A PRE-FILTER FOR RECURSIVE CONTINUOUS-TIME-MODEL PARAMETER ESTIMATION

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Wan-Hing SIEW

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ABSTRACT

This thesis reports the development of a prefiltering technique for estimating the parameters of continuous-time models given by differential equations. This technique is based on some special integrals, named the Fixed Interval Integrals (FII), which result from multiple integrations over intervals of fixed length. An estimation method using this prefiltering technique has several significant features:

- it is capable of estimating both system delay and the parameters of a second order dynamical model.
- it is able to be implemented on discrete-time digital devices.
- it is independent of initial conditions, hence these do not need to be known.
- it uses well-established discrete-time estimation algorithms.

This development starts with the definition of a new operator notation system. It then studies in detail the properties of the fixed interval integration and its relationship to traditional calculus operations. Several possible methods to realize the FII operation using analog and digital filters are also given.

Using these results and some simulation examples, the use of FII in parameter and delay estimation of continuous-time models is demonstrated.

Since the FII is likely to be useful in engineering and mathematics beyond parameter estimation, some other possible applications of the FII are outlined.

TABLE OF CONTENT

page

ACKNOWLEDGEMENT v

LIST OF CONTRIBUTIONS vi

LIST OF MATHEMATICAL STATEMENTS vii

LIST OF FIGURES x

LIST OF TABLES xi

1. INTRODUCTION 1
   - 1.1 Scope of This Thesis 2
   - 1.2 Continuous-Time-Model Parameter Estimation 4
   - 1.3 Some Advantages of using Continuous-time-models to Design control Systems 6
   - 1.4 Organization of this thesis 7

2. LITERATURE SURVEY 8
   - 2.1 Continuous-time parametric Models 9
   - 2.2 A Survey of Methods for Estimating Continuous-time Models 11
     - 2.2.1 General Classification 11
     - 2.2.2 Deterministic Output Error Methods 12
     - 2.2.3 Stochastic Output Error Methods 14
     - 2.2.4 Equation Error Methods 17
   - 2.3 A Survey of Methods for Estimating the Time Delay and Parameters in Continuous-time Models 22
   - 2.4 Summary 24

3. FIXED INTERVAL INTEGRALS AND DEFINITE INTEGRALS 25
   - 3.1 Introduction 26
   - 3.2 A Notational System for Operators 27
     - 3.2.1 Definition of Some Calculus Operators 29
     - 3.2.2 Advantages of the Operator Notation 35
   - 3.3 An elementary algebra of derivative and integral operators 37
   - 3.4 Some General Comments 44

3.5 Properties of FII 45

3.6 summary 51

4. REALIZATION OF FIXED INTERVAL INTEGRALS 52

4.1 Introduction 53

4.2 Frequency Response Method 53
4.2.1 Analog Filters 54
4.2.2 Digital Filters Response Methods 56
4.2.3 Comments on Frequency Response Methods 58

4.3 Hybrid Method 59

4.4 Numerical Method 62
4.4.1 Numerical Method I 64
4.4.2 Numerical Method II 68
4.4.3 Numerical Method III 71
4.4.4 Frequency Response 73
4.4.5 Selecting the Appropriate Numerical Method 88

4.5 Summary 92

5. APPLICATION OF FIXED INTERVAL INTEGRALS 93

5.1 Introduction 94

5.2 FII system descriptions 96

5.3 Parameter Estimation of continuous-time-model 101
5.3.1 FII Parameterised Model Equations 103
5.3.2 Validity of Assumption on Past System Behaviour 108
5.3.3 Selecting the FII Interval (M) 109
5.3.4 Selecting a Parameter Estimation Algorithm 119

5.4 Simultaneous Estimation of Continuous-time-model Parameter and Pure delay 121
5.4.1 FII of Piece-Wise Defined Functions 123
5.4.2 FII of Piece-Wise Constant Functions 126
5.4.3 Delay and Parameter Estimation Equation 134
5.4.4 Solving for Delays and Parameters 145
5.4.5 General Procedure to Estimate Delay and Model Parameters 155

5.5 Other Possible Applications of The FII 168
5.5.1 Solution of Differential Equation 168
5.5.2 Discrete-time Control for Continuous-time Systems 170

5.6 Summary 170

6. CONCLUSION 172
6.1 Concluding Summary 173
6.2 Further Work 175

REFERENCES 177

APPENDICES

A. MATLAB PROGRAMS FOR SIMULATION EXAMPLES 187
FII.m 188
FIIdu.m 189
backdiff.m 190
PRBS.m 191
spectral.m 191
lsimdu.m 192
eg_system.m 193
pinknoise.m 194
delay_sys.m 194
eg5.3_1.m 194
eg5.3_2.m 195
eg5.3_3.m 196
eg5.4_1.m 196
eg5.4_2.m 199

B1. MODELLING OF PAPER-STOCK CONCENTRATION CONTROL SYSTEM 201
B1.1 INTRODUCTION 202
B1.2 UNITS 204
B1.3 MODELLING STOCK CONCENTRATION 205
B1.3.1 Nonlinear Model 205
B1.3.2 Linearized Models 206
B1.4 MEASUREMENT MODELS 209
B1.5 MODELS OF CONCENTRATION DISTURBANCES 210
B1.5.1 Spectral Analysis 211
B1.5.2 Sampling Time 213
B1.5.3 Black Box Modelling 214
B1.6 DISCRETE-TIME STATE SPACE DESCRIPTION 230
B1.6.1 Disturbances State-Space Description 230
B1.6.2 Augmented System 232

B1.6.3 Controllability 234
B1.6.4 Observability 235

B1.7 summary 236

B2. PERFORMANCE COMPARISON OF PAPER MACHINE STOCK CONCENTRATION CONTROLLERS 238

B3 CONTINUOUS-TIME-MODELSELF TUNING CONTROL 263

B3.1 Introduction 264

B3.2 An Overview of the proposed Continuous-time-model Self-tuning Control System 265

B3.3 System 266

B3.4 Controller 268
B3.4.1 Selecting an Appropriate Control Scheme 269
B3.4.2 Smith Predictor 271
B3.4.3 State-feedback Control 273

B3.5 Control Tuner 274

B3.6 Parameter Estimator 275

B3.7 Simulation Study 276
B3.7.1 Concentration Control System 276
B3.7.2 Simulation 279
B3.7.2 Results and Discussion 280

B3.8 Summary 284

C. MATLAB PROGRAMMES FOR SIMULATION STUDY OF CONTINUOUS-TIME-MODEL SELF-TUNING CONTROL 285
CTSTC.m 286
CTSTC_ini.m 288
CTSTC_system.m 289
CTSTC_control.m 289
CTSTC_pe.m 290
CTSTC_tuner 292
FII.m 292
backdiff.m 294

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LIST OF CONTRIBUTIONS

The major contributions in this thesis are :

1. the development of an operator notation system for some calculus operations which is convenient for both theoretical and implementational work.

2. the determination of the relationships between Fixed Interval Integration and some other calculus operations.

3. the development of several realization methods for Fixed Interval Integration.

4. the development of a prefiltering technique based on the Fixed Interval Integration, for estimating parameters and pure delay of continuous-time models described by differential equations.

LIST OF MATHEMATICAL STATEMENTS

		page
Definition 3.2-0	(Sets of Numbers)	29
Definition 3.2-1a	(General Format of Operator Notation)	30
Definition 3.2-1b	(First Notational Abbreviation)	31
Definition 3.2-1c	(Second Notational Abbreviation)	31
Definition 3.2-2	(Order of an Operation)	31
Definition 3.2-3	(Zero Order Operation)	32
Definition 3.2-4	(Operators)	33
Definition 4.4-1	(Delay Interval of Delay Operator)	63
Definition 5.2-1	(Systems of Interest)	96
Definition 5.4-1	(Systems of Interest With Pure Delay)	121
Definition 5.4-2	(Piece-Wise Defined Functions)	123
Definition 5.4-3	(Piece-Wise Constant Functions)	126
Definition 5.4-4	(Portions of Pure Delay)	135
Assumption 3.2-1	(Linearity of Operators)	32
Assumption 3.3-1	(Values of Past Measurements I)	39
Assumption 3.3-2	(Values of Past Measurements II)	40
Lemma 3.2-1	(i+1th Order Backward Difference Involving the ith Power of a Variable)	34
Lemma 3.2-2	(ith Order Backward Difference Involving the ith Power of a Variable)	34
Lemma 4.4-1	(Sum of FII Interval)	68
Lemma 4.4-2	(Difference of FII Interval)	71
Lemma 5.4-1	(FII Description for Delay Systems)	136
Lemma 5.4-2	(Backward Difference of Polynomials of Delay Operator)	137
Lemma 5.4-3	(Sum of Coefficients of Backward Difference)	145
Lemma 5.4-4	(Formula for Fractional Delay of Second Order Systems with System Zero at Non-zero)	151
Lemma 5.4-5	(Solutions for Fractional Delay of Second Order Systems with System Zero at Non-zero)	153

Theorem 3.3-1	(Commutativity of Operators)	37
Theorem 3.3-2	(Commutativity among δ, η, ∇ and σ)	40
Theorem 3.3-3	(Commutativity among δ, η, ∇ and ζ)	42
Theorem 3.3-4	(Relationship between Operators)	43
Theorem 3.3-5	(A Commutativity Framework for ρ, η, and ζ)	44
Theorem 4.4-1	(Numerical Method I)	65
Theorem 4.4-2	(Numerical Method II)	69
Theorem 4.4-3	(Numerical Method III)	71
Theorem 4.4-4	(Frequency Response of Numerical Method I)	71
Theorem 5.2-1	(FII System Description I)	97
Theorem 5.2-2	(FII System Description II)	97
Theorem 5.2-3	(FII System Description III)	98
Theorem 5.4-1	(FII of Piece-Wise Defined Functions)	124
Theorem 5.4-2	(FII of Piece-Wise Constant Functions)	128
Theorem 5.4-3	(Sum of Coefficients of FII for PCF)	129
Theorem 5.4-4	(FII Estimation Equation for Delay Systems)	139
Theorem 5.4-5	(Sum of Coefficients)	146
Theorem 5.4-6	(Formula for Fractional Delay of Second Order Systems with no System Zero)	150
Theorem 5.4-7	(Formula for ν and b_1 of Second Order Systems with System Zero at Zero)	150
Theorem 5.4-8	(Formula for ν and b_1 of Second Order Systems with System Zero at Non-zero)	153
Corollary 3.5-1	(Independence of Constant Initial Conditions)	45
Corollary 3.5-2	(Boundness of FII)	46
Corollary 3.5-3	(Frequency Response of FII)	47
Corollary 3.5-4	(Constant Input Response of the FII)	49
Corollary 3.5-5	(Impulse Response of the FII Operator)	50
Corollary 4.4-1	(Recursive Numerical Method I)	65
Corollary 4.4-2	(Contributor to Approximation Error)	77
Corollary 4.4-3	(Exact Match for Constant Input)	78
Corollary 4.4-4	(Indication for Phase Error)	81
Corollary 4.4-5	(Approximation Error of FII Gain)	82

Corollary 4.4-6	(Invariance of Approximation Error)	83
Corollary 4.4-7	(Stability of FII Numerical Rules)	88
Corollary 5.2-1	(Independence on Constant Initial Conditions)	99
Corollary 5.2-2	(Independence on Derivatives)	100
Corollary 5.2-3	(Independence on Infinitely Accumulating Terms)	100
Corollary 5.3-1	(Linear-in-the-parameters Vector Form of FII System Descriptions)	103
Corollary 5.3-2	(Augmented Vector Form of FII System Descriptions)	104
Corollary 5.4-1	(Alternative Form of FII Operational Polynomial)	129
Corollary 5.4-2	(Linear-in-the-Parameter Form of FII Estimation Equation for Delay systems)	141
Criteria 3.2-1	(An Appropriate Notation System)	28
Criteria 5.3-1	(Parameter Estimation Criteria for CD Approach)	102
Criteria 5.3-2	(Selecting the Interval of FII)	111
Rule 4.4-1	(Trapezoidal Rule for FII)	66
Rule 4.4-2	(Simpson 1/3 Rule for FII)	67
Rule 5.3-1	(Selecting the FII interval for Systems with White Noise)	113
Rule 5.3-2	(Selecting the FII interval for Systems with Colour Noise)	114

LIST OF FIGURES

page

Figure 3.5-1 Frequency response of the FII operator 49

Figure 4.2-1 Frequency response of an analog approximation to FII operation.. 55

Figure 4.2-2 Frequency response of FII in terms of sampling frequency, f_s. 56

Figure 4.3-1 Hybrid Method I 60

Figure 4.3-2 Hybrid Method II.. 60

Figure 4.4-1a Gain of numerical rules for FII................................. 79

Figure 4.4-1b Gain of numerical rules for FII (cont')......................... 80

Figure 4.4-2 "Blow-out" of Simpson 1/3 rule. 87

Figure 5.3-1 Elements of parameter estimation 102

Figure 5.3-2 Simulation result of Simulation Example 5.3-1 for $a_1 = 3$, $a_0 = 3$, $b = 5$.. 107

Figure 5.3-3 Simulation result of Simulation Example 5.3-2: a system with 5% white noise, M=0.7, $a_1 = 3$, a_0 =2 and $b = 5$.. 115

Figure 5.3-4 Error norm for various FII interval, M: a system with 5% white noise... 116

Figure 5.3-5 Spectrum of the noise in the system of Simulation Example 5.3-3.. 117

Figure 5.3-6 Error norm for various FII interval, M: a system with 5% coloured noise.. 118

Figure 5.4-1 Piece-Wise Defined Function.................................... 123

Figure 5.4-2 Delays and time indices.. 135

Figure 5.4-3 Estimates of parameters and delay for Simulation Example 5.4-1.. 163

Figure 5.4-4 Estimates of parameters and delay for Simulation Example 5.4-2.. 167

LIST OF TABLES

Table 4.4-1a Percentage error of numerical formulae 85

Table 4.4-1b Percentage error of numerical formulae (cont')............ 86

Table 5.3-1 Parameter estimates, percentage errors and error norm for various FII interval, M: a system with 5% white noise.. 116

Table 5.3-2 Parameter estimates, percentage errors and error norm for various FII interval, M: a system with 5% coloured noise and $a_1 = 3$, $a_0 = 2$, $b = 5$.......................... 118

Table 5.4-1 Coefficients β_i in FII description of strictly proper second order delay systems 143

Table 5.4-2 Coefficients β_i in FII description of strictly proper second order delay systems when M=5T_s. 144

Table 5.4-3 Estimated values of parameters a and β_i for Simulation Example 5.4-1 161

Table 5.4-4 Calculated values of S_i, d, b and v for Simulation Example 5.4-1 162

Table 5.4-5 Estimates of coefficients β_i for Simulation Example 5.4-2 165

Table 5.4-6 Estimates of Parameters and Delay for Simulation Example 5.4-2 166

CHAPTER ONE

INTRODUCTION

This chapter outlines the scope of the research work reported in this thesis. It also describes some of the standard approaches to parameter estimation. Some advantages of control system design using continuous-time models are also given. These advantages and the recent advent of low cost and fast computers have been the motivation for this research work of parameter estimation of continuous-time models.

1.1 SCOPE OF THIS THESIS

The development of electronic devices and computers in the last two or three decades has had a profound influence on system identification and modern control engineering. In the last decade, users have been able to progress from interacting with main-frame computers that are relatively unresponsive to individual needs, to minicomputers and then to personal computers based on microchips. Now portable and even laptop computers of phenomenal power and capability are available at relatively modest prices.

This development in computational power accompanied by staggering reductions in the cost per unit of computing has had enormous implications for those intending to use computers to implement algorithms they design. Many designs thought a decade ago likely to make impractical cost and computing demands may now be realized. Gawthrop (1982) proposed the use of continuous-time models instead of discrete-time models in designing discrete-time self-tuning control for continuous-time systems. In his approach, continuous-time models are used to design continuous-time control schemes for dynamical systems. The continuous-time control schemes are then realized directly on digital devices that inherently involve discrete-time sampling. This direct realization is achieved using numerical methods for solving differential equations such as the Runge-Kutta method. As the discrete-time behaviour of the digital controller has not been compensated in the control design, a much faster sampling rate is thus needed when compared to the discrete-time model approach, in order to ensure an acceptable approximation. Also, the digital controller requires higher memory and computational capability because it needs to calculate the numerical solution during each control interval. This continuous-time model approach is now receiving increasing attention, perhaps due to the availability of fast and powerful digital controllers in low cost. The need for on-line parameter estimation techniques based on continuous-time models thus increases.

At the outset of the project reported here, a number of parameter estimation techniques based on continuous-time models had been developed. However these techniques are sensitive to the initial system conditions which are usually not known. They also involve special algorithms which are complicated to implement. Consequently, these techniques could not be used for on-line purposes without difficulty.

For this reason, the objective of this research project was to find a parameter estimation technique for continuous-time models which, besides serving the traditional role of system identification, could also be coupled with an adaptive control scheme to form an explicit self-tuning control system based on a continuous-time model.

During the course of this project, a pre-filter for parameter estimation of continuous-time-models has been developed. A parameter estimation technique can then be formed by "clipping on" the pre-filter to most well-developed discrete-time parameter estimation algorithms. This pre-filter is based on a special integral which will be referred to as Fixed Interval Integral (FII). This technique has the advantages of being computationally simple and being insensitive to noise. Most important of all, perhaps, both the parameters and the pure delay of a continuous-time system can be estimated when this pre-filter is used.

It was found later that, although developed independently, the FII involved the same manipulation of sampled data as the "integrated sampling technique" of Schoukens (1990), and the "linear integral filter" of Sagara and Zhao (1990). However both Schoukens, and Sagara and Zhao did not include the problem of pure delay estimation.

The FII technique is based on an approach termed the "Continuous-time model, discrete-time estimation" approach. More details on this approach are given in Section 1.2.

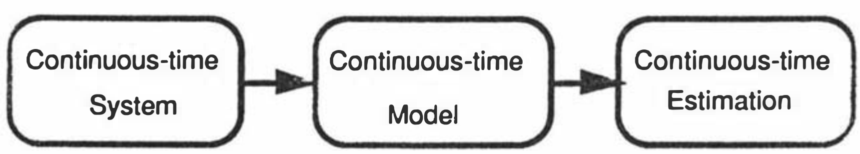
1.2 CONTINUOUS-TIME-MODEL PARAMETER ESTIMATION

There are three possible approaches to parameter estimation of continuous-time systems.

The first is the "Discrete-time model discrete-time estimation" approach (DD). When a system is subjected to a control input that varies only at the beginning of discrete-time intervals, the system, though it might be inherently continuous in time, could be modelled using a discrete-time model and coupled with a discrete-time estimation procedure.

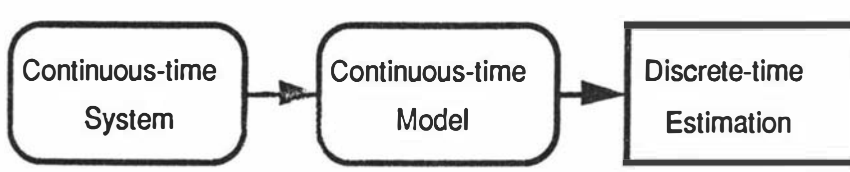
DD parameter Estimation

The second approach might be termed the "Continuous-time model continuous-time estimation" approach (CC). In this approach, the system is first modelled with a continuous-time model, and then the estimates of the parameters in the model are updated in continuous-time.



CC parameter Estimation

The third approach is the "Continuous-time model discrete-time estimation" approach (CD), in which a continuous-time model of the system is estimated in discrete-time rather than in continuous-time.



CD parameter Estimation

In the last two or three decades, the DD approach has dominated the field of system identification. This could be due to two reasons. Firstly, the popularity of digital controllers meant that discrete-time models rather than continuous-time models were the main concern of control system design. Secondly, the most important developments in the area of system identification can often be traced to parallel developments in digital computers; this is no doubt because of the computational complexity involved in system identification and parameter estimation. Therefore, it was logical to go "completely digital": to compute digitally and also to model in discrete-time terms, so that the mathematical characterisation of the system matched the serial processing nature of the digital computer. Consequently, most of the significant advances in system identification of recent decades were concerned with the estimation of discrete-time models based on sampled data (Box and Jenkins 1970, Eykhoff 1974, Goodwin and Payne 1977, Hsia 1977, Åstrom and Wittenmark 1984 and Ljung 1987).

Nevertheless, it seems that the relevance and importance of parameter estimation of continuous-time-models have been increasingly recognized in recent years. This could be due to several advantages of using continuous-time models to design control systems, some of which are gathered together in Section 1.3.

A natural development in parameter estimation of continuous-time models is the CC approach. Many pioneers in parameter estimation of the continuous-time models used 'steepest gradient' techniques (Margolis and Leondes 1959), and 'model-reference-adaptive-system' techniques (Whitaker 1958, Landau 1972, Eykhoff 1974, Petrov and Krutova 1975). These techniques are CC in nature and are mostly implemented on analog devices. A continuous-time equivalent to some fundamental discrete-time estimation techniques such as recursive-least-squares has also been developed in recent years (Gawthrop 1987).

However, as parameter estimation involves considerable computational complexity, it is inevitable from a practical point of view that it be implemented on digital devices which are more flexible and easier in handling computation. The CD approach has been thus receiving increasing interest and it has been a major direction in the recent research of parameter estimation of continuous-time systems. A survey of the related work is presented in Chapter 2. The underlying concept of the CD approach is the transformation of the parameter estimation problem of continuous-time models to a discrete-time-like form, and then most of the ideas used in connection with the estimation of discrete-time models are directly applicable to the estimation of the parameters of the continuous-time system model.

1.3 SOME ADVANTAGES OF USING CONTINUOUS-TIME-MODELS TO DESIGN CONTROL SYSTEMS

There are several advantages of using continuous-time models to design control systems, rather than using the discrete-time models that are either described by shift operators (Ogata 1987) or delta operators (Middleton and Goodwin 1990). These advantages have indirectly motivated this research work on parameter estimation of continuous-time models. Some of these advantages are as follows.

1. The system model relates directly to the physical system. Properties like heat or mass transfer coefficients, tank size, viscosity and time constant can often be observed directly from a continuous-time model. Thus the whole control design procedure can then make more sense in relation to the actual physical system.

2. Artifacts of sampling are avoided. When a continuous-time system with zeros lying inside the stability region is sampled using a zero-order hold, the resultant discrete-time model may have zeros lying outside the stability region. It is well known that unstable zeros limit the performance that can be achieved when controlling a system. Another consequence of zeros is that such systems may be excited by unbounded input pulse sequences giving zero sampled output signal, but the actual continuous-time system may have hidden oscillations, "inter-sample ripple" (Jury 1956), between sampling points. Unstable zeros may occur in systems in which (Åstrom et al 1984):

   a) the number of poles exceeds the number of zeros in the continuous-time system by at least two.

   b) there is a time delay that is not an integral multiple of the sampling time.

   c) the continuous-time system transfer function is improper or non-minimum-phase.

3. The system model and controller design are independent of sampling interval. This is useful for the following reasons:

a) only one common overall continuous-time system model is necessary even though different sub-sections of the identification or the control scheme are implemented at different sampling rates

b) a re-design is not needed when the sampling interval must be changed.

c) in the case of adaptive control, existing system model parameters are not affected by changes in the sampling interval. This means that a new learning phase for parameter estimation is avoided.

1.4 ORGANIZATION OF THIS THESIS

There are six subsequent chapters in this thesis:

Chapter 2 surveys on-line techniques used to estimate the time delay and parameters of continuous-time models.

Chapter 3 develops an operator notation system for use throughout this thesis. It introduces the special Fixed Interval Integrals (FII) and presents a set of operator algebra relating the FII to some traditional calculus operators. It also determines some important properties of the FII.

Chapter 4 determines several possible methods to realize the FII. A detailed analysis is given for the recommended method -- the numerical method.

Chapter 5 develops a parameter estimation technique based on the FII transformation. This technique is capable of estimating simultaneously both the delay and the parameters of a continuous-time model. Other possible applications of FII are also outlined to motivate future study.

Chapter 6 gives the final conclusion of the thesis.

CHAPTER TWO

LITERATURE SURVEY

This chapter surveys methods for on-line time delay and parameter estimation of continuous-time models and categorizes them into different groups. Some features of these methods are compared and contrasted.

2.1 CONTINUOUS-TIME PARAMETRIC MODELS

The systems of interest in this thesis are lumped parameter dynamical systems which:

- are time-invariant;
- have a single output and a single input;
- have an input delay;
- can be described by models in the form of linear ordinary differential equations (ODE).

Thus a description of these systems is the state-space description given by:

$$\frac{dx(t)}{dt} = \mathbf{G}\, x(t) + \mathbf{H}\, u(t-\tau) + \mathbf{M}\, w(t) \tag{2.1-1}$$

$$y(t) = \mathbf{L}\, x(t) + \mathbf{N}\, v(t) \tag{2.1-2}$$

where

$x(t)$ = vector of state variables characterizing the system dynamics.

$y(t)$ = the observed output of the system.

$u(t)$ = the input which is assumed to be known exactly.

τ = the input delay.

$w(t)$ = unmeasurable input disturbances.

$v(t)$ = unmeasurable output disturbances.

The stochastic inputs $w(t)$ and $v(t)$ are taken to be zero-mean white noise independent of $u(t)$. The **G**, **H** and **L** are suitably dimensioned matrices whose elements constitute the unknown system parameters. The **M** and **N** matrices relate the stochastic inputs to the system states. When **M** and **N** are unknown, the unknown input disturbances are,

$$d(t) = \mathbf{M}\, w(t) \tag{2.1-3}$$

and the output disturbances, also unknown, are,

$$n(t) = \mathrm{N}\, v(t) \tag{2.1-4}$$

Another linear representation is the *polynomial matrix description* (PMD) of the general form,

$$y(t) = \frac{\mathrm{A}(\rho)}{\mathrm{B}(\rho)}\, u(t) + \varepsilon(t) \tag{2.1-5}$$

where

$$\mathrm{A}(\rho) = \rho^n + a_{n-1}\,\rho^{n-1} + a_{n-2}\,\rho^{n-2} + \ldots + a_0 \tag{2.1-6}$$

$$\mathrm{B}(\rho) = b_c\,\rho^c + b_{c-1}\,\rho^{c-1} + b_{c-2}\,\rho^{c-2} + \ldots + b_0 \tag{2.1-7}$$

where a_i and b_i are real coefficients some of which may by zero, and ρ^i is the ith derivative operator. The transfer function of the system is thus given by,

$$\frac{\mathrm{A}(\rho)}{\mathrm{B}(\rho)}$$

The $e(t)$ is a vector of stochastic disturbances which accounts for the combined effect of the input and output disturbances $w(t)$ and $v(t)$, respectively, at the output of the system. It is often considered to have rational spectral density and to be of the form,

$$\varepsilon(t) = \frac{\mathrm{D}(\rho)}{\mathrm{C}(\rho)}\,\xi(t) \tag{2.1-8}$$

$$= \mathrm{E}(\rho)\,\xi(t)$$

where

$$\mathrm{E}(\rho) = \frac{\mathrm{D}(\rho)}{\mathrm{C}(\rho)} \tag{2.1-9}$$

is the disturbance transfer function, and $\xi(t)$ is a zero-mean white noise.

The parameter estimation of continuous-time models amounts to estimating the quantities $a_{n-1} \ldots a_0$ and $b_n \ldots b_0$ in Equations (2.1-6) and (2.1-7).

2.2 A SURVEY OF METHODS FOR ESTIMATING DELAY-FREE CONTINUOUS-TIME MODELS

2.2.1 General Classification

The parameters of a lumped linear dynamic system can be estimated either indirectly or directly.

In the case of indirect methods, there are two possible approaches to estimate a continuous-time model, that is,

(i) via non-parametric models
(ii) via discrete-time models

In the first indirect approach, the system is first modelled by a non-parametric description such as its frequency response, impulse response or its step response. A continuous-time parametric model may then be fitted to the non-parametric model (Sanathanan and Koerner 1963, Lawrence and Rogers 1979, Godfery 1980). In contrast the second indirect approach begins by estimating a discrete-time model, and then a continuous-time model is derived from the discrete-time model (Astrom and Eykhoff 1971, Sargan 1974, Unbehauen and Rao 1987, Ogata 1987).

In the case of direct methods for parameter estimation of continuous-time models, the general underlying principle is that the system parameters are chosen to minimize a scalar cost (or loss) function which is formulated in terms of one of a number of possible norms of an error function. This error function reflects the discrepancy between the model and the real system. Based on the nature of the error function, three general classifications of direct methods are possible:

(i) deterministic output error (DOE) methods
(ii) stochastic output error (SOE) methods
(iii) equation error (EE) methods

The output error is the numerical difference between the output of the actual system and the output due to the model. The deterministic methods presume the system is free from unknown disturbances, meanwhile disturbances are taken into account in the stochastic methods. The equation error is generated from the input-output equation of the system. A more in depth discussion of these methods is given in the later sections.

The indirect methods of identifying continuous-time models are generally unsuitable for on-line purposes (Unbehauen and Rao 1987, 1990). This is because they usually involve complex procedures and non-numerical transformations. In view of these, this section concentrates only on the direct methods which are more suitable for the purpose of this study, that is on-line parameter estimation of continuous-time models.

2.2.2 Deterministic Output Error Methods

The deterministic output error (DOE) method is a classical direct approach to the problem of system parameter estimation. Here, the parameters are chosen so that they minimise the error, $e(t)$, between the model output $\hat{y}(t)$, and the observed output $y(t)$. In this deterministic framework, it is assumed that the system output $y(t)$ can be observed exactly, or there is no measurement noise. So the DOE is defined in the form of:

$$e(t) = y(t) - \hat{y}(t) \qquad (2.2\text{-}1)$$

$$= y(t) - \frac{\hat{B}(\rho)}{\hat{A}(\rho)} u(t) \qquad (2.2\text{-}2)$$

where $\hat{B}(\rho)$ and $\hat{A}(\rho)$ are respectively the estimates of the polynomials $B(\rho)$ and $A(\rho)$.

Research on DOE methods has been strongly tied to self adaptive system design using the "Continuous-time model continuous-time estimation" (CC) mechanism. Therefore, as pointed out by Young (1981), it is not easy to segregate those DOE methods specifically for parameter estimation and those related methods aimed at adjusting control system parameters using the model of the process. Noted pioneers of the DOE approach to parameter estimation include Margolis and Leondes (1959), Whitaker (1958) and Donaldson and Leondes (1963). In these works, the update of parameters was based on a steepest descending-gradient of the output error. These algorithms were thus inherently continuous in time and were implemented using analog devices. Conditions for convergence of the CC algorithms for these DOE methods have been studied and developed by researchers such as Shackcloth and Butchart (1965), Landau (1976) and Ljung (1977). Some general multivariable cases were covered by Anderson (1977 and 1985). Texts by Narendra (1976) and Landau (1979) are useful references on these topics.

A later development of the DOE approach was the "adaptive observer". Although there were several different forms of adaptive observer, there were two essential features in all of these adaptive observers. First, the system input and output were passed through a certain form of the Luenberger (1964) observer or its equivalent to obtain the states. Second, the observed states were used to estimate implicitly or explicitly the system parameters which in turn are applied to update the parameters of the observer.

The adaptive observers of Carroll and Lindorff (1973), and Kudva and Narendra (1973) were derived by means of Lyapunov synthesis techniques. Their development depended on a minimal realization for the unknown system and required injection of certain auxiliary signals to ensure convergence. Later modifications, which rested on a non-minimal canonical form for the representation of the unknown system, can be found in work by Luders and Narendra (1974). A complete discussion of minimal and non-minimal realizations has been given by O'Reilly (1983). The methods based on non-minimal realizations eliminated the need of any auxiliary signal and thereby simplified the synthesis of the observer.

Other modifications were the adaptive observers of Ichikawa (1982) and the parameterized observer of Kreisselmeier (1977). These observers converged exponentially and were usually called "fast-convergence observers", as they converged faster than other observers, which converged asymptotically.

Research on the DOE approach was very much limited to the period between the 1960's and 1970's. This could be due to the realization that whilst the DOE methods have yielded interesting simulated results, the concentration on the theoretical basis within a restrictive and somewhat artificial deterministic framework has limited the practical application of these methods (Young 1981). Another reason could be the popularity and the superiority of digital devices set the trend upon "equation error" approaches (see later Section 2.2.4), on which most discrete-time parameter estimation techniques are based.

Nevertheless, the DOE methods formed an important foundation for most modern techniques, especially for adaptive control, because the DOE methods are inherently to be implemented on line. As pointed out by Young (1979), there is a direct link between the adaptive observers and the stochastic EE estimation procedures based on "state variable filters". Further details concerning the EE approaches are given in Section 2.2.4.

2.2.3 Stochastic Output Error Methods

Parameter estimation techniques in this category can usually be identified as one of the following three:

a) Prediction error (PE) Methods
b) Maximum likelihood (ML) Methods
c) Bayesian methods

The prediction error is the discrepancy between the actual system output and the best prediction of the system output given all the current and past information on the system *and* the system disturbance. This usually involves concurrent and explicit estimation of system and system disturbance model parameters. The PE method commonly involves the minimisation of $e(t)$ defined as,

$$e(t) = \frac{\hat{C}}{\hat{D}}\left(y(t) - \frac{\hat{B}}{\hat{A}} u(t) \right) \tag{2.2-3}$$

where $\hat{A}$ and $\hat{B}$ are estimates of the system polynomials A and B given in Section 2.1. $\hat{C}$ and $\hat{D}$ are the estimates of the disturbance polynomials C and D.

The ML method (Kendall and Stuart 1961) is based on the definition of an error function of the PE type, but the formulation is restricted by an additional assumption that the system disturbances are specified by probability distributions.

The Bayesian method (Young 1968) can be considered as an extension of the PE or the ML methods. It arises from the application of the Bayes rule for linking *a priori* to *a posteriori* probability statements and so allows for the inclusion of *a priori* information into the solution of the estimation problem. The most well known example of this kind is a recursive algorithm in the case of discrete-time update; here the covariance of a previous estimate is used to obtain the covariance of the present estimate and the estimate itself (Ljung 1986).

Before proceeding further, it is useful to note that the aforementioned groups of methods are not mutually exclusive. In fact some researchers, such as Young (1981), considered the PE methods as the general set for the Stochastic Output Error (SOE) approaches, the ML methods as a subset of the PE methods, and the Bayesian methods as a subset of ML methods. However, in order to include some modern methods and also, for generality, the set of discrete-time-model methods, it seems to be more convenient to consider these three methods as three separate but mutually inclusive groups.

Early research of SOE originated from the "continuous-time-model discrete-time estimation" (CD) approach, using off-line and batch-processing ML techniques for systems with zero-mean measurement noise. These were the sum-of-square-errors techniques of Box and Coutie (1956) and Box (1960), and the modification by Rosenbrock and Storey (1966) using weighted least squares.

Some extensions of the ML work to enable it to handle more general cases with input and output noises were the PE-ML type CC techniques of Astrom and Kallstrom (1973), Stepner and Mehra (1973), and Balakrishnan (1973). A CD version was later given by Kallstrom *et al* (1976). Basically these were the stochastic version of the "adaptive observer" discussed in the previous section, but with two major differences. The Kalman type filter (Kalman 1960, Kalman and Bucy 1961) was used instead of the Luenberger type observer, and the output error was replaced by the square of the output error.

Mehra and Tyler (1973) and Sastry and Gauvrit (1978) gave a Bayesian form of these observers. Here, in a CD case, the difference between the estimated output from a predictive type Kalman filter and the measured output, were used in a Bayesian type algorithm to update the system states, parameters and the covariance of the estimates. Although the convergence of this approach was not fully dealt with until much later, by Yashin (1986), it had been used successfully in some application work (Mehra and Tyler 1973).

More recently, a different approach to the SOE adaptive observer was given by Chen and Tomizuka (1988). Instead of estimating both the states and the parameters, their observer estimated only the parameters, using some implicit states. The implicit states required for the parameter estimation were generated from the input and output of the system using some pre-determined filters. This approach amounts in fact to an SOE formulation of an EE approach based on state variable filters. More details on this EE approach are given later in Section 2.2.4.

Another popular SOE method is based on the Extended Kalman Filter (EKF). It was first used for a continuous-time system by Kopp and Orford in 1963 (Young 1981).There is one major difference between the EKF and the aforementioned adaptive observer. The adaptive observer employs two separate but dependent estimation routines, used respectively for estimating the system states and the system parameters. In contrast the EKF employs only a single routine to estimate an augmented state vector, $\overset{*}{x}$, which has the general form of,

$$\overset{*}{x} = \begin{bmatrix} x \\ \theta \end{bmatrix} \tag{2.2-4}$$

where x is a vector of the original system states as given in Equation (2.1-1) and θ is the vector of system parameters. As the θ is the multiplier (coefficient) of x, the augmented state equations are thus nonlinear. A number of different methods have been used to solve this nonlinear estimation problem. These included direct linearization (Jazwinski 1970,Young 1974), quasi-linearization (Bellman and Kalaba 1965, Burns and Cliff 1980), invariant imbedding (Bellman *et al* 1960, Bellman and Kalaba 1964) and generalized partitioning (Eulrich *et al* 1980). They all involved some form of linearization about the current estimates. Some asymptotic behaviour of the EKF were given by Ljung (1979) and Westerlund and Tysso (1980).

The EKF had been a popular SOE method prior to 1980 and had reasonable success in application (Young 1981). It is inherently recursive and the system states are estimated concurrently. Thus it can be used for adaptive state-feedback control. However, as pointed out by Young (1981), the EKF is only an approximation to the true PE methods due to the underlying linearization implied by the algorithm. This may have accounted for the rather poor performance of EKF in certain circumstances and the problem of convergence and low statistical efficiency (Ljung 1979, Young 1981). This perhaps is also the reason for the reduced use of the EKF from 1980 on.

2.2.4 Equation Error Methods

The equation error (EE) is a linear algebraic function of the unknown parameters. It has a basic definition of,

$$e(t) = \hat{A}\, y(t) - \hat{B}\, u(t)\ ) \tag{2.2-5}$$

where $\hat{A}$ and $\hat{B}$ are estimates of the system polynomials A and B given in Section 2.1, and $e(t)$ is the EE. As $\hat{A}$ and $\hat{B}$ are polynomials of derivatives in a continuous-time model, this implies the generation of time derivatives of the system output $y(t)$, and input $u(t)$. To avoid the problems that arise from differentiation of a noisy signal, an alternative form of EE is usually used, that is,

$$e(t) = \mathrm{F}\left\{ \hat{A}\, y(t) \right\} - \mathrm{F}\left\{ \hat{B}\, u(t) \right\} \tag{2.2-6}$$

where F{•} is a certain transformation (This form of EE was termed the "Generalized equation error" by some researchers such as Young, 1981).

Early works on using EE methods for continuous-time models can be traced back to Kendall and Stuart (1961) in a deterministic setting. However a large number of different techniques have since been developed in both deterministic and stochastic settings. The EE methods have become the most popular on-line technique in recent years (Young 1981, Unbehauen and Rao 1987, Ljung and Gunnarsson 1990). This could be due to their simplicity and the ease with which they may be developed and implemented.

The EE methods for continuous-time models can be identified in terms of,

- the kind of F{•} transformation that is used.
- the estimation technique or algorithm used to estimate the system parameters using the equation error.

The F{•} transformation can generally be divided into three groups,

- derivative approximation.
- special characterization.
- integral transformation.

The derivative approximation is a straightforward approach in which the derivatives of the system input and output are approximated from the measurements of the input and output. The approximation techniques applied include the use of generalized orthogonal polynomial (Chang *et al* 1986) and block pulse function (Kraus and Schaufelberger 1990). Development of this approach seems very limited. This may be due to the fact the problems arise from differentiation of noisy signal are hard to overcome in this approach. However this approach has the advantage of having the same parameterization as the system equation.

The techniques in the category of special characterization are those involving a pre-determined filtering. This special filtering serves the purpose of eliminating the need for differentiating noisy signals. The resulting model equation depends on the filtering applied. An early technique is the use of state variable filters (Kaya and Yamamura 1962, Valstar 1963, Kohr 1963, Young 1965, 1969, 1976, Young *et al* 1978). In more recent years the state variable filter approach has been developed into the use of the Poisson moment functional (Saha and Rao 1980,1981, 1982, 1983; Sivakumar and Rao 1982). Other techniques include the use of modal function (El-Shafey and Bohn 1987, Kraus and Schaufelberger 1990). A disadvantage of this approach is that the parameters of the resulting model equation differ from the original system parameters. The parameters of the resulting model equation are generally functions of the filter parameters and the system parameters.

A popular approach in recent years is the integral transformation. The integral transformation approach starts by transforming the differential equation of the system model into an integral equation model using multiple integrations. Consider a simple second order differential equation model,

$$\frac{d^2y}{dt^2}(t) + a_1\frac{dy}{dt}(t) + a_0y(t) = bu(t) \tag{2.2-7}$$

The integral equation model of the system is obtained by integrating repeatedly the system's differential equation, that is,

$$\int_0^t\int_0^t \frac{d^2y}{dt^2}(t)\,dt\,dt + a_1\int_0^t\int_0^t \frac{dy}{dt}(t)\,dt\,dt + a_0\int_0^t\int_0^t y(t)\,dt\,dt$$

$$= b\int_0^t\int_0^t u(t)\,dt\,dt \tag{2.2-8}$$

This yields,

$$y(t) + a_1\int_0^t y(t)\,dt + a_0\int_0^t\int_0^t y(t)\,dtdt - t\left(\frac{dy}{dt}(0)+a_1y(0)\right)$$

$$= b\int_0^t\int_0^t u(t)\,dtdt \tag{2.2-9}$$

This method has the advantages that the differentiation of noisy signals is avoided. Furthermore, the integral equation has the same parameters (the a_1, a_0 and b in Equations (2.2-7) and (2.2-9)) as the original differential equation model.

The question now is how to realize the integrals in this integral equation. A large number of techniques have been used. An early technique is the use of orthogonal functions such as the Walsh function (Mathew and Fairman 1974, Rao and Sivakumar 1981). Palanisamy and Bhattacharya (1981) later introduced the use of another orthogonal function, the block pulse function. As the block pulse function was easier generated than Walsh function, it soon became a common approach in early 1980's (Rao 1983, Hwang and Guo 1984, Jiang and Schaufelberger 1985). An extension to state estimation using block pulse function was given by Sinha and Zhou (1984). Mukhopadhyay and Rao (1991) extended this use further to a joint state and parameter estimation in multiple input and output systems. Some related works include the use of orthogonal polynomials such as the Legendre polynomial (Hwang and Guo 1984).

In recent years, the numerical methods are also applied to approximate the integrals. This includes the use of the trapezoidal integration rule (Whitefield and Messali 1987) and Simpson's integration rule (Chao el. al 1987). With availability of low cost and fast digital devices, the numerical approximation seems to be a more attractive technique as the complicated mathematical formulation which is involved in the use of orthogonal functions or polynomials can be avoided.

A major draw-back of the integral transformation approach is that some additional terms corresponding to the initial conditions result from the multiple integration. In the previous second order example, this additional term is the,

$$t\left(\frac{\mathrm{d}y}{\mathrm{d}t}(0)+a_1y(0)\right)$$

in Equation (2.2-9) which consists of the initial system conditions, $\frac{\mathrm{d}y}{\mathrm{d}t}(0)$ and $y(0)$. The complexity of the estimation problem thus increases as the initial conditions are also unknown quantities and need to be estimated concurrently with the system parameters.

Another problem with the integral transformation approach is that the integral equation consists of quantities that might accumulate indefinitely. These quantities are the integrals of the system input and output, and also the time variable t associating with the initial conditions as shown in the above example. As a result, the estimation routine needs to be reset at regular intervals and a learning phase needs to be allowed for after each reset.

In order to overcome the problem of initial conditions in the integral transformation, two techniques have been developed in recent years. They are the "integrated sampling technique" (IST) of Schoukens (1990) and "linear integral filter" (LIF) of Sagara and Zhao (1990). These techniques involve a special integration defined by,

$$\int_{t-M}^{t} y(t)\,\mathrm{d}t$$

where M is a constant determining the interval of integration. The same manipulation of data is involved in the parameter estimation technique to be presented later in this thesis. However, this special integral is referred to as the "fixed interval integral" (FII) as the length of the integration interval is constant.

Almost all of these transformations result in a model description similar to the standard linear-in-the-parameters description (Ljung 1987, Gawthrop 1982) of discrete-time models. In the case of the previous example of integral transformation, the resulting integral model given by Equation (2.2-9) can be written as,

$$x_4(t) + p_3 x_3(t) + p_2 x_2(t) + p_1 x_1(t) = p_0 x_0(t) \tag{2.2-10}$$

where

$$x_4(t) = y(t) \tag{2.2-11}$$

$$x_3(t) = \int_0^t y(t)\, dt \tag{2.2-12}$$

$$x_2(t) = \int_0^t \int_0^t y(t)\, dt dt \tag{2.2-13}$$

$$x_1(t) = t \tag{2.2-14}$$

$$x_0(t) = \int_0^t \int_0^t u(t)\, dt dt \tag{2.2-15}$$

and,

$$p_3 = a_1 \tag{2.2-16}$$

$$p_2 = a_0 \tag{2.2-17}$$

$$p_1 = -\left(\frac{dy}{dt}(0) + a_1 y(0)\right) \tag{2.2-18}$$

$$p_0 = b \tag{2.2-19}$$

Note that the Equation (2.2-10) is linear in terms of the parameters $p_3 \ldots p_0$. The variables $x_4 \ldots x_0$ are quantities to be realized. Therefore Equation (2.2-10) is in the form of a standard linear-in-the-parameters description or a standard regression equation. The problem of estimating the parameters of the continuous-time model thus becomes the estimation of the parameters $p_3 \ldots p_0$ given the quantities $x_4 \ldots x_0$.

The transformed model description can generally be sampled with discrete time intervals. Consequently, most of the reported works used directly the well-established EE estimation algorithm for discrete-time models (Ljung 1987), to estimate the parameters of the transformed model descriptions. However some reported work incorporated specially developed algorithms to suit their purposes. These algorithms included the refined instrumental variable of Young (1976) and the bias compensating least square of Zhao *et al* (1991).

2.3 A SURVEY OF METHODS FOR ESTIMATING THE TIME DELAY AND PARAMETERS IN CONTINUOUS-TIME MODELS

In the last twenty years, several methods have been developed to estimate the unknown time delay and parameters in continuous-time models. These methods can be divided into two general classes, that are the direct methods and the indirect methods. The direct methods base directly on continuous-time models. The indirect methods require a discrete-time approximation of the continuous-time model.

The direct methods can be further divided into two groups. The first group involves two separate stages of estimation routine. The second group uses only one stage of estimation routine.

In the two-stage direct methods, the delay and parameters are estimated separately in two different stages. In the first stage, the parameters are estimated using an assumed value of the delay. These parameter estimates are passed to the second stage to calculate a cost or error function with respect to the time delay. A suitable minimization routine is then applied to find a better estimate of the delay. These two stages are iterated to form the time delay and parameter estimation process. These methods include the moment functional method of Rao and Sivakumar (1976) and their later work (1979) using Walsh functions. The method of Pearson and Wuu (1984) also consists of two separate stages. However, their routine estimates the decoupled delay first using a spline approximation, and then estimates the system parameters.

The one-stage direct methods usually start by approximating the delay term with a rational function. The resulting model is a model of higher order, with the delay value appears as a normal parameter or a combination of the parameters. The delay and parameters are then estimated simultaneously using one of the methods for delay-free systems. Following this principle, Gabay and Merhav (1976), and Agarwal and Canudas (1987) proposed using Pade type approximation; while Gawthrop and Nihtila (1985) introduced an all poles approximation. All these methods used the equation-error approach described in the previous section.

Comparing with the two-stage methods, the one-stage methods require less computation as they involve only one estimation routine. The parameters and delay estimates also converge faster in terms of the number of iterations. The one-stage methods are thus more ideal for on-line purposes.

The indirect methods generally are limited to time delays that are integral multiples of the sampling interval. An exception is the method of Ferretti *el al* (1991) which is based on zero of sampling systems. The indirect methods can also be divided into the two-stage methods and the one-stage methods. The two-stage methods include the correlation method of Zheng and Feng (1990), and the method of Ferretti *et al* (1991) mentioned earlier.

The common one-stage indirect methods are the extended B-polynomial methods. These include the work of Biswas and Singh (1978) and Kurz and Goedecke (1981). In these methods, the numerator of a shift operator model is expanded to include the possible time delay terms. The time delay is then derived by analysing the significance of the estimated numerator coefficients. For example, consider a simple delay system that is described in discrete-time by,

$$y(k) = b\, u(k\text{-}2) \qquad (2.3\text{-}1)$$

where k is the discrete-time index and b is the model parameter.

When the time delay is not known, above description is expanded to give,

$$y(k) = b_0\, u(k) + b_1\, u(k\text{-}1) + b_2\, u(k\text{-}2) + b_3\, u(k\text{-}3) \qquad (2.3\text{-}2)$$

The parameters b_0 ... b_3 are then estimated using an equation-error routine for delay free systems. If the conditions for consistent parameter estimation are fulfilled, the parameters b_0, b_1 and b_3 will converge to zero. The time delay and model parameter can thus be identified simultaneously.

The major requirement of the extended B-polynomial methods is that the possible range of time delay must be *a priori*. Also the range of possible time delay can not be too large, otherwise the computational load will be too high to be practical. However, with the availability of powerful digital devices in recent years, this limitation is becoming less and less significant.

The time delay and parameters estimation method developed in this thesis follows the extend B-polynomial approach. However, it is based directly on continuous-time models rather than discrete-time models. This method will be presented later in Chapter 5.

2.4 SUMMARY

This chapter has surveyed the methods used for the parameter estimation of continuous-time models. The survey concentrated on the methods which will be most suitable for on-line estimation. It was found that among the many different methods that have been employed to estimate on-line the parameters of continuous-time systems, the most popular methods are the equation error methods. Reasons for this are that these methods are easy to develop and to implement on digital computers. Most of the equation error methods can be coupled directly with a well-established discrete-time-model estimation algorithm.

CHAPTER THREE

FIXED INTERVAL INTEGRALS AND DEFINITE INTEGRALS

This chapter presents a special integral transformation called the Fixed Interval Integral transformation, using a modified operator notation. This special integration is developed to overcome some of the problems encountered in using the classical definite integration. The relationship between this special integral and the classical definite integral is then given using an algebra for some related calculus operators.

3.1 INTRODUCTION

The previous chapter outlined some of the many different techniques employed in parameter estimation of continuous-time models. Most techniques involve pre-processing or special transformation of the system signals in order to avoid computation of the derivative terms occurring in the continuous-time model. A popular class of techniques constitutes the Equation-Error approach based on an "integral equation". Among the advantages of this approach are that it is,

1. simple,
2. relatively easy to implement,
3. inherently immune to noise.

However, as mentioned in the previous chapter, these techniques have three undesirable properties which make them difficult to use for practical adaptive control. These undesirable properties are that the techniques,

1. involve terms that accumulate indefinitely with time.
2. depend on some initial conditions at an arbitrary initial instant.
3. depend on all the past quantities back to the arbitrary initial instant.

The infinite accumulation (the first property) means a reset mechanism is needed at a regular interval. Due to the second and third properties above, the problem of initial conditions has to be dealt with at every reset.

These problems motivated the use of a special integral in this research work. This integral has the form:

$$\int_{t-\mathrm{M}}^{t} f(v)\,\mathrm{d}v \qquad (3.1\text{-}1)$$

It is named the Fixed-Interval-Integral (FII) because, at any time instant t, the length of integration interval is fixed by the argument M. The use of this FII to overcome the aforementioned problems will be discussed in detail in Chapter 5. The purpose of this chapter is to establish a convenient algebra for later work, and to determine the relationship of the FII to the derivative operation and to the classical definite integral operation. It is determined later, in Chapters 4 and 5, that this relationship is important in order to derive a suitable realization for the FII, and to derive a delay and parameter estimation technique based on the FII operation.

There are four subsequent sections in this chapter:

Section 3.2 develops a new notational system for some calculus operations.

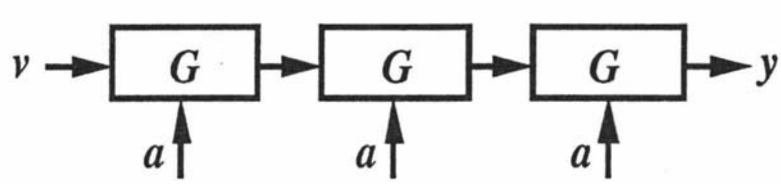
Section 3.3 establishes an operator algebra for these calculus operations. It also determines the relationship of the FII to the derivative and classical definite integral operations.

Section 3.4 gives some general comments about the operator algebra.

Section 3.5 determines other important properties of the FII.

3.2 A NOTATIONAL SYSTEM FOR OPERATORS

Most operations in electrical and electronic engineering can be summarized as the manipulation of signals. This manipulation of signals often consists of a repetition or a cascade of similar elements. Furthermore, these operations might have some auxiliary parameters or variables that need to be pre-assigned or updated from time to time. A popular description of such operations in engineering is the graphical block diagram of the form:



where v is the input signal feeding into a cascade of physical devices, each denoted by G; a is an auxiliary parameter, and the output y is the desired transformation of v. In this description it is very clear:

1. how many times the elementary operation is repeated.
2. which is the input, which is the output and which are the other parameters for the cascade of operations.
3. which are the significant variables for the cascade of operations as a whole. (The intermediate outputs from the first two G blocks are not significant in this case.)

However in mathematics, the common description for these signal manipulations is the functional notation. For the system given in the above block diagram, its description in the functional notation is:

$$G(\,G(\,G(v,a)\,,\,a)\,,a)$$

This description can become very complicated, for example, when a cascade of integrations is represented using Leibnitz's or Newton's integral notation (Spiegel 1974). For example, the difference between two triple integrals is represented by:

$$y(t) = \int_0^t \int_0^{v_3} \int_0^{v_2} f(v_1)\; dv_1 dv_2 dv_3 \;-\; \int_0^t \int_0^{v_3} \int_0^{v_2} g(v_1)\; dv_1 dv_2 dv_3$$

It can be seen from this equation that the functional notation system does not offer the three advantages of block diagrams mentioned above. This notation can become too complicated to enable easy manipulation of terms when there are more elements involved. Furthermore, it makes the argument rather difficult to follow.

Consequently, an alternative notation system is developed in this work. Such a system should fulfill the following criteria:

Criteria 3.2-1 (An Appropriate Notation System)

1. be convenient for manipulation.
2. be convenient for implementation
3. display explicitly all significant variables and parameters.
4. be general in form.

A notational system intended to satisfy these criteria is defined in the next subsection for use throughout this work. It is a system developed from the more conventional operator notation used by a number of authors (Martin 1981, Lee 1987, Marchenko 1988).

3.2.1 Definition of Some Calculus Operators

We will want to represent the derivative operation,

$$\frac{d}{d\nu} f(\nu)$$

by the operator, ρ. We could thus write,

$$\rho \, f(\nu) = \frac{d}{d\nu} f(\nu) \tag{3.2-1}$$

However we may want to evaluate the operation at a particular value of υ, for example at $\nu = t$:

$$\left. \frac{d}{d\nu} f(\nu) \right|_t$$

To represent this, a subscript for the operator notation is thus introduced:

$$\rho_{(\nu=t)} \, f(\nu) = \left. \frac{d}{d\nu} f(\nu) \right|_t \tag{3.2-2}$$

In this section a number of different operators will also be defined, among them the FII given by Equation (3.1-1). The general notational structure for these operators is given in Definition 3.2-1.

Definition 3.2-0 (Sets of numbers)

W is the set of positive integers including zero, { 0,1,2.... }

R is the set of real numbers ♦♦

Definition 3.2-1a (General format of operator notation)

The operations on a function, f, may be represented in the general format,

$$\Theta^{k}_{(\nu_1 = x\ ;\ a_1, a_2, \ldots)}\ \mathrm{f}(\nu_1, \nu_2, \ldots)$$

where

Θ represents a particular operator.

$\mathrm{f}(\nu_1, \nu_2, \ldots.)$ is the operand, which is a function of ν_1, ν_2 ....

$k \in \mathbf{W}$ is the order of the operation defined in Definition 3.2-2.

ν_1 the first argument in the subscribed parenthesis, is the operational variable. It indicates which input variable the operator Θ is to operate on.

x is the output variable of the operation, or the value that ν_1 takes for the operation.

$a_1, a_2 \ldots \in \mathbf{R}$ (unless otherwise specified) are the auxiliary parameters or arguments of the operation. Note that they are separated from the operational variables by a semi-colon. ♦♦

So using this notation, an operator, σ, for the FII given in Equation (3.1-1), may be defined as:

$$\sigma_{(\nu = t\ ;\ \mathrm{M})}\ \mathrm{f}(\nu) \ = \ \int_{t-\mathrm{M}}^{t} f(\nu)\, \mathrm{d}\nu \tag{3.2-3}$$

Though for single integral operations, the advantages of this apparently complex notation may not be apparent at first sight, they will be amply demonstrated in the case of operations involving multiple integrals.

The second operational argument, x, in Definition 3.2-1a, makes for flexible use of the operator. It allows the output variable to be different from the input variable. In the example of the FII operator of Equation (3.2-3), ν is the input variable and t is the output variable. Also it gives a means of identifying sampled information. For example:

$\Theta_{(v=x_1)}$ means the output of the operation is sampled at $v=x_1$

$\Theta_{(v=x_2)}$ means the output of the operation is sampled at $v=x_2$

However there are cases where it may not be necessary to use this second operational argument, for example, when the variable that is to be operated on is obvious, especially when the input is a function of time. Similarly the intermediate outputs in a cascade of similar operations will usually have no significance, as they are simply dummy variables to be passed to the next operation. Consequently, for simplicity in representing a cascade of operations, the following three definitions are useful.

Definition 3.2-1b (First notational abbreviation)

The operator notation employed when the input variable does not need explicit reference is as follows,

$$\Theta_{(x\,;\,a)}\,\mathrm{f}(t) = \Theta_{(t=x\,;\,a)}\,\mathrm{f}(t) \qquad \blacklozenge\blacklozenge$$

Definition 3.2-1c (Second notational abbreviation)

When none of the arguments or variables needs to be specified, we will write,

$$\Theta_1\,\Theta_2 = \Theta_{1(x\,;\,a_1)}\,\Theta_{2(x\,;\,a_2)} \qquad \blacklozenge\blacklozenge$$

Definition 3.2-2 (Order of an operation)

The order of an operation is the number of times the operation is repeated. That is, a kth order operation is defined as,

$$\Theta^k\,\mathrm{f} = \underbrace{\Theta(\Theta\cdots(\Theta(\Theta}_{k}\mathrm{f}))\cdots) \qquad \blacklozenge\blacklozenge$$

Using these abbreviations, a cascade of similar operations represented formally as,

$$\Theta_{(x_2=x\ ;\ a)}\ \Theta_{(x_1=x_2\ ;\ a)}\ \Theta_{(t=x_1\ ;\ a)}\ \mathrm{f}(t)$$

can be more simply represented, without ambiguity, as,

$$\Theta^3_{(x\ ;\ a)}\ \mathrm{f}(t)$$

Definition 3.2-3 (Zero order operation)

The operand is defined to be invariant under a zero order operation, ie:

$$\Theta^0\ \mathrm{f} = \mathrm{f}$$ ♦♦

Assumption 3.2-1 (Linearity of operators)

We will restrict attention to operators that are linear in the following senses:

$$(\Theta_1 + \Theta_2 + \ldots. + \Theta_i)\ \mathrm{f} = \Theta_1\ \mathrm{f} + \Theta_2\ \mathrm{f} + \ldots. + \Theta_i\ \mathrm{f}$$

$$\Theta\ (\mathrm{f}_1 + \mathrm{f}_2 + \ldots. + \mathrm{f}_i) = \Theta\ \mathrm{f}_1 + \Theta\ \mathrm{f}_2 + \ldots. + \Theta\ \mathrm{f}_i$$

Using these definitions, a set of operators is defined for the operations used later in this work. These are given in the next definition.

Definition 3.2-4 (Operators)

Operation	Definition
$\delta_{(v=t;\tau)}\,f(v)$	$f(t-\tau)$
$\rho_{(v=t)}\,f(v)$	$\left.\frac{d}{dv}f(v)\right\vert_t$
$\eta_{(v=t)}\,f(v)$	$\int_0^t f(v)dv$
$\zeta_{(v=t;t_0)}\,f(v)$	$\int_{t_0}^t f(v)dv$
$\sigma_{(v=t;M)}\,f(v)$	$\int_{t-M}^t f(v)dv$
$\nabla_{(v=t;M)}\,f(v)$	$f(t) - f(t-M)$

By these definitions, it is clear that

$\delta_{(v=t;\tau)}\,f(v)$ gives the value of $f(v)$ backward shifted by τ units of time, and evaluated at time t.

$\rho_{(v=t)}\,f(v)$ gives the derivative of $f(v)$ evaluated at time t.

$\eta_{(v=t)}\,f(v)$ gives the (definite) integral of $f(v)$ evaluated between 0 and t.

$\zeta_{(v=t;\,t_0)}\,f(v)$ gives the (definite) integral of $f(v)$ evaluated between t_0 and t.

$\sigma_{(v=t;\,M)}\,f(v)$ gives the fixed-interval integral of $f(v)$ evaluated at time t, with fixed interval of M units of time.

$\nabla_{(v=t;\,M)}\,f(v)$ gives the backward difference of $f(v)$ evaluated at time t, with fixed difference interval of M units of time.

The ζ operator is a generalization of η. These two operators are effectively the same when $t_0 = 0$, ie:

$$\zeta_{(t\ ;\ 0)}\, f(v) = \eta_{(t)}\, f(v)$$

However, they are given as two distinct operators here in order to emphasize the inherent reset mechanism of ζ, at time t_0.

The basic properties of differences can be found in a standard text on finite element differences such as Richardson (1954). Two important properties of differences for this work are given in Lemma 3.2-1 and Lemma 3.2-2.

Lemma 3.2-1

(i+1th order backward difference involving the ith power of a variable)

For all positive integers i, the following holds (Richardson 1954):

$$\nabla_{(x_{i+1}\ ;\ M_{i+1})} \cdots\cdots \nabla_{(x_2\ ;\ M_2)} \nabla_{(v=x_1\ ;\ M_1)}\ v^i = 0$$

where v^i means v to the power of i.

Therefore if,

$$M_{i+1} = M_i = \ldots\ldots = M_1 = M$$

then the above equation can be simplified to:

$$\nabla^{i+1}_{(v=x;\ M)}\ v^i = 0$$

Lemma 3.2-2

(ith order backward difference involving the ith power of a variable)

For all positive integers i, the following holds (Richardson 1954):

$$\nabla^{i}_{(v=x;\ M)}\ v^i = i\,!$$

3.2.2 Advantages of the Operator Notation

The operator notation system defined in the previous section has several advantages that are important to the work presented in this thesis. This section will demonstrate these advantages by using some quantities that will be dealt with often in the later chapters. These quantities are the backward differences of integrals. Let consider the following two quantities,

$$x_1(t_1) = \int_0^{t_1} f(t)\,\mathrm{d}t - \int_0^{t_1-\mathrm{M}} f(t)\,\mathrm{d}t \tag{3.2-4}$$

$$x_2(t_1) = \int_0^{v=t_1} \int_0^{v} f(t)\,\mathrm{d}t\,\mathrm{d}v - 2\int_0^{v=t_1-\mathrm{M}} \int_0^{v} f(t)\,\mathrm{d}t\,\mathrm{d}v$$

$$+ \int_0^{v=t_1-2\mathrm{M}} \int_0^{v} f(t)\,\mathrm{d}t\,\mathrm{d}v \tag{3.2-5}$$

where $x_1(t_1)$ is the first order backward difference of a first order integral, and $x_2(t_1)$ is the second order backward difference of a second order integral. Both of them are to be evaluated at time t_1.

Using the operator notation system defined in the previous section, these two quantities can be rewritten as,

$$x_1(t_1) = \nabla_{(t=t_1;\ \mathrm{M})}\ \eta_{(t)}\, f(t) \tag{3.2-6}$$

$$x_2(t_1) = \nabla^2_{(t=t_1;\ \mathrm{M})}\ \eta^2_{(t)}\, f(t) \tag{3.2-7}$$

The advantages of the operator notation are now examined using Criteria 3.2-1.

<u>1. Be convenient for manipulation</u>

Comparing Equations (3.2-4) and (3.2-5) with Equations (3.2-6) and (3.2-7), it is obvious that the equations using the operator notation are much shorter and simpler. This is more obvious when the order of the difference and integral is high. Therefore, the quantities involved can be bring forward easier to other equations when the operator notation is used.

2. Be convenient for implementation

It is quite clear from Equations (3.2-6) and (3.2-7) that, $x_1(t_1)$ and $x_2(t_1)$ can be realised by cascading backward difference devices and integrals. Meanwhile, careful inspection is required to find out the operations needed to implement Equations (3.2-4) and (3.2-5).

3. Display explicitly all significant variables and parameters

The variables such as t, v and M appear in several places in Equations (3.2-4) and (3.2-5). Careful inspection is needed to comprehend their purposes and significance. However, it is clear from Equations (3.2-6) and (3.2-7) that M is common to all the difference operations, and all the operations operate on the time variable t.

4. Be general in form

Using the operator notation, a general form can be written for $x_1(t_1)$ and $x_2(t_1)$, that is,

$$x_i(t_1) = \nabla^{i}_{(t=t_1;\ M)} \ \eta^{i}_{(t)} f(t) \tag{3.2-8}$$

However, Equations (3.2-4) and (3.2-5) can not be arranged into a simple and general form. ♦♦

Although the work presented in this thesis can be developed using the traditional Leibnitz's integral notation, the use of the operator notation system has enable some of the results to be derived easier. Furthermore, these results can be presented in a simpler and clearer form when the operator notation is used.

3.3 AN ELEMENTARY ALGEBRA OF DERIVATIVE AND INTEGRAL OPERATORS

Using the notation defined in the previous section, an operator algebra representing some elementary properties and relationships of the operators ρ, η, ζ, σ, and ∇ are given in this section. The relevant properties and relationships are grouped together and presented in five theorems, Theorems 3.3-1 to 3.3-5.

Theorem 3.3-1 shows the commutativity for the differential operator ρ, the FII operator σ, the shift operator δ, and also the backward difference operator ∇. Then using Assumptions 3.3-1 and 3.3-2, the commutativity property is extended to include the classical integral operators, η and ζ in Theorems 3.3-2 and 3.3-3. This follows by some relationships between operators in Theorem 3.3-4. These relationships and the commutativity properties are important to the development of the parameter estimation technique in this thesis. The last theorem, Theorem 3.3-5, is included largely for theoretical interest. It presents a framework to enable the differential operator to be commutative with the classical integral operators.

In most of these theorems, the commutativity is shown only for the first order operators. The applicability of these theorems for operators of higher order and some other general comments are given Section 3.4.

These theorems are given in the following. For simplicity, the operands are omitted in most cases.

<u>**Theorem 3.3-1**</u> (Commutativity of operators)

(a) $\rho\,\delta = \delta\,\rho$

(b) $\sigma\,\delta = \delta\,\sigma$

(c) $\nabla\,\rho = \rho\,\nabla$

(d) $\nabla\,\sigma = \sigma\,\nabla$

(e) $\sigma\,\rho = \rho\,\sigma$ ♦♦

Proof

(a)

$$
\begin{aligned}
\rho_{(t)}\delta_{(v;\tau)}.\mathrm{f}(v) &= \rho_{(t)}.\mathrm{f}(v-\tau) \\
&= \frac{d}{dv}\mathrm{f}(v-\tau)|_{v=t} \\
&= \frac{d\,\mathrm{f}(v-\tau)}{d(v-\tau)}.\frac{d(v-\tau)}{dv}|_{v=t} \\
&= \frac{d\,\mathrm{f}(v-\tau)}{d(v-\tau)}|_{v=t} \\
&= \frac{d\,\mathrm{f}(\omega)}{d\omega}\bigg|_{\omega=t-\tau} \qquad ,\omega = \mathrm{v} - \tau \\
&= \frac{d\,\mathrm{f}(v)}{dv}\bigg|_{v=t-\tau} \\
&= \rho_{(t-\tau)}.\mathrm{f}(v) \\
&= \delta_{(t-\tau)}.\rho_{(t)}\mathrm{f}(v)
\end{aligned}
$$

(b)

$$
\begin{aligned}
\sigma_{(t;M)}\delta_{(v;\tau)}.\mathrm{f}(v) &= \sigma_{(t,M)}.\mathrm{f}(v-\tau) \\
&= \int_{t-M}^{t} \mathrm{f}(v-\tau)dv \\
&= \int_{t-\tau-M}^{t-\tau} \mathrm{f}(\omega)d\omega \qquad ,\omega = v-\tau \\
&= \int_{(t-\tau)-M}^{(t-\tau)} \mathrm{f}(v)\,dv \\
&= \sigma_{(t-\tau;M)}.\mathrm{f}(v) \\
&= \delta_{(t;\tau)}\sigma_{(t;M)}.\mathrm{f}(v)
\end{aligned}
$$

(c)

$$
\begin{aligned}
\nabla_{(t;M)}\,\rho_{(t)}\,\mathrm{f}(t) &= \rho_{(t)}\,\mathrm{f}(t) - \rho_{(t-M)}\,\mathrm{f}(t) \\
&= \rho_{(t)}\,\mathrm{f}(t) - \rho_{(t)}\,\mathrm{f}(t-M) \qquad ,\text{by theorem 3.3-1a} \\
&= \rho_{(t)}\nabla_{(t;M)}\,\mathrm{f}(t)
\end{aligned}
$$

(d)

$$
\begin{aligned}
\nabla_{(t;N)}\,\sigma_{(t;M)}\,\mathrm{f}(t) &= \sigma_{(t;M)}\,\mathrm{f}(t) - \sigma_{(t-N;M)}\,\mathrm{f}(t) \\
&= \sigma_{(t;M)}\,\mathrm{f}(t) - \sigma_{(t;M)}\,\mathrm{f}(t-N) \qquad ,\text{by theorem 3.3-1b} \\
&= \sigma_{(t;M)}\nabla_{(t;N)}\,\mathrm{f}(t)
\end{aligned}
$$

(e)

First we have,

$$\sigma_{(t;M)}\rho_{(t)}\,\mathrm{f}(v) \;=\; \int_{t-M}^{t}\frac{d\,\mathrm{f}(v)}{dv}\,dv$$
$$=\; \mathrm{f}(t) \;-\; \mathrm{f}(t-M)$$
$$=\; \nabla_{(t;M)}\,\mathrm{f}(v) \qquad (3.3-1)$$

Now using Leibnitz's rule:

$$\frac{d}{dt}\left[\int_b^a \mathrm{f}(t)\,dt\right] \;=\; \mathrm{f}(a)\frac{da}{dt} \;-\; \mathrm{f}(b)\frac{db}{dt}$$

we have,

$$\rho_{(t)}\sigma_{(t;M)}\,\mathrm{f}(v) \;=\; \frac{d}{dt}\left[\int_{t-M}^{t}\mathrm{f}(v)\,dv\right]$$
$$=\; \mathrm{f}(t)\frac{dt}{dt} \;-\; \mathrm{f}(t-M)\frac{d(t-M)}{dt}$$
$$=\; \mathrm{f}(t) \;-\; \mathrm{f}(t-M)$$
$$=\; \nabla_{(t;M)}\,\mathrm{f}(v) \qquad (3.3-2)$$

Finally, with Equations (3.3-1) and (3.3-2), we have,

$$\rho_{(t)}\sigma_{(t;M)}\,\mathrm{f}(v) \;=\; \sigma_{(t;M)}\rho_{(t)}\,\mathrm{f}(v)$$ ♦ ♦

The above two theorems can be extended to include the η and ζ operator using the following assumptions.

Assumption 3.3-1

f(t) is an integrable function such that :

$\mathrm{f}(t) \;=\; 0$ for all $t < 0$

and thus

$\mathrm{f}(t\text{-M}) \;=\; 0$ for all $t <$ M ♦ ♦

Assumption 3.3-2

$f(t)$ is an integrable function such that:

$$f(t) = 0 \qquad \text{for all } t < t_0$$

and thus

$$f(t-M) = 0 \qquad \text{for all } t < t_0 + M$$ ♦♦

Theorem 3.3-2 (Commutativity among δ, η, ∇ and σ)

If Assumption 3.3-1 holds, then the following are true.

(a) $\eta\,\delta = \delta\,\eta$

(b) $\nabla\,\eta = \eta\,\nabla$

(c) $\sigma\,\eta = \eta\,\sigma$

(d) The above results (a) to (c) are also true for η^k, for all $k \in \mathbf{W}$ ♦♦

Proof

(a)

$$\begin{aligned}
\eta_{(t)}\,f(v-\tau) &= \int_0^t f(v-\tau)\,dv \\
&= \int_{-\tau}^{t-\tau} f(w)\,dw && ,w = v-\tau \\
&= \int_0^{t-\tau} f(w)\,dw && ,\text{by assumption } 3.3\text{-}1 \\
&= \int_0^{t-\tau} f(v)\,dv \\
&= \eta_{(t-\tau)}\,f(v)
\end{aligned}$$

(b)

$$\begin{aligned}
\nabla_{(t;M)}\,\eta_{(t)}\,f(t) &= \eta_{(t)}\,f(t) - \eta_{(t-M)}\,f(t) \\
&= \eta_{(t)}\,f(t) - \eta_{(t)}\,f(t-M) && ,\text{by theorem } 3.3\text{-}2\text{a} \\
&= \eta_{(t)}\nabla_{(t;M)}\,f(t)
\end{aligned}$$

(c) $\eta_{(t)}\,\sigma_{(t;M)}\,\mathrm{f}(v)$

$$
\begin{aligned}
&= \int_0^{t_2} \int_{t_1-M}^{t_1} \mathrm{f}(v)\,dv\,dt_1 \\
&= \int_0^{t_2} \int_0^{t_1} \mathrm{f}(v)\,dv\,dt_1 - \int_0^{t_2} \int_0^{t_1-M} \mathrm{f}(v)\,dv\,dt_1 \\
&= \int_0^{t_2} \int_0^{t_1} \mathrm{f}(v)\,dv\,dt_1 - \int_{-M}^{t_2-M} \int_0^{\omega} \mathrm{f}(v)\,dv\,dw \qquad , \omega = t_1 - M \\
&= \int_0^{t_2} \int_0^{\tau} \mathrm{f}(v)\,dv\,d\tau - \int_0^{t_2-M} \int_0^{\tau} \mathrm{f}(v)\,dv\,d\tau \qquad , \text{by assumption 3.3-1} \\
&= \int_{t_2-M}^{t_2} \int_0^{\tau} \mathrm{f}(v)\,dv\,d\tau \\
&= \sigma_{(t;M)}\,\eta_{(t)}.\mathrm{f}(v)
\end{aligned}
$$

(d)

Because of the inherent associativity of the operator, we can consider a η^{k+1} operation as a η operation, with the result of η^k operation as the operand, that is:

$$\eta_{(t)}^{k+1}\,\mathrm{f}(t) = \eta_{(t)}^{1}\left(\eta_{(t)}^{k}\,\mathrm{f}(t)\right) = \eta_{(t)}\,\mathrm{f}_k(t)$$

where

$$\mathrm{f}_k(t) = \eta_{(t)}^{k}\,\mathrm{f}(t)$$

Therefore Theorems 3.3-2a to 3.3-2c, which require the operand to follow Assumption 3.3-1, are also true for η^{k+1}, if $\mathrm{f}_k(t)$ follows Assumption 3.3-1.

Now to prove that the $\mathrm{f}_k(t)$ follows Assumption 3.3-1 for all k, let:

$k = 1$ and $\mathrm{f}(t)$ obeying Assumption 3.3-1

then

$$\mathrm{f}_k(t) = \eta_{(t)}^{k}\,\mathrm{f}(t) = \eta_{(t)}^{1}\,\mathrm{f}(t) = 0 \qquad , \text{for all } t < 0$$

so by induction, $\mathrm{f}_k(t)$ follows Assumption 3.3-1 for all k. As a result, Theorem 3.3-2a to 3.3-2c are also true for $k = 2, 3, \ldots$.

◆◆

Theorem 3.3-3 (Commutativity among δ, η, ∇ and ζ)

If Assumption 3.3-2 holds then the following are true for all $\tau \in \mathbf{R}$

(a) $\zeta\,\delta = \delta\,\zeta$

(b) $\nabla\,\zeta = \zeta\,\nabla$

(c) $\sigma\,\zeta = \zeta\,\sigma$

(d) $\eta\,\zeta = \zeta\,\eta$

(e) The above results (a) to (c) are also true for ζ^k, for all $k \in \mathbf{N}$ ♦♦

Proof

(a), **(b)**, **(c)** and **(e)** are proved in the same way as Theorems 3.3-2 a to 3.3-2d, but using Assumption 3.3-2.

(d)

$$\eta_{(t)}\zeta_{(t;\,t_0)}\mathrm{f}(t) = \int_0^t \int_{t_0}^{t_2} \mathrm{f}(t_1)\,dt_1 dt_2$$
$$= \int_0^t \left[\int_0^{t_2} \mathrm{f}(t_1)\,dt_1 - \int_0^{t_0} \mathrm{f}(t_1)\,dt_1\right] dt_2$$

but by Assumption 3.3-2,

$$\int_0^{t_0} \mathrm{f}(t_1)\,dt_1 = 0$$

so,

$$\eta_{(t)}\zeta_{(t;t_0)}\mathrm{f}(t) = \int_0^t \int_0^{t_2} \mathrm{f}(t_1)\,dt_1 dt_2$$
$$= \int_{t_0}^t \int_0^{t_2} \mathrm{f}(t_1)\,dt_1 dt_2 = \zeta_{(t;t_0)}\eta_{(t)}\mathrm{f}(t)$$

as $\int_0^{t_0} \int_0^{t_2} \mathrm{f}(t_1)\,dt_1 dt_2 = 0$ by Assumption 3.3-2 . ♦♦

Theorem 3.3-4 (Relationship between operators)

(a) $\sigma = \nabla \eta = \nabla \zeta$

(b) $\sigma \rho = \nabla$

(c) $\rho \eta = \rho \zeta = 1$ ♦

Proof

(a)

$$\begin{aligned} \sigma \mathrm{f} &= \int_{t-M}^{t} \mathrm{f}(v)\,dv \\ &= \int_{t_0}^{t} \mathrm{f}(v)\,dv - \int_{t_0}^{t-M} \mathrm{f}(v)\,dv \\ &= \nabla \zeta \mathrm{f} \end{aligned}$$

By assigning $t_0 = 0$, it can be proven in the same way that, $\sigma = \nabla \eta$.

(b)

$$\begin{aligned} \sigma \rho \mathrm{f} &= \int_{t-M}^{t} \frac{d\mathrm{f}}{dt} dt \\ &= \mathrm{f}(t) - \mathrm{f}(t-M) = \nabla \mathrm{f} \end{aligned}$$

(c) Using the First principle of calculus or Leibnitz's rule,

$$\rho \zeta \mathrm{f} = \frac{d}{dt}\left\{\int_{t_0}^{t} \mathrm{f}(v)\,dv\right\} = \mathrm{f}$$

By assigning $t_0 = 0$, it can be proven in the same way that, $\rho \eta = 1$.

♦ ♦

Theorems 3.3-1 to 3.3-4 show the properties of the ∇, ρ, η, ζ and σ operators. Note that some properties or relationships are true for some operators but not for some others. This is due to the lack of commutativity between the ρ and η and, between the ρ and ζ operators. However, restricted commutativities between these operators can be achieved when the operators are used following the backward-difference:

Theorem 3.3-5 (A commutativity framework for ρ, η, and ζ)

(a) $\nabla \eta \rho = \nabla \rho \eta = \nabla$

(b) $\nabla \zeta \rho = \nabla \rho \zeta = \nabla$

Proof

(a) and **(b)** :

Using Theorems 3.3-4a and 3.3-4b, it can be shown that,

$$\nabla \eta \rho = \nabla \zeta \rho = \sigma \rho = \nabla$$

and using Theorem 3.3-4c,

$$\nabla \rho \eta = \nabla \rho \eta = \nabla$$

♦ ♦

3.4 SOME GENERAL COMMENTS

Three general comments about these results may be made. Firstly, the commutativity given in Theorem 3.3-1 and the inherent associativity imply that the theorem is also true for operators of any order. For example, we can show

$$\nabla \rho^2 = \rho^2 \nabla$$

using the following procedure,

$$\begin{aligned} \nabla \rho^2 = \nabla \rho \rho &= (\nabla \rho) \rho \\ &= (\rho \nabla) \rho && \text{, by Theorem 3.3-1c} \\ &= \rho (\nabla \rho) \\ &= \rho (\rho \nabla) && \text{, by Theorem 3.3-1c} \\ &= \rho^2 \nabla \end{aligned}$$

However the commutativity of the first order η and ζ operators given by Theorems 3.3-2 and 3.3-3 is restricted respectively by Assumption 3.3-1 and 3.3-2. It is thus necessary to prove that these operators of higher order, η^k and ζ^k, have the same commutativity when the assumptions are followed. In view of these, Theorems 3.3-2d and 3.3-3e are presented in order to justify the commutativity of η^k and ζ^k respectively.

Secondly, Theorems 3.3-1 to 3.3-3 mean that:

(i) ρ, σ and ∇ are always commutative with respect to each other.

(ii) η is commutative with respect to σ and ∇, if Assumption 3.3-1 holds.

(iii) ζ is commutative with respect to σ, ∇ and η, if Assumption 3.3-2 holds.

Note that ρ and η, and ρ and ζ are not generally commutative with respect to each other. The commutativity relationships between these operators that are specified in Theorem 3.3-5 are provided largely for theoretical interest.

Finally, the FII relationships given by Theorems 3.3-4a and 3.3-4b are the most important results for the remainder of this thesis.

3.5 PROPERTIES OF FII

Some important properties of the FII are now presented.

<u>Corollary 3.5-1</u> (Independence of Constant Initial Conditions)

The FII, of any order, is independent of any constant initial conditions.

♦♦

Proof

This is directly due to Theorem 3.3-4a; that is because

$$\sigma = \nabla \eta$$

the constant initial conditions that result from the classical definite integration, η, are eliminated by the difference operation ∇. ♦♦

Corollary 3.5-2 (Boundness of FII)

If f(t) is bounded for all time t, then the FII of f(t), of any order is also bounded for all time. ♦♦

Proof

A bounded function, f(t), can be represented by a Fourier cosine series over a period $0 \le t \le L$ for an arbitrarily large L,

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L} t) \tag{3.5-1}$$

where a_0 and a_n are the Fourier half-range cosine coefficients.

Noting the identity:

$$\sin(t) = \cos\left(\frac{\pi}{2} - t\right) \tag{3.5-2}$$

the kth definite integral over the interval $[0, t]$ is thus of the form:

$$\eta^k_{(t)} f(t) = q_0 \frac{t^k}{k!} + \sum_{n=1}^{k-2} b_n \frac{t^n}{n!} + \sum_{n=1}^{\infty} c_n \cos(g_n(t)) \tag{3.5-3}$$

where b_n and c_n are some appropriate constants, and the functions $g_n(t)$ are some appropriate functions of t.

As $\sigma = \nabla\eta$ (Theorem 3.3-4a), $\sigma^k f(t)$ is thus the kth backward difference of Equation (3.5-3). Noting that,

$$\begin{aligned} \cos(t) - \cos(t - M) &= -2\sin\left(\tfrac{2t-M}{2}\right)\sin\left(\tfrac{M}{2}\right) \\ &= K\sin\left(\tfrac{2t-M}{2}\right) \qquad , K = \text{constant} \\ &= K\cos\left(\tfrac{M+\pi-2t}{2}\right) \end{aligned}$$

and using Lemma 3.2-1 and 3.2-2, we get

$$\sigma^k_{(t;M)} f(t) = q_0 + \sum_{n=1}^{\infty} d_n \cos(h_n(t)) \tag{3.5-4}$$

where d_n are some appropriate constants and $h_n(t)$ are some appropriate functions of t.

As both terms in the right-hand side of Equation (3.5-4) do not accumulate with time, the FII is thus also bounded in the interval $[0,L]$. ♦♦

Corollary 3.5-3 (Frequency response of FII)

The frequency domain equivalent of the FII operator, σ{ω} is:

$$\sigma\{\omega\} = \frac{2}{\omega}\sin(\frac{\omega \mathrm{M}}{2})\, \mathrm{e}^{-\frac{j\omega \mathrm{M}}{2}}$$

where M is the fixed-interval of the FII, and ω is the frequency variable.

♦♦

Proof

The backward difference operator ∇ is defined in Definition 3.2-4 as:

$$\nabla_{(t;\mathrm{M})} = 1 - \delta_{(t;\mathrm{M})} \tag{3.5-5}$$

So its frequency domain equivalent, ∇{s}, is given by:

$$\nabla\{s\} = (\ 1 - \mathrm{e}^{-\mathrm{M}s}\ ) \tag{3.5-6}$$

where s is the Laplace transform variable. Also, the integral operator η has its frequency domain equivalent of:

$$\eta\{s\} = \frac{1}{s} \tag{3.5-7}$$

It has been determined from Theorem 3.3-4a that $\sigma = \nabla\eta$. So the frequency domain equivalent of the FII operator, σ{s}, is given by:

$$\begin{aligned}\sigma\{s\} &= \nabla\{s\}\eta\{s\} \\ &= (\ 1 - \mathrm{e}^{-\mathrm{M}s}\ )\frac{1}{s} \\ &= \frac{1}{j\omega}(\ 1 - \cos(-\omega\mathrm{M}) - j\sin(-\omega\mathrm{M})\ )\end{aligned} \tag{3.5-8}$$

and j is the imaginary variable, $\sqrt{-1}$.

Using the identity,

$$\cos(A) = 1 - 2\sin^2(\tfrac{A}{2})$$

and

$$\sin(A) = 2\sin(\tfrac{A}{2})\cos(\tfrac{A}{2})$$

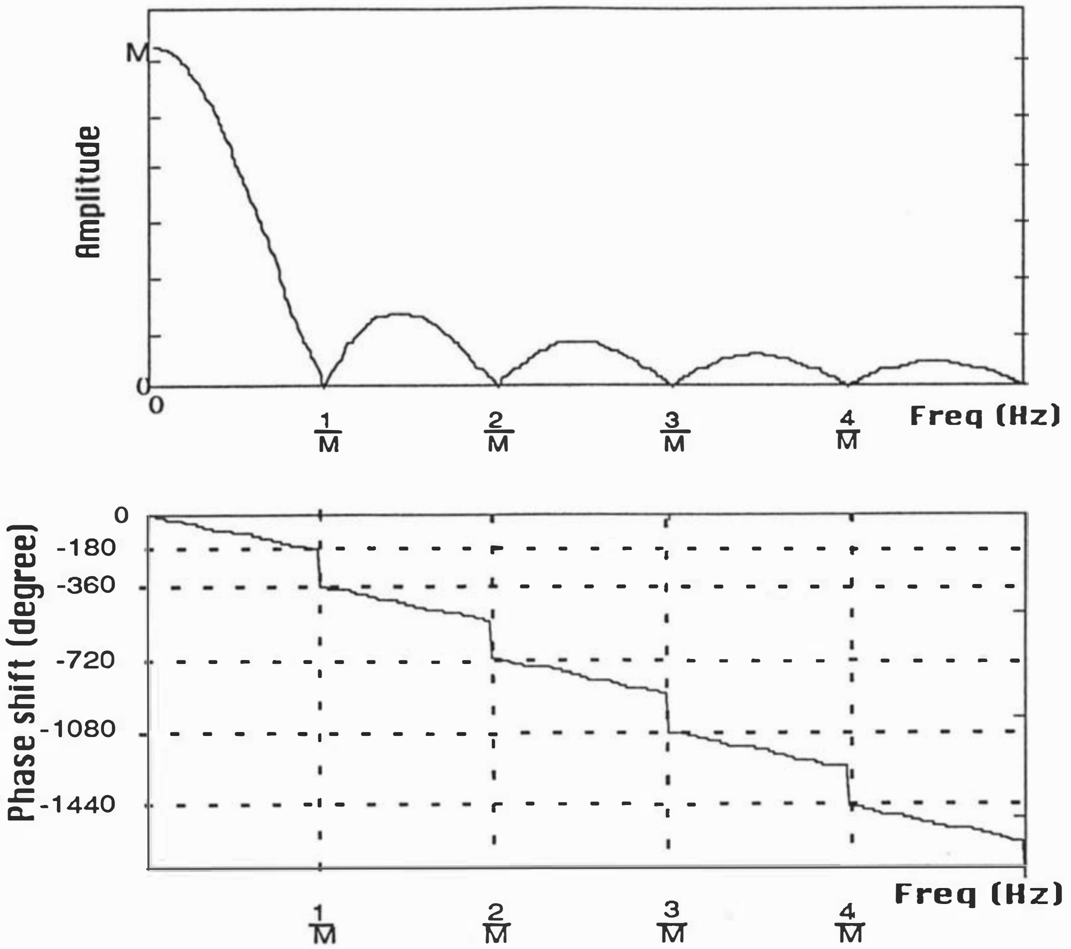
Equation (3.5-8) above becomes,

$$\begin{aligned}
\sigma\{s\} &= \frac{1}{j\omega}\Big(2\sin^2(-\tfrac{\omega M}{2}) - 2j\sin(-\tfrac{\omega M}{2})\cos(-\tfrac{\omega M}{2})\Big) \\
&= \frac{1}{j\omega}2\sin(-\tfrac{\omega M}{2})\Big(\sin(-\tfrac{\omega M}{2}) - j\cos(-\tfrac{\omega M}{2})\Big) \\
&= -\frac{2}{j\omega}\sin(\tfrac{\omega M}{2})\Big(\sin(-\tfrac{\omega M}{2}) - j\cos(-\tfrac{\omega M}{2})\Big) \\
&= \frac{2}{\omega}\sin(\tfrac{\omega M}{2})\Big(-\tfrac{1}{j}\sin(-\tfrac{\omega M}{2}) + \cos(-\tfrac{\omega M}{2})\Big) \\
&= \frac{2}{\omega}\sin(\tfrac{\omega M}{2})\Big(j\sin(-\tfrac{\omega M}{2}) + \cos(-\tfrac{\omega M}{2})\Big) \\
&= \frac{1}{\omega}2\sin(\tfrac{\omega M}{2})\, e^{-\frac{j\omega M}{2}}
\end{aligned}$$

♦ ♦

Corollary 3.5-3 shows that the bandwidth of the FII operation can be manipulated by changing M. A linear plot of the frequency response of the FII is show in Figure 3.5-1.

Figure 3.5-1 Frequency Response of the FII Operator



<u>**Corollary 3.5-4**</u> (Constant input response of the FII)

When the FII operation is subjected to a constant input (that is when ω=0), the response of the FII is given by,

$$\sigma(\omega=0) = M. \qquad \blacklozenge$$

Proof

By Corollary 3.5-3:

$$\sigma\{\omega\} = \frac{2}{\omega}\sin\left(\frac{\omega M}{2}\right) e^{-\frac{j\omega M}{2}}$$

Thus the constant input response is,

$$\sigma\{\omega=0\} = \lim_{\omega \to 0} \frac{2}{\omega} \sin(\frac{\omega M}{2})\, e^{-\frac{j\omega M}{2}} = \frac{2}{\omega}\,\frac{\omega M}{2}$$

$$= M \qquad \blacklozenge\blacklozenge$$

<u>**Corollary 3.5-5**</u> (Impulse response of the FII operator)

The impulse response of the FII operator in the time domain, $S(t)$ is,

$$S(t) = \begin{cases} 1 & \text{, for } 0 \le t \le M \\ 0 & \text{, else} \end{cases} \qquad \blacklozenge$$

Proof

The impulse response of an operator is given by the inverse Laplace transform of the operator's Laplace description (Coughanowr and Koppel 1983). So the impulse response of the FII operator, $S(t)$, is:

$$S(t) = \mathcal{L}^{-1}\{\sigma\{s\}\} = \mathcal{L}^{-1}\left\{(1 - e^{-Ms})\,\frac{1}{s}\right\}$$

$$= \begin{cases} 1 & \text{, for } 0 \le t \le M \\ 0 & \text{, else} \end{cases} \qquad \blacklozenge\blacklozenge$$

Some of these properties are useful for developing possible realizations of the FII operator in the next chapter.

3.6 SUMMARY

In this chapter, a new operator notation system has been developed. This is convenient for mathematical manipulation and for the implementation of the calculus operation in this work. Some results towards an algebra of calculus operators including a special integral, named the Fixed-interval-integral (FII) are then given. The key results for the purposes of the rest of this work are:

(1) $\sigma = \nabla\eta = \nabla\zeta$

(2) $\sigma\rho = \nabla$

This means that the kth order FII is the kth backward difference of a classical kth order integral.

It has also been determined that the FII is:

(1) independent of constant initial conditions.

(2) bounded for all time if the operand is bounded for all time.

The frequency response, constant input response and impulse response of the FII operator are also presented as they are useful later in realizing the FII operation.

CHAPTER FOUR

REALIZATION OF FIXED-INTERVAL-INTEGRALS

This chapter presents several possible methods to implement the Fixed-Interval-Integral operation. These methods include the use of :

(1) frequency response
(2) a hybrid of analog and digital devices
(3) numerical techniques

Some limitations and advantages of these methods are also discussed.

4.1 INTRODUCTION

The last chapter showed several properties of the Fixed Interval Integral (FII). Some of these properties were given purely for theoretical interest while others have significance in parameter estimation and system control. In particular, two of them are useful in realizing the FII operation. These are the relationship between FII and the classical definite integral, determined in Theorem 3.3-5a, and the frequency response determined in the Corollary 3.5-3. Using these two properties, three methods of realization of FII are possible, namely the:

1) frequency response method
2) hybrid method
3) numerical method

The use of these three methods in realizing the FII operation is discussed respectively in Sections 4.2 to 4.4 in this chapter. A summary is then given in section 4.5.

4.2 FREQUENCY RESPONSE METHOD

One way to realize the FII operation is to use analog or digital filters whose frequency responses approximate those of the FII. This is possible because the frequency response of FII, $\sigma\{\omega\}$, has been determined in Corollary 3.5-3 as

$$\sigma\{\omega\} = \frac{2}{\omega}\sin\left(\frac{\omega M}{2}\right) e^{-\frac{j\omega M}{2}} \tag{4.2-1}$$

where, M, is the fixed interval of the FII.

A linear plot of the gain amplitude and phase shift at various frequencies in terms of the fixed interval of FII, M, is shown in Figure 3.51.

The design of analog and digital filters to match this frequency response is discussed in the following Sections 4.2.1 and 4.2.2.

4.2.1 Analog Filters

Some standard techniques for designing analog filters to match frequency requirements can be found in texts such as Hasler and Neirynck (1986), and White (1980). These are usually based on a frequency description in the Laplace domain, which is in the form of a linear polynomial in the Laplace variable, *s*. So, for the purpose of filter design, it is more convenient to use the Laplace domain equivalent of Equation 4.2-1; that is,

$$\sigma\{s\} = \frac{1 - e^{-Ms}}{s} \tag{4.2-2}$$

However, this equation consists of a nonlinear exponential term in the numerator, therefore an approximation such as series expansion is needed to enable it to be used for filter design. Furthermore, Figure 4.2-1 shows that discontinuities occur in both the gain amplitude and the phase shift of the FII operation when the input frequency is a multiple of $\frac{1}{M}$ Hz. These discontinuities are not easily matched using analog filters, therefore the analog realization of FII operation is usually limited to operational frequencies of less than $\frac{1}{M}$ Hz.

For example, using the second order Padè approximation (Middleton and Goodwin 1990), that is,

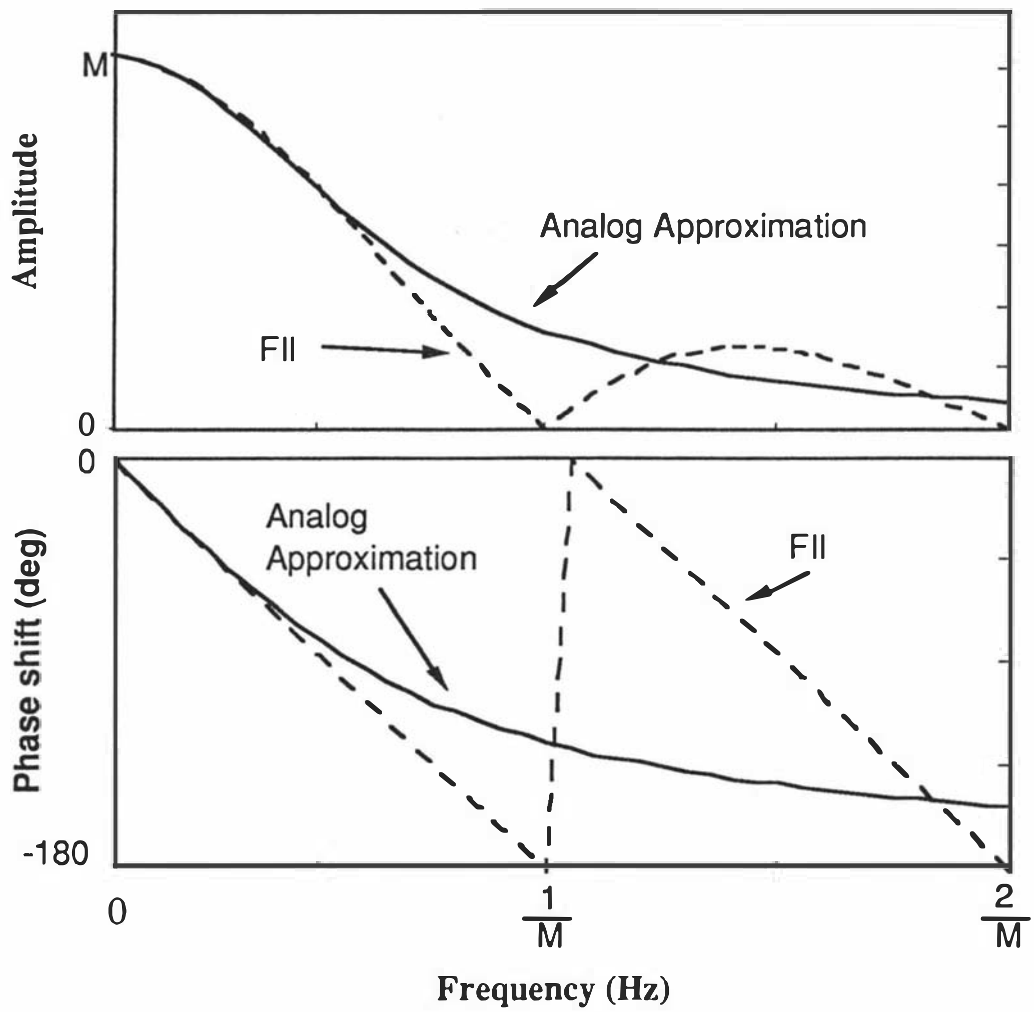
$$e^{-Ms} \approx \frac{1 - \frac{M}{2}s + \frac{M^2}{12}s^2}{1 + \frac{M}{2}s + \frac{M^2}{12}s^2} \tag{4.2-3}$$

Equation (4.2-2) can be rewritten as,

$$\sigma\{s\} \approx \frac{M}{1 + \frac{M}{2}s + \frac{M^2}{12}s^2} \tag{4.2-4}$$

A comparison between this approximation and the actual FII can be seen in the frequency plot in Figure 4.2-1. It shows that this approximation fails to match the discontinuities at $\frac{1}{M}$ Hz and beyond. So its use is limited to frequencies below $\frac{1}{M}$ Hz. The plot also shows the approximation is exact for constant input (that is zero frequency) and the error of approximation increases as the frequency increases.

Figure 4.2-1 Frequency Response of An Analog Approximation to FII Operation.

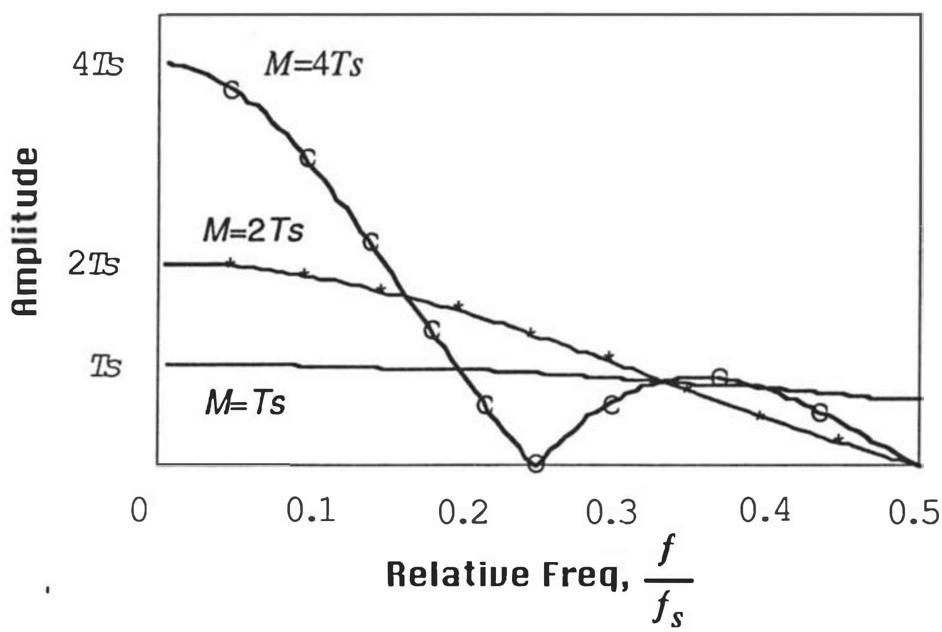


4.2.2 Digital Filters

In digital filter design, it is useful to plot the frequency response at various frequencies in terms of the sampling frequency, f_S, as given in Figure 4.2-2.

Due to the discrete-time nature of digital filters, it is convenient to have the FII interval, M, as a multiple of the sampling interval, T_S. The effect of different values for M can also be seen from Figure 4.2-2.

Figure 4.2-2 Frequency Response of FII in terms of Sampling Frequency, f_S.



Also, in order to avoid the effect of frequency aliasing (Hamming 1977, Bose 1985, Ogata 1987), the operational frequency of a digital filter should be more than the Nyguist Frequency, f_N , where,

$$f_N = \frac{f_S}{2} \tag{4.2-5}$$

There are generally two approaches to designing digital filters to match a given frequency response, namely the:

1) indirect approach
2) direct approach

The techniques used in the indirect approach are based on finding a discrete-time equivalent of a continuous-time filter (Hamming 1977, Bose 1985, Williams 1986, Kuc 1988). These involve transforming the Laplace domain description to a discrete-time description. The discrete-time description is usually in the form of a linear difference equation involving the shift operator (or operational variable), z^{-1} such as,

$$G(z) = \frac{1 + b_1 z^{-1} + b_2 z^{-2} + \ldots}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots} \tag{4.2-6}$$

where G(z) is the transfer function between the input and output of the filter, and a_i and b_i are the filter's coefficients. The transformation is performed using one of a number of transformations between the Laplace operational variable, s, and the discrete-time operational variable, z. The resultant filter is an Infinite-Impulse-Response (IIR) digital filter which is recursive in form. As these techniques rely on the continuous-time description, they suffer the same limitation as the analog filters described in the previous section.

In the case of the direct approach, the resultant filter is usually a non-recursive filter with Finite-Impulse-Response (FIR). The FIR filter has a description similar to Equation (4.2-6) but with no denominator. There are two common techniques for designing a FIR filter (Hamming 1977, Bose 1985, Williams 1986, Kuc 1988). The first involves finding the impulse response of the desired filter and truncating the response using an appropriate window if the impulse response is not finite. The coefficients of the filter are then taken to be the values of the impulse response at various time-shifted instants. However this technique is not suitable for the FII operation, due to the peculiar finite impulse response of the FII operation. As determined in Corollary 3.5-5, the impulse response of the FII is finite and it has a constant value of one over the FII interval. This means the coefficients of the filter are all simply one as well. So for a FII interval of, $M = k\,T_s$, the transfer function in z domain of the ideal FII filter, $\sigma(z)$, is given by,

$$\sigma(z) = 1 + z^{-1} + z^{-2} + \ldots + z^{-k} \tag{4.2-7}$$

which means,

$$\int_{t-kT_s}^{t} f(t)\,\mathrm{d}t = f(t) + f(t-T_s) + f(t-2T_s) + \ldots + f(t-\mathrm{k}T_s) \tag{4.2-8}$$

This is simply a running sum and a very bad realization of the integral even for a very large k.

The second FIR filter design technique is the Frequency-Sampling Method for linear phase filters. It is suitable for the FII operation as Corollary 3.5-3 indicates that the FII operation is also of linear phase. In this method, samples of the gain function are taken at various frequencies. But, rather than calculating the impulse responses using these samples as in the impulse response technique, a special parallel filter structure is used to implement each frequency sample directly. It is capable of implementing frequencies higher than $\frac{1}{M}$ Hz, provided sufficient samples are used. However, the design procedure is rather complicated and a large filter structure usually results since the size of the filter is proportional to the number of samples used.

An alternative approach is the technique given by Schoukens *et al* (1988). Here, the designing of filters was re-formulated as a problem of parameter estimation, and the objective was to estimate a difference-equation description from frequency response data in terms of both phase shift and gain amplitude. In this case, Equation (4.2-1) is used to generate the frequency data. However, as pointed by Schoukens (1991), the discontinuity in the FII frequency response is not easy to match. So this technique is also usually limited to frequencies less than $\frac{1}{M}$ Hz.

4.2.3 Comments on Frequency Response Methods

The previous two sections have outlined several approaches to the realization of the FII operation based on its frequency response. These techniques can only provide limited realization, mostly due to the peculiar frequency and impulse response of the FII operation. This peculiarity can be explained by Theorem 3.3-5a, that is,

$$\sigma = \nabla \eta$$

From this equation, it can be seen that the FII is a hybrid of a discrete-time operation (the backward difference) and a continuous-time operation (the integration). So a realization using analog filters which are inherently continuous in time suffer from the fact than the discrete-time backward difference is difficult to match. Meanwhile the use of a digital filter based on frequency response methods is complicated by the continuous-time integration. In addition, a digital filter design using a frequency response technique usually involves complex and numerous calculations.

In view of the hybrid characteristics of the FII operation, a realization involving both analog and digital components is suggested in the following section.

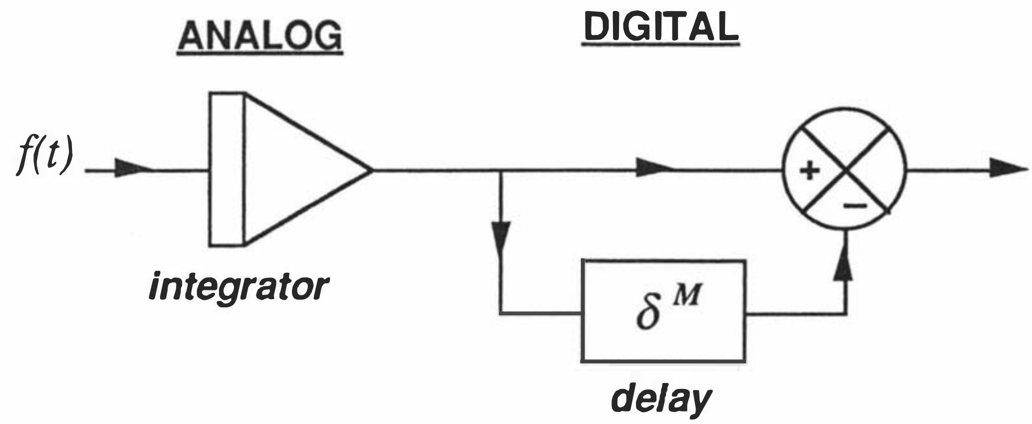
4.3 HYBRID METHOD

Theorem 3.3-5a in the previous chapter established that,

$$\sigma = \nabla \eta \tag{4.3-1}$$

This gives an important undertanding of the FII operation. It means the FII operation can be separated into two sub-operations, backward differencing (denoted by ∇ in Equation 4.3-1) and definite integration (denoted by η). Because it is easier to implement the backward difference opearation in discrete-time and the definite integration operation in continuous-time, a natural way to realize the FII operation is to use a hybrid of digital and analog devices. Therefore a possible implementation is that given in Figure 4.3-1.

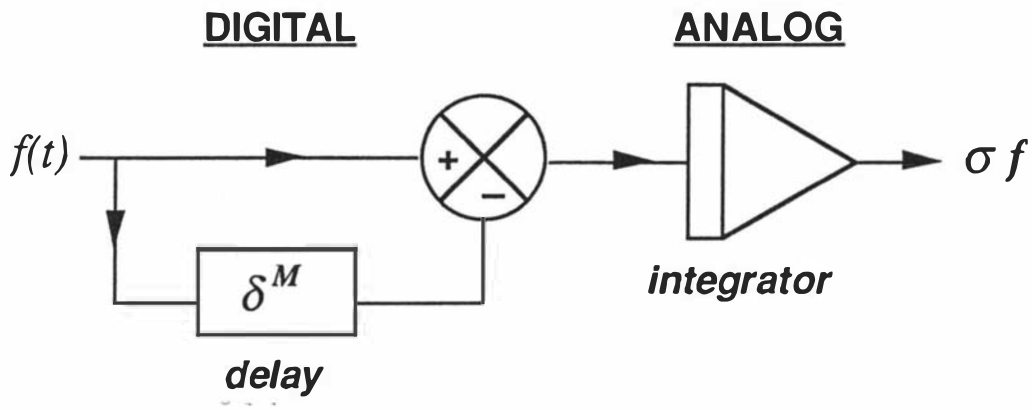
Figure 4.3-1 Hybrid Method I



Here, δ is the delay operator as defined in Chapter 3

If Assumption 3.3-1 or 3.3-2 holds, the commutativity of ∇ and η (determined in Theorem 3.3-2) can be used to yield an alternative arrangement given in Figure 4.3-2,

Figure 4.3-2 Hybrid Method II



The validity of these two assumptions in practice will be discussed in detail in Chapter 5.

Apart from the arrangement of blocks, there is another major implementational difference in these two hybrid configurations. In

Hybrid Method I, the analog integrator integrates the input $f(t)$ directly. So its output may accumulate indefinitely and thus this configuration requires a reset mechanism. In Hybrid Method II, the output of the integrator is also the final result of the FII operation. It has been determined in Corollary 3.5-2 that the FII is bounded for all time if the input $f(t)$ is bounded for all time. So a reset mechanism may not be needed if the boundary value of the FII is within the capability of the integrator.

As both the integrator and the delay components are usually standard devices in a modern control system, the hybrid method provides a very simple and easy means to realize the FII operation. However the capability of this method is limited by the capability of the available integrator.

It is important to note that the arrangements given by the two hybrid methods are valid even if the integrator is a discrete-time approximation. The use of a digital integrator in this arrangment provides the foundation for the next form of FII realization, the numerical method.

4.4 NUMERICAL METHOD

It is well known that the definite integral of a function can be approximated using the Newton-Cotes family of numerical integration formulae or rules (Bajpal et al 1974, Davis and Rabinowitz 1984). These formulae have the general form of

$$\int_a^b f(t)\,\mathrm{d}t = h\,[\,w_0 f(a) + w_1 f(a+h) + \ldots + w_n f(b)\,] \qquad (4.4\text{-}1)$$

where h is a constant step size and w_i is an appropriate weighting. Also the weightings of these formulae are symmetric so that,

$$w_i = w_{n-i} \qquad (4.4\text{-}2)$$

In the context of instrumentation science, it is more convenient to assign the upper limit of integration as the present time, the lower limit as a given initial time and the step size as the sampling interval T_s (Williams 1986, Ogata 1987). This gives

$$\int_{t_0}^{t} f(t)\,\mathrm{d}t = h\,[\,w_0 f(t) + w_1 f(t-T_s) + \ldots + w_n f(t_0)\,] \qquad (4.4\text{-}3)$$

or in a recursive form,

$$\int_{t_0}^{t} f(t)\,\mathrm{d}t = J_n(\,f(t)\,) + \int_{t0}^{t-nT_s} f(t)\,\mathrm{d}t \qquad (4.4\text{-}4)$$

where

$$J_n(\,f(t)\,) = T_s[\,w_0 f(t) + w_1 f(t-T_s) + \ldots + w_n f(t-nT_s)\,] \qquad (4.4\text{-}5)$$

$n \in \mathbf{W}$, and $\mathbf{W} = \{0,1,2,3,\ldots\}$ as defined in definition 3.2-0.

J_n is the "*fundamental formula*" of a Newton-Cotes type numerical integration rule (Davis and Rabinowitz 1984). It consists of the minimum number of moving terms required for the numerical integration rule. The number of terms in J_n depends on the order of the numerical integration rule; an nth order (or $(n+1)$-point) requires $(n+1)$ terms. For example, using the Trapezoidal Rule (Davis and Rabinowitz 1984), which is a first order Newton-Cotes formula, Equation 4.4-3 becomes,

$$\int_{t_0}^{t} f(t)\, dt \approx J_1 + \int_{t_0}^{t-T_s} f(t)\, dt$$

where the fundamental formula, J_n, consists of two terms of form,

$$J_n = J_1 = \frac{T_s}{2} \left[f(t) + f(t-T_s) \right]$$

Before proceeding, it is convenient to make the following definition, which will be used in the sections that follow.

<u>Definition 4.4-1</u> (Delay Interval of Delay Operator)

The delay interval of the delay operator is the sampling interval, T_s, unless otherwise specified, that is,

$$\delta^k = \delta^k(t;\, T_s) = \delta(t;\, kT_s) \qquad \blacklozenge\,\blacklozenge$$

Using this definition and the operators' notation defined in Chapter 3, a Newton-Cotes formula for numerical integration given in Equation (4.4-4) can be rewritten as,

$$\zeta f(t) \approx \left[J_n(\delta) + \delta^n \zeta \right] f(t) \tag{4.4-6}$$

where

$$J_n(\delta) = T_s \left(w_0 + w_1\, \delta + w_2\, \delta^2 + \ldots w_n\, \delta^n \right) \tag{4.4-7}$$

or omitting the operand, $f(t)$,

$$\zeta \approx J_n(\delta) + \delta^n \zeta \tag{4.4-8}$$

Equation (4.4-8) can be expanded in the second term, $\delta^n \zeta$, using Equation (4.4-8) itself to give,

$$\zeta \approx J_n(\delta) + \delta^n \left[J_n(\delta) + \delta^n \zeta \right]$$
$$\approx (1 + \delta^n) J_n(\delta) + \delta^{2n} \zeta$$

This expansion process can be repeated to give,

$$\zeta \approx \sum_{i=1}^{r} \delta^{(i-1)n} J_n(\delta) + \delta^{rn} \zeta \qquad (4.4\text{-}9)$$

for some integer, $r \in \mathrm{W}$

Equation (4.4-9) plays an important role later in deriving a numerical approximation to the FII.

When a FII is approximated by numerical rules, the fixed-interval of the FII is limited to a multiple of the sampling interval because of the discrete-time nature of numerical methods.

Three numerical methods for the realization of FII operations have been developed in this thesis. All these methods assume that, the fixed-interval, M, is a multiple of the sampling interval, T_s, such that:

$$M = m\, T_s \quad , \; m \in \mathrm{W} \qquad (4.4\text{-}10)$$

These methods are different in terms of the number of fundamental numerical formulae which are combined, and the possible orders of the numerical rules. They are described in the following.

4.4.1 Numerical Method I

In this approach, a single fundamental numerical formula is used. The order of this formula, n, is chosen such that,

$$m = r\, n \qquad r, n \in \mathrm{W}$$

where m is defined in Equation (4.4-10). The following theorem gives an approximation of FII operations using this approach.

Theorem 4.4-1 (Numerical Method I)

If $J_n(\delta)$ is the fundamental formula of an nth order or $(n+1)$ point $(n \in \mathbf{W})$ Newton-Cotes rule and, T_s is the sampling interval, then a numerical approximation of a FII with interval, M, such that,

$$M = r\, n\, T_s \qquad , r \in \mathbf{W}$$

is given by,

$$\sigma^k_{(t;\ rnT_s)} \approx \left[\sum_{i=1}^{r} \delta^{(i-1)n}\ J_n(\delta) \right]^k$$ ♦♦

Corollary 4.4-1 (Recursive Numerical Method I)

The recursive version of the numerical rule given in Theorem 4.4.1 is,

$$\sigma^k \approx [\ (1 - \delta^{rn})\ J_n(\delta) + \delta^n\, \sigma\ ]^k$$

Note : the FII interval, $M = r\, n\, T_s$ ♦♦

Proof

By Theorem 3.3-5,

$$\begin{aligned} \sigma_{(t;\ rnT_s)} &= \nabla_{(t;\ rnT_s)}\ \zeta_{(t;\ t0)} \\ &= (\ 1 - \delta^{rn}\ )\ \zeta_{(t;\ t0)} \\ &= \zeta_{(t;\ t0)} - \delta^{rn}\ \zeta_{(t;\ t0)} \end{aligned} \qquad (4.4\text{-}11)$$

and from Equation 4.4-8,

$$\zeta \approx \sum_{i=1}^{r} \delta^{(i-1)n}\, J_n(\delta) + \delta^{rn}\, \zeta \qquad (4.4\text{-}12)$$

Substituting Equation (4.4-12) into the first term of Equation (4.4-11) yields,

$$\sigma_{(t;\ rnT_S)} \approx \sum_{i=1}^{r} \delta^{(i-1)n} J_n(\delta) \ + \ \delta^{rn}\zeta \ - \ \delta^{rn}\zeta$$

$$\approx \sum_{i=1}^{r} \delta^{(i-1)n} J_n(\delta) \tag{4.4-13}$$

So cascading the above equation gives the kth FII numerical approximation in Theorem 4.4-1.

The recursive equation in Corollary 4.4-1 is obtained by noting that the summation term in Theorem 4.4-1,

$$\sum_{i=1}^{r} \delta^{(i-1)n}$$

is a geometric series of δ^n. So using the well known summation formula for a geometric series (Tennent 1971), we get,

$$\sum_{i=1}^{r} \delta^{(i-1)n} J_n(\delta) \ = \ \frac{1-\delta^{rn}}{1-\delta^n} \ J_n(\delta) \tag{4.4-14}$$

Substituting this equation into Theorem 4.4-1 gives,

$$\sigma^k \approx \left[\frac{1-\delta^{rn}}{1-\delta^n} J_n(\delta) \right]^k \tag{4.4-15}$$

Finally, rearranging Equation (4.4-15) yields Corollary 4.4-1. ♦♦

Using this approximation a FII version of the popular Trapezoidal rule and the Simpson rule can be derived and are given as follows

<u>**Rule 4.4-1**</u> (Trapezoidal Rule for FII)

$$\sigma^k_{(t;\ rT_S)} \approx \frac{T_S}{2}[\ 1 + 2\delta + 2\delta^2 + \ldots + 2\delta^{r-1} + \delta^r\ ]^k$$

Proof

The trapezoidal rule is a first order Newton-Cotes rule, that is, $n = 1$, and it has the fundamental formula of:

$$J_1(\delta) = \frac{T_s}{2}(1 + \delta)$$

So, using Theorem 4.4-1, Rule 4.4-1 is derived. ♦♦

Rule 4.4-2 (Simpson 1/3 Rule for FII)

$$\sigma^k_{(t;\ 2rT_s)} \approx \frac{T_s}{3}\left[(1 + 4(\delta + \delta^3 + \ldots + \delta^{2r-1}) + 2(\delta^2 + \delta^4 + \ldots + \delta^{2r-2}) + \delta^{2r}\right]^k$$ ♦♦

Proof

Again by using Theorem 4.4-1 with, n=2, and,

$$J_2(\delta) = \frac{T_s}{3}(1 + 4\delta + \delta^2)$$ ♦♦

The use of these rules are demonstrated in the following example.

Example 4.4.1

Let the sampling interval be 1 second and the FII of interest be second order with interval of 2 second, that is,

$$T_s = 1, \quad k = 2, \quad \text{and } M = 2 = 2T_s$$

(i) for Trapezoidal rule, $r = 2$:

$$\sigma^k_{(t;\ 2)} \approx \frac{1}{2}(1 + 2\delta + \delta^2)^2$$

$$\approx \frac{1}{2}(1 + 2\delta + \delta^2)(1 + 2\delta + \delta^2)$$

$$\approx \frac{1}{2}(1 + 4\delta + 6\delta^2 + 4\delta^3 + \delta^4)$$

so, $\sigma^{k}_{(t;\ 2)} f(t)$

$$\approx \frac{1}{2}\left[\, f(t) + 4f(t\text{-}T_S) + 6f(t\text{-}2T_S) + 4f(t\text{-}3T_S) + f(t\text{-}4T_S)\,\right]$$

(ii) for Simpson 1/3 rule, r=1 :

$$\sigma^{k}_{(t;\ 2)} \approx \frac{1}{3}(\, 1 + 4\delta + \delta^2\,)^2$$

$$\approx \frac{1}{3}(\, 1 + 8\delta + 18\delta^2 + 8\delta^3 + \delta^4\,) \qquad \blacklozenge\blacklozenge$$

Some properties of these rules and a comparison of different orders is discussed in Section 4.4.4.

4.4.2 Numerical Method II

In this approach, two fundamental formulae of n_1th order and n_2th order respectively are combined. The orders are chosen so that,

$$(\, n_1 + n_2\,)T_s \;=\; m\,T_s \;=\; M \quad ; \qquad n_2,\, n_2,\, m \in \mathbf{W}$$

An approximation to the FII using this approach is determined in the following lemma and theorem.

<u>Lemma 4.4-1</u> (Sum of FII Interval)

For all M_1 and $M_2 \in \mathbf{R}$,

$$\sigma_{(t;\ M_1+M_2)} = \sigma_{(t;\ M_1)} + \delta_{(t;\ M_1)}\,\sigma_{(t;\ M_2)} \qquad \blacklozenge\blacklozenge$$

Proof

$$\begin{aligned}
\sigma_{(t;\ M_1+M_2)} f(v) &= \int_{t-M_1-M_2}^{t} f(v)\,\mathrm{d}v \\
&= \int_{t-M_1}^{t} f(v)\,\mathrm{d}v + \int_{t-M_1-M_2}^{t-M_1} f(v)\,\mathrm{d}v \\
&= \sigma_{(t;\ M_1)} f(v) + \sigma_{(t-M_1;\ M_2)} f(v) \\
&= \sigma_{(t;\ M_1)} f(v) + \delta_{(t;\ M_1)} f(v)\, \sigma_{(t;\ M_2)} f(v)
\end{aligned}$$

◆◆

Theorem 4.4-2 (Numerical Method II)

If $J_{n_1}(\delta)$ and $J_{n_2}(\delta)$ are fundamental formulae, respectively of n_1th and n_2th order Newton-Cotes rule, and T_s is the sampling interval, then a numerical approximation of a FII with interval, M, such that,

$$M = (\, r_1 n_1 + r_2 n_2 \,)\, T_s \qquad ;\ r_1, r_2, n_1, n_2 \in \mathbf{W}$$

is given by,

$$\sigma^{k}_{(t;\ M)} \approx \left[\sum_{i=1}^{r} \delta^{(i-1)n_1}\ J_{n_1}(\delta) + \sum_{i=r_1 n_1+1}^{r_1 n_1+r_2} \delta^{(i-1)n_2}\ J_{n_2}(\delta) \right]^{k}$$

◆◆

Proof

By Lemma 4.4-1,

$$\begin{aligned}
&\sigma_{(t;\ r_1 n_1 T_s\ +\ r_2 n_2 T_s)} \\
&\quad = \sigma_{(t;\ r_1 n_1 T_s)} + \delta_{(t;\ r_1 n_1 T_s)}\, \sigma_{(t;\ r_2 n_2 T_s)} \qquad (4.4\text{-}16)
\end{aligned}$$

and using Theorem 4.4-1, the first term in Equation (4.4-16) becomes,

$$\sigma_{(t;\ r_1 n_1 \mathrm{T_s})} = \sum_{\mathrm{i}=1}^{r_1} \delta^{(\mathrm{i}-1)n_1}\ \mathrm{J}_{n_1}(\delta) \qquad (4.4\text{-}17)$$

and the second term in Equation (4.4-16) becomes,

$$\delta_{(t;\ r_1 n_1 T_S)}\sigma_{(t;\ r_2 n_2 T_S)} = \delta_{(t;\ r_1 n_1 T_S)} \sum_{i=1}^{r_2} \delta^{(i-1)n_2}\, J_{n_2}(\delta)$$

$$= \delta^{\,r_1 n_1} \sum_{i=1}^{r_2} \delta^{(i-1)n_2}\, J_{n_2}(\delta)$$

$$= \sum_{i=r_1 n_1+1}^{r_1 n_1+r_2} \delta^{(i-1)n_2}\, J_{n_2}(\delta) \qquad (4.4\text{-}18)$$

Combining Equations (4.4-17) and (4.4-18) yields Theorem 4.4-2 . ♦♦

The following example demonstrates the use of this theorem.

Example 4.4.2

Let $M = 3T_s$. The FII can then be approximated by using, n_1=1 and n_2=2, namely the Trapezoidal and Simpson rule respectively, and by setting r_1=r_2=1. So,

$$\sigma_{(t;\ 3T_s)} = J_1(\delta) + \delta J_2(\delta)$$

$$= \frac{T_s}{2}(1+\delta) + \frac{T_s}{3}\delta(1+4\delta+\delta^2)$$

$$= \frac{T_s}{2} + \frac{5T_s}{6}\delta + \frac{4T_s}{3}\delta^2 + \frac{T_s}{3}\delta^3 \qquad ♦♦$$

Note that the weightings of these combined rules lose the symmetry property given by Equation 4.4-2. Further discussion and some properties of this approach compared with the other two numerical methods are given in Sections 4.4.4 and 4.4.5 later.

4.4.3 Numerical Method III

A similar approach is taken to that in the previous section, that is, two fundamental formulae of respectively, n_1th order and n_2th order are used, but the orders are chosen such that,

$$n_1 - n_2 = m\,T_s = M \; ; \qquad n_2,\, n_2,\, m \in \mathbf{W}$$

An approximation of FII using this approach is determined in the following lemma and theorem.

<u>Lemma 4.4-2</u> (Difference of FII Interval)

For all M_1 and $M_2 \in \mathbf{R}$,

$$\sigma_{(t;\; M_1 - M_2)} = \sigma_{(t;\; M_1)} - \delta_{(t;\; M_1 - M_2)}\, \sigma_{(t;\; M_2)} \qquad \blacklozenge\blacklozenge$$

Proof

$$\sigma_{(t;\; M_1-M_2)} f(v) = \int_{t-M_1+M_2}^{t} f(v)\,\mathrm{d}v$$

$$= \int_{t-M_1}^{t} f(v)\,\mathrm{d}v - \int_{t-M_1}^{t-M_1+M_2} f(v)\,\mathrm{d}v$$

$$= \sigma_{(t;\; M_1)} f(v) - \sigma_{(t-M_1+M_2;\; M_2)} f(v)$$

$$= \sigma_{(t;\; M_1)} f(v) - \delta_{(t;\; M_1-M_2)}\, \sigma_{(t;\; M_2)} f(v)$$

$\blacklozenge\blacklozenge$

<u>Theorem 4.4-3</u> (Numerical Method III)

If $J_{n_1}(\delta)$ and $J_{n_2}(\delta)$ are fundamental formulae respectively of n_1th and n_2th order Newton-Cotes rule and, T_s is the sampling interval, then a numerical approximation of a FII with interval, M, such that,

$$M = (\, n_1 - n_2 \,)\, T_s \qquad ; \; n_1, n_2 \in \mathbf{W}$$

is given by,

$$\sigma^{k}_{(t;\ n_1T_s-n_2T_s\ )} \approx \left[\ J_{n_1}(\delta)\ -\ \delta^{n_1-n_2}\right] J_{n_2}(\delta)\ )^{k} \qquad \blacklozenge\blacklozenge$$

Proof

By Lemma 4.4-2,

$$\sigma_{(t;\ n_1T_s-n_2T_s)} = \sigma_{(t;\ n_1T_s)} - \delta^{n_1-n_2}\,\sigma_{(t;\ n_2T_s)} \qquad (4.4\text{-}19)$$

Using Theorem 4.4-1, with r = 1, the above equation becomes,

$$\sigma_{(t;\ n_1T_s-n_2T_s)} = J_{n_1}(\delta) - \delta^{n_1-n_2}\,J_{n_2}(\delta) \qquad (4.4\text{-}20)$$

Theorem 4.4-3 is then obtained by cascading Equation (4.4-20). ♦♦

The following example demonstrates the use of this theorem.

<u>**Example 4.4.3**</u>

Let $M = T_s$. For this method, a possible choice of orders is, n_1=3 and n_2=2. These correspond to the Simpson 1/3 and 3/8 rules. So,

$$\begin{aligned}\sigma_{(t;\,T_s)} &= J_3(\delta) - \delta J_2(\delta)\\ &= \frac{3T_s}{8}(1+3\delta+3\delta^2+\delta^3) - \frac{T_s}{3}\delta(1+4\delta+\delta^2)\\ &= \frac{3T_s}{8} + \frac{5T_s}{12}\delta - \frac{7T_s}{12}\delta^2 + \frac{T_s}{24}\delta^3\end{aligned} \qquad \blacklozenge\blacklozenge$$

Further discussion of this approach and a comparison with the other two numerical methods are given in Sections 4.4.4 and 4.4.5.

4.4.4 Frequency Response

A useful method of comparing the performance of the numerical methods is the frequency response analysis. Because of the discrete-time nature of the numerical methods, the Shannon (Ogata 1987, Middleton and Goodwin 1990) or the Nyquist (Hamming 1977, Kuc 1988) sampling theorem should be followed when choosing the sampling rate. Following this theorem, the highest useful frequency is limited to half the sampling frequency. Therefore it is more convenient to express the frequency response of the numerical methods in terms of the relative frequency, f_r, given by:

$$f_r = \frac{f}{f_s} \tag{4.4-21}$$

where f is the actual frequency in hertz and f_s is the sampling frequency in hertz. This relative frequency can be related to other common quantities by:

$$f_r = \frac{\omega}{\omega_s} = \frac{\omega T_s}{2\pi} \tag{4.4-22}$$

where, ω and ω_s are, respectively, the actual frequency and sampling frequency in radians, and T_s is the sampling interval.

The following theorem and five corollaries provide some of the properties of the numerical rules given by Numerical Method I. The properties derived are general for all numerical rules used in Numerical Method I and provide an general and easy means to design and analyse these numerical rules. All these properties are due to the symmetry of the coefficients in these rules (see Equation 4.4-2).

As the rules given by Numerical Method II and Numerical Method III are not symmetric in the coefficients, they do not enjoy the neat and general properties of Numerical Method I. However the frequency response of each numerical rule in Numerical Method II and Numerical Method III can be analysed individually in a similar way.

Theorem 4.4-4 (Frequency Response of Numerical Method I)

For an nth order Newton-Cotes type numerical approximation of FII operation, as determined in Theorem 4.4.1 (Numerical Method I), the frequency response, $\hat{\sigma}_n(j\omega)$, is given by,

$$\hat{\sigma}_n(j\omega) = N(j\omega)\, e^{-j\frac{M\omega}{2}}$$

where

$$N(j\omega) = \frac{\sin(\frac{M\omega}{2})}{\sin(\frac{nT_s\omega}{2})}\, G\{\, J_n(j\omega) \,\}$$

and

$$G\{\, J_n(j\omega) \,\} = 2T_s \sum_{i=0}^{(n-1)/2} w_i \cos(\tfrac{\omega T_s}{2}(n-2i)) \qquad \text{, if } n \text{ is odd}$$

$$G\{\, J_n(j\omega) \,\} = 2T_s \left[\ \tfrac{1}{2} w_{n/2} + \sum_{i=0}^{(n-2)/2} w_i \cos(\tfrac{\omega T_s}{2}(n-2i))\ \right] \qquad \text{, if } n \text{ is even}$$

Note : $M = rnT_s$

Proof

From Corollary 4.4-1, the numerical rules given by Numerical Method I have the general form:

$$\frac{1-\delta^{rn}}{1-\delta^{n}}\, J_n(\delta) \tag{4.4-23}$$

where $J_n(\delta)$ is the Newton-Cotes type nth order fundamental formula of the rule and has the form:

$$J_n(\delta) = T_s \sum_{i=0}^{n} w_i\, \delta^i$$

and the FII interval, M, is given by:

$$M = r\, n\, T_s \qquad , r \in W$$

Because of the symmetry in the coefficients w_i (Davis and Rabinowitz 1984), that is:

$$w_i = w_{n-i}$$

J_n can be rewritten as:

$$J_n(\delta) = T_s \sum_{i=0}^{(n-1)/2} w_i \left(\delta^i + \delta^{n-i} \right) \qquad \text{, if } n \text{ is odd}$$

$$J_n(\delta) = T_s \left[w_{n/2}\, \delta^{n/2} + \sum_{i=0}^{(n-2)/2} w_i \left(\delta^i + \delta^{n-i} \right) \right] \qquad \text{, if } n \text{ is even}$$

So if n is odd, the frequency response of J_n is given by:

$$J_n(s) = T_s \sum_{i=0}^{(n-1)/2} w_i \left(e^{-sT_s i} + e^{-sT_s(n-i)} \right)$$

$$= T_s\, e^{-\frac{sT_s n}{2}} \sum_{i=0}^{(n-1)/2} w_i \left(e^{\frac{sT_s(n-2i)}{2}} + e^{-\frac{sT_s(n-2i)}{2}} \right)$$

Using the identity,

$$e^{j\theta} + e^{-j\theta} = 2\cos\theta$$

gives,

$$J_n(j\omega) = 2T_s\, e^{-j\frac{n\omega T_s}{2}} \sum_{i=0}^{(n-1)/2} w_i \cos\left(\frac{\omega T_s(n-2i)}{2}\right) \tag{4.4-24}$$

Defining $G\{J_n(j\omega)\}$ as the gain function of $J_n(j\omega)$, then,

$$G\{J_n(j\omega)\} = 2T_s \sum_{i=0}^{(n-1)/2} w_i \cos\left(\frac{\omega T_s(n-2i)}{2}\right) \tag{4.4-25}$$

Similarly, if n is even, it be can shown that,

$$G\{(J_n(j\omega)\} = 2T_s \left[\tfrac{1}{2} w_{n/2} + \sum_{i=0}^{(n-2)/2} w_i \cos\left(\frac{\omega T_s}{2}(n-2i)\right) \right] \tag{4.4-26}$$

Also the frequency response of $(1-\delta^n)$ is given by,

$$1 - e^{-jn\omega T_s} = e^{-j\frac{n\omega T_s}{2}} \left(e^{j\frac{n\omega T_s}{2}} - e^{-j\frac{n\omega T_s}{2}}\right)$$

$$= 2j\, sin(\frac{n\omega T_s}{2})\, e^{-j\frac{n\omega T_s}{2}} \qquad (4.4\text{-}27)$$

due to the identity,

$$e^{j\theta} - e^{-j\theta} = 2j\, sin\theta$$

So substituting Equations (4.4-24), (4.4-25) and (4.4-27) into Equation (4.4-23) gives,

$$\hat{\sigma}_n(j\omega) = \frac{2jsin(\frac{rn\omega T_s}{2})\, e^{-j\frac{rn\omega T_s}{2}}}{2jsin(\frac{n\omega T_s}{2})\, e^{-j\frac{n\omega T_s}{2}}} e^{-j\frac{n\omega T_s}{2}} G\{J_n(j\omega)\}$$

$$= \frac{sin(\frac{rn\omega T_s}{2})}{sin(\frac{n\omega T_s}{2})}\,)\, e^{-j\frac{rn\omega T_s}{2}} G\{J_n(j\omega)\}$$

Theorem 4.4-4 is then obtained by substituting $r\,n\,T_s$ with M. ♦♦

In Theorem 4.4-4, the frequency response is given in the form of the gain function, $G\{\bullet\}$, and the phase function, $P\{\bullet\}$, that is,

$$\hat{\sigma}_n(j\omega) = G\{\hat{\sigma}_n(j\omega)\}\, P\{\hat{\sigma}_n(j\omega)\} \qquad (4.4\text{-}28)$$

where

$$G\{\hat{\sigma}_n(j\omega)\} = N(j\omega) \qquad (4.4\text{-}29)$$

and $P\{\hat{\sigma}_n(j\omega)\} = e^{-j\frac{M\omega}{2}}$ (4.4-30)

Note that the gain function can have both positive and negative values. This gain and phase functions form of representation (Kuc 1988) is chosen rather than the magnitude and phase shift form (Hostetter 1988) because the frequency response is of linear phase. Also, the gain and phase function form is able to provide the important information on phase jump (Kuc 1988). The phase jump of π in a linear phase response (such as the response of the ideal FII shown in Figure 4.2-1) is due to the change of sign of the filter gain. As the gain function, G{•}, can have both positive and negative values, a sign change can be observed directly and thus a phase jump may be seen. On the other hand, the magnitude has only positive values and thus the phase jump could not be observed directly.

Some important properties of the frequency response of the Numerical Method I are given in the following corollaries.

Corollary 4.4-2 (Contributor to approximation error)

The approximation errors in both the gain magnitude and phase shift of a numerical rule for FII, as defined in Theorem 4.4-4, are solely due to the mismatch in the gain function. ♦♦

Proof

Comparison of the frequency response of the actual FII (Corollary 3.5-3) and the frequency response of Numerical Method I (Theorem 4.4-4) shows that both of them have the same phase function, that is :

$$e^{-j\frac{M\omega}{2}}$$

Both of them therefore have the property of linear phase. As the jumps in phase shift of the actual FII are due to the change of sign in the gain function of the FII, there will be no error in phase shift if the gain function of the numerical rule has the same sign changes in its gain function. So both the approximation errors in phase shift and magnitude are solely due to the mismatch in the gain function. ♦♦

<u>**Corollary 4.4-3**</u> (Exact match for constant input)

At zero frequency (constant input), the response of the numerical rules are exactly the same as the ideal FII. ♦

Proof

It is well known that all Newton-Cotes type formulae are exact for constant functions (Davis and Rabinowitz 1984). Also, an nth order formulae requires n sampling intervals and thus it covers a total interval of nT_s. Therefore the gain of J_n at ω=0 is:

$$G\{\ J_n(\omega=0)\ \} = nT_s$$

So, from Theorem 4.4-4, the gain function of the numerical rule becomes:

$$N(\omega=0) = nT_s \lim_{\omega\to 0} \frac{\sin(\frac{rnT_s\omega}{2})}{\sin(\frac{nT_s\omega}{2})}$$

$$= nT_s \lim_{\omega\to 0} \frac{\frac{rnT_s\omega}{2}}{\frac{nT_s\omega}{2}}$$

$$= r\,n\,T_s = M$$

This gain is exactly the same as the ideal FII gain at ω=0. As it has been determined in Corollary 4.4-2, the gain function is the sole contributor to approximation error; the numerical rule thus has no error in magnitude or phase shift at ω=0 ♦♦

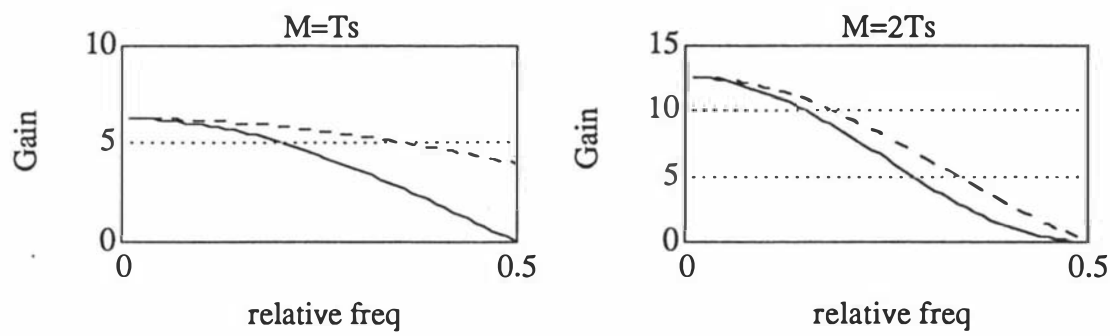
Using Theorem 4.4-4, the frequency responses of numerical rules for Numerical Method I can then be found. These frequency responses are demonstrated in Figures 4.4-1a and 4.4-1b for the first six orders of the numerical rules. The exact match property given by Corollary 4.4-3 can be clearly seen in these plots. Note that these plots are given in terms of the relative frequency f_r.

Figure 4.4-1a : Gain of Numerical Rules for FII

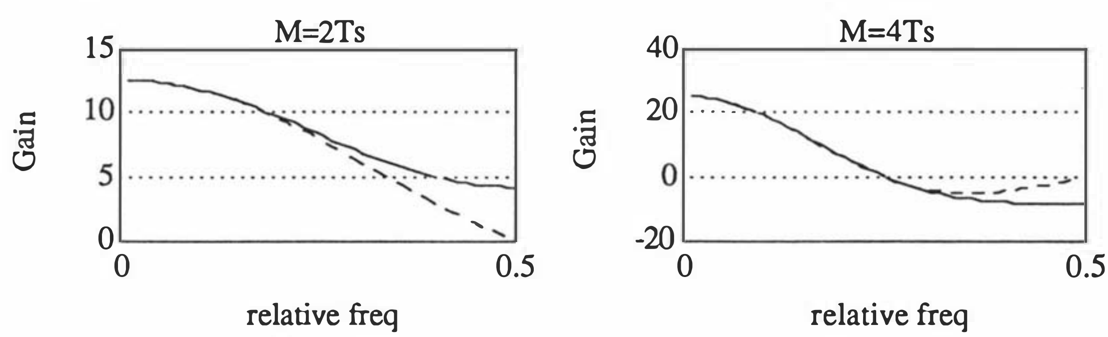
Key:

Solid line : Numerical rule
Dashed line : Ideal FII
Dotted line : Grid

1st Order



2nd order



3rd order

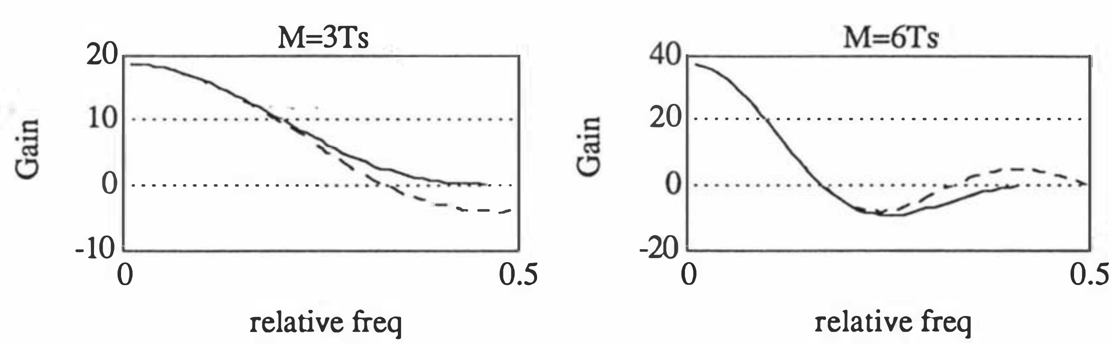
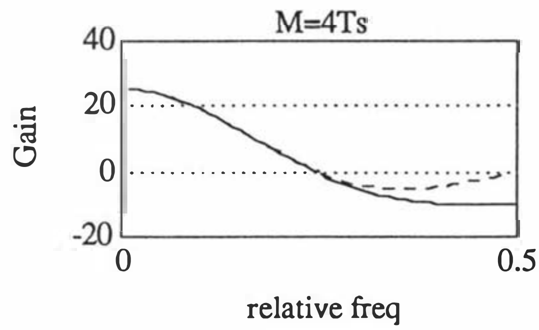


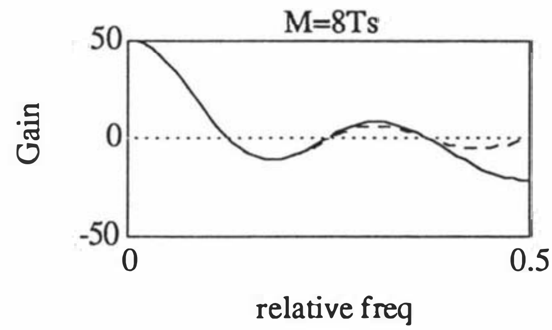
Figure 4.4-1b : Gain of Numerical Rules for FII (cont')

Key :

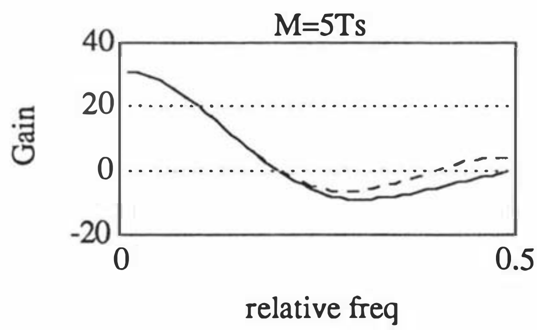
Solid line : Numerical rule
Dashed line : Ideal FII
Dotted line : Grid

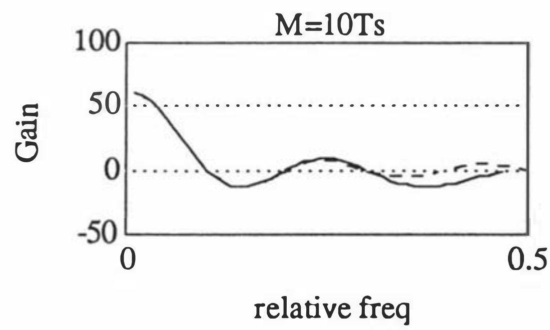
4th order



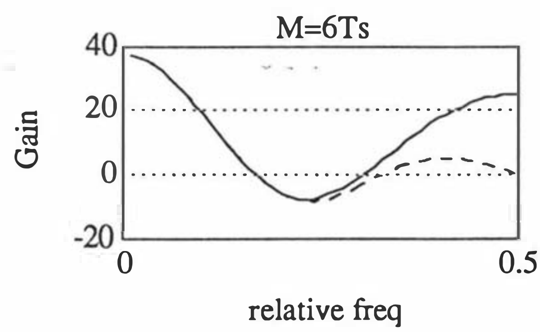


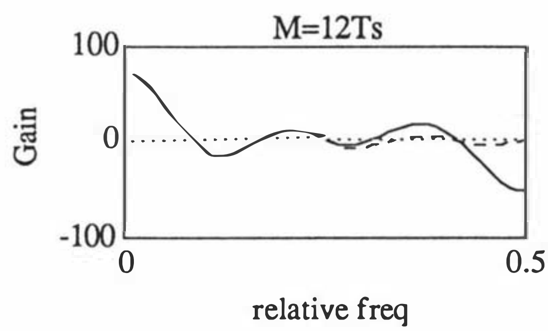
5th order





6th order





A useful measure of the errors in approximating a quantity of varying magnitude, such as the gain of the FII at various frequencies, is the approximation ratio. In our case the approximation ratio for the FII gain is given by,

$$\frac{G\{\hat{\sigma}_n(j\omega)\}}{G\{\sigma(j\omega)\}} \tag{4.4-31}$$

The percentage error can be obtained from the approximation ratio by,

$$\%\text{error} = \left[\frac{G\{\hat{\sigma}_n(j\omega)\}}{G\{\sigma(j\omega)\}} - 1\right] \text{x } 100\% \tag{4.4-32}$$

The gain approximation ratio alone is a sufficient indicator for both the error in phase shift and magnitude because it has been determined in Corollary 4.4-2 that the sole contributor to approximation errors is the gain function. An important relationship between the gain approximation ratio and phase error is determined in the following corollary. This corollary serves as a useful tool in analysing and designing a numerical realization for the FII.

Corollary 4.4-4 (Indication for phase error)

An approximation error in the phase shift occurs when the approximation ratio of the gain function is negative, that is when,

$$\frac{G\{\hat{\sigma}_n(j\omega)\}}{G\{\sigma(j\omega)\}} < 0$$

which is equivalent to when the percentage error is less than negative one hundred percent, that is when,

$$\%\text{error} < -100\%$$

where

$$\%\text{error} = \left[\frac{G\{\hat{\sigma}_n(j\omega)\}}{G\{\sigma(j\omega)\}} - 1\right] \text{x } 100\%$$

♦ ♦

Proof

It has been discussed in the proof of Corollary 4.4-2 that a change of sign in the gain function will result in a phase jump of π. So an error in phase shift occurs when the sign of the numerical rule's gain function is different from the actual FII gain function. A difference in the sign of these two functions will then result in a negative gain approximation ratio and a percentage error of less than (-100%). ♦♦

Corollary 4.4-5 (Approximation error of FII gain)

The gain approximation ratio with respect to relative frequency f_r, for a numerical approximation of FII using Numerical Method I, is given by,

$$\frac{G\{\hat{\sigma}_n(j\omega)\}}{G\{\sigma(j\omega)\}}$$

$$= \begin{cases} \dfrac{2\pi f_r}{\sin(n\pi f_r)} \displaystyle\sum_{i=0}^{(n-1)/2} w_i \cos((n-2i)\pi f_r) & \text{,if } n \text{ odd} \\ \dfrac{2\pi f_r}{\sin(n\pi f_r)} \left[\frac{1}{2} w_{n/2} + \displaystyle\sum_{i=0}^{(n-2)/2} w_i \cos((n-2i)\pi f_r) \right] & \text{, if } n \text{ is even} \end{cases}$$

provided it is defined at the particular relative frequency.

The ratio is not defined at the relative frequencies corresponding to the actual frequencies which are multiples of $\frac{1}{M}$ Hz, that is at,

$$f_r = \frac{i}{rn} \quad , \; i = 1, 2, 3, \ldots$$

Note : n is the order of the numerical rule and $M = r\,nT_s$ ♦♦

Proof

If n is odd, then by Corollary 3.5-3 and Theorem 4.4-4,

$$\frac{G\{\hat{\sigma}_n(j\omega)\}}{G\{\sigma(j\omega)\}} = \left[\frac{\omega}{2\sin(\frac{M\omega}{2})}\right] \frac{2T_s \sin(\frac{M\omega}{2})}{\sin(\frac{nT_s\omega}{2})}) \sum_{i=0}^{(n-1)/2} w_i \cos((\frac{\omega T_s}{2}(n-2i))$$

Using Equation (4.4-22), that is:

$$\omega T_s = 2\pi f_r$$

the ratio becomes:

$$\frac{G\{\hat{\sigma}_n(j\omega)\}}{G\{\sigma(j\omega)\}} = \frac{2\pi f_r \sin(\frac{M\omega}{2})}{\sin(n\pi f_r)\sin(\frac{M\omega}{2})} \sum_{i=0}^{(n-1)/2} w_i \cos((n-2i)\pi f_r)$$

It has been determined in Corollary 3.5-3 that $G\{\sigma(j\omega)\}$, which is the denominator of above ratio, is zero at frequencies:

$$f = \frac{\omega}{2\pi} = \frac{i}{M} = \frac{i}{rnT_s}, \; i = 1, 2, 3, \ldots$$

or the relative frequencies,

$$f_r = f\,T_s = \frac{i}{rn}, \; i = 1, 2, 3, \ldots$$

Therefore the term $\sin(\frac{M\omega}{2})$, which occurs in both the numerator and the denominator, can be cancelled out except at these frequencies. In other words, the ratio is not defined at these frequencies.

At other frequencies, the gain ratio can be rewritten as,

$$\frac{G\{\hat{\sigma}_n(j\omega)\}}{G\{\sigma(j\omega)\}} = \frac{2\pi f_r}{\sin(n\pi f_r)} \sum_{i=0}^{(n-1)/2} w_i \cos((n-2i)\pi f_r)$$

Similarly, the second result can be obtained for even values of n. ♦♦

<u>**Corollary 4.4-6**</u> (Invariance of approximation error)

The approximation ratio, as given by Corollary 4.4-5, for different FII intervals and sampling frequencies are the same at the same relative frequency, provided the approximation ratio can be defined at this relative frequency. ♦ ♦

Proof

The approximation ratio given by Corollary 4.4-5 shows also the relationship between the approximation ratio and the relative frequency f_r. It does not involve the FII interval M nor the sampling frequency f_s. So the ratio is the same for all M and f_s as long as the ratio is defined at a particular f_r.

♦♦

Corollary 4.4-6 provides an important and simple way to design a numerical realization of a FII. It means that the percentage error calculated from convenient values of the sampling interval T_s and the FII interval M can be used to analyse numerical rules of the same order n but different T_s and M.

Tables 4.4-1a and 4.4-1b tabulate the percentage errors at various relative frequencies for the first six Newton-Cotes type formulae for FII. In these tables, the percentage errors are calculated by setting T_s=1 and M=n. However, due to Corollary 4.4-6, they are applicable for all M and T_s.

There are two special entries in these tables. The first is at the relative frequencies where the percentage error is not defined due to division by zero. The second one is more important. It is at the relative frequencies where phase errors occur, that is, where the percentage error is less than –100% (Corollary 4.4-4). In order to minimise the approximation errors, a numerical rule should not be used at these frequencies where phase error occurs.

These tables serve as a reference in selecting appropriate parameters, such as the sampling interval and formula order, for a numerical realization of a FII. An example of the design procedure based on this table is illustrated later in Example 4.4.5.

Some observations from these tables and Figures 4.4-1a and 4.4-1b are now discussed.

Table 4.4-1a Percentage Error of Numerical Formulae

Key

øø : zero divided by zero.

: phase error occurs.

Relative freq.	Order of numerical formula					
	1st	2nd	3rd	4th	5th	6th
0.01	-0.033%	0.000%	0.000%	0.000%	0.000%	0.000%
0.02	-0.132%	0.000%	0.000%	0.000%	0.000%	0.000%
0.03	-0.296%	0.001%	0.002%	0.000%	0.000%	0.000%
0.04	-0.527%	0.002%	0.005%	0.000%	0.000%	0.000%
0.05	-0.824%	0.005%	0.012%	0.000%	0.000%	0.000%
0.06	-1.187%	0.011%	0.026%	-0.001%	-0.001%	0.000%
0.07	-1.617%	0.021%	0.049%	-0.002%	-0.004%	0.000%
0.08	-2.114%	0.037%	0.085%	-0.004%	-0.009%	0.001%
0.09	-2.679%	0.059%	0.138%	-0.008%	-0.020%	0.002%
0.10	-3.312%	0.091%	0.215%	-0.016%	-0.040%	0.005%
0.11	-4.013%	0.134%	0.322%	-0.030%	-0.077%	0.012%
0.12	-4.783%	0.192%	0.467%	-0.054%	-0.144%	0.029%
0.13	-5.623%	0.268%	0.661%	-0.094%	-0.262%	0.070%
0.14	-6.533%	0.366%	0.916%	-0.157%	-0.472%	0.175%
0.15	-7.514%	0.489%	1.248%	-0.259%	-0.850%	0.493%
0.16	-8.567%	0.644%	1.676%	-0.420%	-1.557%	2.110%
0.17	-9.694%	0.835%	2.224%	-0.675%	-2.980%	-7.042%
0.18	-10.894%	1.069%	2.924%	-1.082%	-6.302%	-2.875%
0.19	-12.168%	1.353%	3.814%	-1.741%	-17.497%	-2.636%
0.20	-13.519%	1.697%	4.946%	-2.839%	øø	-2.917%
0.21	-14.948%	2.111%	6.390%	-4.761%	32.442%	-3.506%
0.22	-16.454%	2.605%	8.238%	-8.416%	21.739%	-4.412%
0.23	-18.041%	3.193%	10.619%	-16.565%	19.258%	-5.718%
0.24	-19.709%	3.892%	13.718%	-43.077%	19.054%	-7.579%
0.25	-21.460%	4.720%	17.810%	øø	19.991%	-10.240%

Table 4.4-1b Percentage Error of Numerical Formulae (cont')

Relative freq.	Order of numerical formula					
	1st	2nd	3rd	4th	5th	6th
0.26	-23.296%	5.698%	23.318%	71.131%	21.749%	-14.102%
0.27	-25.218%	6.852%	30.934%	45.247%	24.259%	-19.842%
0.28	-27.229%	8.214%	41.870%	38.160%	27.571%	-28.676%
0.29	-29.331%	9.820%	58.453%	36.031%	31.824%	-42.991%
0.30	-31.525%	11.715%	85.809%	36.138%	37.248%	-68.111%
0.31	-33.814%	13.954%	137.845%	37.624%	44.198%	-81.370%
0.32	-36.201%	16.603%	270.193%	40.179%	53.220%	##
0.33	-38.688%	19.745%	##	43.709%	65.179%	##
0.34	-41.278%	23.486%	##	48.237%	81.524%	##
0.35	-43.975%	27.959%	##	53.867%	104.862%	##
0.36	-46.780%	33.334%	##	60.774%	140.405%	##
0.37	-49.699%	39.839%	##	69.217%	200.307%	##
0.38	-52.734%	47.773%	##	79.557%	321.008%	##
0.39	-55.889%	57.550%	##	92.305%	684.649%	##
0.40	-59.169%	69.748%	##	108.184%	##	##
0.41	-62.579%	85.205%	##	128.247%	##	##
0.42	-66.122%	105.178%	##	154.082%	##	##
0.43	-69.804%	131.646%	##	188.192%	##	##
0.44	-73.631%	167.911%	##	234.755%	##	##
0.45	-77.609%	219.920%	##	301.302%	##	##
0.46	-81.744%	299.570%	##	402.902%	##	##
0.47	-86.043%	434.633%	##	574.731%	##	##
0.48	-90.513%	708.434%	##	922.333%	##	##
0.49	-95.162%	1537.634%	##	1973.476%	##	##
0.50	##	##	##	##	##	##

Effect of Increasing Order of Formulae

The effect of increasing the order of formulae can be seen in Table 4.4-1. At a low relative frequency, say 0.1, increasing formula order generally decreases the percentage error. However moving from an even order formula to the next odd order formula does not reduce the error. This is consistent with the well known property of original numerical formulae (Bajpal el al 1974, Kreyszig 1988, Davis and Rabinowitz 1984).

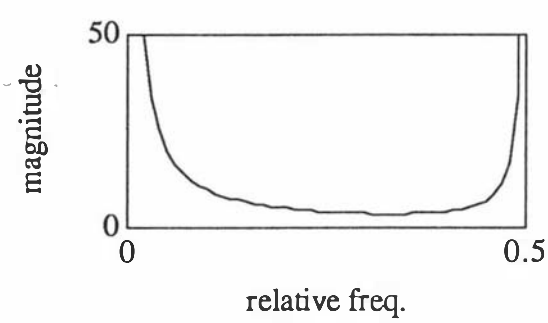
In contrast, the errors of a higher order formula are generally higher at the upper range of the relative frequency. This is because the error of a higher order formula increases more rapidly as the frequency increases. This characteristic is in fact also a property of the original numerical formulae but is less well known (Hamming 1977, Williams 1986).

Also a higher order formula generally encounters a phase shift error at a lower relative frequency.

Response at High Frequency

A major limitation of the original Newton-Cotes type numerical rules for a definite integral is the "blow out" phenomenon at some high frequencies (Hamming 1977, Williams 1986). In other words the gain grows to infinity at these high frequencies. This phenomenon is demonstrated for Simpson 1/3 rule in Figure 4.4-2.

Figure 4.4-2 "Blow-out" of Simpson 1/3 rule.



This phenomenon can be explained using the common stability concept in the discrete-time z domain (Ogata 1987). In the z domain, the original Newton-Cotes rules of order n have the general form of,

$$\zeta(z) = \frac{J_n(z)}{1 - z^n} \tag{4.4-33}$$

This means these rules have n repeated poles at z=1. As a discrete-time system is unstable when there are multiple real poles on the unit circle (Ogata 1987), a Newton-Cotes rule becomes unstable when $n \leq 2$, that is when the rule is of higher order than the Trapezoidal rule.

This problem does not occur with the FII versions. From Theorems 4.4-1 to 4.4-3, the z domain descriptions of these FII version have the general form:

$$\begin{aligned} \hat{\sigma}_n(z) &= T_s \left(w_0 + w_1 z^{-1} + \ldots w_m z^{-m} \right) \\ &= \frac{T_s \left(w_0 z^m + w_1 z^{m-1} + \ldots w_m \right)}{z^m} \end{aligned} \tag{4.4-34}$$

That is all the FII numerical rules have only repeated poles at z=0, and thus they are stable. This is formalized in the following corollary.

<u>**Corollary 4.4-7**</u> (stability of FII numerical rules)

All the numerical rules given by Theorems 4.4-1 to 4.4-3 are stable.

♦♦

4.4.5 Selecting the Appropriate Numerical Method

The previous Sections 4.4.1 to 4.4.3 determined three numerical methods to approximate the FII operation. The equations given in Theorems 4.4-1 to 4.4-3 appear complex but are in fact very easy to use if the underlying concept is understood. That is, a fundamental numerical formula of nth order, J_n, requires (n+1) points or n sampling intervals. So the FII interval that can be approximated by a single fundamental formula is n

sampling intervals. If a FII interval, M, longer than n is needed, it is necessary to cascade this fundamental formula until the required length is reached.

Let us consider the case where $M=5T_s$. The simplest approach is to choose a fundamental formula of fifth order (n=5). If this is more complex than required, a first order fundamental formula can be used by cascading five of first order fundamental formulae. These two approaches are given by Theorem 4.4-1. If n=1 is too inaccurate and n=5 is too complex, Theorem 4.4-2 gives an alternative to use fundamental formulae whose orders are not exact divisors of 5. Using Theorem 4.4-2, two fundamental formulae respectively of second order (n=2) and third (n=3) can be used (as 2+3=5).

Another alternative is given by Theorem 4.4-3 where a fundamental formula of order higher than M can be used. For the previous example, one could use formula of n=11 and n=6. This is because formula of n=6 gives a basic interval of $M=6T_s$ and formula of n=11 gives an interval of $M=11T_s$. So an interval of M=5 can be obtained by taking the difference between these two.

Note that a FII interval of M can always be divided into different subintervals and all the three theorems can be combined in whatever fashion by applying each of them to different subintervals. Of course this may not be much use in practice.

Although all the three numerical methods determined by Theorems 4.4-1 to 4.4-3 can be used to derive a suitable numerical realization for the FII, the Numerical method I is recommended in practice because of the neat properties determined in Section 4.4.4.

Also, as discussed in Section 4.4.4, a higher order formula can only provide a superior approximation at very low relative frequencies. To compensate for the loss of useful bandwidth, a higher sampling rate is then required. Furthermore, it can be seen from Table 4.4-1, the reduction in error becomes less significant at a higher sampling rate. So, unless a very high sampling rate is available and a very high accuracy is required, the Numerical Method I with a low order fundamental formula should be used in practice.

From Table 4.4-1, the Simpson 1/3 formula has the best overall performance. At a relative frequency of 0.25, it has a percentage error of less than 5%. This means that if the sampling rate is four times the highest natural rate of the system, a numerical realization based on the Simpson 1/3 rule will return a maximal error of 5%. A design procedure using Table 4.4-1 is demonstrated in the following example.

Example 4.4.5

Let the highest natural frequency, f_n, in the system be 1 Hz, the required FII interval, M, be 2 seconds, and the sampling interval, T_s, be a multiple of 0.1 second, ie.

$$f_n = 1 \text{ Hz}$$

$$M = 2 \text{ sec}$$

$$T_s = k(0.1) \text{ sec} \qquad , k = 1,2,3,\ldots$$

Also the maximal error allowed is 5% and Numerical Method I is to be applied. So from Table 4.4-1, two possible choices are,

(a) Trapezoidal rule (n=1) with maximal relative frequency, f_r, of 0.12

or (b) Simpson 1/3 (n=2) with maximal f_r of 0.25.

The minimal sampling frequency can be calculated for each case using the equation,

$$f_r = \frac{f_n}{f_s}$$

So the minimal f_s and the maximum T_s for each case is,

(a) $$\min f_s = \frac{1}{0.12} \text{ Hz} = 8.3 \text{ Hz}$$

$$\max T_s = \frac{1}{\min f_s} = 0.12 \text{ second}$$

(b) $$\min f_s = \frac{1}{0.25} \text{ Hz} = 4 \text{ Hz}$$

$$\max T_s = \frac{1}{4 \text{ Hz}} = 0.25 \text{ second}$$

As the FII interval needs to be an exact multiple of nT_S (n is the order of the numerical formula) when Numerical Method I is used and, T_S is restricted to a multiple of 0.1 second, the possible numerical realization are thus:

a) Trapezoidal rule with $T_S = 0.1$ sec. This means, $M = 20\ T_S$ and the index r in Theorem 4.4-1 is 20.

b) Simpson 1/3 rule with $T_S = 0.2$ sec, $M = 10\ T_S$ and $r = 5$.

From Theorem 4.4-1, the size of the numerical realization is given by the FII interval in terms of the sampling interval. Comparing the above two cases, the Simpson 1/3 rule has a longer sampling interval and thus a smaller structure, but yet returns the same accuracy. Therefore the Simpson Rule is preferred to the Trapezoidal rule. ♦♦

4.5 SUMMARY

Three methods of realizing the FII operation have been determined, namely the frequency response method, the hybrid method and the numerical method.

Among these three methods, the simplest is the hybrid method. This is because it uses two basic elements which are commonly available in modern control system, that is an analog integrator (can also be a digital integrator) and a digital backward differentiator. The major disadvantage is that it requires two separate components, and also its capability is limited by the capability of the available integrator.

The recommended method for FII realization is the Numerical Method I. It offers the most flexibility in terms of both accuracy and frequency range. Also its design procedure and realization structure are simple and can be generalized. This is mostly because the nature of numerical methods match the behaviour of the FII operation, as the fundamental formulae of Newton-Cotes numerical rules are inherently a form of fixed interval integration. A reference table on percentage errors has been developed for designing this numerical realization for any sampling interval and FII interval. A drawback of the numerical method is that a reasonably fast sampling rate is usually needed.

The third alternative is the frequency response method, in which analog or digital filters are designed to match the required frequency response. It requires a complicated design procedure. Furthermore this method is usually limited to frequencies of less than $\frac{1}{M}$ Hz, where M is the FII interval. This is because the discontinuities in gain magnitude occurring at multiples of $\frac{1}{M}$ Hz are difficult to match using filter design techniques. Therefore it is not recommended for general cases. However this method is useful when a continuous-time FII operation is needed as it is the only given method suitable for analog filters.

CHAPTER FIVE

APPLICATION OF THE FIXED INTERVAL INTEGRAL

In this chapter, several possible applications of the Fixed Interval Integral (FII) are presented.

It details the formulation of a FII description of a system, which can be employed to estimate in discrete-time, both the continuous-time-model parameters and the pure delay. This approach to continuous-time-model parameter estimation is then demonstrated using several simulation examples.

Other possible applications of FII are also discussed.

5.1 INTRODUCTION

In Chapter 3, a special integral called the Fixed Interval Integral (FII) was introduced. A FII operator, σ, was also defined in Definition 3.2-4, that is:

$$\sigma_{(\nu=t;\ \mathrm{M})}\, f(\nu) \;=\; \int_{t-\mathrm{M}}^{t} f(\nu)\, d\nu$$

Several characteristics of the FII operator were then determined. Of particular importance is its relationship with the derivative operator determined in Theorem 3.3-4b as,

$$\sigma\, \rho \;=\; \nabla$$

where ρ is the differential operator and ∇ is the backward-difference operator as defined in Definition 3.2-4, that is:

$$\rho_{(\nu=t)}\, f(\nu) \;=\; \frac{d}{d\nu} f(\nu)\Big|_{\nu=t}$$

$$\nabla_{(\nu=t;\ \mathrm{M})}\, f(\nu) \;=\; f(t) - f(t-\mathrm{M})$$

This theorem indicates that the FII operator acts as a special kind of inverse derivative operator. Also, as shown in Corollary 3.5-1, the result of this inverse operation is independent of any constant initial conditions. These two characteristics of the FII operation indicate that the FII operation may be useful for transforming a differential equation to an equivalent description which is free of derivatives and independent of constant initial conditions. In fact it is determined in Section 5.2 of this chapter that several transformations of this sort can be achieved easily.

Furthermore, Chapter 4 has established that the FII operation can be readily realized using several different methods. Consequently, the FII description of a system is useful in practice. As determined in Sections 5.3 and 5.4, the FII is especially important in estimating the continuous-time-model parameters and pure delay.

This FII estimation technique for continuous-time models follows the equation-error approach described earlier in Chapter 2. In this approach, continuous-time system models that use differential equations are transformed to system descriptions that can be sampled in discrete-time and do not involve derivatives. This transformation avoids the differentiation of noisy signals and allows the parameters of the continuous-time models be estimated using one of the well established discrete-time algorithms.

However the use of FII is not limited to parameter estimation. Several other possible applications are suggested at the end of this chapter for future study.

There are five subsequent sections in this chapter :

Section 5.2	details the transformation of system descriptions that use differential equations to system descriptions that use the FII.
Section 5.3	shows the use of this FII description for estimating, in discrete-time, a system's continuous-time-model parameters.
Section 5.4	shows an extension of the FII technique given in Section 5.3 to include estimation of continuous-time pure delay when the system is subjected to a piece-wise constant input.
Section 5.5	discusses other possible applications of FII, for future research.
Section 5.6	summaries the key contributions in this chapter.

5.2 FII SYSTEM DESCRIPTIONS

The systems of interest for the parameter estimation technique given in this section are as follows.

Definition 5.2-1 (Systems of Interest)

The systems of interest are single input and single output, lumped parameter and strictly proper dynamical systems, which can be described by models in the form of linear ordinary differential equations (ODEs), that is:

$$\sum_{i=0}^{n} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{c} b_i \frac{d^i u(t)}{dt^i} + e(t)$$

such that $n > c$, and without loss of generality, $a_n = 1$.

Here $y(t)$ is the output, $u(t)$ is the input, $e(t)$ is the model error, and a_i and b_i are the system parameters that need to be estimated.

♦♦

Writing the above ODE using the derivative operator, ρ, which is defined in Chapter 3 gives,

$$A(\rho)\, y(t) = B(\rho)\, u(t) + e(t) \tag{5.2-1}$$

where

$$A(\rho) = \rho^n + a_{n-1}.\rho^{n-1} + a_{n-2}.\rho^{n-2} + ... + a_0 \tag{5.2-2}$$

$$B(\rho) = b_c.\rho^c + b_{c-1}.\rho^{c-1} + b_{c-2}.\rho^{c-2} + ...+ b_0 \tag{5.2-3}$$

n and c are thus respectively the orders of the polynomials $A(\rho)$ and $B(\rho)$.

In order to avoid computation of derivatives, the differential Equation (5.2-1) can be transformed into a FII description using the operators defined in Chapter 3. The following three transformations are possible.

Theorem 5.2-1

For systems given by Definition 5.2-1, the following holds

$$(\nabla^n + a_{n-1} \nabla^{n-1}\sigma + ... + a_{n-k} \nabla^{n-k}\sigma^k + a_0 \sigma^n) y(t)$$

$$= (b_c \nabla^c\sigma^{n-c} + ... + b_{c-k} \nabla^{c-k}\sigma^{n-c+k} + b_0 \sigma^n) u(t) + \sigma^n e(t)$$

♦♦

Proof

Applying the nth order FII operator, σ^n, to both sides of Equation (5.2-1) that is:

$$\sigma^n[\rho^n + a_{n-1} \rho^{n-1} + ... + a_0] y(t)$$

$$= \sigma^n[(b_c \rho^c + ... + b_0) u(t) + e(t)]$$

gives:

$$(\sigma^n\rho^n + a_{n-1} \sigma^n\rho^{n-1} + ... + a_0 \sigma^n) y(t)$$

$$= (b_c \sigma^n\rho^c + ... + b_0 \sigma^n) u(t) + \sigma^n e(t)$$

Then by using Theorem 3.3-4b, that is:

$$\sigma\rho = \nabla$$

and the commutativity of ∇, ρ and σ determined in Theorem 3.3-2, Theorem 5.2-1 is obtained. ♦♦

Theorem 5.2-2

For systems given by Definition 5.2-1, with $y(t)$ and $u(t)$ observing *Assumption 3.3-1,* the following holds,

$$(\nabla^n + a_{n-1} \nabla^n\eta + ... + a_{n-r} \nabla^n\eta^r + a_0 \nabla^n\eta^n) y(t)$$

$$= (b_c \nabla^n\eta^{n-c} + ... + b_{c-r} \nabla^n\eta^{n-c+r} + b_0 \nabla^n\eta^n) u(t) + \nabla^n\eta^n e(t)$$

♦♦

Proof

If Assumption 3.3-1 holds, then by the commutativity of ∇ and η, and Theorem 3.3-4a, that is,

$$\sigma = \nabla \eta$$

a kth order FII operator can be replaced as,

$$\sigma^k = \nabla^k \eta^k$$

So, replacing all the σ^k operators in Theorem 5.2-1 gives Theorem 5.2-2.

♦♦

Theorem 5.2-3

For systems given by Definition 5.2-1, with $y(t)$ and $u(t)$ observing *Assumption 3.3-2*, the following holds,

$$(\nabla^n + a_{n-1} \nabla^n \zeta +\ldots+ a_{n-r} \nabla^n \zeta^r + a_0 \nabla^n \zeta^n) \; y(t)$$

$$= (b_c \nabla^n \zeta^{n-c} +\ldots+ b_{c-r} \nabla^n \zeta^{n-c+r} + b_0 \nabla^n \zeta^n) \; u(t) + \nabla^n \zeta^n e(t)$$

♦♦

Proof

If Assumption 3.3-2 holds, then by the commutativity of ∇ and ζ, and the second identity in Theorem 3.3-4a, that is:

$$\sigma = \nabla \zeta$$

a kth order FII operator can be replaced as:

$$\sigma^k = \nabla^k \zeta^k$$

So, replacing all the σ^k operators in Theorem 5.2-1 gives Theorem 5.2-3.

♦♦

Theorems 5.2-1 to 5.2-3 give three forms of FII description for the continuous-time system. There are two major differences in these three theorems. Firstly, they are different in their dependence on Assumptions 3.3-1 and 3.3-2. Theorem 5.2-1 is the more general theorem as it is not dependent on either of the two assumptions. Theorem 5.2-2 is constrained by Assumption 3.3-1 and is thus weaker than Theorem 5.2-1. Theorem 5.2-3 is the generalization of Theorem 5.2-2, because the initial time in Assumption 3.3-2 can be any time instant besides zero.

Secondly, these theorems imply two different forms of realization of the variable quantities in the transformed system description. Theorem 5.2-1 implies the realization of the kth order FII operator, σ^k. This can be achieved by using the methods determined in Chapter 4. As all these methods rely on the identity determined by Theorem 3.3-4a, that is,

$$\sigma = \nabla \zeta$$

Therefore, the realization of σ^k effectively means cascading the operation, $\nabla\zeta$, k times. In the block diagram form, this is represented as follows:

$$\rightarrow \boxed{\nabla\zeta} \rightarrow \boxed{\nabla\zeta} \cdots\cdots\cdots\cdots \rightarrow \boxed{\nabla\zeta} \rightarrow$$

On the other hand, Theorem 5.2-3 (or 5.2-2) implies the realization of some kth backward differences of kth order integrals, $\nabla^k\zeta^k$, (or $\nabla^k\eta^k$). In block diagram, this is represented as follows:

$$\rightarrow \boxed{\nabla^k} \longrightarrow \boxed{\zeta^k} \rightarrow$$

These two forms are not generally identical, as the equation:

$$(\nabla\zeta)^k = \nabla^k\zeta^k$$

is only true if the system obeys Assumption 3.3-2.

The validity of Assumptions 3.3-1 and 3.3-2 in the case of parameter estimation is discussed later in Section 5.3.2. Three important characteristics of these FII descriptions of Theorems 5.2-1 to 5.2-3 are discussed in the following.

<u>**Corollary 5.2-1**</u> (Independence on Constant Initial Conditions)

The FII descriptions of a system are independent on any constant initial condition. ♦

Proof

The FII descriptions given in Theorems 5.2-1 to 5.2-3 do not consist of any variable with a fixed time-index. ♦♦

<u>**Corollary 5.2-2**</u> (Independence on Derivatives)

The FII descriptions of a system do not involve any derivative term in the system input or output. ♦

Proof

As seen from Theorems 5.2-1 to 5.2-3, the FII descriptions consist of only the backward difference and FII of the system input and output. ♦♦

<u>**Corollary 5.2-3**</u> (Independence on Infinitely Accumulating Terms)

The FII descriptions of a system do not involve any explicit term that accumulates infinitely with time ♦

Proof

As seen from Theorems 5.2-1 to 5.2-3, the FII descriptions do not consist of any term with the time variable, t, as a multiplier. ♦♦

Note that if the system input is bounded and the system is stable then using the Corollary 5.2-3 and Corollary 3.5-2 (Boundness of FII), none of the quantities in the FII description accumulates infinitely with time.

To appreciate these characteristics better, let us consider the popular integral equation described earlier in Chapter 2. In this approach the differential Equation 5.2-1 is integrated repeatedly using the finite integration from a certain initial time to present time. Using the operator notation, this is represented by:

$$\zeta^n[\ (\ \rho^n\ +\ a_{n-1}\ \rho^{n-1}\ +....+\ a_0\ )\ y(t)\ \ ]$$
$$=\ \zeta^n[\ (\ b_c\ \rho^c\ +\ b_{c-1}\ \rho^{c-1}\ +...+\ b_0)\ u(t)\ +\ e(t)\ ] \qquad (5.2\text{-}4)$$

The resultant integral description has a general form of,

$$\sum_{k=0}^{n} a_{n-k}\, \zeta^k\, y(t) + \sum_{k=0}^{n} b_{n-k}\, \zeta^k\, u(t) + \sum_{k=1}^{n} c_k \frac{t^{k-1}}{(k-1)!} = \xi(t) \tag{5.2-5}$$

where $\xi(t)$ is the error term and, c_k relates to the initial conditions such that:

$$c_k = \sum_{i=0}^{k-1} b_{k-1-i}\, \rho^i\, u(t_0) - \sum_{i=0}^{k-1} a_{k-1-i}\, \rho^i\, y(t_0) \tag{5.2-6}$$

This integral equation does not possess the three characteristics of the FII description given by Corollaries 5.2-1 to 5.2-3. Firstly, the integral equation consists of constant initial terms, $u(t_0)$ and $y(t_0)$ in the c_k Secondly it has (constant) derivative terms, $\rho^j u(t_0)$ and $\rho^j y(t_0)$ Finally it has explicit terms that accumulate infinitely with time, namely:

$$\frac{t^{k-1}}{(k-1)!}$$

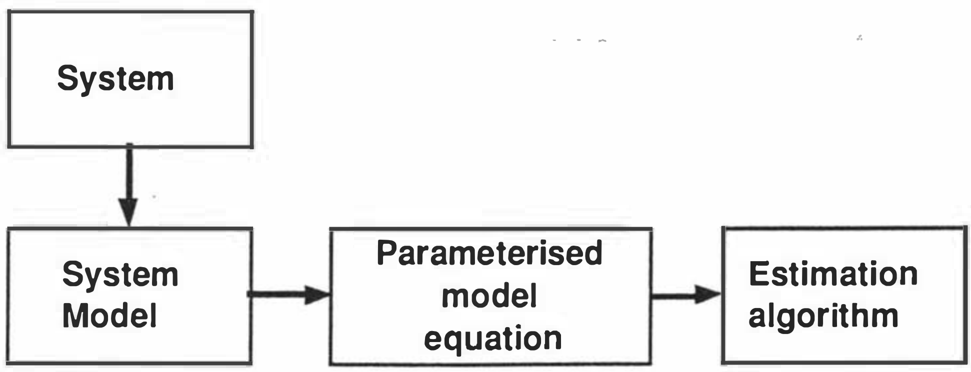
The most important characteristic of the FII descriptions is their independence on initial derivative terms. It is usually not too difficult to obtain initial values of $u(t)$ and $y(t)$, and the infinitely accumulating terms can be constrained by choosing an arbitrary finite period. But the derivative terms are usually not measurable. The characteristics given by Corollaries 5.2-1 to 5.2-3 enable the development of an important parameter estimation technique in the next section, using the FII descriptions of continuous-time systems.

5.3 PARAMETER ESTIMATION OF CONTINUOUS-TIME-MODEL

The exercise of parameter estimation can often be divided into two stages (Ljung 1987, Unbehauen and Rao 1987). The first stage is to derive some *parameterised model equations* from the original model of the system to be identified. The second stage is to estimate the parameters of the system model, directly or indirectly, by estimating the parameters of the parameterised model equations using appropriate estimation algorithms.

So the whole process of parameter estimation usually involves the four elements illustrated in Figure 5.3-1. They are, the system to be identified, a model of the system, parameterised model equations and estimation algorithms.

Figure 5.3-1 Elements of Parameter Estimation



When setting up a parameter estimation mechanism using the "Continuous-time-model discrete-time estimation" (CD) approach discussed in Chapter 2, the parameterised model equations used for estimating the parameters of the system model should meet the following three criteria.

Criteria 5.3-1 (Parameter estimation criteria for CD approach)

a) All the parameters of the system model can be derived easily from the parameterised model equations.

b) All the variable quantities required for the estimation can be readily realized in practice.

c) The parameters can be estimated using a discrete-time algorithm.

♦♦

Three possible forms of parameterised model equations for this purpose are the FII system descriptions derived earlier in Theorems 5.2-1 to 5.2-3.

5.3.1 FII Parameterised Model Equations

The three FII system descriptions given by Theorems 5.2-1 to 5.2-3 can be rewritten in a vector form that is linear in terms of the parameters. For example, the FII system description given by Theorems 5.2-1:

$$\begin{aligned}(\nabla^n + a_{n-1} \nabla^{n-1}\sigma +\ldots+ a_0 \sigma^n) y(t) \\ = (b_c \nabla^c\sigma^{n-c} +\ldots+ b_0 \sigma^n) u(t) + \sigma^n e(t)\end{aligned} \tag{5.3-1}$$

can be rewritten as:

$$\begin{aligned}\nabla^n y(t) = -(a_{n-1} \nabla^{n-1}\sigma +\ldots+ a_0 \sigma^n)y(t) \\ + (b_c \nabla^c\sigma^{n-c} +\ldots+ b_0 \sigma^n) u(t) + \sigma^n e(t)\end{aligned} \tag{5.3-2}$$

or

$$\nabla^n y(t) = \begin{bmatrix} -\nabla^{n-1}\sigma y(t) \\ \vdots \\ -\sigma^n y(t) \\ \nabla^c \sigma^{n-c} u(t) \\ \vdots \\ \sigma^n u(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} a_{n-1} \\ \vdots \\ a_0 \\ b_c \\ \vdots \\ b_0 \end{bmatrix} + \sigma^n e(t) \tag{5.3-3}$$

Note that Equation (5.3-3) is linear in terms of the parameters, $a_{n-1},\ldots,b_0$. This linear-in-the-parameter form of the FII descriptions is formalized in the following corollary.

<u>Corollary 5.3-1</u>
(Linear-in-the-parameter vector form of FII system descriptions)

The FII system descriptions given by Theorems 5.2-1 to 5.2-3 can be written in a vector form which is linear in terms of the system parameters, that is:

$$\psi(t) = \lambda^{\mathrm{T}}(t)\, \theta + \varepsilon(t)$$

where

$$\psi(t) = \nabla^n y(t)$$

$$\lambda(t) = [\ \lambda y_{n-1}\ \ \lambda y_{n-2} \ldots \lambda y_0 \quad \lambda_{uc} \ldots \lambda_{u0}\ ]^T$$

$$\theta = [\ a_{n-1}\ \ a_{n-2} \ldots a_0 \quad b_c \ldots b_0\ ]^T$$

with

$$\lambda_{y_i} = -\nabla^i \sigma^{n-i}\, y(t),\ \ -\nabla^n \eta^{n-i}\, y(t) \quad \text{or} \quad -\nabla^n \zeta^{\,n-i}\, y(t)$$

$$\lambda_{u_i} = \nabla^i \sigma^{n-i}\, u(t),\ \ \nabla^n \eta^{n-i}\, u(t) \quad \text{or} \quad \nabla^n \zeta^{\,n-i}\, u(t)$$

$$\varepsilon(t) = \sigma^n\, e(t),\ \ \nabla^n \eta^n\, e(t) \quad \text{or} \quad \nabla^n \zeta^n\, e(t)$$

♦♦

Corollary 5.3-2 (Augmented vector form of FII system descriptions)

The linear-in-the-parameter vector form of the FII system descriptions given by Corollary 5.3-1 can be augmented to form:

$$\begin{bmatrix} \psi(t_1) \\ \psi(t_2) \\ \vdots \\ \psi(t_n) \end{bmatrix} = \begin{bmatrix} \lambda(t_1) \\ \lambda(t_2) \\ \vdots \\ \lambda(t_n) \end{bmatrix}^T \theta + \begin{bmatrix} \varepsilon(t_1) \\ \varepsilon(t_2) \\ \vdots \\ \varepsilon(t_n) \end{bmatrix}$$

where $t_1 \ldots t_n$ are any sampling time instants that need not to be regularly time-spaced or in the correct sequence. ♦♦

Proof

The vector equation given by Corollary 5.3-1 is an pure algebraic equation that does not consist of dynamics. Therefore it is true for any time instant. ♦♦

The applicability of the FII equations given by Theorems 5.2-1 to 5.2-3 as parameterised model equations in the context of CD approach to parameter estimation, is now examined in terms of the three criteria in Criteria 5.3-1.

a) Obtaining the system parameters easily

The vector θ is the parameter vector of the FII equation. It also consists of all the parameters of the original system model given by Equation 5.2-1. Therefore the parameters of the system model can be obtained directly from these FII parameter estimation equations.

b) Realizing the required variable quantities readily

The vectors $\psi(t)$ and $\lambda(t)$ consist of quantities in the form of the FII of the system output and input. Consequently, if the system output and input are measurable, these quantities can be easily realized using one of the three possible methods established in Sections 4.2 to 4.4.

c) Using discrete-time estimation algorithm

As described in Corollary 5.3-1, the three FII descriptions are linear in the system parameters. Furthermore, as both $\psi(t)$ and $\lambda(t)$ may be constructed from measurable quantities, they may be sampled at different time instants, t_i. So the augmented equation given by Corollary 5.3-2 can be formed. This equation is in the form of a standard discrete-time linear-in-the-parameter equation for parameter estimation. Therefore the parameters of the FII equations can be estimated using a discrete-time algorithm implemented on a digital computer.

In view of these, the FII equations given by Theorems 5.2-1 to 5.2-3 are applicable as a parameter estimation equation to form a CD parameter estimation mechanism.

In practise, it is easier to sample the FII vector equation given by Corollary 5.3-1 with a constant sampling interval, T_s. So a more convenient form of the equation is:

$$\psi(hT_s) = \lambda^{\mathrm{T}}(hT_s)\,\theta + \varepsilon(hT_s) \tag{5.3-4}$$

or

$$\psi(h) = \lambda^{\mathrm{T}}(h)\,\theta + \varepsilon(h) \tag{5.3-5}$$

where $h = 0,1,2,\ldots$, is the discrete-time index.

The question now is, which of the three FII equations given by Theorems 5.2-1 to 5.2-3 should be chosen and, what FII interval and parameter estimation algorithm should be applied. These are discussed in detail in the next three subsections.

Before proceeding to this though, the following simulation example is provided to demonstrate the use of the FII equation given by Theorem 5.2-1, in estimating the continuous-time-model parameters of a simple deterministic system.

Simulation Example 5.3-1

Suppose a system is described by,

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t) = b\,u(t) \tag{5.3-6}$$

that is:

$$(\rho^2 + a_1\rho + a_0)\,y(t) = b\,u(t) \tag{5.3-7}$$

and $a_1 = 3$, $a_0 = 2$, $b = 5$.

Also suppose the sampling interval, T_s, is 0.01 sec and that, the FII interval, M, is 0.5 sec. (A detailed discussion on selecting M is given later in Section 5.3.3).

Applying Theorem 5.2-1 to Equation (5.3-7) yields,

$$(\nabla^2 + a_1\nabla\sigma + a_0\sigma^2)\,y(t) = b\,\sigma^2u(t)$$

or $$\nabla^2y(t) = a_1[-\nabla\sigma y(t)] + a_0[-\sigma^2y(t)] + b[\sigma^2u(t)] \tag{5.3-8}$$

Comparing with Corollary 5.3-1, it can be recognised that Equation (5.3-8) is in the linear-in-the-parameter form:

$$\psi(t) = \lambda^T(t)\,\theta$$

where

$$\psi(t) = \nabla^2\,y(t) \tag{5.3-9}$$

$$\lambda(t) = [\,-\nabla\sigma y(t)\quad -\sigma^2y(t)\quad \sigma^2u(t)\,]^T \tag{5.3-10}$$

$$\theta = [\,a_1\quad a_0\quad b\,]^T \tag{5.3-11}$$

The output of this system when subjected to a Pseudo-random-binary-signal, PRBS (Ljung 1987) input is simulated using MATLAB, a commercial computer aided control systems design package. The FII quantities of the input and output in $\lambda(t)$ are then found using Numerical Method I (Theorem 4.3-1) with Simpson 1/3 formula. The parameters in θ is estimated using a recursive least square algorithm that is described by (Ljung 1987),

$$\hat{\theta}(h) = \hat{\theta}(h\text{-}1) + K(h)\,[\,\psi(h) - \lambda^{\mathrm{T}}(h)\,\hat{\theta}(h\text{-}1)\,] \tag{5.3-12a}$$

$$K(h) = \frac{P(h\text{-}1)\,\lambda(h)}{k_f + \lambda^{\mathrm{T}}(h)\,P(h\text{-}1)\,\lambda(h)} \tag{5.3-12b}$$

$$P(h) = \frac{[\,P(h\text{-}1) - K(h)\,\lambda^{\mathrm{T}}\,P(h\text{-}1)\,]}{k_f} \tag{5.3-12c}$$

where h is the discrete time index, $\hat{\theta}$ is the estimate of θ and k_f is the forgetting factor. The values used in this simulation are,

$$\hat{\theta}(0) = [\ 0\ \ 0\ \ 0\ ]^{\mathrm{T}}$$

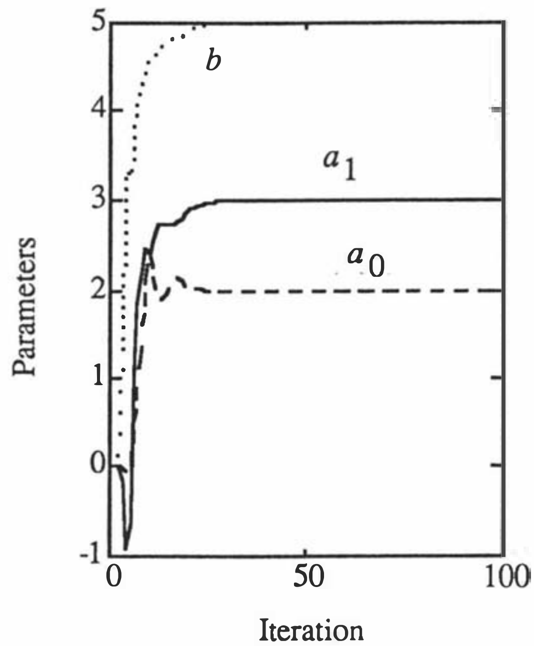
$$P(0) = 10^6$$

$$k_f = 0.99$$

A plot of the estimated parameters can be seen in Figure 5.3-2.

Appendix A gives the details of the related MATLAB functions and customized routines.

Figure 5.3-2 Simulation result of Simulation Example 5.3-1 for $a_1 = 3, a_0 = 2, b = 5$.



5.3.2 Validity of Assumption on Past System Behaviour

The dependence of Theorems 5.2-2 and 5.2-3 on Assumptions 3.3-1 and 3.3-2 means that, both the system's input, $u(t)$, and output, $y(t)$, need to be zero before a parameter estimation procedure based on these corollaries can be started.

To have the natural behaviour of the system following this requirement, it is necessary to apply an input of zero value to the system and to wait until the output settles to a zero value. Alternatively, if the system description is linearized about its steady states, then both the input and the output are required to reach steady states.

In many practical situations this requirement on the natural behaviour of the system is not feasible. Also this requirement limits the use of these techniques in on-line parameter estimation.

However this zero condition can be imposed artificially. This is done by setting all the measurements and related quantities before the starting time to zero. This imposition effectively introduces a step change to the input and output. But it does not disturb the validity of the model described by the differential Equation 5.2-1. This is because the formulation of Theorems 5.2-2 and 5.2-3 requires $u(t)$ and $y(t)$ and their integrals to begin at zero but not their derivatives. Consider the following — Theorem 5.2-1 uses the transformation involving derivatives:

$$\sigma^n \rho^k = \nabla^k \rho^{n-k} \tag{5.3-13}$$

which needs neither Assumption 3.3-1 nor Assumption 3.3-2. Then Theorems 5.2-2 and 5.2-3 are derived from Theorem 5.2-1 using the transformation:

$$\sigma^k = \nabla^k \zeta^k = \nabla^k \eta^k \tag{5.3-14}$$

This transformation only holds subject to the zero condition defined by the two assumptions, but it does not involve any derivative terms. Note that this is possible due to the independence on derivative terms of the FII description as determined in Corollary 5.2-2.

This establishes that Assumptions 3.3-1 and 3.3-2 do not in practice impose any limitation on the three parameter estimation equations given by Theorems 5.2-1 to 5.2-3. They are thus functionally identical and should all produce the same result, although due to the differences in their form, discrepancies might occur resulting from numerical error in implementation. Nevertheless these discrepancies should not be significant if reasonable implementations are applied. In view of this, the three parameter estimation equations are considered interchangeable in the estimation procedures discussed later.

5.3.3 Selecting the FII Interval (M)

Schoukens (1990) suggested a fixed interval, M, of either,

$$\mathrm{M} = T_s \quad \text{or} \quad 2T_s \tag{5.3-15}$$

where T_s is the sampling interval which observes Nyquist's (or Shannon's) Sampling Criterion. His selection of this interval is to avoid the phase jump and zero transmittance of FII at frequencies of multiples of $\frac{1}{M}$ Hz and to have the flattest possible gain behaviour over the Nyquist interval (0 to 0.5 relative frequency). However his presumption of the need for the flattest possible gain response is mistaken. In fact it is also shown later in this section that a FII interval of T_s or $2T_s$ is not applicable in practice.

Another suggestion for M is that of Sagara and Zhao (1990). They suggested M be chosen such that the frequency bandwidth of the FII matches the bandwidth of the system. This choice will be shown later in this section to be appropriate.

To demonstrate that it is unnecessary to have a flat gain response, let us consider the frequency domain equivalent of Equation 5.2-1,

$$y(s) = \frac{B(s)}{A(s)} u(s) + e(s) \tag{5.3-16}$$

where s is the Laplace transform variable and $e(s)$ is the appropriate error term. The FII descriptions in Theorems 5.2-1 to 5.2-3 are obtained by filtering Equation (5.3-16) with an nth order FII filter, $\sigma^n(s)$, that is,

$$\begin{aligned} \sigma^n(s)\, y(s) &= \sigma^n(s) \frac{B(s)}{A(s)} u(s) + \sigma^n(s)\, e(s) \\ &= \frac{B(s)}{A(s)} [\sigma^n(s) u(s)] + \sigma^n(s)\, e(s) \end{aligned} \tag{5.3-17}$$

This shows that the transfer function between $\sigma^n(s)\, y(s)$ and $\sigma^n(s)\, u(s)$ is identical to the original system transfer function. This arises from the property of the FII equations discussed in Section 5.3.1, that is the FII equations have the same parameters as the original system model.

Therefore, on condition that filtered signals, $\sigma^n(s)\, y(s)$ and $\sigma^n(s)\, u(s)$, can still provide sufficient information for the purpose of parameter estimation, the original system transfer function can be identified even though the $\sigma^n(s)$ filter does not have a flat frequency response over the significant range of the system's natural spectrum. Ljung (1987) has determined that a sufficient condition for an nth order system to provide persistently exciting measurements, which will result in consistent and convergent parameter estimates, is that there are n significant frequency components over the significant range of the system's natural spectrum.

This means the zero transmittance at multiples of $\frac{1}{M}$ Hz does not limit the usefulness of the FII parameter estimation equations because not every single frequency in the system's natural spectrum is required to obtain consistent and convergent estimates of parameters.

A small M is not applicable in practice because the resultant parameter estimation technique is sensitive to noise and computational error. This problem can be explained by the identity given by Theorem 3.3-4a, that is:

$$\sigma = \nabla\zeta$$

This identity shows that the FII is the (backward) difference between two integrals. So the smaller the FII interval, the smaller this difference is. Therefore a FII of small M will be sensitive to implementational error and system noise. Also Davis and Rabinowitz (1984) have shown that the smaller the integration interval with respect to the sampling interval, T_s, the larger the error in a numerical realization.

Corollary 3.5-3 (Frequency response of FII) shows that the FII operation is a form of lowpass filter. The larger the FII interval, the faster the gain decays at high frequency. As a result, a large FII interval results good immunization to high frequency noise, but it reduces the number of useful frequencies and also requires a more complex realization.

Consequently, a balance between the following two criteria is needed in selecting the FII interval.

Criteria 5.3-2 (Selecting the Interval of FII)

a) The FII interval, M, should be small enough to preserve the range of significant frequencies of the system's spectrum.

b) The FII interval, M, should be large enough to allow the system's output to change significantly.

♦♦

Considering these criteria, a suitable FII interval may be:

$$M = \frac{1}{\omega_n} = \frac{1}{2\pi f_n} = \iota_n \tag{5.3-18}$$

where ω_n is the natural frequency of the system in radians, f_n is the natural cyclical frequency of the system in hertz and τ_n is the effective time constant of the system.

We now examine the suitability of this value of M using Criteria 5.3-2:

a) Preserving the range of system's significant frequencies

As seen from Figure 3.5-1 (Frequency response of the FII operator), the transmittance of the FII decays more rapidly after the first zero transmittance at $\frac{1}{M}$ Hz. So in order to preserve the significant frequencies of the system, the first zero transmittance of FII should occur at a frequency of reduced significance, that is at a frequency at which the amplitude ratio of the system has significantly decayed.

Now, for the value of M given by Equation (5.3-18), the first zero transmittance of FII occurs at frequency of:

$$\frac{1}{M}\text{ Hz} = 2\pi f_n \text{ Hz} \tag{5.3-19}$$

Also, it is well known that (Marshall 1978, Coughanowr and Koppel 1983, Banks 1986) for a system given by Definition 5.2-1 which has more poles than zeros, the amplitude ratios of frequencies larger than f_n decay at a rate of at least 20dB per decade. Therefore at a frequency of $2\pi f_n$ Hz, the amplitude ratio of the system has decayed at least:

$$\frac{2\pi}{10} \text{ x } 20 \text{ dB} = 12.6 \text{ dB} \tag{5.3-20}$$

This means that with the value of M given by Equation (5.3-18), the first zero transmittance of the FII occurs at a frequency at which the amplitude ratio of the system has significantly decayed and thus, most of the significant frequencies of the system have been preserved.

b) Allowing significant changes in the system's output

The value of M given by Equation (5.3-18) is equal to the time constant of the system. Therefore, significant changes in the system's output generally will occur during this period of M.

Consequently, the value of M given by Equation (5.3-18) is a suitable choice. This guideline for selecting the FII interval M is formalised in the following.

Rule 5.3-1 (Selecting the FII interval for systems with white noise)

A suitable FII interval, M, for systems with white noise is:

$$M = \frac{1}{2\pi f_n} = \frac{1}{\omega_n} = \tau_n$$

where ω_n is the natural frequency of the system in radian, f_n is the natural cyclical frequency of the system in hertz and τ_n is the effective time constant of the system.

♦♦

This selection of M is similar to the conclusion of Sagara and Zhao (1990) which is made from some simulation results. However, they did not considered the cases of non-white noise.

In the case of non-white noise whose bandwidth peaks at a particular frequency, it is proposed here that a different M value should be chosen to minimise the effect of the noise. The proposed FII interval is,

$$M = \frac{k}{f_p} \qquad , k = 1,2,3\ldots \tag{5.3-21}$$

where f_p is the frequency (Hz) at which the noise has the peak magnitude.

This interval is chosen so that zero transmittance of FII at multiples of $\frac{1}{M}$ Hz coincides with the peak frequency of the noise. The optimal value of k depends on the nature of noise and the system. Nevertheless, the k values should be chosen such that the M value is close to the M value given by Rule 5.3-1 so that Criteria 5.3-2 are fulfilled. This second guideline for selecting the FII interval is formalised in the following.

Rule 5.3-2 (Selecting the FII interval for systems with coloured noise)

A suitable FII interval, M, for systems with coloured noise is:

$$M = \frac{k}{f_p} \qquad , k = 1,2,3\ldots$$

provided this value of M is close to the M value given by Rule 5.3-1. Here, f_p, in hertz, is the frequency at which the coloured noise has the peak magnitude.

♦♦

The usefulness of Rules 5.3-1 and 5.3-2, and also the effect of different FII interval length in parameter estimation is demonstrated in the following simulation examples.

Simulation Example 5.3-2

Suppose the system given in Example 5.3-1 is now subjected to a zero-mean Gaussian white-noise that is,

$$(\rho^2 + a_1\rho + a_0)\, y(t) = b\, u(t) + \varepsilon(t)$$

where $a_1 = 3$, $a_0 = 2$, $b = 5$ and $\varepsilon(t)$ is a white-noise term.

A noise-to-signal ratio (NR) is defined as,

$$NR = \frac{\text{standard deviation of } \varepsilon(t)}{\text{standard deviation of } y(t)} \tag{5.3-22}$$

and NR of the white-noise is assumed to be 5%.

Note that the natural frequency of this second order system (Coughanowr and Koppel 1983, Banks 1986) is:

$$\omega_n = \sqrt{a_0} = 1.414 \text{ rad}$$

So using Rule 5.3-1, the FII interval selected is:

$$M = \frac{1}{\omega_n} = 0.7 \text{ sec}$$

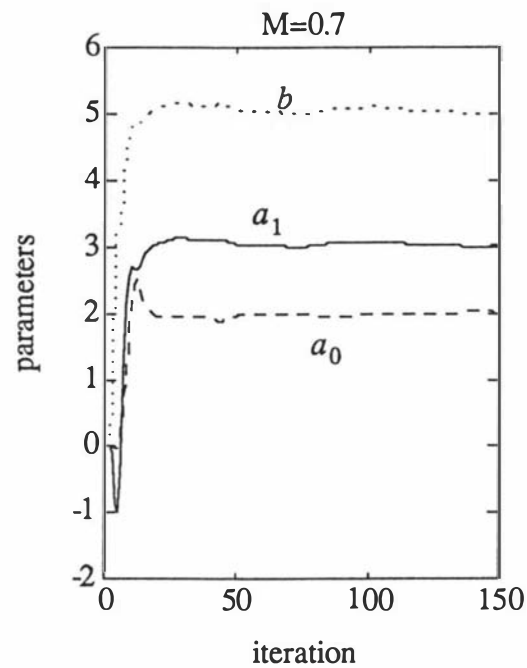
These system parameters are then estimated using the recursive least square algorithm described by Equation (5.3-12) in Simulation Example 5.3-1. The forgetting factor used is 0.99 and the initial values used are,

$$\hat{\theta}(0) = [\ 0\ 0\ 0\ ]^T$$

$$P(0) = 10^6$$

The result of the estimation is shown in Figure 5.3-3.

Figure 5.3-3 Simulation result of Simulation Example 5.3-2: a system with 5% white noise, M=0.7, $a_1 = 3$, $a_0 = 2$ and $b = 5$.



Above figure shows that the estimates of the system parameters converge to the actual values. Therefore, the value of M given by Rule 5.3-1, that is M = 0.7, is a suitable choice.

In order to investigate the effect of different FII interval length, the system parameters are also estimated using different M values.

The performance of the estimators with different M values is then compared using an error norm defined as:

$$||e|| = \frac{||\hat{\theta} - \theta||}{||\theta||} \tag{5.3-23}$$

where θ is the vector of system parameters, $[a_1\ a_0\ \ b]$, $\hat{\theta}$ is the estimated parameters and $||\bullet||$ is the Euclidean norm.

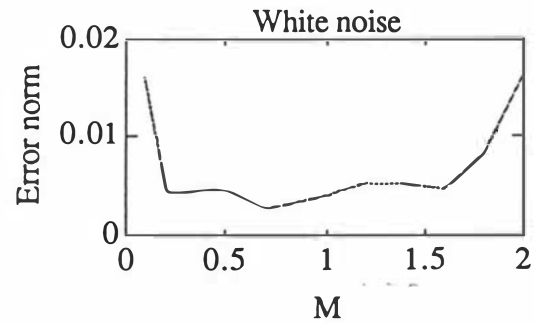
It is found that, generally, stable estimates can be obtained after 150 iterations. Table 5.3-1 shows the parameter estimates, the bias in terms of percentage error and the error norm after 150 iteration.

The error norm after 150 iterations is also plotted in Figure 5.3-4.

Table 5.3-1 Parameter estimates, percentage errors and error norm for various FII interval, M: a system with 5% white noise.

M	Parameter Estimates			Percentage Error			Error Norm
	a_1	a_0	b	a_1	a_0	b	
0.1	2.931	1.960	4.939	-2.29%	-1.98%	-1.21%	0.0162
0.2	2.988	1.996	4.976	-0.38%	-0.20%	-0.47%	0.0043
0.3	2.993	1.996	4.973	-0.20%	-0.15%	-0.52%	0.0044
0.4	2.996	1.997	4.971	-0.11%	-0.15%	-0.57%	0.0048
0.5	2.999	1.998	4.972	-0.00%	-0.08%	-0.55%	0.0045
0.7	3.010	2.004	4.987	0.34%	0.24%	-0.24%	0.0027
1.0	3.022	2.011	5.005	0.74%	0.58%	0.10%	0.0042
1.2	3.027	2.013	5.008	0.92%	0.66%	0.17%	0.0052
1.4	3.029	2.011	5.004	0.99%	0.55%	0.08%	0.0052
1.6	3.024	2.003	4.984	0.81%	0.16%	-0.30%	0.0047
1.8	3.010	1.990	4.951	0.34%	-0.48%	-0.97%	0.0082
2.0	2.987	1.972	4.903	-0.43%	-1.38%	-1.92%	0.0164

Figure 5.3-4 Error norm for various FII interval, M: a system with 5% white noise.



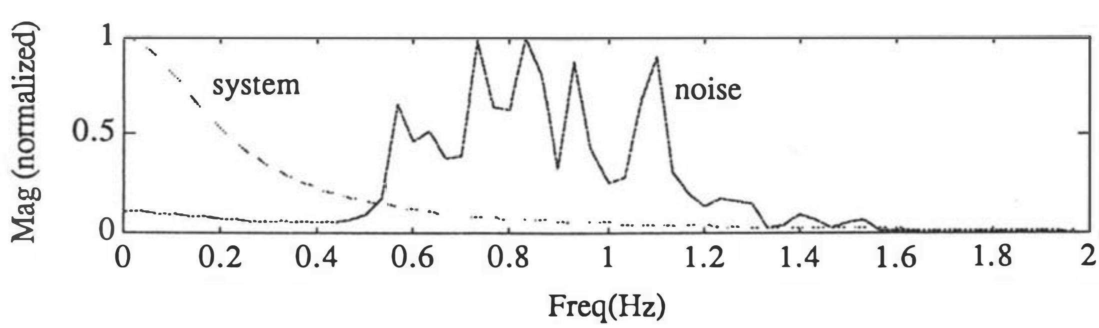
As seen from above figure, the lowest error norm occurs around M=0.7. Therefore, the FII interval, M, value given by Rule 5.3-1 in fact is the optimal choice for the system in this example.

Appendix A gives the software used for the simulation.

Simulation Example 5.3-3

The previous example is now subjected to a "pink" noise whose spectrum is given by Figure 5.3-5.

Figure 5.3-5 Spectrum of the noise in the system of Simulation Example 5.3-3



It can be seen from this figure that the noise peaks at frequencies in the neighbourhood of 0.5 to 1.1 Hz. The noise to signal ratio, as defined in Equation (5.3-22), is assumed to be 5%.

Various M value is then applied to estimate the parameters. The parameters are estimated using the recursive least square algorithm described by Equation (5.3-12) in Simulation Example 5.3-1. The forgetting factor used is 0.99 and the initial values used are,

$$\hat{\theta}(0) = [\ \ 0\ \ 0\ \ 0\ \ ]^T$$

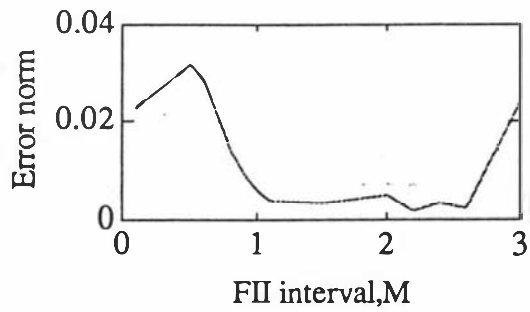
$$P(0) = 10^6$$

It is found that, generally, stable estimates are obtained after 150 iterations. Table 5.3-2 shows the parameter estimates, the bias in terms of percentage error and the error norm after 150 iteration. The error norm is also plotted in Figure 5.3-6.

Table 5.3-2 Parameter estimates, percentage errors and error norm for various FII interval, M: a system with 5% coloured noise and $a_1 = 3$, $a_0 = 2$, $b = 5$.

M	Parameter Estimates			Percentage Error			Error Norm
	a_1	a_0	b	a_1	a_0	b	
0.2	2.893	1.945	4.922	-3.55%	-2.73%	-1.59%	0.023
0.4	2.908	1.953	4.834	-3.05%	-2.33%	-3.32%	0.031
0.6	2.919	1.956	4.849	-2.67%	-2.16%	-3.01%	0.028
0.8	2.962	1.975	4.924	-1.24%	-1.23%	-1.51%	0.014
1.0	2.985	1.989	4.968	-0.48%	-0.54%	-0.63%	0.005
1.2	2.990	1.992	4.979	-0.32%	-0.37%	-0.42%	0.004
1.4	2.991	1.995	4.984	-0.28%	-0.23%	-0.31%	0.003
1.6	2.990	1.992	4.979	-0.32%	-0.38%	-0.41%	0.004
1.8	2.989	1.994	4.979	-0.34%	-0.35%	-0.42%	0.004
2.0	2.986	1.992	4.976	-0.46%	-0.39%	-0.47%	0.004
2.2	2.995	1.996	4.991	-0.16%	-0.17%	-0.18%	0.002
2.4	2.989	1.995	4.982	-0.36%	-0.26%	-0.35%	0.003
2.6	2.993	1.995	4.988	-0.23%	-0.23%	-0.25%	0.002
2.8	2.959	1.979	4.931	-1.36%	-1.01%	-1.30%	0.013
3.0	2.922	1.963	4.879	-2.58%	-1.85%	-2.43%	0.024

Figure 5.3-6 Error norm for various FII interval, M: a system with 5% coloured noise.



This figure shows that the minimal error norm no longer occurs at a FII interval of M=0.7, but in the neighbourhood of 1 to 2.5. Therefore, Rule 5.3-2 is useful here for selecting the FII interval. Using Rule 5.3-2 with k=1, the suitable range of M values is 0.9 to 2. The value of one should be used for the k in Rule 5.3-2 because this k value gives a FII interval closest to the interval given by Rule 5.3-1, that is M=0.7.

Appendix A gives the software used for the simulation.

5.3.4 Selecting a Parameter Estimation Algorithm

It has been established in Corollary 5.3-1 that the FII system equation given by Theorems 5.3-1 to 5.3-3 can be written in standard linear-in-the-parameter form of the Equation-error (EE) structure for discrete-time parameter estimation (Ljung 1987, Young 1980). Therefore all the well established EE estimation algorithms for discrete-time systems (Astrom and Eykhoff 1971, Saha and Rao 1983, Ljung 1987) can be used for the FII technique.

In the case of on-line estimation, the (weighted) recursive least-squares (RLS) and the recursive instrumental-variable (RIV) methods have been found to be the most effective (Young and Jakeman 1979, Whitfield and Messali 1987, Saha et al 1982, Chang et al 1986, Sagara et al 1991).

However, it is important to note that the equation errors of the FII equations in Theorems 5.3-1 to 5.3-3 are effectively given by:

$$\sigma^n \, e(t)$$

where $e(t)$ is the original system noise and σ^n is an n order FII operator. Therefore, the equation error will not be white even if the system noise is white due to the FII filtering. It is well known that the simple least-squares method results biased estimates in a situation with high level of coloured equation error (Saha and Rao 1983, Ljung 1987). Consequently, algorithms for coloured noise such as the instrumental variable methods and the modified or extended least-squares methods should be considered when the noise level is high.

Nevertheless, the best estimation method for the FII estimation technique might be different for systems and noise of different nature. Simulation study should thus be carried out before adopting an estimation method for practical use.

5.4 SIMULTANEOUS ESTIMATION OF CONTINUOUS-TIME-MODEL PARAMETER AND PURE DELAY

The previous section has determined that the FII descriptions are useful in parameter estimation of continuous-time models. It will be shown in this section that when the system is controlled by a discrete-time digital controller, the FII descriptions can be extended for the estimation of pure delay in a continuous-time-model. Although the system input is restricted to a discrete-time input, a delay which is not an exact multiple of the sampling interval can be estimated when the FII descriptions are used. Also, as most modern controllers are digital controllers, an estimation technique based on the FII descriptions is useful in practice. The estimation technique proposed in this section is based on the extended B-polynomial (Biswas and Singh 1978) approach described earlier in Chapter 2.

The system model considered here has a form similar to that of Definition 5.2-1 but with an added delay term in the system input. So the model can be described as follows.

Definition 5.4-1 (Systems of interest with pure delay)

The systems of interest with pure delay are single input and single output, lumped parameter and strictly proper dynamical systems, which have a pure delay term in their input and can be described by models in the form of linear ordinary differential equation (ODE), that is:

$$\sum_{i=0}^{n} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{c} b_i \frac{d^i u(t-\tau)}{dt^i} + e(t)$$

such that $n > c$, and without loss of generality, $a_n = 1$.

Here $y(t)$ is the output, $u(t)$ is the input, τ is the pure delay, $e(t)$ is the model error (or system noise), and a_i and b_i are the model parameters. Quantities that need to be estimated are τ, $a_{n-1} \ldots a_0$, and $b_c \ldots b_0$.

♦♦

Using the operator notations defined in Chapter 3 the above delay-differential equation can be written as:

$$A(\rho)\, y(t) \;=\; B(\rho)\; \delta_{(t;\tau)}\; u(t) \;+\; e(t) \qquad (5.4\text{-}1)$$

where δ is the delay operator and ρ is the differential operator as defined in Definition 3.2-4, and,

$$A(\rho) = \rho^{n} + a_{n-1}.\rho^{n-1} + a_{n-2}.\rho^{n-2} + ... + a_{0} \qquad (5.4\text{-}2)$$

$$B(\rho) = b_{c}.\rho^{c} + b_{c-1}.\rho^{c-1} + b_{c-2}.\rho^{c-2} + ...+ b_{0} \qquad (5.4\text{-}3)$$

Equation (5.4-1) will be used later to develop the FII technique for simultaneous estimation of delay and parameters. To present the FII technique, this section is divided into the following five subsections:

Section 5.4.1 The development of the FII estimation technique for delay system starts with finding the FII of a group of special functions. This group of functions is named here the *Piece-wise Defined Functions* (PDF) because it can be smoothly described in continuous-time only within pieces or intervals of time. In other words, the PDF is a series of functions such that each function in the series is valid only for a certain duration.

Section 5.4.2 Using the result from Section 5.4.1, the FII is found for a subclass of PDF which is named the *Piece-wise Constant Function* (PCF). The PCF is used in later subsections to describe the control signal generated by a digital controller. The FII of this PCF is needed to formulate a FII estimation equation for delay systems that are controlled by digital controllers.

Section 5.4.3 Using the result from Section 5.4.2, an FII estimation equation is formulated for the delay system. The FII estimation equation can be used by a discrete-time estimation method to estimate the delay and parameters of a continuous-time model.

Section 5.4.4 This subsection details the formulae for calculating the delay and other model parameters from the coefficients of the estimation equation given by Section 5.4.3.

Section 5.4.5 This subsection summarizes the steps involved in simultaneous estimation of delay and parameters using the FII equation.

5.4.1 FII of Piece-wise Defined Functions

This subsection presents the FII of a special function named here the piece-wise defined function (PDF). The PDF is defined as follows.

Definition 5.4-2 (Piece-Wise Defined Functions)

A piece-wise defined function (PDF) of t, $r(t)$, is a series of subfunctions. Each of these subfunctions in the series, $r_h(v)$, is defined only within an interval of t with interval length T_s, such that:

$$r(t) = r_h(v) \qquad \text{for} \quad t_h \le t < t_{h+1}$$

where,

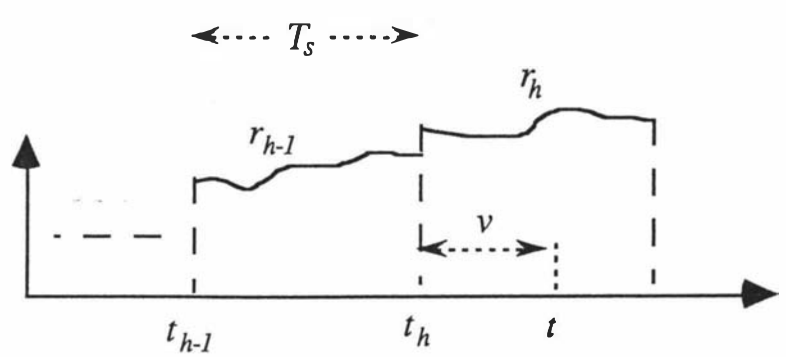
$$t_h = hT_s \quad , \quad h \in \mathrm{W} = \{0,1,2,3,\ldots\}$$

$$v = t - t_h \quad , \quad 0 \le v < T_s$$

♦♦

This PDF is illustrated in Figure 5.4-1.

Figure 5.4-1 Piece-Wise Defined Function.



When the component functions $r_h(v)$ are known, the FII of $r(t)$ can be determined and it is given in the following theorem.

<u>Theorem 5.4-1</u> (FII of Piece-Wise Defined Functions)

If $r(t)$ is a piece-wise defined function as defined in Definition 5.4-2 then the FII of $r(t)$, with FII interval of length mT_s, is given by:

$$\sigma_{(t=t_h+v\,;\,mT_s)}\, r(t)$$

$$= \left[\eta_{(v=v)} + \eta_{(v=T_s)} \sum_{i=1}^{m} \delta^{i}_{(t=t;T_s)} - \eta_{(v=v)}\, \delta^{m}_{(t=t;T_s)}\right] r(t)$$

Where, as defined in Definition 5.4-2,

$$v = t - t_h \quad , \quad 0 \le v < T_s$$

$$t_h = hT_s \quad , \quad h \in \mathbf{W} = \{0,1,2,3,\ldots\}$$

also as in Definition 3.2-4,

$$\eta_{(v=w)} f(v) = \int_0^w f(v)\, dv$$

$$\delta_{(v=w;T_s)} f(v) = f(w-T_s)$$

♦♦

Proof

Applying the FII operator on the PDF $r(t)$:

$$\sigma_{(t=t_h+v;mT_s)}\, r(t)$$

$$= \int_{t_h+v-mT_s}^{t_h+v} r(t)\, dt$$

$$= \int_{t_h}^{t_h+v} r(t)\, dt + \int_{t_{h-1}}^{t_{h-1}+T_s} r(t)\, dt + \ldots + \int_{t_{h-m}+v}^{t_{h-m}+T_s} r(t)\, dt$$

$$= \int_{t_h}^{t_h+v} r_h(v)\, dt + \sum_{i=1}^{m-1}\left[\int_{t_{h-i}}^{t_{h-i}+T_s} r_{h-i}(v)\, dt\right] + \int_{t_{h-m}+v}^{t_{h-m}+T_s} r_{h-m}(v)\, dt$$

In above equation, substituting t with (t_h+v) for the first integral, t with $(t_{h-i}+v)$ for the integrals within the summation and t with $(t_{h-m}+v)$ for the last integral yields,

$$\sigma_{(t=t_h+v\ ;\ mT_s)}\, r(t)$$

$$= \int_0^{v} r_h(v)\, dv + \int_0^{T_s}\left[\sum_{i=1}^{m} r_{h-i}(v)\, dv\right] - \int_0^{v} r_{h-m}(v)\, dv \qquad (5.4\text{-}4)$$

Also as,

$$r(t) = r_h(v) \quad , \ for \ t_h \leq t < t_{h+1}$$

then,

$$r_{h-i}(v) = r(t-iT_s) = \delta^{i}_{(t=t;T_s)}\, r(t) \qquad (5.4\text{-}5)$$

Therefore substituting Equation (5.4-5) into Equation (5.4-4) and expressing the integral with the operator η yields Theorem 5.4-1.

♦♦

It is important to note that the integration operators in Theorem 5.4-1, $\eta_{(v=v)}$ and $\eta_{(v=T_s)}$, operate on the variable v, but the delay operators, $\delta_{(t=t;T_s)}$, operate on the variable t.

An important class of PDFs in the estimation of the pure delay and the parameters of a continuous-time system model is the *piece-wise constant function* (PCF). This is because the output of most digital controllers are driven by a zero-order-hold. Using Theorem 5.4-1, the FII of a PCF is found in the following subsection.

5.4.2 FII of Piece-Wise Constant Functions

A piece-wise constant function is a class of PDFs such that each of the component functions is a constant. The piece-wise constant function is defined as follows.

Definition 5.4-3 (Piece-Wise Constant Functions)

A piece-wise constant function (PCF), $u(t)$, is defined as:

$$u(t) = u_h = u(t_h) \quad , \qquad \text{for } t_h \leq t < t_{h+1}$$

where u_h is constant within the interval of $t_h \leq t < t_{h+1}$ and t_h is as defined in Definition 5.4-2 that is,

$$t_h = hT_s \qquad , \qquad h \in \mathbf{W} = \{0,1,2,3,\ldots\}$$

♦♦

As a PCF is a piece-wise defined function, the FII of a PCF can thus be found by assigning the PCF as the operand in Theorem 5.4-1, that is replacing $r(t)$ with $u(t)$ in Theorem 5.4-1. This gives:

$$\sigma_{(t=t_h+v\ ;\ mT_s)}\ u(t)$$

$$= \left[\eta_{(v=v)} + \eta_{(v=T_s)} \sum_{i=1}^{m} \delta^{i}_{(t=t;T_s)} - \eta_{(v=v)}\ \delta^{m}_{(t=t;T_s)} \right] u(t) \tag{5.4-6}$$

As seen from Definition 5.4-3, $u(t)$ is independent of the variable v, the resultant quantities of,

$$\delta^{i}_{(t=t;T_s)}\ u(t)$$

are thus independent of v as well and thus they are considered as constants in the integration, $\eta_{(v=v)}$ and $\eta_{(v=T_s)}$.

As a result, Equation (5.4-6) becomes (subscripts of δ are omitted for simplicity):

$$\sigma_{(t=t_h+v;\ mT_s)}\ u(t) = \left[v + T_s \sum_{i=1}^{m-1} \delta - (T_s - v)\ \delta^m \right] u(t)$$

$$\triangleq \; P_1(v,\delta)\; u(t) \tag{5.4-7}$$

Equation (5.4-7) gives the first order FII of a PCF. Note that $P_1(v,\delta)$ is a polynomial of δ and the coefficients of δ are functions of variable v. Consequently, the first order FII of a PCF is a function of v and thus has the form of the piece-wise defined function given by Definition 5.4-2. So Theorem 5.4-1 can be used again to find the second order FII of a PCF, by setting the operand to be the first order FII of PCF, $P_1(v,\delta)\, u(t)$. That is:

$$\sigma^2_{(t=t_h+v\;;\;mT_s)}\; u(t)$$

$$= \left[\eta_{(v=v)} + \eta_{(v=T_s)} \sum_{i=1}^{m} \delta^i_{(t=t;T_s)} - \eta_{(v=v)}\; \delta^m_{(t=t;T_s)}\right] P_1(v,\delta)\; u(t) \tag{5.4-8}$$

Again, as $\delta^i_{(t=t;\,T_s)}\, u(t)$ is independent of v, above Equation (5.4-8) becomes,

$$\sigma^2_{(t=t_h+v;\;mT_s)}\; u(t) = \left[\eta_{(v=v)} P_1(v,\delta) + \eta_{(v=T_s)} P_1(v,\delta) \sum_{i=1}^{m} \delta^i_{(t=t;T_s)}\right.$$

$$\left. - \;\eta_{(v=v)}\; P_1(v,\delta)\; \delta^m_{(t=t;T_s)}\right] u(t)$$

or

$$\sigma^2_{(t=t_h+v\;;\;mT_s)}\; u(t) = \sum_{i=0}^{m} p_{i,2}(v,\delta)\; \delta^i\; u(t)$$

$$\triangleq \; P_2(v,\delta)\; u(t) \tag{5.4-9}$$

where,

$$p_{i,2}(v,\delta) = \begin{cases} \eta_{(v=v)}\; P_1(v,\delta) & ,\; i=0 \\ \eta_{(v=T_s)}\; P_1(v,\delta) & ,\; 1 \le i \le (m-1) \\ [\eta_{(v=v)} - \eta_{(v=T_s)}]\; P_1(v,\delta) & ,\; i=m \end{cases} \tag{5.4-10}$$

This process can be continued to find a kth order FII of $u(t)$. So it provides a recursive means to evaluate the FII of PCFs. Note that the polynomial, $P_k(v,\delta)$, serves as an operator to transform the $u(t)$ into its FII. This is formalised in the following theorem.

<u>**Theorem 5.4-2**</u> (FII of Piece-Wise Constant Functions)

If $u(t)$ is a piece-wise constant function as defined in Definition 5.4-3 then the kth order FII of $u(t)$ with FII interval $M = m\,T_s$ is given by:

$$\sigma^{k}_{(t=t_h+v\ ;\ mT_s)}\ u(t) \;=\; P_k(v,\delta)\ u(t)$$

where $P_k(v,\delta)$ is an "operational polynomial" transforming $u(t)$ to its FII. It is given by:

$$P_k(v,\delta)\ u(t) \;=\; \sum_{i=0}^{m} p_{i,k}(v,\delta)\ \delta^i\ u(t)$$

with

$$P_0(v,\delta) \;=\; 1$$

$$p_{i,k}(v,\delta) = \begin{cases} \eta_{(v=v)}\ P_{k\text{-}1}(v,\delta) & ,\ i=0 \\ \eta_{(v=T_s)}\ P_{k\text{-}1}(v,\delta) & ,\ 1 \le i \le (m\text{-}1) \\ [\eta_{(v=v)} - \eta_{(v=T_s)}]\ P_{k\text{-}1}(v,\delta) & ,\ i=m \end{cases}$$

Also, as defined in Definition 5.4-2:

$$v = t - t_h \qquad , \qquad 0 \le v < T_s$$

$$t_h = hT_s \qquad , \qquad h \in \mathbf{W} = \{0,1,2,3,\ldots\}$$

♦♦

Another form of the FII operational polynomial of PCF, $P_k(v,\delta)$, can be observed directly from Theorem 5.4-2 and it is given in the following corollary. This form is more explicit and provides better understanding when $P_k(v,\delta)$ is applied in later subsections to develop an estimation technique for system parameters and delay.

Corollary 5.4-1 (Alternative form of FII operational polynomial)

The FII operational polynomial of PCF, $P_k(v,\delta)$, defined in Theorem 5.4-2 has the form of,

$$P_k(v,\delta) = \sum_{i=0}^{mk} w_{i,k}(v)\ \delta^i_{(t=t;T_s)}$$

where $w_{i,k}(v)$ is an appropriate polynomial of v. Also as defined in Theorem 5.4-2, k is the order of the FII and the FII interval is $m\,T_s$.

♦♦

Proof

Observe from Theorem 5.4-2 that, $P_k(v,\delta)$, is a polynomial of δ. The coefficients of this polynomial of δ are polynomials of v. Also the lowest order of δ in $P_k(v,\delta)$ is zero and the highest order is mk. ♦♦

Corollary 5.4-1 is important because it determines the number of past values of $u(t)$ needed to evaluate the FII of $u(t)$. In other words, it shows that in order to find the kth FII of $u(t)$ with an FII interval of mT_s, a total of $(mk+1)$ measurements of $u(t)$ is required.

The coefficients $w_{i,k}(v)$ are polynomials in the variable v, as they result from multiple integration of the constants u_h with respect to v. The interval T_s is usually pre-defined and thus considered as a constant. An important property of the coefficients $w_{i,k}(v)$ in this context of parameter estimation, is determined in the following theorem.

Theorem 5.4-3 (Sum of coefficients of FII for PCF)

If $w_{0,k}, w_{1,k}, \dots, w_{mk,k}$ are coefficients of the FII operational polynomial $P_k(v,\delta)$, as defined in Corollary 5.4-1, then,

$$\sum_{i=0}^{mk} w_{i,k} = m^k T_s^{\,k}$$

where, as defined in Theorem 5.4-2, k is the order of the FII, T_s is the interval of the PCF and the FII interval is mT_s. ♦♦

Proof

From Theorem 5.4-2,

$$P_k(\nu,\delta)\, u(t) = \left[\eta_{(\nu=\nu)} P_{k-1}(\nu,\delta) + \eta_{(\nu=T_s)} P_{k-1}(\nu,\delta) \sum_{i=1}^{m} \delta^{i}_{(t=t;T_s)} - \eta_{(\nu=\nu)}\, P_{k-1}(\nu,\delta)\, \delta^{m}_{(t=t;T_s)}\right] u(t) \qquad (5.4\text{-}11)$$

In the right-hand side of Equation (5.4-11), the first term,

$$\eta_{(\nu=\nu)} P_{k-1}(\nu,\delta)$$

and the last term,

$$-\,\eta_{(\nu=\nu)}\, P_{k-1}(\nu,\delta)\, \delta^{m}_{(t=t;T_s)}$$

have the same coefficients for δ but with opposite sign. Therefore the coefficients of δ for this two terms sum up to zero.

This means that the sum of the coefficients in $P_k(\nu,\delta)$ is simply equal to the sum of the coefficients in the second term in the right-hand side of Equation (5.4-11), that is:

$$\eta_{(\nu=T_s)} P_{k-1}(\nu,\delta) \sum_{i=1}^{m} \delta^{i}_{(t=t;T_s)} \qquad (5.4\text{-}12)$$

Note that in Equation (5.4-12), the coefficients of δ are decided by the term:

$$\eta_{(\nu=T_s)} P_{k-1}(\nu,\delta)$$

Also, the coefficients of δ in $\eta_{(\nu=T_s)} P_{k-1}(\nu,\delta)$ repeat m times in because of the summation in Equation (5.4-12).

If Theorem 5.4-3 is true, the sum of the coefficients in $P_{k-1}(\nu,\delta)$ will be equal to $m^{k-1}T_s^{k-1}$ and thus the coefficients due to $\eta_{(\nu=T_s)} P_{k-1}(\nu,\delta)$ sum to:

$$\begin{aligned}\eta_{(\nu=T_s)} m^{k-1} T_s^{\,k-1} &= m^{k-1}\, T_s^{\,k-1}\, \eta_{(\nu=T_s)}\, 1 \\ &= m^{k-1}\, T_s^{\,k-1}\, T_s \\ &= m^{k-1}\, T_s^{\,k} \qquad (5.4\text{-}13)\end{aligned}$$

As mentioned earlier, this sum repeats m times for Equation (5.4-12), that is:

$$m\, m^{k-1}\, T_s^{\,k} = m^k\, T_s^{\,k}$$

Therefore, if Theorem 5.4-3 is true for $P_{k-1}(v,\delta)$ then it is also true for $P_k(v,\delta)$.

Now for k=1, it has been found in Equation (5.4-7) that:

$$P_1(v,\delta) = \left[v + T_s \sum_{i=1}^{m-1} \delta - (T_s - v)\, \delta^m \right]$$

So the sum of the coefficients of δ are:

$$\sum_{i=0}^{m} w_{i,k} = v + (m-1)T_s + (T_s - v) = mT_s$$

which means Theorem 5.4-3 is true for k=1. Therefore, by induction Theorem 5.4-3 is true for all k=1,2,3,... ♦♦

The coefficients $w_{i,k}(v)$ of the FII operational polynomial can be found using Theorem 5.4-2. Despite the complicated appearance of the equations, the mechanism to evaluate these coefficients is reasonably simple. This is demonstrated in the following example.

Example 5.4-1

Let M=5 T_s, that is m=5. The first order of FII is found by first evaluating the $p_{i,k}(v,\delta)$ in Theorem 5.4-2:

$$p_{0,1}(v,\delta) = \eta_{(v=v)}\, P_0(v,\delta) = \eta_{(v=v)}\, 1 = \int_0^v 1\, dv = v$$

$$p_{1,1}(v,\delta) = \eta_{(v=T_s)}\, P_0 = p_{0,1}(v,\delta)\Big|_{v=T_s} = T_s$$

$$p_{5,1}(v,\delta) = \left[\, \eta_{(v=v)} - \eta_{(v=T_s)}\, \right] P_0 = p_{1,1}(v,\delta) - p_{0,1}(v,\delta)$$

$$= T_s - v$$

Note that it is necessary to do integration only once.

The coefficients $w_{i,k}(v)$ are then found by shifting $p_{i,k}(v,\delta)$ and summing their elements as illustrated in the following:

$p_{0,1}$:	v					
$p_{1,1}$:		T_s				
$p_{2,1}$:			T_s			
$p_{3,1}$:				T_s		
$p_{4,1}$:					T_s	
+ $p_{5,1}$:						$T_s - v$
	$w_{0,1}$	$w_{1,1}$	$w_{2,1}$	$w_{3,1}$	$w_{4,1}$	$w_{5,1}$

So,

$$w_{0,1} = v$$

$$w_{1,1} = w_{2,1} = w_{3,1} = w_{4,1} = T_s$$

$$w_{5,1} = T_s - v$$

and

$$P_1(v,\delta) = v + T_s(\delta^1 + \delta^2 + \delta^3 + \delta^4) + (T_s - v)\delta^5$$

This mechanism is more obvious for the second order FII. Again, to find the second order FII of $u(t)$, first evaluating $p_{i,2}(v,\delta)$:

$$p_{0,2}(v,\delta) = \eta_{(v=v)} P_1(v,\delta)$$

$$= \int_0^v [v + T_s(\delta^1+\delta^2+\delta^3+\delta^4) + (T_s-v)\delta^5]dv$$

$$= \tfrac{1}{2}v^2 + vT_s(\delta^1+\delta^2+\delta^3+\delta^4) + (vT_s - \tfrac{1}{2}v^2)\delta^5$$

$$p_{1,2}(v,\delta) = p_{2,2}(v,\delta) = p_{3,2}(v,\delta) = p_{4,2}(v,\delta)$$

$$= \tfrac{1}{2}T_s^2 + T_s^2(\delta^1+\delta+\delta^3+\delta^4) + \tfrac{1}{2}T_s^2\,\delta^5$$

$$p_{5,2}(v,\delta) = \tfrac{1}{2}(T_s^2 - v^2) + (T_s^2 - vT_s)(\delta^1+\delta^2+\delta^3+\delta^4) + \tfrac{1}{2}(T_s - v)^2\,\delta^5$$

To find the coefficients $w_{2,i}$, the $p_{2,i}$ are shifted and tabulated as before,

$p_{0,2}$:	$\frac{v^2}{2}$	vT_s	vT_s	vT_s	vT_s	$vT_s-\frac{v^2}{2}$					
$p_{1,2}$:		$\frac{T_s^2}{2}$	T_s^2	T_s^2	T_s^2	T_s^2	$\frac{T_s^2}{2}$				
$p_{2,2}$:			$\frac{T_s^2}{2}$	T_s^2	T_s^2	T_s^2	T_s^2	$\frac{T_s^2}{2}$			
$p_{3,2}$:				$\frac{T_s^2}{2}$	T_s^2	T_s^2	T_s^2	T_s^2	$\frac{T_s^2}{2}$		
$p_{4,2}$:					$\frac{T_s^2}{2}$	T_s^2	T_s^2	T_s^2	T_s^2	$\frac{T_s^2}{2}$	
+ $p_{5,2}$:						$\frac{T_s^2-v^2}{2}$	$T_s^2-vT_s$	$T_s^2-vT_s$	$T_s^2-vT_s$	$T_s^2-vT_s$	$\frac{(T_s-v)^2}{2}$
	$w_{0,2}$	$w_{1,2}$	$w_{2,2}$	$w_{3,2}$	$w_{4,2}$	$w_{5,2}$	$w_{6,2}$	$w_{7,2}$	$w_{8,2}$	$w_{9,2}$	$w_{10,2}$

The $w_{i,2}$ are then found by summing the elements in respective rows in the table. This gives,

$$w_{0,2} = \frac{v^2}{2} \qquad w_{6,2} = 4\tfrac{1}{2}T_s^2 - vT_s$$

$$w_{1,2} = \tfrac{1}{2}T_s^2 + vT_s \qquad w_{7,2} = 3\tfrac{1}{2}T_s^2 - vT_s$$

$$w_{2,2} = 1\tfrac{1}{2}T_s^2 + vT_s \qquad w_{8,2} = 2\tfrac{1}{2}T_s^2 - vT_s$$

$$w_{3,2} = 2\tfrac{1}{2}T_s^2 + vT_s \qquad w_{9,2} = 1\tfrac{1}{2}T_s^2 - vT_s$$

$$w_{4,2} = 3\tfrac{1}{2}T_s^2 + vT_s \qquad w_{10,2} = \tfrac{1}{2}(T_s-v)^2$$

$$w_{5,2} = 4\tfrac{1}{2}T_s^2 + vT_s - v^2$$

♦♦

Theorems 5.4-2 and 5.4-3 for a PCF may be used to develop an estimation technique for the continuous-time-model parameters including the pure delay. This is discussed in the next subsection.

5.4.3 Delay and Parameter Estimation Equation

It has been indicated in Section 5.3 that an appropriate estimation equation is needed before the parameters of a model can be estimated. By combining the results obtained in Sections 5.3.1 and 5.4.2, an estimation equation can be obtained that facilitates simultaneous estimation of parameters and pure delay of the systems given by Definition 5.4-1.

As given by Equation 5.4-1, the system model in operator notation is:

$$A(\rho)\, y(t) = B(\rho)\, \delta_{(t;\tau)}\, u(t) + e(t) \tag{5.4-14}$$

where τ is the pure delay and:

$$A(\rho) = \sum_{i=0}^{n} a_{n-i}\, \rho^{i} \tag{5.4-15}$$

$$B(\rho) = \sum_{i=0}^{c} b_{c-i}\, \rho^{i} \qquad , \; c<n \tag{5.4-16}$$

When the system is controlled by a discrete-time controller with zero-order hold and a sampling interval of T_s, the control signal, $u(t)$, can be described as a piece-wise constant function (PCF), which is defined in Definition 5.4-3 as:

$$u(t) = u_h = u(t_h) \qquad , \text{ for } t_h \le t < t_{h+1} \tag{5.4-17}$$

where u_h is constant within the interval of $t_h \le t < t_{h+1}$ and $t_h$is a sampling time point given by:

$$t_h = hT_s \qquad , \; h \in \mathrm{W} = \{0,1,2,3,...\} \tag{5.4-18}$$

The input delay τ can be divided into the two parts as follows.

Definition 5.4-4 (Portions of Pure Delay)

a) The *integral delay*, τ_d, is defined as the portion of delay which is an exact multiple of the sampling interval, T_s. That is:

$$\tau_d = dT_s \quad , \tau \leq \tau_d < \tau + T_s \quad , d \in \mathrm{W}$$

b) The *fractional delay*, ν is defined as the difference between the pure delay, τ, and the integral delay τ_d. That is:

$$\nu = \tau_d - \tau \quad , 0 \leq \nu < T_s$$

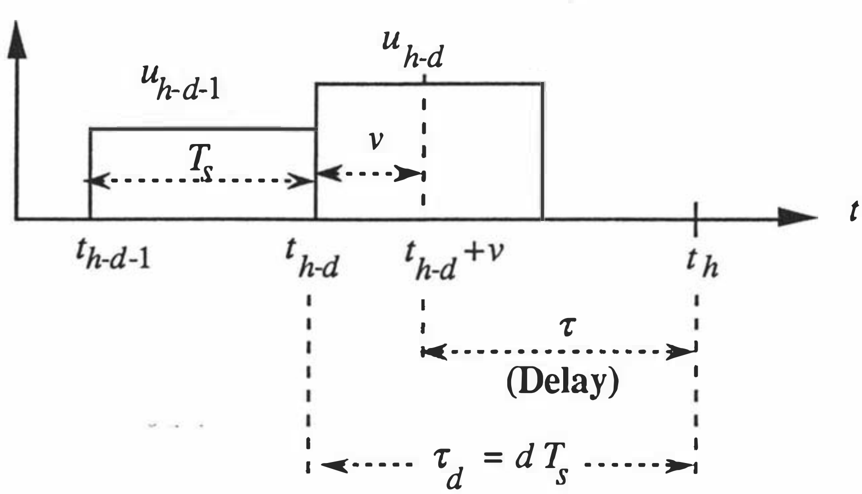
♦♦

Note that by this definition, τ_d is larger than or equal to τ, and ν is smaller that the sampling interval T_s. When the FII description is used with a discrete-time estimation algorithm, it is more convenient to evaluate the FIIs at the sampling time point, t_h. An important time point for later work is the sampling time point t_h delayed by the delay τ. That is:

$$t_h - \tau = t_h - \tau_d + \nu = t_h - dT_s + \nu = t_{h-d} + \nu \tag{5.4-19}$$

These quantities are illustrated in Figure 5.4.3-1.

Figure 5.4-2 Delays and time indices



A FII description for this delay system is now determined.

Lemma 5.4-1 (FII Description for Delay Systems)

For a system described by Definition 5.4-1 with piece-wise constant input having the form of Definition 5.4-3, the following holds:

$$\sum_{i=0}^{n} a_{n-i} \nabla^{n-i}_{(t_h;mT_s)} \sigma^{i}_{(t=t_h;mT_s)} y(t)$$

$$= \sum_{i=0}^{c} b_{c-i} \nabla^{c-i}_{(t_h;mT_s)} \delta^{d}_{(t=t_h;T_s)} P_{n-c+i}\, u(t) + \varepsilon(t)$$

where ε(t) is an error term given by,

$$\sigma^{i}_{(t=t_h;mT_s)} e(t)$$

and P_{n-c+i} is the FII operational polynomial for a (n-c+i)th FII of PCF as defined in Theorem 5.4-2.

♦♦

Proof

Using Theorem 5.2-1 (FII system equation), a FII description, evaluated at time t_h for a system given by Definition 5.4-1 is:

$$\sum_{i=0}^{n} a_{n-1} \nabla^{n-i}_{(t_h;mT_s)} \sigma^{i}_{(t=t_h;mT_s)} y(t)$$

$$= \sum_{i=0}^{c} b_{c-1} \nabla^{c-i}_{(t_h;mT_s)} \sigma^{n-c-i}_{(t=t_h;mT_s)} \delta_{(t;\tau)} u(t) + \varepsilon(t) \qquad (5.4\text{-}20)$$

Then using the commutativity of σ and δ determined in Theorem 3.3-1, the summation term in the right-hand side of Equation (5.4-20) becomes (note the changes in the subscripts of operators):

$$\sum_{i=0}^{c} b_{c-1} \nabla^{c-i}_{(t_h;mT_s)} \sigma^{n-c-i}_{(t=t_h-\tau\,;mT_s)} u(t)$$

$$= \sum_{i=0}^{c} b_{c-1} \nabla^{c-i}_{(t_h;mT_s)} \sigma^{n-c-i}_{(t=t_{h-d}+\nu\,;mT_s)} u(t) \qquad \text{, by Equation (5.4-19)}$$

$$= \sum_{i=0}^{c} b_{c-1} \nabla^{c-i}_{(t_h;mT_s)} \delta^{d}_{(t_h;T_s)} \sigma^{n-c-i}_{(t=t_h+v\ ;mT_s)} u(t) \tag{5.4-21}$$

and by Theorem 5.4-2 (FII of PCFs):

$$\sigma^{n-c-i}_{(t=t_h+v\ ;mT_s)} u(t) = P_{n-c+i}\, u(t) \tag{5.4-22}$$

Lemma 5.4-1 is then obtained by substituting Equation (5.4-22) into Equation (5.4-21).

♦♦

The FII system description in Lemma 5.4-1 can be rearranged to form an estimation equation which is suitable for discrete-time estimation of pure delay and other system parameters. This FII estimation equation is developed in the following.

Lemma 5.4-2 (Backward difference of Polynomial of Delay Operator)

If $P(\delta)$ is a polynomial of the delay operator, $\delta_{(t;T_s)}$, such that:

$$P(\delta) = \sum_{i=i_1}^{i_2} w_i\, \delta^{i}_{(t;T_s)}$$

where w_i are the coefficients of δ such that:

$$w_i = 0 \qquad \text{for} \qquad i < i_1$$

then the backward difference of $P(\delta)$ with respect to t is given by:

$$\nabla^{k}_{(t;mT_s)} \sum_{i=i_1}^{i_2} w_i\, \delta^{i}_{(t;T_s)} = \sum_{i=i_1}^{i_2+mk} \nabla^{k}_{(i;m)} w_i\, \delta^{i}_{(t;T_s)} \tag{5.4-23}$$

♦♦

Proof

Consider the first order ∇:

$$\nabla_{(t;mT_s)} \sum_{i=i_1}^{i_2} w_i\, \delta^{i}_{(t;T_s)} = \left(1 - \delta^{m}_{(t;T_s)}\right) \sum_{i=i_1}^{i_2} w_i\, \delta^{i}_{(t;T_s)}$$

$$= \sum_{i=i_1}^{i_2} w_i \left(\delta^i_{(t;T_s)} - \delta^{i+m}_{(t;T_s)}\right) \tag{5.4-24}$$

Equation (5.4-24) shows that the w_i are the coefficients of δ^i and $(-\delta^{i+m})$ and thus w_{i-m} are the coefficients of δ^{i-m} and $(-\delta^i)$. Therefore, the coefficients of δ^i is:

$$w_i - w_{i-m} = \nabla_{(i;m)} w_i$$

Also the highest order of δ in Equation (5.4-24) is (i_2+m). So:

$$\nabla_{(t;mT_s)} \sum_{i=i_1}^{i_2} w_i \, \delta^i_{(t;T_s)} = \sum_{i=i_1}^{i_2+m} \nabla_{(i;m)} w_i \, \delta^i_{(t;T_s)} \tag{5.4-25}$$

Now for a kth order ∇:

$$\nabla^k_{(t;mT_s)} \sum_{i=i_1}^{i_2} w_i \, \delta^i$$

$$= \nabla^{k-1}_{(t;mT_s)} \nabla_{(t;mT_s)} \sum_{i=i_1}^{i_2} w_i \, \delta^i$$

$$= \nabla^{k-1}_{(t;mT_s)} \sum_{i=i_1}^{i_2+m} \nabla_{(i;m)} w_i \, \delta^i \qquad \text{, by Equation } (5.4\text{-}25)$$

$$= \nabla^{k-2}_{(t;mT_s)} \nabla_{(t;mT_s)} \sum_{i=i_1}^{i_2+m} \nabla_{(i;m)} w_i \, \delta^i$$

$$= \nabla^{k-2}_{(t;mT_s)} \sum_{i=i_1}^{i_2+2m} \nabla^2_{(i;m)} w_i \, \delta^i \qquad \text{, by Equation } (5.4\text{-}25)$$

Continuing this process gives Lemma 5.4-2

♦♦

Note that the ∇ and δ operators in the left-hand side of Equation (5.4-23) operate on variable t. In contrast, the ∇ in the right hand side operates on the index i. Lemma 5.4-2 effectively shows that the coefficients of δ^i in the $\nabla^k P(\delta)$ are $\nabla^k_{(i;m)} w_i$.

Theorem 5.4-4 (FII Estimation Equation for Delay Systems)

For a system described by Definition 5.4-1 with piece-wise constant input having the form of Definition 5.4-3, the following holds:

$$\sum_{i=0}^{n} a_{n-i} \nabla^{n-i}_{(t_h;mT_s)} \sigma^{i}_{(t=t_h;T_s)} y(t) = \sum_{i=d}^{d+nm} \beta_i \ \delta^{i}_{(t=t_h;T_s)} u(t) + \varepsilon(t)$$

where, T_s is the sampling interval, $t_h = hT_s$, $h \in \mathbf{W}$, and (as defined in Definition 5.4-4) v and d are given by:

$$\tau = d\ T_s - v \qquad , \ \tau \text{ is the pure delay of the system and } d \in \mathbf{W}$$

where β_i is an appropriate function of v and the system parameters $b_0, b_1,..,b_c$ that is given by:

$$\beta_i = \beta_{d+j}(v, b_0, \ldots, b_c) = \sum_{l=0}^{c} b_{c-l} \nabla^{c-l}_{(j;m)} w_{j,\ l+n-c} \tag{5.4-26}$$

with,

$$w_{j,l} = \begin{cases} w_{j,l}(v) & , \text{for } 0 \le j \le lm \\ 0 & , \text{for other values of } j \end{cases}$$

and $w_{j,l}(v)$ is the jth coefficient of the lth FII of PCF as defined in Corollary 5.4-1.

♦♦

Proof

It has been determined in Lemma 5.4-1 that the right-hand side of the FII description of Delay system is (omitting the error term):

$$\sum_{l=0}^{c} b_{c-l} \nabla^{c-l}_{(t;mT_s)} \delta^{d}_{(t;T_s)} P_{l+n-c}\, u(t)$$

$$= \sum_{l=0}^{c} b_{c-l} \delta^{d}_{(t;T_s)} \nabla^{c-l}_{(t;mT_s)} P_{l+n-c}\, u(t) \tag{5.4-27}$$

Also by Corollary 5.4-1, P_{l+n-c} is a polynomial of δ operator:

$$P_{l+n-c} = \sum_{j=0}^{m(l+n-c)} w_{j,l+n-c}(v)\ \delta^{j}_{(t;T_s)}$$

So by Lemma 5.4-2:

$$\nabla^{c-l}_{(t;mT_s)} P_{l+n-c} = \sum_{j=0}^{mn} \nabla^{c-l}_{(j;m)}\ w_{j,l+n-c}(v)\ \delta^{j}_{(t;T_s)} \tag{5.4-28}$$

So substituting Equations (5.4-28) into (5.4-27) yields:

$$\sum_{l=0}^{c} b_{c-l}\ \delta^{d}_{(t;T_s)} \sum_{j=0}^{mn} \nabla^{c-l}_{(j;m)}\ w_{j,l+n-c}(v)\ \delta^{j}_{(t;T_s)}\ u(t)$$

$$= \sum_{l=0}^{c} b_{c-l} \sum_{j=0}^{mn} \nabla^{c-l}_{(j;m)}\ w_{j,l+n-c}(v)\ \delta^{d+j}_{(t;T_s)}\ u(t)$$

As the upper and lower limits of the second summation is independent of the index l of the first summation, the order of the two summations can be interchanged to give:

$$\sum_{j=0}^{mn} \sum_{l=0}^{c} b_{c-l}\ \nabla^{c-l}_{(j;m)}\ w_{j,l+n-c}(v)\ \delta^{d+j}_{(t;T_s)}\ u(t)$$

substituting index $(j+d)$ with index i gives the theorem.

♦♦

An alternative form of the FII estimation equation in Theorem 5.4-4 is given in the following corollary in order to examine the suitability of this FII estimation equation for the continuous-time model discrete-time estimation approach.

Corollary 5.4-2 (Linear-in-the-Parameter Form of FII Estimation Equation for Delay systems)

The FII estimation equation given by Theorem 5.4-4 can be written in a linear-in-the-parameter vector form:

$$\psi(t) = \lambda(t)^{\mathrm{T}}\,\theta + \varepsilon(t)$$

where

$$\psi(t) = \nabla^{n}_{(t_h;mT_s)}\, y(t)$$

$$\lambda(t) = \begin{bmatrix} -\nabla^{n-1}_{(t_h;mT_s)}\sigma^{1}_{(t=t_h;mT_s)}\, y(t) \\ -\nabla^{n-2}_{(t_h;mT_s)}\sigma^{2}_{(t=t_h;mT_s)}\, y(t) \\ \vdots \\ -\sigma^{n}_{(t=t_h;mT_s)}\, y(t) \\ u^{*}(t_h) \end{bmatrix}$$

$$\theta = [\, a_{n-1}\;\, a_{n-2}\, \ldots\, a_0\;\, b^{*}\,]^{\mathrm{T}}$$

and

$$u^{*}(t_h) = [\, u(t_{h-d})\, \ldots\, u(t_{h-d-nm})]^{\mathrm{T}}$$

$$b^{*} = [\;\; \beta_d \ldots \beta_{d+nm} \;\;]$$

♦♦

This shows that the FII estimation equation in Theorem 5.4-4 has the standard form of a equation-error type estimation equation for discrete-time estimation. The system parameters, $a_{n-1}\ldots a_0$ can be obtained directly from the estimate of θ. Also, it has been found in Theorem 5.4-4 that β_i is a function of the system delay τ and parameters $b_c\ldots b_0$. Consequently, the FII estimation equation fulfils the Criteria 5.3-1 for continuous-time-model discrete-time-estimation (CD) approach. The FII estimation equation can thus be used in a CD approach to estimate the continuous-time model parameters and delay.

When τ_d (the integral delay as given by Definition 5.4-4) and thus the index d is not known, the extended B polynomial technique (Biswas and Singh

1978, Kurz and Goedecke 1981) can be applied to identify the index d. Following this technique, the u^* and b^* vectors are expanded to form,

$$u^*(t_h) \quad = \left[\, u(t_{h-L}) \ldots u(t_{h-H+nm}) \,\right]^{\mathrm{T}} \quad , H>L \qquad (5.4\text{-}29)$$

$$b^* \quad = \left[\, \beta_L \quad \cdots \quad \beta_{H+nm} \,\right] \qquad (5.4\text{-}30)$$

where L is the lower limit of d and H is the higher limit. When the conditions of consistent estimation is fulfilled (Ljung 1987) the elements of β_i other that $\beta_d \ldots \beta_{d+nm}$ should approach zero and thus d can be identified.

The use of β_i to solve for τ and $b_c \ldots b_0$ is discussed in the next subsection. Also, more details on the overall estimation process are given later in Section 5.4.5.

The following example demonstrates the use of Theorem 5.4-4 to find the FII estimation equation for a delay system.

Example 5.4-2 (FII description of 2nd order delay system)

For a strictly proper second order delay system of Definition 5.4-1, we have the order of polynomial $A(\rho)$, n=2 and the order of the polynomial $B(\rho)$, c=1. Then from Theorem 5.4-4, there are $(nm+1)$, that is $(2m+1)$, non-zero β_i given by:

$$\beta_{d+j}(v, b_0, \ldots, b_c) = \sum_{k=0}^{c} b_{c-k} \, \nabla^{c-k}_{(j;m)} \, w_{j,\,k+n-c} \qquad , \; j=0\ldots 2m$$

$$= \sum_{k=0}^{1} b_{1-k} \, \nabla^{1-k}_{(j;m)} \, w_{j,\,k+1}$$

Thus when j=0:

$$\beta_{d+j} = \beta_d = b_1 \left[\, w_{0,1} - w_{-m,1} \,\right] + b_0 \left[\, w_{0,2} \right]$$

$$= b_1 \left[\, w_{0,1} \,\right] + b_0 \left[\, w_{0,2} \right] \qquad , \text{as } w_{j,k} = 0 \;\text{ for } j < 0,$$

This process can be repeated to find β_{d+1} to β_{d+m-1}.

For j=m, we have:

$$\beta_{d+m} = b_1 [w_{m,1} - w_{m-m,1}] + b_0 [w_{m,2}]$$

$$= b_1 [w_{m,1} - w_{0,1}] + b_0 [w_{m,2}]$$

Consider now the case of j=m+1:

$$\beta_{d+m+1} = b_1 [w_{m+1,1} - w_{1,1}] + b_0 [w_{m+1,2}]$$

$$= b_1 [-w_{1,1}] + b_0 [w_{m+1,2}] \qquad \text{, as } w_{j,k} = 0 \text{ for } j > km,$$

Similarly can find β_{d+m+2} to β_{d+2m}. Table 5.4-1 summarises these results.

When the FII interval, M=5 T_s, that is when $m = \frac{M}{T_s} = 5$, the β_i can be found using the $w_{i,k}$ obtained in Example 5.4-1. These β_i are tabulated in Table 5.4-2.

Table 5.4-1 Coefficients β_i in FII description of strictly proper 2nd order delay systems

β_d	$b_1\ w_{0,1} + b_0\ w_{0,2}$
β_{d+1}	$b_1\ w_{1,1} + b_0\ w_{1,2}$
:	:
β_{d+m}	$b_1 [w_{m,1} - w_{0,1}] + b_0\ w_{1,2}$
β_{d+m+1}	$-b_1\ w_{1,1} + b_0\ w_{m+1,2}$
:	:
β_{d+2m}	$-b_1\ w_{m,1} + b_0\ w_{2m,2}$

Table 5.4-2 Coefficients β_i in FII description of strictly proper 2nd order delay systems when M=5 T_s.

β_d	$b_1\, v + b_0 \frac{v^2}{2}$
β_{d+1}	$b_1\, T_s + b_0 [\, \frac{1}{2} T_s^2 + v\, T_s\,]$
β_{d+2}	$b_1\, T_s + b_0 [\, 1\frac{1}{2} T_s^2 + v\, T_s\,]$
β_{d+3}	$b_1\, T_s + b_0 [\, 2\frac{1}{2} T_s^2 + v\, T_s\,]$
β_{d+4}	$b_1\, T_s + b_0 [\, 3\frac{1}{2} T_s^2 + v\, T_s\,]$
β_{d+5}	$b_1 [\, T_s - 2v\,] + b_0 [\, 4\frac{1}{2} T_s^2 + v\, T_s - v^2\,]$
β_{d+6}	$-b_1\, T_s + b_0 [\, 4\frac{1}{2} T_s^2 - v\, T_s\,]$
β_{d+7}	$-b_1\, T_s + b_0 [\, 3\frac{1}{2} T_s^2 - v\, T_s\,]$
β_{d+8}	$-b_1\, T_s + b_0 [\, 2\frac{1}{2} T_s^2 - v\, T_s\,]$
β_{d+9}	$-b_1\, T_s + b_0 [\, 1\frac{1}{2} T_s^2 - v\, T_s\,]$
β_{d+10}	$b_1 [\, v - T_s\,] + b_0 [\, \frac{1}{2} (T_s - v)^2\,]$

5.4.4 Solving for Delays and Parameters

It has been determined that the FII estimation equation given by Theorem 5.4-4 can be used for discrete-time parameter estimation. The quantities obtained directly from the estimation routine are estimates of the FII estimation equation's coefficients which are given in the vector θ defined by Corollary 5.4-2, that is:

$$\theta \qquad = [\ a_{n-1}\ \ a_{n-2} \cdots a_0\ \ \beta_d \cdots \beta_{d+nm}\ ]^{\mathrm{T}}$$

The system parameters can be obtained directly from θ. However, the pure delay τ, and system parameters $b_c \ldots b_0$ need to be found from the β_i which are functions of the fractional delay v and parameters $b_c \ldots b_0$.

Before proceeding to find the formulae for calculating the pure delay and other system parameters, it is useful to establish the following results about the sum of the coefficients β_i.

Lemma 5.4-3 (Sum of Coefficients of Backward Difference)

Let $X(\delta)$ be a polynomial of the delay operator $\delta_{(t)}$ given by:

$$X(\delta) = (\ k_0 + k_1 \delta_{(t)}^1 + k_2 \delta_{(t)}^2 + k_3 \delta_{(t)}^3 + \ldots\ )$$

where $k_0, k_1, k_2, \ldots$ are coefficients of $\delta_{(t)}$ in the polynomial $X(\delta)$ such that they are invariant under the operation of $\delta_{(t)}$.

Then the sum of the coefficients of $\delta_{(t)}$ resulting from the operation:

$$\nabla_{(t)}\, X(\delta)$$

is zero. ♦♦

Proof

According to the definition in this lemma, $X(\delta)$ can be written in the following vector form:

$$X(\delta) \ = \ K^{\mathrm{T}} D(t) \tag{5.4-31}$$

$$K = [\ k_0\ \ k_1\ \ k_2\ \ \ldots\ ]^{\mathrm{T}} \tag{5.4-32}$$

$$D(t) = [\ 1\ \ \delta_{(t)}^1\ \ \delta_{(t)}^2 \cdots\ ]^{\mathrm{T}} \tag{5.4-33}$$

So:

$$\nabla_{(t)} X(\delta) = (1 - \delta_{(t)}) K^{T} D(t) \quad (5.4\text{-}34)$$

By the definition in this lemma, K^{T} is invariant under the operation of $\delta_{(t)}$. Therefore Equation (5.4-34) becomes:

$$\nabla_{(t)} X(\delta) = K^{T} D(t) - K^{T} \delta_{(t)} D(t) \quad (5.4\text{-}35)$$

Note that the coefficients of $\delta_{(t)}$ for both terms in the right-hand side of Equation (5.4-35) are given by K^{T}. Therefore the sum of the coefficients is:

$$(k_0+k_1+k_2+k_3 \ldots) - (k_0+k_1+k_2+k_3 \ldots) = 0$$ ♦♦

Theorem 5.4-5 (Sum of Coefficients)

If β_i are the coefficients of a FII description for a delay system, as defined in Theorem 5.4-4 , then:

$$\sum_{i=0}^{d+nm} \beta_i = m^n T_s^{\,n}\, b_0$$ ♦♦

Proof

It can be seen from Lemma 5.4-1 and Theorem 5.4-4 that the coefficients β_i result from expanding:

$$\sum_{i=0}^{c} b_{c-i}\, \nabla^{c-i}_{(t;mT_s)}\, \delta^{d}_{(t;T_s)}\, P_{n-c+i}(v,\delta) \quad (5.4\text{-}36)$$

Note that the elements with index i=0 to i=c-1 in the summation given by Equation (5.4-36) involve the operation:

$$\nabla_{(t;M)}\, P_k(v,\delta)$$

As defined in Corollary 5.4-1, $P_k(v,\delta)$ is a polynomial in $\delta_{(t)}$, that is:

$$P_k(v,\delta) = \sum_{i=0}^{mk} w_{i,k}(v)\, \delta^{d}_{(t;T_s)} \quad (5.4\text{-}37)$$

The coefficients of $\delta_{(t)}$ in this polynomial are given by $w_{i,k}(v)$, which are invariant under the operation of $\delta_{(t)}$. So, by Lemma 5.4-3, all the coefficients of $\delta_{(t)}$ in these elements with summation index i=0 to i=c-1 will sum up to zero.

Consequently, and the sum of coefficients of $\delta_{(t)}$ in Equation (5.4-36) is solely due to the element with index i=c, that is:

$$\delta_{(t)}^{d}\, b_0\, P_n(\nu,\delta)$$

As it has been determined in Theorem 5.4-3 that the coefficients of $\delta_{(t)}$ in $P_n(\nu,\delta)$ sum to $m^n\, T_s^{\ n}$, the coefficients of $\delta_{(t)}$ in Equation (5.4-36) thus sum to $m^n T_s^{\ n} b_0$.

♦♦

We now proceed to establish suitable formulae for finding the pure delay τ and parameters $b_0 \ldots b_c$ from β_i.

The pure delay and parameters $b_0 \ldots b_c$ should be found in the following three steps:

a) identifying the integral delay, τ_d
b) calculating b_0.
c) solving simultaneously for the fractional delay ν, and parameters $b_1 \ldots b_c$.

These steps are discussed as follows.

a) Identifying the integral delay, τ_d

As discussed earlier, when the integral delay is not known we should use the extended vector of β_i given by Equation (5.4-30), that is:

$$b^* \;=\; [\;\; \beta_L \;\; \cdots \;\; \beta_{H+nm} \;\;]$$

where L is the lower limit of index d and H is the higher limit. The index d is related to the integral delay τ_d by Definition 5.4-4, that is:

$$\tau_d = d\; T_s \qquad (5.4\text{-}38)$$

There are three possible means to identify the index d and thus the integral delay.

i) Identify, i_s, the smallest value of i for which a statistically significant β_i exists. As already determined in Theorem 5.4-4, the first non-zero β_i is β_d, so the index d is given by:

$$d = i_s \tag{5.4-39}$$

ii) Identify, i_l, the largest value of i for which a statistically significant β_i exists. The last non-zero β_i is β_{d+nm} and thus:

$$d = i_l - nm \tag{5.4-40}$$

where, as defined in Theorem 5.4-4, n is the order of the system model and the FII interval is $M{=}m\,T_s$.

iii) Calculate the sum of absolute values for $(nm+1)$ consecutive β_i, that is:

$$S_i = \sum_{j=i}^{i+nm} |\beta_j| \tag{5.4-41}$$

where $|\bullet|$ means the absolute value. Then identify the value of i giving the largest value of S_i.

It has been determined in Theorem 5.4-4 that there are $(nm+1)$ number of non-zero consecutive β_i (or nm consecutive β_i if v=0 or v= T_s). So the maximal value of S_i occurs when S_i=S_d as it contains all the non-zero β_i.

In practice the third method is preferred. This is because it is difficult to set an arbitrary level of significance for β_i, as the value of β_i depends on the unknown quantities, v and $b_1 \ldots b_c$.

b) Calculating b_0

It has been determined in Theorem 5.4-5 that the sum of the coefficients β_i is, $m^n T_s^{\,n} b_0$. So after the index d of integral delay is identified using the method described earlier, b_0 can be found by:

$$b_0 = \frac{1}{m^n T_s^{\,n}} \sum_{i=d}^{d+nm} \beta_i \qquad (5.4\text{-}42)$$

c) Solving simultaneously for the fractional delay v and parameters $b_1 ... b_c$

Theorem 5.4-5 shows that β_i are functions of v and $b_0 ... b_c$. As b_0 has been obtained using Equation (5.4-42), v and $b_1 ... b_c$ can be found by solving $(c+1)$ number of simultaneous equations that are relating β_i to v and $b_1 ... b_c$.

However there is not a general solution because the relationship of β_i to v and $b_1 ... b_c$ depends on the value of n and c which are respectively the order of the system polynomial $A(\rho)$ and $B(\rho)$ given by Equations (5.4-2) and (5.4-3). The highest power of v in β_i is n. Therefore, an nth order system generally involves the solution of an nth order algebraic equation.

The following demonstrates the derivation of appropriate formulae to find v and b_1 for second order systems.

There are three possible forms of second order systems that can be described by Definition 5.4-1. These are the:

1) systems with no system zero, that is when $b_1 = 0$ and $b_0 \neq 0$
2) systems with system zero at zero, that is when $b_1 \neq 0$ and $b_0 = 0$.
3) system with system zero at non-zero, that is when $b_1 \neq 0$ and $b_0 \neq 0$.

For each of these forms of second order systems, the next three theorems respectively present a suitable formula for finding the fractional delay v and the parameter b_1.

Theorem 5.4-6 (Formula for Fractional Delay of Second Order Systems with no System Zero)

For second order systems that can be described by:

$$(\rho^2 + a_1\rho + a_0)\, y(t) \;=\; b_0\delta_{(t;\; dT_s+v)}\, u(t)$$

the fractional delay, v, can be found from:

$$v \;=\; \frac{\beta_{d+1}}{b_0 T_s} \;-\; \tfrac{1}{2}T_s$$

where, as defined in Theorem 5.4-4, T_s is the sampling interval and β_{d+1} is the second significant coefficient of the FII estimation equation.

♦ ♦

Proof

From Table 5.4-2, the second significant coefficient of the FII estimation Equation for second order systems is:

$$\beta_{d+1} \;=\; b_1 T_s \;+\; b_0[\, \tfrac{1}{2}T_s^{\,2} + T_s v \,]$$

When b_1=0, this becomes:

$$\beta_{d+1} \;=\; \tfrac{1}{2}b_0 T_s^{\,2} + b_0 T_s v$$

Rearranging gives:

$$v \;=\; \frac{\beta_{d+1}}{b_0 T_s} \;-\; \tfrac{1}{2}T_s$$

♦ ♦

Theorem 5.4-7 (Formula for v and b_1 of Second Order Systems with System Zero at Zero)

For second order systems that can be described by:

$$(\rho^2 + a_1\rho + a_0)\, y(t) \;=\; (b_1\rho)\, \delta_{(t;\; dT_s+v)}\, u(t)$$

the parameter b_1 can be found by:

$$b_1 \;=\; \frac{\beta_{d+1}}{T_s}$$

and when b_1 is found, the fractional delay v can be obtained from:

$$v = \frac{\beta_d}{b_1}$$

where, as defined in Theorem 5.4-4, T_s is the sampling interval and, β_d and β_{d+1} are the first two significant coefficients of the FII estimation equation.

♦♦

Proof

From Table 5.4-2, when b_0=0 the first two significant coefficients of the FII estimation equation for second order systems are:

$$\beta_d = b_1 v$$

$$\beta_{d+1} = b_1 T_s$$

Rearranging these two equations respectively gives :

$$v = \frac{\beta_d}{b_1}$$

$$b_1 = \frac{\beta_{d+1}}{T_s}$$

♦♦

So far, formulae have been established for second order systems that consist of either the parameter b_0 or b_1. These formulae are obtained using only simple algebraic substitution and rearrangement. However it will be shown in the following that more complicated manipulation is needed when the second order system consists of both b_0 and b_1. This is because variables of higher power are involved. Also special conditions are needed to ensure the existence of a unique solution.

Lemma 5.4-4

For second order systems that can be described by:

$$(\rho^2 + a_1\rho + a_0) y(t) = (b_1\rho + b_0) \delta_{(t;\ dT_s+v)} u(t)$$

the fractional delay, v, can be obtained from:

$$v = \frac{-k_2 \pm \sqrt{k_2^2 - 4k_1k_3}}{2k_1} \quad , \quad 0 \le v < T_s \qquad (5.4\text{-}43)$$

where,

$$k_1 = \tfrac{1}{2} b_0 T_s \qquad (5.4\text{-}44)$$

$$k_2 = \tfrac{1}{2} b_0 T_s^2 - \beta_{d+1} \qquad (5.4\text{-}45)$$

$$k_3 = T_s \beta_d \qquad (5.4\text{-}46)$$

As defined in Theorem 5.4-4 β_d and β_{d+1} are the first two significant coefficients of the FII estimation equation for the system.

♦♦

Proof

For a strictly proper second order system, the first two coefficients of the FII estimation equation can be obtained from Table 5.4-2:

$$\beta_d = b_1 v + \tfrac{1}{2} b_0 v^2 \qquad (5.4\text{-}47)$$

$$\beta_{d+1} = b_1 T_s + b_0 [\, \tfrac{1}{2} T_s^2 + T_s v \,] \qquad (5.4\text{-}48)$$

Rearranging Equation (5.4-47) gives:

$$b_1 = \frac{\beta_d}{v} - \tfrac{1}{2} b_0 v \qquad (5.4\text{-}49)$$

Substituting (5.4-49) into (5.4-48) and then by rearranging gives:

$$\left(\tfrac{1}{2} b_0 T_s\right) v^2 + \left(\tfrac{1}{2} b_0 T_s^2 - \beta_{d+1}\right) v + T_s \beta_d = 0 \qquad (5.4\text{-}50)$$

Equation (5.4-50) is a quadratic equation of form:

$$k_1 v^2 + k_2 v + k_3 = 0$$

Therefore v is given by Lemma 5.4-4 which is the solution of the quadratic equation (5.4-50).

♦♦

Note that there are two solutions for Equation (5.4-43). The appropriate solution for v is the solution that lies within the range of $0 \le v \le T_s$. However, it is possible that both the two solutions of Equation (5.4-43) are within this acceptable range of v. Therefore, Lemma 5.4-4 itself is not sufficient and thus a condition needs to be established to identify the right solution for v. This is established in the following lemma and theorem.

Lemma 5.4-5

The solutions of Equation (5.4-43), v_1 and v_2, are:

$$v_1 = v$$

$$v_2 = v + 2\frac{b_1}{b_0}$$

where v is the actual fractional delay and, b_0 and b_1 are the system parameters as given in Lemma 5.4-4.

♦♦

Proof

Substituting Equations (5.4-47) and (5.4-48) back to (5.4-45) and (5.4-46) gives:

$$k_2 = -(b_1 T_s + b_0 T_s v) \quad (5.4\text{-}51)$$

$$k_3 = b_1 T_s v + \frac{1}{2} b_0 T_s v^2 \quad (5.4\text{-}52)$$

So using Equations (5.4-51) and (5.4-52) to solve Equation (5.4-43) gives:

$$v_1 = \frac{b_1 T_s + b_0 T_s v - b_1 T_s}{b_0 T_s} = v$$

$$v_2 = \frac{b_1 T_s + b_0 T_s v + b_1 T_s}{b_0 T_s} = v + 2\frac{b_1}{b_0}$$

♦♦

Theorem 5.4-8 (Formula for v and b_1 of Second Order Systems with System Zero at Non-zero)

For second order systems that can be described by:

$$(\rho^2 + a_1 \rho + a_0) y(t) = (b_1 \rho + b_0) \delta_{(t;\ dT_s + v)} u(t)$$

provided the sampling interval, T_s, satisfies the condition of:

$$T_s < 2\left|\frac{b_1}{b_0}\right|$$

then the fractional delay, v, can be obtained from:

$$v = \frac{-k_2 \ \pm \ \sqrt{k_2{}^2 - 4k_1k_3}}{2k_1} \qquad \text{and} \qquad 0 \le v < T_s$$

where,

$$k_1 \ = \tfrac{1}{2}\, b_0 T_s$$

$$k_2 \ = \tfrac{1}{2}\, b_0 T_s^{\,2} - \beta_{d+1}$$

$$k_3 \ = T_s\, \beta_d$$

As defined in Theorem 5.4-4 β_d and β_{d+1} are the first two significant coefficients of the FII estimation equation for the system.

Also when v is found, the parameter b_1 can be obtained from:

$$b_1 \ = \ \frac{\beta_{d+1}}{T_s} \ - \ b_0[\ \tfrac{1}{2}T_s + v\ ]$$

♦♦

Proof

From Lemma 5.4-5, the two possible solutions for v are:

$$v_1 \ = \ v \qquad \text{and} \qquad v_2 \ = \ v + 2\frac{b_1}{b_0}$$

This means the undesirable solution is v_2. In order to distinguish v_2 from v_1, we need to have v_2 lies outside the acceptable range of v defined in Definition 5.4-4, that is:

$$0 \le v < T_s$$

Therefore we need to have:

$$v_2 < 0 \qquad \text{or} \qquad v_2 \ge T_s$$

that is:

$$v + 2\frac{b_1}{b_0} < 0 \tag{5.4-53}$$

or
$$v + 2\frac{b_1}{b_0} \ge T_s \tag{5.4-54}$$

Now consider the two end values of v. When $v=T_s$, from Equation (5.4-53):

$$T_s + 2\frac{b_1}{b_0} < 0 \qquad \Leftrightarrow \qquad T_s < -2\frac{b_1}{b_0} \tag{5.4-55}$$

When v=0, from Equation (5.4-54):

$$T_s \le 2\frac{b_1}{b_0} \tag{5.4-56}$$

As we need to have T_s>0, the common solution for inequation (5.4-55) and (5.4-56) is:

$$T_s < 2\left|\frac{b_1}{b_0}\right|$$

This is thus the condition to have only the right solution of v lies within the acceptable range.

When v is found, b_1 can then be obtain from rearranging Equation (5.4-48).

♦♦

5.4.5 General Procedure to Estimate Delay and Model Parameters

In view of all the previous results, the sequential procedures to estimate the system delay and parameters should be:

1) Decide on the appropriate FII interval, M, and thus the index, m, such that, $M=mT_s$. Use Rules 5.3-1 and 5.3-2 as guideline.

2) Find the FII coefficients of piece-wise-constant functions, $w_{i,k}$. An nth order system generally requires $w_{i,k}$ up to $k=n$. Use Theorem 5.4-2 and see also Example 5.4-1.

3) Formulate the FII estimation equation of the delay system. See Theorem 5.4-4 and Example 5.4-2.

4) Determine the lower limit, L, and upper limit, H, of the index d for the integral delay.

5) Using L and H, expand the FII estimation equation from step (3) to formulate an appropriate estimation equation. Use Corollary 5.4-2, Equations (5.4-29) and (5.4-30), that is:

$$\psi(t) = \lambda(t)^{\mathrm{T}}\,\theta + \varepsilon(t)$$

where,

$$\psi(t) = \nabla^{n}_{(t_h;mT_s)}\, y(t)$$

$$\lambda(t) = \begin{bmatrix} -\nabla^{n-1}_{(t_h;mT_s)}\sigma^{1}_{(t=t_h;mT_s)}\, y(t) \\ \vdots \\ -\sigma^{n}_{(t=t_h;mT_s)}\, y(t) \\ u^*(t_h) \end{bmatrix}$$

$$\theta = [\, a_{n-1} \;\; a_{n-2} \cdots a_0 \;\; b^* \,]^{\mathrm{T}}$$

and,

$$u^*(t_h) = [\, u(t_{h-L}) \ldots u(t_{h-H+nm}) \,]^{\mathrm{T}} \quad , H>L$$

$$b^* = [\;\beta_L \;\; \cdots \;\; \beta_{H+nm}\;]$$

6) Derive the formulae for fractional delay v and parameters $b_c \ldots b_1$ using β_i by assuming the index d for integral delay and parameter b_0 are known. For second order systems, use one of Theorems 5.4-6 to 5.4-8.

7) Estimate the direct parameter θ for the FII estimation equation using an appropriate discrete-time parameter estimation algorithm. See Section 5.3.4 on selecting a parameter estimation algorithm.

8) Obtain $a_{n-1} \ldots a_0$ directly from θ.

9) Calculate S_i using Equation (5.4-41) that is:

$$S_i = \sum_{j=i}^{i+nm} |\beta_j|$$

10) Find index d by identifying the maximal S_i. See Section 5.4.4.

11) Calculate the integral delay τ_d using Definition 5.4-4, that is:

$$\tau_d = d\, T_s.$$

12) Calculate b_0 using Equation (5.4-42) that is:

$$b_0 = \frac{1}{m^n T_s^{\,n}} \sum_{i=d}^{d+nm} \beta_i$$

13) Find the fractional delay v, using the appropriate formula derived in step (6).

14) Calculate the system delay τ using Definition 5.4-4 that is:

$$\tau = \tau_d + v.$$

15) Find the rest of the system parameters $b_1 ... b_c$ using the appropriate formulae derived in step (6).

Two simulation examples using the FII description to estimate the delay and parameters in continuous-time models are presented next.

Simulation Example 5.4-1

Suppose a first order system is described by,

$$\frac{dy(t)}{dt} + a\ y(t) = b\ u(t-t) + e(t) \tag{5.4-57}$$

where,

a and b	are the system parameters.
t	is the input delay.
$y(t)$	is the system output.
$u(t)$	is the system input whose signal is piece-wise constant.
e(t)	is a white noise term.

Using the notation defined in Definition 3.2-4, this description can be rewritten as,

$$(\rho + a)\ y(t) = b\ u(t-\tau) + \varepsilon(t) \tag{5.4-58}$$

It is assumed here that the delay can be divided into the two portions defined by Definition 5.4-4, that is,

$$\tau = \tau_d - v = d\ T_s - v$$

where $\tau_d = d\,T_s$ is the integral delay and v is the fractional delay.

Now let,

- $a = 1$
- $b = 10$
- $\tau = 0.15$ sec
- τ is known to be in between 0 sec and 0.4 sec. Thus the lower limit and upper limit of the integral delay are respectively $L = 0$ and $H = 4$.

A noise-to-signal ratio (NR) is defined as,

$$\text{NR} = \frac{\text{standard deviation of } \varepsilon(t)}{\text{standard deviation of } y(t)}$$

and NR of the white noise is assumed to be 5%.

Note that the time constant of the first order system (Banks 1986) is,

$$\tau_n = \frac{1}{a} = 1 \text{ sec}$$

Therefore, using Shannon Sampling Theorem (Ogata 1987), an appropriate sampling interval is,

$$T_s = 0.1 \text{ sec}$$

The integral delay and the fractional delay are now respectively given by,

$$\tau_d = 2\, T_s = 0.2 \text{ sec}$$

$$v = 0.05 \text{ sec}$$

and the index d is given by,

$$d = \tau_d \,/\, T_s = 2$$

Using Rule 5.3-1, the FII interval selected is,

$$M = \tau_n = 1 \text{ sec} = 10\, T_s$$

and the index m is thus,

$$m = M \,/\, T_s$$

A FII description of this system can now be obtained using Theorem 5.4-4. This gives,

$$\nabla\, y(t) + a\,\sigma\, y(t) = \sum_{i=2}^{2+10} \beta_i\, \delta^i\, u(t) + \varepsilon(t) \tag{5.4-59}$$

where,

$$\beta_2 = \beta_d = b\,v = 10 \times 0.05 = 0.5 \tag{5.4-60}$$

$$\beta_i = b\,T_s = 10 \times 0.1 = 1, \quad \text{for } i = 3 \text{ to } 11 \tag{5.4-61}$$

$$\beta_{12} = b\,(T_s - v) = 10 \times 0.05 = 0.5 \tag{5.4-62}$$

Note that when the parameter b and index d of integral delay are found, the fractional delay can be obtained from Equation (5.4-60), that is by,

$$v = \beta_d \,/\, b \tag{5.4-63}$$

As the delay is known to be between 0 sec and 4 sec, above description is extended to give the following parameterised model equation for a discrete-time estimation routine,

$$\nabla\, y(t) + a\, \sigma\, y(t) = \sum_{i=0}^{4+10} \beta_i\, \delta^i\, u(t) + \varepsilon(t) \qquad (5.4\text{-}64)$$

The output of this system, when driven by a pseudo-random-binary system (Ljung 1987), is simulated using MATLAB. The parameters are then estimated using the recursive least square algorithm described by Equation (5.3-12) in Simulation Example 5.3-1. The forgetting factor used is 0.99, the initial guesses of all the parameters are zero and,

$$P(0) = 10^6$$

Table 5.4-3 gives the value of parameters a and β_i that are obtained from the recursive least-square routine.

The index d of integral delay can be identified using Equation (5.4-41), that is by calculating the sum of absolute values for $(m+1)$ consecutive β_i,

$$S_i = \sum_{j=i}^{i+10} |\beta_j|$$

then identify the value of i giving the largest value of S_i.

These S_i values are presented in Table 5.4-4, along with the identified index d of the integral delay.

The estimated values of b and v are also given in Table 5.4-4. The parameter b is calculated using Theorem 5.4-5, that is,

$$b = \sum_{i=d}^{d+10} |\beta_i| \qquad (5.4\text{-}65)$$

Finally, the fractional delay v is calculated using Equation (5.4-63).

Note that the values of S_i, d, τ_d, b and v can be obtained on-line during each iteration of the RLS routine.

Figure 5.4-3 shows the estimates of the system parameters a and b, and the delay τ.

The computer software routines for this simulation are given in Appendix A.

Table 5.4-3 Estimated Values of Parameters a and β_i

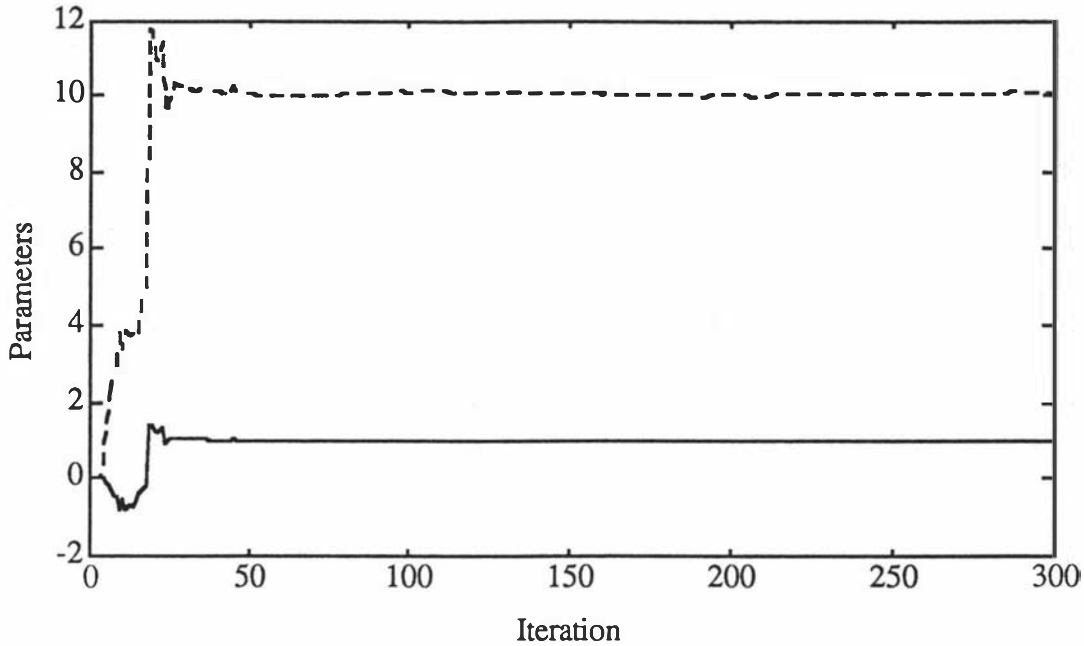
Iteration	a	β_0	β_1	β_2	β_3	β_4	β_5	β_6
0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
20	1.239	0.009	0.062	0.579	0.991	1.063	1.087	1.112
40	1.024	-0.002	0.014	0.508	1.013	1.028	1.045	1.063
60	0.997	-0.015	0.019	0.521	0.997	1.006	1.005	1.001
80	1.003	-0.002	0.008	0.528	1.020	0.995	1.006	1.012
100	1.009	0.003	-0.007	0.515	1.042	0.993	1.007	1.022
120	1.015	-0.002	0.001	0.505	1.025	0.993	0.999	1.021
140	1.018	0.002	0.002	0.504	1.022	0.993	0.999	1.022
160	1.004	0.002	0.011	0.518	1.015	0.992	1.014	1.010
180	1.001	-0.002	0.008	0.508	1.015	0.992	1.012	1.006
200	0.999	0.003	0.014	0.515	1.016	0.987	1.006	0.993
220	1.001	0.010	0.013	0.511	1.021	0.990	1.012	0.989
240	0.999	-0.001	0.011	0.503	1.017	0.992	1.009	0.987
260	1.010	0.003	0.011	0.503	1.012	0.999	1.007	0.990
280	1.008	0.001	0.006	0.503	1.014	1.000	1.007	0.995
300	1.010	0.004	0.005	0.503	1.008	1.004	0.998	0.996

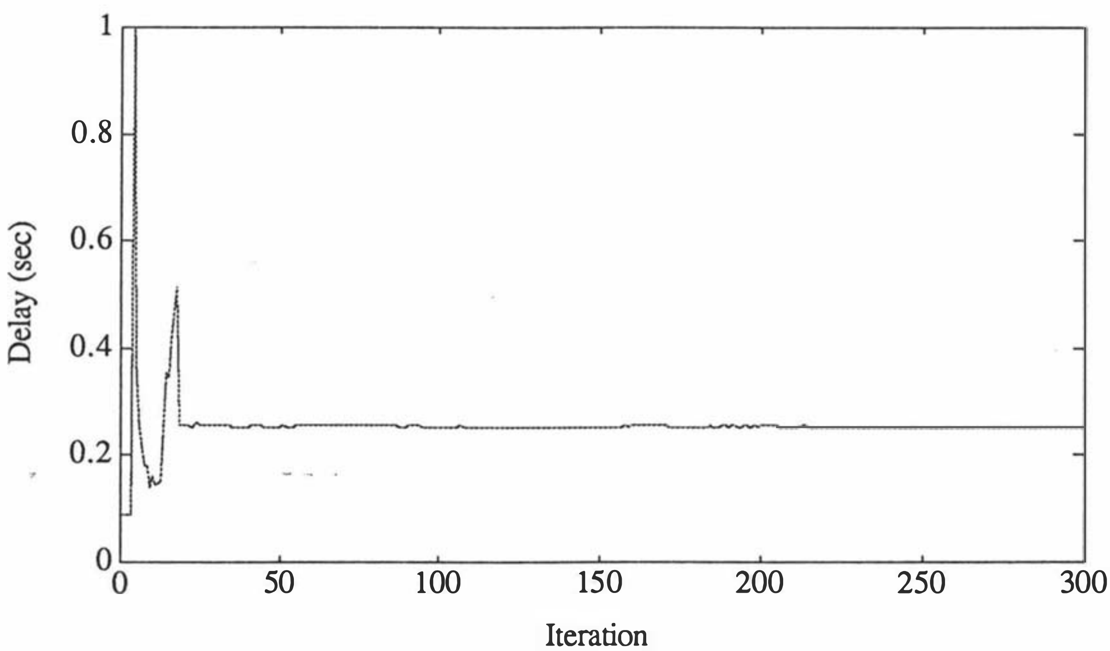
Iteration	β_7	β_8	β_9	β_{10}	β_{11}	β_{12}	β_{13}	β_{14}
0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
20	1.164	1.138	1.179	1.107	1.052	0.510	0.163	0.061
40	1.062	1.035	1.031	0.971	0.951	0.453	-0.028	-0.001
60	1.025	0.982	1.025	0.976	0.988	0.483	-0.018	-0.031
80	1.014	0.992	1.026	0.967	0.998	0.499	-0.023	-0.020
100	1.028	0.990	1.028	0.985	0.990	0.509	-0.021	-0.024
120	1.023	1.013	1.022	0.982	1.016	0.497	-0.025	-0.018
140	1.021	1.013	1.023	0.978	1.018	0.494	-0.009	-0.014
160	1.016	1.000	1.013	0.977	0.998	0.483	-0.017	-0.013
180	1.012	0.999	0.993	0.989	1.006	0.478	-0.007	-0.030
200	1.006	0.996	0.992	0.992	1.001	0.480	-0.001	-0.015
220	1.002	0.989	0.996	1.000	1.000	0.474	0.010	-0.010
240	1.015	0.989	1.002	0.999	1.004	0.482	-0.006	-0.006
260	1.020	0.999	1.001	1.004	1.007	0.497	0.003	-0.003
280	1.016	0.994	0.997	1.005	0.998	0.498	0.003	0.003
300	1.021	1.005	1.001	1.009	0.999	0.502	-0.007	0.007

Table 5.4-4 Calculated Values of S_i, d, b and v

Iteration	S_0	S_1	S_2	S_3	S_4	d	b	v
0	0.000	0.000	0.000	0.000	0.000	0	0.000	0.000
20	9.491	10.534	10.983	10.566	9.637	2	10.983	0.053
40	8.773	9.722	10.161	9.680	8.669	2	10.161	0.050
60	8.573	9.546	10.010	9.507	8.541	2	10.010	0.052
80	8.568	9.564	10.055	9.550	8.550	2	10.055	0.052
100	8.619	9.607	10.108	9.615	8.597	2	10.108	0.051
120	8.586	9.600	10.096	9.616	8.609	2	10.096	0.050
140	8.579	9.595	10.087	9.593	8.584	2	10.087	0.050
160	8.567	9.563	10.036	9.535	8.533	2	10.036	0.052
180	8.536	9.540	10.010	9.509	8.523	2	10.010	0.051
200	8.521	9.518	9.984	9.470	8.469	2	9.984	0.052
220	8.532	9.522	9.984	9.483	8.471	2	9.984	0.051
240	8.525	9.528	10.000	9.503	8.492	2	10.000	0.050
260	8.548	9.552	10.038	9.538	8.528	2	10.038	0.050
280	8.539	9.536	10.029	9.528	8.517	2	10.029	0.050
300	8.553	9.548	10.045	9.549	8.547	2	10.045	0.050

Figure 5.4-3 Estimates of Parameters and Delay





Simulation Example 5.4-2

The system given by Example 5.3-2 is now subjected to an input delay of, $\tau = 0.35$, that is,

$$(\ p^2 + a_1 p + a_0\ )\ y(t) = b\ u(t\text{-}0.35) + \varepsilon(t) \tag{5.4-66}$$

where,

a_1 $= 3$
a_0 $= 2$
b $= 5$
$\varepsilon(t)$ is a white noise signal with noise-to-signal ratio of 5%.

Let
- the sampling interval be, $T_s = 0.1$ sec,
- the FII interval be, $M = 5T_s = 0.5$ sec.
- the delay, τ, is known to be in between 0.1 sec and 0.5 sec. Thus the lower and upper limits of integral delay are respectively L=1 and H=5.

Using Theorem 5.4-4, the FII description of the system is,

$$\nabla^2\ y(t) + 3\nabla\sigma y(t) + 2\sigma^2 y(t) = \sum_{i=1}^{5+10} \beta_i\ \delta^i\ u(t) + \varepsilon(t) \tag{5.4-67}$$

As the system has no system zero, Theorem 5.4-6 can be used to find the fraction delay v.

The coefficients β_i are then estimated using the recursive least square algorithm described by Equation (5.3-12) in Simulation Example 5.3-1. The forgetting factor used is 0.99, the initial guesses of all the parameters are zero and,

$$P(0) = 10^6$$

Table 5.4-5 gives these estimates of β_i . The resulting estimates of system parameters and delay are given in Table 5.4-6. These system estimates are also shown in Figure 5.4-4.

The computer software routines for this simulation are given in Appendix A.

Table 5.4-5 Estimates of coefficients β_i

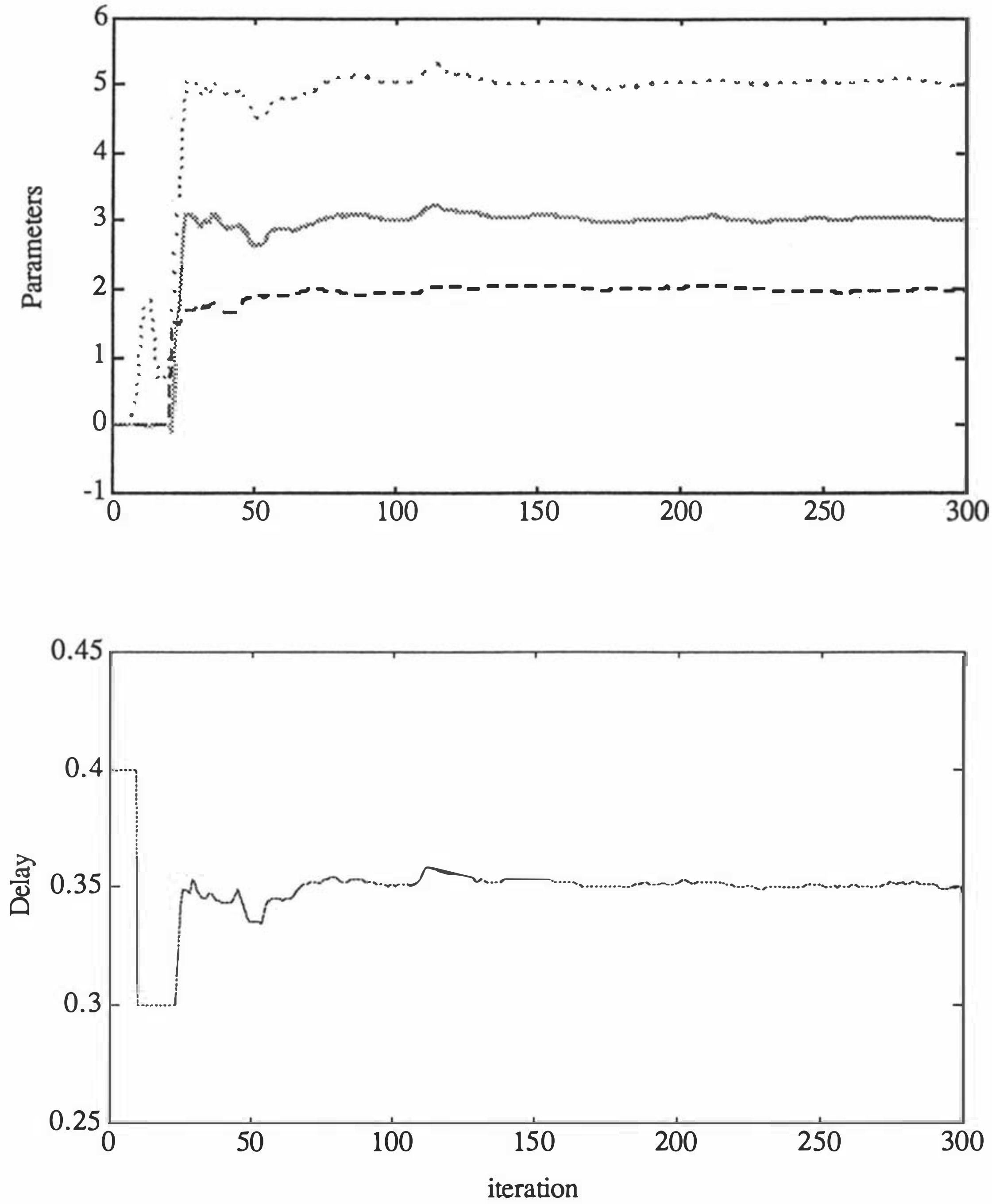
Iteration	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
20	-0.0017	0.0007	0.0051	0.0082	0.0387	0.0813	0.1001	0.0984
40	0.0024	0.0036	0.0035	0.0096	0.0531	0.1015	0.1499	0.1977
60	0.0016	0.0009	-0.0003	0.0056	0.0482	0.0980	0.1472	0.1951
80	0.0008	-0.0000	-0.0002	0.0055	0.0494	0.1007	0.1512	0.2016
100	0.0028	0.0027	0.0018	0.0074	0.0509	0.1011	0.1505	0.2001
120	0.0004	-0.0002	-0.0008	0.0055	0.0494	0.1003	0.1515	0.2032
140	-0.0001	-0.0012	-0.0022	0.0038	0.0477	0.0983	0.1492	0.2006
160	-0.0005	-0.0014	-0.0022	0.0039	0.0480	0.0988	0.1500	0.2015
180	-0.0019	-0.0022	-0.0026	0.0038	0.0479	0.0984	0.1492	0.1997
200	-0.0010	-0.0009	-0.0011	0.0049	0.0493	0.0994	0.1505	0.2015
220	0.0002	0.0001	0.0002	0.0066	0.0506	0.1009	0.1512	0.2014
240	0.0000	-0.0002	-0.0000	0.0066	0.0509	0.1011	0.1515	0.2016
260	0.0010	0.0008	0.0013	0.0074	0.0514	0.1016	0.1517	0.2016
280	0.0008	0.0007	0.0010	0.0077	0.0516	0.1020	0.1526	0.2034
300	0.0024	0.0012	0.0023	0.0082	0.0533	0.0994	0.1486	0.2013

Iteration	β_9	β_{10}	β_{11}	β_{12}	β_{13}	β_{14}	β_{15}
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
20	0.0988	0.0662	-0.0417	-0.0867	-0.0749	-0.0553	-0.2112
40	0.2331	0.1929	0.1423	0.0932	0.0441	0.0013	-0.0032
60	0.2305	0.1916	0.1422	0.0936	0.0457	0.0034	-0.0011
80	0.2403	0.2034	0.1545	0.1056	0.0554	0.0111	0.0029
100	0.2372	0.1997	0.1501	0.1009	0.0515	0.0077	0.0010
120	0.2424	0.2063	0.1568	0.1067	0.0561	0.0110	0.0026
140	0.2389	0.2021	0.1526	0.1027	0.0522	0.0076	0.0007
160	0.2399	0.2029	0.1532	0.1030	0.0523	0.0076	0.0004
180	0.2373	0.1996	0.1497	0.0997	0.0495	0.0055	-0.0011
200	0.2397	0.2024	0.1526	0.1018	0.0512	0.0063	-0.0001
220	0.2389	0.2011	0.1512	0.1007	0.0502	0.0060	-0.0004
240	0.2391	0.2012	0.1511	0.1008	0.0502	0.0059	-0.0004
260	0.2393	0.2017	0.1511	0.1007	0.0496	0.0049	-0.0013
280	0.2412	0.2034	0.1532	0.1026	0.0516	0.0066	-0.0004
300	0.2332	0.1992	0.1448	0.0949	0.0442	0.0040	-0.0011

Table 5.4-6 Estimates of Parameters and Delay

Iteration	a_1	a_0	b	τ
0	0.000	0.000	0.000	0.000
20	-0.003	0.124	0.933	0.300
40	2.901	1.648	4.875	0.343
60	2.876	1.907	4.805	0.345
80	3.078	1.954	5.115	0.354
100	3.026	1.925	5.028	0.350
120	3.147	2.025	5.157	0.355
140	3.054	2.033	5.022	0.352
160	3.063	2.024	5.044	0.352
180	2.983	1.987	4.962	0.350
200	3.016	2.011	5.039	0.352
220	3.009	2.014	5.035	0.350
240	3.008	1.961	5.040	0.350
260	3.028	1.950	5.044	0.350
280	3.059	1.978	5.103	0.351
300	2.935	1.957	4.924	0.343

Figure 5.4-4 Delay and Parameter Estimates



5.5 OTHER POSSIBLE APPLICATIONS OF THE FII

It has been explained in Section 5.2 that FII descriptions of a system have three convenient characteristics. Of particular importance is the absence of any derivative terms. Also, Section 5.3 shows that the FII descriptions have three further properties which are useful in applying them:

- their parameters are exactly the same as those in the original continuous-time differential equation model.
- all their required operations can be realized relatively easily.
- the FII description can be sampled in discrete-time.

In view of these, there seems to be considerable potential for the application of FII in both system science and engineering. The following outlines two other possible applications of FII, to motivate future research.

5.5.1 Solution of Differential Equation

The FII transformation can be applied to find a solution for a differential equation. This solution has the advantage over the classical integral method because it does not depend on any prior knowledge of constant initial conditions and derivative terms.

This application of FII can be demonstrated with the following example:

> "An object is driven by a constant force, F. Find the distance travelled, x, at time t, assuming there is no other force acting. "

By Newton's second law of motion, the system can be described by:

$$\frac{d^2x(t)}{dt^2} = F \tag{5.5-1}$$

Using classical integration methods, the solution can be found by integrating twice from time zero to t, and the result is:

$$x(t) = F\frac{t^2}{2} + t\frac{dx}{dt}(0) + x(0) \tag{5.5-2}$$

This classical solution requires the initial conditions $x(0)$ and $\frac{dx}{dt}(0)$. The initial velocity $\frac{dx}{dt}(0)$ might not be measurable or be measured accurately in practice. However the FII solution does not has this problem.

Applying the second order FII transformation on both sides of Equation (5.5-1) yields,

$$\nabla^2 x(t) = FT_s^2$$

that is:

$$x(t) - 2x(t-T_s) + x(t-2T_s) = FT_s^2$$

or

$$x(t) = FT_s^2 + 2x(t-T_s) - x(t-2T_s) \tag{5.5-3}$$

where T_s is some constant time interval. This solution has no derivative term and thus does not have the problem presented by Equation (5.5-2). Comparison of Equations (5.5-2) and (5.5-3) shows that the FII solution effectively replaces the constant initial conditions in the integral solution with some moving initial conditions.

5.5.2 Discrete-time Control for Continuous-time System

It has been shown that when a linear continuous-time system is subjected to a discrete-time input, a possible description of the behaviour of the system is the FII description given by:

$$\nabla^n y(t) + a_{n-1}\nabla^{n-1}\sigma^1 y(t) + \ldots + a_0\sigma^n y(t)$$
$$= \beta_0 u(t-T_S) + \ldots + \beta_i u(t-iT_S)$$

This description can be considered as,

$$x_n(t) + a_{n-1}x_{n-1}(t) + \ldots + a_0x_0(t) = \beta_0u(t-T_s) + \ldots + \beta_iu(t-iT_s)$$

where $x_i(t)$ is a realizable (or observable) state of the system corresponding to the $\nabla^i\sigma^{n-i}y(t)$ quantity.

This description is an exact description of the continuous-time system and it is true for all time t. The classical z domain description (Ogata 1987) is only true at the sampling instants and it is usually an approximation due to the irrational exponential terms involved.

In view of these, the FII description offers a possible description for the design of feedback control using some continuous-time states. However further research is required to determine the feasibility of this approach.

5.6 SUMMARY

In this chapter, a technique to estimate both the parameters and delay term of a strictly proper continuous-time model is established. This technique makes use of the FII transformation of the original differential-equation model. When the system is subjected to a discrete-time control input, the technique is able to estimate both the input delay and system parameters simultaneously.

The major advantages of this technique include:

-- it makes use of existing discrete-time estimation algorithms.

-- it is able to estimate an input delay which is not an exact multiple of the sampling interval.

A summary of this technique is given in Section 5.4.5.

However a disadvantage of the technique is that general formulae for all system orders can not be established. Formulae for second order systems have been found in this thesis work. Several simulations have shown that the technique is applicable for some second order systems. Nevertheless the usefulness of the technique in other situations is yet to be fully examined.

It is likely that there are other potential applications in system science and engineering because of several convenient and useful features of the FII description. Two other possible applications of FII are also given to motivate future research.

CHAPTER 6

CONCLUSION

This chapter presents a concluding summary of the thesis and suggests several areas for future work.

6.1 CONCLUDING SUMMARY

There is an increasing need for on-line techniques for estimating the parameters of continuous-time models, mostly due to the increasing popularity of the continuous-time-model approach to digital controller design. Using continuous-time models instead of discrete-time models to design digital controllers has several important advantages but requires digital devices of higher capability. However this hardware requirement has become insignificant due to the recent availability of fast and powerful computers in low cost. Some of the advantages of the continuous-time-model approach are that the artefacts of sampling in discrete-time models can be avoided, and also the controller design process makes more sense in relation to the actual physical systems.

In Chapter 2, surveys of existing parameter estimation techniques show that the equation-error methods are the most popular techniques in recent years. This could be due to their simplicity and the ease with which they may be developed and implemented. The ease and simplicity of using the equation-error approach is apparent in this thesis. Using the equation-error approach, the development of parameterised model equations can be segregated from the development of estimation algorithms. Furthermore, the estimation equation in an equation-error form is only a linear summation and thus can be easily manipulated.

This thesis work has successfully developed an recursive technique which is capable of estimating the delay and other parameters of continuous-time models. It is based on the equation-error approach and a special integral which has an integration interval of constant length. This integral is named the Fixed Interval Integral.

A major contributor to the success of this work is the calculus operator notation system developed in Chapter 3. This operator notation system consists of three major components; a notation represents the operation, a superscript indicates the repetition of the operation and some subscripts identify the operational variables and parameters. The use of this operator notation system allows high order calculus operations to be manipulated easily. It also enables the results to be presented clearly in a simple form.

Using the operator notation, several important properties of the Fixed

Interval Integral (FII) were found in Chapter 3. The FII is a backward difference of the traditional definite integral. The FII operator is a perfect anti-derivative operator; it eliminates all derivatives in time varying forms and derivative terms in the initial conditions, and it is applicable on derivatives of any order. An operator algebra was established to present the relationship between the FII and other calculus operations. The algebra allows the calculus operations of any order to be manipulated in a similar and simple manner.

Three methods of implementing the FII operation are determined in Chapter 4, namely the frequency response method, the hybrid method and numerical method. The frequency and impulse response of FII given in Chapter 3 are useful for designing analog and digital filters using the frequency response method. However, it is found that the frequency response method is applicable only for a limited range of low frequencies. This is because of the peculiar frequency and impulse response of FII, which includes zero transmittance and phase jumps at the frequencies of multiples of $\frac{1}{M}$Hz (M is the length of FII integration interval). This peculiarity is due to the FII operation is a combination of an inherently discrete-time backward shift operation and an inherently continuous-time integration. This understanding of the FII characteristics resulted the hybrid method that uses analog integrators and digital backward differencing devices. The hybrid method is simplest and easiest to use as integrators and delay elements are usually available as standard units in modern devices for system control.

However, the numerical methods provide greater flexibility in realising the FII operation. Three numerical methods are presented in Chapter 4. The Numerical Method I is recommended. Because of the symmetry in coefficients, the design and analysis procedure of the Numerical Method I can be generalised. A table of percentage errors has been established as a guide for selecting the sampling interval, order of numerical formula and frequency range.

In Chapter 5, by applying the FII operator to continuous-time models described in differential equations, two forms of FII system descriptions were established respectively for systems without delay elements and strictly proper systems with input delay. These FII system descriptions are linear in terms of the parameters and are independent on any derivatives, constant initial conditions and infinitely accumulating terms. Furthermore, they can be sampled at any time instant to form discrete-time descriptions without any further transformation. Consequently, the FII system descriptions can be used as the parameter estimation equations for discrete-time estimation. A 'continuous-time model discrete-time estimation' technique is thus formed when the FII equations are coupled with any of the well-established discrete-time estimation algorithms.

The FII system description for delay systems is applicable only for systems controlled by digital controllers with zero-order hold. However, the use of this FII system description allows the input delay to be estimated simultaneously with other system parameters. Both fractions of the delay that are and are not exact multiples of sampling interval can be estimated simultaneously. Detailed formulae and procedures of the FII parameter and delay estimation technique were established for second order strictly proper systems. Simulation results show that the FII estimation technique is feasible in practice.

In view of these, the reported work has achieved its objective of developing a pre-filter for recursive parameter estimation of continuous-time models. However, there are several areas where further work is required before the FII technique can be fully applied in practice. These areas are suggested in the following.

6.2 FURTHER WORK

Only for second order systems, this thesis has presented in details the formulae that relate the system parameters and delay to the estimated coefficients. Although these formulae can be obtained for other system orders using the suggested procedure, general formulae for any system order have not be established. The development of the general formulae is thus a priority in further work. If the general formulae do not exist, formulae for some higher order systems need to be established in order to increase the usefulness of the FII estimation technique in practice.

Study of convergence and asymptotic behaviour needs to be added in the future. The number of iterations required to achieve a certain accuracy should be established, in order to decide the appropriate learning interval to be allowed for.

The error in implementing the FII operation might have significant effect on the accuracy of the parameter estimation. The relationship between the realization error and the estimation error needs to be established. This will assist in selecting the appropriate realization of FII. The realization error might be able to be compensated for in the parameter estimation.

To fully explore the usefulness of the FII pre-filtering technique, more simulations involving systems of different structure and order are required. The comparison of estimation algorithms needs to be extended to include other modern algorithms. The interaction between the length of FII integration interval and the estimation algorithm should also be analysed.

As the FII description is free of any derivative terms and its discrete-time form is an exact description of the continuous-time system, there seems considerable potential for the application of the FII in system science and engineering. As suggested in Chapter 5, the FII might be useful in solving differential equations and in designing discrete-time control for continuous-time systems. Further study should thus be carried out to explore the application of the FII besides in parameter estimation.

REFERENCES

Agarwal M. and Canudas C. (1987). On-line estimation of time delay and continuous-time process parameters. *International Journal of Control*, vol:46, p295-311.

Anderson B.D.O. (1977). An approach to multivariable system identification. *Automatica*, vol:13, p401-408.

Anderson B.D.O. (1985). Adaptive forgetting in recursive identification through multiple models. *International Journal of Control*, vol:42, p1175-1193.

Aström K.J. and Eykhoff P.E. (1971). System identification - a survey. *Automatica*, vol:7, p123-162.

Aström K.J. and Kallstrom C.G. (1973). Application of system identification techniques to the determination of ship dynamics. In Eykhoff P. (Ed) *Identification on System Parameter Estimation*, p415-424, North-Holland, Amsterdam.

Aström K.J. and Wittenmark B (1984). *Computer Controlled Systems*, Prentice-Hall.

Aström K.J. and Wittenmark B. (1990). *Computer Controlled Systems: Theory and Design*, Prentice-Hall Inc., USA.

Aström K.J., Hagander P. and Sternby J. (1984). Zeros of sampled systems. *Automatica*, vol:20, p31-38.

Bajpal A.C, Mustoe L.R. and Walker D. (1974). *Engineering Mathematics*, John Wiley and Son Ltd, England.

Balakrishnan A.V. (1973). *Stochastic Differential Systems I*, Springer, New York.

Banks S.P (1986). *Control Systems Engineering*, Prentice-Hall International Series in Systems and Control Engineering, Series Editor: Grimble,M.J., Prentice-Hall International (UK) Ltd..

Bellman R. and Kalaba R.E. (1964). Dynamic Programming, invariant imbedding and quasilinearization: Comparisons and interconnections. In Balakrishnan A. and Neustadt L. (Eds) *Computing Methods in Optimization Problems*, Academic Press, New York, p135-145.

Bellman R. and Kalaba R.E. (1965). *Quasilinearization and Nonlinear Boundary Value Problems*, American Elsevier, New York.

Bellman R., Kalaba R.E. and Wing G.M. (1960). Invariant imbedding and mathematical physics. *Journal of Mathematics and Physics*, vol:1, p280.

Biswas K.K. and Singh G. (1978). Identification of stochastic time-delay systems. *IEEE Transactions on Automatic Control*, vol:AC-23, p504-505.

Blackman P.F. (1977). *Introduction to State-variable Analysis*, Macmillan Press Ltd., UK.

Bose N.K. (1985). *Digital Filters - Theory and Applications*, North-Holland (Elsevier Science Publishing Co.).

Box G.E.P. (1960). Fitting empirical data. *Annals of New York Academy of Science*, vol:8, p792.

Box G.E.P. and Coutie G.A. (1956). Application of digital computers in the exploration of functional relationships. *Proceedings of IEEE*, vol:103B, Supp. 1-3, p100-107.

Box G.E.P. and Jenkins G.M. (1970). *Time Series Analysis, Forecasting and Control*, Holden Day, San Fransisco.

Burns J.A. and Cliff E.M. (1980). An abstract quasi-linearization algorithm for estimating parameters in hereditary systems. *IEEE Transactions on Automatic Control*, vol:AC-25, p126-129.

Cappellini V., Constantinides A.G. and Emiliani P. (1978). *Digital Filters and their Applications*, Academic Press, London..

Carroll R.L. and Lindorf D.P. (1973). An adaptive observer for single-input single-output linear systems. *IEEE Transactions on Automatic Control*, vol:AC-18, p428-435.

Chang R.Y., Yang S.Y. and Wang M.L. (1986). A new approach for parameter identification of time-varying systems via generalized orthogonal polynomial.. *International Journal of Control*, vol:44, p1747-1755.

Chao Y.C., Chen C.L. and Huang H.P. (1987). Recursive parameter estimation of transfer function matrix models via Simpson's integrating rules. *International Journal of System Science*, vol:18, p901-911.

Chen Y.H. and Tomizuka M. (1988). Design of adaptive observer for plant under input disturbance and measurement noise. *International Journal of Control*, vol:47, p625-632.

Coughanowr D.R. and Koppel L.B. (1983). *Process Systems Analysis and Control*, International Student Edition, McGraw Hill , p67, 311.

Davis P.J. and Rabinowitz P. (1984). *Methods of Numerical Integration*, 2nd Edition., Academic Press, New York.

Delchamps D.F. (1988). *State Space and Input-Output Linear Systems*, Springer-Verlag, New York.

Donaldson D.D and Leondes C.T. (1963). A model referenced parameter tracking technique for adaptive control systems - part I and II . *IEEE Transactions on Industry Applications*, vol:82, p241.

El-Shafey M.H. and Bohn E.V. (1987). Use of modal functions for continuous-time system identification and state observer design. *International Journal of Control*, vol:45, p1723-1736.

Eulrich B.J., Andrisani D. and Lainiotis D.G. (1980). Partitioning identification algorithms. *IEEE Transactions on Automatic Control*, vol:AC-25, p521-528.

Eykhoff P. (1974). *System Identification*, Wiley, New York.

Ferrar W.L. (1958). *Integral Calculus*, Oxford University Press, London.

Ferretti G., Maffezzoni C. and Scattolini R. (1991). Recursive estimation of time delay in sampled systems. *Automatica*, vol:27, p653-661.

Gabay E. and Merhav S.J. (1976). Identification of linear systems with time-delay operating in a closed loop in the presence of noise. *IEEE Transactions on Automatic Control*, vol:AC-21, p711-716.

Gawthrop P.J. (1982). A continuous-time approach to discrete-time self-tuning control,. *Optimal Control Application and Methodology*, vol:3, p399-414.

Gawthrop P.J. (1987). *Continuous-time Self-tuning Control. Volume I: Design.*, John Wiley and Sons Inc..

Gawthrop P.J. (1989). *Continuous-time Self-tuning Control. Volume II: Implementation*, John Wiley and Sons Inc..

Gawthrop P.J. and Nihtila M.T. (1985). Identification of time delays using a polynomial identification method. *Systems and Control Letters*, vol:5, p267-271.

Godfrey K.R. (1980). Correlation methods. *Automatica*, vol:16, p527-534.

Goodwin G.C. and Payne R.L. (1977). *Dynamic System Identification: Experiment Design and Data Analysis.*, Academic Press, New York.

Halmos P.R. and Sunder V.S. (1978). *Bounded Integral Operators on L2 Spaces*, Springer-Verlag, Berlin.

Hamming R.W. (1977). *Digital Filters*, Prentice-Hall Inc., USA.

Harris C.J. and Billings S.A. (1981). *Self-tuning and Adaptive Control: Theory and Applications*, Peter Peregrinus.

Hasler M. and Neirynck J. (1986). *Electric Filters*, Artech House Inc, USA.

Hostetter G.H. (1988). *Digital Control System Design*, 1st Edition, Holt,Rinehart and Winston Inc.

Hsia T.C. (77). *Identification: Least Squares Methods*, Lexington Books.

Huang H.P., Chen C.L. and Chao Y.C. (1987). Augmented hybrid method for continuous process identification from sampled data with coloured noise. *International Journal of Control*, vol:46, p1373-1390.

Hwang C. and Guo T.Y. (1984). Transfer-function matrix identificaiton in MIMO systems via shifted Legendre polynomials. *International Journal of Control*, vol:39, p807-814.

Hwang C. and Guo T.Y. (1984). Identification of lumped linear time-varying systems via block functions. *International Journal of Control*, vol:40, p571-583.

Ichikawa K. (1982). Continuous-time adaptive identification and control algorithm via newly developed adaptive laws. *International Journal of Control,* vol:36, p819-831.

Jazwinski A.H. (1970). *Stochastic Processes and Filtering Theory*, Academic Press, New York.

Jiang Z.H. and Schaufelberger W. (1985). Recursive formula for multiple integral using blockpulse functions. *International Journal of Control*, vol:41, p271-279.

Jury E.I. (1956). Hidden oscillations in sampled-data control systems. *Transactions on AIEE*, vol:75, p391.

Kallstrom C.G., Essebo T. and Aström K.J. (1976). A computer program fro maximum likelihood identification of linear multivariable stochastic systems. *Proceedings of 4th IFAC Symposium on Identification and System Parameter Estimation, Tbilisi.*

Kalman R.E. (1960). A new approach to linear filtering and prediction problems.. *ASME Transactions: Journal of. Basic Engineering*, vol:82D, p35.

Kalman R.E. and Bucy R.S. (1961). New results in linear filtering and prediction theory. *ASME Transactions: Journal of. Basic Engineering*, vol:83D, p95.

Kaya Y. and Yamamura S. (1962). A self-adaptive system with a variable-parameter PID controller. *Transactions of AIEE*, p378-379.

Kendall M.G. and Stuart A. (1961). *The Advanced Theory of Statistics*, vol:2, Griffin, London.

Kohr R.H. (1963). A method for the determination of a differential equation model for simple nonlinear systems. *IEEE Transactions on Electron. Computers* , vol:EC-12, p394.

Kopp R.E. and Orford R.J. (1963). Linear regression applied to system identification for adaptive control systems. *AIAA Journal.*, vol:1, p2300.

Kraus F. and Schaufelberger W. (1990). Identificaiton with block pulse functions, modulating functions and differential operators. *International Journal of Control*, vol:51, p931-942.

Kreisselmeier G. (1977). Adaptive Observers with exponential rate of convergence. *IEEE Transactions on Automatic Control*, vol:22, p2-8.

Kreyszig E. (1988). *Advanced Engineering Mathematics, 6th Edition*, John Wiley and Sons Ltd, USA.

Kuc R (1988). *Introduction to Digital Signal Processing*, McGraw-Hill Co., USA.

Kudva P. and Narendra K.S. (1973). Synthesis of an adaptive observer using Lyapunov's direct method. *International Journal of Control*, vol:18, p1201-1210.

Kuo B.C (1980). *Digital Control Systems*, New York, Holt, Rinehart and Winston, Inc.

Kurz H. and Goedecke W. (1981). Digital parameter-adaptive control of processes with unknown dead time. *Automatica*, vol:17, p245-252.

Kwakernaak H. and Sivan R. (1972). *Linear Optimal Control Systems,* 1st Edition, John Wiley and Sons Inc.

Landau I.D. (1972). Model reference adaptive systems-A survey. *ASME Transactions: Journal of Dynamic. Systems*, vol:94, p119.

Landau I.D. (1976). Unbiased recursive identification using model reference adaptive techniques. *IEEE Transactions on Automatic Control*, vol:AC-21, p194.

Landau I.D. (1979). *Adaptive Control: the Model Reference Approach*, Marcel Dekker, New York.

Lawrence P.J. and Rogers G.J. (1979). Sequential transfer function synthesis from measure data. *IEE Proceedings D*, vol:126, p104-106.

Lee S.J. (editor) (1987). *Operator Methods for Optimal Control Problems*, Lecture Notes in Pure and Applied Mathematics, vol:108, Marcel Dekker Inc., New York..

Lewis F.L. (1986). *Optimal Control*, 1st Ed.,John Wiley and Sons Inc.

Ljung L. (1977). On positive real functions and convergence of some recursive schemes. *IEEE Transactions on Automatic Control*, vol:22, p539-551.

Ljung L. (1979). Convergence of recursive estimator. *In Isermann R. (Ed) Identification and System Parameter Estimation*, Pergamon, Oxford, p131-144.

Ljung L. (1986). *System Identification Toolbox for use with MATLABTM : User's Guide*, The Mathworks Inc. .

Ljung, L. (1987). *System Identification: Theory for the user* , 1st Edition, Prentice Hall.

Ljung L. and Gunnarsson S. (1990). Adaptive and tracking in system identification - A survey. *Automatica*, vol:26, p7-21.

Luders G. and Narendra K.S. (1974). A new canonical form for an adaptive observer. *IEEE Transactions on Automatic Control*, vol:19, p117-119.

Luenberger D.G. (1964). Observing the state of a linear system. *IEEE Transactions on Military Electronics*, vol:MIL-18, p74-80.

Malek-zavarei M. and Jamshidi M. (1987). *Time-delay Systems: Analysis, Optimization and Applications*, North-Holland Systems & Control Series, vol.9, North-Holland, Amsterdam (Elsevier Science Co.).

Malvino A.P. and Leach D.P. (1981). *Digital Principles and Applications*, Tata Mcgraw Hill Company, New Delhi.

Marchenko V. (1988). *Nonlinear Equations and Operator Algebra*, D. Reidel Publishing Company, Holland.

Margolis M. and Leondes C.T. (1959). A parameter tracking servo for adaptive control systems, *IRE Transaction on Automatic Control*, vol:AC-4, p100.

Marshall J.E. (1979). *Control of Time-delay Systems*, IEE Control Engineering Series, vol:10, Peregrinus, New York.

Marshall S.A. (1978). *Introduction to Control Theory*, Macmillan Press Ltd., UK.

Martin S. (1981). *Operator Methods in Quantum Mechanics*, North Holland, New York.

Mathew A.V. and Fairman F.W. (1974). Transfer function matrix identification. *IEEE Transaction*, vol:CAS-21, p584-588.

Mehra R.K. and Tyler J.S. (1973). Case studies in aircraft parameter identification. *In Eykhoff P. (Ed), Identification and System Parameter Estimation*, North-Holland.

Middleton R.H. and Goodwin G.C. (1990). *Digital Control and Estimation: A Unified Approach*, Prentice-Hall .

Moler C., Little J. and Bangert S. (1988). *Pro-MATLAB User's Guide*, The Mathworks Inc. .

Mukhopadhyay S. and Rao G.P. (1991). Integral equation approach to joint state and parameter estimation in continuous-time MIMO systems. *IEE Proceedings-D*, vol:138, p93-102.

Narendra (1976). Stable identification scheme. In Mehra R.K. and Lainiotis (Eds). *System Identification: Advances and Case Studies*, Academic Press, New York.

O'Reilly J. (1983). *Observers for Linear Systems*, Academic Press, New York.

Ogata K. (1987). *Discrete-time Control Systems*, 1st Ed., Prentice Hall International, Inc..

Palanisamy K.R. and Bhattacharya D.K. (1981). System identification via block-pulse functions. *International Journal of System Science*, vol:12, p643-647.

Patel R.V. and Munro N. (1982). *Multivariable System Theory and Design*, International Series on Systems and Control, vol:4, Pergamon Press.

Pearson A.E. and Wuu C.Y. (1984). Decoupled delay estimation in the identification of differential delay systems. *Automatica*, vol:20, p761-772.

Petrov B.N. and Krutova P. (1975). Non-searching self adaptive systems: theoretical foundations and prospects. *Proceedings of 6th IFAC Work Congress, Boston*, ISA, New York.

Power H.M. and Simpson R.J. (1978). *Introduction to Dynamics and Control*, McGraw-Hill Book Co., UK..

Rao G.P. (1983). *Piecewise Constant Orthogonal Functions and their Application to Systems and Control*, Springer, Berlin.

Rao G.P. and Sivakumar L. (1976). Identification of deterministic time-lag systems. *IEEE Transaction on Automatic Control*, vol:21, p527-529.

Rao G.P. and Sivakumar L. (1979). Identification of time lag systems via Walsh functions. *IEEE Transactions on Automatic Control*, vol:AC-24, p806-808.

Rao G.P. and Sivakumar L. (1981). Transfer function matrix identification in MIMO systems via Walsh functions. *IEE E Proceedings*, vol:69, p465-466.

Richardson C.H. (1954). *An Introduction to the Calculus of Finite Differences* , D. Van Nostrand, New Jersey.

Rosenbrock H.H. and Storey C. (1966). *Computational Techniques for Chemical Engineers*, Pergamon, Oxford.

Sagara S. and Zhao Z.Y. (1989). Recursive identification of transfer function matrix in continuous system via linear integral filter. *International Journal of Control*, vol:50, p457-477.

Sagara S. and Zhao Z.Y. (1990). Numerical integration approach to on-line identification on continuous-time systems. *Automatica,* vol:26, p63-74.

Saha D.C. and Rao G.P. (1980). Identification of lumped linear time-varying parameter systems via Poisson moment functions. *International Journal of Control*, vol:32, p709-721.

Saha D.C. and Rao G.P. (1981). Identification of lumped linear systems in the presence of small unknown delays-the Poisson moment functional approach. *International Journal of Control*, vol:33, p945-951.

Saha D.C. and Rao G.P. (1982). Transfer function matrix identification in MIMO systems via Poisson moment functionals. *International Journal of Control*, vol:35, p727-738.

Saha D.C. and Rao G.P. (1983). *Identification of Continuous Dynamical Systems - The Poisson Moment Functional (PMF) Approach*, Springer, Berlin.

Saha D.C., Rao B.B.P. and Rao G.P. (1982). Structure and parameter identification in linear continuous lumped systems-the Poisson moment functional approach. *International Journal of Control*, vol:36, p477-491.

Sanathanan C.K. and Koerner J. (1963). Transfer function synthesis as a ratio of two complex polynomial. *IEEE Transactions on Automatic Control*, vol:AC-8, p56-58.

Sargan J.D. (1974). Some discrete approximations to continuous-time stochastic models. *Journal of Royal. Statistical Society, Series B: Methodological*, vol:36, p74-80.

Sastry D. and Gauvrit M. (1978). Some simplified algorithms for the Bayesian identification of aircraft parameters. *International Journal of System Science*, vol:9, p1215-1221.

Schoukens J. (1990). Modelling of continuous-time systems using a discrete-time representation. *Automatica*, vol:26, p579-583.

Schoukens J, Pintelon R. and Renneboog J. (1988). A maximum likelihood estimator for linear and nonlinear systems. *IEEE Transaction on Instruments and Measurements.*, vol:37, p10-18.

Shackcloth B. and Butchart R.L. (1965). Synthesis of model reference adaptive systems by Liapunov's second method. *Theory of Self Adaptive Control Systems*, Plenum Press, New York.

Sinha N.K. and Zhou Q.J. (1984). State estimation using block-pulse functions. *International Journal of System Science*, vol:15, p341-350.

Sinha N.K. (1972). Estimation of transfer function of continuous system from sample data. *IEE Proceedings D*, vol:119, p612-614.

Sivakumar L. and Rao G.P. (1982). Parameter identification in lumped linear continuous systems in a noisy environment via Kalman-filtered Poisson moment functionals. *International Journal of Control*, vol:35, p509-519.

Smith O.J.M. (1957). Closer Control of Loops with Dead Time. *Chemical Engineering Progress*, vol:53, p217.

Soderstrom T. and Stoica B. (1989). *System Identification*, Prentice-Hall, London.

Spiegel M.R. (1974). *Theory and Problems of Advanced Calculus, Schaum's Outline Series*, McGraw-Hill, Great Britain.

Stepner D.E. and Mehra R.K. (1973). Maximum likelihood identification and optimal input design for identifying aircraft stability and control derivatives. *NASA Report*, No. NASA CR-2200.

Tennet R.M. (1971), editor. *Science Data Book*. Oliver and Boyd.

Unbehauen H. and Rao G.P. (1987). *Identification of Continuous Systems*, North-Holland, Amsterdam.

Valstar J.E. (1963). In flight dynamic checkout. *IEEE Transactions on Aerospace*, vol:AS-1, p2.

Wellstead P.E., Zanker P., and Edmunds J.M. (1979.). Self-tuning pole or zero assignment regulators. *International Journal of Control*, vol:30, p1-26.

Westerlund T. and Tysso A. (1980). Remarks on "Asymptotic behaviour of the extended Kalman filter as a parameter estimator for linear systems". *IEEE Transactions on Automatic Control*, vol:AC-25, p1011-1012.

Whitaker H.P (1958). Design of a model reference adaptive control system for aircraft.. *Instrum. Laboratory*, R-164, MIT, Cambridge.

White D.R.J. (1980). *Electrical Filters Synthesis, Design and Applications, 2nd Edition.*, Don White Consultants Inc, USA..

Whitefield A.H. and Messali N. (1987). Integral-equation approach to system identification. *International Journal of Control*, vol:45, p1431-1445.

Whitefield A.H. (1986). Transfer function synthesis using frequency response data. *International Journal of Control*, vol:43, p1413-1426.

William C.S.. (1986). *Designing Digital Filters,* Prentice-Hall Information and System Sciences Series, Series Editor: Kailath,T, Prentice-Hall Inc, USA..

Yashin A.I. (1986). Continuous-time adaptive filtering. *IEEE Transactions on Automatic Control*, vol:31, p776-779.

Young P.C. (1965). Process parameter estimation and self adaptive control. In Hammond P.H. (Ed) *Theory of Self Adaptive Control Systems*, Plenum Press, New York.

Young P.C. (1968). Process parameter estimation. *Control and Automation Progress*, vol:12, p125-131.

Young P.C. (1970). An instrumental variable method for real-time identification of a noisy process. *Automatica*, vol:6, p271-283.

Young P.C. (1974). Recursive approaches to time-series analysis. *Bulletin of Institute of Applied Mathematics*, vol:10, p209-217.

Young P.C. (1976). Some observations on instrumental variable methods of time-series analysis. *International Journal of Control*, vol:23, p593-612.

Young P.C. (1979). Self adaptive Kalman filter. *Electronics Letter*, vol:15, p358-369.

Young P.C. (1981). Parameter estimation for continuous-time Models - A survey. *Automatica*, vol:17, p23-39.

Young P.C. and Jakeman A. (1979). Refined instrumental variable methods recursive time-series analysis. Part I. Single input single output systems. *International Journal of Control*, vol:29, p1-30.

Young P.C. and Jakeman A. (1980). Refined instrumental variable methods recursive time-series analysis. Part III. Extensions. *International Journal of Control*, vol:31, p741-764.

Young P.C., Jakeman A.J. and McMurtrie R. (1980). An instrumental variable method for model structure identification. *Automatica*, vol:16, p281-294.

Zhao Z.Y., Sagara S. and Wada K. (1991). Bias-compensating least squares method for identification of continuous-time systems from sampled data. *International Journal of Control*, vol:53, p445-461.

Zheng W.X. and Feng C.B. (1990). Identification of stochastic time lag systems in the presence of coloured noise. *Automatica*, vol:26, p769-779.

APPENDIX A

MATLAB Programs for Simulation Examples

Custom Functions

FII.m	Function to find Fixed Interval Integral of a continuous-time variable using numerical approximation.
FIIdu.m	Function to find Fixed Interval Integral of a piece-wise-constant variable using numerical approximation.
backdiff.m	Fucntion to find backward difference
PRBS.m	Function to generate PRBS signal.
spectral.m	Function to find spectral magnitude
lsimdu.m	Function to generate output of a system that is driven by piece-wise-constant control input

Batch Files

eg_system.m	Batch routines to generate deterministic system output of simulation examples
pinknoise.m	Batch routines to generate pinknoise for simulation Example 5.3-3
delay_sys.m	Batch routines to generate delay system output of simulation examples

eg5.3_1.m Batch routines for Simulation Example 5.3-1 (Deterministic System)

eg5.3_2.m Batch routines for Simulation Example 5.3-2 (Effect of different FII interval length -- system with white noise)

eg5.3_3.m Batch routines for Simulation Example 5.3-3 (Effect of different FII interval length -- system with colour noise)

eg5.4_1.m Batch routines for Simulation Example 5.4-1 (Simultaneous delay and system parameters estimation of a first order system)

eg5.4_2.m Batch routines for Simulation Example 5.4-1 (Simultaneous delay and system parameters estimation of a second order system)

FII.m

```
function yfii = FII(y,T,M,method)
% FII Returns numerical approximation of Fixed-Interval-Integral
%  y = function input-- capable for muitivariable
%                each column is a variable
%  T = sampling interval of y (per unit time)
%  M = interval of FII (per unit time)
%  method = order of Newton-Cotes numerical method to approximate the
FII
%       = 1 : trapezoidal rule
%       = 2 : Simpson's 1/3 rule
%       = 3 : Simpson's 3/8 rule
%       = 4 : Boole's rule
%
% Assumption : All past values of y is zero
% by W.H.Siew 1992.  Reference: Chapter4 of WanHing Siew's thesis.

if abs( (M/T)-round(M/T))>1e3*eps,
   error('FII interval must be multiple of sampling interval');end
mm=M/T;

if nargin<4, method=2; end;      %Simpson1/3 by default

if method==1,               % Trapezoidal rule
  n=1; Jn=T/2*[1 1]; end
```

```
if method==2,                 % simpson 1/3 rule
  n=2; Jn=T/3*[1 4 1]; end
if method==3,                 % simpson 3/8 rule
  n=3; Jn= T*3/8*[1 3 3 1]; end
if method==4,                  % Boole's rule
  n=4; Jn= 2*T/45*[7 32 12 32 7]; end

order = int2str(n);
if rem(mm,n),
    error(..['FII interval must be multiple of ',order,' for ',order,' order rule']);
end

r=mm/n;
w=zeros(1,mm+1);
for k=0:r-1, w(k*n+1:(k+1)*n+1)=w(k*n+1:(k+1)*n+1)+Jn; end

den =zeros(w) ; den(1) = 1;
yfii=zeros(y) ; yfii = dlsim(w,den,y);
return
%Alternative
[ry cy] = size(y); y=[zeros(mm,cy);y];
for k=1:ry,
   yfii(k,:)=w*y(k:k+mm,:);
end
return
```

FIIdu.m

```
function [ufii1, ufii2]= FIIdu(u,T,M)
% FII Returns analytic 1st and 2nd order
%    Fixed-Interval-Integral of Piese-wise Constant u
%  u = function input-- each column is a variable
%  T = sampling interval of u (per unit time)
%  M = interval of FII (per unit time)
%
% Assumption : All past values of u is zero
% by W.H.Siew 1992.  Reference: Chapter5 of WanHing Siew's thesis.
%

if abs( (M/T)-round(M/T))>1e3*eps,
   error('FII interval must be multiple of sampling interval');end

mm=M/T;

num1= T*ones(1,mm);       den1=zeros(1,mm);

num2=zeros(1,2*mm);       den2=zeros(1,2*mm);
```

```
num2(1:mm)     =(T^2)/2*(1:2:2*mm-1);
num2(mm+1:2*mm)=(T^2)/2*(2*mm-1:-2:1);

ufii1=dlsim([0 num1],[1 den1],u);
ufii2=dlsim([0 num2],[1 den2],u);
return
```

backdiff.m

```
function [dx ,DD ]= backdiff(x, shift,order)
%BACKDIFF          Return backward difference of x
%
% dx = x(k) - n. x(k-T) + n.x(k-2T) - ..... x(k-nT)
%
% dx = backdiff(x, shift,order)
% [dx, DD] = backdiff(x, shift,order)
%               order = n = order of backward difference
%                          shift = T = number of data shift
%                                      x                 = data
%                                 dx               = backward difference
of x
%    DD  = Matrix of  [x  -n.x(k-T)  n.x(k-2T) ....]
%  W.H. Siew

if nargin ==2, shift =1; end
n =order; T=shift;
x=x(:);

[rx,cr] = size(x) ;
dd = zeros(rx, n+1);

for i = 1 : n+1
  m = (i-1)*T ;
  if m+1>rx, break, end
  dd(m+1:rx,i) = (-1)^(i-1)* x( 1:rx-m);
end

if n>=2,    dd(:,2:n) = n*dd(:,2:n);  end
DD=dd;
dx = sum(dd')';
return
```

PRBS.m

```
function x = prbs(r,c,subdivision)
%PRBS      Generating descrete-time Pseudo Random Binary Signal
%          X = PRBS(A)
%      X = PRBS(r,c)
%          X = PRBS(r,c, subdivision)
%
% A and X are column vector(s)
%
% W.H. Siew

[nr,nc] = size(r);

if any( [nr nc]~=[1 1]),
  if nargin ~= 2, subdivision = 1;
  else subdivision = c;
  end
  r=nr; c=nc;
else
  if nargin ~= 3, subdivision=1; end;
end

if r==1, x = zeros(r, c*subdivision);
else x = zeros(r*subdivision , c);
end

xx = ones(subdivision,1);
rand('normal')
n = sign( rand(r,c) );
n = xx * n(:)' ;
x(:) = n;
return
```

spectral.m

```
function [M,fq]=spectral(y,T)
%
% SPECTRAL find the spectral magnitude of Y
%      ** NOT the power spectral density
%
%      [M,FQ] = SPECTRAL(Y,T)
%           Y = data
%            T = sampling time (not frequency)
%      M = Spectral magnitude of Y
%     FQ = corresponding frequency for M.
%          Maximum frequency is half the sampling frequency.
%
```

```
% plot(fq,M) give the spectral magnitude plot
% plot(fq,M.^2) give the Power spectral density plot
%
% SPECTRAL(Y,T) ie no output arguements returns spectral magnitude
plot
%
% W.H. Siew 5th Sep 1990

if nargin==1,
  fprintf('Warning : Sampling time not specfied! Default of 1 used.\007\n');
 T=1;
end

fsampling=1/T;

ty=fft(y);
M = abs(ty) ;

[r,c]=size(ty);
M = M(1:r/2,:);
fq = fsampling*( 0:(r/2-1) )'/r;

if nargout==0,
plot(fq,M) ; xlabel('freq (Hz)') ; ylabel('Spectral Mag');
 M =[];
end
return
```

lsimdu.m

```
function [y,x] = lsimdu(num,den,u,Tu,t)
%LSIMDU Simulation of continuous-time linear systems
%       y(s) = NUM(s)/DEN(s) u(s)
% subjected to PIECE-WISE CONSTANT Inputs with regular interval, Tu
% -- Scalar only
%               [y,x] = lsimdu(NUM,DEN,u,Tu,t)
% By W.H. Siew

Ts=t(2)-t(1); t=t(:); u=u(:);

if Tu<Ts,
error('Input interval must be equal or larger that sampling interval'); end
if rem(Tu,Ts)>eps,
error('Input interval must be multiple of sampling interval'); end

%Expanding the input vector
 div=Tu/Ts; u= ones(div,1)*u' ; u=u(:);
```

```
nt=length(t); u=u(1:nt);

% Convert to state space
        [a,b,c,d] = tf2ss(num,den);

% Get equivalent zero order hold discrete system
 [Ad,Bd] = c2d(a,b,Ts);
 x = ltitr(Ad,Bd,u);
 y = x * c.' + u * d.';
return
```

eg_system.m

```
%System deterministic input and output
fname='eg53_system'

Ts=0.01; tt=(0:Ts:30-Ts)';
Tu=0.1;  u=prbs(30/Tu,1);

%Continuous-time system
num=5; den=[1 3 2];
yy=lsimdu(num,den,u,Tu,tt);
stdy=std(yy);
thm= [ den(2:length(den)) num]';

%Discrete-time system
[numd dend]=c2dm(num,den,Tu);
thmd= [ dend(2:length(dend)) numd(2:3)]';

[Magy,fy]=spectral(yy,Ts);

vars='Ts Tu tt u num den thm yy stdy numd dend thmd Magy fy';

eval(['save ',fname, '.mat ' ,vars])
disp([fname,'.m done'])
return
```

pinknoise.m

```
% Generating pinknoise normalized w.r.t variance of the
% deterministic output of system in example 5.3

load eg53_system.mat
load pink1.mat          % fpeak=1;fvar=0.2;

eep = stdy*pink;        % normalizing w.r.t the variance of y
yye = lsim(1,den,eep,tt);

save eg53_pinknoise1.mat  fpeak fvar eep yye
disp('eg53_pinknoise1.m  done')
return
```

delay_sys.m

```
%Delay System deterministic input and output
load eg53_system.mat
%variable from eg53_system.mat are:
% Ts tt Tu  u num den thm yy stdy numd dend thmd Magy fy

delay = 0.35
nndelay = delay/Ts;   nyy=length(yy);
yydelay=zeros(yy); yydelay(nndelay+1:nyy)=yy(1:nyy-nndelay);
clear nyy
return
```

eg5.3_1.m

```
Ts=0.01;        t=(0:Ts:30-Ts)';
Tu=0.1;         u=prbs(300,1);
y=lsimdu(5,[1 3 2],u,Tu,t);
M=0.1;

[ufii1 ,ufii2]=FIIdu(u,Tu,M);
yfii1=FII(y,Ts,M);
yfii2=FII(yfii1,Ts,M);

ly=length(y);
y=y(1:Tu/Ts:ly);
yfii1=yfii1(1:Tu/Ts:ly);
yfii2=yfii2(1:Tu/Ts:ly);

mm=M/Tu;
yy=[0; backdiff(y,mm,2)];     % added unit delay for algorithm
yy(length(yy))=[];
```

```
uu1=-backdiff(yfii1,mm,1);
uu2=-yfii2;
uu3=ufii2;

 na=0; nb=[1 1 1]; nk=[1 1 1]; nn=[na nb nk ];
 lamda = 0.9                %forgetting factor
 thm_M01=rarx([yy uu1 uu2 uu3],nn,'ff',lamda);
return
```

eg5.3_2.m

```
M=0.1;
noiseratio=0.5 ;   lamda=0.99;

load eg53_system.mat
load eg53_whtnoise1.mat   ; disp(' white noise')

yy = yy+sqrt(noiseratio)*yye;     %adding noise

%Calculating FII
[ufii1 ,ufii2]=FIIdu(u,Tu,M);
yfii1=FII(yy,Ts,M);  yfii2=FII(yfii1,Ts,M);

y=yy(1:Tu/Ts:length(tt));  t=tt(1:Tu/Ts:length(tt));
yfii1=yfii1(1:Tu/Ts:length(tt) );
yfii2=yfii2(1:Tu/Ts:length(tt) );

%Parameter  estimation
mm=M/Tu;
xy=[0; backdiff(y,mm,2)];    % added unit delay for algorithm
xy(length(y))=[];          %
xu1=-backdiff(yfii1,mm,1);
xu2=-yfii2;
xu3=ufii2;

 na=0; nb=[1 1 1]; nk=[1 1 1];
 thmRLS=rarx([xy xu1 xu2 xu3],[na nb nk],'ff',lamda);

enorm300  =  norm(  thmRLS(300,:)'-thm)/norm(thm);
enorm150  =  norm(  thmRLS(150,:)'-thm)/norm(thm);
pe150     = 100*( thmRLS(150,:)'-thm) ./ thm;
pe300     = 100*( thmRLS(300,:)'-thm) ./ thm;
return
```

eg5.3_3.m

```
M= 0.1;          % FII interval

noiseratio=0.5;    % var(noise)/var(y)
lamda=0.99;        % forgeting factor of algo

load eg53_system.mat
load eg53_pinknoise1.mat  ; disp('pinknoise')
yy = yy+sqrt(noiseratio)*yye;     %adding noise

%Calculating FII
[ufii1 ,ufii2]=FIIdu(u,Tu,M);
yfii1=FII(yy,Ts,M);  yfii2=FII(yfii1,Ts,M);

y=yy(1:Tu/Ts:length(tt));  t=tt(1:Tu/Ts:length(tt));
yfii1=yfii1(1:Tu/Ts:length(tt) );
yfii2=yfii2(1:Tu/Ts:length(tt) );

%Parameter  estimation
mm=M/Tu;
xy=[0; backdiff(y,mm,2)];    % added unit delay for algorithm
xy(length(y))=[];          %
xu1=-backdiff(yfii1,mm,1);
xu2=-yfii2;
xu3=ufii2;

 na=0; nb=[1 1 1]; nk=[1 1 1];
 thmRLS=rarx([xy xu1 xu2 xu3],[na nb nk],'ff',lamda);

enorm300  =  norm(  thmRLS(300,:)'-thm)/norm(thm);
enorm150  =  norm(  thmRLS(150,:)'-thm)/norm(thm);
pe150     = 100*( thmRLS(150,:)'-thm) ./ thm;
pe300     = 100*( thmRLS(300,:)'-thm) ./ thm;
return
```

eg5.4_1.m

```
%Generate system input and output
%~~~~~~~~~~~~~~~~~~~~~~~~~~~~
Tb = 0.01;              tt = (0:Tb:30-Tb)';
Ts = 0.1;               u = prbs(300,1);
delay = 0.15;

yy = lsimdu(10,[1 1],u,Ts,tt);

[ryy,cyy]= size(yy);   yydelay = zeros(ryy,cyy);
```

```
%Shift the yy to form delayed output
%~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
idelay = 1+ delay/Tb  ;
yydelay(idelay:ryy) = yy(1: ryy-idelay+1);

yClean = yydelay(1:Ts/Tb:ryy);

save eg541sys.mat Ts Tb u yy yydelay yClean
clear ryy cyy tt idelay


%Estimate direct coefficients using RLS
%~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
load eg541sys

m = 10;                %FII interval M = m Ts

%Add white noise
%~~~~~~~~~~~~~~~~~~
NoiseRatio = 0.05          %in term of std. dev.

e = randn(size(yClean));
y = yClean + NoiseRatio *std(yClean)*e;

%Form FII of output
%~~~~~~~~~~~~~~~~~~~~~
yFII = FII(y,Ts,m*Ts);

%Form delay array of input
```

```
L= 0 ;H = 4;
nBeta = 1*m + (L - H + 1)  ;

xy = [0;backdiff(y,m,1) ];        %add unit delay required by algo
xy(length(xy))=[];

xu1= -yFII;
xu2 = u;

na=0;          nb = [1, nBeta ];      nk = 1+ [0, dmin ];
lamda = 0.99;

thm = rarx([xy xu1 xu2],[na nb nk], 'ff', lamda);
[rthm, cthm] = size(thm);

a = thm(:,1);          beta = thm(:,2:cthm);
save eg541rls.mat m NoiseRatio e y a beta lamda

%Calculate parameters and delays
%~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
[rbeta,cbeta] = size(beta);

% Find integral delay
%~~~~~~~~~~~~~~~~~~~~~~~~~
numSi = cbeta - (m+1) + 1;        Si = zeros(rbeta, numSi);

for k = 1 : numSi
    ii = k + (0:m);
    Si(:, k) = sum( abs(beta(:,ii))' )' ;
end

maxSi  =  max(Si')';
d = zeros(rbeta,1);
for k = 1:rbeta
    d(k) = min(find(Si(k,:) == maxSi(k))) - 1;
end

%Find  other
%~~~~~~~~~~~
b = sum( beta(:, 3+(0:m) )' )';
v = beta(:,3) ./ b;

save eg541cal Si b v d
```

eg54_2.m

```
NoiseRatio =0.05 ;   lamda=0.99;
load eg541_system.mat
load eg533_whtnoise1.mat    ; disp(' white noise')

M=0.5;  mTu = M/Tu;    %mTu=5
ndelayarray=mTu*2+5    % = 15
L = 1; H = 5;  % Lower and higher limit of delay in terms of sampling
               % interval
nBeta = 2*mTu + (H-L) + 1

if nBeta<(2*mTu+1), error('Not enough delay array');end
yydelay = yydelay+NoiseRatio *yye;      %adding noise

%%%%%parameter  and  delay  estimation%%%%%%%
%Calculating FII  of output
ydelayfii1=FII(yydelay,Ts,M);  ydelayfii2=FII(ydelayfii1,Ts,M);
ydelay = yydelay(1:Tu/Ts:length(tt));  t=tt(1:Tu/Ts:length(tt));
ydelayfii1 = ydelayfii1(1:Tu/Ts:length(tt) );
ydelayfii2 = ydelayfii2(1:Tu/Ts:length(tt) );

%Parameter  estimation
mTu=M/Tu;
xy=[0; backdiff(ydelay,mm,2)];     % added unit delay for algorithm
xy(length(ydelay))=[];          %

xu1=-backdiff(ydelayfii1,mm,1);
xu2=-ydelayfii2;
xu3= u;

 na=0;          nb=[1  nBeta ];         nk= 1+ [0 0 L];
 thmRLS=rarx([xy xu1 xu2 xu3],[na nb nk],'ff',lamda);
clear xy xu1 xu2 xu3 xu4

a1hat=thmRLS(:,1); a0hat=thmRLS(:,2);
beta=thmRLS(:, 2+(1:nBeta) );

nctot= ndelayarray-2*mTu;  tot=zeros(length(beta), nctot );
for h=1:nctot
  tot(:,h)=sum(beta(:,h:2*mTu+h)')' ;
end

dd=find( tot(300,:)==max(tot(300,:)) );
bhat=[ sum(beta(:,dd:dd+2*mTu)')/mTu^2/Tu^2  ]';

df1= sqrt(2*thmRLS(:,6)./bhat);
df2= Tu-sqrt( abs(thmRLS(:,16) ./bhat));
```

```
df3= (beta(:,dd+2)-beta(:,dd+9))/2/Tu ./bhat;
hh=find(df3<0); if any(hh), df3(hh)=zeros(length(hh),1);  end;
hh=find(df3>Tu); if any(hh), df3(hh)=Tu*ones(length(hh),1); end;

df4= (beta(:,dd+3)-beta(:,dd+8))/2/Tu ./bhat;
hh=find(df4<0); if any(hh), df4(hh)=zeros(length(hh),1);  end;
hh=find(df4>Tu); if any(hh), df4(hh)=Tu*ones(length(hh),1); end;

df5= (beta(:,dd+4)-beta(:,dd+7))/2/Tu ./bhat;
hh=find(df5<0); if any(hh), df5(hh)=zeros(length(hh),1);  end;
hh=find(df5>Tu); if any(hh), df5(hh)=Tu*ones(length(hh),1); end;

%delayhat=(dd+ddmin)*Tu-(df3+df4+df5)/3;
delayhat=(dd+ddmin)*Tu-df5;

h=300; param300 = [a1hat(h) a0hat(h) bhat(h) delayhat(h)];
h=150; param150 = [a1hat(h) a0hat(h) bhat(h) delayhat(h)];
enorm300 = norm( param300-[thm' delay])/norm([thm' delay]);
enorm150 = norm( param150-[thm' delay])/norm([thm' delay]);
pe300    = 100*( param300-[thm' delay]) ./ [thm' delay];
pe150    = 100*( param150-[thm' delay]) ./ [thm' delay];
pe= [a1hat a0hat bhat delayhat] - ones(a1hat)*[3 2 5 0.35];
pe=100*pe ./ ( ones(a1hat)*[3 2 5 0.35] );

clear nctot
return
```

APPENDIX B1

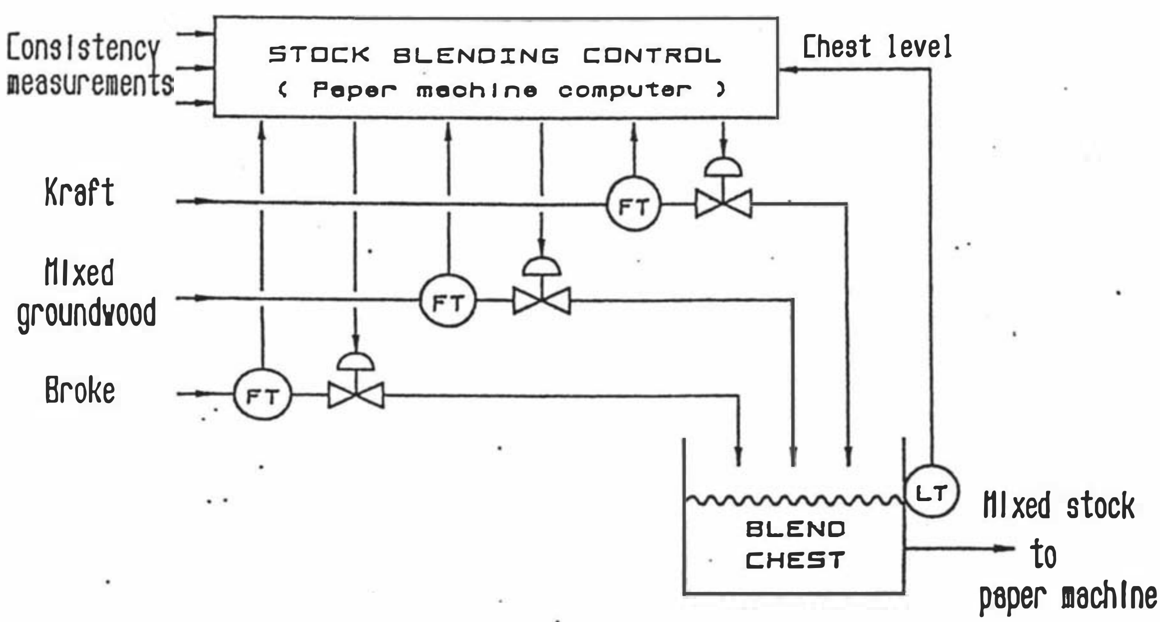
MODELLING OF PAPER-STOCK CONCENTRATION CONTROL SYSTEM

This appendix introduces an industry case study on paper-stock concentration control system. It details the modelling of the control system and several system disturbances

B1.1 INTRODUCTION

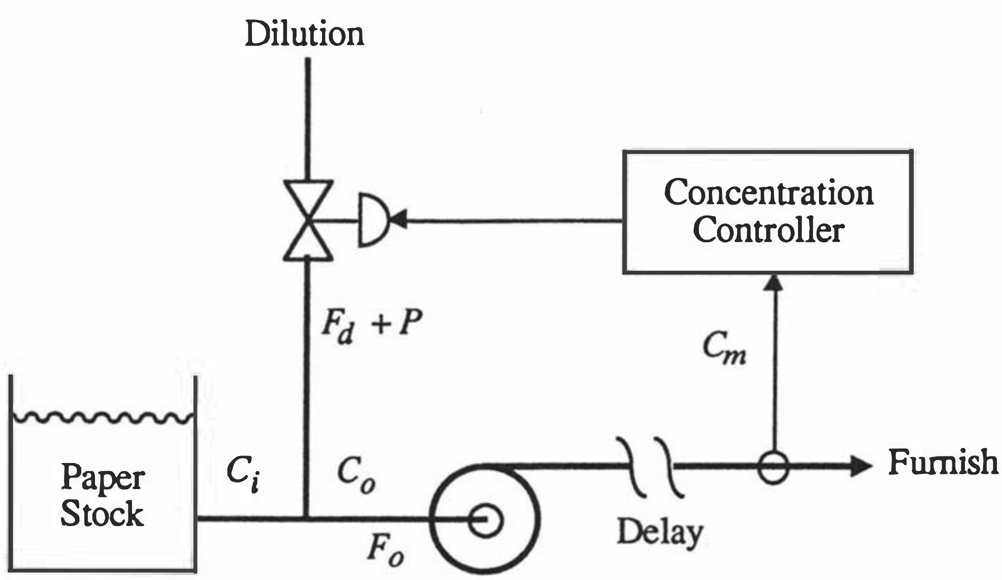
The three paper machines at Tasman produce newsprint from blends of semi-bleached Kraft pulp, mixed groundwood (refiner plus stone groundwood) and broke. These three components, which are collectively named the *"furnish components"*, are combined in the desired ratios in the blend chests at the wet end of each paper machine. The general stock blending arrangement is illustrated in Figure B1-1. The objective of the stock blending control is to ratio the components on a dry fibre basis, even though it is the total stock flows (fibre plus water) that are manipulated. The flows are controlled via the blend chest level and are corrected for changes in concentration on an infrequent basis, using manual concentration test results entered by operators (every two or four hours). To maintain a uniform blend it is therefore essential that any variability in stock concentration is minimized through good concentration control.

Figure B1-1 Schematic layout of stock blending process



The concentration of each furnish component is controlled upstream of the blend chest by adding dilution water prior to each stock pump (see Figure B1-2). Concentration transmitters are located some distance downstream from the dilution points, causing a delay of roughly 25 seconds in the concentration control loop. Pneumatic two-term (PI) controllers are currently used as concentration controllers, but a new distributed control system is about to replace all the wet-end controls on two of the paper machines. This provides the perfect opportunity to improve concentration control through the implementation of a more effective control strategy.

Figure B1-2 Schematic layout of concentration control system



The notation used in Figure B1-2 and subsequently, is as follows:

C_i = concentration of incoming pulp from stock chest

C_d = concentration of dilution water

C_o = concentration of mixed (furnish) stream at mixing point (as if there were instant perfect mixing)

C_m = measured concentration

F_i = flowrate of incoming pulp

F_o = flowrate of pulp at mixing point.

F_m = measured flowrate

F_d = desired flowrate of dilution water

P = variation in dilution flow due to changes in pressure

B1.2 UNITS

Units common to the pulp and paper industry, rather the SI units, are used in this chapter for variables and constants. These are, percentage of fibre (%) for stock concentration and gallon per minute (gpm) for flowrate. The variation in pressure has the same units as flowrate, gpm, because it is expressed as the resultant change in flow.

It has been found that the numerical values of concentration are small relative to the flowrate and thus the computation for design and analysis involves the division of numbers with a large difference in magnitude. This causes significant computational errors in digital computer and complication in handling the numerical results. To avoid these difficulties, units of 0.001% is used for the calculation of all parameters and indices. The units used are summarized as the following :

Stock Concentration -- Parameter calculation

0.001% , one thousandth percentage of fibre in pulp stock by mass.

Stock Concentration -- Presentation

%, percentage of fibre in pulp stock by mass.

Flowrate

gpm, gallon per minute. = x 4.4 Litre per minute
= x 264 Litre per second

Effect of Variation in Pressure

gpm, gallon per minute.

B1.3 MODELLING STOCK CONCENTRATION

In order to simulate the behaviour of the stock control system and to determine suitable controllers, an appropriate model of the system must first be determined. It is determined in the following section that this model is nonlinear. For design of the controllers, a linearized version of this model is required and it is detailed in Section 6.3.2.

B1.3.1 Nonlinear Model

The system illustrated in Figure B1-2 can be modelled as the mixing of two incompressible fluid streams. The system is thus governed by the following mass balance equations :

$$F_i(t) + F_d(t) + P(\mathrm{t}) = F_o(t) \qquad \text{(B1-1)}$$

$$F_i(t).C_i(t) + [\ F_d(t) + P(t)\ ]\ C_d(t) = F_o(t).C_o(t) \qquad \text{(B1-2)}$$

The first is a balance of volumetric flowrates and the second is a balance of pulp stock. Assuming the concentration of the dilution water is negligible, that is,

$$C_d = 0 \qquad \text{(B1-3)}$$

these equations yield a relationship between the controlled output, C_o, the control input, F_d, and the system disturbances, C_i, F_o and P :

$$C_o(t) = C_i(t)\left[1 - \frac{F_d(t) + P(t)}{F_o(t)}\right] \tag{B1-4}$$

As the concentration is measured some distance downstream and the pulp fluid assumed incompressible, the measured concentration and flow are given by :

$$C_m(\mathrm{t}) = C_o(\mathrm{t\text{-}L}) \tag{B1-5}$$

$$F_m(\mathrm{t}) = F_o(\mathrm{t}) \tag{B1-6}$$

where L is the pure time delay between the mixing point and the measuring point. Substituting Equations (B1-5) and (B1-6) into Equation (B1-4), gives an equation for the measured concentration:

$$C_m(\mathrm{t}) = C_i(\mathrm{t\text{-}L})\left[1 - \frac{F_d(\mathrm{t\text{-}L}) + P(\mathrm{t\text{-}L})}{F_m(\mathrm{t\text{-}L})}\right] \tag{B1-7}$$

B1.3.2 Linearized Models

Equation (B1-7) describing the behaviour of measured concentration is nonlinear due to the multiplication and division of variables in the equation. Most methods of controller design presume the system equations have been linearized. To linearize Equation (B1-7), the actual values of the variables (upper case symbols) are written as deviations around the steady state values of these variables. Thus deviation variables (lower case symbols) are defined as follows :

$c_i(t) = C_i(t) - C_{iss}$, C_{iss} = steady state value of $C_i(t)$

$c_m(t) = C_m(t) - C_{mss}$, C_{mss} = steady state value of $C_m(t)$

$f_o(t) = F_o(t) - F_{oss}$, F_{oss} = steady state value of $F_o(t)$

$f_d(t) = F_d(t) - F_{dss}$, F_{dss} = steady state value of $F_d(t)$

$p(t) = P(t) - P_{ss}$, P_{ss} = steady state value of $P(t)$

The use of these variables, in conjunction with suitable Taylor's Series expansions (Coughanowr and Koppel 1983), enables the following linearized continuous-time system description to be determined :

$$c_m(t) = k_1.c_i(t\text{-}L) + k_2.f_o(t\text{-}L) + k_3.p(t\text{-}L) + k_3.f_d(t\text{-}L) \qquad \text{(B1-8)}$$

where

$$k_1 = 1 - \frac{F_{dss} + P_{ss}}{F_{oss}} \qquad \text{(B1-9)}$$

$$k_2 = C_{iss}\frac{F_{dss} + P_{ss}}{F_{oss}} \qquad \text{(B1-10)}$$

$$k_3 = -\frac{C_{iss}}{F_{oss}} \qquad \text{(B1-11)}$$

Notice that the three parameters k_1, k_2 and k_3 of this simple linearized model are completely determined by the steady state values of the system.

In this dilution process, the manipulated dilution flow (f_d) is considered as the controlled input and the diluted or regulated concentration of mixed stream (c_o) is the system output. As far as the dilution process is concerned, the concentration of incoming stock (c_i), the stock flow rate (f_o) and the variation of dilution flow due to pressure fluctuation (p) are uncontrollable quantities. So they are considered as the system disturbances. Consequently the concentration of the mixed stream can be modelled as the weighted sum of the control input and disturbances described by Equation (B1-8).

The company's furnish concentration target or setpoint of 3% is taken as the steady state value of the furnish consistency, ie :

$$C_{mss} = C_{oss} = \text{setpoint} = 3\% = 3000 \times 0.001\%$$

To decide on the steady state values of the incoming concentration and furnish flow, measurements are taken from the plant during some typical runs. The mean values of these disturbances are then evaluated and taken as the steady state values. The mean value of variation in flow due to pressure, P_{ss}, could not be obtained from the plant as it was not measured during this work. So, for simplicity, P_{ss}, is taken to be zero.

The steady state value of dilution flow, F_{dss} can be obtained by rearranging Equation B1-7 :

$$F_{dss} = F_{oss}\left(1 - \frac{C_{mss}}{C_{iss}} \right) - P_{ss} \qquad \text{(B1-12)}$$

A suitable linearized model is derived for each of the three streams. The steady state values about which these models are linearized, and the resultant linearized model parameters are tabulated in Table B1-1.

Table B1-1 Parameters and Steady States of Linearized Model

Broke Stream	**Kraft Stream**	**Groundwood Stream**
$C_{oss} = C_{mss} = 3.000$ %	$C_{oss} = C_{mss} = 3.000$ %	$C_{oss} = C_{mss} = 3.000$ %
$C_{iss} = 3.8635$ %	$C_{iss} = 3.255$ %	$C_{iss} = 3.439$ %
$F_{oss} = 501.10$ gpm	$F_{oss} = 359.07$ gpm	$F_{oss} = 1431.64$ gpm
$P_{ss} = 0$ gpm	$P_{ss} = 0$ gpm	$P_{ss} = 0$ gpm
$F_{dss} = 112.00$ gpm	$F_{dss} = 28.18$ gpm	$F_{dss} = 182.60$ gpm
k1 = +0.7765	k1 = +0.9215	k1 = +0.8724
k2 = +1.7232	k2 = +0.7116	k2 = +0.3064
k3 = -7.7101	k3 = -9.0666	k3 = -2.4019

Laplace Transform of Linearized Model

Taking the Laplace transform of the linearized continuous-time system equation gives the following model in the frequency domain :

$$c_m(s) = \{ k_1 c_i(s) + k_2 f_o(s) + k_3 p(s) + k_3 f_d(s) \} e^{-Ts} \qquad \text{(B1-13)}$$

output disturbances input pure delay

Linearized Discrete-time Model

Assuming that the pure time delay, L, is an integer multiple of the sampling interval, T, that is :

$$L = mT \tag{B1-14}$$

the linearized discrete-time model is given by,

$$c_m(k) = k_1 c_i(k-m) + k_2 f_o(k-m) + k_3 p(k-m) + k_3 f_d(k-m) \tag{B1-15}$$

output disturbances control input

where k is the discrete time variable, that is k stands for kT seconds.

B1.4 MEASUREMENT MODELS

Four different measurements can be made of the flow mixing process. They are,

1. concentration of the furnish stock, C_m
2. concentration of the incoming stock, C_i
3. flow rate of the furnish stock, F_m
4. variation in dilution flow due to changes in pressure, P.

A control strategy based on only the first measurement results in a purely feedback control loop. The use, in addition, of one or more of the other three measurements will lead to a feedforward-feedback control system. Several measurement models were formulated in order to investigate the effect of increasing the number of measurements and the effect of different combinations of measurements on control performance.

For easy reference, a simple binary code was employed to indicate which measurements are made. The left most digit represents the first measurement, C_m, the second represents the second measurement, C_i, and so on. A "1" indicates the measurement is made and a "0" indicates it is not. For example a feedforward-feedback control system based on a measurement y(t) of furnish concentration and furnish flow,

$$y(t) = [\ c_m(t)\ \ f_m(t)\ ]^{\mathrm{T}}$$

would be coded as 1010 because it is based on measurements 1 and 3. Each of the control strategies used in the study is determined for each of the six following measurement schemes,

$$y(t) = [\ c_m(t)\ ]^{\mathrm{T}} \qquad \text{code : 1000} \tag{B1-16}$$

$$y(t) = [\ c_m(t)\ f_m(t)\ ]^{\mathrm{T}} \qquad \text{code : 1010} \tag{B1-17}$$

$$y(t) = [\ c_m(t)\ f_m(t)\ p(t)\ ]^{\mathrm{T}} \qquad \text{code : 1011} \tag{B1-18}$$

$$y(t) = [\ c_m(t)\ c_i(t)\ ]^{\mathrm{T}} \qquad \text{code : 1100} \tag{B1-19}$$

$$y(t) = [\ c_m(t)\ c_i(t)\ f_m(t)\ ]^{\mathrm{T}} \qquad \text{code : 1110} \tag{B1-20}$$

$$y(t) = [\ c_m(t)\ c_i(t)\ f_m(t)\ p(t)\ ]^{\mathrm{T}} \qquad \text{code : 1111} \tag{B1-21}$$

Here a superscripted "T" after a vector means transpose of the vector.

B1.5 MODELS OF CONCENTRATION DISTURBANCES

A knowledge of the dynamics of the system disturbances, namely C_i, F_o and P, is essential for the design and simulation of controllers for this system. The disturbance dynamics determine the appropriate sampling time and provide realistic disturbance sequences for simulation.

The disturbances, incoming concentration, C_i, and mixed stream flowrate, F_o, were sampled on-line from the furnish stream to Tasman's number 2 paper machine. The incoming concentration could not be measured directly because concentration transmitters were not installed at the incoming streams. Therefore the incoming concentration was measured using the concentration transmitters at the furnish stream by keeping the dilution valves closed. Measurement noise was presumed to be a small fixed fraction of the variations in the measurements.

It was found that the steady state values C_{iss} and F_{oss} are the dominant components in both incoming concentration and furnish flow. Therefore in order to identify the variability precisely, these constant values were subtracted from the data for all subsequent analysis.

B1.5.1 Spectral Analysis

Power spectra of the disturbance processes were obtained from the raw data which is sampled at 1 second intervals. Their highest natural frequencies then detected. This was done by taking the Fourier Transformation of the sampled data using the commercial computer package, MATLAB^{TM}.

Figures B1-3a to B1-3c show the power spectra of incoming concentration and furnish flow (after subtracting the steady state values) for the three streams. The fastest measured variability of each was determined from the following highest frequencies shown by the power spectrum plots:

Broke stream incoming concentration --	2.5 cycle/min
Broke stream furnish flow --	4.0 cycle/min
Kraft stream incoming concentration --	1.5 cycle/min
Kraft stream furnish flow --	1.0 cycle/min
Groundwood stream incoming concentration --	1.0 cycle/min
Groundwood stream furnish flow --	1.0 cycle/min

The highest frequency present is 4 cycle/min or 15 sec/cycle, in the case of the broke stream flow. Thus, by the Shannon sampling theorem (Hostetter 1988), the maximum sampling time, T, should be 7 sec.

Figure B1-3a Power Spectrum of Incoming Concentration and Furnish Flow in Broke Stream

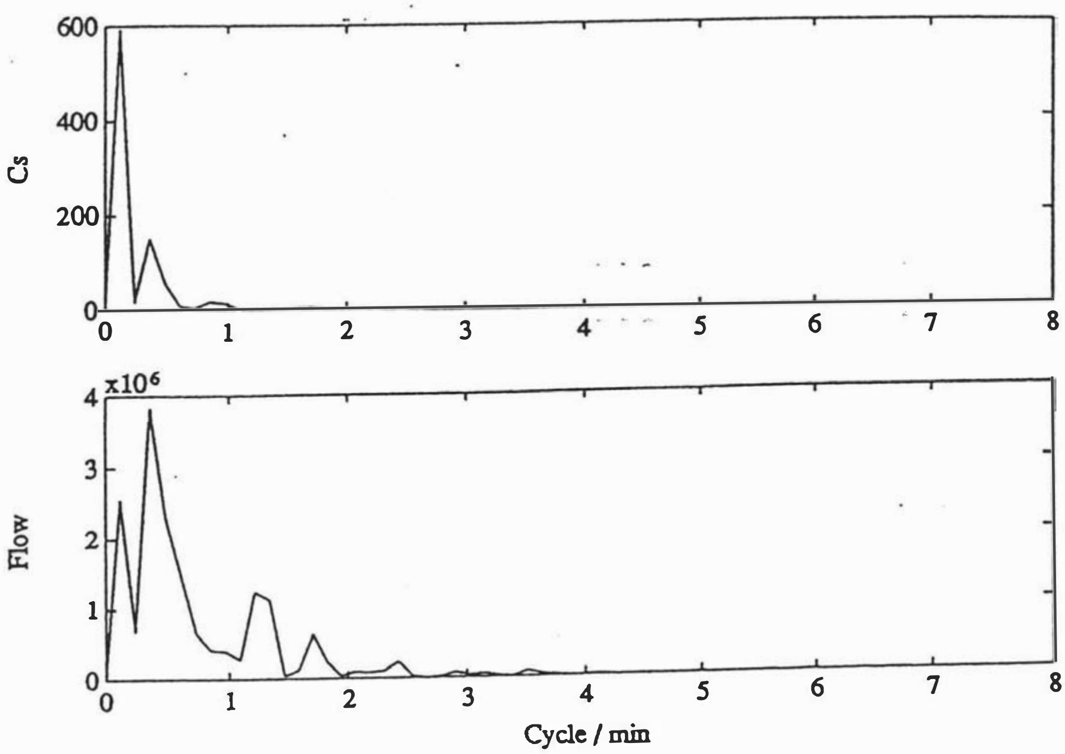


Figure B1-3b Power Spectrum of Incoming Concentration and Furnish Flow in Kraft Stream

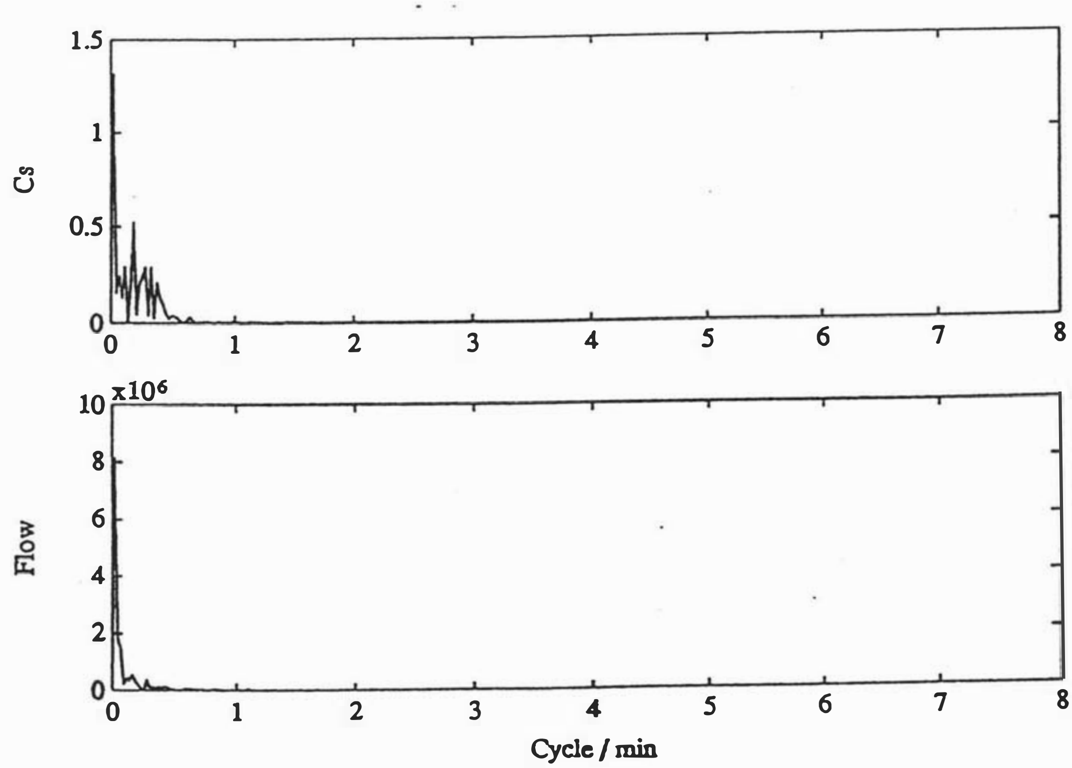
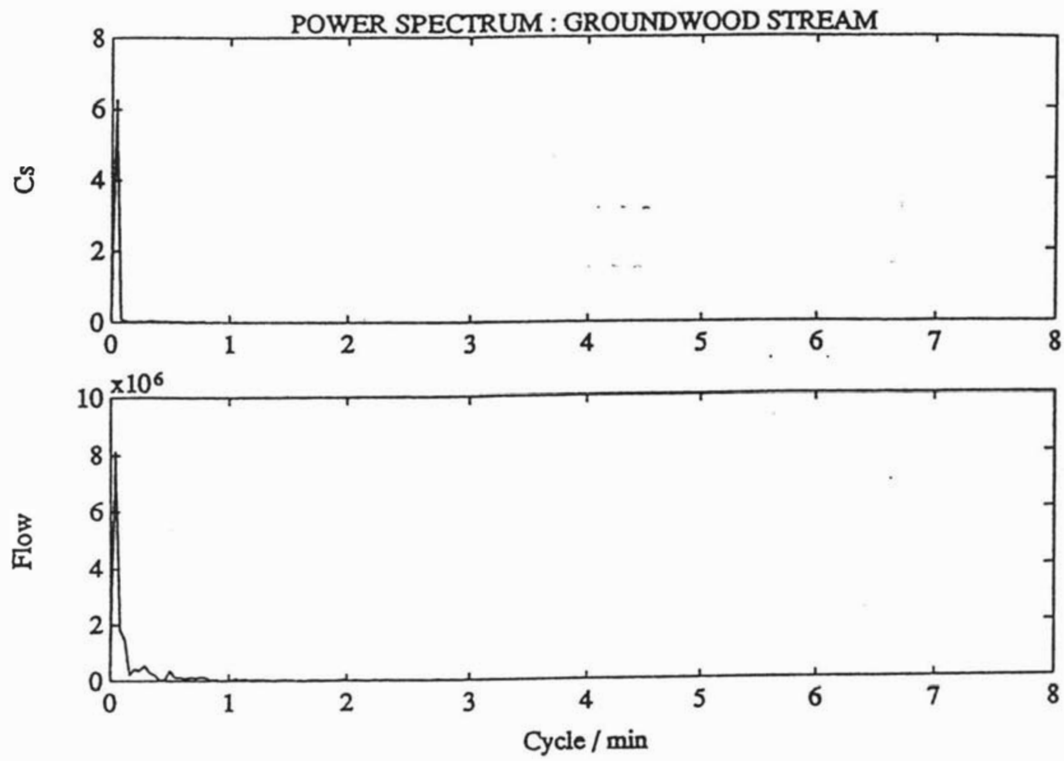


Figure B1-3c Power Spectrum of Incoming Concentration and Furnish Flow in Groundwood Stream



B1.5.2 Sampling Time

There are two considerations to be taken into account when deciding on the sampling time. On one hand the sampling interval should be kept small to avoid the effects of aliasing. But on the other hand, complications arise if the sampling time is much smaller than the dead time, or pure delay. This results from the fact that matrices and vectors increase in size by the number of sampling intervals in the dead time thereby increasing computer processing time. Furthermore, control algorithms gain much simplicity by having the dead time an integral number of sampling intervals. A dead time of 25 seconds and a maximum sampling interval of 7 seconds therefore leads to the choice of 5 seconds as a suitable interval.

B1.5.3 Black Box Modelling

Black box modelling techniques were applied to find the deviation variable models of the disturbances sampled at Tasman's number 2 paper machine. Because there are no easily measured 'inputs' that can be said to be causing the disturbance variations, only time series models driven by random input terms can be considered (Ljung 1987). The time series models considered were of the linear form :

$$\begin{aligned} y(k) + a_1.y(k\text{-}1) + a_2.y(k\text{-}2) &+ \ldots \\ = e(k) + b_1.e(k\text{-}1) + b_2.e(k\text{-}1) &+ \ldots \end{aligned} \qquad \text{(B1-22)}$$

where y is measurement or quantity to be modelled and e is the random input term or the "noise".

The left hand side of Equation (B1-22) is the autoregressive (AR) part, with constant coefficients a_1, a_2 ..., while the right-hand-side is the moving average part (MA) with constant coefficients b_1, b_2 ... . A model of this form is called an ARMA model, or Autoregressive model with coloured noise driving terms (Ljung 1987). This model reduces to an AR model (or autoregressive model with white noise driving term) when all the coefficients, b_1, b_2, ... in the MA part are zero.

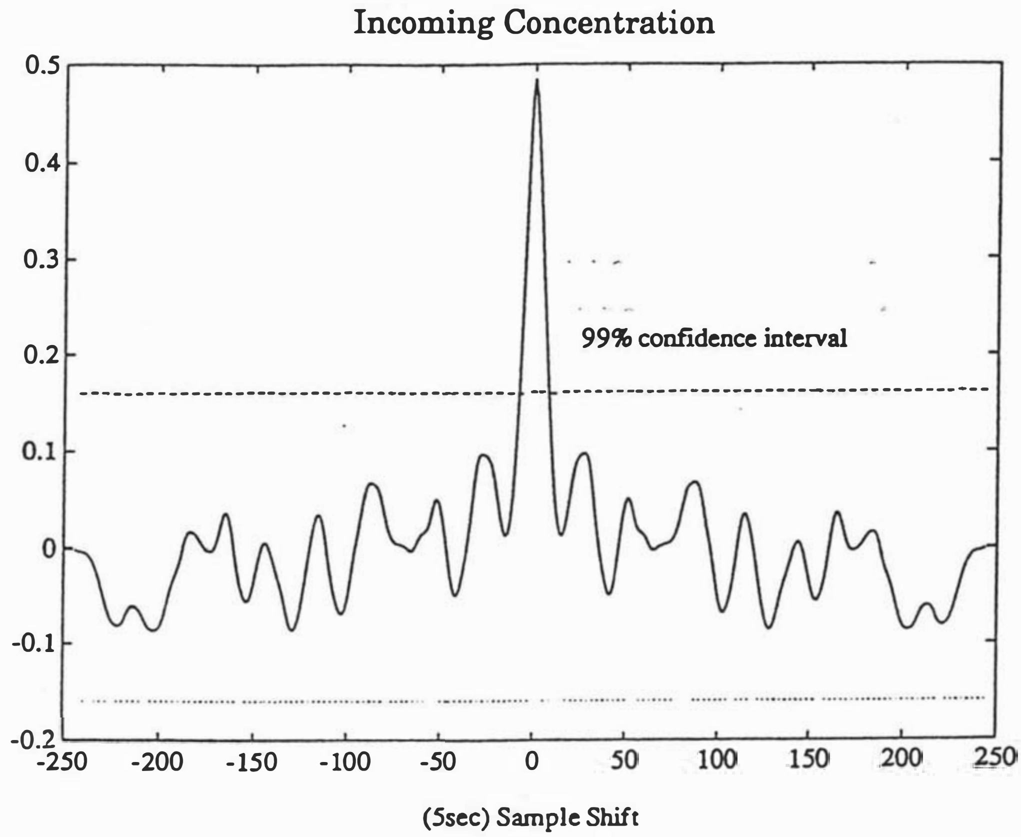
The significant orders of the AR part of the models can be estimated from the autocorrelation plots of the disturbances as given Figures B1-4a to B1-4c. All these autocorrelation plots show only a single spike peaking at zero sample shift. This indicates that all the significant quantities are in the region of zero sample shift and, in general, the significance of time-shifted or past quantities decreases with the shift interval. Also it means that these disturbances do not exhibit significant periodic behaviour. The highest significant order or sample shift for each disturbance, at 99% confidence, is seen from the lower close-up plot in each of these figures and they are given as the following :

Broke stream incoming concentration -- 7
Broke stream furnish flow -- 15

Kraft stream incoming concentration -- 12
Kraft stream furnish flow -- 20

Groundwood stream incoming concentration -- 17
Groundwood stream furnish flow -- 5

Figure B1-4a Autocorrelation: Broke stream



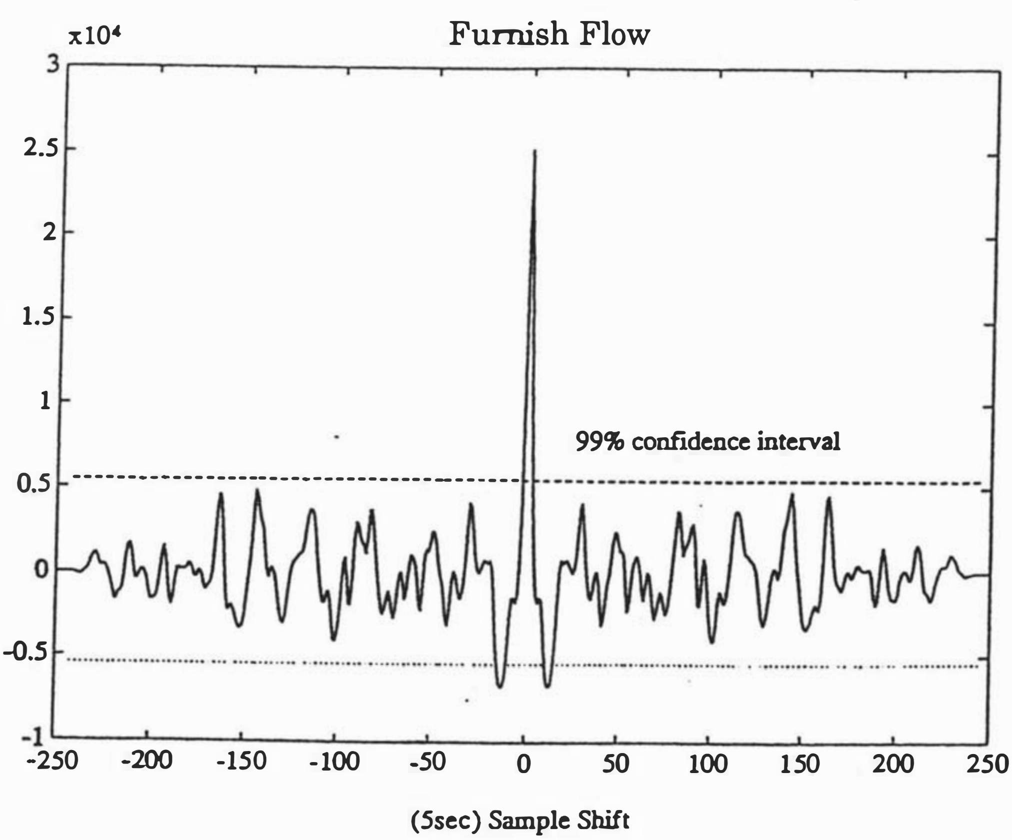
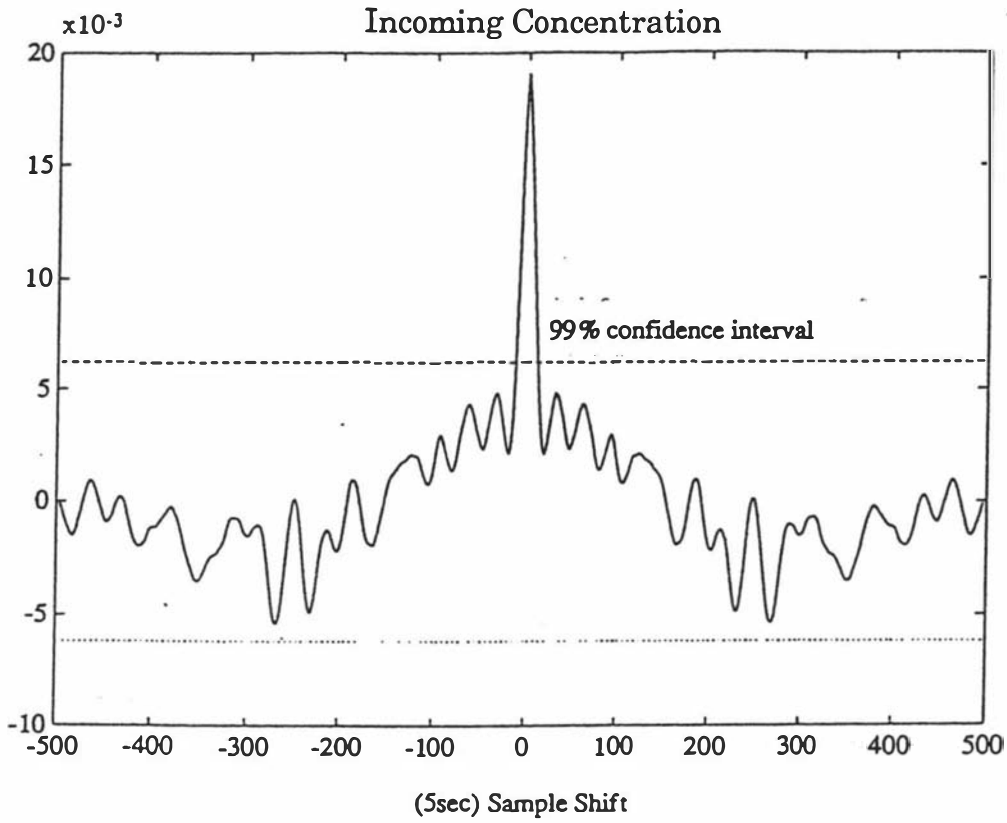


Figure B1-4b Autocorrelation: Kraft stream



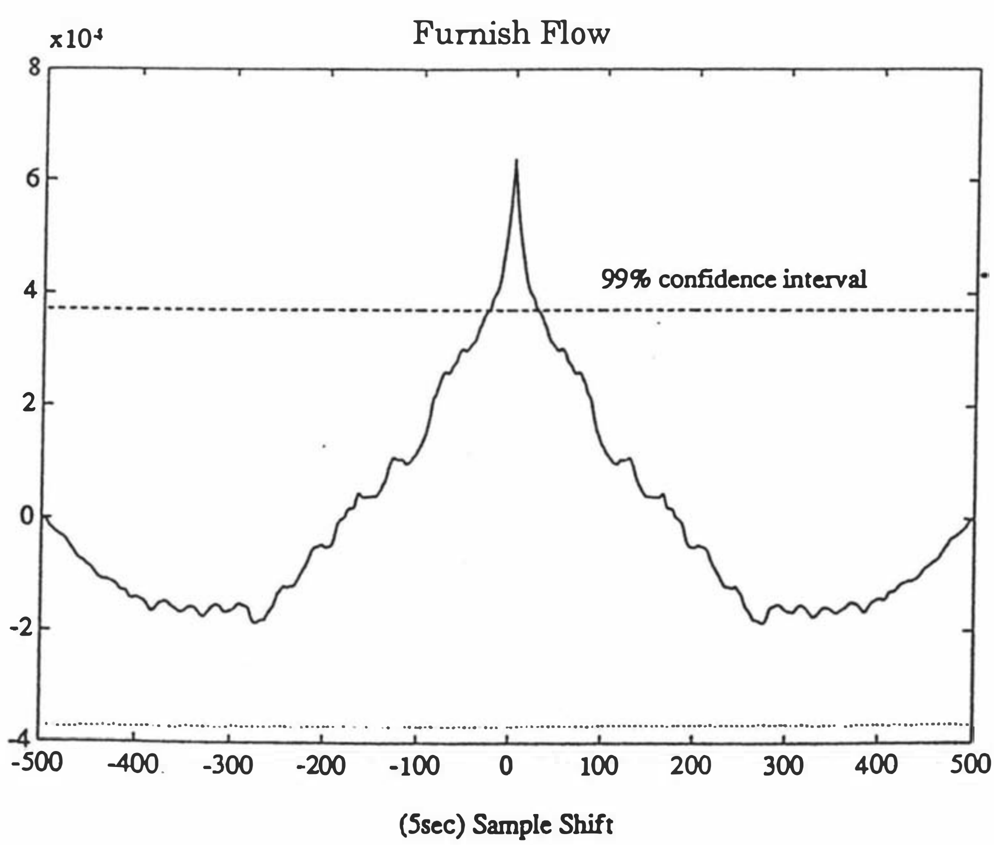
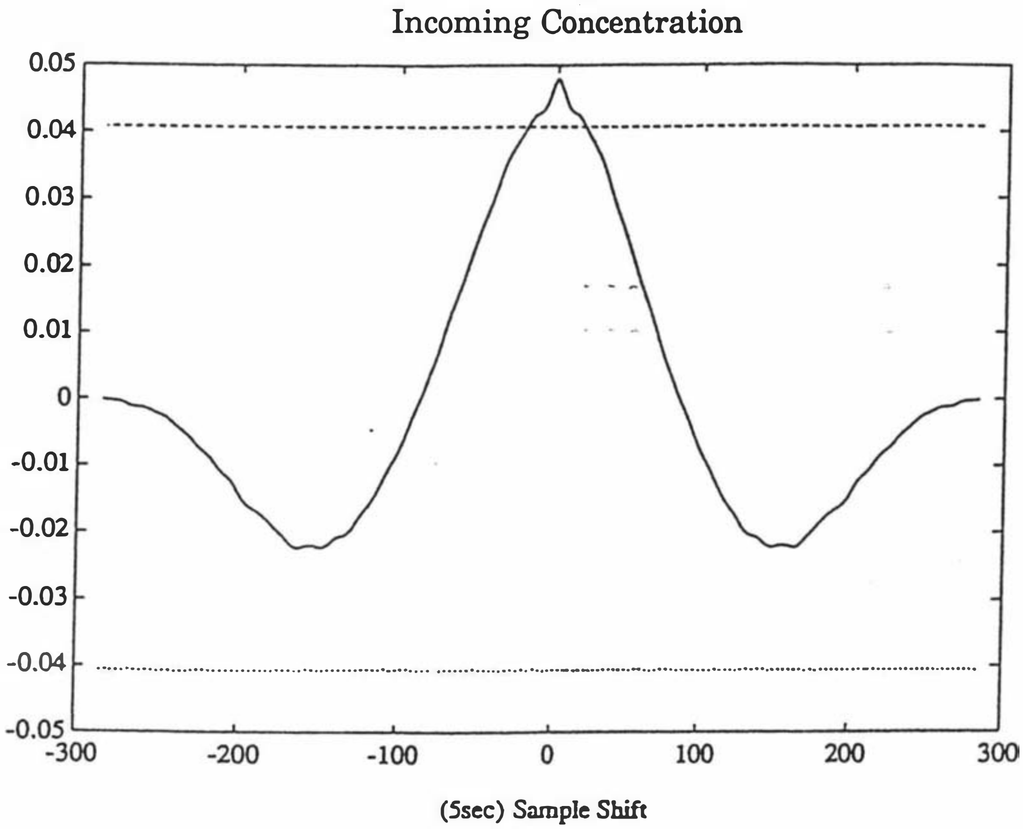
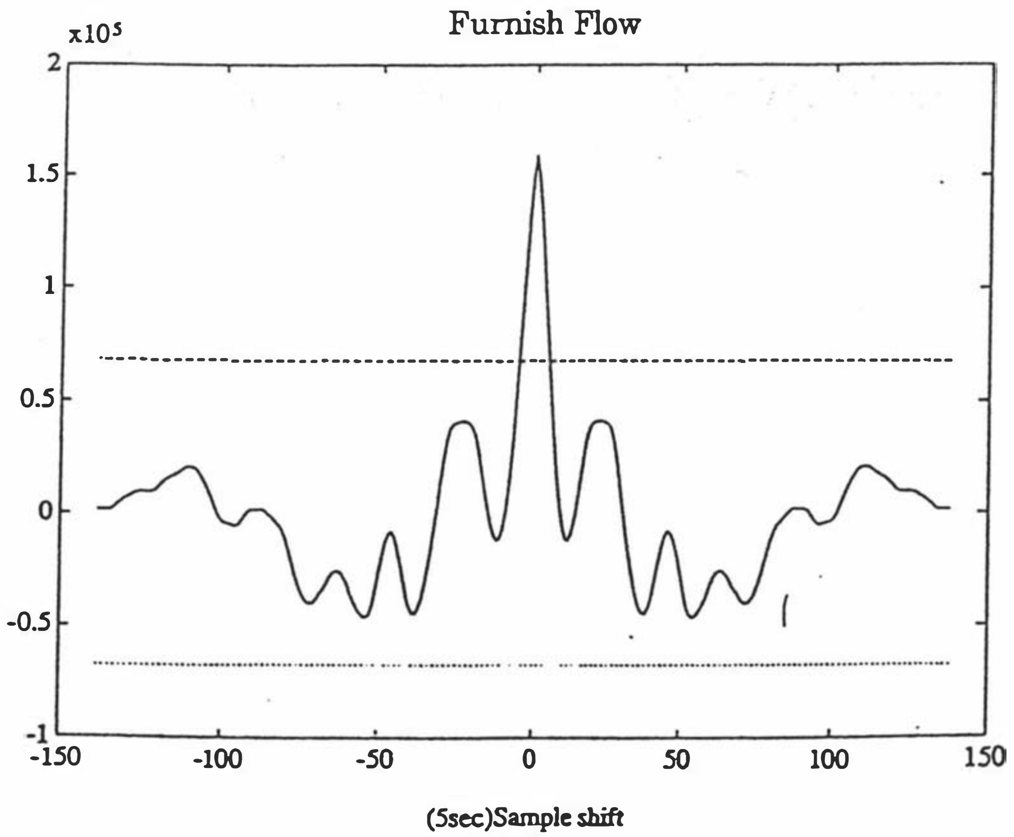


Figure B1-4c Autocorrelation: Groundwood stream





Based on these estimates for the order of the AR part, several ARMA models with different orders for AR and MA parts were fitted to the data using the minimum-prediction-error method.

The Mean-Square-Error (MSE) of models of different order are tabulated in part (a) of Tables B1-2 to B1-7. (Note that an ARMA model with zero order MA part is equivalent to an AR model). All these tables show that the MSE decreases rapidly when the order of the AR part increases to two, but further increase of the order of the AR part gives only small decrease in MSE. Also, the use of MA orders higher than zero does not result in a significant decrease in MSE when the orders of the AR part are two or more. Therefore it can be concluded that a pure AR model (ie. with zero order MA part) of order two or higher is appropriate for all the three streams.

The Final-Prediction-Error, FPE, (Ljung 1987) criteria which penalizes the use of insignificant higher orders is given in part (b) of Tables B1-2 to B1-7. It can be observed that the second order AR model results in a minimum or near minimum value of FPE in all these tables. This suggests the choice of a second order AR model rather than a higher order model for the sake of simplicity.

As a consequence of these results it was decided that the adequate and appropriate models for the two disturbances, that is the incoming concentration and furnish flow, in all three streams were second order autoregressive models with a random and uncorrelated white noise driving term, i.e. a second order AR model. Thus the deviation variable disturbance models for each of these components of the furnish are of the form :

$$c_i(k+1) = V_{c1}.c_i(k) + V_{c2}.c_i(k-1) + w_c(k) \qquad \text{(B1-23)}$$

$$f_o(k+1) = V_{f1}.f_o(k) + V_{f2}.f_o(k-1) + w_f(\mathrm{k}) \qquad \text{(B1-24)}$$

where V_{c1}, V_{c2}, V_{f1} and V_{f2} are constants determined by the modelling, and w_c(k) and w_f(k) represent respectively the white noise driving terms for the incoming concentration model and the furnish flow model.

Table B1-2a MSE Of ARMA Models -- Broke Incoming Concentration.

Order MA / AR	0	1	2	3	4	5
1	9.944e-05	4.158e-05	3.117e-05	2.789e-05	2.797e-05	2.781e-05
2	3.348e-05	2.745e-05	2.720e-05	2.745e-05	2.751e-05	2.749e-05
3	2.824e-05	2.736e-05	2.741e-05	2.685e-05	2.742e-05	2.713e-05
4	2.746e-05	2.747e-05	2.745e-05	2.755e-05	2.705e-05	2.726e-05
5	2.759e-05	2.761e-05	2.708e-05	2.678e-05	2.702e-05	2.747e-05
6	2.772e-05	2.759e-05	2.657e-05	2.738e-05	2.662e-05	2.706e-05
7	2.679e-05	2.679e-05	2.701e-05	2.570e-05	2.532e-05	2.544e-05

Table B1-2b FPE Of ARMA Models -- Broke Incoming Concentration

Order MA / AR	0	1	2	3	4	5
1	1.002e-04	4.227e-05	3.194e-05	2.882e-05	2.914e-05	2.921e-05
2	3.403e-05	2.813e-05	2.810e-05	2.860e-05	2.889e-05	2.911e-05
3	2.894e-05	2.827e-05	2.856e-05	2.820e-05	2.903e-05	2.896e-05
4	2.837e-05	2.862e-05	2.883e-05	2.917e-05	2.887e-05	2.933e-05
5	2.874e-05	2.899e-05	2.867e-05	2.859e-05	2.908e-05	2.981e-05
6	2.911e-05	2.921e-05	2.836e-05	2.947e-05	2.889e-05	2.960e-05
7	2.837e-05	2.860e-05	2.907e-05	2.789e-05	2.770e-05	2.806e-05

Table B1-3a MSE Of ARMA Models -- Broke Furnish Flow

Order AR \ MA	0	1	2	3	4	5
1	4.146e+01	3.241e+01	3.146e+01	3.160e+01	3.171e+01	3.092e+01
2	3.264e+01	3.162e+01	3.150e+01	3.037e+01	3.174e+01	3.047e+01
3	3.181e+01	3.168e+01	3.160e+01	3.027e+01	3.049e+01	3.056e+01
4	3.170e+01	3.169e+01	3.139e+01	3.051e+01	3.114e+01	2.924e+01
5	3.180e+01	3.177e+01	3.143e+01	3.129e+01	3.148e+01	2.950e+01
6	3.167e+01	3.167e+01	3.091e+01	2.945e+01	2.941e+01	2.932e+01
7	3.168e+01	3.008e+01	2.980e+01	3.015e+01	2.995e+01	3.001e+01
8	3.162e+01	3.119e+01	3.023e+01	3.004e+01	3.013e+01	2.944e+01
9	3.058e+01	2.971e+01	2.970e+01	2.944e+01	2.942e+01	2.942e+01
10	3.000e+01	3.001e+01	2.986e+01	2.943e+01	2.895e+01	2.963e+01
11	2.998e+01	2.967e+01	2.931e+01	2.924e+01	2.926e+01	2.923e+01
12	2.991e+01	3.025e+01	2.962e+01	2.928e+01	2.974e+01	2.915e+01
13	2.988e+01	2.953e+01	2.892e+01	2.891e+01	2.875e+01	2.963e+01
14	2.957e+01	2.957e+01	2.928e+01	2.906e+01	2.923e+01	2.924e+01
15	2.971e+01	3.100e+01	2.952e+01	2.860e+01	2.918e+01	2.906e+01

Table B1-3b FPE Of ARMA Models -- Broke Furnish Flow

Order AR \ MA	0	1	2	3	4	5
1	4.180e+01	3.295e+01	3.224e+01	3.265e+01	3.303e+01	3.248e+01
2	3.318e+01	3.241e+01	3.255e+01	3.164e+01	3.333e+01	3.226e+01
3	3.260e+01	3.273e+01	3.291e+01	3.179e+01	3.228e+01	3.262e+01
4	3.275e+01	3.301e+01	3.296e+01	3.231e+01	3.324e+01	3.147e+01
5	3.313e+01	3.336e+01	3.328e+01	3.340e+01	3.388e+01	3.201e+01
6	3.326e+01	3.353e+01	3.299e+01	3.170e+01	3.191e+01	3.208e+01
7	3.354e+01	3.211e+01	3.208e+01	3.272e+01	3.277e+01	3.310e+01
8	3.375e+01	3.357e+01	3.281e+01	3.286e+01	3.323e+01	3.274e+01
9	3.291e+01	3.224e+01	3.249e+01	3.248e+01	3.272e+01	3.298e+01
10	3.255e+01	3.283e+01	3.294e+01	3.273e+01	3.246e+01	3.349e+01
11	3.280e+01	3.273e+01	3.260e+01	3.279e+01	3.307e+01	3.332e+01
12	3.299e+01	3.364e+01	3.321e+01	3.310e+01	3.389e+01	3.350e+01
13	3.323e+01	3.312e+01	3.269e+01	3.296e+01	3.304e+01	3.433e+01
14	3.316e+01	3.343e+01	3.337e+01	3.339e+01	3.386e+01	3.416e+01
15	3.358e+01	3.533e+01	3.392e+01	3.314e+01	3.409e+01	3.423e+01

Table B1-4a MSE Of ARMA Models -- Kraft Incoming Concentration

Order MA / AR	0	1	2	3	4	5
1	1.197e-06	9.970e-07	9.583e-07	9.436e-07	9.348e-07	9.257e-07
2	9.340e-07	9.092e-07	8.989e-07	9.010e-07	8.994e-07	8.997e-07
3	9.226e-07	9.021e-07	9.013e-07	9.009e-07	8.962e-07	8.986e-07
4	9.144e-07	9.003e-07	8.977e-07	9.015e-07	9.025e-07	8.956e-07
5	9.027e-07	8.997e-07	8.985e-07	8.940e-07	8.939e-07	8.913e-07
6	9.003e-07	8.975e-07	9.086e-07	8.900e-07	8.861e-07	8.711e-07
7	8.969e-07	8.963e-07	8.962e-07	8.892e-07	8.796e-07	8.807e-07
8	8.974e-07	8.969e-07	8.894e-07	8.892e-07	8.793e-07	8.855e-07
9	8.968e-07	8.969e-07	8.908e-07	8.850e-07	8.815e-07	8.814e-07
10	8.970e-07	8.966e-07	8.963e-07	8.915e-07	8.896e-07	8.900e-07
11	8.984e-07	8.875e-07	8.869e-07	8.814e-07	8.943e-07	9.030e-07
12	9.003e-07	8.870e-07	8.874e-07	8.868e-07	8.745e-07	8.761e-07

Table B1-4b FPE Of ARMA Models -- Kraft Incoming Concentration

Order MA / AR	0	1	2	3	4	5
1	1.202e-06	1.005e-06	9.699e-07	9.589e-07	9.538e-07	9.483e-07
2	9.415e-07	9.203e-07	9.134e-07	9.193e-07	9.214e-07	9.254e-07
3	9.338e-07	9.167e-07	9.196e-07	9.230e-07	9.218e-07	9.280e-07
4	9.292e-07	9.186e-07	9.197e-07	9.273e-07	9.320e-07	9.286e-07
5	9.211e-07	9.217e-07	9.241e-07	9.232e-07	9.269e-07	9.279e-07
6	9.223e-07	9.231e-07	9.384e-07	9.229e-07	9.225e-07	9.106e-07
7	9.225e-07	9.256e-07	9.293e-07	9.257e-07	9.194e-07	9.243e-07
8	9.268e-07	9.300e-07	9.259e-07	9.294e-07	9.228e-07	9.331e-07
9	9.299e-07	9.337e-07	9.311e-07	9.288e-07	9.289e-07	9.324e-07
10	9.338e-07	9.372e-07	9.407e-07	9.394e-07	9.411e-07	9.454e-07
11	9.391e-07	9.315e-07	9.345e-07	9.325e-07	9.500e-07	9.631e-07
12	9.449e-07	9.346e-07	9.389e-07	9.420e-07	9.326e-07	9.381e-07

Table B1-5a MSE Of ARMA Models -- Kraft Furnish Flow

Order AR \ MA	0	1	2	3	4	5
1	5.211e+00	4.826e+00	4.827e+00	4.816e+00	4.797e+00	4.735e+00
2	4.835e+00	4.829e+00	4.739e+00	4.643e+00	4.645e+00	4.704e+00
3	4.818e+00	4.648e+00	4.654e+00	4.659e+00	4.633e+00	4.639e+00
4	4.799e+00	4.645e+00	4.645e+00	4.633e+00	4.644e+00	4.658e+00
5	4.787e+00	4.731e+00	4.708e+00	4.635e+00	4.639e+00	4.632e+00
6	4.746e+00	4.679e+00	4.612e+00	4.624e+00	4.645e+00	4.605e+00
7	4.747e+00	4.687e+00	4.676e+00	4.660e+00	4.655e+00	4.727e+00
8	4.713e+00	4.712e+00	4.704e+00	4.696e+00	4.607e+00	4.636e+00
9	4.722e+00	4.724e+00	4.682e+00	4.683e+00	4.657e+00	4.689e+00
10	4.728e+00	4.678e+00	4.679e+00	4.661e+00	4.623e+00	4.647e+00
11	4.732e+00	4.676e+00	4.725e+00	4.700e+00	4.625e+00	4.626e+00
12	4.732e+00	4.677e+00	4.692e+00	4.647e+00	4.637e+00	4.632e+00
13	4.735e+00	4.779e+00	4.598e+00	4.752e+00	4.642e+00	4.602e+00
14	4.735e+00	4.696e+00	4.682e+00	4.625e+00	4.610e+00	4.605e+00
15	4.679e+00	4.691e+00	4.639e+00	4.626e+00	4.616e+00	4.607e+00

Table B1-5b FPE Of ARMA Models – Kraft Furnish Flow

Order AR \ MA	0	1	2	3	4	5
1	5.232e+00	4.865e+00	4.885e+00	4.894e+00	4.895e+00	4.850e+00
2	4.874e+00	4.887e+00	4.816e+00	4.738e+00	4.759e+00	4.839e+00
3	4.877e+00	4.724e+00	4.748e+00	4.773e+00	4.765e+00	4.791e+00
4	4.877e+00	4.739e+00	4.758e+00	4.766e+00	4.796e+00	4.830e+00
5	4.884e+00	4.847e+00	4.843e+00	4.787e+00	4.810e+00	4.823e+00
6	4.862e+00	4.813e+00	4.763e+00	4.795e+00	4.836e+00	4.814e+00
7	4.882e+00	4.841e+00	4.849e+00	4.851e+00	4.866e+00	4.961e+00
8	4.867e+00	4.885e+00	4.897e+00	4.909e+00	4.835e+00	4.885e+00
9	4.896e+00	4.918e+00	4.894e+00	4.914e+00	4.907e+00	4.961e+00
10	4.923e+00	4.889e+00	4.911e+00	4.911e+00	4.891e+00	4.936e+00
11	4.946e+00	4.908e+00	4.979e+00	4.973e+00	4.913e+00	4.934e+00
12	4.967e+00	4.928e+00	4.964e+00	4.936e+00	4.945e+00	4.960e+00
13	4.990e+00	5.056e+00	4.884e+00	5.068e+00	4.971e+00	4.948e+00
14	5.010e+00	4.989e+00	4.994e+00	4.953e+00	4.957e+00	4.971e+00
15	4.971e+00	5.003e+00	4.968e+00	4.974e+00	4.984e+00	4.994e+00

Table B1-6a MSE Of ARMA Models -- Groundwood Incoming Concentration

Order MA / AR	0	1	2	3	4	5
1	1.430e-06	1.048e-06	9.408e-07	9.390e-07	9.417e-07	9.264e-07
2	9.486e-07	9.470e-07	9.361e-07	9.383e-07	9.413e-07	9.403e-07
3	9.474e-07	9.316e-07	9.390e-07	9.330e-07	9.361e-07	9.427e-07
4	9.454e-07	9.441e-07	9.279e-07	9.316e-07	9.398e-07	8.951e-07
5	9.454e-07	9.319e-07	9.290e-07	9.281e-07	9.204e-07	8.843e-07
6	9.477e-07	9.218e-07	9.180e-07	9.232e-07	9.175e-07	8.938e-07
7	9.078e-07	8.973e-07	8.902e-07	8.902e-07	8.766e-07	8.767e-07
8	9.091e-07	9.049e-07	8.944e-07	8.891e-07	8.879e-07	8.870e-07
9	8.753e-07	8.740e-07	8.740e-07	8.718e-07	8.588e-07	8.557e-07
10	8.746e-07	8.709e-07	8.704e-07	8.699e-07	8.619e-07	8.675e-07
11	8.774e-07	8.736e-07	8.736e-07	8.460e-07	8.589e-07	8.619e-07
12	8.739e-07	8.733e-07	8.541e-07	8.624e-07	8.589e-07	8.489e-07
13	8.753e-07	8.726e-07	8.673e-07	8.638e-07	8.654e-07	8.360e-07
14	8.608e-07	8.482e-07	8.451e-07	8.458e-07	8.365e-07	8.208e-07
15	8.467e-07	8.465e-07	8.471e-07	8.467e-07	8.095e-07	8.226e-07
16	8.491e-07	8.469e-07	8.459e-07	8.400e-07	8.447e-07	8.258e-07
17	8.485e-07	8.478e-07	8.477e-07	8.463e-07	8.307e-07	8.217e-07

Table B1-6b FPE Of ARMA Models -- Groundwood Incoming Concentration

Order MA / AR	0	1	2	3	4	5
1	1.440e-06	1.063e-06	9.608e-07	9.657e-07	9.753e-07	9.663e-07
2	9.620e-07	9.671e-07	9.628e-07	9.718e-07	9.818e-07	9.877e-07
3	9.676e-07	9.582e-07	9.726e-07	9.731e-07	9.832e-07	9.972e-07
4	9.723e-07	9.779e-07	9.678e-07	9.785e-07	9.941e-07	9.535e-07
5	9.792e-07	9.719e-07	9.757e-07	9.817e-07	9.804e-07	9.486e-07
6	9.884e-07	9.682e-07	9.710e-07	9.834e-07	9.843e-07	9.656e-07
7	9.536e-07	9.491e-07	9.482e-07	9.550e-07	9.470e-07	9.538e-07
8	9.616e-07	9.639e-07	9.595e-07	9.605e-07	9.659e-07	9.718e-07
9	9.324e-07	9.376e-07	9.441e-07	9.485e-07	9.409e-07	9.441e-07
10	9.382e-07	9.409e-07	9.469e-07	9.531e-07	9.509e-07	9.639e-07
11	9.479e-07	9.504e-07	9.571e-07	9.334e-07	9.544e-07	9.644e-07
12	9.507e-07	9.568e-07	9.423e-07	9.583e-07	9.611e-07	9.566e-07
13	9.589e-07	9.627e-07	9.637e-07	9.665e-07	9.752e-07	9.487e-07
14	9.498e-07	9.425e-07	9.457e-07	9.531e-07	9.493e-07	9.380e-07
15	9.407e-07	9.472e-07	9.546e-07	9.609e-07	9.251e-07	9.468e-07
16	9.501e-07	9.543e-07	9.600e-07	9.600e-07	9.722e-07	9.572e-07
17	9.561e-07	9.621e-07	9.688e-07	9.741e-07	9.629e-07	9.592e-07

Table B1-7a MSE Of ARMA Models -- Groundwood Furnish Flow

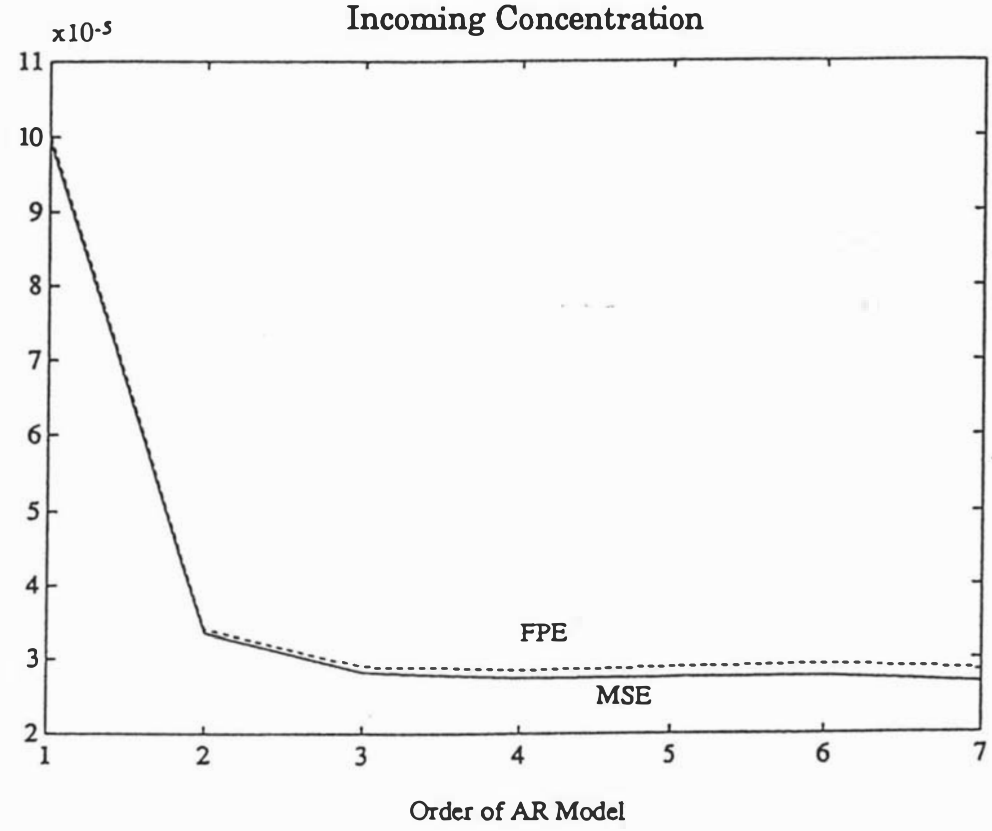
Order AR \ MA	0	1	2	3	4	5
1	1.222e+02	8.524e+01	8.417e+01	8.165e+01	8.055e+01	8.140e+01
2	8.169e+01	8.153e+01	8.015e+01	8.066e+01	8.051e+01	8.122e+01
3	8.245e+01	8.122e+01	8.070e+01	7.951e+01	8.057e+01	8.118e+01
4	8.167e+01	8.091e+01	7.532e+01	7.599e+01	7.683e+01	7.658e+01
5	8.119e+01	8.085e+01	7.874e+01	7.508e+01	7.574e+01	7.582e+01

Table B1-7b FPE Of ARMA Models – Groundwood Furnish Flow

Order AR \ MA	0	1	2	3	4	5
1	1.239e+02	8.773e+01	8.788e+01	8.649e+01	8.656e+01	8.875e+01
2	8.407e+01	8.513e+01	8.490e+01	8.668e+01	8.778e+01	8.984e+01
3	8.609e+01	8.603e+01	8.673e+01	8.669e+01	8.911e+01	9.110e+01
4	8.651e+01	8.695e+01	8.212e+01	8.405e+01	8.622e+01	8.718e+01
5	8.725e+01	8.815e+01	8.709e+01	8.425e+01	8.623e+01	8.758e+01

The adequacy of second order models can be seen clearly from the plots of AR model order against Mean-Square-prediction-Error (MSE) and Final-Prediction-Error (FPE) in Figures B1-5a to B1-5c.

Figure B1-5a MSE & FPE vs AR Model Order: Broke Stream



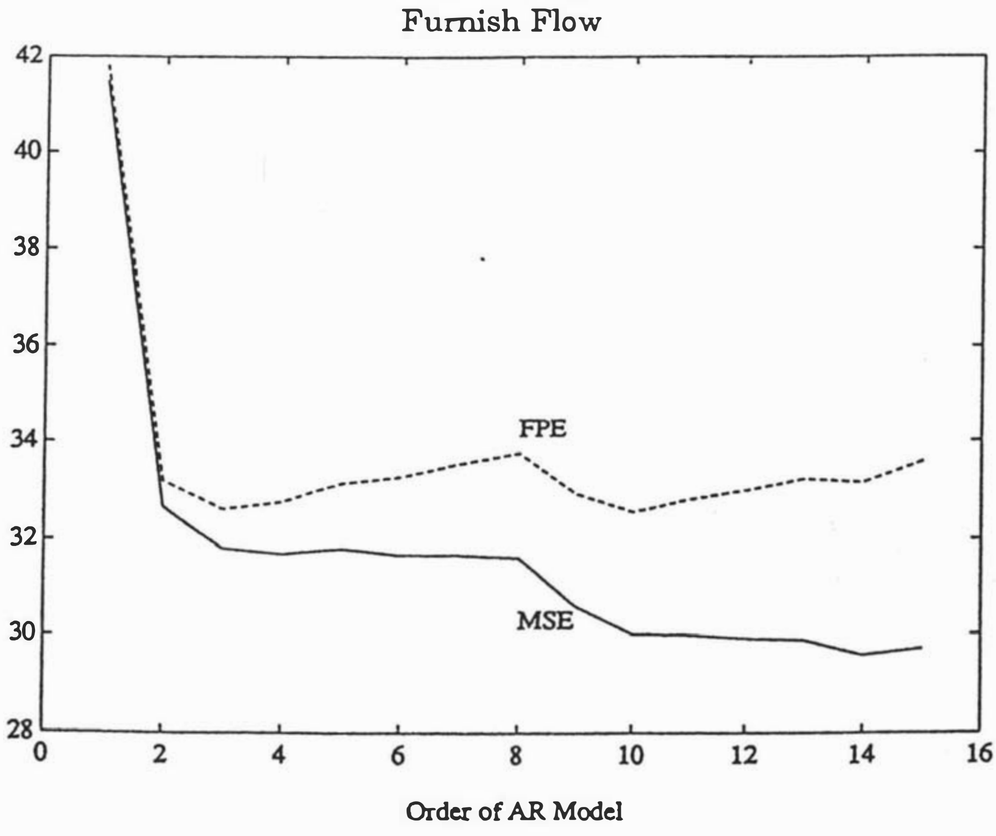
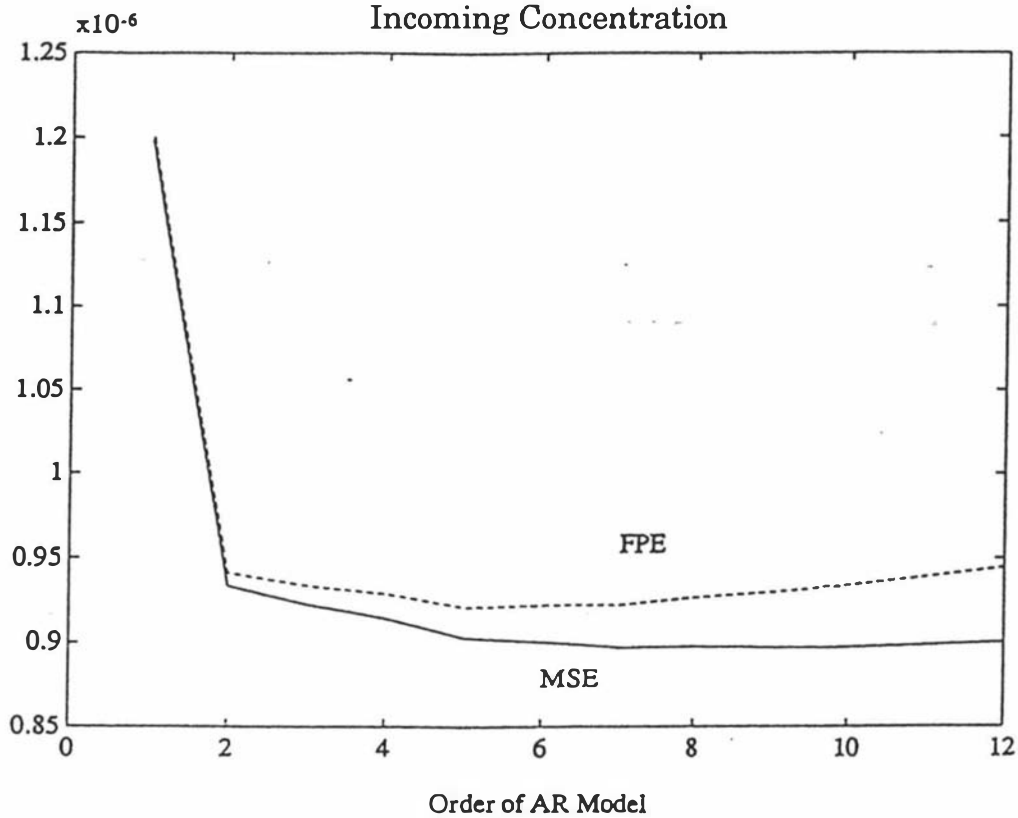


Figure B1-5b MSE & FPE vs AR Model Order: Kraft Stream



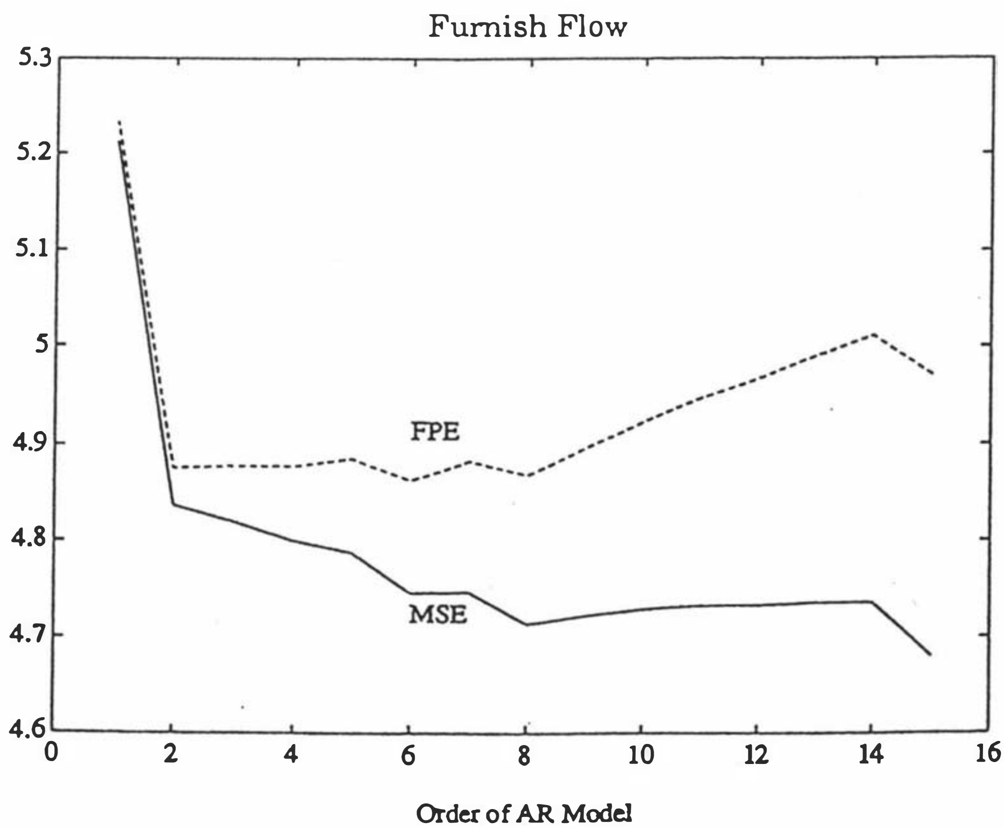
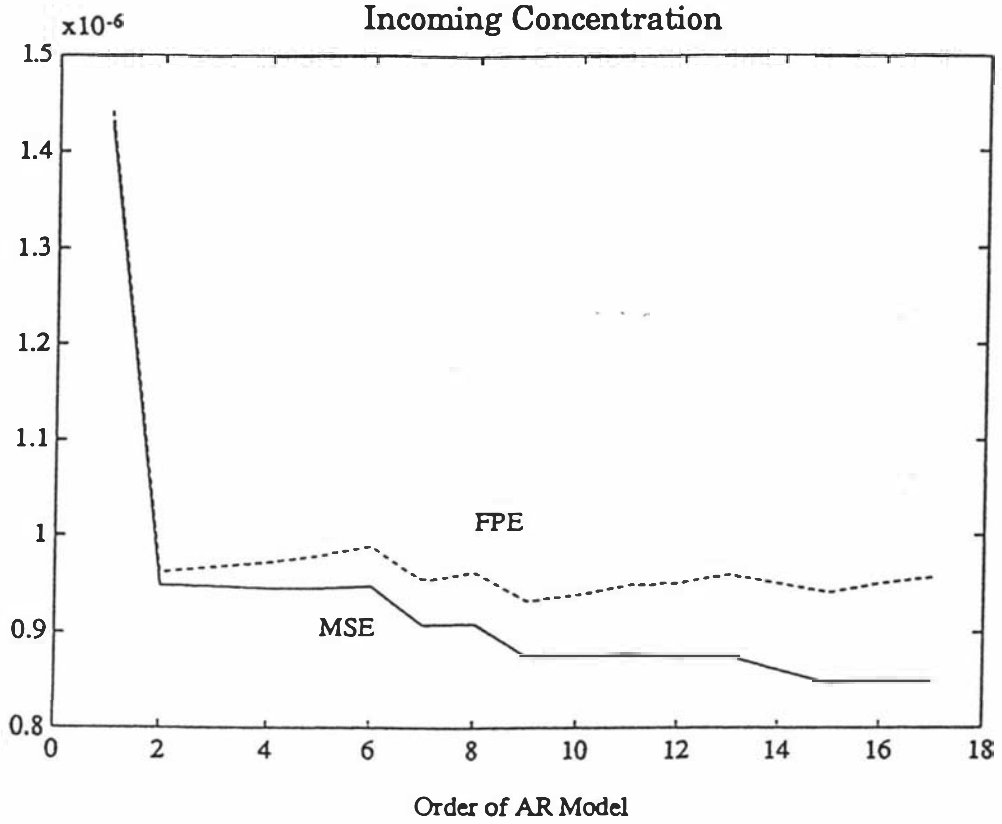
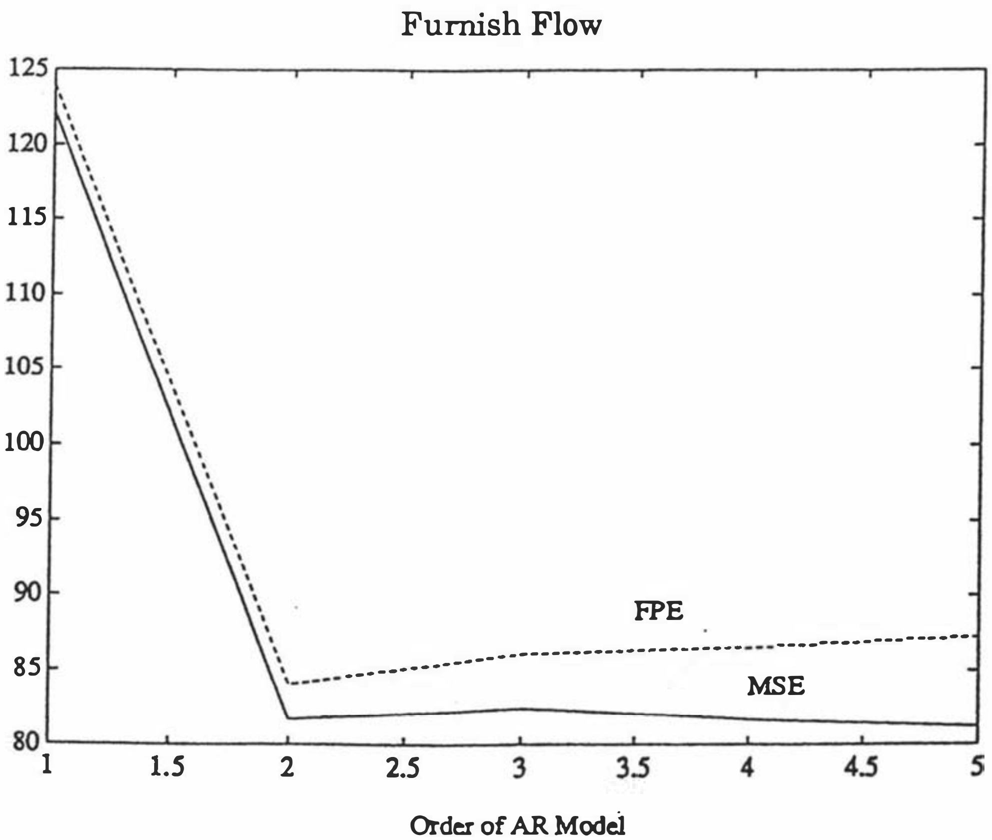


Figure B1-5c MSE & FPE vs AR Model Order: Groundwood Stream





Due to the fact that, p, the effect of variations in dilution flow pressure on pulp flow, could not be measured at the time of this study, the model of the effect of pressure variation on dilution is estimated to have the same form as the furnish flow :

$$p(k+1) = V_{p1}.p(k) + V_{p2}.p(k\text{-}1) + w_p(k) \qquad \text{(B1-25)}$$

where $w_p(k)$ represents the white noise driving term of the model, and the constants V_{p1} and V_{p2}, in the absence of better information, have been taken as V_{f1} and V_{f2} respectively.

The coefficients and steady state values for each of the models are given in Table B1-8.

Table B1-8 Parameters of Disturbances' Model

Incoming Cs	V_{c1}	V_{c2}	**Steady State**
Broke	1.775	-0.812	3.863 %
Kraft	1.441	-0.464	3.255 %
Groundwood	1.565	-0.573	3.438 %

Furnish flow	V_{f1}	V_{f2}	**Steady State**
Broke	1.110	-0.423	501.1 gpm
Kraft	1.185	-0.203	359.1 gpm
Groundwood	1.548	-0.586	1431.6 gpm

B1.5.4 Stability of Disturbance Models

As given in Table B1-8, the deviation models of the concentration disturbances are :

Broke Cs : $y(k) - 1.775\,y(k-1) + 0.812\,y(k-2) = e(t)$
Flow : $y(k) - 1.110\,y(k-1) + 0.423\,y(k-2) = e(t)$

Kraft Cs : $y(k) - 1.441\,y(k-1) + 0.464\,y(k-2) = e(t)$
Flow : $y(k) - 1.185\,y(k-1) + 0.203\,y(k-2) = e(t)$

Groundwood Cs : $y(k) - 1.565\,y(k-1) + 0.573\,y(k-2) = e(t)$
Flow : $y(k) - 1.548\,y(k-1) + 0.586\,y(k-2) = e(t)$

Using these equations, the corresponding eigenvalues can be found and are given in Table B1-9. None of the absolute magnitudes of these eigenvalues are larger than one, and so all these models are stable (Ogata 1987).

Table B1-9 Eigenvalues of Disturbances' Model

	Eigenvalues	Absolute magnitude
Broke Cs	0.885 + 0.181i	0.903
	0.885 - 0.181i	0.903
Broke Flow	0.566 + 0.381i	0.683
	0.566 - 0.381i	0.683
Kraft Cs	0.953	0.953
	0.493	0.493
Kraft Flow	0.961	0.961
	0.300	0.300
Groundwood Cs	0.983	0.983
	0.589	0.589
Groundwood Flow	0.886	0.886
	0.661	0.661

B1.6 DISCRETE-TIME STATE SPACE DESCRIPTION

For the purpose of system analysis and controller design, it is convenient to have the discrete-time models of the control system and the system disturbances in the form of a state space description (Coughanowr and Koppel 1983, Ogata 1987). For simplicity and the reasons discussed in Section B1.5.2, the following assumption is made for the subsequent state-space models.

Assumption B1-1

The pure delay, L, is assumed to be an exact multiple of the sampling interval, T, that is:

$$L = mT \qquad , m = 0,1,2,3\ldots \qquad \blacklozenge$$

B1.6.1 Disturbances State-Space Description

Defining the disturbances state vector, x_n :

$$x_n(k) = \begin{bmatrix} c_i(k) \\ c_i(k\text{-}1) \\ \vdots \\ c_i(k\text{-}m+1) \\ f_o(k) \\ f_o(k\text{-}1) \\ \vdots \\ f_o(k\text{-}m+1) \\ p(k) \\ p(k\text{-}1) \\ \vdots \\ p(k\text{-}m+1) \end{bmatrix} \quad , x_n(k) \in \mathrm{R}^{3m} \tag{B1-26}$$

and the white noise driving vector, w :

$$w(k) = \begin{bmatrix} w_c(k) \\ w_f(k) \\ w_p(k) \end{bmatrix} \quad , \ w(k) \in \mathrm{R}^3 \tag{B1-27}$$

the disturbance models described in Section B1.5.3 can then be combined and written in state-space form as :

$$x_n(k+1) = \begin{bmatrix} c_i(k+1) \\ c_i(k) \\ \vdots \\ c_i(k-m) \\ f_o(k+1) \\ f_o(k) \\ \vdots \\ f_o(k-m) \\ p(k+1) \\ p(k) \\ \vdots \\ p(k-m) \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{N_{ci}} & \emptyset & \emptyset \\ \emptyset & \mathrm{N_f} & \emptyset \\ \emptyset & \emptyset & \mathrm{N_p} \end{bmatrix} x_n(k) + \begin{bmatrix} \mathrm{E_{ci}} \\ \mathrm{E_{fo}} \\ \mathrm{E_p} \end{bmatrix} + w(k) \qquad \text{(B1-28)}$$

where

$$\mathrm{N_{ci}} = \begin{bmatrix} \mathrm{V_{c1}} & \mathrm{V_{c2}} & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad \text{dimension } m \times m$$

$$\mathrm{N_{fo}} = \begin{bmatrix} \mathrm{V_{f1}} & \mathrm{V_{f2}} & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad \text{dimension } m \times m$$

$$\mathrm{N_{p}} = \begin{bmatrix} \mathrm{V_{p1}} & \mathrm{V_{p2}} & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad \text{dimension } m \times m$$

$$\mathrm{E_{ci}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \quad \text{dimension } m \times 3$$

$$E_{fo} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \quad \text{dimension } m \text{ x } 3$$

$$E_{p} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \quad \text{dimension } m \text{ x } 3$$

and Ø is the zero matrix of appropriate dimension.

B1.6.2 Augmented System

An augmented state-space description incorporating all the disturbances is chosen in order to allow for the design of control strategies including feedforward elements,

Defining two more state vectors, the concentration system state, $x_s(k)$, and control-input state vector, $x_c(k)$ as :

$$x_s(k) = c_m(k) \qquad , x_s(k) \in \mathrm{R}^1 \qquad \text{(B1-29)}$$

$$x_c(k) = \begin{bmatrix} f_d(k\text{-}1) \\ f_d(k\text{-}2) \\ \vdots \\ f_d(k\text{-}m+1) \end{bmatrix} \qquad , x_c(k) \in \mathrm{R}^{m\text{-}1} \qquad \text{(B1-30)}$$

a possible augmented system state vector, $x(k)$, is thus,

$$x(k) = \begin{bmatrix} x_s(k) \\ x_n(k) \\ x_c(k) \end{bmatrix} \qquad , x(k) \in \mathrm{R}^{4m} \qquad \text{(B1-31)}$$

The dynamics of the system can then be described by an augmented state-space system description of the form,

$$x(k+1) = \begin{bmatrix} \emptyset & A_{ci} & A_{fo} & A_p & A_{fd} \\ \emptyset & N_{ci} & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & N_{fo} & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & N_p & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & N_{fd} \end{bmatrix} x(k)$$

$$+ \begin{bmatrix} \emptyset \\ \emptyset \\ \emptyset \\ \emptyset \\ B_{fd} \end{bmatrix} f_d(k) + \begin{bmatrix} \emptyset \\ E_{ci} \\ E_{fo} \\ E_p \\ \emptyset \end{bmatrix} w(k)$$

$$= A \,.\, x(k) + B \,.\, f_d(k) + E \,.\, w(k) \qquad \text{(B1-32)}$$

$x(k) \in R^{4m}$; $f_d(k) \in R^1$; $w(k) \in R^3$

where N_{ci}, N_{fo}, N_p, E_{ci}, E_{fo} and E_p are defined as in the previous section and

$A_{ci} = [\ 0 \dots 0\ \ k_1\ ]$, dimension $1 \times m$

$A_{fo} = [\ 0 \dots 0\ \ k_2\ ]$, $1 \times m$

$A_p = [\ 0 \dots 0\ \ k_3\ ]$, $1 \times m$

$A_{fd} = [\ 0 \dots 0\ \ k_3\ ]$, $1 \times (m\text{-}1)$

$$N_{fd} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad , (m\text{-}1) \times (m\text{-}1)$$

$$B_{fd} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad , (m\text{-}1) \times 1$$

The parameter k_1, k_2 and k_3 are the coefficients of the linearized system equation defined in Equation B1-8. The values of these parameters are determined totally by the steady state values of the system.

Following this state-space description, the measurement vector, $y(k)$, is given by,

$$y(k) = \mathrm{C}\, x(k) + \mathrm{D}\, v(k) \tag{B1-33}$$

where, $v(k)$, is a vector of measurement noises. The structures of the matrices C and D are dependent on the measurement scheme employed.

B1.6.3 Controllability

For the a system described by Equation (B1-32):

$$x(k+1) = \mathrm{A}\, x(k) + \mathrm{B}\, f_d(k) + \mathrm{E}\, w(k)$$

the controllability matrix (Ogata 1987), P_c is given by :

$$P_c = \left[\; \mathrm{B} \mid \mathrm{A}\,\mathrm{B} \mid \mathrm{A}^2\,\mathrm{B} \mid \ldots\ldots \mid \mathrm{A}^{4m-1}\,\mathrm{B} \;\right]$$

Consequently the controllability of our system is given by :

$$
Pc = \begin{bmatrix}
\emptyset & A_{fd}\cdot B_{fd} & A_{fd}\cdot N_{fd}\cdot B_{fd} & \cdots & A_{fd}\cdot N_{fd}^{4m-2}\cdot B_{fd} \\
\emptyset & \emptyset & \emptyset & \cdots & \emptyset \\
\emptyset & \emptyset & \emptyset & \cdots & \emptyset \\
\emptyset & \emptyset & \emptyset & \cdots & \emptyset \\
B_{fd} & N_{fd}\cdot B_{fd} & N_{fd}^{2}\cdot B_{fd} & \cdots & N_{fd}^{4m-1}\cdot B_{fd}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & k_3 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}
$$

$\leftarrow \quad m\text{-}1 \quad \rightarrow$

where k_3 is the non-zero parameter given in Table B1-1 for each stream.

The matrix is of rank m. The controllability matrix indicates that the first state and the last (m-1) states are controllable. This controllability characteristics is expected for the augmented system, as the first state corresponds to the original system state, x_s, and the last (m-1) states are the states of control input, x_c. The uncontrollable states correspond to the $3m$ disturbance states, x_n, described in Section B1.5. Since it has been found that the disturbance model is stable, the augmented system is stabilizable (Ogata 1987).

B1.6.4 Observability

For a measurement vector, $y(k)$, and measurement matrix, C, such that:

$$y(k) = C\,x(k)$$

the observability matrix (Ogata 1987), Q_o, of the system described by Equation (B1-32) is given by :

$$Q_o = \begin{bmatrix} C \\ C\cdot A \\ C\cdot A^2 \\ \vdots \\ \vdots \\ C\cdot A^{(4m-1)} \end{bmatrix}$$

Therefore, the observability matrix of the augmented system can be obtained for the six different measurement schemes. Using the computer package MATLAB, it is found that the observability matrices of all the measurement schemes are of full rank. Thus all the states are observable when any of the six measurement schemes is employed or, in other words, the concentration system given by Equation (B1-32) is completely observable. This property of complete observability is important in the design of an optimal multivariable control which will be discussed in detail in the next chapter.

B1.7 SUMMARY

The problem of paper stock concentration control has been outlined in this chapter. The overall concentration control system consists of three separate but structurally identical systems, one for each of the broke, kraft and groundwood streams of paper stock.

Several models involved in these control systems are established. It is found that each of these systems is inherently described by a purely algebraic (or zero order) model. The system model is found to be driven by one controllable input, F_d. It has one output requiring regulation, C_m and three uncontrollable disturbances, C_i, F_m and P. This model is nonlinear and contains a pure delay term, as given by Equation B1-7:

$$C_m(t) = C_i(t\text{-}L)\left[1 - \frac{F_d(t\text{-}L) + P(t\text{-}L)}{F_m(t\text{-}L)}\right]$$

For the purpose of controller design, a linearized discrete-time version of this model is determined in Equation B1-15 as:

$$c_m(k) = k_1 c_i(k-m) + k_2 f_o(k-m) + k_3 p(k-m) + k_3 f_d(k-m)$$

The appropriate model parameters for these system models are tabulated in Table B1-1.

Using measurements from the actual plant, three models were also determined respectively for the three disturbances in each stream. This was achieved with the assistance of a computer-aided parameter estimation package.

It was found that the most suitable models for all these three disturbances were the second order auto-regressive models driven by white noise terms. They have the general form of,

$$y(k+1) = V_1.y(k) + V_2.y(k-1) + w(k)$$

where y is the disturbance variable, w is the white noise driving term and V_1 and V_2 are constant parameters. The appropriate model parameters for these disturbances are tabulated in Table B1-8.

Six possible measurement schemes were also assigned. They consist of different combinations of system disturbances (feedforward) and system output (feedback) measurements. For easy reference, some simple binary codes were assigned to each of these schemes, to indicate which measurements were made.

For the purpose of system analysis and controller design, a state-space model was derived to combine the system model and the disturbance models into a single description.

APPENDIX B2

PERFORMANCE COMPARISON OF PAPER MACHINE STOCK CONCENTRATION CONTROLLERS

A paper presented at 44th Annual General Conference of APPITA (1991), Rotorua, New Zealand. It is also published in *APPITA* vol.45 No.4, July 1992.

PERFORMANCE COMPARISON OF PAPER MACHINE STOCK CONCENTRATION CONTROLLERS

P.C. AUSTIN*, W.H. SIEW**, G.E. SHERSON***

* Senior Lecturer, Production Technology Department, Massey University, Palmerston North, New Zealand.

** Research Student, Production Technology Department, Massey University, Palmerston North, New Zealand.

*** Senior Systems Engineer, Tasman Pulp and Paper Co., Kawerau, New Zealand.

SUMMARY

A New Zealand pulp and paper company wishes to improve its stock concentration control, and has needed therefore to investigate the performance of various alternative control systems. The paper reports the results of a simulation study that formed a part of this investigation.

A model of the stock concentration system contains pure delay elements and dominant disturbances. The pulp and paper company is thus interested in applying deadtime compensation, possible combined with one or more feedforward control loops. In this paper, the performance of six control strategies is simulated, including three-term, deadbeat and optimal multi-variable controllers, each with and without the inclusion of a Smith predictor. For each strategy, a pure feedback control solution is calculated, as well as a number of feedforward-feedback control solutions, depending on the disturbances measured. The work was done with the aid of a commercially marketed computer aided control system design package.

INTRODUCTION

Tasman Pulp and Paper Company is evaluating the performance of consistency controllers in the paper mill stock blending area. It currently uses standard three-term (PID) controllers to regulate the consistency of the individual furnish components, namely groundwood, kraft and broke. The company wishes to install an improved consistency control system, and has needed therefore to investigate the performance of various alternative control systems. The paper reports the results of a simulation study that formed a part of this investigation.

STOCK CONSISTENCY CONTROL AT TASMAN PULP AND PAPER

The three paper machines at Tasman produce newsprint from blends of semi-bleached Kraft pulp, mixed groundwood (refiner plus stone groundwood), and broke. These furnish components are combined in the desired ratios in the blend chests at the wet end of each paper machine. The general stock blending arrangement is illustrated in Figure 1. The objective of the stock blending control is to ratio the components on a dry fibre basis, even though it is the total stock flows (fibre plus water) that are manipulated. The flows are controlled by the blend chest level and are corrected for changes in consistency on an infrequent basis, using manual consistency test results entered by operators (every two or four hours). To maintain a uniform blend it is therefore essential that any variability in stock consistency is minimized through good consistency control.

The consistency of each furnish component is controlled upstream of the blend chest by adding dilution water prior to each stock pump (see Figure 2). Consistency transmitters are located some distance downstream from the dilution points, causing a delay of the order of 25 seconds in the consistency control loop. The consistency controllers used currently are pneumatic two-term (PI) controllers, but on two of the paper machines, all the wet end controls are about to be replaced with a new distributed control system. This provides the perfect opportunity to improve consistency control through the implementation of a more effective control strategy.

Fig. 1 - Schematic layout of stock blending process

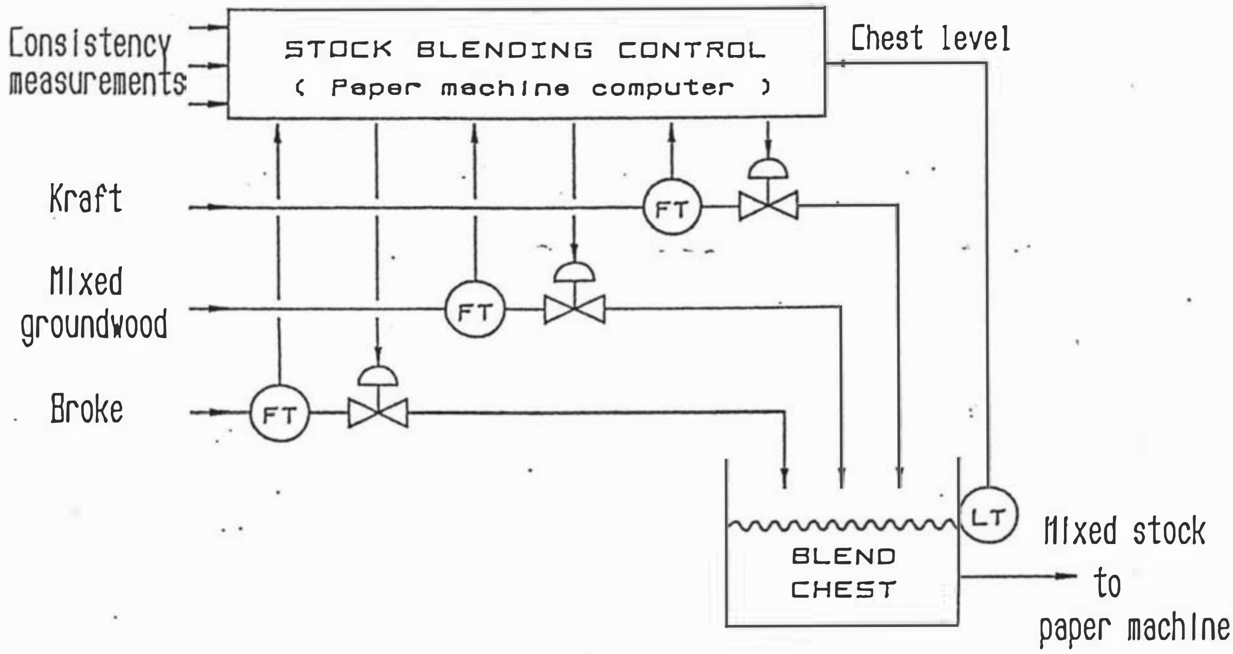
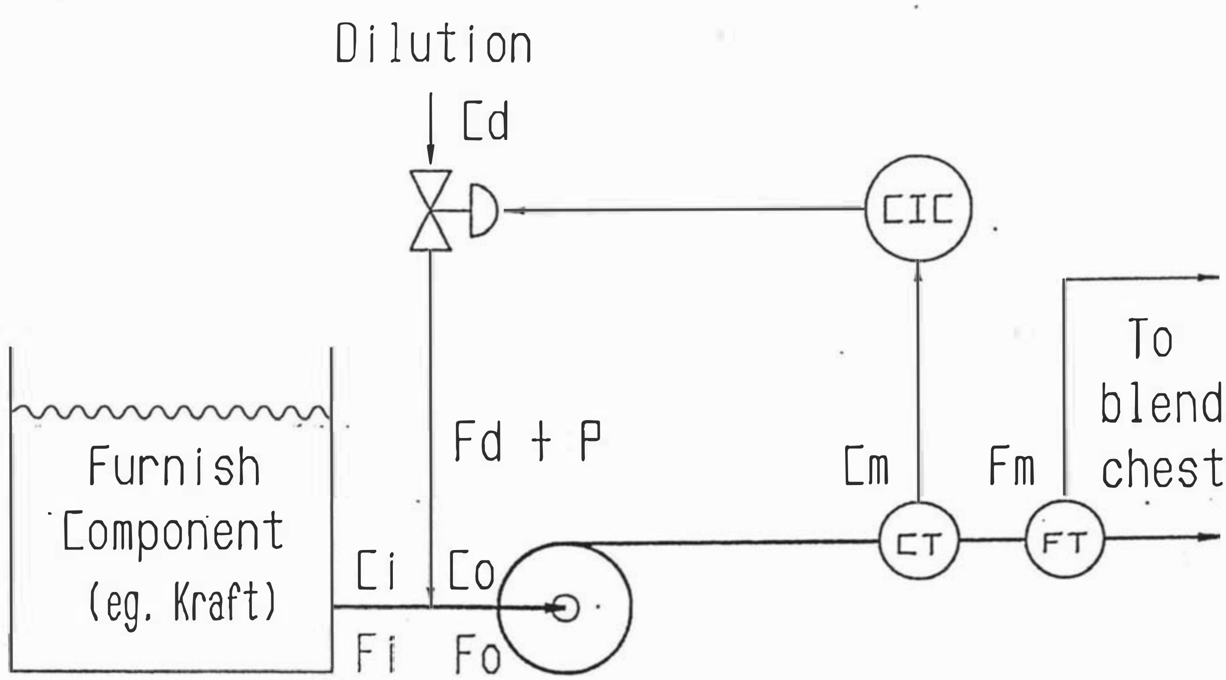


Fig. 2 - Schematic layout of consistency control system



The notation used in Figure 2 and subsequently, is as follows:

C_i = consistency of incoming pulp from stock chest

C_d = consistency of dilution water

C_o = consistency of mixed (furnish) stream at mixing point (as if where were instant perfect mixing)

C_m = measured consistency

F_i = flowrate of incoming pulp

F_o = flowrate of pulp at mixing point.

F_m = measured flowrate

F_d = intended or ideal flowrate of dilution water

P = variation in dilution flow due to changes in pressure

MODELLING STOCK CONSISTENCY

In order to simulate the behaviour of the stock control system and to determine controllers for it, an appropriate model of the system must first be determined. This model is nonlinear. For the design of the controllers, a linearized version of this model is required.

Nonlinear Model

The system illustrated in Figure 2 can be modelled as the mixing of two incompressible fluid streams. The system is thus governed by the following mass balance equations:

$$Fi(t) + Fd(t) + P(t) = Fo(t) \tag{1}$$

$$F_i(t).C_i(t) + [\ F_d(t) + P(t)\ ]\ C_d(t) = F_o(t).C_o(t) \tag{2}$$

The first is a balance of volumetric flowrates and the second is a balance of pulp stock. Assuming the consistency of the dilution water is negligible, that is $C_d = 0$, these equations yield a relationship between the controlled output, C_o, the control input, Fd, and the system disturbances, C_i, F_o and P:

$$Co(t) = Ci(t).(1 - (Fd(t) + P(t))/ Fo(t)) \tag{3}$$

As the consistency is measured some distance downstream and because the pulp fluid has been assumed incompressible, the measured consistency and flow are given by :

$$Cm(t) = Co(t-L) \tag{4}$$

$$Fm(t) = Fo(t) \tag{5}$$

where L is the pure time delay between the mixing point and the measuring point. Substituting equations (4) and (5) into equation (3), gives an equation for the measured consistency:

$$Cm(t) = Ci(t-L).[1 - (Fd(t-L) + P(t-L))/ Fm(t-L)] \tag{6}$$

Linearized discrete model

Equation (6) describing the behaviour of measured consistency is nonlinear due to the multiplication and division of variables in the equation. Most methods of controller design presume the system equations have been linearized. To linearize equation (6), assume first that the pure time delay L is an exact integer multiple of the sampling interval, T, that is $L = mT$. Then the actual values of the variables (upper case symbols) are written as deviations around the steady state values of these variables. Thus deviation variables (lower case symbols) are defined as follows :

$c_i(k) = C_i(k) - C_{iss}$, C_{iss} = steady state value of $C_i(k)$

$c_m(k) = C_m(k) - C_{mss}$, C_{mss} = steady state value of $C_m(k)$

$f_o(k) = F_o(k) - F_{oss}$, F_{oss} = steady state value of $F_o(k)$

$f_d(k) = F_d(k) - F_{dss}$, F_{dss} = steady state value of $F_d(k)$

$p(k) = P(k) - P_{ss}$, P_{ss} = steady state value of $P(k)$

The use of these variables, in conjuction with suitable Taylor's Series expansions, see [1], enables the following discretized linearized system description to be determined :

$$c_m(k) = k_1.c_i(k-m) + k_2.f_o(k-m) + k_3.p(k-m) + k_3.f_d(k-m) \tag{7}$$

where k is the discrete time variable, that is k stands for kT seconds, and:

$$k_1 = 1 - (F_{dss} + P_{ss}) / F_{oss} \tag{8}$$

$$k_2 = C_{iss} (F_{dss} + P_{ss}) / F^2_{oss} \tag{9}$$

$$k_3 = - C_{iss} / F_{oss} \tag{10}$$

Notice that the three parameters k_1, k_2 and k_3 of this simple linearized model are completely determined by the steady state values of the system.

Measurement Models

Four different measurements can be made of the flow mixing process. They are :

1. consistency of the furnish stock, C_m
2. consistency of the incoming stock, C_i
3. flow rate of the furnish stock, F_m
4. variation in dilution flow due to changes in pressure, P.

A control strategy based on only the first measurement results in a purely feedback control loop. The use, in addition, of one or more of the other three measurements will lead to a feedforward-feedback control system. In order to investigate the effect on controlled performance of increasing the number of measurements and the effect of different combinations of measurements, several measurement models are formulated. For easy reference, a simple binary code is employed to indicate which measurements are made. The left most digit represents the first measurement, C_m, the second represents the second measurement, C_i, and so on. A "1" indicates the measurement is made and a "0" indicates it is not. For example a feedforward- feedback control system based on a measurement $y(t)$ of furnish consistency and furnish flow :

$$y(t) = [\ c_m(t) \quad f_m(t)\ ]^T \tag{11}$$

is coded as 1010 because it is based on measurements 1 and 3. Each of the control strategies used in the paper is determined for each of the following measurement schemes :

1000, 1100, 1010, 1110, 1110 and 1111

MODELS OF CONSISTENCY DISTURBANCES

A knowledge of the dynamics of the system disturbances for each stream, namely C_i, F_o and P is essential for the design and simulation of controllers. The disturbance dynamics determine the appropriate sampling time and provide realistic disturbance sequences for simulation. In this paper, for the sake of brevity, only the details of modelling the broke stream are presented - this stream exhibits the greatest variation in consistency and flow. However, the disturbance models of the kraft and groundwood streams are determined in the same way and have exactly the same form as the broke stream disturbance models, only the parameters and constants have different values.

The disturbance processes C_i and F_m for the furnish to Tasman's number 2 paper machine were sampled on-line at 1 second intervals. Incoming consistency is measured by closing the dilution valve. Measurement noises are presumed to be a small fixed fraction of the variations in the measurements.

The steady state values C_{iss} and F_{oss} are the dominant components in both incoming consistency and furnish flow. Therefore in order to identify the variability precisely, these constant values are subtracted from the data for all subsequent analysis.

Fig. 3 - Power spectrum of broke incoming consistency and furnish flow

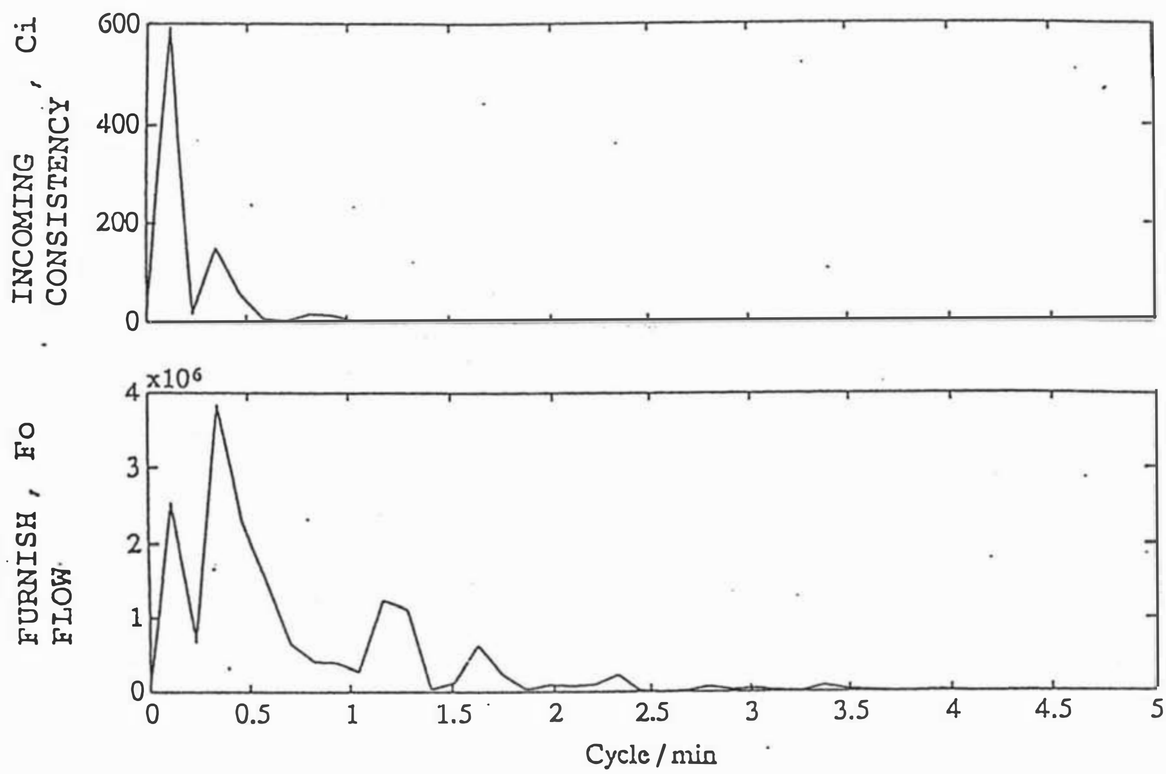


Figure 3 shows the power spectra of incoming broke consistency and furnish flow (after subtracting the steady state values). The fastest measured variability of each is determined from the following highest frequencies shown by the power spectrum plots:

Broke stream incoming consistency -- 2.5 cycle/min

Broke stream flow -- 4 cycle/min

The fastest variability has frequency 4 cycle/min or 15 sec/cycle, in the case of the broke stream flow. By Shannon sampling theorem, see [2], the maximum sampling time should enable at least two samples to be taken within on cycle of the fastest mode. Therefore here, the maximum sampling rate should be at least one sample in 7.5 seconds. Since the dead time is known to be 25 sec, a sampling time of 5 second is used in this study for the digital control systems designed..

Black box modelling techniques have been applied to find the deviation variable models of the disturbances sampled at Tasman's number 2 paper machine. Because there are no easily measured 'inputs' that can be said to be causing the disturbance variations, only autoregressive time series models driven by random inputs terms can be considered [3]. By using a commercial computer aided control system design package called MATLAB™ [4],[5], several linear time series models were fitted to the data using the minimum-prediction-error method. After considerable investigation of a range of possible models, it was found that a second order time series model with a random and uncorrelated white noise driving term was appropriate and adequate for each of the disturbances. Thus the deviation variable disturbance models for each of these components of the furnish are of the form :

$$c_i(k+1) = V_{c1}.c_i(k) + V_{c2}.c_i(k-1) + w_c(k) \tag{12}$$

$$f_o(k+1) = V_{f1}.f_o(k) + V_{f2}.f_o(k-1) + w_f(k) \tag{13}$$

where V_{c1}, V_{c2}, V_{f1} and V_{f2} are constants determined by the modelling, and $w_c(k)$ and $w_f(k)$ represent respectively the white noise driving terms for the incoming consistency model and the furnish flow model.

Figure 4 gives the plots of model order against Mean-Square-prediction-Error (MSE). It shows that the MSE reduces dramatically as the model order increases from on to two, but any further increase in model order results only in a smal reduction in MSE. Therefore, with control system in mind, a second order model is chosen for both furnish flow and incoming stock concentration.

Autocorrelation plots of the prediction error, Figure 5, shows prediction errors are uncorrelated with past and future prediction errors. This indicates the appropriateness of the white noise models. Figure 6 shows a comparison of actual data with one-step-ahead predictions using the above models.

Fig. 4 - MSE for various orders of model

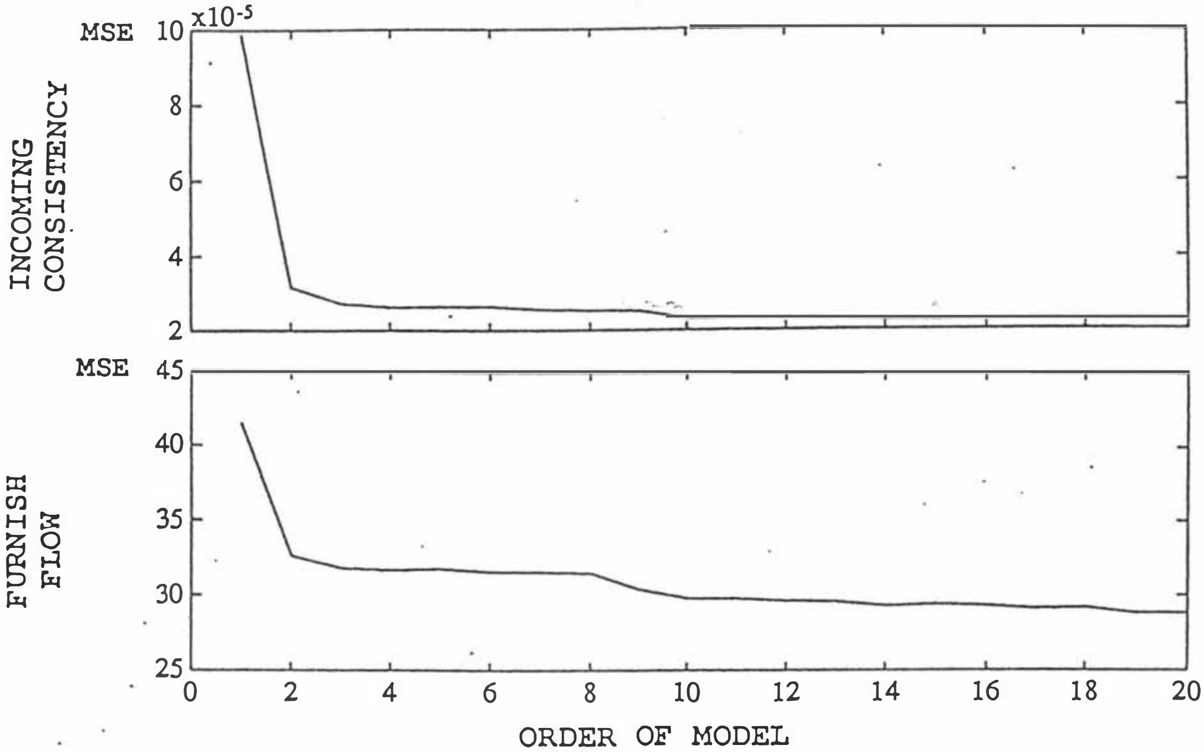


Fig. 5 - Autocorrelation of model error

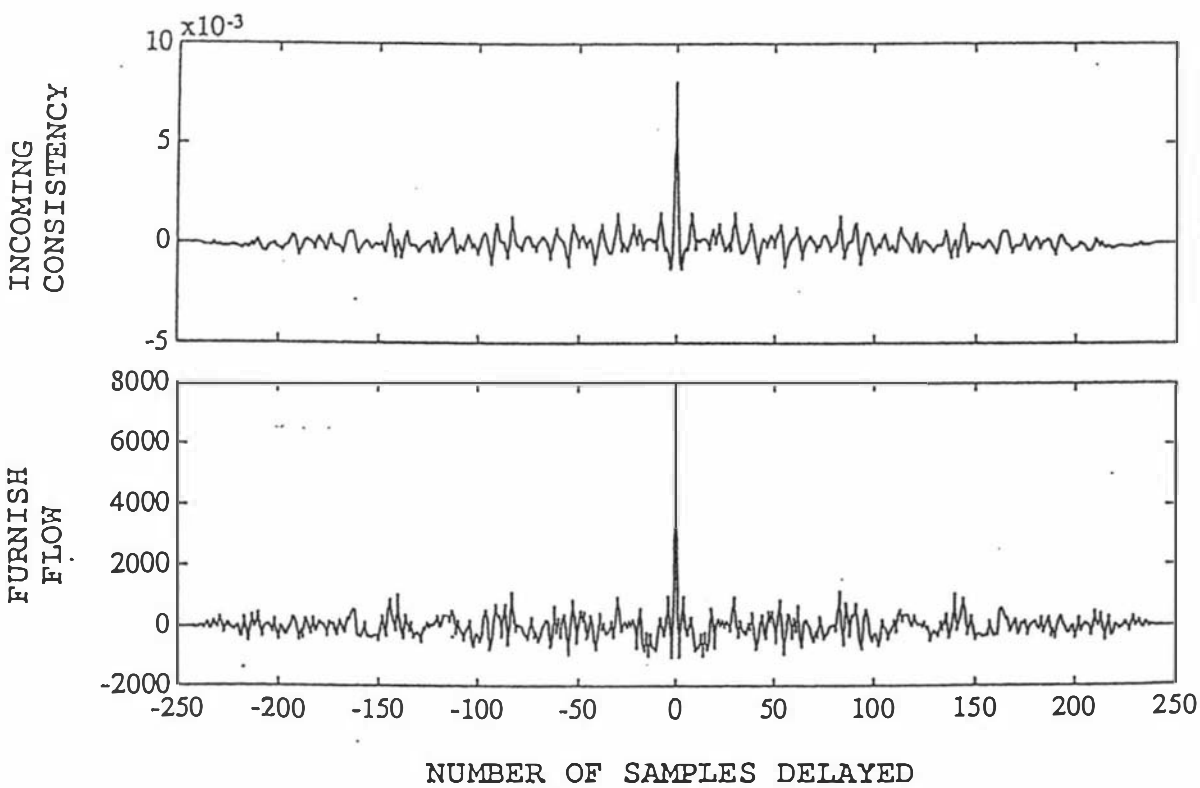
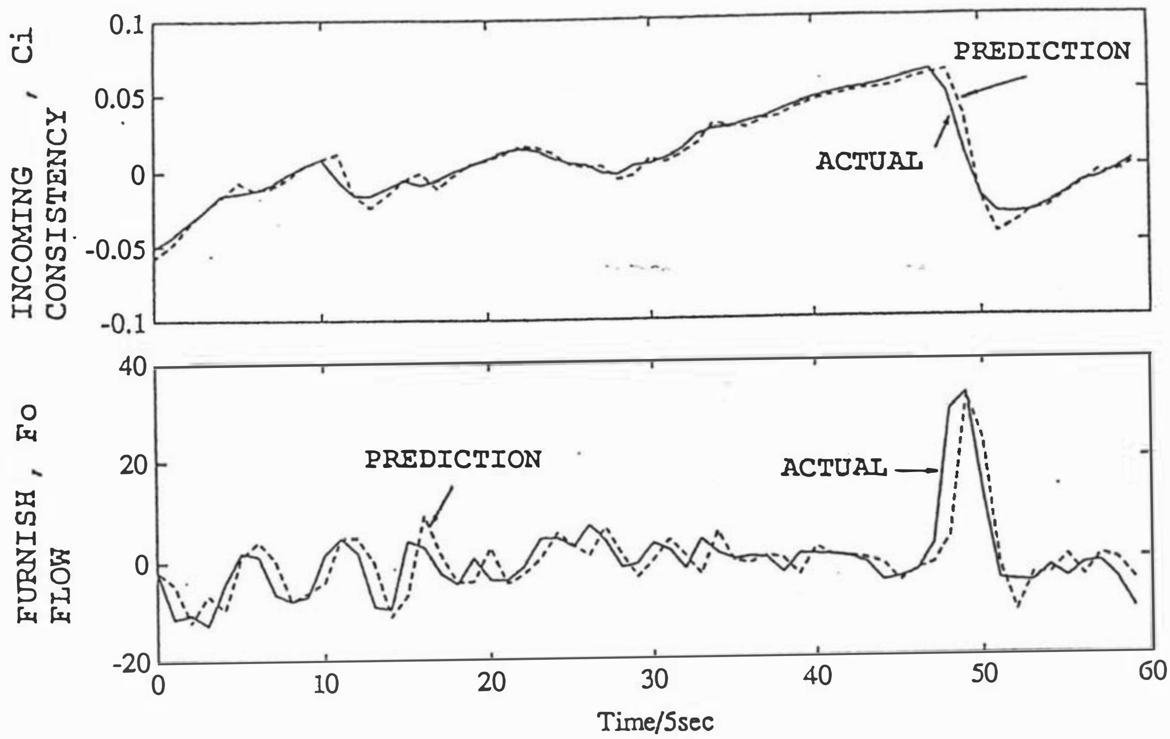


Fig. 6 - Comparison of one-step-ahead predictions with actual data



Due to the fact that P, the effect on flow of variations in dilution flow pressure, was not able to be measured at the time of this study, the model of the effect of pressure variation on dilution is estimated to have the same form as furnish flow :

$$p(k+1) = V_{p1}.p(k) + V_{p2}.p(k-1) + w_p(k) \tag{14}$$

where $w_p(k)$ represents the white noise driving term of the model, and the constants V_{p1} and V_{p2}, in the absence of better information, have been taken as V_{f1} and V_{f2} respectively.

The coefficients and steady state values for each of the models are given in Table 1.

Table 1 - Parameters for broke stream

Steady states	System model	Disturbance model
C_{oss} = 3.000%	k_1 = 0.7765	V_{c1} = 1.7700
C_{iss} = 3.863%	k_2 = 1.7233X10^{-3}	V_{c2} = -0.8160
F_{oss} = 2204.8 L/min	k_3 = -7.7101X10^{-3}	V_{f1} = 1.1335
F_{dss} = 492.8 L/min		V_{f2} = -0.4668
P_{ss} = 0 L/min		V_{p1} = 1.1335
		V_{p2} = -0.4668

DESIGN OF POSSIBLE CONTROL SYSTEMS

In order to obtain realistic simulations, the two hundred and forty data points obtained for each disturbance are divided into two halves. The first half of the data is used for controller design and tuning. The second half is then used for simulation.

It is the control objective to minimize the error between the furnish stock concentration, C_o, and its setpoint, S. However a controller, giving the minimum error alone, may employ an impractically large dilution flow, F_d. Thus the controllers are designed to minimize the error in furnish flow stock concentration while keeping a check on the magnitude of the dilution flow. The design criterion can thus be expressed in the performance index:

$$J = \sum_{k=0}^{\infty} \left\{ \mathrm{Q} \, [\, C_o(k) - S \,]^2 + \mathrm{R} \, [\, F_d(k) - F_{dss} \,]^2 \right\} \tag{15}$$

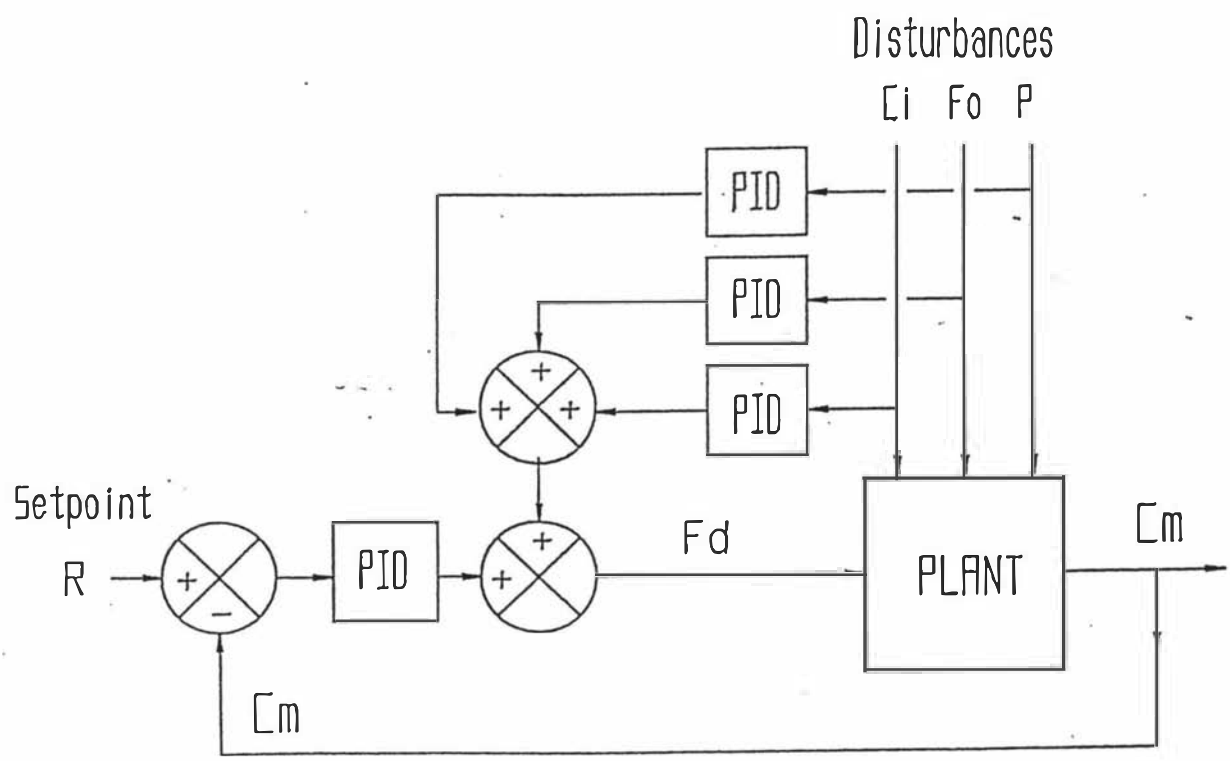
The two weightings Q and R are set by the designer to ensure that the control system delivers control action of appropriate magnitude. A large Q/R ratio will result in a controller giving only small variations in furnish consistency but it may employ large amounts of control action. Reducing the Q/R ratio will reduce the magnitude of the control action called for but will increase the variation in furnish consistency. The effect of changing this Q/R ratio can be seen in Figures 10a and 10b, which will be discussed in further detail in the following sections.

PID Controllers

Currently at Tasman Pulp and Paper Company, pulp stock consistency is controlled by a PI controller using only the measured furnish consistency as a feedback signal. In order to compare the best possible performance of traditional control technology with more modern control strategies, the derivative action (D term) is used in the simulation work, in addition to the PI action. The performance of the purely feedback PID controller, coded as PID1000, is used subsequently as a baseline for the comparison of controller performance. In terms of the measurement coding system described in the previous section, the five other designs of PID controller evaluated are PID1010, PID1011, PID1100, PID1110 and PID1111. The configuration of PID1111 is illustrated in Figure 7.

These PID controllers are tuned to minimize the performance index of equation (15). The PID gains that minimize this index were found using an optimization routine based on the Nelder- Mead simplex algorithm [4] provided in the MATLAB™ package, with Cohen Coon settings [6] used as initial values.

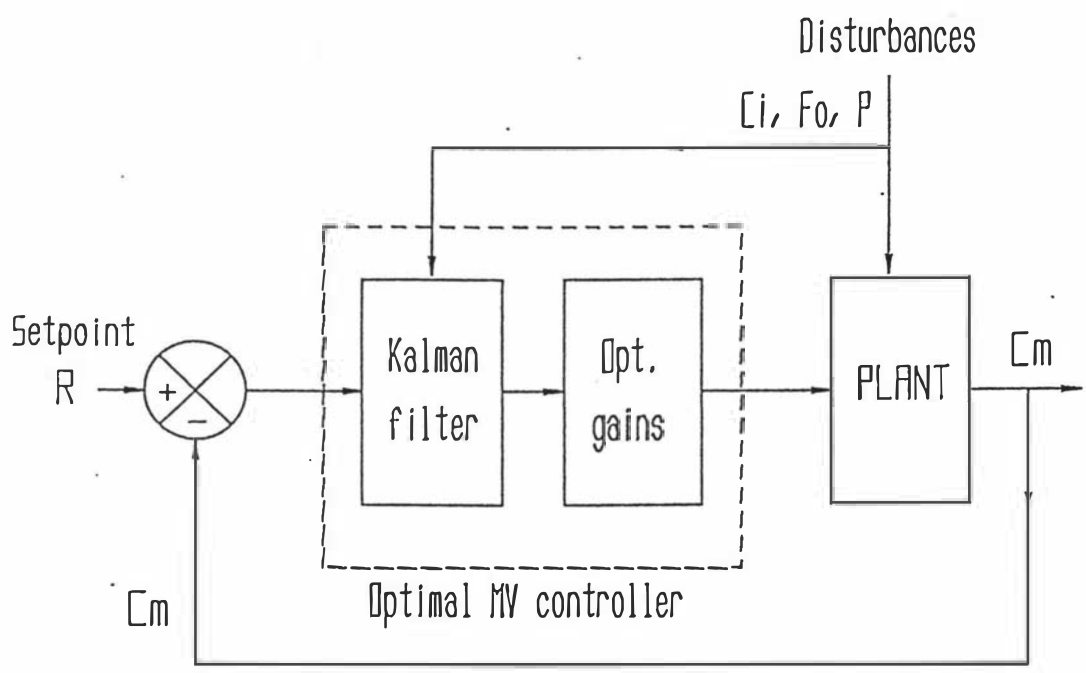
Fig. 7 : A feedforward-feedback PID control system



Optimal Multivariable Controller

The structure of an optimal multivariable controller [7]-[9] is determined optimally to minimize the performance index equation (15). The optimal controller consists of a Kalman filter [8], [9], which filters the measurement noise and estimates the quantities required by the rest of the controller. Based on the different measurement schemes, the optimal multivariable controllers evaluated are MV1000, MV1010, MV1011, MV1100, MV1110 and MV1111. Figure 8 shows the general configuration of optimal multivariable controllers.

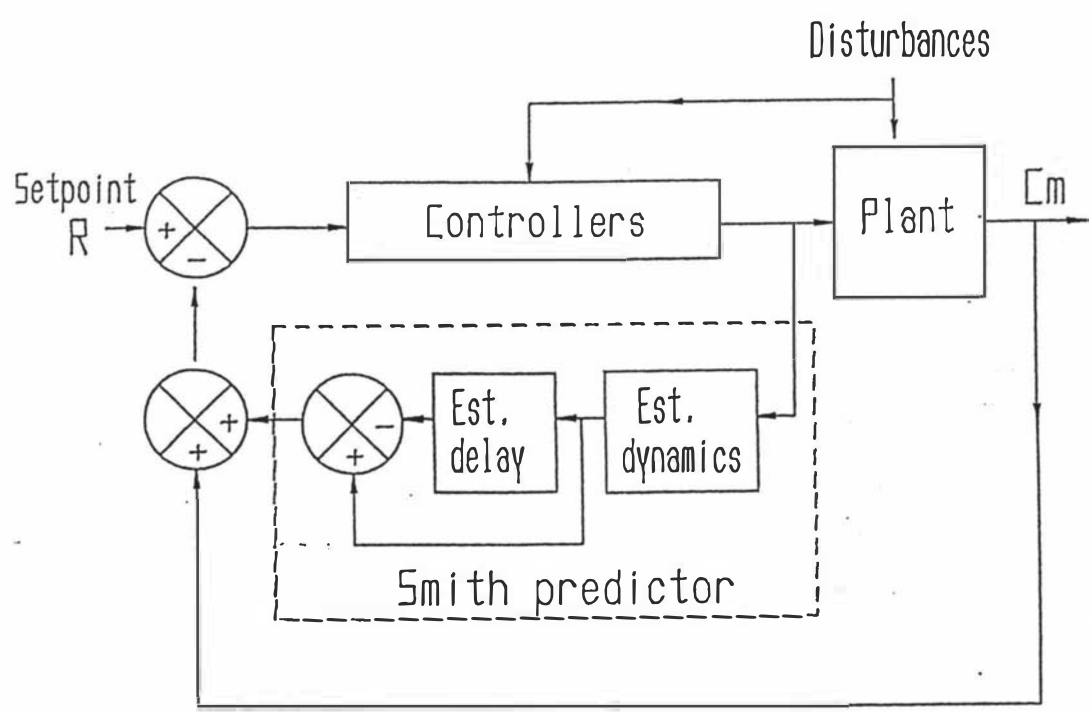
Fig. 8 : Optimal multivariable control system



Smith Predictor

To overcome the complication of pure delay, a Smith predictor (10) can be used. The Smith predictor enables any feedback controller to control the system as if there was no pure delay. A loop is added between the control signal and the feedback signal as illustrated in Figure 9, from which it can be seen that the Smith predictor consists of a disturbance- free model of the system dynamics to estimate the output consistency without pure delay and an estimation of pure delay to predict the actual measured consistency output. This idea provides two further types of control strategy for this study. They result from using the Smith predictor with firstly PID controllers and secondly with optimal multivariable controllers. These controllers are again determined so as to minimize the performance index of equation (15).

Fig. 9 : Configuration of Smith predictor



SIMULATION OF CONTROLLED CONSISTENCY

The non-linear system model is used to simulate the behaviour of consistency under the twenty four different control systems designed using the linearized model, as described in the previous section. The discrete-time non-linear simulation is carried out over a simulation period of 900 sec.

A number of measures of the performance of the various control systems designed are considered :

Mean-square-error of the furnish consistency: This measures the capability of a controller to achieve the required furnish stock concentration,

$$mse(C_o) = 1/N \sum_{k=0}^{k=N-1} \{ C_o(k) - R \}^2 \tag{16}$$

where N = number of points in the simulation

R = furnish consistency setpoint

Minimum and maximum values of furnish consistency:

$$\min(C_o) = \text{minimum of } C_o(k) \tag{17}$$

$$\max(C_o) = \text{maximum of } C_o(k) \;, \quad 0 \le k \le N\text{-}1 \tag{18}$$

Mean-square-value of the dilution flow: This measures the variation in dilution flow,

$$mse(F_d) = 1/N \sum_{k=0}^{k=N-1} \{ F_d(k) - F_{dss} \}^2 \tag{19}$$

Maximum and minimum values of dilution flow: Recording these maximum and minimum values enables a check to be made that dilution flow is always within the range of the valve,

$$\min(F_d) = \text{minimum of } F_d(k) \tag{20}$$

$$\max(F_d) = \text{maximum of } F_d(k) \;, \quad 0 \le k \le N\text{-}1 \tag{21}$$

CONTROLLER COMPARISON AND DISCUSSION

Variation in dilution flow

The simulations show that across all the control schemes, the variations in the maximum and minimum values of dilution flow for different controllers are quite small, though the dilution flow mean-square-value varies significantly. The highest dilution flow recorded is 577 L/min and the lowest is 447 L/min. These values are well within the range of the dilution flow valve. Because of this, and space limitation in this paper, the maximum and minimum values of dilution flow are not be presented here.

Effect of varying weightings in optimal performance index

Figures 10a and 10b illustrate the mean-square-values of error in furnish consistency and of dilution flow, for different values of the Q/R ratio. From these simulations, certain characteristics of the performance of the resulting controllers are evident.

Firstly, as the ratio Q/R is decreased, the deviations from the setpoint of furnish flow increase, while the amount of dilution flow used decreases. This behaviour, which was expected, is true for all PID and all multivariable controllers.

Secondly, the greater the number of measurements that are made and hence the more feedforward control loops that are employed by the controller, the greater is the impact on mean square furnish error of varying the ratio Q/R. For controllers 1111, the mean square furnish error is highly sensitive to Q/R:

	Q/R=1000	**Q/R=0.001**
PID	MSE = 16	MSE = 1165
multivariable	MSE = 16	MSE = 776

while for controller 1000 it is very insensitive. Noting that even for Q/R ratios of 1000, the amount of dilution flow demanded is always well within the range of the dilution flow valve, one can conclude that for all but controller 1000 (for which a very small Q/R values is acceptable), the value Q/R=1000 should be used.

Fig. 10a - Effect of varying weightings in performance index on PID controllers

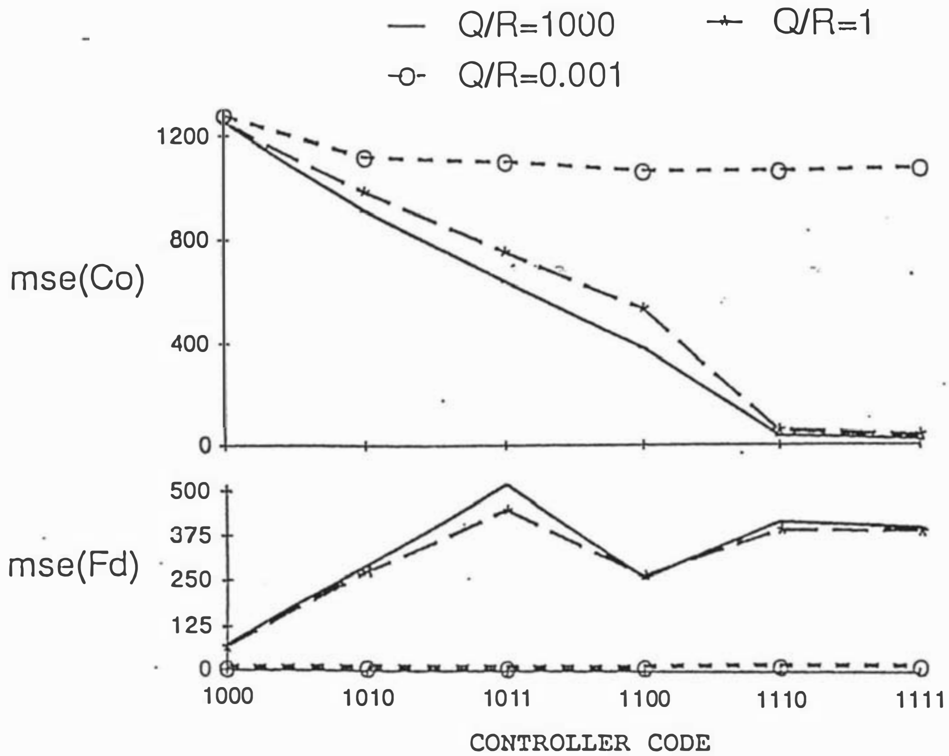
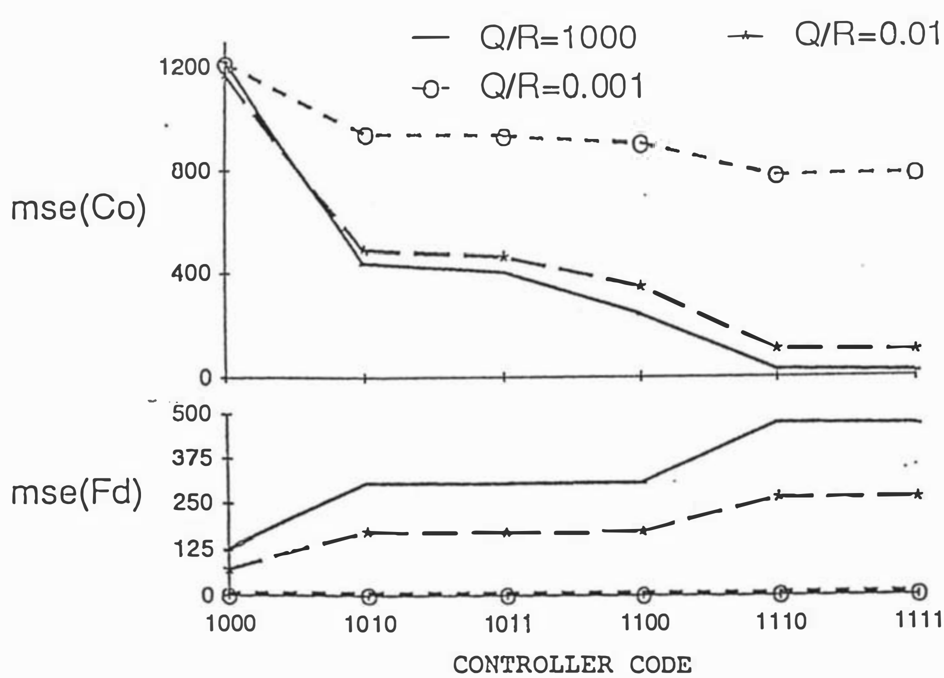


Fig. 10b - Effect of varying weightings in performance index on optimal multivariable controllers

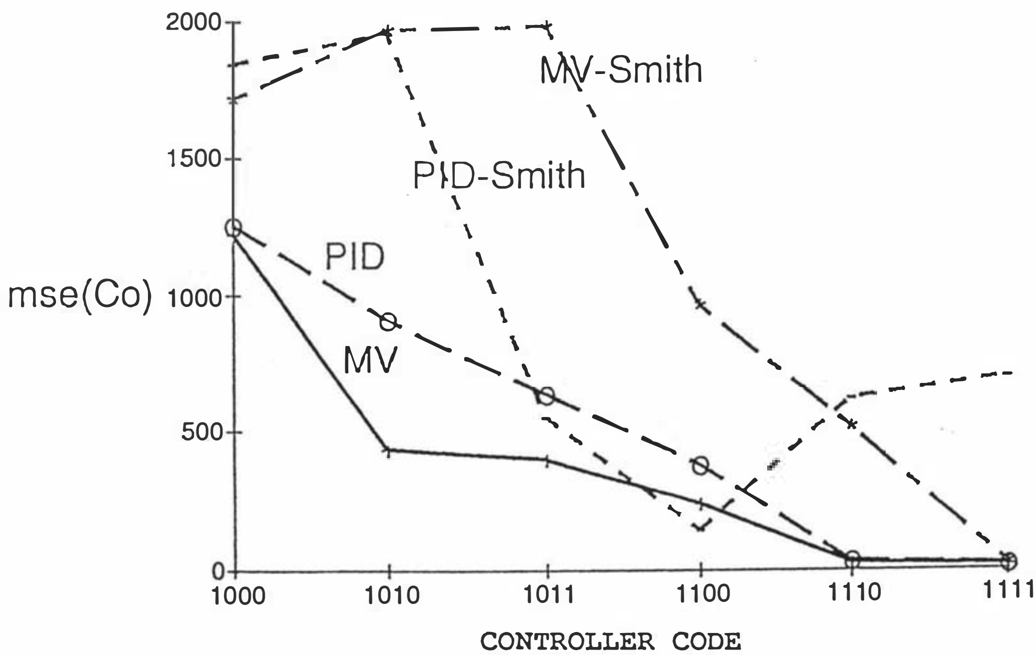


Effect of Smith predictor

Figures 11 shows the performance comparison using Q/R=1000 of the four control strategies - PID, optimal multivariable, and their corresponding Smith predictor type-controllers.

Even when used with the real nonlinear system, the Smith predictor evidently does not generally improve the controllers' performance. As shown in Figure 9, the Smith predictor is determined from a disturbance-free model of the process. However, the stock consistency control problem being studied here is strongly driven by significant disturbances. Thus in this situation, the Smith predictor is unable to take appropriate action, not having the necessary information about the disturbances.

Fig. 11 - Comparison of the controllers' performance



PID controllers and multivariable controllers

As illustrated in Figure 11, the performance of the purely feedback PID controller is comparable with that of the purely feedback optimal multivariable controller. But when feedforward terms are introduced, the multivariable controllers with large weight on furnish consistency give superior performance. On average, the mean-square-error in consistency is 25% better for designs with Q/R larger than one. Furthermore obtaining the optimal tuning parameters for the PID controllers is a very difficult task even in the computer aided simulation, especially when feedforward loops are used, because there are a large number of correlated parameters that need to be tuned. The actual plant performance may not be as good as the simulated plant performance because the actual tuning parameters may not be the optimal ones determined in this simulation study.

As already observed, the variations in dilution flow called for by each controller design are quite acceptable.

Effect of the number of feedforward controls

The mean square error of the furnish consistency resulting from optimal multivariable and PID controllers for different measurement models can be seen from Figures 10a and 10b. The furnish consistency resulting from multivariable and PID controllers improves with each feedforward loop that is introduced . Among the three disturbances fed forward, using the measurement of incoming consistency for feedforward control by itself resulted in the greatest improvement, because it reduces the mean square error in furnish consistency to between 20% to 31% of what it was. Furnish flow is the next most significant measurement, by itself reduces the mean square error to between 36% and 73%. A controller with all three feedforward loops has the best performance, resulting in a mean square error of only about 2% of the mean square error resulting from purely feedback control. The question of which controllers to use (1000, 1011 or 1111 or some other) is to be answered by comparing the cost of installing appropriate sensors to make the required feedforward measurements with the value to the company of the improved control of furnish that results.

Figures 10a and 10b also illustrate the increase in the mean- square-value of the dilution flow as the number of feedforward terms increases. An increase in control action is a natural consequence of increasing the number of control loops. However considering the improvement in furnish consistency, this increase is small and easily accommodated by the dilution flow valve.

The importance of feeding forward measurements of incoming consistency and of furnish flow can be seen from the linearized consistency equation (7). Using the steady state values for the broke stream given in Table 1, equation (7) becomes (time index omitted):

$$c_m = 0.7765\, c_i + 0.00039\, f_o - 0.00175\, p - 0.00175\, f_d \tag{30}$$

As the standard deviations of the three disturbances, c_i, f_o and p are 0.09%, 66 L/min and 1.32 L/min respectively, their contributions to the variation in furnish consistency are in the ratio of

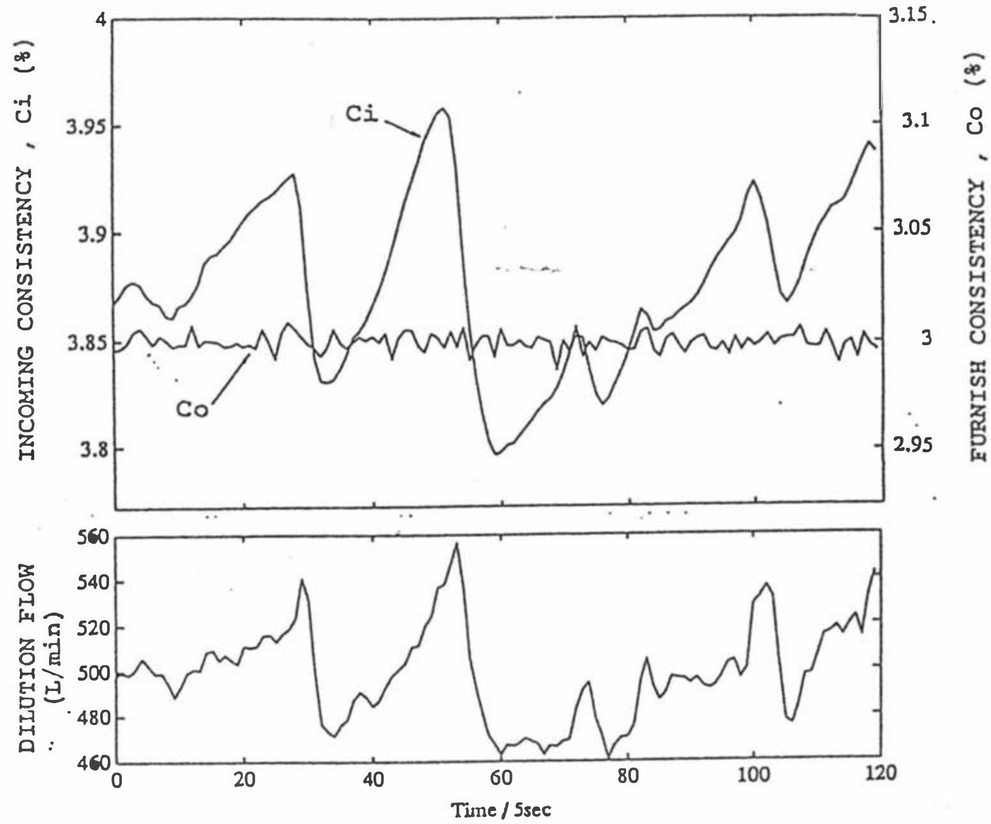
0.7765 x 0.09 : 0.00039 x 66 : 0.00175 x 1.32

or 70 : 26 : 2

when the manipulated dilution flow is maintained constant. Thus the variation in incoming consistency, c_i, has the dominant effect, followed by the variation in furnish flow, f_o.

Figure 12 shows the incoming and the simulated furnish consistency and the control input, dilution flow, when the optimal multivariable controller with Q/R=1000 is used with all three feedforward loops over the simulation period.

Fig. 12 - Incoming, controlled consistency and dilution flow: optimal multivariable controller



CONCLUSION

A simulation has been conducted to compare the performance of a number of possible different control schemes for stock consistency control Tasman Pulp and Paper's second paper machine. Six different measurement schemes for each of four different control strategies have been considered. In comparing the four different control strategies, it has been found that optimal multivariable controllers have performance superior to that of three term controllers, in regulating the furnish consistency. Three term controllers have a pre-set structure whereas optimal multivariable controllers have their structure determined optimally by the plant model and are thus better fitted to the actual plant. Furthermore, optimal values of the three term controller tunning parameters are very time consuming to obtain, whereas optimal multivariable controllers are much more straigtforwardly determined

The question was considered of whether to use Smith predictors with either three term controllers or multivariable controllers. Standard Smith predictors were found to give poorer performance, because of the significance for the system of the disturbances, which a standard Smith predictor is unable to compensate for.

In considering the question of which measurements of the system to make for use by the controller, it has been shown that the most dominant disturbance is incoming consistency and the next most dominant is furnish flowrate. Compared with the purely feedback controlled consistency alone, the addition of a feedforward loop based on a measurement of incoming consistency reduces the resulting mean square error in controlled consistency to between 20% and 31% of what it was with feedback alone. The addition of a feedforward loop based on a measurement of furnish flowrate reduces the mean square error to between 36% and 73%. Using dilution pressure in a feedforward loop offers a small improvement.

REFERENCES

[1] D.R. Coughanowr and L.B. Koppel - *Process Systems Analysis and Control*, International Student Edition, P. 67, McGraw Hill (1983).

[2] G.H. Hostetter, *Digital Control System Design*, 1st Ed., P.108, Holt,Rinehart and Winston Inc. (1988).

[3] L. Ljung, *System Identification : Theory for the user*, 1st Ed., P.69 , Prentice Hall (1987).

[4] C. Moler, J. Little and S. Bangert, *Pro-MATLAB™ User's Guide*, The Mathworks Inc. (1988).

[5] L. Ljung, *System Identification Toolbox for use with MATLAB™* : User's Guide, The Mathworks Inc. (1986).

[6] D.R. Coughanowr and L.B. Koppel - *Process Systems Analysis and Control*, International Student Edition, P. 311, McGraw Hill (1983).

[7] F.L. Lewis, *Optimal Control*, 1st Ed. P.25, John Wiley and Sons Inc. (1986).

[8] K. Ogata, *Discrete-time Control Systems*, 1st Ed., P.668, Prentice Hall (1987).

[9] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*, 1st Ed., P.488, John Wiley and Sons Inc. (1972).

[10] Smith O.J.M. - *Chemical Engineering Progress* 53(5):217, Closer Control of Loops with Dead Time, (1957).

APPENDIX B3

CONTINUOUS-TIME-MODEL SELF TUNING CONTROL

This appendix demonstrates the use of the FII parameter estimation technique in a self-tuning control scheme for a system with delay. The main features of this scheme are that the continuous-time model of the system is used and the system delay is estimated simultaneously with other system parameters. The estimated continuous-time model updates the tuning of a continuous-time controller incorporating a delay compensator (Smith predictor). The continuous-time controller is then realized directly as a discrete-time controller. A simulation example is given based on an industrial case study of paper-pulp concentration system.

B3.1 INTRODUCTION

In the Chapter 5 of this thesis, a parameter estimation technique was developed for continuous-time systems with delay elements. This technique is based on a special integral called the Fixed Interval Integral (FII). Two important features of this FII estimation technique are that:

- it is able to estimate simultaneously both the system delay and system parameters;
- it is suitable for on-line tuning with slow time-varying systems.

In view of this, it is possible to form an explicit self-tuning controller by coupling the FII estimation technique with an appropriate control scheme based on a continuous-time model. This follows directly from the continuous-time-model approach to self-tuning control of Gawthrop (1987 and 1989).

This appendix proposes a self-tuning control scheme for a system with delay. It is based on a state-feedback control scheme incorporating a Smith predictor. It will be shown that this control scheme matches the characteristics of the FII technique and makes use of a delay element which is not a necessarily exact multiple of the sampling time (that is the "fractional delay" defined in Chapter 5). It will also be shown that this continuous-time control scheme can easily be implemented directly on digital devices.

There are eight major sections in this appendix,

Section B3.2	gives an overview of the structure and components involved in the proposed explicit continuous-time-model self-tuning control system.
Section B3.3	describes the assumptions made on the delay system to be controlled.
Sections B3.4 to B3.6	detail the structure and the implementation of the major components in the self-tuning controller.

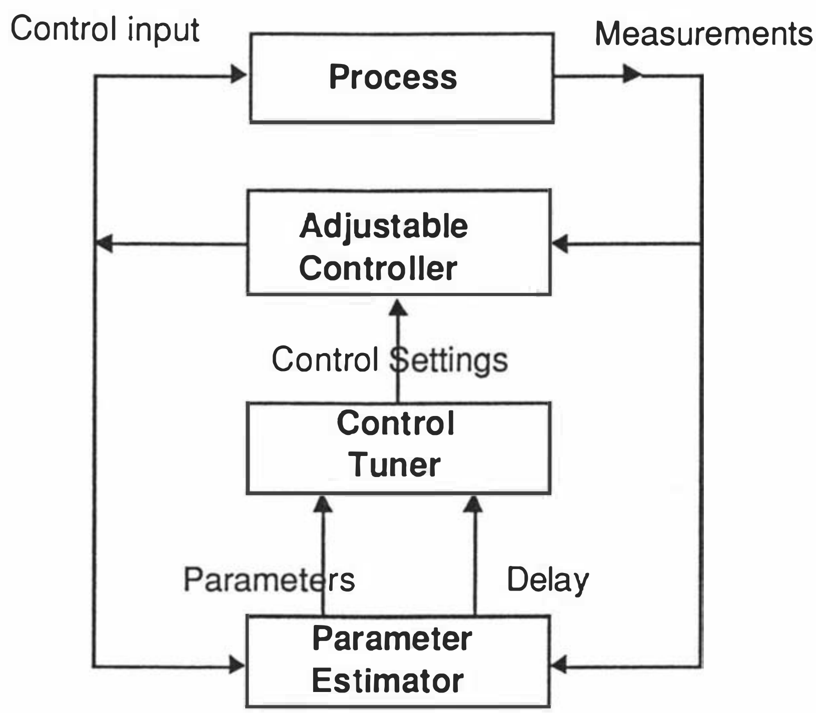
Section B3.7	demonstrates the use of this continuous-time-model self-tuning control scheme on a simulated industrial case study of paper-pulp concentration control.
Section B3.8	summarizes the key contributions in this chapter.

B3.2 AN OVERVIEW OF THE PROPOSED CONTINUOUS-TIME-MODEL SELF-TUNING CONTROL SYSTEM

The proposed continuous-time-model self-tuning control system can be divided into the three major blocks, as illustrated in Figure B3.2-1. That is the,

a) controller,
b) control tuner, and
c) parameter estimator.

Figure B3.2-1 Overview of the Continuous-time-model Self-tuning Control System



These three blocks can be implemented on either a single digital device or on three separate digital devices. The sampling time in each block can be different since they are all based on the same continuous-time model which is valid for all sampling intervals. This flexibility of sampling is also true for the sub-blocks or elements inside each block.

The main function of the controller block is to provide an appropriate control action to the system. Its action is based on control settings which can be adjusted externally without disrupting the operation of the controller. In tuning the controller, the updated control settings are provided by the control tuner. Using the most current estimates of the system parameters found by the parameter estimator, the control tuner calculates the appropriate settings for the controller to satisfy some pre-defined criteria.

In practice the controller should be able to function with the absence of the other two blocks. Its function and structure is kept simple to enable rapid control of the system. Therefore it is recommended that the controller be implemented on a separate digital device with a fast rate of operation and sampling. Most of the complex computations are allocated to the parameter estimator and the control tuner. Therefore, digital devices with high computational capability are required for these two blocks. More details on these three blocks and the delay system to be controlled are given in the following sections.

B3.3 SYSTEM

It is assumed here that the system or process to be controlled can be described by Definition 5.4-1, that is:

$$\sum_{i=0}^{n} a_i \, \rho^i \, y(t) = \sum_{i=0}^{n-1} b_i \, \rho^i \, \delta_{(t;\tau)} \, u(t) \tag{B3.3-1}$$

where the delay is τ and the vector of parameters is given by,

$$\theta = [\, a_n \;\; a_{n-1} \, \ldots \, a_0 \;\; b_{n-1} \, \ldots \, b_0 \,]^{\mathrm{T}} \tag{B3.3-2}$$

This system model has the equivalent state-space description of,

$$\rho\, x(t) = \mathbf{A}x(t) + \mathbf{B}u(t-\tau) \tag{B3.3-3}$$

$$y(t) = \mathbf{C}x(t) \tag{B3.3-4}$$

where $x(t)$ is a vector of some appropriate states. The matrices, **A**, **B** and **C** can be formed easily by using the elements in the parameter vector, θ, when the controllable or observable canonical form (Blackman 1977, Banks 1986, Delchamps 1988) of the state-space description is applied. In the case of the controllable canonical form, these matrices can be written as:

$$\mathbf{A} = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \bullet & -a_1 & -a_0 \\ 1 & 0 & \bullet & 0 & 0 \\ 0 & 1 & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & 1 & 0 \\ 0 & 0 & \bullet & 0 & 0 \end{bmatrix} \tag{B3.3-6}$$

$$\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ \bullet \\ \bullet \\ 0 \end{bmatrix} \tag{B3.3-7}$$

$$\mathbf{C} = [b_{n-1} \quad b_{n-2} \quad \bullet\bullet \quad b_0] \tag{B3.3-8}$$

or in the case of observable form,

$$\mathbf{A} = \begin{bmatrix} -a_{n-1} & 1 & 0 & \bullet & 0 & 0 \\ -a_{n-2} & 0 & 1 & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ -a_1 & 0 & 0 & \bullet & 1 & 0 \\ -a_0 & 0 & 0 & \bullet & 0 & 0 \end{bmatrix} \tag{B3.3-9}$$

$$\mathbf{B} = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \bullet \\ b_0 \end{bmatrix} \tag{B3.3-10}$$

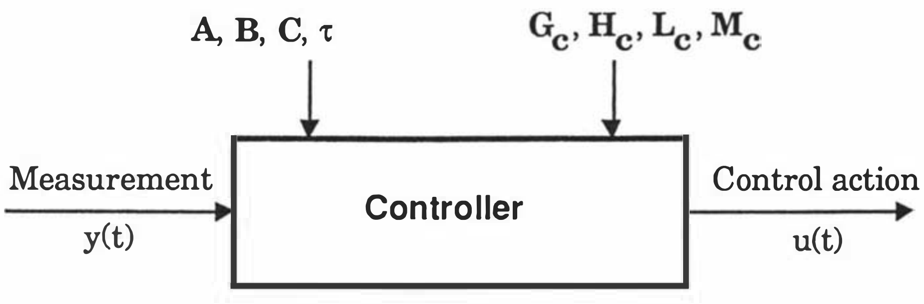
$$\mathbf{C} = [\ 1 \quad 0 \quad 0 \quad \bullet \quad \bullet \quad 0\ ] \tag{B3.3-11}$$

Note that these matrices can be formed without any numerical operations. The ease in transforming the delay-differential equation into a state-space description is important for the control scheme discussed in the next section.

B3.4 CONTROLLER

The controller used here is a state-feedback controller (Kwakernaak and Siven 1972, Lewis 1986, Banks 1986) incorporating a Smith predictor (Smith 1957, Marshall 1979). It is chosen for reasons discussed later in this section. The controller block consists of one primary input signal, several auxiliary input signals for tuning and one output signal as illustrated in Figure B3.4-1.

Figure B3.4-1 Input and Output Signals of the Controller



The primary input of the controller is the measurement from the system, y(t). The controller output is the control action, u(t), which is passed to the system. The auxiliary input, **A**, **B** and **C** are (estimates of) the state-space system matrices. These matrices and the delay are required for the operation of the Smith Predictor. The auxiliary input $\mathbf{G_c}$, $\mathbf{H_c}$, $\mathbf{L_c}$ and $\mathbf{M_c}$ are the state-space description matrices for the state-feedback controller. The reasons for selecting this kind of controller and its structure are now discussed.

B3.4.1 Selecting an Appropriate Control Scheme

Two major factors were considered in selecting an appropriate control scheme for the proposed continuous-time-model self-tuning control system. Firstly, as determined in Chapter 5, the FII technique can estimate the parameters of a model given by differential equations. Furthermore the technique can estimate the delay element including the portion which is not an exact multiple of the controller's sampling interval. This gives two primary criteria for the appropriate control scheme, the control scheme should:

- be based on the parameters of the differential equations when finding appropriate control parameters and control action. For simplicity in implementation, the parameters of the differential equations should be applied without, or with few, additional numerical operations.

- utilize the fractional delay in order to fully exploit the capability of the FII technique.

The second consideration was the realization of continuous-time control using discrete-time digital devices. The method used here is the "direct approximation" approach given by Gawthrop (1989). In his approach the continuous-time transfer function of the controller is first written in the state-space description (Coughanowr and Koppel 1983), that is,

$$\frac{\mathrm{d}x_c(t)}{\mathrm{d}t} = \mathbf{G}_c\, x_c(t) + \mathbf{H}_c\, y(t) \qquad \text{(B3.4-1)}$$

$$u(t) = \mathbf{L}_c\, x_c(t) + \mathbf{M}_c\, y(t) \qquad \text{(B3.4-2)}$$

where $y(t)$ is the measurement of the system, $u(t)$ is the desired control input, $x_c(t)$ are some appropriate states and, $\mathbf{G_c}$, $\mathbf{H_c}$, $\mathbf{L_c}$ and $\mathbf{M_c}$ are some appropriate matrices for the controller. This effectively reduces the original high order transfer function into a set of first order differential equations. Therefore when the measurement $y(t)$ is sampled and the values of $y(t)$ between samples are interpolated (usually with a first order approximation), the values of $u(t)$ can be approximated using any of the well known numerical solution methods for first order differential equations (Bajpal et. al. 1974, Kreyszig 1988).

In view of this, a third criterion for the control scheme is established,

- the controller should be given in the state-space form of Equation B3.4-1 and B3.4-2. to simplify implementation.

An example algorithm for the direct approximation approach to realizing a continuous-time controller is given in Figure B3.4-2. It is based on the fourth order Runge-Kutta method (Kreyszig 1988). Here, r, is the step size of the approximation.

Figure B3.4-2 Direct Approximation of Continuous-time Transfer Function using Runge-Kutta method

INPUT : $y(t\text{-}r)$, $y(t)$, $x_c(t\text{-}r)$

DEFINE FUNCTION: $\mathbf{F}(x,y) = \mathbf{G_c}\,x + \mathbf{H_c}\,y$

MAIN PROGRAM :

$$k_1 = r\,\mathbf{F}(\,x_c(t\text{-}r)\,,\,y(t\text{-}r)\,)$$

$$k_2 = r\,\mathbf{F}\Big(\,x_c(t\text{-}r) + \tfrac{1}{2}k_1\,,\,\frac{y(t\text{-}r) + y(t)}{2}\Big)$$

$$k_3 = r\,\mathbf{F}\Big(\,x_c(t\text{-}r) + \tfrac{1}{2}k_2\,,\,\frac{y(t\text{-}r) + y(t)}{2}\Big)$$

$$k_4 = r\,\mathbf{F}(\,x_c(t\text{-}r) + k_3\,,\,y(t)\,)$$

$$x_c(t) = x_c(t\text{-}r) + \tfrac{1}{2}(\,k_1 + 2k_2 + 2k_3 + k_4\,)$$

$$u(t) = \mathbf{L_c}\,x_c(t) + \mathbf{M_c}\,y(t)$$

OUTPUT : $u(t)$ and $x_c(t)$.

$x_c(t)$ is fedback as $x_c(t\text{-}r)$ for next iteration

An obvious requirement of this approach is a relatively small sampling interval for $y(t)$. However the step size, r, needs not to be equal to the sampling interval as the values of $y(t)$ between samples are interpolated.

Considering the two factors mentioned earlier, a state-feedback control scheme (Kwakernaak and Siven 1972, Lewis 1986, Banks 1986) incorporating a Smith predictor (Smith 1957, Marshall 1979) was chosen for the proposed self-tuning control system. Figure B3.4-3 illustrates the general structure of this control scheme. The appropriateness and implementation of this control scheme is discussed in the next two subsections.

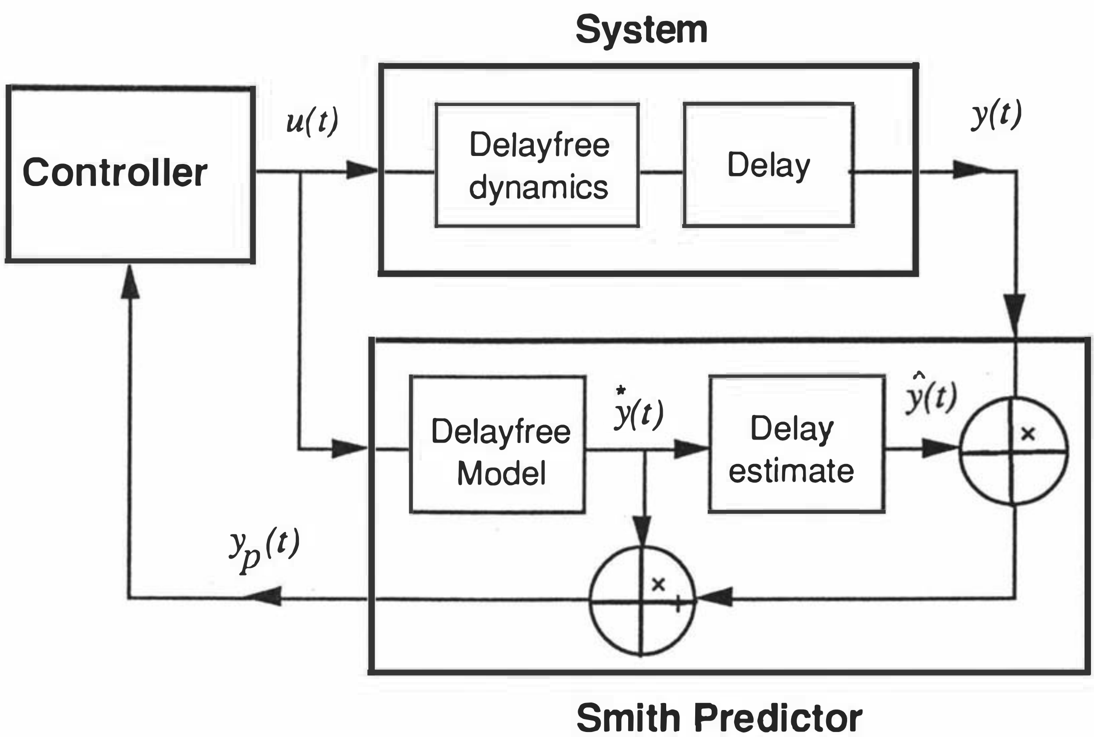
B3.4.2 Smith Predictor

As seen from Figure B3.4-3, the Smith predictor effectively replaces the measurement, $y(t)$ with a predicted measurement, $y_p(t)$ in the feedback loop. The predicted measurement is given by (Smith 1957),

$$y_p(t) = \overset{*}{y}(t) + y(t) - \hat{y}(t) \tag{B3.4-3}$$

where $\overset{*}{y}(t)$ is the estimated measurement without delay and, $\hat{y}(t)$ is the estimated measurement with delay. Therefore the realization of a Smith predictor involves the approximation of these two quantities.

Figure B3.4-3 General Structure of Smith Predictor Control Scheme.



Once the system state-space matrices or their estimates are known, the undelayed measurement, $\overset{*}{y}(t)$, can be approximated using the aforementioned direct approximation method, that is,

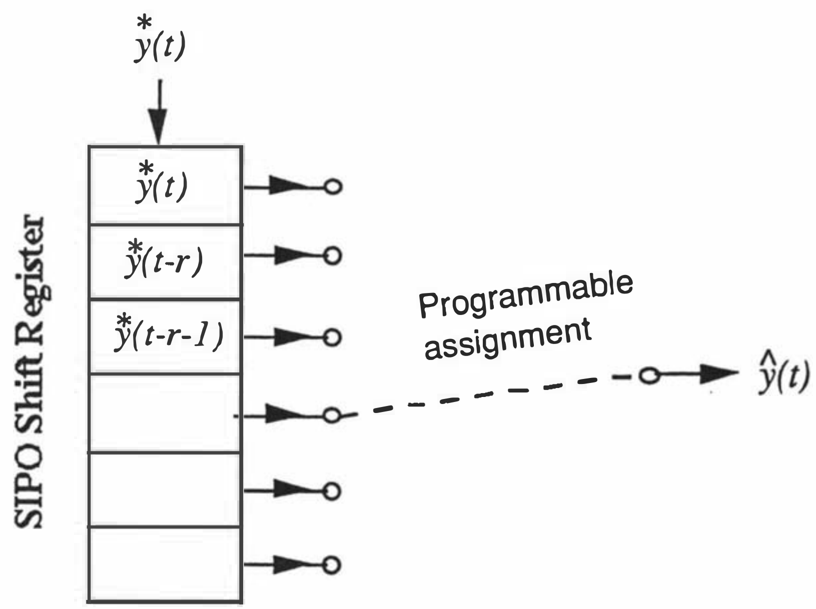
$$\frac{d\overset{*}{x}(t)}{dt} = \mathbf{A}\overset{*}{x}(t) + \mathbf{B}u(t) \tag{B3.4-4}$$

and $\overset{*}{y}(t) = \mathbf{C}\overset{*}{x}(t)$ (B3.4-5)

Here, the control input, $u(t)$ is maintained constant within each sampling interval to match the discrete-time nature of the controller. As discussed in Section B3.3, the matrices **A**, **B** and **C** in a state-space description can be obtained easily from the parameters of a differential equation. Therefore the parameters estimated by the FII estimation technique can be applied here without complicating the controller's structure.

To generate the delayed estimate, $\hat{y}(t)$, it is proposed that a Serial-in Parallel-out (SIPO) Shift-Register (Malvino and Leach 1981) be used. Using this mechanism, the value of $\overset{*}{y}(t)$ at each step of approximation is stored in a stack of memory units. When the most current value of $\overset{*}{y}(t)$ is found, it is stored at the top of the stack while all the old values are shifted downward (and the oldest value is discarded). This is illustrated in Figure B3.4-4.

Figure B3.4-4 Realization of Delayed Measurement



This mechanism also allows the value of delay to be adjusted, because the output signal from the memory stack, that is the $\hat{y}(t)$, can be assigned to a different memory unit in the stack.

Note that the time difference between two values in consecutive memory units is the step size of approximation, *r*. As mentioned earlier, this value is independent of the interval of $u(t)$ or any sampling interval in the system. So it can be set as an arbitrarily small value within the capability of the digital device. In other words the control scheme can make use of the fractional delay up to a precision of *r* units of time.

B3.4.3 State-feedback control

It is well known that the transfer function of an optimal state-feedback controller can be written as (Kwakernaak and Siven 1972, Lewis 1986),

$$\frac{\mathrm{d}x_c(t)}{\mathrm{d}t} = \left[\mathbf{A} - \mathbf{BK_o} - \mathbf{K_eC}\right] x_c(t) + \mathbf{K_e}y(t) \tag{B3.4-6}$$

and $u(t) = \mathbf{K_o}x_c(t)$ (B3.4-7)

where $\mathbf{K_o}$ is the optimal feedback gain matrix and $\mathbf{K_e}$ is the state estimation gain matrix. Both $\mathbf{K_o}$ and $\mathbf{K_e}$ are found using the system state-space matrices, **A**, **B** and **C**.

Comparing Equations (B3.4-6) and (B3.4-7) with Equation (B3.4-1) and (B3.4-2) we obtain:

$$\mathbf{G_c} = \left[\mathbf{A} - \mathbf{BK_o} - \mathbf{K_eC}\right] \tag{B3.4-8}$$

$$\mathbf{H_c} = \mathbf{K_e} \tag{B3.4-9}$$

and $\mathbf{L_c} = \mathbf{K_o}$ (B3.4-10)

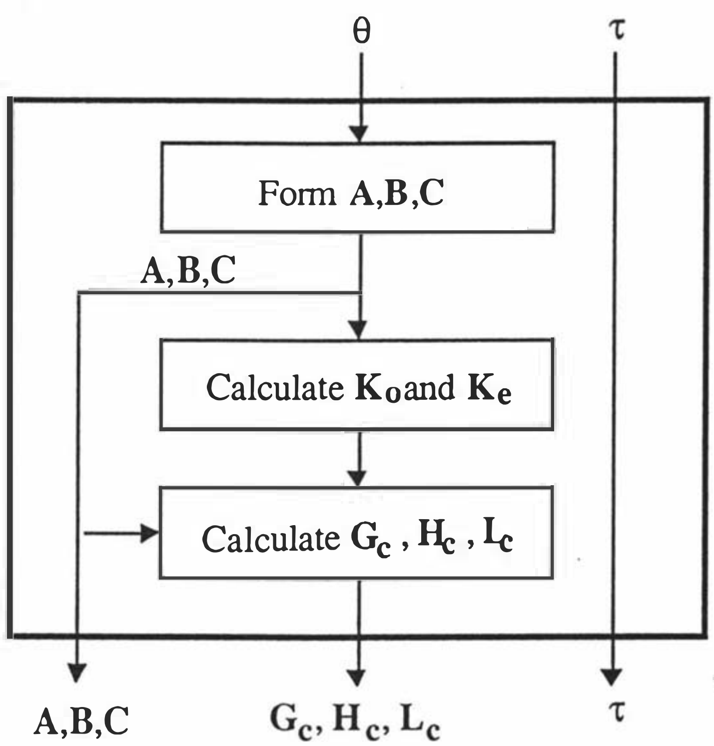
Therefore when all the **A**, **B**, **C**, $\mathbf{K_e}$ and $\mathbf{K_o}$ matrices are given, the continuous-time state-feedback control can be realized easily using the direct approximation method mentioned earlier. The required matrices for this control block are generated by the control tuner block that is discussed in the next section.

B3.5 CONTROL TUNER

The function of the control tuner is to find the matrices required for the controller in the previous section. It uses estimates of the system parameters in θ provided by the parameter estimator.

As mentioned earlier in Section B3.3, the **A, B** and **C** matrices are formed directly from parameters in θ. Once these matrices have been established, the optimal feedback gain, $\mathbf{K_o}$, and state-observation gain, $\mathbf{K_e}$, can be found using methodologies given in standard texts on optimal control such as Kwakernaak and Siven (1972) and Lewis (1986). The state-space description of the controller given by $\mathbf{G_c}$, $\mathbf{H_c}$ and $\mathbf{L_c}$ are then formed using Equations B3.4-8 to B3.4-10. The functions of the control tuner and its main input and output signals can be seen in Figure B3.5-1. Note that the estimate of the delay is not used in this block and is passed directly to the block output.

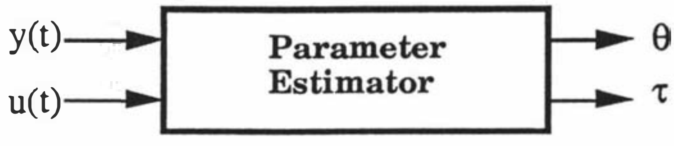
Figure B3.5-1 Functional Structure of the Control Tuner



B3.6 PARAMETER ESTIMATOR

The parameter estimator provides estimates of the system parameters and delay which are then used by the control tuner block. Its main input and output signals are given in Figure B3.6-1.

Figure B3.6-1 Major input and output signals of the Parameter Estimator

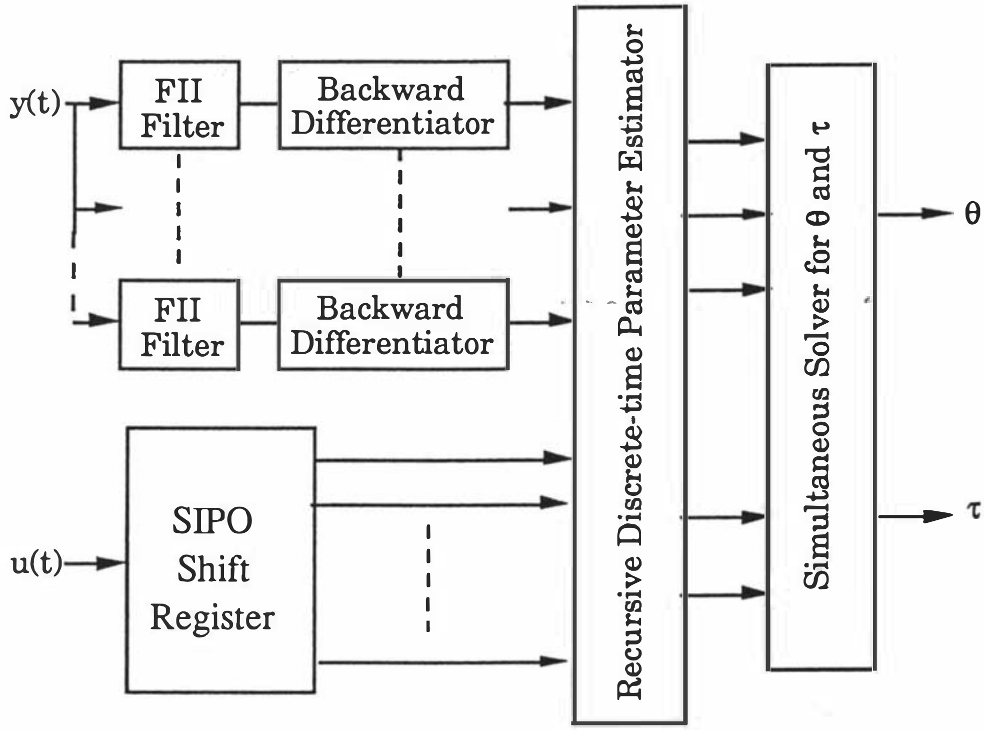


The configuration of the parameter estimator is based on the FII estimation technique for continuous-time models, which was developed in Chapter 5. An overview of the functional structure is shown in Figure B3.6-2.

For flexibility and accuracy, the FII filter in the parameter estimator should be implemented using Numerical Method I developed in Chapter 4.

The SIPO shift register in Figure B3.6-2 works on the mechanism mentioned in Section B3.4. Its function is to provide the array of delayed $u(t)$ required for the FII estimation technique. Details on other elements in the FII parameter estimator have been given in Chapter 5.

Figure B3.6-2 Structure of FII Parameter Estimator



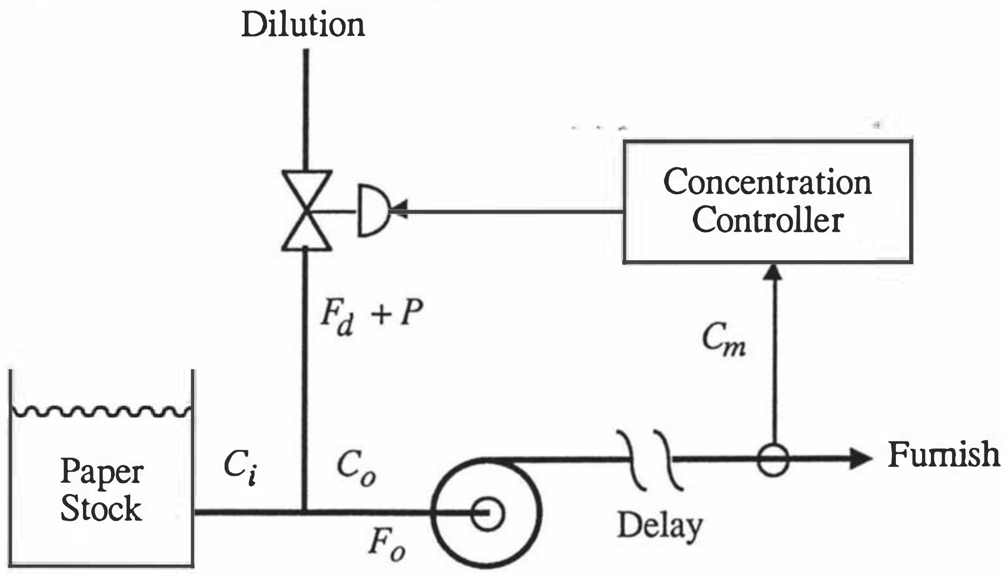
B3.7 SIMULATION STUDY

The previous sections have proposed a continuous-time-model self-tuning control scheme for delay systems. In this section, the feasibility of this self-tuning control system will be studied using a simulation of paper-pulp concentration control. This simulation is based on an industrial case study of paper machines at Tasman Paper and Pulp Company, Rotorua, New Zealand. The details of the modelling work in this study are given in Appendix B1. A publication about the performance comparison of stock concentration controllers is attached in Appendix B2.

B3.7.1 Concentration Control System

In Tasman, the concentration of the paper pulp is controlled by adding dilution water prior to each stock pump (see Figure B3.7-1). Concentration transmitters are located some distance downstream from the dilution points, causing a delay in the concentration control loop.

Figure B3.7-1 Schematic Layout of the Concentration Control System



The notations used in Figure B3.7-1 and subsequently, are as follows:

C_i = concentration of incoming pulp from stock chest

C_o = concentration of diluted (furnish) stream at dilution point (as if there were instant, perfect mixing)

C_m = measured concentration

F_o = flowrate of furnish stream

F_d = flowrate of dilution water

P = variation in dilution flow due to changes in pressure

The flowrate of the diluted pulp, termed here the *furnish flowrate*, is usually maintained constant downstream by an independent controller. However the furnish flowrate is occasionally changed by plant operators to follow different production criteria. It has been found in Appendix B1 that the dynamics of the concentration system depends on the furnish flowrate. In other words, these step changes in flowrate result in step changes in the parameters and delay of the concentration control system. Consequently the concentration controller needs to be re-tuned each time the flowrate is changed in order to obtain optimal performance. A self-tuning scheme is therefore a possible solution to this problem.

It was found in Appendix B1 that a linearized equation for the measured concentration is,

$$c_m(t) = k_1\, c_i(t-\tau) + k_2 f_o(t-\tau) + k_3\, p(t-\tau) + k_3 f_d(t-\tau) \qquad \text{(B3.7-1)}$$

where the deviation variables (lower case symbols are defined as follows,

$$c_i(t) = C_i(t) - C_{iss} \quad , \; C_{iss} = \text{steady state value of } C_i(t) \qquad \text{(B3.7-2)}$$

$$c_m(t) = C_m(t) - C_{mss} \quad , \; C_{mss} = \text{steady state value of } C_m(t) \qquad \text{(B3.7-3)}$$

$$f_o(t) = F_o(t) - F_{oss} \quad , \; F_{oss} = \text{steady state value of } F_o(t) \qquad \text{(B3.7-4)}$$

$$f_d(t) = F_d(t) - F_{dss} \quad , \; F_{dss} = \text{steady state value of } F_d(t) \qquad \text{(B3.7-5)}$$

$$p(t) = P(t) - P_{ss} \quad , \; P_{ss} = \text{steady state value of } P(t) \qquad \text{(B3.7-6)}$$

and k_1, k_2 and k_3 are some appropriate constants.

However this original model is not appropriate for this simulation study because it does not involve any derivative terms. The FII technique has been developed for systems of higher order. A simulation study using this original system will not be able to demonstrate the capability of the FII technique for systems with derivatives. Consequently a second order dynamics representing the dilution flow valve is added in this simulation study. The modified concentration system is given by,

$$c_m(t) = b f_d(t-\tau) + \varepsilon(t) \qquad \text{(B3.7-7)}$$

$$(\rho^2 + a_1\rho + a_0)\, f_d(t) = u(t) \qquad \text{(B3.7-8)}$$

where b, a_0 and a_1 are the system parameters, $u(t)$ is the control signal driving the dilution valve and $\varepsilon(t)$ is the disturbance of the system given by,

$$c_m(t) = k_1\, c_i(t-\tau) \qquad \text{(B3.7-9)}$$

The deviation of furnish flow, f_o, does not appear in Equation B3.7-7 as the furnish flow is assumed constant. The variation in pressure, p, is omitted in this simulation for simplicity.

B3.7.2 Simulation

In the industrial study mentioned earlier, three separate concentration control systems were studied. Each of these systems are based on different streams of paper pulp. These different kinds of paper pulp are the termed by the company as kraft pulp, mixed groundwood pulp and broke pulp. However it was found that the broke and groundwood systems are not suitable for the proposed Smith predictor type control scheme because the disturbances dominate the behaviour of the system. In view of this, the simulation study given here is based on the kraft pulp system. The actual concentration of incoming pulp is applied in the simulation.

The values of system parameter and delay used in the simulation are,

$b = -9.048 \qquad a_1 = 2$

$\tau = 13.14 \qquad a_2 = 2$

Other details of the kraft concentration control system are given in Appendix B1.

The FII interval used in these simulations is 0.8 seconds, which was chosen according to the recommendation in Chapter 5. For simplicity, a single sampling interval of 0.2 seconds is used here for all the blocks in the self-tuning control system. Both the simulation step for the delay system and the approximation step for Smith predictor are set as 0.02 seconds. A recursive least squares algorithm with a forgetting factor of 0.99 is applied in the parameter estimator.

The simulation was performed using a commercial computer package called MATLAB. The programme written for this simulation is given in Appendix C.

B3.7.2 Results and Discussion

Three possible cases are simulated in this study:

1) The self tuning controller starts with zero values as initial guesses for the system parameters and delay.

2) As for case (1) but after 500 iterations (or 100 sec) the furnish flow is subjected to a step change, which effectively changes the value of parameter b to -5. The delay is maintained constant in this case.

3) As for case (1) but after 500 iterations the parameter b and the delay are changed to -5 and 7.32 seconds respectively.

For the purpose of comparison, the performance of a controller with fixed parameters is also simulated in each case. The fixed parameter controller is designed using the actual parameters of the original system.

Both the fixed parameter and self-tuning controllers are tuned to minimize the performance index, J, given by:

$$J = \sum_{k=0}^{\infty} \left\{ 1000 \left[C_o(k) - S \right]^2 + \left[F_d(k) - F_{dss} \right]^2 \right\}$$

where S is the desired furnish concentration.

In all these cases the variation in dilution flow is limited to ±25 litre per minute around steady state.

The simulation results for the three cases are shown in Figures B3.7-2 to B3.7-4. In these figures,

- part (a) shows the resultant furnish concentration of the fixed parameter controller.

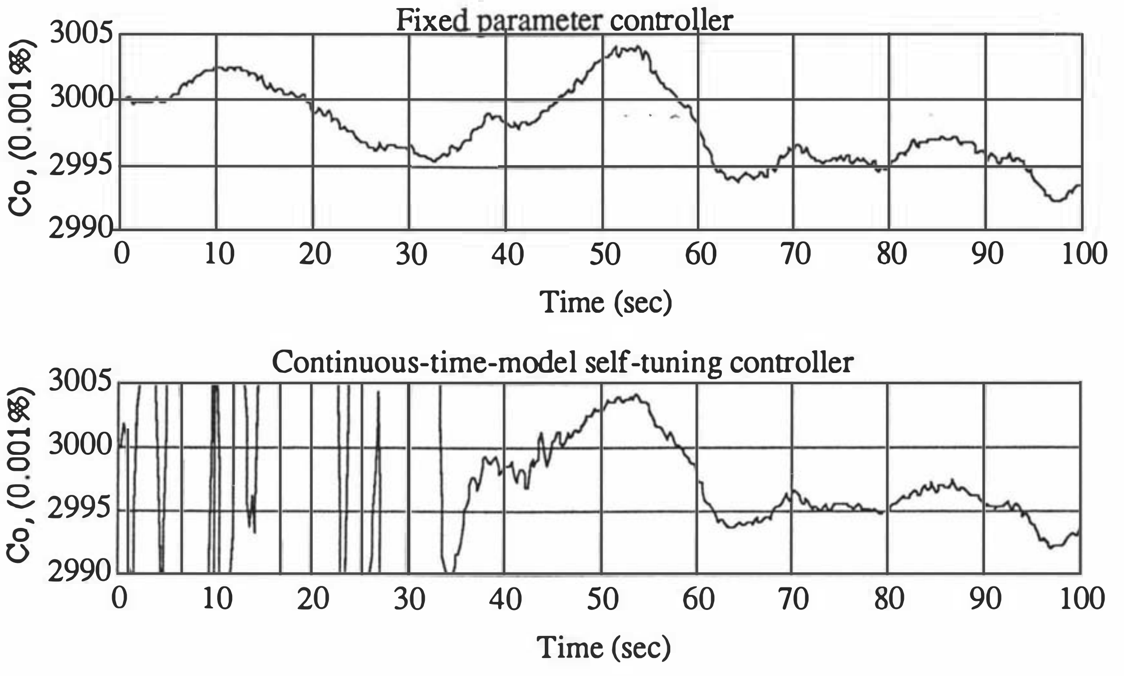
- part (b) shows the resultant furnish concentration of the self-tuning controller.

- part (c) shows the percentage error in estimates of the parameters.

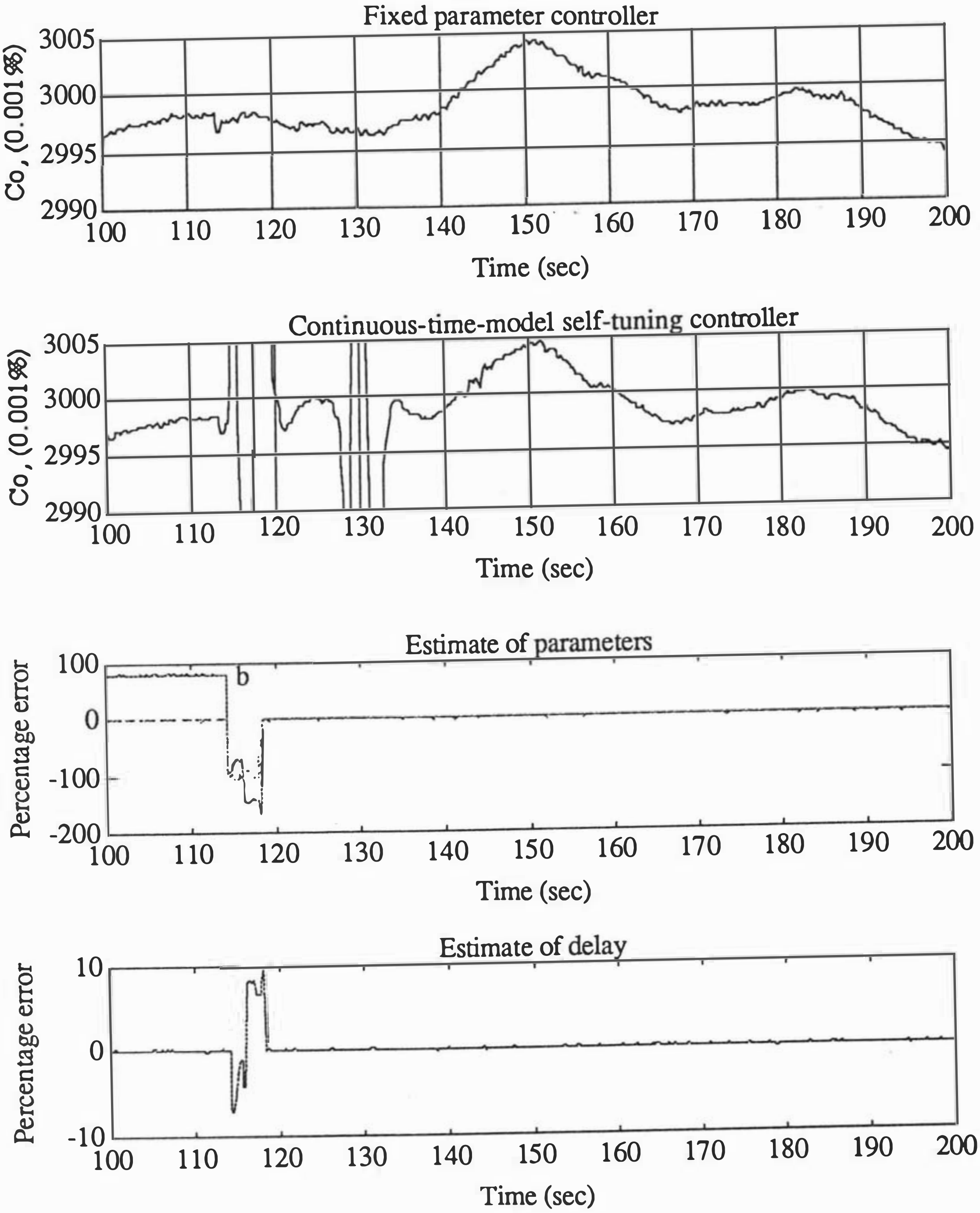
- part (d) shows the percentage error in estimates of the delay.

The unit of concentration in the figures is one thousandth percentage of fibre in paper stock by mass (0.001%).

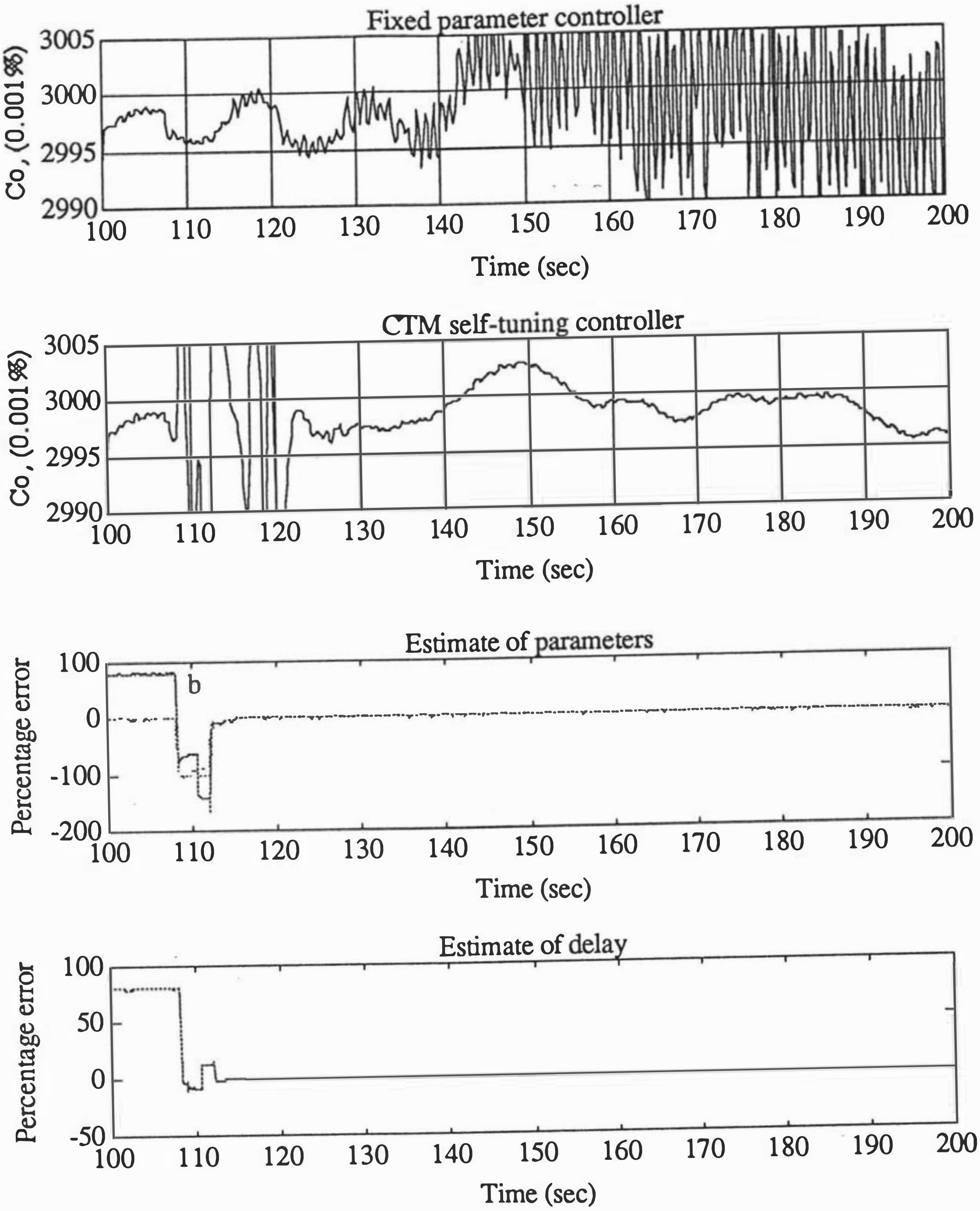
Figures B3.7-2 Simulation result of Case (1): zero initial guesses



Figures B3.7-3 Simulation result of Case (2): parameter b changes by -45% after 100 sec.



Figures B3.7-4 Simulation result of Case (3): parameter b changes by -45% and delay changes by -50% after 100 sec.



The feasibility of the proposed self-tuning control system for the delay system can be clearly seen from Figure B3.7-2. This figure shows that the performance of the self-tuning controller approaches the performance of the fixed parameter controller after a learning phase of about 25 second.

The replacement of a fixed parameter controller with the self-tuning controller in the concentration system might be hard to justify if the delay element is constant and known. It is shown in Figure B3.7-3 that the fixed parameter controller performs reasonably well, even though the system gain (parameter b) has changed by about 45%. Meanwhile the self-tuning controller needs to undergo a learning phase of about 15 second. Furthermore its eventual performance is very similar to the fixed parameter controller which is designed using the original parameter of $b = -9.048$.

However the usefulness of the self-tuning controller is highlighted in Figure B3.7-4 when the delay is changed. Here the fixed parameter controller becomes unstable, while the self-tuning controller manages to adjust itself after a learning period.

B3.8 SUMMARY

In this chapter a continuous-time-model self-tuning control scheme was developed for a class of delay systems. This self-tuning control system was formulated by coupling the on-line FII parameter and delay estimation technique with an appropriate control scheme. It is found that an appropriate control scheme for this purpose is a state-feedback control incorporating a Smith predictor, because such a control scheme couples nicely with the FII estimation technique and can make full use of the delay estimated by the FII technique. Furthermore this continuous-time control can be implemented easily on digital devices.

The performance of the self-tuning controller is demonstrated using a simulation study based on an industrial case study of paper stock concentration control. It is found that a self-tuning controller is most useful for the system with both uncertain delay and system parameters.

APPENDIX C

MATLAB PROGRAMMES FOR SIMULATION STUDY OF CONTINUOUS-TIME-MODEL SELF-TUNING CONTROL

Programmes List

CTSTC	Main programme for simulating continuous-time-model self-tuning control
CTSTC_ini	Sub-routine called by CTSTC to initialize variables needed for CTSTC
CTSTC_system	Sub-routine called by CTSTC to simulate output of concentration system.
CTSTC_control	Sub-routine called by CTSTC to find control action.
CTSTC_pe	Sub-routine called by CTSTC to estimate parameters and delay of the concentration system.
CTSTC_tuner	Sub-routine called by CTSTC to calculate control setting needed to tuner the controller.
FII	Function called by CTSTC_pe to find Fixed Interval Integral using numerical approximation.
BACKDIFF	Function called by CTSTC_pe to find backward difference.

CTSTC.m

```
%CTSTC.m          M files for continuous-time Self-tuning Control
% cm = vector of measured furnish pulp concentration (deviation)
% fd = vector of the control input, dilution flow (deviation)
% cohat =    vector estimated delay-free furnish concentration
%            by Smith Predictor
% cmhat = vector of estmated cm by Smith Predictor
%
% xs = iterative system states
% xsp = iterative Smith predictor states
% xc = iterative control states
%
% numhat = vector of estimated numerator
% denhat = vector of estimated denominator
% delayhat = vector of estimated delay
%
% Ae,Be,Ce,De = iterative estimate of system matrices
% delaye = iterative estimate of delay

% Defining paper-stock concentration system
Foss = 359 ;  Ciss = 3.2555e3 ;  k3 = -Ciss/Foss ;

num=[ k3 ]; den=[ 1 2 2 ];  delay = 13.14 ;
cmi = 0 % 0.1e3     ;  % initial output of system
E = [1;0];

%Defining Implementational parameters
Tbase = 0.05;
Tu =  0.5    ;   % Sampling interval for controller
no_subinterval = 10 ;  % No. of subinterval between Tu (must be =< 1 ),
                % it defines the base unit of fractional delay

ddmin = 0     ; % minimum no. of "exact delay"
ddmax = 5    ; % max. no. of "exact delay"
Tfii = Tu      ;
M =  6*Tfii  ;   % FII interval
lamda = 0.98  ;  % Forgettin factor for Parameter estimation
```

```
% Initialize Simulation
tend =  300*Tu       ; %ending time of simulation
Tsim = Tu     ; %simulation step ##must be exact divider of tend
CTSTC_ini

%Defining initial guess of system parameter and delay
numhat(1,:) = [  0 ];
denhat(1,:) = [ 0 0 0 ] ;
delayhat(1) = 0     ;

% Calculate initial control parameters
QQ=1e6*eye(2);      RR=1;
WW=1;   VV=1;
h=1 ; CTSTC_tuner

fprintf('Simulating :')
for h=2 : tend/Tsim
  %Continuous-time delay system with discrete-time control
  % @ return cm(h) and xs
  CTSTC_system

  %Calculate DT estimate of CT control.
  % need Ae,Be,Ce,De,delaye,Kc,Ke, no_subinterval,lcohat, Tsim, Tu,
  % xsp & xc     @@ return fd(h)
  CTSTC_control

  % parameter estimation @ return numhat(h), denhat(h) and
  CTSTC_pe

  % update control parameters
  %## need numhat(h), denhat(h) and delayhat(h)
  % @Return Ae,Be,Ce,De,delaye,Kc and Ke
  CTSTC_tuner
end
disp('Simulation  end')
return
```

CTSTC_ini.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%CTSTC_ini   M file to initialize simulation of CTSTC
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

[A,B,C,D] = tf2ss(num,den);
xs=zeros(length(A),1)   ;
xs(2) = cmi   ;          %initial states of the system
rsys = max(delay/Tbase+1, Tu/Tbase+1 ); ;
sys_buffer = zeros(rsys,1);

xsp = zeros(length(A),1);   %   - Smith predictor
xc = zeros(xsp) ;           %   - controller

cm=zeros(tend/Tsim,1) ;
lcohat = ddmax ;
cohat  =  zeros(lcohat,no_subinterval);

ysp=zeros(2,1)       ;  % input vector for estimating DT approx of CT fd(h)
fd=zeros(tend/Tsim,1) ;
cm=zeros(fd)  ;  cm(1) = cmi ;
numhat=zeros(tend/Tsim,length(num)  );
denhat=zeros(tend/Tsim,length(den)  );
return
```

CTSTC_system.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%CTSTC_system  M file to simulate delay system with discrete-time
%input  for CTSTC
%SWH Feb 1992
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
lud = Tu/Tbase +1 ;   ud = fd(h-1)*ones(lud,1);
[y,xs]=lsim(A, B ,C,D  ,ud, (0:Tbase:Tu),xs);
xs = xs(lud,:)';
sys_buffer(1:rsys-lud+1) = sys_buffer(lud :rsys) ;
sys_buffer(rsys-lud+1:rsys) =  y ;
cm(h)=sys_buffer(1) + ci(h);
return
```

CTSTC_control.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%CTSTC_control
% Mfiles to find DT estimate of CT optimal state feedback
% control with state estimator (Kalman filter)
%
% Ae,Be,Ce,De = pre-defined estimate of system parameters
% Kc      = pre-defined control gain
% Ke      = pre-defined state-estimator gain
% no_subinterval, lcohat, Tsim, Tu, xsp, xc
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Smith Predictor
%%% Estimate delay free co
[Ade,Bde] = c2d(Ae,Be,Tu/no_subinterval);
lud = no_subinterval+1;   ud = fd(h-1)*ones(lud,1);
[y,xsp]=dlsim(Ade,Bde,Ce,De,ud,xsp);
xsp=xsp(lud,:)' ;

%%% Storing into delay buffer
cohat(1:lcohat-1,:) = cohat(2:lcohat,:) ;
```

```
cohat(lcohat,:)= fliplr( y(2:lud)' )    ;

%% Separate delay estimate into two parts :
%% - exact delay in terms of no. of Tu
%% - fractional delay in terms of no of Tu/no_subinterval
exactdelay = fix(delaye/Tu) ;
fractdelay = round( rem(delaye,Tu) * (Tu/no_subinterval) ) ;
if (fractdelay == no_subinterval),  exactdelay = exactdelay+1; fractdelay=0;
end

%%% estimate cm (with delay)
cme =cohat(lcohat-exactdelay, fractdelay+1);
ysp(1) = ysp(2);  ysp(2) = cohat(lcohat) + cm(h) - cme ;

% state feedback control with state estimator, as if no delay
[Afb,Bfb,Cfb,Dfb] = reg(Ae,Be,Ce,De,Kc,Ke);
[fde, xc] = lsim(Afb,Bfb,Cfb,Dfb, ysp, (0:Tsim:Tsim)', xc );
xc=xc(2,:)' ; fd(h) = -1*fde(2);
return
```

CTSTC_pe.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%CTSTC_pe   M files to estimate system parameters
% ufii1 , ufii2 = iterative FII of piece-wise constant u
%
% SWH Oct 1991, Ref : Wanhing Siew's thesis
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

mTu=M/Tfii; sys_order=2 ;
ndelayarray = sys_order*mTu + ddmax - ddmin +1 ;

%Calculating FII
yfii1=FII( cm(1:h) ,Tfii,M);
yfii2=FII( yfii1   ,Tfii,M);

%Parameter estimation
clear xy xu1 xu2 xu3
```

```
xy=backdiff( [0;cm(1:h-1)] ,mTu,2) ;    % added unit delay for algorithm
xu1=-backdiff( yfii1(1:h),mTu,1 );
xu2=-yfii2(1:h);
xu3=   fd(1:h)*ones(1,ndelayarray);

na=0; nb=ones(1,2+ndelayarray);
nk= [1 1 [1+1+ddmin:ndelayarray+1+ddmin] ];

if h<=2,
   disp('initialize parameter estimation')
   thm=zeros(ndelayarray+2,1);
   phi=zeros(na+sum(nb+nk-1),1);
  P=1e4*eye(ndelayarray+2);
end

[thm,yhat,P,phi] =rarx([xy(h) xu1(h) xu2(h) xu3(h,:)], ..
        [na nb nk],'ff',lamda,thm,P,phi);
thm=thm';
a1hat=thm(1); a0hat=thm(2);
beta=thm(3:2+ndelayarray);

nctot= ndelayarray-2*mTu;  tot=zeros(nctot,1 );
for kk=1:nctot
  tot(kk)=sum( beta(kk:kk+2*mTu) );
end

dd=find( tot==max(tot) );
dd=dd(1);   % estimate of exact delay
bhat= sum( beta(dd:dd+2*mTu) )/mTu^2/Tu^2  ; % estimate b

df3= (beta(dd+1+3)-beta(dd+2*mTu-3))/2/Tu /bhat;
if (df3<0), df3=0;  end
if (df3>Tu), df3=Tu ; end
delaye=  (dd+ddmin)*Tu-df3;

numhat(h,:)= bhat          ;
denhat(h,:)= [ 1 a1hat a0hat ]          ;
delayhat(h) = delaye      ;
return
```

CTSTC_tuner

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% CTSTC_tuner   M file to calculate control parameters given
%               estimate of system parameters
%
% numhat = predefined estimate of system numeritor
% denhat = predefined estimate of system denominator
% QQ,RR =  predefined control weightings
% WW  =  predefined  system noise covariance
% VV  =  predefined measurement noise covariance
%
% Return Ae,Be,Ce,De, Kc, Ke and delaye
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
[Ae,Be,Ce,De]= tf2ss(numhat(h,:),denhat(h,:));
Kc = lqr(Ae,Be,QQ,RR);
Ke = lqe(Ae,E,Ce,WW,VV);
delaye = delayhat(h) ;
return
```

FII.m

```
function yfii = FII(y,T,M,method)
% FII Returns numerical approximation of Fixed-Interval-Integral
%  yfii = FII(y,T,M,method)
%  y = function input-- capable for muitivariable
%             each column is a variable
%  T = sampling interval of y (per unit time)
%  M = interval of FII (per unit time)
%  method = order of Newton-Cotes numerical method to approximate
%           the FII
%      = 0 : rectangular rule
%      = 1 : trapezoidal rule
%      = 2 : Simpson's 1/3 rule
%      = 3 : Simpson's 3/8 rule
%      = 4 : Boole's rule
%
```

```
% Assumption : All past values of y is zero
% SWH Feb 1992 Reference: WanHing Siew's thesis.

if ( abs( (M/T)-round(M/T) )>1e3*eps ),
   error('FII interval must be multiple of sampling interval');end
mm=M/T;

if nargin<4, method=2; end;      %Simpson1/3 by default
if method==0,
  n=0; Jn=T ;  end
if method==1,              % Trapezoidal rule
  n=1; Jn=T/2*[1 1]; end
if method==2,              % simpson 1/3 rule
  n=2; Jn=T/3*[1 4 1]; end
if method==3,              % simpson 3/8 rule
  n=3; Jn= T*3/8*[1 3 3 1]; end
if method==4,              % Boole's rule
  n=4; Jn= 2*T/45*[7 32 12 32 7]; end

if n==0,
   numFII = T*ones(1,mm+1) ; numFII(1) = 0;
else
  order = int2str(n);
  if ( abs( (mm/n)-round(mm/n) )>1e3*eps ), error(..
  ['FII interval must be multiple of ',order,' for ',order,' order rule']);
  end

  r=mm/n; w=zeros(1,mm+1);
   for k=0:r-1, w(k*n+1:(k+1)*n+1)=w(k*n+1:(k+1)*n+1)+Jn; end
  numFII=w;
end

denFII = [1 zeros(1,mm)] ;
yfii=filter(numFII, denFII, y);
%End of FII
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

backdiff.m

```
function dx = backdiff(x, shift,order)
%BACKDIFF          Return backward difference of x
%       dx = backdiff(x, shift,order)
%       order = order of backward difference
%       shift = number of data shift
%       x = data
%       dx = backward difference of x

if nargin ==2, shift =1; end
Jn=zeros(1,shift+1) ; Jn(1)=1; Jn(shift+1)=-1;
num=Jn;
for k=2:order,
    num=conv(Jn,num);
end

dx = filter(num,[1 zeros(1,order*shift) ] ,x);
%End of Backdiff
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```