Differential Geometry
of
Projectively Related
Finsler Spaces

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This Thesis is Dedicated to the Memory of

My Loving Mother

and

My Loving Father

Abstract

The aim of this thesis is to study the theory of Finsler spaces by considering the following main objectives.

(i) To present the basic concepts of Finsler geometry including connections, flag curvature, projective changes, Randers spaces and Finsler spaces with other types of (α, β)-metric, where α is a Riemannian metric and β is a one-form.

(ii) To introduce a Riemannian space of non-zero constant sectional curvature by considering a locally projectively flat Finsler space. The requirement for the Riemannian connection to be metric compatible gives a system of partial differential equations. Further, we compute two standard Riemannian metrics of non-zero constant sectional curvature by choosing two solutions of this system of partial differential equations.

(iii) To give two examples of locally projectively flat Randers metrics of scalar curvature by using a Riemannian metric computed in (ii) to illustrate the fact that some locally projectively flat Randers metrics of scalar curvature do not have isotropic S-curvature. We also prove that the scalar curvature of a Randers metric is not necessarily a constant if the metric has isotropic S-curvature and closed one-form by using an example.

(iv) To find necessary and sufficient conditions for Finsler spaces with various types of (α,β)-metric to be locally projectively flat and determine whether the conditions, a Riemannian metric (α) is locally projectively flat and a one-form (β) is closed, can occur at the same time in the locally projectively flat Finsler spaces with various types of (α,β)-metric.

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Notation

Throughout this dissertation the following notation is used.

M : - Differentiable manifold of dimension *n*.

p : - A point on M.

$(x^1,...,x^n)$: - Local coordinates of p denoted by x.

T_pM : - Tangent space of M at p.

y : - A tangent vector at p.

$(y^1,...,y^n)$: - Vector components of y.

$\left\{\frac{\partial}{\partial x^1},...,\frac{\partial}{\partial x^n}\right\}$: - A basis in T_pM.

$y = y^1\frac{\partial}{\partial x^1}+...+y^n\frac{\partial}{\partial x^n} = y^i\frac{\partial}{\partial x^i}$: - The Einstein summation convention.

$TM = \bigcup_{p\in M} T_pM$: - Tangent bundle of M.

(x,y) : - Local coordinates of a point in TM.

$\mathfrak{I}^n = (M,F)$: - A Finsler space of dimension *n*.

$F = F(x,y)$: - Finsler metric on M.

$l = (l^1,...,l^n)$: - Normalized supporting element of F, where $l^i = y^i/F$.

$l_i = \frac{\partial F}{\partial y^i}$: - Partial derivative of F with respect to y^i.

$g_{ij} = g_{ij}(x, y)$: - Finsler metric tensor of F (page 9, chapter 2).

$h_{ij} = h_{ij}(x, y)$: - Angular metric tensor of F (page 15, chapter 2).

$G^i = G^i(x, y)$: - Geodesic coefficients of F (page 19, chapter 2).

$G^i_{jk} = G^i_{jk}(x, y) = \dfrac{\partial^2 G^i}{\partial y^j \partial y^k}$: - Coefficients of the Berwald connection (page 19, chapter 2).

$\alpha = \alpha(x, y)$: - Riemannian metric defined on M.

$\Gamma^i_{jk} = \Gamma^i_{jk}(x)$: - Christoffel symbols of the Riemannian connection (page 17, chapter 2).

$\beta = \beta(x, y)$: - Differentiable one-form defined on M.

R : - Riemann curvature (page 31, chapter 3).

K : - Scalar curvature (page 36, chapter 3).

τ : - Distortion (page 40, chapter 3).

C : - Cartan torsion (page 40, chapter 3).

I : - Mean Cartan torsion (page 41, chapter 3).

S : - S-curvature (page 42, chapter 3).

E : - E-curvature (page 43, chapter 3).

L : - Landsberg curvature (page 48, chapter 3).

J : - Mean Landsberg curvature (page 49, chapter 3).

R^n : - *n*-dimensional real vector space.

All lower case Latin letters of the Einstein summation run from 1 to *n*.

That is, $i, j, k, r, s... = 1,...,n$.

Chapter 1

Introduction

Finsler geometry is an emerging branch of Mathematics, which appeared in 1918, after the pioneering work done by Paul Finsler, a German, who was the founder of Finsler geometry [Fi]. Finsler studied a geometry of a space with a generalized metric, which is called a Finsler space. The geometry of a Finsler space is called Finsler geometry. But Finsler did not investigate the geometric properties of the Finsler spaces very broadly. Soon after his graduation, he changed his research work to set theory. After P. Finsler, in the middle of 1920's L. Berwald [Be1], J. L. Synge [Sy] and J. H. Taylor [Ta] developed the theory of Finsler spaces as a generalization of Riemannian geometry.

In 1926 L. Berwald [Be2] was the first who introduced the concept of connection in Finsler geometry and defined a connection called the Berwald connection. In 1933, E. Cartan introduced a connection called the Cartan connection. The relation between the Berwald connection and the Cartan connection was given by his paper [Ca]. Next in 1943, S. S. Chern introduced a connection called the Chern connection [Ch1] [BaCh].

Since then, important contributions to Finsler geometry have resulted one after another. However, we have only a few books referring to Finsler geometry. The following mathematicians have published some books ([Ru] [Bu] [Ma1] [BaChSh] [Sh1] [Sh2]) in Finsler geometry by, H. Rund, H. Busemann, M. Matsumoto, S. S. Chern, Z. Shen, D. Bao, etc. Finsler geometry has applications in many areas of Mathematics in biology, physics, geology, etc. (See [AIM] [AbPa] [Mi] [MiAn] [Bj]). During the past decade, there has been further work on Finsler spaces with constant curvature, Finsler spaces with scalar curvature and isotropic S-curvature.

In chapter 2, we introduce some basic concepts of Finsler geometry, such as Finsler spaces and connections. There are many connections in Finsler geometry such as the Berwald, Cartan, Rund and Chern connections. Here, I am mainly concerned with the Berwald connection.

In chapter 3, we discuss flag curvature, for Finsler metrics. L. Berwald first successfully extended the notion of Riemann curvature to Finsler spaces [Be3] [Be4]. The flag curvature in Finsler geometry is an extension of the sectional curvature in Riemannian geometry [Be5] [Be6] [BaChSh]. For a Finsler space, at each point on the manifold the flag curvature is a function of a tangent plane and a vector in the plane called the pole vector. We say that a Finsler metric is of scalar curvature if the flag curvature is independent of the tangent planes containing the pole vectors. So the scalar curvature is a function on the tangent bundle. In dimension two, every Finsler metric is of scalar curvature. If the flag curvature is constant then the Finsler metric is said to be of constant flag curvature. One of the important problems in Finsler geometry is to characterize Finsler metrics of scalar curvature. In this chapter we also discuss several important non-Riemannian curvatures, some of them are the S-curvature, E-curvature Landsberg curvature and mean Landsberg curvature, which all vanish for Riemannian metrics. S-curvature was defined by Z. Shen in 1996 for a volume comparison theorem [Sh3]. Further details about S-curvature are given in [Sh1] [Sh2].

In chapter 4, we discuss projective geometry of Finsler metrics and introduce two essential projective invariants, the Douglas tensor and Weyl tensor, which play important roles in understanding the projective properties of Finsler metrics [Dg1] [Dg2] [We]. A Finsler space where both of these tensors vanish is characterized as a locally projectively flat Finsler space [Ma3]. In 1977 Z. I. Szabo proved that a Finsler space is of scalar curvature if and only if the Weyl tensor vanishes identically [Sz1]. The well-known problem in Finsler geometry is to study Finsler metrics with common geodesics, which are curves that minimize the arc length. Finsler metrics in which the geodesics are straight lines are said to be projective Finsler metrics or locally projectively flat (cf. page 55). The well-known Hilbert Fourth Problem in Finsler geometry is to study and characterize metrics on an open subset in the n-dimensional real vector space, whose geodesics are straight lines [Hi]. It is well known that every projective Finsler metric is of scalar curvature [Be5][Be6]. In 1961, A. Rapcsak found the necessary and sufficient conditions that a Finsler space is projective to another Finsler space [Rap]. This theorem is called the Rapcsak theorem, which is one of the most important theorems in projective Finsler geometry and will be used frequently throughout this dissertation.

In the early 20[th] century, P. Funk discovered the projective Finsler metrics on the unit ball in an n-dimensional real vector space, which are called Funk metrics [Fk1] [Fk2]. In 1983, T. Okada proved that the Funk metrics have constant negative flag curvature, -1/4, [Ok]. In 2001, Z. Shen constructed a projective Finsler metric with constant curvature zero defined on the unit ball in an n-dimensional real vector space [Sh4]. He also classified locally projectively flat Finsler metrics of constant flag curvature [Sh5].

The last section of this chapter (4.3) contains my original research work. The main purpose of this dissertation is to study the projective changes of Finsler metrics and construct some examples of the locally projectively flat Finsler metrics. M. Matsumoto's paper, [Ma3], is mainly concerned with the projective changes of Finsler metrics and locally projectively flat Finsler metrics. The remark in [Ma3] says that a locally projectively flat Finsler space of non-zero constant flag curvature may be a Riemannian space of non-zero constant sectional curvature. Motivated by his remark, in Proposition 4.3.1, we introduce a Riemannian space, which is of non-zero constant sectional curvature, using a locally projectively flat Finsler space. Using the requirement for the Riemannin connection to be metric compatible, we introduce a new system of partial differential equations for a Riemannian metric of non-zero constant sectional curvature. By considering a special class of solutions called radially symmetric solutions, we construct examples of Riemannian metrics of non-zero constant sectional curvature in Examples 4.3.1.

In chapter 5, we introduce Randers metrics, which are very special Finsler metrics with many interesting geometric properties. Randers metrics appear in both theory and applications and are built from just two pieces of familiar data, a Riemannian metric and a differential one-form. Both are defined on a common underlying smooth manifold. These metrics were introduced by G. Randers in 1941 in the context of general relativity [Ra]. Later on these metrics were applied to the theory of the electron microscope by R. S. Ingarden in 1957 [AIM]. Since then, mathematicians have made efforts to investigate the geometric properties of Randers metrics, particularly the conditions under which a Randers metric is of constant curvature [YS] [SSAY] [Ma2] [BaRo] [MS]. Recently this problem was solved by D. Bao, C. Robles and Z. Shen using Zermelo's navigation on Riemannian manifolds [BaRoSh]. It is also well known that a Randers metric is locally projectively flat if and only if the Riemannian metric is of constant sectional curvature and the differential one-form is closed. This follows from a result in [BaMa2] and the Beltrami theorem on locally projectively flat Riemannian metrics, which says that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. In 2001, Z. Shen classified all locally projectively flat Randers metrics with constant flag curvature [Sh8]. In 2003, X. Chen, X. Mo and Z. Shen also classified locally projectively flat Randers metrics with isotropic S-curvature [ChMoSh].

This chapter also contains my original work. In Lemma 5.1.7 and Lemma 5.2.5, we use the Riemannian metric found in Example (i) of Examples 4.3.1 to show that the Funk metrics on the unit ball in Euclidean space are locally projectively flat Randers metrics of constant flag curvature with constant S-curvature. In the literature, all known Randers metrics are of scalar curvature with isotropic S-curvature. Motivated by this, in Example 5.3.2, we show that some Randers metrics of scalar curvature do not have isotropic S-curvature. Finally we also give answer to the following open problem announced on the workshop held in Mathematical Sciences Research Institute in Berkeley, California, USA in 2002, by using an example given in [ChMoSh]. Can the scalar curvature of a Randers metric be a constant if it has isotropic S-curvature and one-form is closed?

Starting with chapter 6, we study other types of Finsler spaces with (α,β)–metric, where α is a Riemannian metric and β is a one-form defined on a differentiable manifold. Finsler spaces with (α,β)–metric were introduced by several authors in [Kr] [Ma4]

[Ma5]. The projective geometry of these spaces has been studied in [Ma6] [HaIc] [Sh] [BaMa1] [PaLe]. But the classification of locally projectively flat Finsler spaces with (α,β)–metric with constant flag curvature is still open [Sh10]. Therefore, in this chapter, we study locally projectively flat Finsler spaces with (α,β)–metric. This chapter contains my original work except for Theorem 6.3.1, 6.3.2 and 6.3.3. In Theorem 6.2.2, 6.2.4 and 6.2.6, we obtain necessary and sufficient conditions for each Finsler space with (α,β)–metric to be locally projectively flat. In Theorem 6.2.3, 6.2.5 and 6.2.7, we find the scalar curvature of each type of locally projectively flat Finsler spaces with (α,β)–metric when α is locally projectively flat. Then in Theorem 6.2.1, we investigate whether the conditions, α being locally projectively flat and β being closed, can be present in the locally projectively flat Finsler spaces with (α,β)–metric.

Next we consider Finsler spaces with (α,β)–metric, where α is locally projectively flat and β is closed. In Theorem 6.2.10, we give necessary and sufficient conditions for these Finsler spaces to be not locally projectively flat. Another important problem in Finsler geometry is to classify Finsler metrics of constant flag curvature, which are not locally projectively flat. This problem turns out to be very difficult. The first step is to construct as many examples as possible. However, in the literature, D. Bao and Z. Shen have constructed some Finsler metrics with constant flag curvature, which are not locally projectively flat [BaSh], [Sh6] and [Sh7]. So far all of them are of Randers type. Motivated by this in Example 6.2.1 and 6.2.2, we introduce some Finsler spaces with (α,β)–metric which are not locally projectively flat using the Riemannian metrics and one-forms given in Lemma 5.1.7 and Example 5.3.2.

Finally in chapter 6.3, we study Finsler spaces with (α,β)–metric of Douglas type and in examples 6.3.1, we investigate whether the not locally projectively flat Finsler metrics given in Example 6.2.1 and 6.2.2 can be used to define Finsler spaces with (α,β)–metric of Douglas type. In Lemma 6.3.4, we also introduce a Finsler space with (α,β)–metric which is not locally projectively flat using a Randers metric with isotropic S-curvature. In Examples 6.3.2, we also construct examples of Finsler spaces with (α,β)–metric which are not locally projectively flat using Lemma 6.3.4.

The final chapter gives the summary and conclusions of the research. Note that the following are my own original work.

In chapter 4,

Proof of Lemma 4.1.3 (pages 55-57), Proposition 4.3.1 (pages 69-76), Radially symmetric solutions (pages 76-77), Lemma 4.3.2 (page 77), Corollary 4.3.3 (page 78), Lemma 4.3.4 (page 78), Theorem 4.3.5 (page 78-79), Solutions of the equation (4-61) (pages 79-80), 4.3.1 Examples of Riemannian metrics of non-zero constant sectional curvature (pages 80-81), Lemma 4.3.6 (pages 81-83) and Examples 4.3.2 (pages 84-86).

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Chapter 2

Preliminaries

Before we go into details, we need to introduce some basic definitions, results and notations in Finsler geometry. The purpose of this chapter is to review the basic language of Finsler geometry and to provide a quick reference. Further details can be found in the following sources: [Sh1] [Sh2] [BaChSh] [AIM] [Ma1]

2.1 Finsler Spaces

The theory of Finsler spaces has its origins in the calculus of variations. Finsler geometry is the geometry of integrals of the form

$$s = \int_a^b F(x^i, dx^i/dt)dt. \tag{2-1}$$

Here, s is a generalised notion of arc length of a curve γ described by $x(t) = (x^i(t), i = 1,\ldots,n),\ t \in [a,b]$ on a differentiable manifold of dimension *n*, and F is a positive smooth function of the x^i and dx^i/dt for $i = 1,\ldots,n$. For any $t \in [a,b]$, the $x^i(t)$ correspond to local coordinates of a point p on γ, and the dx^i/dt evaluated at p correspond to components of the tangent vector to γ at p. Now, s must be independent of the choice of the parameter; hence, we require that F is a positively homogeneous function of degree one in the variables dx^i/dt.

That is, for $\lambda > 0$,

$$F(x^i, \lambda y^i) = \lambda F(x^i, y^i),$$

where $y^i = dx^i / dt$. Finsler spaces are smooth manifolds equipped with a Finsler metric. Before we define a Finsler metric, we will give the following definitions [Sh1].

Definition 2.1.1

A differentiable manifold of dimension n is a set together with a family of injective mappings $x_a : U_a \subset R^n \to M$ of open sets U_a of R^n into M such that

(i) $\bigcup_a x_a(U_a) = M$

(ii) For any pair a, b with $x_a(U_a) \cap x_b(U_b) = W \neq \Phi$, the sets $x_a^{-1}(W)$, $x_b^{-1}(W)$ are open sets in R^n and the mappings $x_a^{-1} \circ x_b$, $x_b^{-1} \circ x_a$ are differentiable.

A family $\{(U_a, x_a)\}$ satisfying (i) and (ii) is called a **differentiable structure** on M.

Definition 2.1.2

Let M be an n-dimensional differentiable manifold. The tangent bundle TM consists of all tangent vectors on M with the natural manifold structure. Let $\pi : TM \to M$ denote the natural projection. Let (U, φ) be a local coordinate system in M. Namely, U is an open subset of M and

$$\varphi = (\varphi^1, \dots, \varphi^n) : U \to R^n$$

is a diffeomorphism onto an open subset of R^n. φ^i's are functions on U and their values $x^i = \varphi^i(x)$ at a point $p \in U$ are called the coordinates of p. Such a map φ is called a coordinate map on M. The coordinate map φ induces a map

$$\hat{\varphi} = (\hat{\varphi}^1, \dots, \hat{\varphi}^{2n}) : \hat{U} := \pi^{-1}(U) \to R^n \times R^n$$

which is defined by

$$\hat{\varphi}(y) = (x^1, \dots, x^n, y^1, \dots, y^n),$$

where $y = y^i \frac{\partial}{\partial x^i} \in T_pM$ and $\varphi(x) = (x^1,\dots,x^n)$. $\hat{\varphi}$ is a diffeomorphism from $\hat{U}$ onto an open subset in R^{2n}. We call $(\hat{U},\hat{\varphi})$ the **standard local coordinate system** in TM. For simplicity, we usually let (x^i) stand for (U,φ) and (x^i,y^i) for $(\hat{U},\hat{\varphi})$. The coordinate map φ induces n linearly independent vector fields on U denoted by $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^{n}$, and the standard local coordinate map $\hat{\varphi}$ induces $2n$ linearly independent vector fields on $\hat{U}$ denoted by $\left\{\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^i}\right\}_{i=1}^{n}$. For a scalar function F on TM, $F(x,y)$ is viewed as a function of (x^i,y^i) in a standard local coordinate system.

This thesis is concerned with the local geometry of Finsler spaces and standard local coordinate systems are used extensively. We also use the Einstein summation convention: repeated indices indicate the summation running from 1 to n. Now we define a Finsler metric on a manifold.

Definition 2.1.3

Let M be an n-dimensional differentiable manifold. A **Finsler metric** on M is a function $F : TM \to R$ which has the following properties.

(i) $F(x,y) > 0$,

(ii) $F(x,\lambda y) = \lambda F(x,y), \qquad \lambda > 0,$

(iii) $F(x,y)$ is C^∞ on $TM \setminus \{0\}$,

(iv) The **Finsler metric tensor,**

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} \tag{2-2}$$

is positive definite for all $(x,y) \in TM \setminus \{0\}$.

(M,F) is called a **Finsler space of dimension *n*** and is denoted by $\mathfrak{I}^n$.

Since F is a positively homogeneous function of degree one in y, we use the following theorem for homogeneous functions frequently.

Theorem 2.1.1 (Euler's Theorem)

Suppose a real-valued function H on R^n is differentiable away from the origin of R^n, where R^n is the *n*-dimensional real vector space. Then the following two statements are equivalent.

(i) H is positively homogeneous of degree r, i.e., for all $\lambda > 0$

$$H(\lambda y) = \lambda^r H(y),$$

where $y = (y^1, \ldots, y^n)$.

(ii) $y^i \dfrac{\partial H(y)}{\partial y^i} = rH(y)$.

The proof of this result can be found in [BaChSh].

Definition 2.1.4

For a non-zero vector $y \in T_pM \setminus \{0\}$, F induces an **inner product** on T_pM defined by

$$g_y(u,v) = g_{ij}(x,y)u^i v^j, \tag{2-3}$$

where the fundamental form g_y on T_pM is a bilinear symmetric form, $u = (u^1, \ldots, u^n) \in T_pM$ and $v = (v^1, \ldots, v^n) \in T_pM$.

Since the $g_{ij}(x,y)$ are positively homogeneous functions of degree zero in y, the $g_{ij}(x,y)$ are invariant under $y \to \lambda y$ for all $\lambda > 0$, and there is exactly one inner product for each direction. Let us show that g_y is independent of the choice local coordinates. When we change the local coordinate system (x^i) in the base manifold M, to another local coordinate system $(\hat{x}^i)$, the corresponding standard local coordinate system (x^i, y^i) in TM changes subject to the following coordinate transformation

$$\hat{x}^p = \hat{x}^p(x^1,...,x^n)$$

$$\hat{y}^p = \frac{\partial \hat{x}^p}{\partial x^i} y^i.$$

Then, we first show that g_{ij} given in equation (2-2) transforms to the following rule. Let $\hat{g}_{pq}$ be the Finsler metric tensor subject to the new coordinate system. Now, equation (2-2) gives

$$\hat{g}_{pq} = \frac{1}{2} \frac{\partial^2 F^2}{\partial \hat{y}^p \partial \hat{y}^q}.$$

Using the above coordinate transformation, we have

$$\hat{g}_{pq} = \frac{1}{2} \frac{\partial}{\partial \hat{y}^p} \left(\frac{\partial x^j}{\partial \hat{x}^q} \frac{\partial F^2}{\partial y^j} \right)$$

$$= \frac{1}{2} \frac{\partial x^j}{\partial \hat{x}^q} \frac{\partial}{\partial y^i} \left(\frac{\partial F^2}{\partial y^j} \right) \frac{\partial y^i}{\partial \hat{y}^p}$$

$$= \frac{1}{2} \frac{\partial x^j}{\partial \hat{x}^q} \frac{\partial^2 F^2}{\partial y^i \partial y^j} \frac{\partial x^i}{\partial \hat{x}^p}.$$

Hence equation (2-2) implies

$$\hat{g}_{pq} = \frac{\partial x^i}{\partial \hat{x}^p} \frac{\partial x^j}{\partial \hat{x}^q} g_{ij}.$$

Now, contracting the above expression with $\hat{u}^p$ and $\hat{v}^q$, we have

$$\hat{g}_{pq} \hat{u}^p \hat{v}^q = \frac{\partial x^i}{\partial \hat{x}^p} \frac{\partial x^j}{\partial \hat{x}^q} g_{ij} \hat{u}^p \hat{v}^q,$$

where $\hat{u}^p$ and $\hat{v}^q$ are the vector components of u and v subject to the new coordinate system respectively. Then from the coordinate transformation, the above expression implies

$$\hat{g}_{pq}\hat{u}^p\hat{v}^q = \frac{\partial x^i}{\partial \hat{x}^p}\frac{\partial x^j}{\partial \hat{x}^q} g_{ij} \frac{\partial \hat{x}^p}{\partial x^k} u^k \frac{\partial \hat{x}^q}{\partial x^l} v^l .$$

This simplifies to

$$\hat{g}_{pq}\hat{u}^p\hat{v}^q = g_{ij}u^i v^j .$$

Hence g_y is independent of the choice local coordinates.

From now on we write g_{ij} instead of $g_{ij}(x,y)$ and F instead of $F(x,y)$ for our convenience. Using equation (2-2), we now show that $F^2 = g_{ij}y^i y^j$. Contracting g_{ij} given in equation (2-2) with y^i and y^j gives

$$g_{ij}y^i y^j = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} y^i y^j .$$

Since F^2 is a homogeneous function of degree two in y and $\partial F^2 / \partial y^j$ is a homogeneous function of degree one in y, Euler's Theorem implies

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F^2}{\partial y^j}\right) y^i = \frac{\partial F^2}{\partial y^j}$$

and

$$\frac{\partial F^2}{\partial y^j} y^j = 2F^2 ;$$

therefore,

$$F^2 = g_{ij}y^i y^j . \tag{2-4}$$

If $u = v = y$, then from equation (2-3)

$$g_y(y,y) = g_{ij}y^i y^j = F^2 .$$

Let

$$l^i = \frac{y^i}{F} , \tag{2-5}$$

where $l^i = l^i(x, y)$ are vector fields on $TM \setminus \{0\}$. Then equation (2-4) shows that

$$g_{ij} l^i l^j = 1,$$

and hence the vector field $l = (l^1, ..., l^n)$ has unit length with respect to the fundamental form g_y. That is, $|l| = 1$. The vector field l is called the **normalized supporting element**. Note that $F(x, l) = 1$.

The absolute length $|y|$ of a tangent vector $y \in T_pM$ is defined as $|y| = F(x, y)$. The function $F(x, y)$ is called the **Minkowski norm** on $TM \setminus \{0\}$.

We note here two simple results of interest.

(i) Let

$$l_i = \frac{\partial F}{\partial y^i}. \qquad (2\text{-}6)$$

Euler's Theorem implies

$$l_i y^i = \frac{\partial F}{\partial y^i} y^i = F.$$

Relation (2-5) implies

$$l^i l_i = 1. \qquad (2\text{-}7)$$

(ii) Let $y_i = g_{ij} y^j$. Then from equation (2-2)

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} = \frac{1}{2}\frac{\partial}{\partial y^i}\left(\frac{\partial F^2}{\partial y^j}\right) = \frac{\partial}{\partial y^i}\left(F\frac{\partial F}{\partial y^j}\right)$$

$$= \frac{\partial F}{\partial y^i}\frac{\partial F}{\partial y^j} + F\frac{\partial^2 F}{\partial y^i \partial y^j}.$$

Contracting g_{ij} with y^j and using the property of homogeneity of F, we have

$$g_{ij} y^j = F\frac{\partial F}{\partial y^i}.$$

Since $y_i = g_{ij}y^j$, the above expression gives

$$y_i = F\frac{\partial F}{\partial y^i}.$$

From relation (2-6), we can see that an alternative expression for l_i is given by

$$l_i = \frac{y_i}{F}.$$

In addition, it is clear that

$$y_i y^i = F^2.$$

Special Cases

We have the following important special cases for Finsler spaces.

(i) If the g_{ij}' s are independent of y, then

$$F = \sqrt{g_{ij}(x)y^i y^j} = |\, y\,|$$

and the resulting space is called **Riemannian**.

(ii) If there exists a local coordinate system (x) on M such that in terms of the corresponding local coordinates (x,y) on TM, $F(x,y)$ is a function of y only then the resulting space is called **Locally Minkowski Space.** In this case

$$F = \sqrt{g_{ij}(y)y^i y^j} = |\, y\,|$$

(iii) If $g_{ij} = \delta_{ij}$, then

$$F = \sqrt{\delta_{ij}y^i y^j} = \sqrt{(y^1)^2 + \ldots + (y^n)^2} = \sqrt{\langle y, y\rangle} = |\, y\,|,$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. In this case $|\, y\,|$ is the standard Euclidean norm.

Definition 2.1.5

For a non-zero vector $y \in T_pM \setminus \{0\}$, define

$$h_y(u,v) = h_{ij}(x,y)u^i v^j, \tag{2-8}$$

where $u=(u^1,...,u^n)\in T_pM$, $v=(v^1,...,v^n)\in T_pM$ and

$$h_{ij}(x,y)=g_{ij}-l_il_j. \tag{2-9}$$

h_y is called the **angular metric form** on T_pM associated with y, and $h_{ij}=h_{ij}(x,y)$ is called the **angular metric tensor.**

Lemma 2.1.2

The angular metric has the following properties:

(i) $h_y(y,y)=0$;

(ii) $h_{ij}=F\dfrac{\partial^2 F}{\partial y^i \partial y^j}$;

(iii) $h_i^k=\delta_i^k-l_il^k$.

Proof:

(i)

If $u=v=y$, then (2-8) and (2-9) imply

$$h_y(y,y)=h_{ij}y^iy^j=g_{ij}y^iy^j-l_il_jy^iy^j.$$

Equations (2-4) and (2-6) give

$$h_y(y,y)=F^2-\frac{\partial F}{\partial y^i}\frac{\partial F}{\partial y^j}y^iy^j,$$

and Euler's Theorem implies

$$h_y(y,y)=0.$$

(ii)

Recall that (cf. page 13)

$$g_{ij}=\frac{\partial F}{\partial y^i}\frac{\partial F}{\partial y^j}+F\frac{\partial^2 F}{\partial y^i \partial y^j}.$$

Using the definition of l_i, equation (2-6), we see that

$$g_{ij} = l_i l_j + F \frac{\partial^2 F}{\partial y^i \partial y^j}.$$

Relation (2-9) thus shows that

$$h_{ij} = F \frac{\partial^2 F}{\partial y^i \partial y^j}. \tag{2-10}$$

(iii)

We know that the indices on objects are lowered and raised by (g_{ij}) and its inverse matrix (g^{ij}) respectively. Therefore, contracting h_{ij} given in (2-9) with g^{jk} gives

$$h_{ij} g^{jk} = g_{ij} g^{jk} - l_i l_j g^{jk},$$

and since $g_{ij} g^{jk} = \delta_i^k$, where $\delta_i^k = 0$ if $k \neq i$ and $\delta_i^k = 1$ if $k = i$,

$$h_i^k = \delta_i^k - l_i l^k. \tag{2-11}$$

Definition 2.1.6

Let $\Im^n = (M, F)$ and $\overline{\Im}^n = (M, \overline{F})$ be two Finsler spaces on a common underlying manifold M of dimension *n*. F and $\overline{F}$ are said be **conformal** if there exists a factor $\psi(x, y)$ such that

$$\overline{g}_{ij} = \psi(x, y) g_{ij}, \tag{2-12}$$

where g_{ij} and $\overline{g}_{ij}$ are Finsler metric tensors of F and $\overline{F}$ respectively.

If F and $\overline{F}$ are conformal then contracting $\overline{g}_{ij}$ with y^i and y^j and using relations (2-12) and (2-4) gives,

$$\overline{F}^2 = \psi(x, y) F^2. \tag{2-13}$$

Lemma 2.1.3

(i) If a Finsler space $\mathfrak{F}^n$ admits a conformal correspondence with a Riemannian space then $\mathfrak{F}^n$ must be a Riemannian space.

(ii) The geodesics of $\mathfrak{F}^n$ with respect to the Finsler metric F are not, in general, geodesics with respect to the Finsler metric $\overline{F}$, which is conformal to F. However, the conformal transformation leaves geodesics invariant if and only if ψ is a constant.

The above results can be found in [Ru].

Special Case

In the case of Riemannian metric tensors, which are conformal to each other, ψ depends only on x and hence equation (2-12) reduces to

$$\overline{g}_{ij}(x) = \psi(x) g_{ij}(x). \tag{2-14}$$

If ψ =constant, then the Riemannian metrics are said to be **homothetic.** Homothetic transformations were studied by S. Shanks in Riemannian geometry [Sa].

2.2 Connections

To introduce geometric quantities, we need to first introduce connections. Various connections in Finsler geometry, were developed by J. L. Synge (1925), J. H. Taylor (1925), L. Berwald (1926), E. Cartan (1934), S. S. Chern (1948), among others. In Riemannian geometry, there is a unique connection, called the **Riemannian connection**, which was introduced by Levi-Civita using Christoffel symbols. It has two remarkable properties.

(1) The connection is compatible with the metric. In other words, the covariant derivative $g_{ij|k}$ of the metric tensor vanishes. This means

$$g_{ij|k} = \frac{\partial g_{ij}(x)}{\partial x^k} - g_{rj}(x)\Gamma^r_{ik}(x) - g_{ir}(x)\Gamma^r_{jk}(x) = 0. \tag{2-15}$$

Here Γ^i_{jk} denote the **Christoffel symbols**

$$\Gamma^i_{jk}(x) = \frac{1}{2} g^{ri}(x) \left(\frac{\partial g_{rj}(x)}{\partial x^k} + \frac{\partial g_{rk}(x)}{\partial x^j} - \frac{\partial g_{jk}(x)}{\partial x^r} \right). \tag{2-16}$$

(2) Torsion is zero. This means that $\Gamma^i_{jk}(x) = \Gamma^i_{kj}(x)$.

In Finsler geometry, a connection cannot have both these properties. There are several linear connections in Finsler geometry such as the Berwald connection [Be2] [Be3], Cartan connection [Ca], Chern connection [Ch1] [Ch2] [BaCh]. L. Berwald was the first to introduce the concept of connection in Finsler geometry. Berwald started his theory from the equations of geodesics to define the Berwald connection [Be2]. The Berwald connection is not metric compatible, but its torsion is zero. The Cartan connection is metric compatible, but its torsion is non-zero. The relation between the Berwald connection and the Cartan connection is given by [Ca]. The Chern connection is not metric compatible but its torsion is zero. Before we introduce the Berwald connection, we give the following definition given in [Sh11].

Definition 2.2.1

A C^∞ curve in a Finsler space is a **geodesic** if it has constant speed and is locally minimizing. Thus a geodesic in (M, F) is a curve $c: I = [a,b] \to M$ with $F(c(t), \dot{c}(t)) =$ constant and for any $t_0 \in I$, there is a small number $\varepsilon > 0$ such that c is minimizing on $[t_0 - \varepsilon,\ t_0 + \varepsilon] \cap I$.

Using the calculus of variations, one can show that geodesics in a Finsler space $\Im^n = (M, F)$ are determined by a system of second order ordinary differential equations. If a geodesic is represented locally by the equations $x = x^i(t),\ i = 1, \ldots, n$ for an arbitrary parameter t, then the equations of a geodesic of $\Im^n$ are given by

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt} \right) = \psi(t) \frac{dx^i}{dt},$$

where $\psi(t)=\left(\dfrac{d^2 s}{dt^2}\right)\Big/\left(\dfrac{ds}{dt}\right)$, s is defined by the equation (2-1) and $G^i = G^i(x,y)$ are local functions on $TM \setminus \{0\}$ defined by

$$G^i = \frac{1}{4} g^{ik}\left\{\left(\frac{\partial^2 F^2}{\partial y^k \partial x^j}\right) y^j - \frac{\partial F^2}{\partial x^k}\right\}. \tag{2-17}$$

The G^i are called the **geodesic coefficients** of F, and are positively homogeneous functions of degree two in y.

The Berwald Connection

Throughout this thesis we use the Berwald connection, which is easily expressed in terms of the geodesic coefficients and is most commonly used in the literature on projective Finsler geometry. Let

$$G^i_k = G^i_k(x,y) = \frac{\partial G^i}{\partial y^k}$$

and

$$G^i_{kj} = G^i_{kj}(x,y) = \frac{\partial^2 G^i}{\partial y^j \partial y^k}.$$

The G^i_{jk} are called the **coefficients of the Berwald connection.**

Note that G^i_j and G^i_{jk} are local functions on $TM \setminus \{0\}$. Then from the coordinate transformation given in page 11, the quantities G^i_j transform according to the following rule given in [BaChSh].

$$\hat{G}^p_q = \frac{\partial \hat{x}^p}{\partial x^i}\frac{\partial x^j}{\partial \hat{x}^q} G^i_j + \frac{\partial \hat{x}^p}{\partial x^i}\frac{\partial^2 x^i}{\partial \hat{x}^q \partial \hat{x}^r}\hat{y}^r. \tag{2-17a}$$

Differentiating the above expression with respect to $\hat{y}^r$ gives

$$\hat{G}^p_{qr} = \frac{\partial \hat{x}^p}{\partial x^i}\frac{\partial x^j}{\partial \hat{x}^q}\frac{\partial x^k}{\partial \hat{x}^r}G^i_{jk} + \frac{\partial \hat{x}^p}{\partial x^i}\frac{\partial^2 x^i}{\partial \hat{x}^q \partial \hat{x}^r}. \tag{2-17b}$$

Note that the second term on the right hand side of the above expression is independent of (y^i). Differentiating the above with respect to y^l yields

$$\frac{\partial \hat{G}^p_{qr}}{\partial \hat{y}^s}\frac{\partial \hat{x}^s}{\partial x^l} = \frac{\partial \hat{x}^p}{\partial x^i}\frac{\partial x^j}{\partial \hat{x}^q}\frac{\partial x^k}{\partial \hat{x}^r}\frac{\partial G^i_{jk}}{\partial y^l}. \tag{2-17c}$$

This implies that $\frac{\partial G^i_{jk}}{\partial y^l}$ are the coefficients of a tensor on $TM \setminus \{0\}$.

Definition 2.2.2

Let $q \in TM \setminus \{0\}$ with local coordinates $(x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^n)$ and

$$T(TM \setminus \{0\}) = \bigcup_{q \in TM \setminus \{0\}} T_q(TM \setminus \{0\}).$$

Then $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\right\}$ is a basis of $T_q(TM \setminus \{0\})$.

Let

$$HTM = \text{span}\left\{\frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n}\right\}$$

and

$$VTM = \text{span}\left\{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\right\},$$

where

$$\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - G^j_k \frac{\partial}{\partial y^j}, \tag{2-18}$$

The above argument gives a direct decomposition of the tangent bundle of $TM \setminus \{0\}$. This means that

$$T(TM \setminus \{0\}) = HTM \oplus VTM.$$

The spaces HTM and VTM are called the **horizontal tangent bundle** and the **vertical tangent bundle** over $TM \setminus \{0\}$ respectively.

We prove the following important lemmas.

Lemma 2.2.1 ([Sh1])

For any Finsler metric on a manifold,

$$\frac{1}{2}\left\{\frac{\partial g_{ij}}{\partial x^k} - g_{il}G^l_{jk} - g_{jl}G^l_{ik} - 2C_{ijl}G^l_k\right\}$$

$$= 2G^l \frac{\partial C_{ijk}}{\partial y^l} + C_{ljk}G^l_i + C_{ilk}G^l_j + C_{ijl}G^l_k - y^l \frac{\partial C_{ijk}}{\partial x^l}, \qquad (2\text{-}19)$$

where

$$C_{ijk}(x,y) = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}. \qquad (2\text{-}20)$$

Proof:

Let us find G^i given in equation (2-17) in terms of g_{ij}. Equation (2-4) gives

$$F^2 = g_{lr}y^l y^r.$$

Differentiating above expression with respect to x^j, we have

$$\frac{\partial F^2}{\partial x^j} = \frac{\partial g_{lr}}{\partial x^j} y^l y^r.$$

Then differentiating this expression with respect to y^k and simplifying, we have

$$\frac{\partial}{\partial y^k}\left(\frac{\partial F^2}{\partial x^j}\right) = \frac{\partial}{\partial x^j}\left(\frac{\partial g_{lr}}{\partial y^k} y^l y^r\right) + 2\frac{\partial g_{rk}}{\partial x^j} y^r.$$

Using equation (2-2) gives

$$\frac{\partial}{\partial y^k}\left(\frac{\partial F^2}{\partial x^j}\right) = \frac{1}{2}\frac{\partial}{\partial x^j}\left\{\frac{\partial}{\partial y^l}\left(\frac{\partial^2 F^2}{\partial y^k \partial y^r}\right) y^l y^r\right\} + 2\frac{\partial g_{rk}}{\partial x^j} y^r.$$

Using the homogeneity of F gives

$$\frac{\partial}{\partial y^k}\left(\frac{\partial F^2}{\partial x^j}\right) = 2\frac{\partial g_{rk}}{\partial x^j}y^r.$$

Hence equation (2-17) implies

$$G^i = \frac{1}{4}g^{il}\left\{2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right\}y^j y^k, \tag{2-21}$$

which can be recast as

$$g_{jl}G^l = \frac{1}{4}\left\{2\frac{\partial g_{jr}}{\partial x^l} - \frac{\partial g_{rl}}{\partial x^j}\right\}y^r y^l. \tag{2-22}$$

Differentiating (2-22) with respect to y^i, using (2-20) and the homogeneity of F gives

$$g_{jl}G^l_i + 2C_{ijl}G^l = \frac{1}{2}\left\{\frac{\partial g_{ij}}{\partial x^r} + \frac{\partial g_{jr}}{\partial x^i} - \frac{\partial g_{ir}}{\partial x^j}\right\}y^r. \tag{2-23}$$

Differentiating (2-23) with respect to y^k and using (2-20) yields

$$g_{jl}G^l_{ki} = -2C_{kjl}G^l_i - 2C_{ijl}G^l_k - 2G^l\frac{\partial C_{ijl}}{\partial y^k} + \frac{1}{2}\left\{\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right\} + y^l\frac{\partial C_{ijk}}{\partial x^l}. \tag{2-24}$$

Now $g_{il}G^l_{jk}$ is given by (2-24) with i and j interchanged, and adding (2-24) and $g_{il}G^l_{jk}$ gives (2-19).

Lemma 2.2.2 ([Sh1])

For any Finsler metric F on a manifold

$$\frac{\delta F}{\delta x^k} = 0,$$

where $\delta/\delta x^k$ is given by equation (2-18).

Proof:

Contracting equation (2-20) with y^i gives

$$C_{ijk}y^i = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}y^i.$$

Equation (2-2) and the homogeneity of F imply

$$C_{ijk}y^i = 0. \tag{2-25}$$

Contracting equation (2-19) with y^i and y^j gives

$$\frac{\partial g_{ij}}{\partial x^k}y^i y^j = 2g_{il}y^i G^l_k, \tag{2-26}$$

where we have used equation (2-25). Differentiating equation (2-4) with respect to x^k gives

$$2F\frac{\partial F}{\partial x^k} = \frac{\partial g_{ij}}{\partial x^k}y^i y^j, \tag{2-27}$$

and differentiating equation (2-4) with respect to y^l gives

$$2F\frac{\partial F}{\partial y^l} = \frac{\partial g_{ij}}{\partial y^l}y^i y^j + 2g_{il}y^l. \tag{2-28}$$

Equations (2-20), (2-25) and (2-28) imply

$$F\frac{\partial F}{\partial y^l} = g_{il}y^i. \tag{2-29}$$

Now,

$$\frac{\delta F}{\delta x^k} = \frac{\partial F}{\partial x^k} - G^l_k\frac{\partial F}{\partial y^l},$$

and equations (2-27), (2-29) and (2-26) thus yield

$$\begin{aligned}\frac{\delta F}{\delta x^k} &= \frac{1}{2F}\frac{\partial g_{ij}}{\partial x^k}y^i y^j - \frac{1}{F}G^l_k g_{il}y^i \\ &= \frac{1}{2F}2g_{il}y^i G^l_k - \frac{1}{F}G^l_k g_{il}y^i = 0,\end{aligned}$$

and hence the horizontal covariant derivative of F on $\mathfrak{I}^n$ is zero.

In 1981, T. Okada and S. Numata established an important result about the Berwald connection as follows [OkNu].

Theorem 2.2.3

The Berwald connection is uniquely determined from the Finsler metric F of $\mathfrak{I}^n$ by the following five axioms.

(i) For $k = 1,\ldots,n$,

$$F_{|k} = \frac{\partial F}{\partial x^k} - G_k^i \frac{\partial F}{\partial y^i} = 0,$$

where $G_k^i = \dfrac{\partial G^i}{\partial y^k}$ and $F_{|k}$ denotes the horizontal covariant derivative with respect to the Berwald connection.

(ii) $G_{kj}^i = \dfrac{\partial G_j^i}{\partial y^k}$.

(iii) $y^k G_{kj}^i = G_j^i$.

(iv) $G_{jk}^i = G_{kj}^i$.

(v) $C^i_{jk} = 0$,

where $C^i_{jk} = g^{ir} C_{rjk}$.

Special Cases

(i) Riemannian Space

We show that the G^i_{jk} are the Christoffel symbols when F is Riemannian.

Recall that if F is Riemannian then

$$F^2 = g_{ij}(x) y^i y^j,$$

so that relation (2-17) becomes

$$G^i = \frac{1}{4} g^{ik}(x) \left\{ 2 \frac{\partial g_{lk}(x)}{\partial x^j} - \frac{\partial g_{lj}(x)}{\partial x^k} \right\} y^l y^j. \tag{2-30}$$

Differentiating equation (2-30) with respect to y^r, y^q we have

$$G^i_{rq} = \frac{1}{2} g^{ik}(x) \left\{ \frac{\partial g_{rk}(x)}{\partial x^q} + \frac{\partial g_{qk}(x)}{\partial x^r} - \frac{\partial g_{qr}(x)}{\partial x^k} \right\};$$

hence by the definition of Γ^i_{jk}, equation (2-16),

$$G^i_{jk} = \Gamma^i_{jk}(x). \tag{2-31}$$

We now show that

$$G^i = \frac{1}{2} \Gamma^i_{jk}(x) y^j y^k. \tag{2-32}$$

Contracting equation (2-31) with y^j and y^k gives

$$G^i_{jk} y^j y^k = \Gamma^i_{jk}(x) y^j y^k,$$

and since G^i_k is a homogeneous function of degree one in y,

$$G^i_k y^k = \Gamma^i_{jk}(x) y^j y^k.$$

Relation (2-32) thus follows from the above expression since G^i is a homogeneous function of degree two in y.

(ii) **Locally Minkowski Space**

Recall that for locally Minkowski space,

$$F^2 = g_{ij}(y) y^i y^j.$$

Equation (2-17) thus gives

$$G^i = 0; \text{ consequently, } G^i_j = 0 \quad \text{and} \quad G^i_{jk} = 0.$$

Definition 2.2.3

A Finsler metric F is called a **Berwald metric,** if in a standard local coordinate system (x,y), the geodesic coefficients G^i are quadratic in y^i for all $p \in M$, that is, if there are local functions $G^i_{jk}(x)$ on M such that

$$G^i = \frac{1}{2} G^i_{jk}(x) y^j y^k.$$

If F is a Berwald metric, the space $\mathfrak{I}^n = (M,F)$ is called a **Berwald space**. In other words, if $G^i_{jk} = G^i_{jk}(x)$ then $\mathfrak{I}^n$ is a Berwald space.

If F is a Riemannian metric, then from equation (2-32),

$$G^i = \frac{1}{2} \Gamma^i_{jk}(x) y^j y^k,$$

and hence F is a Berwald metric. The converse, however, is not true. There are many non-Riemannian Berwald metrics (we give examples in sections 5.1 and 6.1 for non-Riemannian Berwald metrics). Riemannian metrics are a special class of Berwald metrics. The classification of Berwald metrics was done by Z. I. Szabo in 1981 [Sz2]. We quote the following result for a Berwald metric in the two-dimensional case.

Lemma 2.2.4

Every positive definite two-dimensional Berwald metric is either locally Minkowskian or Riemannian.

Covariant derivatives with respect to the Berwald connection

In terms of the Berwald connection, we give the following definitions for the covariant derivatives of quantities on $TM \setminus \{0\}$ in the usual way.

(i) For a scalar field $X(x,y)$,

$$X_{|i} = \frac{\partial X}{\partial x^i} - G^k_i \frac{\partial X}{\partial y^k},$$

$$X|_i = \frac{\partial X}{\partial y^i}$$

(ii) For a covariant vector field $X_i(x,y)$,

$$X_{i|j} = \frac{\partial X_i}{\partial x^j} - \frac{\partial X_i}{\partial y^k} G^k_j - X_k G^k_{ij},$$

$$X_i|_j = \frac{\partial X_i}{\partial y^j}.$$

(iii) For a contravariant vector field $X^i(x,y)$,

$$X^i{}_{|j} = \frac{\partial X^i}{\partial x^j} - \frac{\partial X^i}{\partial y^k} G^k_j + X^k G^i_{kj}, \tag{2-33}$$

$$X^i\big|_j = \frac{\partial X^i}{\partial y^j}.$$

Covariant derivatives of tensor fields are natural extensions of these. For example, a mixed tensor field $X^i_j(x,y)$,

(iv)

$$X^i_{j|k} = \frac{\partial X^i_j}{\partial x^k} - \frac{\partial X^i_j}{\partial y^l} G^l_k + X^l_j G^i_{lk} - X^i_l G^l_{jk},$$

$$X^i_j\big|_k = \frac{\partial X^i_j}{\partial y^k}.$$

Note that the above covariant derivatives are tensors on $TM \setminus \{0\}$. Here, we give the proof only for definition (i). Differentiating $X(x,y)$ with respect to $\hat{x}^p$ and $\hat{y}^p$ by using the coordinate transformation given in page 11, we have

$$\frac{\partial X}{\partial \hat{x}^p} = \frac{\partial X}{\partial x^k}\frac{\partial x^k}{\partial \hat{x}^p} + \frac{\partial X}{\partial y^k}\frac{\partial^2 x^k}{\partial \hat{x}^p \partial \hat{x}^r}\hat{y}^r,$$

$$\frac{\partial X}{\partial \hat{y}^p} = \frac{\partial x^k}{\partial \hat{x}^p}\frac{\partial X}{\partial y^k}. \tag{2-33a}$$

Let us calculate $\frac{\partial X}{\partial \hat{x}^p} - \hat{G}_p^q \frac{\partial X}{\partial \hat{y}^q}$ by using the above expressions and equation (2-17a).

$$\frac{\partial X}{\partial \hat{x}^p} - \hat{G}_p^q \frac{\partial X}{\partial \hat{y}^q} = \frac{\partial X}{\partial x^k}\frac{\partial x^k}{\partial \hat{x}^p} + \frac{\partial X}{\partial y^k}\frac{\partial^2 x^k}{\partial \hat{x}^p \partial \hat{x}^r}\hat{y}^r -$$
$$\left(G_j^l \frac{\partial \hat{x}^q}{\partial x^l}\frac{\partial x^j}{\partial \hat{x}^p} + \frac{\partial \hat{x}^q}{\partial x^l}\frac{\partial^2 x^l}{\partial \hat{x}^p \partial \hat{x}^r}\hat{y}^r \right)\frac{\partial x^k}{\partial \hat{x}^q}\frac{\partial X}{\partial y^k}.$$

Since $\frac{\partial \hat{x}^q}{\partial x^l}\frac{\partial x^k}{\partial \hat{x}^q} = \delta_l^k$, the above expression simplifies to

$$\frac{\partial X}{\partial \hat{x}^p} - \hat{G}_p^q \frac{\partial X}{\partial \hat{y}^q} = \left(\frac{\partial X}{\partial x^i} - G_i^k \frac{\partial X}{\partial y^k} \right)\frac{\partial x^i}{\partial \hat{x}^p}.$$

This implies that

$$\hat{X}_{|p} = X_{|i}\frac{\partial x^i}{\partial \hat{x}^p}.$$

It is also clear from equation (2-33a)

$$\hat{X}\big|_p = X\big|_i \frac{\partial x^i}{\partial \hat{x}^p}.$$

The above two expressions show that $X_{|i}$ and $X\big|_i$ transform as vectors under coordinate changes. Similarly, we can show that the covariant derivatives given in (ii), (iii) and (iv) are tensors using (2-17b).

Now, let us use (2-33) to show that $y^i{}_{|j} = 0$ on $\mathfrak{I}^n$.

Proof:

From equation (2-33),

$$y^i{}_{|j} = \frac{\partial y^i}{\partial x^j} - \frac{\partial y^i}{\partial y^r}G_j^r + y^k G_{kj}^i,$$

and since $\dfrac{\partial y^i}{\partial x^j} = 0$ and $\dfrac{\partial y^i}{\partial y^r} = \delta^i_r$, we have

$$y^i|_j = -\delta^i_r G^r_j + y^k G^i_{kj}.$$

Now, $\delta^i_r G^r_j = G^i_j$, and hence

$$y^i|_j = -G^i_j + y^k G^i_{kj}.$$

The third axiom in Theorem 2.2.3 thus shows that

$$y^i|_j = 0. \tag{2-34}$$

Similarly, we can also show that

$$l^i|_j = 0$$

and

$$l_{i|j} = 0.$$

Remark:

In contrast with the Riemannian connection, it should be remarked that the Berwald connection on a Finsler space is not generally metric compatible. Since

$$g_{ij|k} = \frac{\partial g_{ij}}{\partial x^k} - g_{il}G^l_{jk} - g_{jl}G^l_{ik} - \frac{\partial g_{ij}}{\partial y^l}G^l_k,$$

equations (2-19) and (2-20) imply that

$$g_{ij|k} = -2C_{ijk|m}y^m \neq 0$$

and

$$g_{ij}\big|_k = 2C_{ijk} \neq 0$$

in general.

If F is Riemannian we know that $g_{ij} = g_{ij}(x)$, $C_{ijk} = 0$ and $G^i_{jk} = \Gamma^i_{jk}$ then

$$g_{ij|k} = \frac{\partial g_{ij}(x)}{\partial x^k} - g_{rj}(x)\Gamma^r_{ik} - g_{ir}(x)\Gamma^r_{jk} = 0,$$

and

$$g_{ij}\Big|_k = \frac{\partial g_{ij}(x)}{\partial y^k} = 0.$$

Note that when the metric is Riemannian the Berwald connection reduces to the Riemannian connection, which is metric compatible.

The Ricci identities

The commutation laws of the covariant differentiations with respect to the Berwald connection are called the Ricci identities. These identities can be written in the form

(i) $X^i{}_{|j|k} - X^i{}_{|k|j} = X^r H^i_{rjk} - X^i\Big|_r R^r_{jk}$;

(ii) $X^i{}_{|j}\Big|_k - X^i\Big|_{k|j} = X^r G^i_{rjk}$;

(iii) $X^i\Big|_j\Big|_k - X^i\Big|_k\Big|_j = 0$;

where the tensors H^i_{rjk}, R^i_{jk} and G^r_{ijk} are given by

$$R^i_{jk} = \frac{\partial G^i_j}{\partial x^k} - \frac{\partial G^i_k}{\partial x^j} - G^i_{jm} G^m_k + G^i_{km} G^m_j, \tag{2-35}$$

$$H^i_{rjk} = R^i_{jk}\Big|_r = \frac{\partial R^i_{jk}}{\partial y^r}, \tag{2-36}$$

and

$$G^i_{jkr} = \frac{\partial G^i_{jk}}{\partial y^r}. \tag{2-37}$$

See [Ma1] for more details.

Chapter 3

Curvature

3.1 Riemann Curvature

Riemann curvature is the central concept in Riemannian geometry introduced by B. Riemann in 1854 for Riemannian metrics. In 1926, L. Berwald extended the notion of Riemann curvature to Finsler spaces as a generalization of the sectional curvature of a Riemannian space [Be3] [Be4]. The **Riemann curvature** of a Finsler space, $\mathfrak{F}^n = (M,F)$, is a family of linear transformations on tangent spaces defined as follows:

$$R = \{R_y : T_pM \to T_pM \,/\, y \in T_pM \setminus \{0\}, p \in M\},$$

where

$$R_y = R^i_k dx^k \frac{\partial}{\partial x^i}. \tag{3-1}$$

Here, $R^i_k = R^i_k(x,y)$ denotes the coefficients of the Riemann curvature of F given by

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}, \tag{3-2}$$

and $\{\partial/\partial x^1,\dots,\partial/\partial x^n\}$ is a local coordinate basis of T_pM. We call R the **Riemann curvature** of $\Im^n$. The Riemann curvature depends only on the geodesic coefficients G^i given by the equation (2-17).

Lemma 3.1.1

R_y has following properties:

(i) $R_y(y)=0$;

(ii) R_y is self-adjoint with respect to g_y, i,e., $g_y(R_y(u),v)=g_y(u,R_y(v))$,

where $u,v\in T_pM$ and g_y is the fundamental form on T_pM.

Proof:

(i)

Using the linear operator defined in (3-1), we have

$$R_y(y)=R^i_k dx^k(y)\frac{\partial}{\partial x^i}.$$

Since $dx^k(y)=dx^k(y^j\frac{\partial}{\partial x^j})=y^j\delta^k_j=y^k$,

$$R_y(y)=R^i_k y^k\frac{\partial}{\partial x^i}. \tag{3-3}$$

Contracting (3-2) with y^k gives

$$R^i_k y^k=2\frac{\partial G^i}{\partial x^k}y^k-y^j\frac{\partial^2 G^i}{\partial x^j\partial y^k}y^k+2G^j\frac{\partial^2 G^i}{\partial y^j\partial y^k}y^k-\frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}y^k.$$

We know that G^i is a homogeneous function of degree two in y; therefore, $\partial G^i/\partial x^j$ is also a homogeneous function of degree two in y. Euler's Theorem 2.1.1 implies

$$\frac{\partial^2 G^i}{\partial x^j\partial y^k}y^k=\frac{\partial}{\partial y^k}\left(\frac{\partial G^i}{\partial x^j}\right)y^k=2\frac{\partial G^i}{\partial x^j},$$

and since $\partial G^i / \partial y^j$ is a homogeneous function of degree one in y,

$$\frac{\partial^2 G^i}{\partial y^j \partial y^k} y^k = \frac{\partial}{\partial y^k}\left(\frac{\partial G^i}{\partial y^j}\right) y^k = \frac{\partial G^i}{\partial y^j}.$$

Since G^i is a homogeneous function of degree two in y,

$$\frac{\partial G^j}{\partial y^k} y^k = 2G^j,$$

so that

$$R^i_k y^k = 2\frac{\partial G^i}{\partial x^k} y^k - 2y^j \frac{\partial G^i}{\partial x^j} + 2G^j \frac{\partial G^i}{\partial y^j} - 2G^j \frac{\partial G^i}{\partial y^j} = 0;$$

therefore, $R_y(y) = 0$.

(ii)

Equation (3-3) implies

$$R_y(u) = R^i_k u^k \frac{\partial}{\partial x^i},$$

and hence

$$g_y(R_y(u), v) = g_{ij} R^i_k u^k v^j. \tag{3-4}$$

It can be shown that

$$g_{ij} R^j_k = g_{kj} R^j_i,$$

(cf. [Sh2], section 8-1). Contracting this expression with u^k and v^i gives

$$g_{ij} R^j_k u^k v^i = g_{kj} R^j_i u^k v^i.$$

It is clear that from equation (3-4)

$$g_y(R_y(u), v) = g_y(u, R_y(v)).$$

3.2 Flag Curvature

The flag curvature is an important quantity in Finsler geometry. The notion of flag curvature in Finsler geometry is first introduced by L. Berwald [Be3] [Be4]. One of the fundamental problems in Finsler geometry entails the study of Finsler metrics having constant flag curvature. The flag curvature, which generalizes the sectional curvature in Riemannian geometry, does not depend on whether one is using the Berwald, Chern, or Cartan connection.

Definition 3.2.1

Using a two-dimensional plane, we define the flag curvature as follows. Let $\mathfrak{I}^n = (M, F)$ be a Finsler space of dimension *n* and $p \in M$. For a tangent plane $P = \text{span}\{y, u\} \subset T_pM$,

let

$$K(P, y) = \frac{g_y(R_y(u), u)}{g_y(y, y)g_y(u, u) - g_y(y, u)g_y(y, u)}, \tag{3-5}$$

where $y, u \in T_pM$. The quantity $K(P, y)$ is called the **flag curvature** of $\{P, y\}$. The vector y is called the **flagpole.** The flag curvature, $K(P, y)$ is a function of the tangent plane $P = \text{span}\{y, u\} \subset T_pM$.

Note that the denominator of the equation (3-5) is non-zero (cf. page 36). Choosing a different vector v from T_pM gives a different plane $Q = \text{span}\ \{y, v\}$ so that, in general, $K(P, y) \neq K(Q, y)$.

For a Finsler surface (M, F), $n = 2$, the tangent plane P at each point is the whole tangent space T_pM. In this case $K(P, y) = K(T_pM, y)$ is a scalar function on $TM \setminus \{0\}$. The function $K = K(x, y)$ is called the **Gauss curvature**.

Note that when F is Riemannian $K(P,y)=K(P)$ is independent of $y \in P$, in which case $K(P)$ is called the **sectional curvature** of the section $P \subset T_pM$. Hence equation (3-5) gives

$$K(P)=\frac{g(R_y(u),u)}{g(y,y)g(u,u)-g(y,u)g(y,u)},$$

where g is given by $g(y,y)=g_{ij}(x)y^i y^j$. (See [Sh1], [Ps] for further details).

Let us compute the flag curvature of $\{P,y\}$. Equation (3-4) implies

$$g_y(R_y(u),u)=g_{ij}R^i_k u^j u^k.$$

Using Definition 2.1.4 gives

$$g_y(u,u)=g_{ij}u^i u^j,$$

$$g_y(y,y)=g_{ij}y^i y^j=F^2,$$

$$g_y(y,u)=g_{ij}y^i u^j,$$

and

$$g_y(y,u)g_y(y,u)=g_{ij}y^i u^j g_{hk}y^h u^k.$$

Hence equation (3-5) simplifies to

$$K(P,y)=\frac{g_{ij}R^i_k u^j u^k}{F^2 g_{ij}u^i u^j - g_{ij}g_{hk}y^i y^h u^j u^k}. \tag{3-6}$$

Now, equations (2-29) and (2-6) imply

$$g_{ij}y^i = Fl_j.$$

Therefore, equation (3-6) gives

$$K(P,y)=\frac{g_{ij}R^i_k u^j u^k}{F^2 g_{ij}u^i u^j - F^2 l_j l_k u^k u^j}.$$

Now, $g_{ij}u^i u^j = g_{jk}u^j u^k$, so that

$$K(P,y)=\frac{g_{ij}R^i_k u^j u^k}{F^2(g_{jk}-l_j l_k)u^j u^k}.$$

Note that the above $g_{jk}-l_j l_k \neq 0$. Since, if $g_{jk}-l_j l_k = 0$ then equation (2-9) gives $h_{jk}=0$. Hence equation (2-10) shows that $\det(g_{jk})=0$. This contradicts the condition (iv) in Definition 2.1.3. Thus, the above expression gives

$$R^i_k g_{ij} u^j u^k = K(P,y)F^2(g_{jk}-l_j l_k)u^j u^k.$$

Definition 3.2.2

If, for every fixed $y \in T_pM$, the flag curvature is independent of the tangent plane $P=\text{span}\{y,u\}\subset T_pM$, then $\mathfrak{F}^n$ is said to be of **scalar curvature** at the point p. If $\mathfrak{F}^n$ has scalar curvature at all points in M then $\mathfrak{F}^n$ is a Finsler space of scalar curvature. In this case K is independent of the plane, but it does depend on where the flagpole starts, i.e.,

$$K=K(x,y).$$

The scalar curvature of F is given by

$$R^i_k g_{ij} = KF^2(g_{jk}-l_j l_k). \tag{3-7}$$

The above expression for K can be simplified as follows. Contracting (3-7) with g^{jr} gives

$$R^i_k g_{ij} g^{jr} = KF^2(g_{jk}g^{jr}-l_j l_k g^{jr}),$$

and since $g_{ij}g^{jr}=\delta^r_i$ and $l_j g^{jr}=l^r$,

$$R^i_k \delta^r_i = KF^2(\delta^r_k - l_k l^r).$$

Now $R^i_k \delta^r_i = R^r_k$, and thus

$$R^r_k = KF^2(\delta^r_k - l_k l^r).$$

Equations (2-5) and (2-6) imply

$$R^r_k = K\left(F^2\delta^r_k - F\frac{\partial F}{\partial y^k}y^r\right), \tag{3-8}$$

where K is a homogeneous function of degree zero in y.

Definition 3.2.3

The trace of the Riemann curvature R^i_i is called the **Ricci curvature** and denoted by $Ric(x, y)$. Equation (3-8) implies

$$Ric(x, y) = (n-1)KF^2. \tag{3-9}$$

Note that $Ric(x, y)$ is a scalar function on $TM \setminus \{0\}$ with the following homogeneity property

$$Ric(x, \lambda y) = \lambda^2\, Ric(x, y), \quad \lambda > 0.$$

Formulae for the scalar curvature

We state the following well-known formulae for a Finsler space $\mathfrak{I}^n$ of scalar curvature K for $n > 2$ given in [Ma3].

$$R^i_{jk} = K_j h^i_k - K_k h^i_j. \tag{3-10}$$

$$K_i = \frac{F^2}{3}\frac{\partial K}{\partial y^i} + Ky_i. \tag{3-11}$$

$$K_i y^i = KF^2. \tag{3-12}$$

$$H^r_{ijk} = h^r_k \frac{\partial K_j}{\partial y^i} + \frac{(h^r_i l_j + h_{ij} l^r)K_k}{F} - h^r_j \frac{\partial K_k}{\partial y^i} - \frac{(h^r_i l_k + h_{ik} l^r)K_j}{F}. \tag{3-13}$$

$$H_{ij} = (n-2)\frac{\partial K_j}{\partial y^i} + \frac{3K_i l_j}{F} + K(h_{ij} - 2l_i l_j). \tag{3-14}$$

$$H_i = (n+1)K_i. \tag{3-15}$$

$$\frac{\partial K_i}{\partial y^j} - \frac{\partial K_j}{\partial y^i} = \frac{l_i K_j - l_j K_i}{F}. \tag{3-16}$$

The above functions h^i_j, R^i_{jk} and H^r_{ijk} are defined in (2-11), (2-35) and (2-36) respectively. Note that

$$R^i_{jk} y^j = R^i_k, \tag{3-17}$$

$$H_{ij} = H^r_{ijr},$$

and

$$H_i = \frac{(nH_{ji} + H_{ij}) y^j}{(n-1)}, \tag{3-18}$$

provided $n \neq 1$.

Definition 3.2.4

A Finsler space is said to be of **constant flag curvature** if there is a constant λ such that $K(P,y) = \lambda$, for all $y \in P \subset T_pM$, $p \in M$.

Equation (3-8) indicates that

$$R^i_j = \lambda\left(F^2\delta^i_j - F\frac{\partial F}{\partial y^j} y^i\right). \tag{3-19}$$

Remark

Definitions 3.2.1, 3.2.2 indicate that every two-dimensional Finsler space is of scalar curvature. We now state the formula for the scalar curvature of $\mathfrak{I}^n$, when $n = 2$. (cf. [Ma3]). We use the Berwald frame (l^i, m^i), $(i = 1,2)$, where l^i is given by the equation (2-5) and m^i is the unit vector orthogonal to l^i. The scalar curvature of $\mathfrak{I}^n$ $(n = 2)$ is

$$R^i_{jk} = FKm^i(l_j m_k - m_j l_k), \tag{3-20}$$

where

$$m_i = g_{ij} m^j,$$

$$l_i = g_{ij} l^j,$$

$$l^i l_j + m^i m_j = \delta^i_j, \tag{3-21}$$

$$m_i l^i = 0 = l_i m^i, \tag{3-22}$$

$$m^i m_i = 1 = l^i l_i, \tag{3-23}$$

R^i_{jk} is given by (2-35) and K is the Gauss curvature of $\Im^n$, when $n = 2$.

We can simplify the right hand side of (3-20). Contracting (3-20) with m_i, l^j and m^k and using (3-22) and (3-23) gives

$$R^i_{jk} m_i l^j m^k = KF,$$

and equation (2-5) implies

$$R^i_{jk} m_i y^j m^k = KF^2.$$

Equation (3-17) indicates that

$$K = \frac{R^i_k m_i m^k}{F^2}, \tag{3-24}$$

where K is also a homogeneous function of degree zero in y.

3.3 Non-Riemannian Curvatures

Besides the flag curvature, there are other important quantities in Finsler geometry, which are S-curvature, E-curvature and Landsberg curvature. These quantities vanish for Riemannian metrics, and hence they are called non-Riemannian curvatures. Before discussing these quantities, however, we need to introduce a few definitions.

Definition 3.3.1

Let B^n_x be the unit ball in a Finsler space centered at $p \in M$, i.e.,

$$B^n_x = \{y \in R^n, | y |= F(x,y) < 1\}$$

and B^n be the unit ball in Euclidean space centered at the origin, i.e.,

$$B^n = \{x \in R^n, |x| < 1\},$$

where R^n is *n*-dimensional real vector space and $|x| = \sqrt{\delta_{ij}x^i x^j}$. Let

$$\sigma_F(x) = \frac{volume(B^n)}{volume(B_x^n)}. \qquad (3\text{-}25)$$

For a Finsler space, $\mathfrak{I}^n = (M, F)$, we define the **distortion**, $\tau = \tau(x, y)$, by

$$\tau(x, y) = \ln\left(\frac{\sqrt{\det(g_{ij})}}{\sigma_F(x)}\right), \qquad (3\text{-}26)$$

The distortion $\tau(x, y)$ is a scalar function defined on $TM \setminus \{0\}$ that satisfies the homogeneity condition

$$\tau(x, \lambda y) = \lambda\tau(x, y), \quad \lambda > 0.$$

According to [Sh2], when $F = \sqrt{g_{ij}(x) y^i y^j}$ is Riemannian,

$$\sigma_F(x) = \sqrt{\det(g_{ij}(x))}.$$

Hence equation (3-26) implies that $\tau = 0$, when F is a Riemannian metric.

Definition 3.3.2

Let $\mathfrak{I}^n = (M, F)$ be a Finsler space of dimension *n*, and fix a local frame $\{\partial/\partial x^1, ..., \partial/\partial x^n\}$ for T_pM. The **Cartan torsion** C_y on T_pM is a trilinear symmetric form defined by

$$C_y\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = C_{ijk},$$

where

$$C_{ijk} = \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k}.$$

In terms of the g_{ij}, equation (2-2) implies

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

Note that if F is Riemannian, then $g_{ij} = g_{ij}(x)$; consequently,

$$C_{ijk} = 0.$$

Definition 3.3.3

The **mean Cartan torsion** I_y is a linear form on T_pM defined by

$$I_y\left(\frac{\partial}{\partial x^i}\right) = I_i(x,y) = g^{jk}C_{ijk}. \tag{3-27}$$

Note that if F is Riemannian, then $C_{ijk} = 0$, and hence $I_y = 0$.

We show that the vertical covariant derivative of τ is also equal to the mean Cartan torsion. Now,

$$\begin{aligned}\frac{\partial \tau}{\partial y^i} &= \frac{\partial}{\partial y^i}\left\{\ln\left(\frac{\sqrt{\det(g_{ij})}}{\sigma_F(x)}\right)\right\} \\ &= \frac{\partial}{\partial y^i}\left(\ln\sqrt{\det(g_{ij})}\right) - \frac{\partial}{\partial y^i}(\ln(\sigma_F(x)) \\ &= \frac{\partial}{\partial y^i}\left(\ln\sqrt{\det(g_{ij})}\right)\end{aligned}$$

It can be shown (cf. [Sh2]) that

$$\frac{\partial}{\partial y^i}\left(\ln\sqrt{\det(g_{ij})}\right) = g^{jk}C_{ijk};$$

therefore,

$$I_i = \frac{\partial \tau}{\partial y^i},$$

where we have used (3-27).

We are now in a position to define S-curvature.

Definition 3.3.4

Let $\Im^n = (M,F)$ be a Finsler space and $y \in T_pM \setminus \{0\}$, the function $S = S(x,y)$ is defined by

$$S = \left(\frac{d}{dt} \left\{ \tau(\gamma(t), \frac{d\gamma}{dt}) \right\} \right)_{t=0},$$

where $\gamma(t)$ is the geodesic with $x = \gamma(0)$ and $y = \left(\frac{d\gamma}{dt} \right)_{t=0}$. S is called **S-curvature** of F [Sh1] [Sh2]. Note that this quantity is also called the mean covariation in [Sh3] and mean tangent curvature in [Sh9]. The S-curvature measures the rate of change of the distortion along geodesics.

It can be shown (cf. [Sh2]) that in a standard local coordinate system in TM, the S-curvature is given by

$$S = \frac{\partial G^i}{\partial y^i} - \frac{y^i}{\sigma_F(x)} \frac{\partial \sigma_F(x)}{\partial x^i}, \tag{3-28}$$

where G^i denote the geodesic coefficients and $\sigma_F(x)$ is given in (3-25). The S-curvature satisfies the homogeneity condition

$$S(x, \lambda y) = \lambda S(x,y), \qquad \lambda > 0.$$

If F is Riemannian, then $\tau = 0$, and therefore

$$S = 0.$$

It can also be proved that $S = 0$, if F is a Berwald metric [Sh3].

Definition 3.3.5

Let $\Im^n = (M,F)$ be a Finsler space. If there is a scalar function $c = c(x)$ on M such that

$$S = (n+1)cF$$

then F is said to have **isotropic S-curvature.** If $c =$ constant then F is said to have **constant S-curvature.**

The relationship between the scalar curvature and S-curvature of Finsler metrics was studied by X. Chen, X. Mo and Z. Shen [ChMoSh].

Theorem 3.3.1 ([ChMoSh])

Let $\mathfrak{I}^n = (M, F)$ be a Finsler space of dimension n with scalar curvature K. Suppose that the S-curvature is isotropic. Then there is a scalar function $\hat{\sigma}(x)$ on M such that

$$K = 3\frac{c_{x^i}(x)y^i}{F} + \hat{\sigma}(x),$$

where $c_{x^i} = \partial c / \partial x^i$. Further, $c =$ constant if and only if $K = K(x)$ is a scalar function on M.

Definition 3.3.6

Let $\mathfrak{I}^n = (M, F)$ be a Finsler space. For a non-zero vector $y \in T_pM \setminus \{0\}$, define $E_y = E_{ij}(x, y)dx^i \otimes dx^j : T_pM \times T_pM \to R$ by

$$E_{ij} = \frac{1}{2}\frac{\partial^3 G^m}{\partial y^m \partial y^i \partial y^j}(x, y). \tag{3-29}$$

The **E-curvature** of the Finsler space $E = \{E_y\}_{y \in TM \setminus \{0\}}$, is a symmetric bilinear form on T_pM, $E_y(u, v) = E_{ij}u^i v^j$, where $u = u^i \dfrac{\partial}{\partial x^i}$ and $v = v^i \dfrac{\partial}{\partial x^i} \in T_pM$.

Note that from (2-17c), E_{ij} is a tensor on $TM \setminus \{0\}$.

We show that $E_y(y, v) = 0$. Now,

$$E_y(y, v) = E_{ij}y^i v^j = \frac{1}{2}\frac{\partial}{\partial y^i}\left(\frac{\partial^2 G^r}{\partial y^j \partial y^r}\right)y^i v^j,$$

and since $\left(\dfrac{\partial^2 G^r}{\partial y^j \partial y^r}\right)$ is a homogeneous function of degree zero in y,

$$E_y(y,v)=0,$$

for any $v \in T_pM$. Thus, $E_y(y,y)=0$.

Note that E_y has the homogeneity property

$$E_{\lambda y}(u,v)=\lambda^{-1}E_y(u,v), \quad \lambda>0.$$

If F is Riemannian, then $G^k_{ij}=\Gamma^k_{ij}(x)$; therefore $G^r_{rij}=0$, and hence $E_y=0$.

The E- curvature is also called the **mean Berwald curvature**. Clearly, for any Berwald metric, $E_y=0$. In general, the converse is not true. Finsler metrics with vanishing E-curvature are called weakly Berwald metrics.

We now find a relationship between the E-curvature and S-curvature. Differentiating equation (3-28) with respect to y^j gives

$$S_{y^j}=\frac{\partial S}{\partial y^j}=\frac{\partial}{\partial y^j}\left(\frac{\partial G^i}{\partial y^i}\right)-\frac{\delta^i_j}{\sigma_F(x)}\frac{\partial \sigma_F(x)}{\partial x^i}, \tag{3-30}$$

and differentiating equation (3-30) with respect to y^k gives

$$S_{y^j y^k}=\frac{\partial^2 S}{\partial y^j \partial y^k}=\frac{\partial^2}{\partial y^j \partial y^k}\left(\frac{\partial G^i}{\partial y^i}\right)=2E_{ij},$$

where we have used (3-29). Thus,

$$E_{ij}=\frac{1}{2}S_{y^i y^j}, \tag{3-31}$$

i.e.,

$$E_y(u,v) = \frac{1}{2} S_{y^i y^j} u^i v^j.$$

Note that equation (3-31) is independent of $\sigma_F(x)$, and hence the E-curvature is independent of the volume form even though the S-curvature depends on the volume form.

Definition 3.3.7 ([Sh8])

F is said to have **constant E-curvature** μ, denoted by $E = (n+1)\mu F^{-1}h$, if for any $p \in M$ and any non-zero vector $y \in T_pM$,

$$E_y(u,v) = (n+1)\mu F^{-1} h_y(u,v),$$

where $u, v \in T_pM$, h_y is an angular metric form on T_pM associated with y given by (2-8) and μ is a constant.

Relationship between the Constant E-Curvature and Constant S-Curvature

Suppose that F has constant S-curvature,

$$S = (n+1)cF, \tag{3-32}$$

where c is a constant. Differentiating (3-32) with respect to y^i and y^j, we get

$$S_{y^i y^j} = (n+1)cF_{y^i y^j}$$

and equations (2-10) and (3-31) imply

$$2E_{ij} = (n+1)cF^{-1}h_{ij},$$

i.e.,

$$E_y(u,v) = \frac{1}{2F}(n+1)ch_y(u,v).$$

We thus see that if $S = (n+1)cF$, then $E = \dfrac{1}{2F}(n+1)ch$.

X. Chen and Z. Shen proved that a Randers metric has constant E-curvature if and only if it has constant S-curvature [ChSh2]. Further details about S-curvature and E-curvature are given in [Sh1] and [Sh2].

Well-known examples for constant S-curvature and constant E-curvature are the Funk metrics [Fk1] [Fk2] on the unit ball, B^n. These metrics are defined by

$$F(x,y)=\frac{\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}}{1-|x|^2}\pm\frac{\langle x,y\rangle}{1-|x|^2}, \tag{3-33}$$

where $y\in T_pB^n\cong R^n$, p is a point on B^n, the local coordinates of p are $x=(x^1,...,x^n)$, R^n is the n-dimensional real vector space, and $|\,.\,|$ and $\langle\ ,\ \rangle$ denote the standard Euclidean norm and inner product, respectively. Funk metrics have the following properties:

(i) These metrics are locally projectively flat Randers metrics with $K=-1/4$. where K is the constant curvature;

(ii) $S=\pm\frac{1}{2}(n+1)F$;

(iii) $E=\pm\frac{1}{4F}(n+1)h$;

(iv) $J\pm\frac{1}{2}FI=0$.

We prove these relations in chapter 5.

Motivated by the results for Funk metrics, in 2001 Z. Shen proved that Finsler metrics of the form

$$F=\frac{\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}}{1-|x|^2}\pm\frac{\langle x,y\rangle}{1-|x|^2}\pm\frac{\langle a,y\rangle}{1+\langle a,x\rangle},$$

where $y\in T_pR^n$ and $a\in R^n$ is a constant vector with $|a|<1$, have the same properties as Funk metrics [Sh8]. He thus classified all locally projectively flat Randers metrics with constant curvature. (See chapter 5.1 of this dissertation for the definition of Randers metrics).

In 2002, Z. Shen also proved that the Finsler metric on the cylindrical domain $\Omega := \{p = (x, y, \overline{p}) \in R^2 \times R^{n-2} / x^2 + y^2 < 1\}$ defined by

$$F(\hat{y}) = \frac{\sqrt{(-yu + xv)^2 + |\hat{y}|^2 (1 - x^2 - y^2)} - (-yu + xv)}{1 - x^2 - y^2},$$

where $\hat{y} = (u, v, \overline{y}) \in T_p\Omega = R^n$, $n \geq 2$ and $p = (x, y, \overline{p}) \in \Omega$, has zero flag curvature, zero S-curvature and zero E-curvature [Sh6].

Once given a connection on a manifold, one can define the covariant derivatives of tensor fields along a curve.

Definition 3.3.8

Let $\mathfrak{I}^n = (M, F)$ be a Finsler space and $c : [a, b] \to M$ be a C^∞ curve. We define the **covariant derivatives** of a vector field $u = u(t)$, $t \in [a, b]$, along c by

$$D_{\dot{c}(t)} u(t) = \left\{ \frac{du^i}{dt}(t) + u^j(t) \frac{\partial G^i}{\partial y^j}\left(c(t), \dot{c}(t)\right) \right\} \frac{\partial}{\partial x^i},$$

where $u(t) = u^i(t) \dfrac{\partial}{\partial x^i}$ and $\dot{c}(t) = \dot{c}^i(t) \dfrac{\partial}{\partial x^i}$.

Definition 3.3.9

A vector field $v = v(t)$ along c is said to be **parallel** if

$$D_{\dot{c}(t)} v(t) = 0.$$

Note that if c is a geodesic then

$$D_{\dot{c}(t)} \dot{c}(t) = 0.$$

Definition 3.3.10

Let $\Im^n = (M,F)$ be a Finsler space. For a non-zero vector $y \in T_pM \setminus \{0\}$ define

$$L_y(u,v,w) = \left[\frac{d}{dt}\left(C_{\dot{\gamma}(t)}(u(t),v(t),w(t))\right)\right]_{t=0},$$

where $\gamma(t)$ is the geodesic with $\gamma(0)=x$, $\dot{\gamma}(0)=y$, and $u(t)$, $v(t)$, $w(t)$ are parallel vector fields along $\gamma(t)$ with $u(0)=u$, $v(0)=v$, $w(0)=w$ and $C_{\dot{\gamma}(t)}$ is the Cartan torsion. We call L_y the **Landsberg curvature** of the Finsler space. The Landsberg curvature measures the rate of change of the Cartan torsion along geodesics.

Thus in a standard local coordinate system (x,y) in TM, the Landsberg curvature $L_y = L_{ijk}(x,y)dx^i \otimes dx^j \otimes dx^k$ is given by

$$L_{ijk} = C_{ijk|m}y^m$$

(cf. [Sh11]). This implies that

$$L_{ijk} = 2G^l \frac{\partial C_{ijk}}{\partial y^l} + C_{ljk}G^l_i + C_{ilk}G^l_j + C_{ijl}G^l_k - y^l \frac{\partial C_{ijk}}{\partial x^l}. \tag{3-34}$$

It can be shown (cf. [Sh11]) that from above equation

$$L_{ijk} = -\frac{1}{2} y^r g_{rl} \frac{\partial^3 G^l}{\partial y^i \partial y^j \partial y^k}. \tag{3-35}$$

Hence from the equations (2-17c) and (3-35), L_{ijk} is a tensor on $TM \setminus \{0\}$.

A Finsler metric is called a **Landsberg metric** if $L = 0$ [La1] [La2]. The Cartan torsion is thus constant along geodesics in a Landsberg space.

Definition 3.3.11

Let $\Im^n = (M,F)$ be a Finsler space. For a non-zero vector $y \in T_pM \setminus \{0\}$ and define

$$J_y(u) = \left[\frac{d}{dt} \left(I_{\dot{\gamma}(t)} u(t) \right) \right]_{t=0},$$

where $\gamma(t)$ is the geodesic with $x = \gamma(0)$ $y = \dot{\gamma}(0)$ and $u(t)$ is a parallel vector field along $\gamma(t)$ with $u(0) = u$. We call J_y the **mean Landsberg curvature** of the Finsler space. The mean Landsberg curvature measures the rate of change of the mean Cartan torsion along geodesics. In a standard local coordinate system (x, y) in TM, the $J_y = J_i dx^i$ is given by

$$J_i = I_{i|m} y^m$$

(cf. [Sh11]). This gives

$$J_i = y^j \frac{\partial I_i}{\partial x^j} - I_j \frac{\partial G^j}{\partial y^i} - 2G^j \frac{\partial I_i}{\partial y^j}. \qquad (3\text{-}36)$$

By a direct computation (from equations (3-27), (3-34) and (3-36)), one can show that

$$J_i = g^{jk} L_{ijk}. \qquad (3\text{-}37)$$

Then equations (3-35) and (3-37) imply

$$J_i = -\frac{1}{2} F g^{jk} \frac{\partial F}{\partial y^r} \frac{\partial^3 G^r}{\partial y^i \partial y^j \partial y^k}. \qquad (3\text{-}38)$$

Hence from the equations (2-17c) and (3-38), J_i is a tensor on $TM \setminus \{0\}$.

A Finsler metric is called a **weak Landsberg metric** if $J = 0$.

The ratio, J/I is the relative rate of change of the mean Cartan torsion along geodesics. There are many examples that satisfy $J/I = -c(x)F$, where $c(x)$ is a scalar function on M. The Funk metrics are two of them. In these cases $c = \pm 1/2$. X. Chen and Z. Shen showed that for a Randers metric on an n-dimensional manifold M, $J + c(x)FI = 0$ if and only if $S = (n+1)c(x)F$ and the one-form of the Randers metric is closed [ChSh2].

The next two theorems give interesting relationships between the scalar curvature and non-Riemannian curvatures of a Finsler space. The proofs of these results can be found in [ChMoSh].

Theorem 3.3.2

Let $\mathfrak{I}^n = (M, F)$ be an n-dimensional Finsler space of scalar curvature. Suppose that the mean Landsberg curvature satisfies

$$J + c(x)FI = 0,$$

where $c = c(x)$ is a scalar function on M. Then the flag curvature $K = K(x, y)$ and the distortion $\tau = \tau(x, y)$ satisfy

$$\frac{n+1}{3}K_{y^i} + \left(K + c(x)^2 - \frac{c_{x^j} y^j}{F}\right)\tau_{y^i} = 0,$$

where $K_{y^i} = \partial K / \partial y^i$, $c_{x^j} = \partial c / \partial x^j$ and $\tau_{y^i} = \partial \tau / \partial y^i$.

(a) If $c =$ constant, then there is a scalar function $\rho(x)$ on M such that

$$K = -c^2 + \rho(x) e^{-\frac{3\tau}{n+1}}.$$

(b) Suppose that F is non-Riemannian on any open subset of M. If $K = K(x)$ is a scalar function on M, then $c(x) = c$ is a constant, in which case $K = -c^2 \le 0$.

Theorem 3.3.3

Let $\mathfrak{I}^n = (M, F)$ be an n-dimensional Finsler space of scalar curvature. Suppose that the S-curvature and the mean Landsberg curvature satisfy

$$S = (n+1)c(x)F$$

and

$$J + c(x)FI = 0,$$

where $c = c(x)$ is a scalar function on M. Then the flag curvature $K = K(x, y)$ can be written in the form

$$K(x,y)=3\frac{c_{x^i}(x)y^i}{F}+\sigma(x)=-\frac{3c(x)^2+\sigma(x)}{2}+\mu(x)e^{-\frac{2\tau}{n+1}},$$

where $\sigma(x)$ and $\mu(x)$ are scalar functions on M.

(a) Suppose that F is non-Riemannian on any open subset of M. If $c(x)=c$ is a constant, then $K=-c^2$, $\sigma(x)=-c^2$ and $\mu(x)=0$.

(b) If $c(x)\neq$ constant, then the distortion is given by

$$\tau=\ln\left[\frac{2\mu(x)F}{6c_{x^i}(x)y^i+3F(\sigma(x)+c(x)^2)}\right]^{\frac{n+1}{2}}.$$

Chapter 4

Projective Finsler Geometry

4.1 Projective Change

In projective Finsler geometry, we have a remarkable theorem called Rapcsak theorem, which plays an important role in the projective geometry of Finsler spaces. This theorem gives the necessary and sufficient conditions that a Finsler space is projective to another Finsler space and will be presented in this section. By considering two Finsler spaces on a common underlying manifold of dimension *n*, we give the definition of the projective change as follows.

Definition 4.1.1

A projective change is a mapping from $\Im^n = (M,F)$ to $\overline{\Im}^n = (M,\overline{F})$, which is a diffeomorphism and maps geodesics of $\Im^n$ to geodesics of $\overline{\Im}^n$. If any geodesic of $\Im^n$ is a geodesic of $\overline{\Im}^n$ and the converse is also true, then the change $F \to \overline{F}$ of the metric is called a **projective change** and $\Im^n$ is said to be **projective** to $\overline{\Im}^n$.

Let $c: x(t) = \{x^i(t), i = 1,\dots,n\}$ be a curve in M which is a geodesic of both $\mathfrak{F}^n$ and $\overline{\mathfrak{F}}^n$. In page 18 in chapter 2, we know that the geodesic c on $\mathfrak{F}^n$ satisfies

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = \psi(t)\frac{dx^i}{dt}, \tag{4-1}$$

and the geodesic c on $\overline{\mathfrak{F}}^n$ satisfies

$$\frac{d^2x^i}{dt^2} + 2\overline{G}^i\left(x, \frac{dx}{dt}\right) = \overline{\psi}(t)\frac{dx^i}{dt}, \tag{4-2}$$

where $\overline{\psi}(t) = \left(\dfrac{d^2\overline{s}}{dt^2}\right) \Big/ \left(\dfrac{d\overline{s}}{dt}\right)$, $\overline{s} = \int \overline{F}(x(t), dx/dt)dt$, and

$$\overline{G}^i = \frac{1}{4}\overline{g}^{ik}\left\{\left(\frac{\partial^2 \overline{F}^2}{\partial x^r \partial y^k}\right)y^r - \frac{\partial \overline{F}^2}{\partial x^k}\right\}.$$

Equations (4-1) and (4-2) show that

$$2\overline{G}^i\left(x, \frac{dx}{dt}\right) - 2G^i\left(x, \frac{dx}{dt}\right) = (\overline{\psi}(t) - \psi(t))\frac{dx^i}{dt},$$

and hence

$$\overline{G}^i(x,y) = G^i(x,y) + P(x,y)y^i,$$

where $P(x,y)$ is a positively homogeneous function of degree one in y. This establishes one part of the following theorem.

Theorem 4.1.1 ([Kn])

A Finsler space $\mathfrak{F}^n$ is projective to another Finsler space $\overline{\mathfrak{F}}^n$ if and only if there exists a positively homogeneous scalar field of degree one in y, $P(x,y)$; such that

$$\overline{G}^i(x,y) = G^i(x,y) + P(x,y)y^i. \tag{4-3}$$

The scalar field $P = P(x,y)$ is called the **projective factor** of the projective change.

The proof of the converse of the above theorem is given in [Sh1].

In 1961, A. Rapcsak [Rap] proved that

$$\overline{G}^i = G^i + \frac{\overline{F}_{|k} y^k}{2\overline{F}} y^i + \frac{\overline{F}}{2} \overline{g}^{il} \left\{ \frac{\partial \overline{F}_{|k}}{\partial y^l} y^k - \overline{F}_{|l} \right\}, \tag{4-4}$$

where $\overline{F}_{|k}$ denotes the horizontal covariant derivative of $\overline{F}$ on $\mathfrak{F}^n = (M, F)$ given by

$$\overline{F}_{|k} = \frac{\partial \overline{F}}{\partial x^k} - \frac{\partial G^r}{\partial y^k} \frac{\partial \overline{F}}{\partial y^r}.$$

Note that equation (4-4) holds in general and does not require (4-3). Then Rapcsak theorem follows from relation (4-4).

Theorem 4.1.2 (Rapcsak Theorem [Rap])

Let $\mathfrak{F}^n = (M, F)$ and $\overline{\mathfrak{F}}^n = (M, \overline{F})$ be two Finsler spaces on a common underlying manifold M of dimension *n*. The change $F \to \overline{F}$ of the metric is a projective change if and only if $\overline{F}$ satisfies

$$\overline{F}_{|i} - \frac{\partial \overline{F}_{|k}}{\partial y^i} y^k = 0, \tag{4-5}$$

where " | " denotes the horizontal covariant derivative of $\overline{F}$ on $\mathfrak{F}^n$. We say that $\mathfrak{F}^n$ is projective to $\overline{\mathfrak{F}}^n$ with the projective factor given by

$$P = \frac{\overline{F}_{|k} y^k}{2\overline{F}}. \tag{4-6}$$

Definition 4.1.2

If there exists a projective change $F \to \overline{F}$ of a Finsler space $\mathfrak{F}^n = (M, F)$ such that the Finsler space $\overline{\mathfrak{F}}^n = (M, \overline{F})$ is a locally Minkowski space, then $\mathfrak{F}^n$ is called **locally projectively flat**.

If $\overline{\mathfrak{I}^n}$ is a locally Minkowski space then recall that $\overline{G^i} = 0$, and equation (4-3) thus implies that

$$G^i = -Py^i, \tag{4-7}$$

where P is the projective factor of the projective change $F \to \overline{F}$.

According to [Sh11], a Finsler metric F on a differentiable manifold M is locally projectively flat if any point $p \in M$, there is a local coordinate system (x^i) in M such that every geodesic $c = c(t)$ is straight, i.e., $x^i(t) = f(t)a^i + b^i$, where $f(t)$ is a C^∞ function, a^i and b^i are constants. Hence locally projectively flat Finsler spaces are important to us.

The following lemma is very important in Finsler geometry, which gives the requirement for any Finsler metric to be locally projectively flat and will be used to establish our main results in chapter 6. Here the proof given by my self is rather long and the author could not find a shorter proof, which fits into this chapter.

Lemma 4.1.3 ([Ha])

A Finsler space $\mathfrak{I}^n = (M, F)$ is locally projectively flat if and only if

$$\frac{\partial F}{\partial x^i} - \frac{\partial^2 F}{\partial x^k \partial y^i} y^k = 0. \tag{4-8}$$

Proof:

Let $\overline{\mathfrak{I}^n} = (M, \overline{F})$ be a locally Minkowski space. Assume that $\mathfrak{I}^n$ is locally projectively flat. Hence, according to Definition 4.1.2 $\mathfrak{I}^n$ is projective to $\overline{\mathfrak{I}^n}$. From Definition 4.1.1, we know that the inverse map $\overline{F} \to F$ of a projective change is a projective change and $\overline{\mathfrak{I}^n}$ is also projective to $\mathfrak{I}^n$. Hence, Theorem 4.1.2 implies

$$F_{|i} - \frac{\partial F_{|k}}{\partial y^i} y^k = 0, \tag{4-9}$$

where $F_{|k}$ denotes the horizontal covariant derivative of F on $\overline{\Im}^n = (M, \overline{F})$ given by

$$F_{|k} = \frac{\partial F}{\partial x^k} - \frac{\partial \overline{G}^r}{\partial y^k} \frac{\partial F}{\partial y^r}.$$

Since $\overline{\Im}^n$ is a locally Minkowski space we have $\overline{G}^i = 0$, and hence the above expression reduces to

$$F_{|k} = \frac{\partial F}{\partial x^k}. \tag{4-10}$$

Substituting equation (4-10) into (4-9) gives equation (4-8).

For the converse, assume that (4-8) holds. Then we show that $\Im^n$ is projective to $\overline{\Im}^n$. According to Theorem 4.1.2, we have to show that $\overline{F}$ satisfies the equation (4-5). Now, $\overline{F} = \overline{F}(y)$; hence

$$\overline{F}_{|i} = -G_i^r \frac{\partial \overline{F}}{\partial y^r},$$

and similarly

$$\overline{F}_{|k} = -G_k^r \frac{\partial \overline{F}}{\partial y^r}.$$

Differentiating the above expression with respect to y^i and contracting with y^k gives

$$\frac{\partial \overline{F}_{|k}}{\partial y^i} y^k = -G_{ik}^r \frac{\partial \overline{F}}{\partial y^r} y^k - G_k^r \frac{\partial^2 \overline{F}}{\partial y^r \partial y^i} y^k.$$

Since G^i is a homogeneous function of degree two in y, the above expression implies

$$\frac{\partial \overline{F}_{|k}}{\partial y^i} y^k = -G_i^r \frac{\partial \overline{F}}{\partial y^r} - 2G^r \frac{\partial^2 \overline{F}}{\partial y^r \partial y^i}.$$

Hence,

$$\overline{F}_{|i} - \frac{\partial \overline{F}_{|k}}{\partial y^i} y^k = 2G^r \frac{\partial^2 \overline{F}}{\partial y^r \partial y^i}.$$

We now find G^r using equation (2-17).

$$G^r = \frac{1}{4} g^{ri} \left\{ \left(\frac{\partial^2 F^2}{\partial y^i \partial x^k} \right) y^k - \frac{\partial F^2}{\partial x^i} \right\}$$

$$= \frac{1}{4} g^{ri} \left\{ \frac{\partial}{\partial y^i} \left(2F \frac{\partial F}{\partial x^k} \right) y^k - 2F \frac{\partial F}{\partial x^i} \right\}$$

$$= \frac{1}{2} g^{ri} \left\{ \frac{\partial F}{\partial y^i} \frac{\partial F}{\partial x^k} y^k + F \frac{\partial^2 F}{\partial y^i \partial x^k} y^k - F \frac{\partial F}{\partial x^i} \right\},$$

and from equation (4-8),

$$G^r = \frac{1}{2} g^{ri} \left\{ \frac{\partial F}{\partial y^i} \frac{\partial F}{\partial x^k} y^k \right\}.$$

Using equation (2-29), we have

$$G^r = \frac{1}{2F} g^{ri} g_{ji} y^j \frac{\partial F}{\partial x^k} y^k$$

$$= \frac{1}{2F} \delta^r_j y^j \frac{\partial F}{\partial x^k} y^k$$

$$= \frac{1}{2F} \frac{\partial F}{\partial x^k} y^r y^k,$$

consequently,

$$\overline{F}_{|i} - \frac{\partial \overline{F}_{|k}}{\partial y^i} y^k = \frac{y^r y^k}{F} \frac{\partial F}{\partial x^k} \frac{\partial^2 \overline{F}}{\partial y^r \partial y^i}$$

$$= \frac{y^k}{F} \frac{\partial F}{\partial x^k} \frac{\partial}{\partial y^r} \left(\frac{\partial \overline{F}}{\partial y^i} \right) y^r = 0,$$

where we have used the property of homogeneity of $\overline{F}$. Hence $\Im^n$ is projective to $\overline{\Im}^n$. That is, $\Im^n$ is locally projectively flat. The proof is complete.

We now find the projective factor $\overline{P}$ of the projective change $\overline{F} \to F$, where F is locally projectively flat. Equation (4-3) yields the relation

$$G^i = \overline{G}^i + \overline{P}y^i.$$

Since $\overline{G}^i = 0$, we have

$$G^i = \overline{P}y^i, \tag{4-11}$$

Equations (4-6) and (4-10) imply

$$\overline{P} = \frac{1}{2F}\frac{\partial F}{\partial x^k}y^k. \tag{4-12}$$

Equations (4-7) and (4-11) also give

$$\overline{P} = -P.$$

Relationship between the projective factors of any three projective changes

Let $\mathfrak{I}^n = (M,F)$, $\overline{\mathfrak{I}}^n = (M,\overline{F})$ and $\widetilde{\mathfrak{I}}^n = (M,\widetilde{F})$ be three Finsler spaces on a common underlying manifold M of dimension n and G^i, $\overline{G}^i$, $\widetilde{G}^i$ be the geodesic coefficients of F, $\overline{F}$, $\widetilde{F}$ respectively. Let P, $\overline{P}$ and $\widetilde{P}$ be the projective factors of the projective changes $F \to \overline{F}$, $\overline{F} \to \widetilde{F}$ and $\widetilde{F} \to F$ respectively. Equation (4-3) yields the relations

$$\overline{G}^i = G^i + Py^i,$$

$$\widetilde{G}^i = \overline{G}^i + \overline{P}y^i,$$

$$G^i = \widetilde{G}^i + \widetilde{P}y^i,$$

which imply

$$(P + \overline{P} + \widetilde{P})y^i = 0.$$

Differentiating both sides of the above expression with respect to y^k gives

$$(P + \overline{P} + \widetilde{P})\delta^i_k + \left(\frac{\partial P}{\partial y^k} + \frac{\partial \overline{P}}{\partial y^k} + \frac{\partial \widetilde{P}}{\partial y^k}\right)y^i = 0.$$

Let $k = i$, then $\delta^i_i = n$, and therefore

$$(P+\overline{P}+\widetilde{P})n+\left(\frac{\partial P}{\partial y^i}+\frac{\partial \overline{P}}{\partial y^i}+\frac{\partial \widetilde{P}}{\partial y^i}\right)y^i=0.$$

Since P, $\overline{P}$ and $\widetilde{P}$ are homogeneous functions of degree one in y,

$$P+\overline{P}+\widetilde{P}=0.$$

4.2 Projectively Invariant Tensors

In projective Finsler geometry, there are two important projectively invariant tensors, namely the Weyl tensor and the Douglas tensor. These tensors provide us with some important information about the projective properties of Finsler metrics. For example, it is shown that these tensors vanish simultaneously in a locally projectively flat Finsler space. Z. I. Szabo [Sz1] proved that the condition for a Finsler space to be of scalar curvature is that the Weyl tensor vanishes. In this section, we show that these tensors are invariant under any projective change.

Definition 4.2.1

The **Douglas tensor** is defined in terms of G^i as follows.

$$D^h_{ijk}=G^h_{ijk}-\frac{(y^h G_{ijk}+\varepsilon_{(ijk)}\{\delta^h_i G_{jk}\})}{(n+1)},$$

where

$$G^h_{ijk}=\frac{\partial G^h_{jk}}{\partial y^i},$$

$$G_{ij}=G^r_{ijr},$$

$$G_{ijk}=\frac{\partial G_{ij}}{\partial y^k},$$

and

$$\varepsilon_{(ijk)}\{\delta^h_i G_{jk}\}=\delta^h_i G_{jk}+\delta^h_j G_{ki}+\delta^h_k G_{ij}.$$

Theorem 4.2.1

The Douglas tensor is invariant under any projective change.

Proof:

Let $\Im^n = (M, F)$ and $\overline{\Im}^n = (M, \overline{F})$ be two Finsler spaces on a common underlying manifold M of dimension n and $\Im^n$ be projective to $\overline{\Im}^n$. We denote quantities of $\overline{\Im}^n$ by putting a bar over the symbol. Differentiating equation (4-3) with respect to y^j, y^k and y^h, we obtain

$$\overline{G}^i_j = G^i_j + P_j y^i + P\delta^i_j, \tag{4-13}$$

$$\overline{G}^i_{jk} = G^i_{jk} + P_{jk} y^i + P_j\delta^i_k + P_k\delta^i_j, \tag{4-14}$$

$$\overline{G}^i_{hjk} = G^i_{hjk} + P_{jkh} y^i + P_{jk}\delta^i_h + P_{jh}\delta^i_k + P_{kh}\delta^i_j, \tag{4-15}$$

where $P_i = \partial P / \partial y^i$, $P_{ij} = \partial P_i / \partial y^j$ and $P_{ijk} = \partial P_{ij} / \partial y^k$. Substituting $i = k = r$ into (4-15) gives

$$\overline{G}^r_{hjr} = G^r_{hjr} + P_{hjr} y^r + P_{hj}\delta^r_r + P_{hr}\delta^r_j + P_{jr}\delta^r_h.$$

Let $G_{ij} = G^r_{ijr}$ and $\overline{G}_{ij} = \overline{G}^r_{ijr}$. Since $\delta^r_r = \delta^1_1 + \ldots + \delta^n_n = 1 + \ldots + 1 = n$, $P_{hr}\delta^r_j = P_{hj}$ and $P_{hj} = P_{jh}$ we have

$$\overline{G}_{hj} = G_{hj} + P_{hjr} y^r + nP_{hj} + 2P_{hj}.$$

Now, P is a homogeneous function of degree one in y; hence,

$$P_{hjr} y^r = -P_{hj},$$

and therefore

$$\overline{G}_{hj} = G_{hj} + (n+1)P_{hj}. \tag{4-16}$$

Differentiating (4-16) with respect to y^k yields

$$\overline{G}_{hjk} = G_{hjk} + (n+1)P_{hjk}, \tag{4-17}$$

where $\overline{G}_{hjk} = \partial\overline{G}_{hj} / \partial y^k$ and $G_{hjk} = \partial G_{hj} / \partial y^k$.

Let $\varepsilon_{(ijk)}\{\delta_i^h P_{jk}\} = \delta_i^h P_{jk} + \delta_j^h P_{ki} + \delta_k^h P_{ij}$. Then equation (4-15) is

$$\overline{G}_{ijk}^h = G_{ijk}^h + y^h P_{ijk} + \varepsilon_{(ijk)}\{\delta_i^h P_{jk}\}. \tag{4-18}$$

Equations (4-16), (4-17) and (4-18) imply

$$\overline{G}_{ijk}^h = G_{ijk}^h + y^h \frac{(\overline{G}_{ijk} - G_{ijk})}{(n+1)} + \varepsilon_{(ijk)}\left\{\delta_i^h \frac{(\overline{G}_{jk} - G_{jk})}{(n+1)}\right\}.$$

The above expression can be recast as

$$\overline{G}_{ijk}^h - \frac{(y^h \overline{G}_{ijk} + \varepsilon_{(ijk)}\{\delta_i^h \overline{G}_{jk}\})}{(n+1)} = G_{ijk}^h - \frac{(y^h G_{ijk} + \varepsilon_{(ijk)}\{\delta_i^h G_{jk}\})}{(n+1)}.$$

Let

$$D_{ijk}^h = G_{ijk}^h - \frac{(y^h G_{ijk} + \varepsilon_{(ijk)}\{\delta_i^h G_{jk}\})}{(n+1)}$$

and

$$\overline{D}_{ijk}^h = \overline{G}_{ijk}^h - \frac{(y^h \overline{G}_{ijk} + \varepsilon_{(ijk)}\{\delta_i^h \overline{G}_{jk}\})}{(n+1)},$$

then

$$\overline{D}_{ijk}^h = D_{ijk}^h.$$

The above equation shows that D_{ijk}^h is a projectively invariant tensor.

Definition 4.2.2

If the Douglas tensor is equal to zero, a Finsler space is called a **Douglas space** and its metric is called a **Douglas metric.**

Definition 4.2.3

The **Weyl tensor** is defined as follows.

$$W_{jk}^i = R_{jk}^i + \frac{(H_{jk} - H_{kj})y^i + \delta_j^i H_k - \delta_k^i H_j}{(n+1)},$$

where R^i_{jk} is given in equation (2-35),

$$H^h_{ijk} = \frac{\partial R^h_{jk}}{\partial y^i},$$

$$H_{ij} = H^r_{ijr},$$

and

$$H_i = (nH_{ki} + H_{ik})y^k / (n-1).$$

Theorem 4.2.2

The Weyl tensor is also invariant under any projective change.

Before giving the proof, however, we state some formulae given in [FuYa] and [Ma3].

$$\overline{R}^i_{jk} = R^i_{jk} + y^i Q_{jk} + \delta^i_j Q_k - \delta^i_k Q_j, \tag{4-19}$$

where

$$Q_i = P_{|i} - PP_i,$$

and

$$Q_{ij} = P_{i|j} - P_{j|i}.$$

$$\overline{H}^h_{ijk} = H^h_{ijk} + y^h Q_{jk}\big|_i + \delta^h_i Q_{jk} + \delta^h_j Q_k\big|_i - \delta^h_k Q_j\big|_i, \tag{4-20}$$

where

$$Q_{jk}\big|_i = \frac{\partial Q_{jk}}{\partial y^i},$$

and

$$Q_k\big|_i = \frac{\partial Q_k}{\partial y^i}.$$

$$Q_{ij} = Q_j\big|_i - Q_i\big|_j. \tag{4-21}$$

$$H^i_{ijk} = H^i_{kji} - H^i_{jki}. \tag{4-22}$$

$$R^r_{ir} = H_{ki} y^k, \tag{4-23}$$

where

$$H_{ki} = H^r_{kir}. \tag{4-24}$$

Let us show that the Weyl tensor is invariant.

Proof:

Substituting $h = i = r$ into equation (4-20) gives

$$\overline{H}^r_{rjk} = H^r_{rjk} + y^r Q_{jk}\Big|_r + \delta^r_r Q_{jk} + \delta^r_j Q_k\big|_r - \delta^r_k Q_j\Big|_r.$$

Since Q_{jk} is a homogeneous function of degree zero in y, $Q_{jk}\Big|_r y^r = 0$; therefore,

$$\overline{H}^r_{rjk} = H^r_{rjk} + nQ_{jk} + Q_k\big|_j - Q_j\Big|_k.$$

Equation (4-21) implies

$$\overline{H}^r_{rjk} = H^r_{rjk} + (n+1)Q_{jk},$$

and equations (4-22) and (4-24) give

$$\overline{H}_{kj} - \overline{H}_{jk} = H_{kj} - H_{jk} + (n+1)Q_{jk}. \tag{4-25}$$

Contracting equation (4-25) with y^k we get

$$\overline{H}_{kj} y^k - \overline{H}_{jk} y^k = H_{kj} y^k - H_{jk} y^k + (n+1)Q_{jk} y^k. \tag{4-26}$$

Substituting $i = k = r$ to equation (4-19) gives

$$\overline{R}^r_{jr} = R^r_{jr} + y^r Q_{jr} + \delta^r_j Q_r - \delta^r_r Q_j.$$

Since $\delta^r_r = n$ and $y^r Q_{jr} = y^k Q_{jk}$, using equation (4-23), the above expression gives

$$\overline{H}_{kj} y^k = H_{kj} y^k + y^k Q_{jk} - (n-1)Q_j.$$

Substituting for $y^k Q_{jk}$ from (4-26) gives

$$\overline{H}_{kj} y^k = H_{kj} y^k + \frac{(\overline{H}_{kj} - \overline{H}_{jk}) y^k}{(n+1)} - \frac{(H_{kj} - H_{jk}) y^k}{(n+1)} - (n-1)Q_j,$$

so that

$$(n\overline{H}_{kj}+\overline{H}_{jk})y^k=(nH_{kj}+H_{jk})y^k-(n^2-1)Q_j. \tag{4-27}$$

Using equation (3-18) and putting $(n-1)H_i=(nH_{ki}+H_{ik})y^k$ and $(n-1)\overline{H}_i=(n\overline{H}_{ki}+\overline{H}_{ik})y^k$ into (4-27), we have

$$\overline{H}_j=H_j-(n+1)Q_j. \tag{4-28}$$

Replacing j by k from (4-28),

$$\overline{H}_k=H_k-(n+1)Q_k. \tag{4-29}$$

Substituting for Q_{jk} from (4-25), Q_j from (4-28) and Q_k from (4-29) into (4-19) yields

$$\overline{R}^i_{jk}=R^i_{jk}+y^i\frac{(H_{jk}-H_{kj})-(\overline{H}_{jk}-\overline{H}_{kj})}{(n+1)}+\delta^i_j\frac{(H_k-\overline{H}_k)}{(n+1)}-\delta^i_k\frac{(H_j-\overline{H}_j)}{(n+1)},$$

and therefore

$$\overline{R}^i_{jk}+\frac{(\overline{H}_{jk}-\overline{H}_{kj})y^i+\delta^i_j\overline{H}_k-\delta^i_k\overline{H}_j}{(n+1)}=R^i_{jk}+\frac{(H_{jk}-H_{kj})y^i+\delta^i_jH_k-\delta^i_kH_j}{(n+1)}.$$

Let

$$\overline{W}^i_{jk}=\overline{R}^i_{jk}+\frac{(\overline{H}_{jk}-\overline{H}_{kj})y^i+\delta^i_j\overline{H}_k-\delta^i_k\overline{H}_j}{(n+1)}$$

and

$$W^i_{jk}=R^i_{jk}+\frac{(H_{jk}-H_{kj})y^i+\delta^i_jH_k-\delta^i_kH_j}{(n+1)}. \tag{4-30}$$

The above calculations show that

$$\overline{W}^i_{jk}=W^i_{jk},$$

i.e., W^i_{jk} is another projectively invariant tensor.

Special Cases

(i) Locally Minkowski space

We know that $G^i=0$ in a locally Minkowski space, and therefore equation (2-35) indicates that

$$R^h_{jk} = 0.$$

Hence

$$H^h_{ijk} = 0.$$

Then equation (4-24) implies $H_{ij} = 0$. Hence $D^h_{ijk} = 0$ and $W^h_{ij} = 0$ in a locally Minkowski space.

(ii) Locally projectively flat Finsler space

By definition a locally projectively flat Finsler space is projective to a locally Minkowski space and the Douglas tensor and Weyl tensor are invariant under any projective change. Therefore, $\overline{D}^h_{ijk} = D^h_{ijk} = 0$ and $\overline{W}^h_{ij} = W^h_{ij} = 0$. Hence both the Weyl tensor and Douglas tensor are equal to zero in a locally projectively flat Finsler space.

Theorem 4.2.3 ([Sz1])

A Finsler space $\mathfrak{I}^n = (M, F)$ is of scalar curvature if and only if its Weyl tensor is zero.

Proof:

Consider a Finsler space $\mathfrak{I}^n$ of scalar curvature K, where $n > 2$. Since $\mathfrak{I}^n$ is of scalar curvature, equation (3-10) and equation (2-11) can be used to show that

$$R^k_{ij} = K_i \delta^k_j - K_j \delta^k_i - l^k (K_i l_j - K_j l_i). \tag{4-31}$$

Equation (3-14) implies

$$H_{ij} - H_{ji} = (n-2)\left(\frac{\partial K_j}{\partial y^i} - \frac{\partial K_i}{\partial y^j}\right) + \frac{3}{F}(K_i l_j - K_j l_i),$$

and hence equation (3-16) gives

$$H_{ij} - H_{ji} = \frac{(n+1)}{F}(K_i l_j - K_j l_i). \tag{4-32}$$

Multiplying (4-32) with y^k yields

$$(H_{ij} - H_{ji})y^k = \frac{(n+1)}{F}(K_i l_j - K_j l_i)y^k,$$

and using equation (2-5) gives

$$\frac{(H_{ij} - H_{ji})y^k}{n+1} = (K_i l_j - K_j l_i)l^k. \qquad (4\text{-}33)$$

Now, equation (3-15) implies

$$H_j \delta_i^k = (n+1)K_j \delta_i^k,$$

and similarly

$$H_i \delta_j^k = (n+1)K_i \delta_j^k;$$

hence,

$$\delta_i^k H_j - \delta_j^k H_i = (n+1)(\delta_i^k K_j - \delta_j^k K_i),$$

i.e.,

$$\frac{\delta_i^k H_j - \delta_j^k H_i}{n+1} = \delta_i^k K_j - \delta_j^k K_i \qquad (4\text{-}34)$$

Adding (4-31), (4-33) and (4-34) thus gives

$$R_{ij}^k + \frac{(H_{ij} - H_{ji})y^k + \delta_i^k H_j - \delta_j^k H_i}{(n+1)} = 0,$$

i.e.,

$$W_{ij}^k = 0.$$

A proof of the converse can be found in [Ma3]. If $n = 2$, then every two-dimensional Finsler space is of scalar curvature and the Weyl tensor is equal to zero [Ma3].

Scalar curvature of a locally projectively flat Finsler space

Note that Theorem 4.2.3 implies that locally projectively flat Finsler metrics are of scalar curvature (cf. [Be5] [Be6]). In this section, we determine the scalar curvature of a locally projectively flat Finsler space in terms of the Finsler metric and the projective factor. Let $\mathfrak{F}^n = (M, F)$ be a locally projectively flat Finsler space and G^i be the geodesic coefficients of F. Substituting equation (4-11) to equation (3-2) and taking $i = k$, gives

$$R_i^i = (n-1)\overline{P}^2 - (n-1)\frac{\partial \overline{P}}{\partial x^i} y^i, \tag{4-35}$$

where $\overline{P}$ is the projective factor given by equation (4-12). For $r = k = i$, equation (3-8) implies

$$R_i^i = K(n-1)F^2, \tag{4-36}$$

where K is the scalar curvature of $\mathfrak{I}^n$. Equations (4-35) and (4-36) thus imply

$$K = \frac{1}{F^2}\left(\overline{P}^2 - \frac{\partial \overline{P}}{\partial x^k} y^k\right). \tag{4-37}$$

We give three well-known theorems for the Finsler spaces of scalar curvature.

Theorem 4.2.4 ([FuYa])

A Finsler space of scalar curvature is a space of constant curvature if and only if one of the following conditions holds:

(i) $H_{ij} = H_{ji}$;

(ii) $H_{ji} y^j = H_{ij} y^j$;

(iii) H_i is proportional to y_i.

Theorem 4.2.5 ([Ma3])

A Finsler space $\mathfrak{I}^n = (M, F)$ is locally projectively flat if and only if

(i) $n > 2$: $W^i_{jk} = 0$ and $D^h_{ijk} = 0$,

(ii) $n = 2$: $D^h_{ijk} = 0$ and $K_{ij} = 0$,

where

$$K_{ij} = F\left\{3K_{|r} m^r - \left(F\frac{\partial K}{\partial y^j} m^j\right)_{|k} l^k\right\}(l_i m_j - m_j l_i).$$

K is the Gauss curvature given in (3-24) and " | " denotes the horizontal covariant derivative in $\mathfrak{I}^n$.

Theorem 4.2.6 ([Ba])

A Finsler space $\mathfrak{I}^n = (M, F)$ is a projectively flat Berwald space if and only if it belongs to one of the following classes.

(i) $n \geq 3$

(a) locally Minkowski spaces,

(b) Riemannian spaces of constant curvature,

(ii) $n = 2$

(a) locally Minkowski spaces,

(b) Riemannian spaces of constant curvature,

(c) one-form metric spaces with $F = \dfrac{\beta^2}{\gamma}$, where β and γ are independent one-forms in y^i.

(See [AIM] for further details about one-form metric spaces).

4.3 Examples of Locally Projectively Flat Finsler Spaces

This section contains my original research work. One of the remarks in [Ma3] is that a locally projectively flat Finsler space of non-zero constant curvature may be a Riemannian space of constant curvature. Motivated by this remark, we introduce a system of partial differential equations for some Riemannian spaces of non-zero constant sectional curvature by using a locally projectively flat Finsler space. From the solutions of this system of partial differential equations we obtain two particular examples of locally projectively flat Finsler spaces, which are Riemannian spaces of non-zero constant sectional curvature.

Proposition 4.3.1

Let $\mathfrak{I}^n = (M, F)$ be a Finsler space having geodesic coefficients of the form $G^k = \rho_r y^r y^k \neq 0$, where $\rho = \rho(x)$ is a scalar function on M and $\rho_r = \partial\rho / \partial x^r$. Then:

(a) $\mathfrak{I}^n$ is locally projectively flat;

(b) The projective factor is $-\rho_r y^r$;

(c) $\mathfrak{I}^n$ is of constant curvature;

(d) $\mathfrak{I}^n$ is a Riemannian space of non-zero constant sectional curvature given by

$$K = \frac{1}{F^2}(\rho_i \rho_j - \rho_{ij}) y^i y^j; \tag{4-38}$$

(e) ρ satisfies the following system of partial differential equations.

$$2(\rho_{ij}\rho_k + \rho_{jk}\rho_i + \rho_{ik}\rho_j) = \rho_{ijk} + 4\rho_i \rho_j \rho_k, \tag{4-39}$$

where $\rho_{ij} = \partial\rho_i / \partial x^j$ and $\rho_{ijk} = \partial\rho_{ij} / \partial x^k$.

Proof:

(a)

Using Theorem 4.1.2, we first show that $\mathfrak{I}^n$ with $G^k = \rho_r y^r y^k$ is a locally projectively flat Finsler space. Let $\overline{\mathfrak{I}}^n = (M, \overline{F})$ be a locally Minkowski space. We have to show that

$$\overline{F}_{|i} - \frac{\partial \overline{F}_{|k}}{\partial y^i} y^k = 0,$$

where

$$\overline{F}_{|i} = \frac{\partial \overline{F}}{\partial x^i} - G_i^k \frac{\partial \overline{F}}{\partial y^k}.$$

Since $\overline{F} = \overline{F}(y)$, we know that

$$\frac{\partial \overline{F}}{\partial x^i} = 0;$$

consequently,

$$\overline{F}_{|i} = -G_i^k \frac{\partial \overline{F}}{\partial y^k}. \tag{4-40}$$

Now, we know that

$$G^k = \rho_r y^r y^k. \tag{4-41}$$

Differentiating (4-41) with respect to y^i gives

$$G_i^k = \rho_i y^k + \rho_r y^r \delta_i^k, \tag{4-42}$$

where have used $\rho_r \delta_i^r = \rho_i$. Substituting (4-42) into (4-40) gives

$$\overline{F}_{|i} = -(\rho_i y^k + \rho_r y^r \delta_i^k) \frac{\partial \overline{F}}{\partial y^k},$$

i.e.,

$$\overline{F}_{|i} = -\rho_i y^k \frac{\partial \overline{F}}{\partial y^k} - \rho_r y^r \frac{\partial \overline{F}}{\partial y^i}.$$

Using the property of homogeneity of $\overline{F}$, we have

$$\overline{F}_{|i} = -\rho_i \overline{F} - \rho_r y^r \frac{\partial \overline{F}}{\partial y^i}. \tag{4-43}$$

Replacing i by k in equation (4-43) gives

$$\overline{F}_{|k} = -\rho_k \overline{F} - \rho_r y^r \frac{\partial \overline{F}}{\partial y^k}. \tag{4-44}$$

Differentiating equation (4-44) with respect to y^i and contracting with y^k gives

$$\frac{\partial \overline{F}_{|k}}{\partial y^i} y^k = -\rho_k \frac{\partial \overline{F}}{\partial y^i} y^k - \rho_i \frac{\partial \overline{F}}{\partial y^k} y^k - \rho_r y^r \frac{\partial^2 \overline{F}}{\partial y^i \partial y^k} y^k.$$

Using the property of homogeneity of $\overline{F}$, we have

$$\frac{\partial \overline{F}_{|k}}{\partial y^i} y^k = -\rho_k \frac{\partial \overline{F}}{\partial y^i} y^k - \rho_i \overline{F}. \tag{4-45}$$

Equations (4-43) and (4-45) show that

$$\overline{F}_{|i} - \frac{\partial \overline{F}_{|k}}{\partial y^i} y^k = 0$$

so that by Theorem 4.1.2, $\mathfrak{I}^n$ is projective to $\overline{\mathfrak{I}}^n$. Since $\overline{\mathfrak{I}}^n$ is a locally Minkowski space, from Definition 4.1.2, $\mathfrak{I}^n$ is a locally projectively flat Finsler space.

(b)

We use relation (4-6) to find the projective factor. Contracting equation (4-44) with y^k gives

$$\overline{F}_{|k} y^k = -\rho_k \overline{F} y^k - \rho_r y^r \frac{\partial \overline{F}}{\partial y^k} y^k.$$

Using the property of homogeneity of $\overline{F}$, we have

$$\overline{F}_{|k} y^k = -2\rho_k \overline{F} y^k,$$

and substituting this expression to (4-6) (replacing k by r) gives

$$P = -\rho_r y^r.$$

(c)

We show that $\mathfrak{I}^n$ is of constant curvature, for $n > 2$. For $n = 2$ see part (d). Let W_{ij}^k be the Weyl tensor of $\mathfrak{I}^n$. Since $\mathfrak{I}^n$ is locally projectively flat, from Theorem 4.2.5, we have $W_{ij}^k = 0$. Hence, Theorem 4.2.3 indicates that $\mathfrak{I}^n$ is of scalar curvature. We use Theorem 4.2.4 (i) to show that $\mathfrak{I}^n$ is of constant curvature. We need to show that $H_{ij} = H_{ji}$. Let us calculate H_{ij} in terms of ρ. We know that $H_{ijk}^r = \partial R_{jk}^r \big/ \partial y^i$, where R_{jk}^r is given by (2-35) (with $r = i$). We use equation (4-42) to calculate R_{jk}^r. Let us find the first two terms of R_{jk}^r. Equation (4-42) implies

$$G_j^r = \rho_j y^r + \rho_i y^i \delta_j^r \tag{4-46}$$

Differentiating equation (4-46) with respect to x^k gives

$$\frac{\partial G_j^r}{\partial x^k} = \frac{\partial \rho_j}{\partial x^k} y^r + \frac{\partial \rho_i}{\partial x^k} y^i \delta_j^r, \tag{4-47}$$

and interchanging j and k in equation (4-47) gives

$$\frac{\partial G_k^r}{\partial x^j} = \frac{\partial \rho_k}{\partial x^j} y^r + \frac{\partial \rho_i}{\partial x^j} y^i \delta_k^r. \tag{4-48}$$

Differentiating equation (4-46) with respect to y^m, we have

$$G_{jm}^r = \rho_j \delta_m^r + \rho_m \delta_j^r, \tag{4-49}$$

and from equation (4-46),

$$G_k^m = \rho_k y^m + \rho_i y^i \delta_k^m.$$

Multiplying G_{jm}^r and G_k^m gives

$$\begin{aligned} G_{jm}^r G_k^m &= (\rho_j \delta_m^r + \rho_m \delta_j^r)(\rho_k y^m + \rho_i y^i \delta_k^m) \\ &= (\rho_j y^r \rho_k + \rho_j y^i \rho_i \delta_k^r + \rho_m \rho_k \delta_j^r y^m + \rho_m \rho_i y^i \delta_j^r \delta_k^m). \end{aligned} \tag{4-50}$$

Interchanging j and k in equation (4-50) yields

$$G_{km}^r G_j^m = (\rho_k \rho_j y^r + \rho_k \rho_i y^i \delta_j^r + \rho_m \rho_j \delta_k^r y^m + \rho_m \rho_i y^i \delta_k^r \delta_j^m). \tag{4-51}$$

Substituting equations (4-47), (4-48), (4-50) and (4-51) into equation (2-35), with $r = i$, we get

$$R_{jk}^r = \left(\frac{\partial \rho_j}{\partial x^k} - \frac{\partial \rho_k}{\partial x^j}\right) y^r + \left(\frac{\partial \rho_i}{\partial x^k} \delta_j^r - \frac{\partial \rho_i}{\partial x^j} \delta_k^r\right) y^i + (\rho_j \delta_k^r - \rho_k \delta_j^r) \rho_i y^i,$$

and since $\dfrac{\partial \rho_j}{\partial x^k} = \dfrac{\partial \rho_k}{\partial x^j}$,

$$R_{jk}^r = \left(\frac{\partial \rho_i}{\partial x^k} \delta_j^r - \frac{\partial \rho_i}{\partial x^j} \delta_k^r\right) y^i + (\rho_j \delta_k^r - \rho_k \delta_j^r) \rho_i y^i. \tag{4-52}$$

Differentiating (4-52) with respect to y^i yields

$$\frac{\partial R^r_{jk}}{\partial y^i} = H^r_{ijk} = \left(\frac{\partial \rho_i}{\partial x^k}\delta^r_j - \frac{\partial \rho_i}{\partial x^j}\delta^r_k\right) + (\rho_j\delta^r_k - \rho_k\delta^r_j)\rho_i,$$

which for taking $k = r$, gives

$$H_{ij} = \frac{\partial \rho_i}{\partial x^j} - n\frac{\partial \rho_i}{\partial x^j} + (n-1)\rho_i\rho_j.$$

Here we have used the relations $\delta^r_r = n$ and $H_{ij} = H^r_{ijr}$. Since $\dfrac{\partial \rho_i}{\partial x^j} = \rho_{ij}$ the above expression reduces to

$$H_{ij} = (n-1)(\rho_i\rho_j - \rho_{ij}) = H_{ji}. \tag{4-53}$$

Theorem 4.2.4 shows that $\mathfrak{I}^n$ is a locally projectively flat Finsler space of constant curvature.

(d)

We first show that $K \neq 0$. Suppose that $K = 0$. We know from equation (4-49) that $G^i_{jk} = G^i_{jk}(x)$; hence from Definition 2.2.3, we can say that F is a Berwald metric. It is known that every Berwald metric with zero curvature must be locally Minkowski space [AIM] [BaChSh]. Therefore, $\mathfrak{I}^n$ is a locally Minkowski space, which indicates that $G^i = 0$. This contradicts a hypothesis of Proposition 4.3.1. Hence, $K \neq 0$.

To establish the formula for K we consider two cases.

Case (i): $n > 2$.

Equation (3-18) and equation (4-53) imply

$$H_i = (n+1)(\rho_i\rho_j - \rho_{ij})y^j,$$

and equation (3-15) thus shows that

$$K_i = (\rho_i\rho_j - \rho_{ij})y^j.$$

Contracting the above expression with y^i and using equation (3-12) gives

$$K = \frac{1}{F^2}(\rho_i \rho_j - \rho_{ij}) y^i y^j .$$

Since $K \neq 0$,

$$F^2 = \frac{1}{K}(\rho_i \rho_j - \rho_{ij}) y^i y^j = g_{ij}(x) y^i y^j .$$

Hence the metric tensor is

$$g_{ij}(x) = \frac{1}{K}(\rho_i \rho_j - \rho_{ij}). \tag{4-54}$$

Recall that if $g_{ij} = g_{ij}(x)$, then F is Riemannian. Hence $\mathfrak{I}^n$ is a Riemannian space of non-zero constant sectional curvature given by

$$K = \frac{1}{F^2}(\rho_i \rho_j - \rho_{ij}) y^i y^j .$$

Case (ii): $n = 2$.

From Definition 3.2.1, we know that any two-dimensional Finsler space is of scalar curvature. Contracting equation (3-20) with m_i, l^j, m^k and using (3-22) and (3-23) gives

$$KF = R^i_{jk} m_i l^j m^k ,$$

and equation (4-52) yields

$$KF = \{(\rho_{rk}\delta^i_j - \rho_{rj}\delta^i_k) y^r + (\rho_j \delta^i_k - \rho_k \delta^i_j)\rho_r y^r\} m_i l^j m^k ,$$

which simplifies to

$$KF = (\rho_{rk} m_j l^j m^k - \rho_{rj} m_k l^j m^k) y^r + (\rho_j m_k l^j m^k - \rho_k m_j l^j m^k)\rho_r y^r .$$

Relations (3-22) and (3-23) further simplify this expression to

$$KF = -\rho_{rj} l^j y^r + \rho_j l^j \rho_r y^r .$$

The l^j can be eliminated from the above expression by use of equation (2-5) so that

$$K = \frac{1}{F}\left(-\rho_{rj}\frac{y^j}{F} y^r + \rho_j \frac{y^j}{F}\rho_r y^r \right),$$

i.e.,

$$K = \frac{1}{F^2}(\rho_i \rho_j - \rho_{ij}) y^i y^j,$$

where $i, j = 1,2$.

Now we show that $\mathfrak{I}^n$ is a Riemannian space of non-zero constant sectional curvature when $n = 2$. Since F is a Berwald metric, Lemma 2.2.4 shows that F is either locally Minkowskian or Riemannian. Recall that if F is locally Minkowskian then $G^i = 0$, which contradicts a hypothesis of Proposition 4.3.1. Hence F is a Riemannian metric. Further in part (a), we have proved that F is locally projectively flat, which implies that F is a locally projectively flat Berwald metric. Hence from Theorem 4.2.6, F is a Riemannian metric of non-zero constant sectional curvature.

(e)

Since $\mathfrak{I}^n$ is a Riemannian space, it must be metric compatible with the Riemannian connection. This means that

$$\frac{\partial g_{ij}}{\partial x^k} - g_{rj}\Gamma^r_{ik} - g_{ir}\Gamma^r_{jk} = 0 \tag{4-55}$$

(equation (2-15)), and the coefficients of the Berwald connection coincide with the Christoffel symbols of the Riemannian connection. Equation (4-49) thus implies

$$\Gamma^i_{jk} = G^i_{jk} = \rho_k \delta^i_j + \rho_j \delta^i_k.$$

Equation (4-55) can thus be written

$$\frac{\partial g_{ij}}{\partial x^k} - g_{rj}(\rho_i \delta^r_k + \rho_k \delta^r_i) - g_{ir}(\rho_k \delta^r_j + \rho_j \delta^r_k) = 0,$$

i.e.,

$$\frac{\partial g_{ij}}{\partial x^k} - 2g_{ij}\rho_k - g_{kj}\rho_i - g_{ik}\rho_j = 0. \tag{4-56}$$

Differentiating equation (4-54) with respect to x^k,

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{1}{K}(\rho_{ik}\rho_j + \rho_{jk}\rho_i - \rho_{ijk}),$$

where $\rho_{ijk} = \partial\rho_{ij} / \partial x^k$. From equation (4-54), we can also write

$$g_{kj} = \frac{1}{K}(\rho_k \rho_j - \rho_{kj}),$$

$$g_{ik} = \frac{1}{K}(\rho_i \rho_k - \rho_{ik}),$$

so that in terms of ρ equation (4-56) can be written as

$$2(\rho_{ij}\rho_k + \rho_{jk}\rho_i + \rho_{ik}\rho_j) = \rho_{ijk} + 4\rho_i\rho_j\rho_k.$$

The proof of Proposition 4.3.1 is complete.

Radially Symmetric Solutions

Proposition 4.3.1 can be used to construct examples of Riemannian metrics of constant sectional curvature. Specifically, we can obtain such examples from solutions ρ to the system of partial differential equations (4-39). In this section we consider a special class of solutions called radially symmetric solutions.

Let

$$T_{ijk}[\rho] = \rho_{ijk} + 4\rho_i\rho_j\rho_k - 2(\rho_{ij}\rho_k + \rho_{ik}\rho_j + \rho_{jk}\rho_i). \tag{4-57}$$

Here, $i, j, k \in \{1,2,\ldots,n\}$, $\rho_i = \partial\rho / \partial x^i$, $\rho_{ij} = \partial\rho_i / \partial x^j$, $\rho_{ijk} = \partial\rho_{ij} / \partial x^k$,

and it is assumed that ρ has continuous partial derivative of at least third order. Now, we seek solutions to

$$T_{ijk}[\rho] = 0 \tag{4-58}$$

of the form $\rho = \rho(\phi)$, where

$$\phi = \frac{1}{2}\delta_{ij}x^i x^j = \frac{1}{2}|x|^2,$$

i.e., radially symmetric solutions. We thus seek radially symmetric solutions ρ to a system of third order partial differential equations (4-58) such that ρ is a smooth function of ϕ and equation (4-58) is satisfied for all values of $i, j, k \in \{1,2,\ldots,n\}$.

Now,

$$\rho_i = \rho'(\phi)\frac{\partial\phi}{\partial x^i} = \rho'(\phi)\delta_{im}x^m = \rho'(\phi)x^i,$$

$$\rho_{ij} = \rho''(\phi)\delta_{im}x^m\delta_{jl}x^l + \rho'(\phi)\delta_{ij},$$

$$= \rho''(\phi)x^i x^j + \rho'(\phi)\delta_{ij},$$

$$\rho_{ijk} = \rho'''(\phi)x^i x^j x^k + \rho''(\phi)\{x^i\delta_{jk} + x^j\delta_{ik} + x^k\delta_{ij}\},$$

where $\rho'(\phi)$ denotes $d\rho/d\phi$, so that

$$T_{ijk}[\rho] = x^i x^j x^k\{\rho'''(\phi) + 4\rho'(\phi)^3 - 6\rho''(\phi)\rho'(\phi)\}$$
$$+ (x^i\delta_{jk} + x^j\delta_{ik} + x^k\delta_{ij})\{\rho''(\phi) - 2\rho'(\phi)^2\}. \tag{4-59}$$

Lemma 4.3.2

If $\rho(\phi)$ is a solution to (4-58) for all $i, j, k \in \{1,2,...,n\}$ then ρ is a solution to the third order ordinary differential equation

$$\rho'''(\phi) + 4\rho'(\phi)^3 - 6\rho''(\phi)\rho'(\phi) = 0. \tag{4-60}$$

Proof:

We know that the coefficients of $x^i x^j x^k$ and $(x^i\delta_{jk} + x^j\delta_{ik} + x^k\delta_{ij})$ in (4-59) are invariant under index permutations. If $n > 2$ then we make the special choice $i \neq j \neq k$ so that $\delta_{jk} = \delta_{ij} = \delta_{ik} = 0$ and hence equations (4-59) and (4-58) imply

$$x^i x^j x^k\{\rho'''(\phi) + 4\rho'(\phi)^3 - 6\rho''(\phi)\rho'(\phi)\} = 0$$

for all $x^i\ x^j\ x^k$ in some neighbourhood of $x_0 \in R^n$. Since ρ is a smooth function of ϕ, the above expression gives

$$\rho'''(\phi) + 4\rho'(\phi)^3 - 6\rho''(\phi)\rho'(\phi) = 0.$$

For $n = 2$, we take $i = j = k = 1$. Then equations (4-59) and (4-58) imply

$$(x^1)^2\{\rho'''(\phi) + 4\rho'(\phi)^3 - 6\rho''(\phi)\rho'(\phi)\} + 3\{\rho''(\phi) - 2\rho'(\phi)^2\} = 0.$$

Similarly, taking $i = j = 1$ and $k = 2$ gives

$$(x^1)^2\{\rho'''(\phi) + 4\rho'(\phi)^3 - 6\rho''(\phi)\rho'(\phi)\} + \{\rho''(\phi) - 2\rho'(\phi)^2\} = 0.$$

The above two expressions yield (4-60).

Corollary 4.3.3

If $\rho(\phi)$ is a solution to (4-58) for all $i, j, k \in \{1,2,...,n\}$ then

$$\rho''(\phi) - 2\rho'(\phi)^2 = 0. \tag{4-61}$$

Proof:

The differential equation (4-61) immediately follows from (4-60) since equation (4-59) reduces to

$$(x^i \delta_{jk} + x^j \delta_{ik} + x^k \delta_{ij})\{\rho''(\phi) - 2\rho'(\phi)^2\} = 0$$

and we can choose i, j, k such that $i = j \neq k$ as a special case exploiting the invariance of the coefficients.

Lemma 4.3.4

Let ρ be a three times differentiable solution to equation (4-61). Then ρ is also a solution to equation (4-60).

Proof:

Equation (4-61) implies

$$\rho''(\phi)\rho'(\phi) = 2\rho'(\phi)^3,$$

and since ρ is three times differentiable we have also that

$$\rho'''(\phi) - 4\rho''(\phi)\rho'(\phi) = 0,$$

by differentiating both sides of (4-61) with respect to ϕ. Therefore,

$$\begin{aligned}&\rho'''(\phi) + 4\rho'(\phi)^3 - 6\rho''(\phi)\rho'(\phi) \\ &= 4\rho''(\phi)\rho'(\phi) + 2\rho''(\phi)\rho'(\phi) - 6\rho''(\phi)\rho'(\phi) = 0.\end{aligned}$$

In summary, we have the following result:

Theorem 4.3.5

If $\rho(\phi)$ is a solution to equation (4-58) for all $i, j, k \in \{1,2,...,n\}$ then $\rho(\phi)$ is a solution to equation (4-61). Moreover, if $\rho(\phi)$ is a solution to equation (4-61) then $\rho(\phi)$ is a solution to equation (4-58).

Note that the final statement follows immediately since the coefficients are identically zero.

Solutions of the equation (4-61)

We now solve equation (4-61). Evidently $\rho'(\phi)=0$ is a solution, though it will not lead to a useful metric for our problem. Suppose that $\rho'(\phi)\neq 0.$ Then equation (4-61) can be recast in the form

$$\frac{\rho''(\phi)}{\rho'(\phi)}=2\rho'(\phi);$$

consequently,

$$\ln|\rho'(\phi)|=2\rho(\phi)+c,$$

where c is a constant of integration. The above relation implies

$$\rho'(\phi)=\tilde{c}e^{2\rho(\phi)},$$

where $\tilde{c}=e^{c}\neq 0$ and hence

$$\frac{1}{\tilde{c}}e^{-2\rho}d\rho=d\phi,$$

i.e.,

$$-\frac{1}{2\tilde{c}}e^{-2\rho}=\phi+c_1,$$

where c_1 is a constant of integration. Solving the above equation for ρ gives

$$\rho=-\frac{1}{2}\ln(A\phi+B), \tag{4-62}$$

where $A(\neq 0)$ and B are constants of integration. Substituting $2\phi=|x|^2$ and differentiating (4-62) with respect to x^i, we also have

$$\rho_i=-\frac{\delta_{ij}x^j}{|x|^2+\lambda},$$

where $\lambda=2B/A.$ Note that since ρ is a smooth function of $\phi,$ from equation (4-62) $|x|^2+\lambda\neq 0.$

Now we calculate F using equation (4-38). We have

$$\rho_i\rho_j-\rho_{ij}=\rho'(\phi)^2\delta_{im}x^m\delta_{jl}x^l-\left(\rho''(\phi)\delta_{im}x^m\delta_{jl}x^l+\rho'(\phi)\delta_{ij}\right)$$
$$=\left(\rho'(\phi)^2-\rho''(\phi)\right)\delta_{im}x^m\delta_{jl}x^l-\rho'(\phi)\delta_{ij},$$

so that from equation (4-61) we see that

$$\rho_i\rho_j-\rho_{ij}=-\rho'(\phi)^2\delta_{im}x^m\delta_{jl}x^l-\rho'(\phi)\delta_{ij},$$

and hence

$$F^2=\frac{-\rho'(\phi)}{K}\{\delta_{ij}+\rho'(\phi)\delta_{im}x^m\delta_{jl}x^l\}y^iy^j$$
$$=\frac{-\rho'(\phi)}{K}\{|y|^2+\rho'(\phi)\langle x,y\rangle^2\}.$$

Differentiating (4-62) with respect to ϕ gives

$$\rho'(\phi)=-\frac{1}{2\phi+\lambda}.$$

Hence

$$F^2=\frac{1}{K}\left\{\frac{|y|^2}{|x|^2+\lambda}-\frac{\langle x,y\rangle^2}{(|x|^2+\lambda)^2}\right\},$$

where we have used $2\phi=|x|^2$.

4.3.1 Examples of Riemannian metrics of non-zero constant sectional curvature

Example (i)

If $K=-1$ and $\lambda=-1$ then we have

$$F^2=\frac{|y|^2}{1-|x|^2}+\frac{\langle x,y\rangle^2}{(1-|x|^2)^2},$$

i.e.,

$$F(x,y)=\frac{\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}}{1-|x|^2},$$

where $y \in T_pB^n \cong R^n$. This is the Klein metric defined on the unit ball B^n in R^n stated in [Sh2].

Example (ii)

If $K = 1$ and $\lambda = 1$ then we have

$$F^2 = \frac{|y|^2}{1+|x|^2} - \frac{\langle x,y\rangle^2}{(1+|x|^2)^2},$$

i.e.,

$$F(x,y) = \frac{\sqrt{|y|^2 + (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1+|x|^2},$$

where $y \in T_pR^n \cong R^n$ This is the Riemannian metric given by [Sh2].

The next lemma provides an alternative method of calculating geodesic coefficients of some Riemannian metrics of non-zero constant sectional curvature rather than using (2-17), which involves finding the inverse Finsler metric tensor.

Lemma 4.3.6

Let $\mathfrak{I}^n = (M,F)$ be a Riemannian space defined on a underlying manifold M of dimension n and let

$$F^2 = \mu(\rho_i\rho_j - \rho_{ij})y^iy^j, \tag{4-63}$$

where μ is a non-zero constant, and $\rho = \rho(x)$ is a scalar field defined on M that satisfies equation (4-39). Then

$$G^i = \rho_r y^r y^i \tag{4-64}$$

and the curvature of $\mathfrak{I}^n$ is $1/\mu$.

Proof:

Recall that

$$G^i = \frac{1}{4} g^{ik} \left\{ \left(\frac{\partial^2 F^2}{\partial x^r \partial y^k} \right) y^r - \frac{\partial F^2}{\partial x^k} \right\} \tag{4-65}$$

(equation (2-17)). Equation (4-63) shows that

$$\frac{\partial F^2}{\partial x^k} = \mu(\rho_{ik}\rho_j + \rho_i \rho_{jk} - \rho_{ijk}) y^i y^j , \tag{4-66}$$

and

$$\frac{\partial F^2}{\partial y^k} = \mu(\rho_i \rho_j - \rho_{ij})(\delta^i_k y^j + \delta^j_k y^i)$$

$$= 2\mu(\rho_k \rho_j - \rho_{kj}) y^j .$$

Now,

$$\frac{\partial}{\partial x^r} \left(\frac{\partial F^2}{\partial y^k} \right) = 2\mu(\rho_{kr}\rho_j + \rho_k \rho_{jr} - \rho_{rkj}) y^j ,$$

and contracting this expression with y^r gives

$$\frac{\partial}{\partial x^r} \left(\frac{\partial F^2}{\partial y^k} \right) y^r = 2\mu(\rho_{kr}\rho_j + \rho_k \rho_{jr} - \rho_{rkj}) y^j y^r . \tag{4-67}$$

Replacing i by r in (4-66) then subtracting (4-66) from (4-67) gives

$$\frac{\partial}{\partial x^r} \left(\frac{\partial F^2}{\partial y^k} \right) y^r - \frac{\partial F^2}{\partial x^k} = \mu(\rho_{kr}\rho_j + 2\rho_k \rho_{jr} - \rho_r \rho_{jk} - \rho_{rkj}) y^j y^r$$

and since $\rho_{kr}\rho_j y^j y^r = \rho_r \rho_{jk} y^j y^r$,

$$\frac{\partial}{\partial x^r} \left(\frac{\partial F^2}{\partial y^k} \right) y^r - \frac{\partial F^2}{\partial x^k} = \mu(2\rho_k \rho_{jr} - \rho_{rkj}) y^j y^r . \tag{4-68}$$

Substituting (4-68) into (4-65) yields

$$G^i = \frac{1}{4} \mu g^{ik} (2\rho_k \rho_{jr} - \rho_{rkj}) y^j y^r .$$

The function ρ_{rkj} can be eliminated from the above expression by use of equation (4-39). We thus get

$$G^i = \frac{\mu}{2} g^{ik}(2\rho_r\rho_k\rho_j - \rho_{jk}\rho_r - \rho_{rk}\rho_j)y^j y^r$$

$$= \mu g^{ik}(\rho_r\rho_k\rho_j - \rho_{jk}\rho_r)y^j y^r. \qquad (4\text{-}69)$$

We know that the metric tensor of $\mathfrak{I}^n$ is given by

$$g_{kr} = \mu(\rho_k\rho_r - \rho_{kr}),$$

and contracting g_{kr} with g^{ik} gives

$$g_{kr}g^{ik} = \mu(\rho_k\rho_r - \rho_{kr})g^{ik}.$$

Since $g_{kr}g^{ik} = \delta^i_r$, we have

$$\delta^i_r = \mu\rho_k\rho_r g^{ik} - \mu\rho_{kr}g^{ik};$$

hence,

$$\rho_k\rho_r g^{ik} = \frac{1}{\mu}\delta^i_r + \rho_{kr}g^{ik}.$$

The above expression and equation (4-69) imply

$$G^i = \mu\left\{\rho_j\left(\frac{1}{\mu}\delta^i_r + g^{ik}\rho_{kr}\right) - g^{ik}\rho_{jk}\rho_r\right\}y^j y^r$$

$$= \rho_j y^i y^j + \mu g^{ik}(\rho_j\rho_{kr} - \rho_{jk}\rho_r)y^j y^r$$

$$= \rho_r y^r y^i.$$

Hence equations (4-38) and (4-63) imply

$$K = \frac{1}{\mu}.$$

We use Lemma 4.3.6 to find geodesic coefficients of the Riemannian metrics found in Examples 4.3.1.

Examples 4.3.2

Example (i)

We consider the Riemannian metric found in Example (i) of Examples 4.3.1, where

$$F^2 = \frac{|y|^2}{1-|x|^2} + \frac{\langle x,y\rangle^2}{(1-|x|^2)^2}$$

$$= \frac{\delta_{ij} y^i y^j}{1-|x|^2} + \frac{\delta_{ki}\delta_{rj} x^k x^r y^i y^j}{(1-|x|^2)^2}.$$

For this example, note that

$$\rho_i = \frac{\delta_{ki} x^k}{1-|x|^2}.$$

Now,

$$\rho_{ij} = \frac{\delta_{ij}}{1-|x|^2} + 2\rho_i \rho_j,$$

so that

$$F^2 = (\rho_{ij} - 2\rho_i\rho_j + \rho_i\rho_j) y^i y^j$$

$$= -(\rho_i\rho_j - \rho_{ij}) y^i y^j. \tag{4-70}$$

Lemma 4.3.6 implies $\mu = -1$, and

$$G^i = \rho_r y^r y^i = \frac{\delta_{rk} x^k y^r y^i}{1-|x|^2};$$

therefore,

$$G^i = \frac{\langle x,y\rangle y^i}{1-|x|^2}. \tag{4-71}$$

The sectional curvature is $K = \frac{1}{\mu} = -1$.

Example: (ii)

Similarly, the Riemannian metric in Example (ii) of Examples 4.3.1 gives

$$F^2 = (\rho_i \rho_j - \rho_{ij}) y^i y^j, \tag{4-72}$$

where

$$\rho_i = -\frac{\delta_{ri} x^r}{1 + |x|^2}.$$

Lemma 4.3.6 implies $\mu = 1$, and

$$G^i = -\frac{\langle x, y\rangle y^i}{1 + |x|^2},$$

and the sectional curvature is $K = 1$.

Example: (iii)

Consider the following Riemannian metric.

$$F = \frac{\sqrt{|y|^2 - \varepsilon(|x|^2 |y|^2 - \langle x, y\rangle^2)}}{1 - \varepsilon |x|^2},$$

where $\varepsilon > 0$ and $|x|^2 < 1/\varepsilon$. Then

$$\begin{aligned} F^2 &= \frac{\varepsilon\langle x, y\rangle^2}{(1 - \varepsilon|x|^2)^2} + \frac{|y|^2}{1 - \varepsilon |x|^2} \\ &= \frac{1}{\varepsilon}\left(\frac{\varepsilon^2\langle x, y\rangle^2}{(1 - \varepsilon|x|^2)^2} + \frac{\varepsilon|y|^2}{1 - \varepsilon|x|^2}\right) \\ &= \frac{1}{\varepsilon}\left(\frac{\varepsilon^2 \delta_{im} x^m \delta_{jr} x^r}{(1 - \varepsilon|x|^2)^2} + \frac{\varepsilon\delta_{ij}}{1 - \varepsilon|x|^2}\right) y^i y^j. \end{aligned}$$

For this example,

$$\rho_i = \frac{\varepsilon\delta_{im} x^m}{1 - \varepsilon|x|^2}$$

and

$$\rho_{ij} = \frac{\varepsilon \delta_{ij}}{1 - \varepsilon |x|^2} + 2\rho_i \rho_j.$$

Then

$$F^2 = -\frac{1}{\varepsilon}(\rho_i \rho_j - \rho_{ij}) y^i y^j.$$

Using Lemma 4.3.6, we have

$$G^i = \frac{\varepsilon \delta_{jk} x^j y^k y^i}{1 - \varepsilon |x|^2} = \frac{\varepsilon \langle x, y \rangle y^i}{1 - \varepsilon |x|^2}.$$

Since $\mu = -\frac{1}{\varepsilon}$,

$$K = \frac{1}{\mu} = -\varepsilon,$$

where K is the non-zero constant sectional curvature of F.

Chapter 5

Randers Space

5.1 Randers Metric

The Randers metric is a special Finsler metric that arises in general relativity. This metric was introduced by G. Randers in 1941 [Ra]. The Randers metric has the following form

$$F = F(x,y) = \alpha(x,y) + \beta(x,y),$$

where $\alpha(x,y) = \sqrt{\alpha_{ij}(x)y^i y^j}$ is a Riemannian metric on the n-dimensional differentiable manifold M, and $\beta(x,y) = b_i(x)dx^i$ is a one-form on M. Here, $b_i(x)$ is a covariant vector field defined on M. We also need to have the following condition for Randers metric tensor to be positive definite.

$$\sqrt{\alpha^{ij}b_i b_j} < 1, \tag{5-1}$$

where (α^{ij}) is the inverse of (α_{ij}). The space $\mathfrak{I}^n = (M, F = \alpha + \beta)$ is called a **Randers space** of dimension n, and the space (M,α) is called its **associated Riemannian space**. The Funk metrics on the unit ball given by (3-33) are examples of Randers metrics, where α is the Klein metric given by example (i) of Examples 4.3.1. Further details are given in [BaChSh], [Sh1], [Sh2] and [AIM].

Definition 5.1.1

The one-form β is said to be **parallel** with respect to α if

$$b_{i|j} = \frac{\partial b_i}{\partial x^j} - b_k \Gamma_{ij}^k = 0,$$

where Γ_{ij}^k denote the Christoffel symbols of α.

In 1979, S. Kikuchi proved that a Randers metric $F = \alpha + \beta$ is a Berwald metric if and only if β is parallel with respect to α [Ki]. This characterization is useful for determining the Randers metrics that are also Berwald metrics.

Definition 5.1.2

The one-form β is said to be **closed** if $b_{i|j} - b_{j|i} = 0$. In other words, β is closed if

$$\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i} = 0. \tag{5-2}$$

In 1997, S. Bacso and M. Matsumoto proved that a Randers metric $F = \alpha + \beta$ is a Douglas metric if and only if β is closed [BaMa2].

Relationship between the geodesic coefficients of the Randers space and its associated Riemannian space

Let $\overline{G}^i$ and G^i be the geodesic coefficients of $F = \alpha + \beta$ and α respectively. Then equation (4-4) gives the general relation between $\overline{G}^i$ and G^i as follows.

$$\overline{G}^i = G^i + \frac{F_{|k} y^k}{2F} y^i + \frac{F}{2} g^{ij} \left\{ \frac{\partial F_{|k}}{\partial y^j} y^k - F_{|j} \right\}, \tag{5-3}$$

where $F_{|k}$ denotes the horizontal covariant derivative of F on the Riemannian space (M, α).

Now, taking the horizontal covariant derivative of F with respect to α gives

$$F_{|k} = \alpha_{|k} + \beta_{|k}.$$

According to Lemma 2.2.2, $\alpha_{|k} = 0$ on (M,α); moreover, equation (2-34) shows that

$$y^i{}_{|k}=0.$$

Since $\beta = b_i y^i$, we thus have

$$F_{|k} = b_{i|k} y^i. \tag{5-4}$$

Differentiating (5-4) with respect to y^j and contracting with y^k gives

$$\frac{\partial F_{|k}}{\partial y^j} y^k = b_{j|k} y^k. \tag{5-5}$$

Replacing k by j and i by k in (5-4) gives

$$F_{|j} = b_{k|j} y^k,$$

and subtracting (5-5) and the above expression yields

$$\frac{\partial F_{|k}}{\partial y^j} y^k - F_{|j} = b_{j|k} y^k - b_{k|j} y^k = (b_{j|k} - b_{k|j}) y^k. \tag{5-6}$$

Hence, the relationship between $\overline{G}^i$ and G^i is

$$\overline{G}^i = G^i + \frac{b_{j|k} y^j y^k}{2F} y^i + \frac{F}{2} g^{ij} (b_{j|k} - b_{k|j}) y^k. \tag{5-7}$$

Special Cases

(i)

If β is closed, i.e., $b_{j|k} - b_{k|j} = 0$, then from equation (5-6) and Theorem 4.1.2, (M,α) is projective to (M,F), and equation (5-7) implies

$$\overline{G}^i = G^i + \frac{b_{j|k} y^j y^k}{2F} y^i. \tag{5-8}$$

The projective factor of the projective change $\alpha \to F$ is thus

$$P = \frac{b_{j|k} y^j y^k}{2F}. \tag{5-9}$$

(ii)

If β is parallel with respect to α, i.e., $b_{i|j} = 0$, then

$$\overline{G}^i = G^i = \frac{1}{2}\Gamma^i_{jk} y^j y^k,$$

where Γ^i_{jk} are the Christoffel symbols of α. Definition 2.2.3 shows that in this case $F = \alpha + \beta$ is also a Berwald metric.

In 1980, M. Hashiguchi and Y. Ichijyo found the following necessary and sufficient conditions that a Randers space is projective to its associated Riemannian space [HaIc].

Theorem 5.1.3 ([HaIc])

For a Randers metric $F = \alpha + \beta$, β is closed if and only if (M, F) is projective to (M, α).

Theorem 5.1.4 ([BaMa2])

A Randers metric $F = \alpha + \beta$ is a Douglas metric if and only if β is closed.

The next theorem is a direct consequence of Theorem 5.1.4. Note that the proof is provided by the author.

Theorem 5.1.5

A Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if β is closed and α is locally projectively flat.

Proof:

Suppose that $F = \alpha + \beta$ is locally projectively flat. Equation (4-8) implies

$$\frac{\partial F}{\partial x^i} - \frac{\partial^2 F}{\partial y^i \partial x^k} y^k = 0. \tag{5-10}$$

Now,

$$\frac{\partial F}{\partial x^i} = \frac{\partial}{\partial x^i}(\alpha + \beta) = \frac{\partial \alpha}{\partial x^i} + \frac{\partial b_j}{\partial x^i} y^j, \tag{5-11}$$

and similarly

$$\frac{\partial F}{\partial x^k} = \frac{\partial \alpha}{\partial x^k} + \frac{\partial b_j}{\partial x^k} y^j.$$

Differentiating the above expression with respect to y^i gives

$$\frac{\partial^2 F}{\partial y^i \partial x^k} = \frac{\partial^2 \alpha}{\partial y^i \partial x^k} + \frac{\partial b_j}{\partial x^k} \delta_i^j, \tag{5-12}$$

and equations (5-10), (5-11) and (5-12) thus imply

$$\frac{\partial \alpha}{\partial x^i} + \frac{\partial b_k}{\partial x^i} y^k - \left(\frac{\partial^2 \alpha}{\partial y^i \partial x^k} + \frac{\partial b_i}{\partial x^k} \right) y^k = 0,$$

$$\frac{\partial \alpha}{\partial x^i} - \frac{\partial^2 \alpha}{\partial y^i \partial x^k} y^k + \left(\frac{\partial b_k}{\partial x^i} - \frac{\partial b_i}{\partial x^k} \right) y^k = 0. \tag{5-13}$$

By hypothesis F is locally projectively flat; consequently, the Douglas tensor of F is equal to zero by Theorem 4.2.5. Therefore F is a Douglas metric, and Theorem 5.1.4 shows that β must be closed so that equation (5-13) reduces to

$$\frac{\partial \alpha}{\partial x^i} - \frac{\partial^2 \alpha}{\partial y^i \partial x^k} y^k = 0,$$

hence Lemma 4.1.3 shows that α is locally projectively flat. The converse also follows easily.

Remark: In Riemannian geometry, Beltrami showed that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. Combining this with Theorem 5.1.5 we can conclude that a Randers metric is locally projectively flat if and only if the Riemannian metric is of constant sectional curvature and β is closed.

Notation

The following notation will be used frequently in our discussion of Randers metrics:

$$b_{i|j} = \frac{\partial b_i}{\partial x^j} - b_k \Gamma^k_{ij},$$

where Γ^k_{ij} are the Christoffel symbols of α;

$$s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}),$$

$$s^i_j = \alpha^{ik} s_{kj},$$

where (α^{ik}) is the inverse of (α_{ik});

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}),$$

$$e_{ij} = r_{ij} + b_i s_j + b_j s_i,$$

where $s_j = b_i s^i_j$;

$$\| b \| = \sqrt{\alpha^{ij} b_i b_j} < 1.$$

Theorem 5.1.6 ([Si])

Let $\Im^n = (M, F = \alpha + \beta)$ be a Randers space of dimension $n \geq 2$. Then,

(a) $\Im^n$ has constant negative flag curvature K and $s_i = 0$ if and only if

(1) The Riemannian space (M, α) is of negative constant sectional curvature $\mu = -\lambda^2$, where λ is a non-zero constant,

(2) $b_{i|j} = \lambda(\alpha_{ij} - b_i b_j)$.

In this case we have $K = -\frac{\lambda^2}{4}$ and the Randers space is called an RCG-space.

(b) $\Im^n$ is flat (i.e., $K = 0$) and $s_i = 0$ if and only if it is a locally Minkowski space.

(c) $\Im^n$ has constant positive flag curvature K and $s_i = 0$ if and only if

(1) The Riemannian curvature tensor R of the associated Riemannian space (M,α) satisfies the relation

$$R_{hijk} = K(1-\|b\|^2)(\alpha_{ij}\alpha_{hk} - \alpha_{ik}\alpha_{hj}) + K(\alpha_{ij}b_h b_k - \alpha_{ik}b_h b_j)$$
$$- K(\alpha_{hj}b_i b_k - \alpha_{hk}b_i b_j) - b_{i|j}b_{h|k} + b_{i|k}b_{h|j} + 2b_{h|i}b_{j|k}.$$

(2) $\|b\|$ is a constant.

(3) $b = b_i(x)$ is a non parallel Killing vector field.

In this case Randers space is called an RCT-space.

Remark:

The " RCG " stands for Randers space of constant curvature with gradient; the " RCT " stands for Randers space of constant curvature with translation.

As we mentioned in chapter 3.3, the Funk metrics on the unit ball in Euclidean space are Randers metrics with constant flag curvature [Sh8]. However the author could not find the proof in there. Therefore we prove this result using example (i) of Examples 4.3.1 and Theorem 5.1.6.

Lemma 5.1.7

Let

$$\alpha = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2},$$

and

$$\beta = \pm\frac{\langle x,y\rangle}{1-|x|^2}.$$

Then $\Im^n = (M, F = \alpha \pm \beta)$ are locally projectively flat Randers spaces of constant flag curvature defined on the unit ball $B^n = \{x \in R^n, |x| < 1\}$.

Proof:

Now, recall that α is the Riemannian metric with constant sectional curvature, -1 (cf. example (i) of Examples 4.3.1). Hence the equation (4-70) gives the Riemannian metric tensor

$$\alpha_{ij} = -(\rho_i\rho_j - \rho_{ij}), \tag{5-14}$$

where

$$\rho_i = \frac{\delta_{ik}x^k}{1-|x|^2}. \tag{5-15}$$

It is also clear that

$$\beta = b_i y^i = \pm\rho_i y^i. \tag{5-16}$$

Let us show that $\mathfrak{I}^n$ is of constant flag curvature using the part (a) in Theorem 5.1.6. Since α is the Riemannian metric with constant sectional curvature, $\mu = -1$, the condition (1) of (a) in Theorem 5.1.6 implies

$$\lambda = \pm 1.$$

We first consider the case, where $\lambda = 1$ and $b_i = \rho_i$. We now show that the condition (2) of (a) in Theorem 5.1.6 is valid. The $b_{i|j}$ can be expressed as

$$b_{i|j} = \frac{\partial b_i}{\partial x^j} - b_k\Gamma_{ij}^k,$$

where Γ_{ij}^k are the Christoffel symbols of α. From (4-71), we have geodesic coefficients of α given by

$$G^k = \rho_r y^r y^k.$$

Differentiating this expression with respect to y^i and y^j gives (cf. equation (2-31))

$$G_{ij}^k = \rho_i\delta_j^k + \rho_j\delta_i^k = \Gamma_{ij}^k.$$

Hence

$$b_{i|j} = \rho_{ij} - 2\rho_i\rho_j. \tag{5-17}$$

Since $\lambda = 1$ and $b_i = \rho_i$, from (5-14),

$$\lambda(\alpha_{ij} - b_i b_j) = \rho_{ij} - 2\rho_i\rho_j,$$

and we thus have condition (2) of (a) in Theorem 5.1.6. Note that for $\lambda = -1$ and $b_i = \rho_i$, the condition (2) of (a) in Theorem 5.1.6 is not valid. Therefore we next consider the case $\lambda = -1$ and $b_i = -\rho_i$. Similarly, for $\lambda = -1$ and $b_i = -\rho_i$,

$$\begin{aligned} b_{i|j} &= -\rho_{ij} + \rho_k(\rho_i\delta_j^k + \rho_j\delta_i^k), \\ &= 2\rho_i\rho_j - \rho_{ij}. \end{aligned} \tag{5-18}$$

Since $\lambda = -1$ and $b_i = -\rho_i$, from (5-14),

$$\lambda(\alpha_{ij} - b_i b_j) = 2\rho_i\rho_j - \rho_{ij}.$$

Then from equation (5-18) and the above expression give

$$b_{i|j} = \lambda(\alpha_{ij} - b_i b_j).$$

Since conditions (1) and (2) hold in (a) of Theorem 5.1.6, this shows that $\Im^n$ is of constant flag curvature, $K = -\frac{1}{4}$, and

$$s_i = 0. \tag{5-19}$$

We know that from (5-15) $\frac{\partial \rho_i}{\partial x^j} - \frac{\partial \rho_j}{\partial x^i} = 0$, hence (5-2) shows that β is closed. Since α has constant sectional curvature and β is closed, the remark of Theorem 5.1.5 indicates that F is locally projectively flat given by

$$F(x,y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2} \pm \frac{\langle x,y\rangle}{1-|x|^2}. \tag{5-20}$$

We now consider the condition (5-1) for a Randers metric tensor to be positive definite, namely,

$$\sqrt{\alpha^{ij}\rho_i\rho_j} < 1. \tag{5-21}$$

To simplify the above condition first we need to find α^{ij}. Equation (5-14) gives

$$\alpha_{ij} = \rho_{ij} - \rho_i\rho_j.$$

Differentiating (5-15) with respect to x^j gives

$$\rho_{ij} = \frac{\delta_{ij}}{1-|x|^2} + 2\rho_i\rho_j,$$

and therefore

$$\alpha_{ij} = \frac{\delta_{ij}}{1-|x|^2} + \rho_i\rho_j.$$

Equation (5-15) yields

$$\alpha_{ij} = \frac{1}{1-|x|^2}\left(\delta_{ij} + \frac{\delta_{ri}\delta_{mj}x^r x^m}{1-|x|^2}\right).$$

Let

$$\alpha^{ij} = (1-|x|^2)(\delta^{ij} - x^i x^j), \tag{5-22}$$

where (δ^{ij}) is the inverse of (δ_{ij}). We show that (α^{ij}) given by (5-22) is the inverse of (α_{ij}), i.e.,

$$\alpha_{ij}\alpha^{ik} = \delta_j^k.$$

Multiplying α_{ij} by α^{ik} gives

$$\alpha_{ij}\alpha^{ik} = \left(\delta_{ij} + \frac{\delta_{ri}\delta_{mj}x^r x^m}{1-|x|^2}\right)(\delta^{ik} - x^i x^k).$$

i.e.,

$$\alpha_{ij}\alpha^{ik} = \delta_{ij}\delta^{ik} + \frac{\delta_{ri}x^r\delta_{mj}x^m\delta^{ik}}{1-|x|^2} - \delta_{ij}x^i x^k - \frac{\delta_{ri}x^r x^i\delta_{mj}x^m x^k}{1-|x|^2}.$$

Since $\delta_{ij}\delta^{ik} = \delta_j^k$, $\delta_{ri}\delta^{ik} = \delta_r^k$, $\delta_r^k x^r = x^k$ and $\delta_{mj}x^m x^j = |x|^2$,

$$\alpha_{ij}\alpha^{ik} = \delta_j^k + \frac{\delta_{mj}x^m x^k}{1-|x|^2} - \delta_{mj}x^m x^k - \frac{|x|^2\,\delta_{mj}x^m x^k}{1-|x|^2}.$$

$$= \delta_j^k;$$

hence, (α^{ij}) is the inverse of (α_{ij}). Using equations (5-15) and (5-22), we see that

$$\alpha^{ij}\rho_i\rho_j = (\delta^{ij} - x^i x^j)\frac{\delta_{ri}x^r\delta_{kj}x^k}{1-|x|^2},$$

and since $\delta^{ij}\delta_{ri} = \delta^j_r$ and $\delta^j_r x^r = x^j$, this expression simplifies to

$$\alpha^{ij}\rho_i\rho_j = \frac{\delta_{kj}x^k x^j - \delta_{ri}x^r x^i \delta_{kj}x^k x^j}{1-|x|^2}.$$

In addition, $\delta_{kj}x^k x^j = \delta_{ri}x^r x^i = |x|^2$, so that the above expression reduces to

$$\alpha^{ij}\rho_i\rho_j = \frac{|x|^2 - |x|^4}{1-|x|^2} = |x|^2.$$

Hence from equation (5-21), we require that

$$\sqrt{\alpha^{ij}\rho_i\rho_j} = |x| < 1.$$

The resulting Randers metric, F is thus defined on the unit ball $B^n = \{x \in R^n, |x| < 1\}$.

Similarly, for $\beta = -\rho_i y^i$, we can show that $F = \alpha - \beta$, is also defined on the unit ball B^n.

5.2 Non-Riemannian Curvatures of Randers Metrics

The non-Riemannian curvature properties of Randers metrics were studied by Z. Shen and X. Chen in [Sh1][Sh2] [ChSh1] [ChSh2]. Throughout this dissertation we use the following notations. E, J, I denote the E-curvature, the mean Landsberg curvature, the mean Cartan torsion respectively, which are defined in chapter 3.3, and h is the angular metric of F given in Definition 2.1.5. According to [Sh1], the S-curvature of $F = \alpha + \beta$ is given by

$$S = (n+1)\left\{\frac{e_{ij}y^i y^j}{2F} - (s_i y^i + \gamma_i y^i)\right\}, \tag{5-23}$$

where

$$\gamma_i = \frac{1}{2}\frac{\partial}{\partial x^i}\{\ln(1-\alpha^{kj}b_k b_j)\}. \tag{5-24}$$

We summarize some key results that relate non-Riemannian curvatures of Randers metrics in this section. The proof of these results can be found in [ChSh2].

Theorem 5.2.1

Let $F=\alpha+\beta$ be a Randers metric on an *n*-dimensional manifold M, and let $c=c(x)$ be a scalar function on M. The following are equivalent:

(a) $J+cFI=0$;

(b) $S=(n+1)cF$ and β is closed.

(c) $E=\frac{1}{2F}(n+1)ch$ and β is closed.

(d) $e_{ij}y^i y^j = 2c(\alpha^2-\beta^2)$ and β is closed.

As we mentioned in chapter 3.3, the Funk metrics are well-known examples for constant S-curvature and constant E-curvature. We prove this result by using above Theorem. Note that the following proof is provided by the author.

Lemma 5.2.2

The Funk metrics defined in (5-20) have the following properties:

(i) $S=\pm\frac{1}{2}(n+1)F$;

(ii) $E=\pm\frac{1}{4F}(n+1)h$;

(iii) $J\pm\frac{1}{2}FI=0$.

Proof:

We first show these results for the case

$$F(x,y)=\frac{\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}}{1-|x|^2}+\frac{\langle x,y\rangle}{1-|x|^2}.$$

Let us show that the condition (d) in Theorem 5.2.1 is valid for the above F. Using the notation of chapter 5.1, recall that

$$e_{ij}(x)=r_{ij}+b_is_j+b_js_i,$$

where

$$r_{ij}=\frac{1}{2}(b_{i|j}+b_{j|i}).$$

Equations (5-17) and (5-19) imply

$$e_{ij}=\rho_{ij}-2\rho_i\rho_j. \tag{5-25}$$

Now,

$$2c(\alpha^2-\beta^2)=2c\{\alpha_{ij}y^iy^j-b_iy^ib_jy^j\}. \tag{5-26}$$

Substituting (5-14) and (5-16) into (5-26) gives

$$\begin{aligned}2c(\alpha^2-\beta^2)&=2c\{(\rho_{ij}-\rho_i\rho_j)y^iy^j-\rho_iy^i\rho_jy^j\}\\&=2c(\rho_{ij}-2\rho_i\rho_j)y^iy^j;\end{aligned}$$

hence equation (5-25) implies

$$e_{ij}y^iy^j=2c(\alpha^2-\beta^2)$$

if and only if $c=1/2$. Recall that β is closed (cf. Lemma 5.1.7) and we thus have the condition (d) in Theorem 5.2.1. Therefore, Theorem 5.2.1 shows that

$$S=\frac{1}{2}(n+1)F,$$

$$E=\frac{1}{4F}(n+1)h$$

and

$$J+\frac{1}{2}FI=0.$$

Similar arguments can be used to show these relations with negative sign for the other Funk metric.

5.3 Projectively Flat Randers Metrics with Isotropic S-Curvature

It is well known that all locally projectively flat Finsler metrics have scalar curvature [Be5] [Be6]. Since Randers metrics are special Finsler metrics, all locally projectively flat Randers metrics also have scalar curvature. In Lemma 5.1.7 and Lemma 5.2.2, we proved that the Funk metrics are locally projectively flat Randers metrics with constant flag curvature and constant S-curvature. It has been proved that all Randers metrics of constant flag curvature have constant S-curvature [BaRo]. However, not all locally projectively flat Randers metrics are of constant flag curvature. Some of them are of scalar curvature with isotropic S-curvature. This problem was studied by X. Chen, X. Mo and Z. Shen in 2003. They classified locally projectively flat Randers metrics with isotropic S-curvature [ChMoSh]. But locally projectively flat Randers metrics with scalar curvature may or may not have isotropic S-curvature. Example 5.3.2 illustrates such a metric that has scalar curvature but does not have isotropic S-curvature.

Scalar curvature of a locally projectively flat Randers metric

Let $F=\alpha+\beta$ be a locally projectively flat Randers metric and K be the scalar curvature of F. Since F is locally projectively flat, from the remark of Theorem 5.1.5, we know that α is a Riemannian metric with constant sectional curvature and β is closed. Let $\overline{G}^i$ and G^i be the geodesic coefficients of F and α respectively. Using the equation (3-2) and replacing G^i by $\overline{G}^i$, we have

$$\overline{R}^i_k = 2\frac{\partial \overline{G}^i}{\partial x^k} - y^j \frac{\partial^2 \overline{G}^i}{\partial x^j \partial y^k} + 2\overline{G}^j \frac{\partial^2 \overline{G}^i}{\partial y^j \partial y^k} - \frac{\partial \overline{G}^i}{\partial y^j}\frac{\partial \overline{G}^j}{\partial y^k}.$$

Since β is closed, using equation (5-8) the above expression simplifies to the following result given in [Sh1].

$$\overline{R}^i_k = R^i_k + \left\{\frac{3}{4}\left(\frac{\phi}{F}\right)^2 - \frac{\psi}{2F}\right\}h^i_k + \eta_k y^i, \tag{5-27}$$

where

$$\phi = b_{k|j}y^k y^j,$$

$$\psi = b_{i|j|k}y^i y^j y^k,$$

$$h^i_k = \delta^i_k - l_k l^i,$$

$$\eta_k = -(b_{r|j|k} - b_{r|k|j})y^r y^j.$$

Substituting $k = i$ into (5-27), since $h^i_i = n-1$ and $\eta_i y^i = 0$, gives

$$\overline{R}^i_i = R^i_i + \left\{\frac{3}{4}\left(\frac{\phi}{F}\right)^2 - \frac{\psi}{2F}\right\}(n-1). \tag{5-28}$$

Equation (3-8) with $r = k = i$ yields

$$\overline{R}^i_i = (n-1)KF^2,$$

$$R^i_i = (n-1)\mu\alpha^2,$$

and relation (5-28) thus gives

$$KF^2 = \mu\alpha^2 + \frac{3}{4}\left(\frac{\phi}{F}\right)^2 - \left(\frac{\psi}{2F}\right) \tag{5-29}$$

provided $n \neq 1$.

Scalar curvature of a locally projectively flat Randers metric with isotropic S-curvature

Consider a locally projectively flat Randers metric F with isotropic S-curvature, i.e.,

$$S = (n+1)c(x)F,$$

where $c(x)$ is a scalar function on M. We determine ϕ and ψ in terms of α, β and c. Theorem 5.2.1 indicates that

$$e_{ij}y^i y^j = 2c(\alpha^2 - \beta^2), \tag{5-30}$$

where

$$e_{ij} = r_{ij} + b_i s_j + b_j s_i. \tag{5-31}$$

Since β is closed, $b_{i|j} = b_{j|i}$; therefore, $s_{ij} = 0$ and hence $s_i = 0$ and $s_j = 0$. In addition,

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}) = b_{i|j}.$$

Consequently, equation (5-31) implies

$$e_{ij} = b_{i|j},$$

so that

$$e_{ij} y^i y^j = b_{i|j} y^i y^j.$$

Equation (5-30) implies

$$b_{i|j} y^i y^j = 2c(\alpha^2 - \beta^2),$$

and therefore

$$\phi = b_{i|j} y^i y^j = 2c(\alpha^2 - \beta^2). \tag{5-32}$$

Taking the horizontal covariant derivative of ϕ with respect to α and contracting with y^k we get

$$\phi_{|k} y^k = b_{i|j|k} y^i y^j y^k = \psi.$$

From equation (5-32) we also have

$$\phi = 2c(\alpha^2 - \beta^2),$$

so that

$$\phi_{|k} = 2c_{|k}(\alpha^2 - \beta^2) + 2c(\alpha^2 - \beta^2)_{|k}, \tag{5-33}$$

where

$$c_{|k} = \frac{\partial c}{\partial x^k} - \Gamma^i_k \frac{\partial c}{\partial y^i},$$

and

$$\Gamma^i_{kj} y^j = \Gamma^i_k.$$

Since $c = c(x)$,

$$c_{|k} = \frac{\partial c}{\partial x^k} = c_{x^k}. \tag{5-34}$$

Now,

$$(\alpha^2 - \beta^2)_{|k} = \alpha^2{}_{|k} - \beta^2{}_{|k} = 2\alpha\alpha_{|k} - 2\beta\beta_{|k},$$

and since " | " is the horizontal covariant derivative with respect to the Riemannian metric α,

$$\alpha_{|k} = 0,$$

so that

$$(\alpha^2 - \beta^2)_{|k} = -2\beta\beta_{|k} = -2\beta b_{i|k} y^i. \tag{5-35}$$

Substituting (5-34) and (5-35) into (5-33) yields

$$\phi_{|k} y^k = 2c_{x^k} y^k (\alpha^2 - \beta^2) + 2c(-2\beta b_{i|k} y^i y^k),$$

and relation (5-32) gives

$$\psi = \phi_{|k} y^k = 2c_{x^k} y^k (\alpha^2 - \beta^2) - 8c^2 \beta(\alpha^2 - \beta^2). \tag{5-36}$$

Equation (5-29) now enables us to determine the scalar curvature of a locally projectively flat Randers metric with isotropic S-curvature in terms of α, β, c and μ. Substituting (5-32), (5-36) and $F = \alpha + \beta$ into (5-29) gives

$$KF^2 = \mu\alpha^2 + 3c^2(\alpha - \beta)^2 - c_{x^k} y^k (\alpha - \beta) + 4c^2 \beta(\alpha - \beta). \tag{5-37}$$

X. Chen, X. Mo and Z. Shen used equation (5-37) to give explicit expressions for β and K in terms of c. They determined the function c according to the sign of the sectional curvature of α. A key result is the following classification theorem.

Theorem 5.3.1 ([ChMoSh])

Let $F = \alpha + \beta$ be a locally projectively flat Randers metric on an n-dimensional manifold M and let μ denote the constant sectional curvature of α. Suppose that the S-curvature is isotropic so that $S = (n+1)c(x)F$. Then F can be classified as follows.

(a) If $\mu + 4c(x)^2 = 0$, then $c(x) =$ constant and the flag curvature is $K = -c^2$.

(1) if $c = 0$, then F is locally Minkowskian with flag curvature, $K = 0$;

(2) if $c \neq 0$, then after a normalization, F is locally isometric to the following

Randers metrics on the unit ball $B^n \subset R^n$

$$F(x,y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} \pm \langle x,y\rangle}{1-|x|^2} \pm \frac{\langle a,y\rangle}{1+\langle a,x\rangle},$$

where $a \in R^n$ with $|a| < 1$, and $K = -\frac{1}{4}$.

(b) If $\mu + 4c(x)^2 \neq 0$, then F is given by

$$F(x,y) = \alpha(x,y) - \frac{2c_{x^k}(x)y^k}{\mu + 4c(x)^2},$$

and the scalar curvature of $\mathfrak{I}^n$ is given by

$$K(x,y) = 3\left(\frac{c_{x^k}(x)y^k}{F(x,y)} + c(x)^2\right) + \mu.$$

(1) if $\mu = -1$, then

$$\alpha(x,y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2},$$

where $y \in T_pB^n \cong R^n$.

In this case,

$$c(x) = \frac{\lambda + \langle a,x\rangle}{2\sqrt{(\lambda + \langle a,x\rangle)^2 \pm (1-|x|^2)}},$$

where $\lambda \in R$ and $a \in R^n$ with $|a|^2 < \lambda^2 \pm 1$.

(2) if $\mu = 0$, then

$$\alpha(x,y) = |y|,$$

where $y \in T_pR^n \cong R^n$.

In this case,

$$c(x) = \frac{\pm 1}{2\sqrt{k + \langle a, x\rangle + |x|^2}},$$

where $k > 0$ and $a \in R^n$ with $|a|^2 < k$.

(3) if $\mu = 1$, then

$$\alpha(x, y) = \frac{\sqrt{|y|^2 + (|x|^2 |y|^2 - \langle x, y\rangle^2)}}{1 + |x|^2},$$

where $y \in T_p R^n \cong R^n$.

In this case,

$$c(x) = \frac{\varepsilon + \langle a, x\rangle}{2\sqrt{1 + |x|^2 - (\varepsilon + \langle a, x\rangle)^2}},$$

where $a \in R^n$, and $\varepsilon \in R$ is such that $|\varepsilon|^2 + |a|^2 < 1$.

The following example of a locally projectively flat Randers metric with isotropic S-curvature is given in [ChMoSh].

Example 5.3.1

Let $F = \alpha + \beta$ and for an arbitrary number ε with $0 < \varepsilon \leq 1$, define

$$\alpha = \frac{\sqrt{(1-\varepsilon^2)(xu + yv)^2 + \varepsilon(u^2 + v^2)(1 + \varepsilon(x^2 + y^2))}}{1 + \varepsilon(x^2 + y^2)}$$

$$\beta = \frac{\sqrt{1-\varepsilon^2}(xu + yv)}{1 + \varepsilon(x^2 + y^2)},$$

where (u, v) is a vector on R^2 at any point $p = (x, y) \in R^2$. Now,

$$\sqrt{\alpha^{ij} b_i b_j} = \sqrt{1-\varepsilon^2}\sqrt{\frac{x^2 + y^2}{\varepsilon + x^2 + y^2}} < 1,$$

and hence F is a Randers metric on R^2. The scalar curvature of F is

$$K(x,y,u,v)=-\frac{5-\varepsilon^2+4\varepsilon(x^2+y^2)}{2(\varepsilon+x^2+y^2)^2}+\frac{3\left(1+\varepsilon(x^2+y^2)\right)\alpha}{(\varepsilon+x^2+y^2)^2F}.$$

The isotropic S-curvature of F is

$$S=3cF,$$

where

$$c=\frac{\sqrt{1-\varepsilon^2}}{2(\varepsilon+x^2+y^2)}.$$

So far all known Randers metrics with scalar curvature have isotropic S-curvature [Sh11]. Motivated by this we define the following Randers metrics to show that some Randers with scalar curvature do not have isotropic S-curvature.

Example 5.3.2

Let

$$\alpha(x,y)=\frac{\sqrt{|y|^2+(|x|^2|y|^2-\langle x,y\rangle^2)}}{1+|x|^2},$$

and

$$\beta(x,y)=\pm\frac{\langle x,y\rangle}{1+|x|^2}.$$

Let us show that $F=\alpha\pm\beta$ are locally projectively flat Randers metrics of (non-constant) scalar curvature without isotropic S-curvature defined on unit ball $B^n=\{x\in R^n,|x|<1\}$. Recall that α is the Riemannian metric with constant sectional curvature, 1 (cf. Example (ii) of Examples 4.3.1. Then (4-72) gives the Riemannian metric tensor

$$\alpha_{ij}=(\rho_i\rho_j-\rho_{ij}), \tag{5-38}$$

where

$$\rho_i=-\frac{\delta_{ri}x^r}{1+|x|^2}, \tag{5-39}$$

It is also clear that

$$\beta = b_i y^i = \pm \rho_i y^i, \tag{5-40}$$

and $i, j, r... = 1,...,n$. Now,

$$\rho_{ij}(x) = \frac{\partial \rho_i}{\partial x^j} = -\frac{\delta_{ij}}{1+|x|^2} + 2\frac{\delta_{ri}\delta_{mj}x^r x^m}{(1+|x|^2)^2}, \tag{5-41}$$

and it is clear that $\rho_{ij} = \rho_{ji}$. Therefore, β is closed. Equations (5-39) and (5-41) indicate that

$$\alpha_{ij} = \frac{1}{1+|x|^2}\left(\delta_{ij} - \frac{\delta_{ri}\delta_{mj}x^r x^m}{1+|x|^2}\right). \tag{5-42}$$

We now show that F is defined on the unit ball B^n using condition (5-1).
i.e.,

$$\sqrt{\alpha^{ij}\rho_i\rho_j} < 1. \tag{5-43}$$

Let

$$\alpha^{ij} = (1+|x|^2)(\delta^{ij} + x^i x^j), \tag{5-44}$$

where (δ^{ij}) is the inverse of (δ_{ij}). As in Lemma 5.1.7, we can also show that (α^{ij}) given in (5-44) is the inverse of (α_{ij}) by showing that $\alpha_{ij}\alpha^{ik} = \delta_j^k$. Then equations (5-39) and (5-44) give

$$\alpha^{ij}\rho_i\rho_j = \frac{|x|^2 + |x|^4}{1+|x|^2} = |x|^2. \tag{5-45}$$

Condition (5-43) thus reduces to $|x| < 1$. Hence, the resulting Randers metric is defined on the unit ball $B^n \subset R^n$.

Using the remark of Theorem 5.1.5, we know that F is locally projectively flat since α has constant sectional curvature and β is closed. It is well known that locally projectively flat Finsler metrics have scalar curvature [Be5] [Be6]. We can thus conclude that F has scalar curvature.

We now find the scalar curvature of F in terms of α and β. Recall that

$$K=\frac{1}{F^2}\left(\overline{P}^2-\frac{\partial\overline{P}}{\partial x^k}y^k\right), \tag{5-46}$$

where

$$\overline{P}=\frac{1}{2F}\frac{\partial F}{\partial x^k}y^k \tag{5-47}$$

(cf. equation (4-37)). We first calculate $\overline{P}$ in terms of α and β. Now,

$$\frac{\partial F}{\partial x^k}y^k=\frac{\partial\alpha}{\partial x^k}y^k+\frac{\partial\beta}{\partial x^k}y^k, \tag{5-48}$$

and, from equation (5-38)

$$\alpha^2=(\rho_i\rho_j-\rho_{ij})y^iy^j, \tag{5-49}$$

so that

$$2\alpha\frac{\partial\alpha}{\partial x^k}=(\rho_{ik}\rho_j+\rho_{jk}\rho_i-\rho_{ijk})y^iy^j;$$

thus,

$$2\alpha\frac{\partial\alpha}{\partial x^k}y^k=(\rho_{ik}\rho_j+\rho_{jk}\rho_i-\rho_{ijk})y^iy^jy^k.$$

Since $\rho_{ik}\rho_jy^iy^jy^k=\rho_{jk}\rho_iy^iy^jy^k$,

$$2\alpha\frac{\partial\alpha}{\partial x^k}y^k=(2\rho_{ik}\rho_j-\rho_{ijk})y^iy^jy^k. \tag{5-50}$$

Contracting (4-39) with y^i, y^j and y^k gives

$$6\rho_{ij}\rho_ky^iy^jy^k=(\rho_{ijk}+4\rho_i\rho_j\rho_k)y^iy^jy^k.$$

Equation (5-50) and the above expression imply

$$\alpha\frac{\partial\alpha}{\partial x^k}y^k=2(\rho_i\rho_j\rho_k-\rho_{ij}\rho_k)y^iy^jy^k$$

$$=2(\rho_i\rho_j-\rho_{ij})y^iy^j\rho_ky^k.$$

Equations (5-49) and (5-40) simplify the above expression to

$$\frac{\partial \alpha}{\partial x^k} y^k = 2\alpha\beta. \tag{5-51}$$

Differentiating β with respect to x^k, contracting with y^k and using (5-38) and (5-40) gives

$$\frac{\partial \beta}{\partial x^k} y^k = \rho_{ik} y^i y^k = \beta^2 - \alpha^2. \tag{5-52}$$

Equations (5-48), (5-51) and (5-52) imply

$$\frac{\partial F}{\partial x^k} y^k = 2\alpha\beta + \beta^2 - \alpha^2, \tag{5-53}$$

and hence equation (5-47) yields

$$\overline{P} = \frac{2\alpha\beta + \beta^2 - \alpha^2}{2(\alpha + \beta)}.$$

Differentiating $\overline{P}$ with respect to x^k gives

$$\frac{\partial \overline{P}}{\partial x^k} = \frac{1}{2}\left(\frac{(\alpha+\beta)\frac{\partial}{\partial x^k}(2\alpha\beta + \beta^2 - \alpha^2) - (2\alpha\beta + \beta^2 - \alpha^2)\frac{\partial}{\partial x^k}(\alpha+\beta)}{(\alpha+\beta)^2}\right).$$

Contracting the above expression with y^k and using (5-51) and (5-52), we have

$$\frac{\partial \overline{P}}{\partial x^k} y^k = \frac{\left(\beta\frac{\partial \alpha}{\partial x^k} + \alpha\frac{\partial \beta}{\partial x^k} + \beta\frac{\partial \beta}{\partial x^k} - \alpha\frac{\partial \alpha}{\partial x^k}\right) y^k}{(\alpha+\beta)} - \frac{(2\alpha\beta + \beta^2 - \alpha^2)^2}{2(\alpha+\beta)^2}.$$

$$= \frac{3\alpha\beta^2 - 3\alpha^2\beta + \beta^3 - \alpha^3}{\alpha+\beta} - \frac{(2\alpha\beta + \beta^2 - \alpha^2)^2}{2(\alpha+\beta)^2}.$$

Substituting the above expression and $\overline{P}^2$ to equation (5-46) yields

$$K = \frac{6\alpha^2\beta^2 - \beta^4 + 7\alpha^4 - 4\alpha\beta^3 + 4\alpha^3\beta}{4(\alpha+\beta)^4}. \tag{5-54}$$

Let us show that K cannot be a constant. Differentiating K with respect to y^i gives

$$\frac{\partial K}{\partial y^i}=\frac{1}{4(\alpha+\beta)^4}\frac{\partial}{\partial y^i}(6\alpha^2\beta^2-\beta^4+7\alpha^4-4\alpha\beta^3+4\alpha^3\beta)$$

$$-\frac{6\alpha^2\beta^2-\beta^4+7\alpha^4-4\alpha\beta^3+4\alpha^3\beta}{(\alpha+\beta)^5}\frac{\partial}{\partial y^i}(\alpha+\beta).$$

Since, $\frac{\partial}{\partial y^i}(6\alpha^2\beta^2-\beta^4+7\alpha^4-4\alpha\beta^3+4\alpha^3\beta)$

$$=(12\alpha\beta^2+28\alpha^3+12\alpha^2\beta-4\beta^3)\frac{\partial\alpha}{\partial y^i}$$

$$+(12\alpha^2\beta-4\beta^3-12\alpha\beta^2+4\alpha^3)\frac{\partial\beta}{\partial y^i}$$

Expanding gives

$$(12\alpha\beta^2+28\alpha^3+12\alpha^2\beta-4\beta^3)(\alpha+\beta)\frac{\partial\alpha}{\partial y^i}$$

$$=(24\alpha^2\beta^2+28\alpha^4+40\alpha^3\beta+8\alpha\beta^3-4\beta^4)\frac{\partial\alpha}{\partial y^i},$$

and

$$(12\alpha^2\beta-4\beta^3-12\alpha\beta^2+4\alpha^3)(\alpha+\beta)\frac{\partial\beta}{\partial y^i}$$

$$=(16\alpha^3\beta-16\alpha\beta^3+4\alpha^4-4\beta^4)\frac{\partial\beta}{\partial y^i}.$$

By a direct computation,

$$\frac{\partial K}{\partial y^i}=\frac{6\alpha\beta(\alpha^2+\beta^2)\frac{\partial\alpha}{\partial y^i}-6\alpha^2(\alpha^2+\beta^2)\frac{\partial\beta}{\partial y^i}}{4(\alpha+\beta)^5},$$

which simplifies to

$$\frac{\partial K}{\partial y^i}=\frac{6\alpha(\alpha^2+\beta^2)}{(\alpha+\beta)^5}\left(\beta\frac{\partial\alpha}{\partial y^i}-\alpha\frac{\partial\beta}{\partial y^i}\right).$$

Since $\beta \frac{\partial \alpha}{\partial y^i} - \alpha \frac{\partial \beta}{\partial y^i} = \beta^2 \frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)$,

$$\frac{\partial K}{\partial y^i} = \frac{6\alpha(\alpha^2+\beta^2)\beta^2}{(\alpha+\beta)^5}\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right). \tag{5-55}$$

Now, let us find $\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)$. Using α and β given in above we can calculate

$$\frac{\alpha^2}{\beta^2} = \frac{|y|^2(1+|x|^2)}{\langle x,y\rangle^2} - 1.$$

Differentiating the above expression with respect to y^i gives

$$\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right) = \frac{|y|}{\alpha}\frac{\partial}{\partial y^i}\left(\frac{|y|}{\langle x,y\rangle}\right).$$

Now, since $|y| \neq 0$, if $\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right) = 0$ then $\frac{\partial}{\partial y^i}\left(\frac{|y|}{\langle x.y\rangle}\right) = 0$. This implies that there is a function $\lambda(x)$ such that $|y| = \lambda(x)\langle x,y\rangle$. In this case, $\alpha = \beta\sqrt{\mu(x)}$, for $\mu(x) > 0$, where $\mu(x) = \lambda^2(1+|x|^2) - 1$. Therefore,

$$F = \left(1 + \frac{1}{\sqrt{\mu(x)}}\right)\alpha.$$

Hence, according to Definition 2.1.6, F and α are conformal. Thus from Lemma 2.1.3 (i) F is a Riemannian metric. This gives a contradiction, as a Randers space is Riemannian if and only if $\beta = 0$ [BaChSh]. Therefore,

$$\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right) \neq 0. \tag{5-56}$$

Since $\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right) \neq 0$, equation (5-55) shows that

$$\frac{\partial K}{\partial y^i} \neq 0.$$

This implies that $K \neq K(x)$ or a constant. That is, $K = K(x,y)$. Hence F is of (non-constant) scalar curvature.

We now find the S-curvature of F using equations (5-23) and (5-24). Since β is closed, $b_{i|j} = b_{j|i}$; therefore, $s_{ij} = 0$ and hence $s_i = 0$ and $s_j = 0$. Hence,

$$e_{ij} = b_{i|j} = \frac{\partial b_i}{\partial x^j} - b_k \Gamma_{ij}^k,$$

where Γ_{ij}^k are the Christoffel symbols of α. Since $b_i = \rho_i$ and $\Gamma_{ij}^k = G_{ij}^k = \rho_i \delta_j^k + \rho_j \delta_i^k$, we have

$$e_{ij} = b_{i|j} = \frac{\partial \rho_i}{\partial x^j} - \rho_k (\rho_i \delta_j^k + \rho_j \delta_i^k),$$

where ρ_i is given in (5-39). Therefore,

$$e_{ij} = \rho_{ij} - 2\rho_i \rho_j.$$

Contracting the above expression with y^i and y^j gives

$$e_{ij} y^i y^j = (\rho_{ij} - 2\rho_i \rho_j) y^i y^j. \tag{5-57}$$

Equations (5-39) and (5-41) imply

$$e_{ij} y^i y^j = -\frac{|y|^2}{1+|x|^2}. \tag{5-58}$$

Since $b_i = \rho_i$ and $b_j = \rho_j$, equations (5-24) and (5-45) give

$$\gamma_i = \frac{1}{2} \frac{\partial}{\partial x^i} \left(\ln(1-|x|^2) \right)$$

The above expression simplifies to

$$\gamma_i = \frac{-\delta_{ik} x^k}{(1-|x|^2)}. \tag{5-59}$$

Equations (5-58), (5-59) and (5-23) yield

$$S = (n+1)\left\{\frac{\langle x, y\rangle}{1-|x|^2} - \frac{|y|^2}{2F(1+|x|^2)}\right\}.$$

We now show that F does not have isotropic S-curvature. Recall that if F has isotropic S-curvature then

$$S = (n+1)c(x)F,$$

and hence Theorem 5.2.1 indicates that

$$e_{ij}y^i y^j = 2c(x)(\alpha^2 - \beta^2),$$

Equation (5-57) and the above expression imply

$$(\rho_{ij} - 2\rho_i\rho_j)y^i y^j = 2c(x)(\alpha^2 - \beta^2).$$

Since $\beta = \rho_i y^i = \rho_j y^j$ and $\beta^2 - \alpha^2 = \rho_{ij} y^i y^j$ (cf. equation (5-52)), we have

$$-(\alpha^2 + \beta^2) = 2c(x)(\alpha^2 - \beta^2),$$

i.e.,

$$c(x) = \frac{1}{2}\left(\frac{\alpha^2 + \beta^2}{\beta^2 - \alpha^2}\right).$$

Differentiating $c(x)$ with respect to y^i, since $c = c(x)$, gives

$$\frac{(\beta^2 - \alpha^2)\left(\alpha\frac{\partial\alpha}{\partial y^i} + \beta\frac{\partial\beta}{\partial y^i}\right) - (\alpha^2 + \beta^2)\left(\beta\frac{\partial\beta}{\partial y^i} - \alpha\frac{\partial\alpha}{\partial y^i}\right)}{(\beta^2 - \alpha^2)^2} = 0,$$

i.e.,

$$\alpha\beta\left(\beta\frac{\partial\alpha}{\partial y^i} - \alpha\frac{\partial\beta}{\partial y^i}\right) = 0.$$

Now, $\left(\beta\frac{\partial\alpha}{\partial y^i} - \alpha\frac{\partial\beta}{\partial y^i}\right) = \beta^2\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)$ and thus

$$\alpha\beta^3\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right) = 0.$$

Since $\alpha \neq 0$ and $\beta \neq 0$, the above expression implies that

$$\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)=0.$$

We have already shown that this leads to a contradiction (cf. equation (5-56)). Therefore F does not have isotropic S-curvature.

Similarly, for $\beta=-\rho_i y^i$, we have the following locally projectively flat Randers metric defined on unit ball B^n.

$$\overline{F}(x,y)=\frac{\sqrt{|y|^2+(|x|^2|y|^2-\langle x,y\rangle^2)}}{1+|x|^2}+\frac{\langle x,y\rangle}{1+|x|^2}.$$

Replacing β by $(-\beta)$ in (5-54) gives the scalar curvature of $\overline{F}$ as follows.

$$\overline{K}=\frac{6\alpha^2\beta^2-\beta^4+7\alpha^4+4\alpha\beta^3-4\alpha^3\beta}{4(\alpha-\beta)^4}.$$

Replacing β by $(-\beta)$ in (5-55) gives,

$$\frac{\partial\overline{K}}{\partial y^i}\neq 0.$$

This implies that $\overline{K}\neq\overline{K}(x)$ or a constant. That is, $\overline{K}=\overline{K}(x,y)$. Hence $\overline{F}$ is of (non-constant) scalar curvature.

Using the same process, as above we can calculate the S-curvature of $\overline{F}$. Since $b_i=-\rho_i$, we have

$$b_{i|j}=2\rho_i\rho_j-\rho_{ij},$$

and hence

$$e_{ij}y^i y^j=\frac{|y|^2}{1+|x|^2}.$$

Therefore, the S-curvature of $\overline{F}$ is

$$\overline{S} = (n+1)\left\{\frac{\langle x, y\rangle}{1-|x|^2} + \frac{|y|^2}{2\overline{F}(1+|x|^2)}\right\}.$$

As before, we can show that $\overline{F}$ does not have isotropic S-curvature.

Next we are going to give the answer to the open problem which appeared from a theorem in [ChSh1], using an example given in [ChMoSh]. We first state the theorem in [ChSh1].

Theorem 5.3.2

Let $F = \alpha + \beta$ be a Randers metric on a differentiable manifold M of dimension *n*. Suppose that

(i) flag curvature, $K = K(x)$ is independent of $y \in T_pM$.

(ii) $S = (n+1)c(x)F$ and β is closed.

Then F is locally projectively flat and $K = \text{constant} = -c^2 \leq 0$.

5.3.3 Open Problem

The first condition in Theorem 5.3.2 requires that the flag curvature is independent of y. Chen and Shen have raised the question as to whether this condition can be relaxed to scalar curvature $K(x, y)$. In other words, let $F = \alpha + \beta$ be a Randers metric on a differentiable manifold M of dimension *n*. Suppose that $K = K(x, y)$, $S = (n+1)c(x)F$ and β is closed. Does this still imply that $K = \text{constant}$?

We show that the condition (i) is essential in Theorem 5.3.2 by providing a counterexample using a Randers metric given in [ChMoSh].

Solution:

Consider the following Randers metric given in [ChMoSh].

$$F(x,y)=\frac{|y|\sqrt{1+|x|^2}+\langle x,y\rangle}{\sqrt{1+|x|^2}},$$

where $y\in T_pR^n$. It is easy to verify that F satisfies equation (4-8). Thus it is a locally projectively flat Randers metric on R^n. Hence F has scalar curvature. We first calculate the scalar curvature. Since $F=\alpha+\beta$, we know that

$$\alpha=|y|,$$

and

$$\beta=\frac{\langle x,y\rangle}{\sqrt{1+|x|^2}}. \tag{5-60}$$

Since F is locally projectively flat, equation (4-12) gives

$$\overline{P}=\frac{1}{2F}\frac{\partial F}{\partial x^k}y^k.$$

Since $F=\alpha+\beta$, and $\alpha=|y|$, the above expression reduces to

$$\overline{P}=\frac{1}{2F}\frac{\partial \beta}{\partial x^k}y^k.$$

Differentiating β with respect to x^k and contracting with y^k gives

$$\frac{\partial\beta}{\partial x^k}y^k=\frac{\delta_{kj}y^jy^k}{\sqrt{1+|x|^2}}-\frac{\delta_{ij}x^iy^j\delta_{kr}x^ry^k}{\sqrt{(1+|x|^2)^3}}.$$

Since $|y|^2=\delta_{kj}y^ky^j$ and $\langle x,y\rangle=\delta_{ij}x^iy^j$, the above expression implies

$$\frac{\partial\beta}{\partial x^k}y^k=\frac{|y|^2}{\sqrt{1+|x|^2}}-\frac{\langle x,y\rangle^2}{\sqrt{(1+|x|^2)^3}}.$$

In terms of α and β, we have

$$\frac{\partial\beta}{\partial x^k}y^k=\frac{\alpha^2}{\sqrt{1+|x|^2}}-\frac{\beta^2}{\sqrt{1+|x|^2}}, \tag{5-61}$$

and hence

$$\overline{P} = \frac{1}{2}\frac{(\alpha-\beta)}{\sqrt{1+|x|^2}}.$$

Differentiating $\overline{P}$ with respect to x^k gives

$$\frac{\partial \overline{P}}{\partial x^k} = \frac{1}{2}\left((\alpha-\beta)\frac{\partial}{\partial x^k}\left(\frac{1}{\sqrt{1+|x|^2}}\right) + \frac{1}{\sqrt{1+|x|^2}}\left(\frac{\partial \alpha}{\partial x^k} - \frac{\partial \beta}{\partial x^k}\right)\right).$$

Since $\frac{\partial \alpha}{\partial x^k} = 0$, we have

$$\frac{\partial \overline{P}}{\partial x^k} = -\frac{1}{2}\left(\frac{(\alpha-\beta)\delta_{ik}x^i}{\sqrt{(1+|x|^2)^3}} + \frac{1}{\sqrt{1+|x|^2}}\frac{\partial \beta}{\partial x^k}\right).$$

Contracting the above expression with y^k and using (5-61), we get

$$\frac{\partial \overline{P}}{\partial x^k} y^k = \frac{1}{2}\left(\frac{2\beta^2 - \alpha\beta - \alpha^2}{1+|x|^2}\right).$$

Substituting the above expression and $\overline{P}$ into equation (4-37), we have the scalar curvature of F.

$$K = \frac{3(\alpha-\beta)}{4(1+|x|^2)(\alpha+\beta)}. \tag{5-62}$$

We now find the S-curvature. Since β is closed, recall that the S-curvature of F is

$$S = (n+1)\left\{\frac{b_{i|j}y^i y^j}{2F} - \gamma_i y^i)\right\}. \tag{5-63}$$

Let us calculate $\gamma_i y^i$ using equation (5-24). Since $\alpha = |y|$, i.e., $\alpha^2 = \delta_{ij}y^i y^j$, hence

$$\alpha_{ij} = \delta_{ij}.$$

Therefore, the inverse of α_{ij} is

$$\alpha^{ij} = \delta^{ij}.$$

Now,

$$\alpha^{ij} b_i b_j = \delta^{ij} \frac{\delta_{ik} x^k}{\sqrt{1+|x|^2}} \frac{\delta_{jr} x^r}{\sqrt{1+|x|^2}}.$$

Since $\delta^{ij}\delta_{ik} = \delta^j_k$, $\delta^j_k \delta_{jr} = \delta_{rk}$ and $\delta_{rk} x^r x^k = |x|^2$, we get

$$\alpha^{ij} b_i b_j = \frac{|x|^2}{1+|x|^2}.$$

Hence,

$$\gamma_i = \frac{1}{2}\frac{\partial}{\partial x^i}\left(\ln\left(\frac{1}{1+|x|^2}\right)\right).$$

Contracting the above expression with y^i and simplifying, we get

$$\gamma_i y^i = -\frac{\langle x, y\rangle}{1+|x|^2}. \tag{5-64}$$

Next we calculate $b_{i|j} y^i y^j$. Now,

$$b_{i|j} = \frac{\partial b_i}{\partial x^j} - b_k \Gamma^k_{ij},$$

where Γ^k_{ij} are the Christoffel symbols of α. Since $\alpha_{ij} = \delta_{ij}$, equation (2-16) gives

$$\Gamma^k_{ij} = 0.$$

Hence,

$$b_{i|j} = \frac{\partial b_i}{\partial x^j}.$$

Contracting the above expression with y^i and y^j gives

$$b_{i|j} y^i y^j = \frac{\partial b_i}{\partial x^j} y^i y^j.$$

Since $\beta = b_i y^i$,

$$b_{i|j}y^i y^j = \frac{\partial\beta}{\partial x^k}y^k.$$

Using (5-61), we have

$$b_{i|j}y^i y^j = \frac{\alpha^2-\beta^2}{\sqrt{1+|x|^2}}. \tag{5-65}$$

Relations (5-63), (5-64) and (5-65) imply

$$S=(n+1)\left\{\frac{(\alpha-\beta)}{2\sqrt{1+|x|^2}}+\frac{\langle x,y\rangle}{1+|x|^2}\right\}.$$

Using equation (5-60) the above expression simplifies to

$$S=(n+1)\left\{\frac{(\alpha+\beta)}{2\sqrt{1+|x|^2}}\right\}.$$

Since $F=\alpha+\beta$,

$$S=\frac{(n+1)F}{2\sqrt{1+|x|^2}}.$$

Hence from Definition 3.3.5 the above expression implies that F has isotropic S-curvature, where

$$c(x)=\frac{1}{2\sqrt{1+|x|^2}}.$$

Therefore, F is a Randers metric of scalar curvature with isotropic S-curvature and hence the conditions in open problem are satisfied by F. Let us investigate whether or not $K=$ constant. Differentiating K given in (5-62) with respect to y^i gives

$$\frac{\partial K}{\partial y^i}=\frac{3}{4(1+|x|^2)}\frac{\partial}{\partial y^i}\left(\frac{\alpha-\beta}{\alpha+\beta}\right).$$

$$=\frac{3}{2(1+|x|^2)(\alpha+\beta)^2}\left(\beta\frac{\partial\alpha}{\partial y^i}-\alpha\frac{\partial\beta}{\partial y^i}\right)..$$

Since $\beta\frac{\partial\alpha}{\partial y^i}-\alpha\frac{\partial\beta}{\partial y^i}=\beta^2\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)$, we get

$$\frac{\partial K}{\partial y^i} = \frac{3\beta^2}{2(1+|x|^2)(\alpha+\beta)^2} \frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right).$$

Since $\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right) \neq 0$ and $\beta \neq 0$, then $\frac{\partial K}{\partial y^i} \neq 0$. This implies that $K \neq K(x)$ or a constant. That is, $K = K(x, y)$. Hence, according to this counterexample, the conditions in open problem do not imply that F has constant flag curvature.

Chapter 6

Finsler Spaces with (α,β)-Metric

Besides the Randers metrics, there are several Finsler metrics that can be calculated from $\alpha(x,y)=\sqrt{a_{ij}(x)y^i y^j}$ and $\beta(x,y)=b_i(x)y^i$, where α is a Riemannian metric and β is a one-form defined on the n-dimensional differentiable manifold M.

(i) $F=\dfrac{\alpha^2}{\beta}$ is called a Kropina metric introduced by V. K. Kropina [Kr].

(ii) $F=\dfrac{\alpha^2}{\alpha-\beta}$ was introduced by M. Matsumoto [Ma4].

(iii) $F=\alpha+\dfrac{\beta^2}{\alpha}$ was also introduced by M. Matsumoto [Ma5].

Each $\mathfrak{F}^n=(M,F)$ is called a **Finsler space with (α,β)-metric.**

The projective changes between a Finsler space with (α,β)-metric and its associated Riemannian space with the metric α have been studied by several authors [Ma6] [HaIc] [Sh]. The projective changes between two Finsler spaces with (α,β)-metric have been studied by [BaMa1] [PaLe]. However, the classification of locally projectively flat Finsler spaces with (α,β)-metric of constant curvature is still open [Sh10]. In this chapter, we find necessary and sufficient conditions for each Finsler space with (α,β)-metric to be locally projectively flat and calculate scalar curvature of each locally projectively flat Finsler space with (α,β)-metric when α is locally projectively flat. Finally we give examples of not locally projectively flat Finsler spaces with (α,β)-metric using the Riemannian metrics and one-forms defined in Lemma 5.1.7 and Example 5.3.2.

6.1 Geodesic Coefficients of Finsler Spaces with (α,β)-Metric

Here we obtain the relationship between the geodesic coefficients of F and α by considering each Finsler space with (α,β)-metric separately. Let $\overline{G}^i$ and G^i be the geodesic coefficients of F and α respectively. Equation (4-4) implies that

$$\overline{G}^i = G^i + \frac{F_{|k}y^k}{2F}y^i + \frac{F}{2}g^{ij}\left(\frac{\partial F_{|k}}{\partial y^j}y^k - F_{|j}\right), \tag{6-1}$$

where $F_{|k}$ denotes the horizontal covariant derivative of F on (M,α).

We now find the relationship between $\overline{G}^i$ and G^i for each Finsler space with (α,β)-metric when β is closed since we will require this condition in section 6.2.

Case: (i) $F = \dfrac{\alpha^2}{\beta}$.

Taking the horizontal covariant derivative of F on (M,α) gives

$$F_{|k} = \left(\frac{\alpha^2}{\beta}\right)_{|k} = \frac{\beta\alpha^2{}_{|k} - \alpha^2\beta_{|k}}{\beta^2} = \frac{2\beta\alpha\alpha_{|k} - \alpha^2\beta_{|k}}{\beta^2}.$$

Recall that $\beta_{|k} = b_{i|k}y^i$ and $\alpha_{|k} = 0$ on (M,α).

Hence,

$$F_{|k} = -\frac{\alpha^2}{\beta^2}b_{i|k}y^i,$$

and similarly

$$F_{|j} = -\frac{\alpha^2}{\beta^2} b_{i|j} y^i.$$

Differentiating $F_{|k}$ with respect to y^j and contracting with y^k gives

$$\frac{\partial F_{|k}}{\partial y^j} y^k = -b_{j|k} \left(\frac{\alpha}{\beta}\right)^2 y^k - b_{i|k} y^i y^k \frac{\partial}{\partial y^j}\left(\frac{\alpha}{\beta}\right)^2.$$

Subtracting $F_{|j}$ from from the above expression yields

$$\frac{\partial F_{|k}}{\partial y^j} y^k - F_{|j} = (b_{k|j} - b_{j|k})\left(\frac{\alpha}{\beta}\right)^2 y^k - b_{i|k} y^i y^k \frac{\partial}{\partial y^j}\left(\frac{\alpha}{\beta}\right)^2. \tag{6-2}$$

If β is closed then equation (6-2) implies

$$\frac{\partial F_{|k}}{\partial y^j} y^k - F_{|j} = -b_{i|k} y^i y^k \frac{\partial}{\partial y^j}\left(\frac{\alpha}{\beta}\right)^2,$$

and hence equation (6-1) shows that

$$\overline{G}^i = G^i - \frac{b_{j|k} y^j y^k}{2\beta} y^i - \frac{F}{2} g^{ij} b_{r|k} y^k y^r \frac{\partial}{\partial y^j}\left(\frac{\alpha}{\beta}\right)^2. \tag{6-3}$$

Special Case:

If β is parallel with respect to α then from equation (6-2) and Theorem 4.1.2, (M,α) is projective to (M,F), and equations (6-3) and (2-32) imply

$$\overline{G}^i = G^i = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k,$$

where Γ^i_{jk} are the Christoffel symbols of α. Hence Definition 2.2.3 shows that in this case F is a Berwald metric.

Case: (ii) $F = \dfrac{\alpha^2}{\alpha-\beta}$.

Taking the horizontal covariant derivative of F on (M,α) and simplifying, we have

$$F_{|k} = \left(\frac{\alpha^2}{\alpha-\beta}\right)_{|k} = \frac{\alpha^2\beta_{|k}}{(\alpha-\beta)^2} = \frac{\alpha^2}{(\alpha-\beta)^2} b_{i|k}y^i,$$

where we have used $\beta_{|k} = b_{i|k}y^i$ and $\alpha_{|i} = 0$. Replacing k by j in $F_{|k}$ gives

$$F_{|j} = \frac{\alpha^2}{(\alpha-\beta)^2} b_{i|j}y^i.$$

Differentiating $F_{|k}$ with respect to y^j and contracting with y^k gives

$$\frac{\partial F_{|k}}{\partial y^j} y^k = b_{j|k}y^k \frac{\alpha^2}{(\alpha-\beta)^2} + \frac{2\alpha}{\alpha-\beta} b_{i|k}y^i y^k \frac{\partial}{\partial y^j}\left(\frac{\alpha}{\alpha-\beta}\right).$$

Subtracting $F_{|j}$ from the above expression gives

$$\frac{\partial F_{|k}}{\partial y^j} y^k - F_{|j} = (b_{j|k} - b_{k|j})y^k \left(\frac{\alpha}{\alpha-\beta}\right)^2 + b_{i|k}y^i y^k \frac{\partial}{\partial y^j}\left(\frac{\alpha}{\alpha-\beta}\right)^2.$$

If β is closed then the above expression reduces to

$$\frac{\partial F_{|k}}{\partial y^j} y^k - F_{|j} = b_{i|k}y^i y^k \frac{\partial}{\partial y^j}\left(\frac{\alpha}{\alpha-\beta}\right)^2.$$

Hence equation (6-1) implies

$$\overline{G}^i = G^i + \frac{b_{j|k}y^j y^k}{2(\alpha-\beta)} y^i + \frac{F}{2} g^{ij} b_{r|k}y^r y^k \frac{\partial}{\partial y^j}\left(\frac{\alpha}{\alpha-\beta}\right)^2. \qquad (6\text{-}4)$$

Special Case:

As in case (i), it can be shown that if β is parallel with respect to α then (M,α) is projective to (M,F) and F is a Berwald metric.

Case: (iii) $F=\alpha+\dfrac{\beta^2}{\alpha}$.

Taking the horizontal covariant derivative of F on (M,α) and simplifying gives

$$F_{|k}=\alpha_{|k}+\left(\frac{\beta^2}{\alpha}\right)_{|k}=\frac{2\beta\beta_{|k}}{\alpha}=\frac{2\beta}{\alpha}b_{i|k}y^i.$$

Replacing k by j in $F_{|k}$, we have

$$F_{|j}=\frac{2\beta}{\alpha}b_{i|j}y^i.$$

Differentiating $F_{|k}$ with respect to y^j and contracting with y^k gives

$$\frac{\partial F_{|k}}{\partial y^j}y^k=2\frac{\partial}{\partial y^j}\left(\frac{\beta}{\alpha}\right)b_{i|k}y^i y^k+2\frac{\beta}{\alpha}b_{j|k}y^k.$$

Subtracting $F_{|j}$ from the above expression yields

$$\frac{\partial F_{|k}}{\partial y^j}y^k-F_{|j}=2\frac{\partial}{\partial y^j}\left(\frac{\beta}{\alpha}\right)b_{i|k}y^i y^k+2\frac{\beta}{\alpha}(b_{j|k}-b_{k|j})y^k. \tag{6-5}$$

If β is closed then equation (6-5) shows that

$$\frac{\partial F_{|k}}{\partial y^j}y^k-F_{|j}=2\frac{\partial}{\partial y^j}\left(\frac{\beta}{\alpha}\right)b_{i|k}y^i y^k,$$

and hence equation (6-1) implies

$$\overline{G}^i=G^i+\frac{\beta}{\alpha^2+\beta^2}b_{j|k}y^j y^k y^i+Fg^{ij}\frac{\partial}{\partial y^j}\left(\frac{\beta}{\alpha}\right)b_{r|k}y^r y^k. \tag{6-6}$$

Special Case:

As in case (i) and (ii), if β is parallel with respect to α then (M,α) is projective to (M,F) and F is a Berwald metric.

6.2 Locally Projectively Flat Finsler Spaces with (α,β)-Metric

In Riemannian geometry, the Beltrami theorem says that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. It was mentioned, in the remark of Theorem 5.1.5, that a Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if the Riemannian metric α is of constant sectional curvature and the one-form β is closed. Is this true for other Finsler spaces with (α,β)-metric? Can a Riemannian metric be of constant sectional curvature and β be closed in locally projectively flat Finsler spaces with (α,β)-metric? In other words, can a Riemannian metric be locally projectively flat and β be closed in locally projectively flat Finsler spaces with (α,β)-metric? Motivated by these questions, we prove the following results.

Theorem 6.2.1

Let $\mathfrak{I}^n = (M, F)$ be a locally projectively flat Finsler space with (α,β)-metric, where (a) $F = \dfrac{\alpha^2}{\beta}$, (b) $F = \dfrac{\alpha^2}{\alpha - \beta}$, (c) $F = \alpha + \dfrac{\beta^2}{\alpha}$. If α is locally projectively flat and β is closed then $\mathfrak{I}^n$ is either a locally Minkowski space or a Riemannian space of non-zero constant sectional curvature given by $KF^2 = \mu\alpha^2$, where K and μ are the constant sectional curvature of F and α respectively, F and α are homothetic, $\overline{Ric} = Ric$ and β is parallel with respect to α, where $\overline{Ric}$ and Ric are the Ricci curvatures of F and α respectively.

We defer the proof, which will be given in three separate parts. First we need necessary and sufficient conditions for each Finsler space with (α,β)-metric to be locally projectively flat. These are presented first as theorems.

Theorem 6.2.2

Let $F=\dfrac{\alpha^2}{\beta}$ be a Kropina metric on an n-dimensional differentiable manifold M, where α is a Riemannian metric and β is a one-form defined on M. Then F is locally projectively flat if and only if

$$2\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)\left\{\frac{\partial\alpha}{\partial x^k}-\frac{\alpha}{\beta}\frac{\partial\beta}{\partial x^k}\right\}y^k+\frac{2\alpha}{\beta}\left\{\frac{\partial}{\partial y^i}\left(\frac{\partial\alpha}{\partial x^k}\right)y^k-\frac{\partial\alpha}{\partial x^i}\right\}$$
$$+\frac{\alpha^2}{\beta^2}\left(\frac{\partial b_k}{\partial x^i}-\frac{\partial b_i}{\partial x^k}\right)y^k=0. \tag{6-7}$$

Proof:

Suppose that F is locally projectively flat Kropina metric. Equation (4-8) implies

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right)y^k-\frac{\partial F}{\partial x^i}=0. \tag{6-8}$$

Now,

$$\frac{\partial F}{\partial x^k}=\frac{\partial}{\partial x^k}\left(\frac{\alpha^2}{\beta}\right)=\frac{2\alpha}{\beta}\frac{\partial\alpha}{\partial x^k}-\frac{\alpha^2}{\beta^2}\frac{\partial\beta}{\partial x^k}. \tag{6-9}$$

Differentiating (6-9) with respect to y^i and contracting with y^k gives

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right)y^k=2\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)\frac{\partial\alpha}{\partial x^k}y^k+\frac{2\alpha}{\beta}\frac{\partial}{\partial y^i}\left(\frac{\partial\alpha}{\partial x^k}\right)y^k$$
$$-\frac{2\alpha}{\beta}\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)\frac{\partial\beta}{\partial x^k}y^k-\frac{\alpha^2}{\beta^2}\frac{\partial b_i}{\partial x^k}y^k. \tag{6-10}$$

Replacing k by i in (6-9) and substituting $\beta=b_k(x)y^k$ gives

$$\frac{\partial F}{\partial x^i}=\frac{2\alpha}{\beta}\frac{\partial\alpha}{\partial x^i}-\frac{\alpha^2}{\beta^2}\frac{\partial b_k}{\partial x^i}y^k. \tag{6-11}$$

Equations (6-8), (6-10) and (6-11) thus imply equation (6-7). The converse also follows easily.

Theorem 6.2.3

Let F be a locally projectively flat Kropina metric. Assume that α is locally projectively flat. Then

$$\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)(P-Q)=\frac{1}{4\alpha}\left(\frac{\partial b_k}{\partial x^i}-\frac{\partial b_i}{\partial x^k}\right)y^k,$$

where

$$P=\frac{1}{2\alpha}\frac{\partial \alpha}{\partial x^k}y^k,$$

and

$$Q=\frac{1}{2\beta}\frac{\partial \beta}{\partial x^k}y^k.$$

And the scalar curvature of F is given by

$$KF^2=2\mu\alpha^2+2P^2-4PQ+Q^2+\frac{\partial Q}{\partial x^k}y^k,$$

where μ is the constant sectional curvature of α.

Proof:

Since α is locally projectively flat, equation (4-8) implies

$$\frac{\partial}{\partial y^i}\left(\frac{\partial \alpha}{\partial x^k}\right)y^k-\frac{\partial \alpha}{\partial x^i}=0, \tag{6-12}$$

and equation (4-12) gives the projective factor

$$P=\frac{1}{2\alpha}\frac{\partial \alpha}{\partial x^k}y^k.$$

Since F is locally projectively flat, we have equation (6-7). Hence equations (6-7) and (6-12) imply

$$2\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)\left\{\frac{\partial \alpha}{\partial x^k}-\frac{\alpha}{\beta}\frac{\partial \beta}{\partial x^k}\right\}y^k=-\frac{\alpha^2}{\beta^2}\left(\frac{\partial b_k}{\partial x^i}-\frac{\partial b_i}{\partial x^k}\right)y^k.$$

Using the definitions of P and Q gives

$$4\alpha\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)(P-Q)=-\frac{\alpha^2}{\beta^2}\left(\frac{\partial b_k}{\partial x^i}-\frac{\partial b_i}{\partial x^k}\right)y^k.$$

We now calculate the scalar curvature of F.

Since $\beta^2 \frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right) = -\alpha^2 \frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)$, we have

$$\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)(P-Q) = \frac{1}{4\alpha}\left(\frac{\partial b_k}{\partial x^i} - \frac{\partial b_i}{\partial x^k}\right) y^k. \tag{6-13}$$

Since F is locally projectively flat, equation (4-12) gives the projective factor

$$\overline{P} = \frac{1}{2F}\frac{\partial F}{\partial x^k} y^k.$$

Using equation (6-9), we see that

$$\overline{P} = \left(\frac{1}{\alpha}\frac{\partial \alpha}{\partial x^k} - \frac{1}{2\beta}\frac{\partial \beta}{\partial x^k}\right) y^k = 2P - Q. \tag{6-14}$$

Since F is locally projectively flat, equation (4-37) gives the scalar curvature of F as follows.

$$KF^2 = \overline{P}^2 - \frac{\partial \overline{P}}{\partial x^k} y^k. \tag{6-15}$$

Equation (6-14) and the above expression thus imply

$$KF^2 = (2P-Q)^2 - \frac{\partial}{\partial x^k}(2P-Q) y^k.$$

$$= 4P^2 - 4PQ + Q^2 - 2\frac{\partial P}{\partial x^k} y^k + \frac{\partial Q}{\partial x^k} y^k.$$

Since α is locally projectively flat, equation (4-37) gives the constant sectional curvature of α as follows.

$$\mu\alpha^2 = P^2 - \frac{\partial P}{\partial x^k} y^k. \tag{6-16}$$

The above two expressions show that

$$KF^2 = 2\mu\alpha^2 + 2P^2 - 4PQ + Q^2 + \frac{\partial Q}{\partial x^k} y^k. \tag{6-17}$$

The proof is complete.

We are now in a position to prove Theorem 6.2.1 (a).

Proof of Theorem 6.2.1 (a):

If β is closed then equation (6-13) implies

$$\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)(P-Q)=0. \tag{6-18}$$

For this we consider the following two cases.

Case (i)

Suppose that $P=Q$. Hence equation (6-14) shows

$$\overline{P}=P.$$

Since F and α are locally projectively flat, equation (4-11) gives

$$\overline{G}^i=\overline{P}y^i,$$

and

$$G^i=Py^i,$$

where $\overline{G}^i$ and G^i are the geodesic coefficients of F and α respectively. Since $\overline{P}=P,$ we have

$$\overline{G}^i=G^i.$$

This implies that F is a Berwald metric (cf. case (i) in section 6.1). Hence F is a locally projectively flat Berwald metric. By Theorem 4.2.6, $\mathfrak{I}^n$ is either a locally Minkowski space or a Riemannian space of constant sectional curvature.

If $\mathfrak{I}^n$ is a locally Minkowski space then recall that $\overline{G}^i=0,$ and hence $G^i=0.$ This shows that $\overline{P}=P=0.$ Therefore, equations (6-15) and (6-16) imply $K=\mu=0.$

If $\mathfrak{I}^n$ is a Riemannian space then using equation (6-17) and $P=Q$ gives

$$KF^2=\mu\alpha^2. \tag{6-19}$$

Hence equation (3-9) implies that

$$\overline{Ric}=Ric,$$

where $\overline{Ric}$ and Ric are the Ricci curvatures of F and α respectively.

Since $F = \frac{\alpha^2}{\beta}$, from equation (6-19) we have

$$K = \mu\left(\frac{\beta}{\alpha}\right)^2.$$

Since K is a constant, differentiating K with respect to y^i gives

$$\frac{\partial K}{\partial y^i} = \frac{2\mu\beta}{\alpha}\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right) = 0.$$

Since $\mu \neq 0$, $\alpha \neq 0$ and $\beta \neq 0$, we have

$$\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right) = 0.$$

Hence if $P = Q$ then $\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right) = 0.$

Remark:

If $\mu = 0$ then from equation (6-19) $K = 0$. That is, F is a Riemannian metric of zero curvature. It is known that every Berwald metric with zero curvature must be locally Minkowskian (cf. [AIM] [BaChSh]). Since Riemannian metrics are special Berwald metrics, we can say that $\mathfrak{I}^n$ is a locally Minkowski space. This gives a contradiction. Therefore, $\mu \neq 0$.

Case (ii)

Assume that $\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right) = 0$. That is, there is a function $\lambda(x)$ such that $\beta = \lambda(x)\alpha$.

Then,

$$F = \frac{\alpha^2}{\beta} = \frac{1}{\lambda(x)}\alpha,$$

and hence according to Definition 2.1.6, F is conformal to α. Let

$$F = \psi(x)\alpha.$$

Since $\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)=0$, we have

$$\alpha\frac{\partial\beta}{\partial y^i}-\beta\frac{\partial\alpha}{\partial y^i}=0.$$

The above expression implies that

$$\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)=0.$$

Now, β is closed and $\frac{\partial}{\partial y^j}\left(\frac{\alpha}{\beta}\right)=0$; hence equation (6-2) and Theorem 4.1.2 implies that (M,α) is projective to (M,F). Therefore, Lemma 2.1.3 shows that ψ is a constant and F must be Riemannian.

i.e.,

$$F=\psi\alpha. \tag{6-20}$$

Since F is locally projectively flat, equation (4-12) gives

$$\overline{P}=\frac{1}{2F}\frac{\partial F}{\partial x^k}y^k.$$

Substituting (6-20) into the above expression gives

$$\overline{P}=\frac{1}{2F}\frac{\partial F}{\partial x^k}y^k=\frac{1}{2\alpha}\frac{\partial\alpha}{\partial x^k}y^k=P$$

Therefore, equation (6-14) implies

$$P=Q.$$

Hence if $\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)=0$ then $P=Q$.

Thus from case (i) and (ii), we have both $\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)=0$ and $P=Q$.

Further, using relations (2-2) and (6-20) gives

$$\overline{g}_{ij}(x)=\psi^2 g_{ij}(x),$$

where $\overline{g}_{ij}$ and g_{ij} are metric tensors of F and α respectively, and hence relation (2-14) indicates that F and α are homothetic.

We now show that β is parallel with respect to α.

Since $\overline{G}^i = G^i$ and $\dfrac{\partial}{\partial y^j}\left(\dfrac{\alpha}{\beta}\right) = 0$, equation (6-3) implies

$$b_{j|k} y^j y^k y^i = 0.$$

Contracting the above expression with y_i gives

$$b_{j|k} y^j y^k y^i y_i = 0.$$

Since $y^i y_i = F^2$,

$$b_{j|k} y^j y^k F^2 = 0$$

i.e.,

$$b_{j|k} y^j y^k = 0.$$

Differentiating the above expression with respect to y^i, we have

$$b_{j|k} \delta_i^j y^k + b_{j|k} y^j \delta_i^k = 0,$$

i.e.,

$$b_{i|k} y^k + b_{j|i} y^j = 0.$$

The above expression can also be written as

$$b_{i|k} y^k + b_{k|i} y^k = 0.$$

Since β is closed, we have

$$2b_{i|k} y^k = 0.$$

Differentiating the above expression with respect to y^j,

$$b_{i|j} = 0.$$

Hence β is parallel with respect to α.

Theorem 6.2.4

Let $\mathfrak{I}^n = (M, F)$ be a Finsler space with $F = \dfrac{\alpha^2}{\alpha - \beta}$, where α is a Riemannian metric and β is a one-form defined on a differentiable manifold M of dimension *n*. Then $\mathfrak{I}^n$ is locally projectively flat if and only if

$$\frac{2\beta}{\alpha-\beta}\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\alpha-\beta}\right)\left\{\frac{\alpha}{\beta}\frac{\partial \beta}{\partial x^k}-\frac{\partial \alpha}{\partial x^k}\right\}y^k+\frac{\alpha^2}{(\alpha-\beta)^2}\left(\frac{\partial b_i}{\partial x^k}-\frac{\partial b_k}{\partial x^i}\right)y^k$$
$$+\frac{(\alpha^2-2\alpha\beta)}{(\alpha-\beta)^2}\left\{\frac{\partial}{\partial y^i}\left(\frac{\partial \alpha}{\partial x^k}\right)y^k-\frac{\partial \alpha}{\partial x^i}\right\}=0.$$

Proof:

Assume that $\mathfrak{I}^n$ is locally projectively flat. Equation (4-8) implies

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right)y^k-\frac{\partial F}{\partial x^i}=0. \tag{6-21}$$

Now,

$$\frac{\partial F}{\partial x^k}=\frac{2\alpha}{\alpha-\beta}\frac{\partial \alpha}{\partial x^k}-\frac{\alpha^2}{(\alpha-\beta)^2}\left(\frac{\partial \alpha}{\partial x^k}-\frac{\partial \beta}{\partial x^k}\right). \tag{6-22}$$

Differentiating equation (6-22) with respect to y^i and contracting with y^k gives

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right)y^k=2\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\alpha-\beta}\right)\left\{\frac{\partial \alpha}{\partial x^k}-\frac{\alpha}{\alpha-\beta}\frac{\partial \alpha}{\partial x^k}\right\}y^k$$
$$+\frac{2\alpha}{\alpha-\beta}\left\{\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\alpha-\beta}\right)\frac{\partial \beta}{\partial x^k}+\frac{\partial}{\partial y^i}\left(\frac{\partial \alpha}{\partial x^k}\right)\right\}y^k$$
$$+\frac{\alpha^2}{(\alpha-\beta)^2}\left\{\frac{\partial b_i}{\partial x^k}-\frac{\partial}{\partial y^i}\left(\frac{\partial \alpha}{\partial x^k}\right)\right\}y^k. \tag{6-23}$$

Replacing k by i in equation (6-22) gives

$$\frac{\partial F}{\partial x^i}=\frac{2\alpha}{\alpha-\beta}\frac{\partial \alpha}{\partial x^i}-\frac{\alpha^2}{(\alpha-\beta)^2}\left(\frac{\partial \alpha}{\partial x^i}-\frac{\partial \beta}{\partial x^i}\right). \tag{6-24}$$

Equations (6-21), (6-23) and (6-24) thus give the proof. The converse also follows easily.

Theorem 6.2.5

Let $\mathfrak{I}^n = (M, F)$ be a locally projectively flat Finsler space with $F = \dfrac{\alpha^2}{\alpha - \beta}$.

Assume that α is locally projectively flat. Then

$$\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\alpha-\beta}\right)(P-Q) = \frac{\alpha}{4\beta(\alpha-\beta)}\left(\frac{\partial b_i}{\partial x^k} - \frac{\partial b_k}{\partial x^i}\right)y^k. \tag{6-25}$$

And the scalar curvature of F is given by

$$K\alpha^4 = \mu\alpha^2(\alpha-2\beta)(\alpha-\beta) - (\alpha-\beta)\beta\frac{\partial Q}{\partial x^k}y^k$$
$$+\beta^2(4P^2 - 4PQ - Q^2) - \alpha\beta(3P^2 - 6PQ + 2Q^2), \tag{6-26}$$

where P and Q are given in Theorem 6.2.3 and μ is the constant sectional curvature of α.

Proof:

Since F and α are locally projectively flat, then from Theorem 6.2.4 implies

$$\frac{2\beta}{\alpha-\beta}\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\alpha-\beta}\right)\left\{\frac{\alpha}{\beta}\frac{\partial\beta}{\partial x^k} - \frac{\partial\alpha}{\partial x^k}\right\}y^k + \frac{\alpha^2}{(\alpha-\beta)^2}\left(\frac{\partial b_i}{\partial x^k} - \frac{\partial b_k}{\partial x^i}\right)y^k = 0.$$

Using the definitions of P and Q gives

$$\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\alpha-\beta}\right)(P-Q) = \frac{\alpha}{4\beta(\alpha-\beta)}\left(\frac{\partial b_i}{\partial x^k} - \frac{\partial b_k}{\partial x^i}\right)y^k.$$

We now calculate the scalar curvature of F. Since F is locally projectively flat, equation (4-12) gives the projective factor

$$\overline{P} = \frac{1}{2F}\frac{\partial F}{\partial x^k}y^k.$$

Using equation (6- 22), we have

$$\overline{P}=\frac{\alpha-\beta}{2\alpha^2}\left\{\frac{2\alpha}{\alpha-\beta}\frac{\partial\alpha}{\partial x^k}-\frac{\alpha^2}{(\alpha-\beta)^2}\left(\frac{\partial\alpha}{\partial x^k}-\frac{\partial\beta}{\partial x^k}\right)\right\}y^k$$

$$=\frac{1}{2\alpha}\left\{\frac{\alpha-2\beta}{\alpha-\beta}\frac{\partial\alpha}{\partial x^k}+\frac{\alpha}{\alpha-\beta}\frac{\partial\beta}{\partial x^k}\right\}y^k.$$

In terms of P and Q the above expression simplifies to

$$\overline{P}=\frac{(\alpha-2\beta)P+\beta Q}{\alpha-\beta}. \tag{6-27}$$

Differentiating $\overline{P}$ with respect to x^k gives

$$\frac{\partial\overline{P}}{\partial x^k}=\frac{(\alpha-\beta)\frac{\partial}{\partial x^k}(\alpha P-2\beta P+\beta Q)-(\alpha P-2\beta P+\beta Q)\frac{\partial}{\partial x^k}(\alpha-\beta)}{(\alpha-\beta)^2}.$$

Contracting the above expression with y^k, and using P and Q we get

$$\frac{\partial\overline{P}}{\partial x^k}y^k=\frac{(2\alpha P-4\beta Q)P+2\beta Q^2+(\alpha-2\beta)\frac{\partial P}{\partial x^k}y^k+\beta\frac{\partial Q}{\partial x^k}y^k}{(\alpha-\beta)}-\frac{2\{(\alpha-2\beta)P+\beta Q\}(\alpha P-\beta Q)}{(\alpha-\beta)^2}.$$

Since $\frac{\partial P}{\partial x^k}y^k=P^2-\mu\alpha^2$, calculations yield

$$\frac{\partial\overline{P}}{\partial x^k}y^k=\frac{(\alpha-2\beta)(P^2-\mu\alpha^2)(\alpha-\beta)+(\alpha-\beta)\beta\frac{\partial Q}{\partial x^k}y^k+2\alpha\beta(P-Q)^2}{(\alpha-\beta)^2}.$$

Since F is locally projectively flat, from equation (4-37) we have

$$KF^2=\overline{P}^2-\frac{\partial\overline{P}}{\partial x^k}y^k.$$

Then by a direct computation, we can get equation (6-26).

We are now in a position to prove Theorem 6.2.1 (b).

Proof of Theorem 6.2.1 (b):

If β is closed then equation (6-25) implies

$$\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\alpha-\beta}\right)(P-Q)=0.$$

i.e., either $\dfrac{\partial}{\partial y^i}\left(\dfrac{\alpha}{\alpha-\beta}\right)=0$ or $P=Q$. We know that

$$\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\alpha-\beta}\right)=\frac{1}{(\alpha-\beta)^2}\left(\alpha\frac{\partial\beta}{\partial y^i}-\beta\frac{\partial\alpha}{\partial y^i}\right).$$

Since $\alpha\dfrac{\partial\beta}{\partial y^i}-\beta\dfrac{\partial\alpha}{\partial y^i}=\alpha^2\dfrac{\partial}{\partial y^i}\left(\dfrac{\beta}{\alpha}\right)$, the above expression gives

$$\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\alpha-\beta}\right)=\left(\frac{\alpha}{\alpha-\beta}\right)^2\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right).$$

Hence, if β is closed then either $\dfrac{\partial}{\partial y^i}\left(\dfrac{\beta}{\alpha}\right)=0$ or $P=Q$. Here we also consider the following two cases.

Case: (i)

Assume that $P=Q$. Hence equation (6-27) shows that

$$\overline{P}=P,$$

and hence we can prove that $\mathfrak{I}^n$ is either a locally Minkowski space or a Riemannian space of constant sectional curvature by the same process as in the proof of Theorem 6.2.1 (a). If $\mathfrak{I}^n$ is a locally Minkowski space then we also have $K=\mu=0$.

If $\mathfrak{I}^n$ is a Riemannian space then equation (4-37) shows

$$\mu\alpha^2=P^2-\frac{\partial P}{\partial x^k}y^k,$$

and hence equation (6-26) implies

$$\begin{aligned}K\alpha^4&=\mu\alpha^2(\alpha^2-3\alpha\beta+2\beta^2)+\mu\alpha^2\beta(\alpha-\beta).\\&=\mu(\alpha-\beta)^2\alpha^2.\end{aligned}$$

Therefore,

$$K = \mu\left(\frac{\alpha-\beta}{\alpha}\right)^2.$$

$$KF^2 = \mu\alpha^2.$$

Hence equation (3-9) implies that implies that $\overline{Ric} = Ric$, where $\overline{Ric}$ and Ric are the Ricci curvatures of F and α respectively. Since K is a constant, differentiating K with respect to y^i gives

$$2\mu\left(\frac{\alpha-\beta}{\alpha}\right)\frac{\partial}{\partial y^i}\left(\frac{\alpha-\beta}{\alpha}\right) = 0.$$

Since $\mu \neq 0$ and $\alpha \neq \beta$,

$$\frac{\partial}{\partial y^i}\left(\frac{\alpha-\beta}{\alpha}\right) = 0.$$

We know that

$$\frac{\partial}{\partial y^i}\left(\frac{\alpha-\beta}{\alpha}\right) = \frac{1}{\alpha^2}\left(\beta\frac{\partial\alpha}{\partial y^i} - \alpha\frac{\partial\beta}{\partial y^i}\right) = -\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right),$$

and hence

$$\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right) = 0.$$

We have shown that if $P = Q$ then $\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right) = 0.$

Case: (ii)

Assume that $\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right) = 0$. That is, there is a function $\lambda(x)$ such that $\beta = \lambda(x)\alpha$.

Then

$$F = \frac{\alpha^2}{\alpha-\beta} = \left(\frac{1}{1-\lambda}\right)\alpha,$$

and hence according to Definition 2.1.6, F is conformal to α. As in the proof of Theorem 6.2.1 (a), we can show that if $\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)=0$ then $P=Q$. Thus from case (i) and case (ii), we have both $\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)=0$ and $P=Q$, and hence F and α are homothetic. Further, we can show that β is parallel with respect to α.

Since $\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)=0$, we have

$$\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\alpha-\beta}\right)=0.$$

Since $\overline{G}^i=G^i=Py^i$ and $\frac{\partial}{\partial y^j}\left(\frac{\alpha}{\alpha-\beta}\right)=0$, equation (6-4) gives

$$b_{j|k}y^j y^k y^i=0.$$

This implies

$$b_{j|k}=0.$$

That is, β is parallel with respect to α.

Theorem 6.2.6

Let $\Im^n$ be a Finsler space with $F=\alpha+\frac{\beta^2}{\alpha}$, where α is a Riemannian metric and β is a one-form defined on a differentiable manifold M of dimension *n*. Then $\Im^n$ is locally projectively flat if and only if

$$2\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\left(\frac{\partial\beta}{\partial x^k}-\frac{\beta}{\alpha}\frac{\partial\alpha}{\partial x^k}\right)y^k+\frac{2\beta}{\alpha}\left(\frac{\partial b_i}{\partial x^k}-\frac{\partial b_k}{\partial x^i}\right)y^k$$
$$+\frac{\alpha^2-\beta^2}{\alpha^2}\left\{\frac{\partial}{\partial y^i}\left(\frac{\partial\alpha}{\partial x^k}\right)y^k-\frac{\partial\alpha}{\partial x^i}\right\}=0. \qquad (6\text{-}28)$$

Proof:

Assume that $\Im^n$ is locally projectively flat. Equation (4-8) implies

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right)y^k - \frac{\partial F}{\partial x^i} = 0. \tag{6-29}$$

Now,

$$\frac{\partial F}{\partial x^k} = \frac{\partial \alpha}{\partial x^k} + \frac{2\beta}{\alpha}\frac{\partial \beta}{\partial x^k} - \frac{\beta^2}{\alpha^2}\frac{\partial \alpha}{\partial x^k}. \tag{6-30}$$

Differentiating equation (6-30) with respect to y^i and contracting with y^k gives

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right)y^k = 2\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\left(\frac{\partial \beta}{\partial x^k} - \frac{\beta}{\alpha}\frac{\partial \alpha}{\partial x^k}\right)y^k + \frac{\alpha^2-\beta^2}{\alpha^2}\frac{\partial}{\partial y^i}\left(\frac{\partial \alpha}{\partial x^k}\right)y^k + \frac{2\beta}{\alpha}\frac{\partial b_i}{\partial x^k}y^k.$$

Replacing k by i in equation (6-30) gives

$$\frac{\partial F}{\partial x^i} = \frac{\partial \alpha}{\partial x^i} + \frac{2\beta}{\alpha}\frac{\partial \beta}{\partial x^i} - \frac{\beta^2}{\alpha^2}\frac{\partial \alpha}{\partial x^i}.$$

Substituting the above two expressions to equation (6-29) implies equation (6-28). The converse also follows easily.

Theorem 6.2.7

Let $\mathfrak{F}^n = (M,F)$ be a locally projectively flat Finsler space with $F = \alpha + \frac{\beta^2}{\alpha}$. Assume that α is locally projectively flat. Then

$$\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)(P-Q) = \frac{1}{2\alpha}\left(\frac{\partial b_i}{\partial x^k} - \frac{\partial b_k}{\partial x^i}\right)y^k. \tag{6-31}$$

And the scalar curvature of F is given by

$$KF^2 = \mu\alpha^2 + \frac{4(P-Q)\beta^2(2\alpha^2 Q - \beta^2 Q - 3\alpha^2 P)}{(\alpha^2+\beta^2)^2} + \frac{2\beta^2}{(\alpha^2+\beta^2)}\left(\frac{\partial P}{\partial x^k} - \frac{\partial Q}{\partial x^k}\right)y^k,$$

where P and Q are given in Theorem 6.2.3 and μ is the constant sectional curvature of α.

Proof:

Since F and α are locally projectively flat, equation (6-28) implies

$$\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\left(\frac{\partial\beta}{\partial x^k}-\frac{\beta}{\alpha}\frac{\partial\alpha}{\partial x^k}\right)y^k+\frac{\beta}{\alpha}\left(\frac{\partial b_i}{\partial x^k}-\frac{\partial b_k}{\partial x^i}\right)y^k=0.$$

Using P and Q given in Theorem 6.2.3, we can get the result.

Let us calculate the scalar curvature of F.

Since F is locally projectively flat, equation (4-12) gives the projective factor

$$\overline{P}=\frac{1}{2F}\frac{\partial F}{\partial x^k}y^k.$$

Using equation (6-30), we have

$$\overline{P}=\frac{\alpha}{2(\alpha^2+\beta^2)}\left(\frac{\partial\alpha}{\partial x^k}+\frac{2\beta}{\alpha}\frac{\partial\beta}{\partial x^k}-\frac{\beta^2}{\alpha^2}\frac{\partial\alpha}{\partial x^k}\right)y^k.$$

Using P and Q defined in Theorem 6.2.3 the above expression implies

$$\overline{P}=P+\frac{2\beta^2(Q-P)}{\alpha^2+\beta^2}. \tag{6-32}$$

Differentiating $\overline{P}$ with respect to x^k and contracting with y^k gives

$$\frac{\partial\overline{P}}{\partial x^k}y^k=\frac{\partial P}{\partial x^k}y^k+\frac{8\alpha^2\beta^2(Q-P)^2}{(\alpha^2+\beta^2)^2}+\frac{2\beta^2}{\alpha^2+\beta^2}\left(\frac{\partial Q}{\partial x^k}-\frac{\partial P}{\partial x^k}\right)y^k. \tag{6-33}$$

Since F and α are locally projectively flat, from equation (4-37), we know that

$$KF^2=\overline{P}^2-\frac{\partial\overline{P}}{\partial x^k}y^k, \tag{6-34}$$

and

$$\mu\alpha^2=P^2-\frac{\partial P}{\partial x^k}y^k. \tag{6-35}$$

Equations (6-32), (6-33), (6-34) and (6-35) yield the result.

Proof of Theorem 6.2.1 (c):

From (6-31), if β is closed then either $\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)=0$ or $P=Q$. Following the same process as in the proof of Theorem 6.2.1 (a) and using (6-6) we can show that $\mathfrak{F}^n=(M,F=\alpha+\beta^2/\alpha)$ is either a locally Minkowski space or a Riemannian space of constant sectional curvature given by $KF^2=\mu\alpha^2$, where K and μ are the non-zero constant sectional curvature of F and α respectively, F and α are homothetic, $\overline{Ric}=Ric$, β is parallel with respect to α, where $\overline{Ric}$ and Ric are the Ricci curvatures of F and α respectively. Thus the proof of Theorem 6.2.1 is complete.

Finally we compute the scalar curvature of a locally projectively flat Randers space in terms of P and Q defined in Theorem 6.2.3.

Theorem 6.2.8

Let $\mathfrak{F}^n=(M,F=\alpha+\beta)$ be a locally projectively flat Randers space. Then the scalar curvature K of F is given by the following expression.

$$KF^4=\mu\alpha^4-\alpha\beta\left(2P^2-6PQ+2Q^2+\frac{\partial P}{\partial x^k}y^k+\frac{\partial Q}{\partial x^k}y^k\right)$$
$$+\beta^2\left(Q^2-\frac{\partial Q}{\partial x^k}y^k\right), \qquad (6\text{-}36)$$

where P and Q are defined in Theorem 6.2.3 and μ is the constant sectional curvature of α.

Proof:

Since F is locally projectively flat, equation (4-12) gives the projective factor

$$\overline{P}=\frac{1}{2F}\frac{\partial F}{\partial x^k}y^k=\frac{1}{2(\alpha+\beta)}\left(\frac{\partial\alpha}{\partial x^k}+\frac{\partial\beta}{\partial x^k}\right)y^k.$$

Using P and Q, we have

$$\overline{P} = \frac{\alpha P + \beta Q}{(\alpha + \beta)}. \tag{6-37}$$

Differentiating $\overline{P}$ with respect to x^k and contracting with y^k gives

$$\frac{\partial \overline{P}}{\partial x^k} y^k = \frac{2(\alpha P^2 + \beta Q^2)}{(\alpha + \beta)} - \frac{2(\alpha P + \beta Q)^2}{(\alpha + \beta)^2} + \frac{1}{(\alpha + \beta)} \left(\alpha \frac{\partial P}{\partial x^k} + \beta \frac{\partial Q}{\partial x^k} \right) y^k. \tag{6-38}$$

Since F and α are locally projectively flat, equation (4-37) shows

$$KF^2 = \overline{P}^2 - \frac{\partial \overline{P}}{\partial x^k} y^k, \tag{6-39}$$

and

$$\mu\alpha^2 = P^2 - \frac{\partial P}{\partial x^k} y^k. \tag{6-40}$$

Using equations (6-37), (6-38), (6-39) and (6-40), we can get the result.

Corollary 6.2.9

Let $\Im^n = (M, F = \alpha + \beta)$ be a locally projectively flat Randers space. If $P = Q$ then $\Im^n$ is either a locally Minkowski space or a Riemannian space of constant sectional curvature given by $KF^2 = \mu\alpha^2$, where K and μ are the non-zero constant sectional curvature of F and α respectively, F and α are homothetic, $\overline{Ric} = Ric$ and β is parallel with respect to α, where $\overline{Ric}$ and Ric are the Ricci curvatures of F and α respectively.

Proof:

If $P = Q$ then equation (6-37) implies $\overline{P} = P$. Since F is locally projectively flat, Theorem 5.1.5 implies that α is locally projectively flat, and hence the projective factor is

$$P = \frac{1}{2\alpha} \frac{\partial \alpha}{\partial x^k} y^k,$$

therefore

$$\overline{G}^i = \overline{P}y^i,$$

and

$$G^i = Py^i,$$

where $\overline{G}^i$ and G^i are the geodesic coefficients of F and α respectively. Since $\overline{P} = P$, we have $\overline{G}^i = G^i$, and hence from Definition 2.2.3 F is a Berwald metric. Thus, F is a locally projectively flat Berwald metric. Therefore, Theorem 4.2.6 indicates that $\mathfrak{I}^n$ is either a locally Minkowski space or a Riemannian space of constant sectional curvature.

If $\mathfrak{I}^n$ is a locally Minkowski space then recall that $\overline{G}^i = 0$.

Since $\overline{G}^i = \overline{P}y^i$,

$$\overline{P} = 0.$$

Since $\overline{P} = P$,

$$P = 0.$$

Hence, from equations (6-39) and (6-40), we have

$$K = \mu = 0.$$

If $\mathfrak{I}^n$ is a Riemannian space of constant sectional curvature then from equation (6-36), since $P = Q$,

$$KF^4 = \mu\alpha^4 + 2\alpha\beta\left(P^2 - \frac{\partial P}{\partial x^k}y^k\right) + \beta^2\left(P^2 - \frac{\partial P}{\partial x^k}y^k\right)$$

$$= \mu\alpha^4 + (2\alpha\beta + \beta^2)\left(P^2 - \frac{\partial P}{\partial x^k}y^k\right).$$

Using equation (6-40) gives

$$KF^4 = \mu\alpha^4 + (2\alpha\beta + \beta^2)\mu\alpha^2$$

$$= \mu\alpha^2(\alpha^2 + 2\alpha\beta + \beta^2) = \mu\alpha^2(\alpha + \beta)^2.$$

Since $F = \alpha + \beta$, we have

$$KF^2 = \mu\alpha^2. \tag{6-41}$$

The above expression also implies

$$\overline{Ric} = Ric,$$

where $\overline{Ric}$ and Ric are the Ricci curvatures of F and α respectively. Since $K \neq 0$,

$$F^2 = \frac{\mu}{K}\alpha^2.$$

Using the above expression and (2-2) gives

$$\overline{g}_{ij}(x) = \lambda g_{ij},$$

where $\lambda = \mu/K =$ constant and $\overline{g}_{ij}$ and g_{ij} are metric tensors of F and α respectively. Hence, equation (2-14) implies that F and α are homothetic.

Further, it is known that a Randers metric is a Berwald metric if and only if β is parallel with respect to α [Ki]. This result implies that β is parallel with respect to α since F is Riemannian.

Remark:

Equation (6-41) also follows from equation (5-29) when β is parallel with respect to α. We know that both ϕ and ψ are equal to zero when β is parallel with respect to α. Hence equation (5-29) implies

$$KF^2 = \mu\alpha^2.$$

Motivated by above theorems, now we are going to study Finsler spaces with (α,β)-metric, which are not locally projectively flat.

Theorem 6.2.10

Let $\Im^n = (M, F)$ be a Finsler space with (α,β)-metric, where (a) $F = \dfrac{\alpha^2}{\beta}$, (b) $F = \dfrac{\alpha^2}{\alpha - \beta}$, (c) $F = \alpha + \dfrac{\beta^2}{\alpha}$. Assume that α is locally projectively flat and β is closed. Then F is not locally projectively flat if and only if

$$\left(\frac{\partial\alpha}{\partial x^k}-\frac{\alpha}{\beta}\frac{\partial\beta}{\partial x^k}\right)y^k \neq 0, \tag{6-42}$$

and

$$\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)\neq 0. \tag{6-43}$$

Proof:

We prove this theorem by considering each Finsler space with (α,β)-metric separately.

(a) $F=\dfrac{\alpha^2}{\beta}$.

Assume that (6-42) and (6-43) hold. Subtracting (6-10) and (6-11) gives

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right)y^k-\frac{\partial F}{\partial x^i}=2\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)\left\{\frac{\partial\alpha}{\partial x^k}-\frac{\alpha}{\beta}\frac{\partial\beta}{\partial x^k}\right\}y^k+$$

$$\frac{2\alpha}{\beta}\left\{\frac{\partial}{\partial y^i}\left(\frac{\partial\alpha}{\partial x^k}\right)y^k-\frac{\partial\alpha}{\partial x^i}\right\}+\frac{\alpha^2}{\beta^2}\left(\frac{\partial b_k}{\partial x^i}-\frac{\partial b_i}{\partial x^k}\right)y^k.$$

Since β is closed and α is locally projectively flat, equations (5-2) and (4-8) imply

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right)y^k-\frac{\partial F}{\partial x^i}=2\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)\left\{\frac{\partial\alpha}{\partial x^k}-\frac{\alpha}{\beta}\frac{\partial\beta}{\partial x^k}\right\}y^k.$$

Thus equations (6-42) and (6-43) yield

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right)y^k-\frac{\partial F}{\partial x^i}\neq 0.$$

According to equation (4-8) the above expression implies that F is not locally projectively flat. The converse also follows easily.

(b) $F=\dfrac{\alpha^2}{\alpha-\beta}$.

Assume that (6-42) and (6-43) hold. Subtracting equations (6-23) and (6-24) gives

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right)y^k-\frac{\partial F}{\partial x^i}=\frac{2\beta}{\alpha-\beta}\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\alpha-\beta}\right)\left\{\frac{\alpha}{\beta}\frac{\partial\beta}{\partial x^k}-\frac{\partial\alpha}{\partial x^k}\right\}y^k+$$

$$\frac{\alpha^2}{(\alpha-\beta)^2}\left(\frac{\partial b_i}{\partial x^k}-\frac{\partial b_k}{\partial x^i}\right)y^k+\frac{(\alpha^2-2\alpha\beta)}{(\alpha-\beta)^2}\left\{\frac{\partial}{\partial y^i}\left(\frac{\partial\alpha}{\partial x^k}\right)y^k-\frac{\partial\alpha}{\partial x^i}\right\}.$$

Since β is closed and α is locally projectively flat, equations (5-2) and (4-8) imply

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right)y^k-\frac{\partial F}{\partial x^i}=\frac{2\beta}{\alpha-\beta}\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\alpha-\beta}\right)\left\{\frac{\alpha}{\beta}\frac{\partial\beta}{\partial x^k}-\frac{\partial\alpha}{\partial x^k}\right\}y^k.$$

Since $\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\alpha-\beta}\right)=-\left(\frac{\beta}{\alpha-\beta}\right)^2\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)$, we have

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right)y^k-\frac{\partial F}{\partial x^i}=-2\left(\frac{\beta}{\alpha-\beta}\right)^3\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)\left\{\frac{\alpha}{\beta}\frac{\partial\beta}{\partial x^k}-\frac{\partial\alpha}{\partial x^k}\right\}y^k.$$

Since $\beta\neq 0$, equations (6-42) and (6-43) yield

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right)y^k-\frac{\partial F}{\partial x^i}\neq 0.$$

According to (4-8) the above expression implies that F is not locally projectively flat. The converse also follows easily.

By following the same process for $F=\alpha+\beta^2/\alpha$ we obtain the final part of the theorem.

Example 6.2.1

If

$$\alpha=\frac{\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}}{1-|x|^2},$$

and

$$\beta=\frac{\langle x,y\rangle}{1-|x|^2}$$

Then $F_1=\frac{\alpha^2}{\beta}$, $F_2=\frac{\alpha^2}{\alpha-\beta}$ and $F_3=\alpha+\frac{\beta^2}{\alpha}$ are not locally projectively flat.

Proof:

Note that $\alpha \pm \beta$ are the Funk metrics of Lemma 5.1.7. Recall that with $\mu = -1$

$$\alpha^2 = a_{ij}y^i y^j = (\rho_{ij} - \rho_i\rho_j)y^i y^j, \tag{6-44}$$

and

$$\beta = \rho_i y^i. \tag{6-45}$$

where

$$\rho_i(x) = \frac{\delta_{ik}x^k}{1-|x|^2}.$$

From earlier work we know that α is locally projectively flat and β is closed, we use Theorem 6.2.10 to show that F_1, F_2 and F_3 are not locally projectively flat. We have to prove that equations (6-42) and (6-43) are valid for α and β. We first calculate $\frac{\partial \alpha}{\partial x^k}y^k$ and $\frac{\partial \beta}{\partial x^k}y^k$. Differentiating α^2 with respect to x^k and contracting with y^k gives

$$2\alpha\frac{\partial \alpha}{\partial x^k}y^k = (\rho_{ijk} - \rho_{ik}\rho_j - \rho_{jk}\rho_i)y^i y^j y^k$$

$$= (\rho_{ijk} - 2\rho_{ik}\rho_j)y^i y^j y^k. \tag{6-46}$$

Contracting (4-39) with y^i, y^j and y^k implies

$$6\rho_{ij}\rho_k y^i y^j y^k = (\rho_{ijk} + 4\rho_i\rho_j\rho_k)y^i y^j y^k.$$

Substituting the above expression to (6-46) and dividing by 2 yields

$$\alpha\frac{\partial \alpha}{\partial x^k}y^k = 2(\rho_{ij}\rho_k - \rho_i\rho_j\rho_k)y^i y^j y^k$$

$$= 2(\rho_{ij} - \rho_i\rho_j)y^i y^j \rho_k y^k.$$

Hence, from equations (6-44) and (6-45), we can get

$$\frac{\partial \alpha}{\partial x^k}y^k = 2\alpha\beta.$$

Differentiating β with respect to x^k and using equations (6-44) and (6-45), we also have

$$\frac{\partial \beta}{\partial x^k} y^k = \alpha^2 + \beta^2.$$

Hence,

$$\left(\frac{\partial \alpha}{\partial x^k} - \frac{\alpha}{\beta} \frac{\partial \beta}{\partial x^k} \right) y^k = 2\alpha\beta - \frac{\alpha}{\beta}(\alpha^2 + \beta^2)$$

$$= \frac{\alpha(\beta^2 - \alpha^2)}{\beta}.$$

Substituting the expressions for α and β, we have

$$\beta^2 - \alpha^2 = -\frac{|y|^2}{1-|x|^2} \neq 0,$$

and hence the above two expressions show that (6-42) is valid.

We now show that $\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right) \neq 0.$

Assume that $\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right) = 0.$

From equation (6-20), we have

$$F_1 = \psi\alpha,$$

where ψ is a constant. Then equation (2-2) implies

$$\bar{g}_{ij} = \psi^2 g_{ij},$$

where $\bar{g}_{ij}$ and g_{ij} are Finsler metric tensors of F_1 and α respectively. Notice that the above expression yields

$$\bar{g}^{ij} = \frac{1}{\psi^2} g^{ij}.$$

The above expressions and equation (2-17) yield

$$\bar{G}^i = G^i,$$

where $\overline{G}^i$ and G^i are the geodesic coefficients of F_1 and α respectively and hence equation (6-3) gives

$$b_{j|k}y^j y^k y^i = 0.$$

Recall that the above expression yields

$$b_{j|k} = 0,$$

and hence equation (5-17) yields

$$\rho_{ij} = 2\rho_i\rho_j.$$

This implies that $a_{ij} = \rho_i\rho_j$ and $\det a_{ij} = 0$. That is, α given in (6-44) is not a Riemannian metric. This leads to a contradiction. Hence (6-43) is valid.

Similarly, we can show that equation (6-43) is also true for other Finsler metrics with (α,β)-metric, $F_2 = \dfrac{\alpha^2}{\alpha - \beta}$ and $F_3 = \alpha + \dfrac{\beta^2}{\alpha}$. Therefore, F_1, F_2 and F_3 are not locally projectively flat Finsler metrics with (α,β)-metric.

Similar arguments can be used to prove the following lemma.

Example 6.2.2

If

$$\alpha = \frac{\sqrt{|y|^2 + (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1+|x|^2},$$

and

$$\beta = \frac{\langle x,y\rangle}{1+|x|^2}.$$

Then $\overline{F}_1 = \dfrac{\alpha^2}{\beta}$, $\overline{F}_2 = \dfrac{\alpha^2}{\alpha-\beta}$ and $\overline{F}_3 = \alpha + \dfrac{\beta^2}{\alpha}$ are not locally projectively flat.

6.3 Finsler Spaces with (α,β)-Metric of Douglas Type

This topic has been studied by M. Matsumoto in [Ma5]. He found the necessary and sufficient conditions for each Finsler space with (α,β)-metric to be of Douglas type. We use theorems of M. Matsumoto to investigate whether the Finsler spaces with (α,β)-metric given in Example 6.2.1 and Example 6.2.2 are of Douglas type. Before that we state the following theorems given in [Ma5].

Theorem 6.3.1 ([Ma5])

A Kropina space $\mathfrak{I}^n = (M, F = \alpha^2/\beta)$ $(n > 2)$ with $b^2 = \alpha^{ij}(x)b_i b_j \neq 0$ is of Douglas type, if and only if

$$s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i)$$

is satisfied, where $s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i})$, $s_i = b_j s_i^j$ and $s_i^j = \alpha^{jk} s_{ki}$.

Theorem 6.3.2 ([Ma5])

A Matsumoto space with the metric $F = \dfrac{\alpha^2}{\alpha - \beta}$ is a Douglas space, if and only if

(1) $\alpha^2 \not\equiv 0$ (mod. β): $b_{i|j} = 0$,

(2) $\alpha^2 \equiv 0$ (mod. β): $n = 2$ and $b_{i|j} = k\left(\frac{1}{3}d_i - b_i\right)d_j$,

where $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$ and $k = k(x)$.

Theorem 6.3.3 ([Ma5])

Let $\mathfrak{I}^n$ $(n > 2)$ be a Finsler space with $F = \alpha + \dfrac{\beta^2}{\alpha}$ and suppose that $b^2 = \alpha^{ij} b_i b_j \neq 0, 1$. $\mathfrak{I}^n$ is a Douglas space, if and only if there exists a function $k(x)$ such that

$$b_{i|j} = k(x)\{(1+2b^2)\alpha_{ij} - 3b_i b_j\}. \tag{6-47}$$

Examples 6.3.1

Example (i)

Consider $F = \dfrac{\alpha^2}{\beta}$ and $n > 2$.

Since β is closed in Example 6.2.1 and Example 6.2.2, we know that $s_{ij} = 0,\ s_i = 0$ and $s_j = 0$. Hence from Theorem 6.3.1, F_1 and $\overline{F_1}$ are of Douglas type.

Example (ii)

Consider $F = \dfrac{\alpha^2}{\alpha - \beta}$ and $n > 2$.

Equation (5-17) gives

$$b_{i|j} = \rho_{ij} - 2\rho_i \rho_j. \tag{6-48}$$

If $b_{i|j} = 0$ then $\alpha_{ij} = \rho_i \rho_j$ and $\det(\alpha_{ij}) = 0$. That is, α given in (6-44) is not a Riemannian metric. Hence, we have a contradiction. Therefore, $b_{i|j} \neq 0$. Hence, Theorem 6.3.2 implies F_2 is not of Douglas type when $n > 2$. Similarly, we can show that $\overline{F_2}$ is not of Douglas type when $n > 2$.

Example (iii)

Consider $F = \alpha + \dfrac{\beta^2}{\alpha}$ and $n > 2$.

Assume that equation (6-47) holds. That is,

$$b_{i|j} = k(x)\{(1+2b^2)\alpha_{ij} - 3b_i b_j\}.$$

Using equation (6-48), since $b_i = \rho_i$, we have

$$\rho_{ij} - 2\rho_i \rho_j = k(x)\{(1+2b^2)\alpha_{ij} - 3\rho_i \rho_j\}.$$

Contracting the above expression with y^i and y^j gives

$$(\rho_{ij}-2\rho_i\rho_j)y^i y^j = k(x)\{(1+2b^2)\alpha_{ij}-3\rho_i\rho_j\}y^i y^j.$$

Using (6-44) and (6-45), we can get

$$\alpha^2-\beta^2 = k(x)\{(1+2b^2)\alpha^2-3\beta^2\}.$$

$$k=\frac{\alpha^2-\beta^2}{(1+2b^2)\alpha^2-3\beta^2}.$$

Since $k=k(x)$,

$$\frac{\partial k}{\partial y^i}=0$$

Differentiating k with respect to y^i and using the above expression gives

$$\left((1+2b^2)\alpha^2-3\beta^2\right)\left(\alpha\frac{\partial\alpha}{\partial y^i}-\beta\frac{\partial\beta}{\partial y^i}\right)-$$

$$(\alpha^2-\beta^2)\left\{(1+2b^2)\alpha\frac{\partial\alpha}{\partial y^i}-3\beta\frac{\partial\beta}{\partial y^i}\right\}=0.$$

This simplifies to

$$2\alpha\beta(1-b^2)\left(\alpha\frac{\partial\beta}{\partial y^i}-\beta\frac{\partial\alpha}{\partial y^i}\right)=0.$$

Since $\alpha\frac{\partial\beta}{\partial y^i}-\beta\frac{\partial\alpha}{\partial y^i}==-\beta^2\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)$,

$$2\alpha\beta(1-b^2)\beta^2\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)=0.$$

Since $b^2\neq 1$, $\alpha\neq 0$ and $\beta\neq 0$, the above expression implies that

$$\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)=0.$$

Since F_3 is not locally projectively flat, from Theorem 6.2.10, we have got a contradiction. Hence, (6-47) does not hold. Therefore, F_3 in Example 6.2.1 is not a

Douglas metric. Similarly, we can show that $\overline{F_3}$ in Example 6.2.2 is not a Douglas metric.

Remark:

We can use these results to prove that the Kropina metrics F_1 and $\overline{F_1}$ do not have scalar curvature. Suppose F_1 and $\overline{F_1}$ have scalar curvature then from Theorem 4.2.3 their Weyl tensors are equal to zero. From Example (i) of Examples 6.3.1, we also know that their Douglas tensors are equal to zero. Then using Theorem 4.2.5, we can say that the above Kropina metrics are locally projectively flat. But we know that from Example 6.2.1 and Example 6.2.2, F_1 and $\overline{F_1}$ are not locally projectively flat. Therefore, we have a contradiction. Hence, F_1 and $\overline{F_1}$ do not have scalar curvature. It means that their flag curvatures depend on the planes $P \subset T_pM$ containing $y \in T_pM$, defined in chapter 3.2.

We prove the following result for $\overline{\mathfrak{F}}^n = \left(M, \overline{F} = \alpha + \beta^2/\alpha\right)$ using a Randers metric with isotropic S-curvature and Theorem 6.3.3.

Lemma 6.3.4

Let $\mathfrak{F}^n = (M, F = \alpha + \beta)$, $(n > 2)$, be a Randers space with isotropic S-curvature and β be closed. Suppose that $b^2 = \alpha^{ij} b_i b_j \neq 0, 1$.

Then $\overline{\mathfrak{F}}^n = \left(M, \overline{F} = \alpha + \dfrac{\beta^2}{\alpha}\right)$ is not locally projectively flat.

Proof:

Assume that $\overline{\mathfrak{F}}^n$ is a Douglas space. Hence from Theorem 6.3.3, there exists a function $k(x)$ such that

$$b_{i|j} = k(x)\{(1+2b^2)\alpha_{ij} - 3b_i b_j\}. \tag{6-49}$$

Since F is a Randers metric with isotropic S-curvature, Theorem 5.2.1 implies

$$e_{ij}y^i y^j = 2c(x)(\alpha^2 - \beta^2), \tag{6-50}$$

where

$$e_{ij} = r_{ij} + b_i s_j + b_j s_i,$$

and

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}).$$

Since β is closed, we know that $b_{i|j} = b_{j|i}$, $s_i = 0$ and $s_j = 0$. Therefore,

$$e_{ij} = b_{i|j}.$$

Substituting the above expression to (6-50) gives

$$b_{i|j}y^i y^j = 2c(x)(\alpha^2 - \beta^2). \tag{6-51}$$

Since $\alpha^2 = \alpha_{ij}(x)y^i y^j$ and $\beta^2 = b_i b_j y^i y^j$, equation (6-51) yields

$$b_{i|j}y^i y^j = 2c(x)(\alpha_{ij} - b_i b_j)y^i y^j. \tag{6-52}$$

Differentiating (6-52) with respect to y^k gives

$$b_{i|j}\delta^i_k y^j + b_{i|j}\delta^j_k y^i = 2c(x)\{\alpha_{ij} - b_i b_j\}(\delta^i_k y^j + \delta^j_k y^i).$$

This simplifies to

$$b_{k|j}y^j + b_{i|k}y^i = 2c(x)\{\alpha_{kj}y^j - b_k b_j y^j + \alpha_{ik}y^i - b_i b_k y^i\}$$

Since β is closed, i.e., $b_{k|j} = b_{j|k}$, we have

$$b_{j|k}y^j + b_{i|k}y^i = 2c(x)\{\alpha_{kj}y^j - b_k b_j y^j + \alpha_{ik}y^i - b_i b_k y^i\}.$$

Replacing j by i in the above expression gives

$$2b_{i|k}y^i = 4c(x)\{\alpha_{ik} - b_i b_k\}y^i. \tag{6-53}$$

Differentiating equation (6-53) with respect to y^j implies

$$b_{j|k} = 2c(x)\{\alpha_{jk} - b_j b_k\}.$$

This can be recast as

$$b_{i|j} = 2c(x)\{\alpha_{ij} - b_i b_j\}.$$

Using equation (6-49), we get

$$k(x)\{(1+2b^2)\alpha_{ij}-3b_ib_j\}=2c(x)\{\alpha_{ij}-b_ib_j\}.$$

Contracting this with y^i and y^j yields

$$k(x)\{(1+2b^2)\alpha_{ij}-3b_ib_j\}y^iy^j=2c(x)\{\alpha_{ij}-b_ib_j\}y^iy^j,$$

i.e.,

$$2c(x)=k(x)\left\{\frac{(1+2b^2)\alpha^2-3\beta^2}{\alpha^2-\beta^2}\right\}.$$

Since $c=c(x)$, we have

$$\frac{\partial c}{\partial y^i}=0. \tag{6-54}$$

Differentiating c with respect to y^i and using (6-54) implies

$$k\left\{(1+2b^2)\alpha\frac{\partial\alpha}{\partial y^i}-3\beta\frac{\partial\beta}{\partial y^i}\right\}(\alpha^2-\beta^2)-$$
$$k\{(1+2b^2)\alpha^2-3\beta^2\}\left(\alpha\frac{\partial\alpha}{\partial y^i}-\beta\frac{\partial\beta}{\partial y^i}\right)=0.$$

i.e.,

$$k\left\{3\beta^2\alpha\frac{\partial\alpha}{\partial y^i}-3\beta\alpha^2\frac{\partial\beta}{\partial y^i}+(1+2b^2)\alpha^2\beta\frac{\partial\beta}{\partial y^i}-(1+2b^2)\beta^2\alpha\frac{\partial\alpha}{\partial y^i}\right\}=0.$$

Taking similar terms together and factorizing gives

$$2\alpha\beta k\left(\beta\frac{\partial\alpha}{\partial y^i}-\alpha\frac{\partial\beta}{\partial y^i}\right)(1-b^2)=0.$$

Since $\beta\frac{\partial\alpha}{\partial y^i}-\alpha\frac{\partial\beta}{\partial y^i}=\beta^2\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)$,

$$\alpha\beta^3k\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)(1-b^2)=0.$$

Since $b^2\neq 1$, $\alpha\neq 0$, $\beta\neq 0$ and $\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)\neq 0$,

$$k=0.$$

Thus we have got a contradiction. Therefore, $\overline{\mathfrak{I}}^n$ is not a Douglas space. Hence, $\overline{F}$ is not a Douglas metric. That is, its Douglas tensor is not equal to zero. According to Theorem 4.2.5, $\overline{F}$ is not locally projectively flat. The proof is complete.

Examples 6.3.2

Using the examples given in [ChMoSh] of locally projectively flat Randers metrics with isotropic S-curvature, we give two examples of not locally projectively flat Finsler spaces with (α,β)-metric for $\overline{\mathfrak{I}}^n = \left(M, \overline{F} = \alpha + \dfrac{\beta^2}{\alpha}\right)$, when $n > 2$.

Example (i)

In [ChMoSh], we have the following locally projectively flat Randers metrics with isotropic S-curvature.

$$F(x,y) = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2}\sqrt{(1-|x|^2)+\lambda^2} + \lambda\langle x,y\rangle}{(1-|x|^2)\sqrt{(1-|x|^2)+\lambda^2}},$$

where $y \in T_pB^n$ and $\lambda \in R^n$ is an arbitrary constant. Since $F = \alpha + \beta$, from this we know that

$$\alpha = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2}}{1-|x|^2}$$

and

$$\beta = \frac{\lambda\langle x,y\rangle}{(1-|x|^2)\sqrt{(1-|x|^2 + \lambda^2}}.$$

From Lemma 6.3.4, we have the following not locally projectively flat Finsler metric for $\overline{\mathfrak{I}}^n = \left(M, \overline{F} = \alpha + \dfrac{\beta^2}{\alpha}\right)$, when $n > 2$.

$$\overline{F} = \frac{\{(1-|x|^2)|y|^2 + \langle x,y\rangle^2\}\{(1-|x|^2)+\lambda^2\} + \lambda^2\langle x,y\rangle^2}{\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2}\{(1-|x|^2)+\lambda^2\}(1-|x|^2)}.$$

Example (ii)

In [ChMoSh], we also have the following locally projectively flat Randers metrics with isotropic S-curvature.

$$F(x,y)=\frac{|y|\sqrt{1+|x|^2}+\langle x,y\rangle}{\sqrt{1+|x|^2}},$$

where $y\in T_pR^n$. Since $F=\alpha+\beta$, we know that

$$\alpha=|y|,$$

and

$$\beta=\frac{\langle x,y\rangle}{\sqrt{1+|x|^2}}.$$

Therefore,

$$\overline{F}=\alpha+\frac{\beta^2}{\alpha}=|y|+\frac{\langle x,y\rangle^2}{|y|(1+|x|^2)},$$

and Lemma 6.3.4 implies that $\overline{F}$ is not locally projectively flat.

Chapter 7

Summary and Conclusions

We shall summarize the results obtained throughout the research as follows. Considering a Finsler space, $\Im^n = (M, F)$ with $G^k = \rho_r y^r y^k \neq 0$, where $\rho(x)$ is a scalar function on M and $\rho_r = \partial\rho / \partial x^r$, we discovered that $\Im^n$ is a Riemannian space of non-zero constant sectional curvature given by $K = (\rho_i \rho_j - \rho_{ij}) y^i y^j / F^2$ and $\rho(x)$ satisfies the following system of partial differential equations given by the equation (4-39) in Proposition 4.3.1.

$$2(\rho_{ij}\rho_k + \rho_{jk}\rho_i + \rho_{ik}\rho_j) = \rho_{ijk} + 4\rho_i\rho_j\rho_k.$$

By considering the two solutions of the above system of partial differential equations, we found two standard Riemannian metrics of non-zero constant sectional curvature given by Example (i) and (ii) of Examples 4.3.1. Considering the Riemannian metric given in Example (ii) of Examples 4.3.1, in chapter 5, we were able to introduce two new locally projectively flat Randers metrics of scalar curvature. We proved that these locally projectively flat Randers metrics do not have isotropic S-curvature. Hence from these Examples, we concluded that some locally projectively flat Randers metrics of scalar curvature do not have isotropic S-curvature. Further in chapter 5, we also proved that the scalar curvature of a Randers metric is not necessarily a constant if the metric has isotropic S-curvature and closed one-form by using an example.

In chapter 6, we studied the locally projectively flat Finsler spaces with (α,β)-metric. Considering each type of locally projectively flat Finsler space with (α,β)-metric, in Theorem 6.2.1, we proved that if α is locally projectively flat and β is closed, then all locally projectively flat Finsler spaces with (α,β)-metric are either locally Minkowski spaces or Riemannian spaces of non-zero constant sectional curvature. Note that Theorem 6.2.1 is true for any Finsler spaces with (α,β)-metrics which are not Randers type.

In Theorem 6.2.10, we found the necessary and sufficient conditions for Finsler spaces with (α,β)-metric to be not locally projectively flat when α is locally projectively flat and β is closed. Then we obtained some examples of not locally projectively flat Finsler spaces with (α,β)-metric using the Riemannian metrics and one-forms defined in Lemma 5.1.7 and Example 5.3.2.

Finally we studied the conditions for each type of Finsler space with (α,β)-metric to be of Douglas type. Then we proved that the not locally projectively flat Finsler spaces with (α,β)-metric given in Example (i) of Examples 6.3.1 are Kropina spaces of Douglas type but the not locally projectively flat Finsler spaces with (α,β)-metric given in Example (ii) and (iii) of Examples 6.3.1 are not of Douglas type. Further we concluded that Kropina spaces given in Example (i) of Examples 6.3.1 are not of scalar curvature.

At last we proved that if $\Im^n = (M, F = \alpha + \beta)$ is a Randers space with isotropic S-curvature and β is closed then $\overline{\Im}^n = \left(M, \overline{F} = \alpha + \dfrac{\beta^2}{\alpha}\right)$ is not locally projectively flat when $n > 2$.

However, further research has to be carried out to determine the curvatures of the metrics given by Example (ii) and (iii) of Examples 6.3.1. In addition, we expect to classify locally projectively flat Finsler spaces with (α,β)-metric of constant flag curvature.

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