0.2406 |
| | 1.0 | 0.2291 | 0.2004 | 0.2291 | 0.2004 |
| | 1.5 | 0.1937 | 0.1660 | 0.1937 | 0.1660 |
| | 2.0 | 0.1649 | 0.1378 | 0.1650 | 0.1378 |
| | 2.5 | 0.1412 | 0.1146 | 0.1413 | 0.1146 |
| 0.2 | 0.5 | 0.2724 | 0.2401 | 0.2727 | 0.2406 |
| | 1.0 | 0.2285 | 0.1997 | 0.2289 | 0.2002 |
| | 1.5 | 0.1930 | 0.1653 | 0.1935 | 0.1658 |
| | 2.0 | 0.1642 | 0.1371 | 0.1647 | 0.1376 |
| | 2.5 | 0.1405 | 0.1138 | 0.1409 | 0.1143 |
| 0.4 | 0.5 | 0.2712 | 0.2390 | 0.2720 | 0.2398 |
| | 1.0 | 0.2270 | 0.1982 | 0.2280 | 0.1993 |
| | 1.5 | 0.1914 | 0.1637 | 0.1925 | 0.1648 |
| | 2.0 | 0.1625 | 0.1354 | 0.1637 | 0.1366 |
| | 2.5 | 0.1388 | 0.1121 | 0.1399 | 0.1133 |

**Table 23 :** The expected plug-in error rate of the LDF under the case of " non - equivalence " with $\Sigma = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ or $\Sigma = AR(1)$, (i.e. case P1 or P2 ).

# *Appendix E1*

The following are some values of $\sigma^{ij}$ to explain the behaviour of the expected error rates when $\rho$ is small and negative.

Consider $\Sigma = (1-\rho)\mathbf{I} + \rho\mathbf{J}$ .

ρ=-0.06 and p=4
sigmaInverse =

| | | | |
|---|---|---|---|
| 1.0124 | 0.0690 | 0.0690 | 0.0690 |
| 0.0690 | 1.0124 | 0.0690 | 0.0690 |
| 0.0690 | 0.0690 | 1.0124 | 0.0690 |
| 0.0690 | 0.0690 | 0.0690 | 1.0124 |

ρ=0.40 and p=4
sigmaInverse =

| | | | |
|---|---|---|---|
| 1.3636 | -0.3030 | -0.3030 | -0.3030 |
| -0.3030 | 1.3636 | -0.3030 | -0.3030 |
| -0.3030 | -0.3030 | 1.3636 | -0.3030 |
| -0.3030 | -0.3030 | -0.3030 | 1.3636 |

ρ=-0.06 and p=8
sigmaInverse =

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1.0410 | 0.0976 | 0.0976 | 0.0976 | 0.0976 | 0.0976 | 0.0976 | 0.0976 |
| 0.0976 | 1.0410 | 0.0976 | 0.0976 | 0.0976 | 0.0976 | 0.0976 | 0.0976 |
| 0.0976 | 0.0976 | 1.0410 | 0.0976 | 0.0976 | 0.0976 | 0.0976 | 0.0976 |
| 0.0976 | 0.0976 | 0.0976 | 1.0410 | 0.0976 | 0.0976 | 0.0976 | 0.0976 |
| 0.0976 | 0.0976 | 0.0976 | 0.0976 | 1.0410 | 0.0976 | 0.0976 | 0.0976 |
| 0.0976 | 0.0976 | 0.0976 | 0.0976 | 0.0976 | 1.0410 | 0.0976 | 0.0976 |
| 0.0976 | 0.0976 | 0.0976 | 0.0976 | 0.0976 | 0.0976 | 1.0410 | 0.0976 |
| 0.0976 | 0.0976 | 0.0976 | 0.0976 | 0.0976 | 0.0976 | 0.0976 | 1.0410 |

ρ=0.40 and p=8
sigmaInverse =

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1.4912 | -0.1754 | -0.1754 | -0.1754 | -0.1754 | -0.1754 | -0.1754 | -0.1754 |
| -0.1754 | 1.4912 | -0.1754 | -0.1754 | -0.1754 | -0.1754 | -0.1754 | -0.1754 |
| -0.1754 | -0.1754 | 1.4912 | -0.1754 | -0.1754 | -0.1754 | -0.1754 | -0.1754 |
| -0.1754 | -0.1754 | -0.1754 | 1.4912 | -0.1754 | -0.1754 | -0.1754 | -0.1754 |
| -0.1754 | -0.1754 | -0.1754 | -0.1754 | 1.4912 | -0.1754 | -0.1754 | -0.1754 |
| -0.1754 | -0.1754 | -0.1754 | -0.1754 | -0.1754 | 1.4912 | -0.1754 | -0.1754 |
| -0.1754 | -0.1754 | -0.1754 | -0.1754 | -0.1754 | -0.1754 | 1.4912 | -0.1754 |
| -0.1754 | -0.1754 | -0.1754 | -0.1754 | -0.1754 | -0.1754 | -0.1754 | 1.4912 |

ρ=0.35 and p=8

sigmaInverse =

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1.3824 | -0.1561 | -0.1561 | -0.1561 | -0.1561 | -0.1561 | -0.1561 | -0.1561 |
| -0.1561 | 1.3824 | -0.1561 | -0.1561 | -0.1561 | -0.1561 | -0.1561 | -0.1561 |
| -0.1561 | -0.1561 | 1.3824 | -0.1561 | -0.1561 | -0.1561 | -0.1561 | -0.1561 |
| -0.1561 | -0.1561 | -0.1561 | 1.3824 | -0.1561 | -0.1561 | -0.1561 | -0.1561 |
| -0.1561 | -0.1561 | -0.1561 | -0.1561 | 1.3824 | -0.1561 | -0.1561 | -0.1561 |
| -0.1561 | -0.1561 | -0.1561 | -0.1561 | -0.1561 | 1.3824 | -0.1561 | -0.1561 |
| -0.1561 | -0.1561 | -0.1561 | -0.1561 | -0.1561 | -0.1561 | 1.3824 | -0.1561 |
| -0.1561 | -0.1561 | -0.1561 | -0.1561 | -0.1561 | -0.1561 | -0.1561 | 1.3824 |

ρ=0.50 and p=8

sigmaInverse =

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1.7778 | -0.2222 | -0.2222 | -0.2222 | -0.2222 | -0.2222 | -0.2222 | -0.2222 |
| -0.2222 | 1.7778 | -0.2222 | -0.2222 | -0.2222 | -0.2222 | -0.2222 | -0.2222 |
| -0.2222 | -0.2222 | 1.7778 | -0.2222 | -0.2222 | -0.2222 | -0.2222 | -0.2222 |
| -0.2222 | -0.2222 | -0.2222 | 1.7778 | -0.2222 | -0.2222 | -0.2222 | -0.2222 |
| -0.2222 | -0.2222 | -0.2222 | -0.2222 | 1.7778 | -0.2222 | -0.2222 | -0.2222 |
| -0.2222 | -0.2222 | -0.2222 | -0.2222 | -0.2222 | 1.7778 | -0.2222 | -0.2222 |
| -0.2222 | -0.2222 | -0.2222 | -0.2222 | -0.2222 | -0.2222 | 1.7778 | -0.2222 |
| -0.2222 | -0.2222 | -0.2222 | -0.2222 | -0.2222 | -0.2222 | -0.2222 | 1.7778 |

ρ=0.40 and p=12

sigmaInverse =

Columns 1 through 7

| | | | | | | |
|---|---|---|---|---|---|---|
| 1.5432 | -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 |
| -0.1235 | 1.5432 | -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 |
| -0.1235 | -0.1235 | 1.5432 | -0.1235 | -0.1235 | -0.1235 | -0.1235 |
| -0.1235 | -0.1235 | -0.1235 | 1.5432 | -0.1235 | -0.1235 | -0.1235 |
| -0.1235 | -0.1235 | -0.1235 | -0.1235 | 1.5432 | -0.1235 | -0.1235 |
| -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 | 1.5432 | -0.1235 |
| -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 | 1.5432 |
| -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 |
| -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 |
| -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 |
| -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 |
| -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 |

Columns 8 through 12

| | | | | |
|---|---|---|---|---|
| -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 |
| -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 |
| -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 |
| -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 |
| -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 |
| -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 |
| -0.1235 | -0.1235 | -0.1235 | -0.1235 | -0.1235 |
| 1.5432 | -0.1235 | -0.1235 | -0.1235 | -0.1235 |
| -0.1235 | 1.5432 | -0.1235 | -0.1235 | -0.1235 |
| -0.1235 | -0.1235 | 1.5432 | -0.1235 | -0.1235 |
| -0.1235 | -0.1235 | -0.1235 | 1.5432 | -0.1235 |
| -0.1235 | -0.1235 | -0.1235 | -0.1235 | 1.5432 |

**Important note:

When $\rho$ is positive, the off diagonal elements of the sigma inverse matrix are all negative values. Therefore the sum of all the elements in the inverse matrix is negative.When $\rho$ is negative, the elements of the sigma inverse matrix are all negative values. Therefore the sum of all the elements in the inverse matrix is positive.