Resolving Decomposition By Blowing Up Points And Quasiconformal Harmonic Extensions

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Abstract

In this thesis we consider two problems regarding mappings between various two-dimensional spaces with some constraint on their distortion.

The first question concerns the use of mappings of finite distortion that blow up a point where the distortion is in some L^p class; in particular, we are interested in minimal solutions to the appropriate functional. We first prove some results concerning these minimal solutions for a given radially symmetric metric (in particular the Euclidean and hyperbolic metrics) by proving a theorem which states the conditions under which a minimizer exists, as well as providing lower bounds on the L^p-norm of the function. We then apply these results to the problem of resolving decompositions that arise in the study of Kleinian groups and the iteration of rational maps. Here we prove a result concerning for which values of p we can find a mapping of a particular form which shrinks the unit interval and whose inverse has distortion in the L^p space.

The second is in regards to the Schoen conjecture, which expresses the hope that every quasisymmetric self-mapping of the unit circle extends to a homeomorphism of the disk which is both quasiconformal and harmonic with respect to the hyperbolic metric. The equation for a harmonic map between Riemann surfaces with given conformal structures is a nonlinear second order equation; one wishes to solve the associated boundary value problem. We show here that the existence question can be related to a nonlinear inhomogeneous Beltrami equation and discuss some of the consequences; this result holds in more generality for other conformal metrics as well.

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Chapter 1

Introduction

The theory of quasiconformal mappings began with Grötzsch [16] in 1928, but its importance became more apparent during the 1930s, principally from the work of Ahlfors and Teichmüller. Ahlfors used quasiconformal mappings in his work in value distribution theory, and in 1935 he coined the term "quasiconformal" [3] in work that earned him a Fields medal. Teichmüller used quasiconformal mappings to create a metric on the set of conformally inequivalent compact Riemann surfaces, which started Teichmüller theory [50].

As a brief aside, quasiconformal mappings can be made for not just Riemann surfaces, but Riemann manifolds in general. However, in this thesis we are generally interested solely in the two-dimensional case.

It seems fitting to mention these three here, as all of them have played an active role in forming the background results for the two major problems with which this thesis is concerned.

Quasiconformal mappings have numerous applications in a variety of fields: holomorphic dynamical systems, singular integral operators, inverse problems, the geometry of mappings, and the calculus of variations. Like most such ideas it is founded on a simple concept, namely being a generalised version of a conformal mapping, a very core concept from complex analysis, and many results are known about them. Seeing which of these results we can generalise by the use of quasiconformal mappings is therefore of a great deal of interest.

Alongside quasiconformal mappings there are two other particularly notable and related classes of mappings in this area: the concepts of quasisymmetry and of finite distortion. In Chapter 2 we shall present some more detailed definitions and details on this topic necessary for the remainder of the thesis.

1.1 Resolving Decompositions By Blowing Up Points

In the first part of this thesis we will be investigating functions that blow up a point: we shall define this more precisely in Chapter 3, but for now we will illustrate it with Figure 1.1.1 on page 2. These are orientation-preserving mappings, generally with some control on their distortion in the cases that we are interested in, which transforms a punctured disk into an annulus.

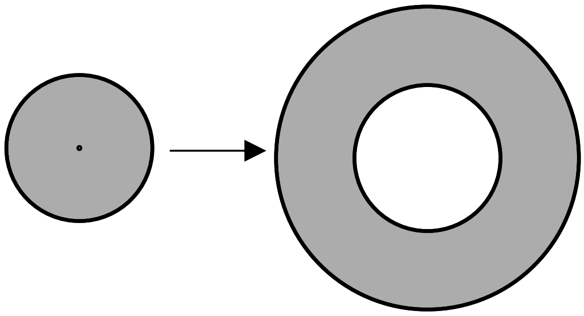


Figure 1.1.1: Blowing up a point: an orientation preserving mapping.

We first want to discuss such mappings in the setting where we aim to minimise some functional of the distortion (say the L^p-norm) and the domain has a metric measure, for instance Euclidean or hyperbolic. We would also like to retain some control in some Sobolev space (for example $W^{1,2}_{loc}$) on the inverse map. In Chapter 3 we first show there are minimisers in L^p for all $p < \infty$ in the Euclidean case (or any flat metric of finite area). In the general curved case the geometry of the metric obviously comes into play and the results are quite different. We handle this case by applying and generalising the work of Martin and McKubre-Jordens [35] to construct suitable bounds on the minimisers.

Having identified the structure of these extremal mappings, we then seek to apply these results to the problem of resolving decompositions of the plane that arise naturally in the study of Kleinian groups (shrinking a curve or lamination on a surface; see Figure 1.1.2 on page 3) and iteration of rational maps (dynamically defined combinatorial correspondences), among other things. We do not identify the extremal mappings in these cases of course, but expect that by basing our constructions around these extremal mappings we obtain optimal (or at least close to optimal) regularity.

To approach these questions analytically we will need a good class of geometric mappings. Such a class consists of the mappings of finite distortion. In this thesis we will assume that our mappings f are (orientation preserving) homeomorphisms, away from a countable set or a small set of dimension less than two - the singular set, and of finite distortion, a precise definition will be given in Section 4.1.

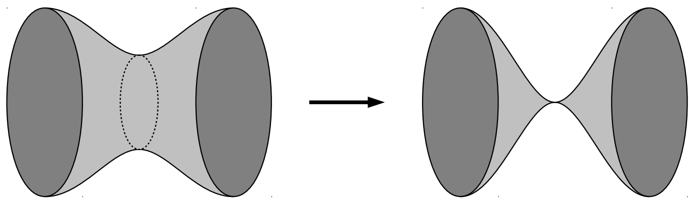


Figure 1.1.2: Shrinking a curve on a surface.

A typical example where one might want to blow up points to resolve a decomposition comes from the theory of dynamical systems where one seeks to resolve the structure of the Julia set of a rational mapping.

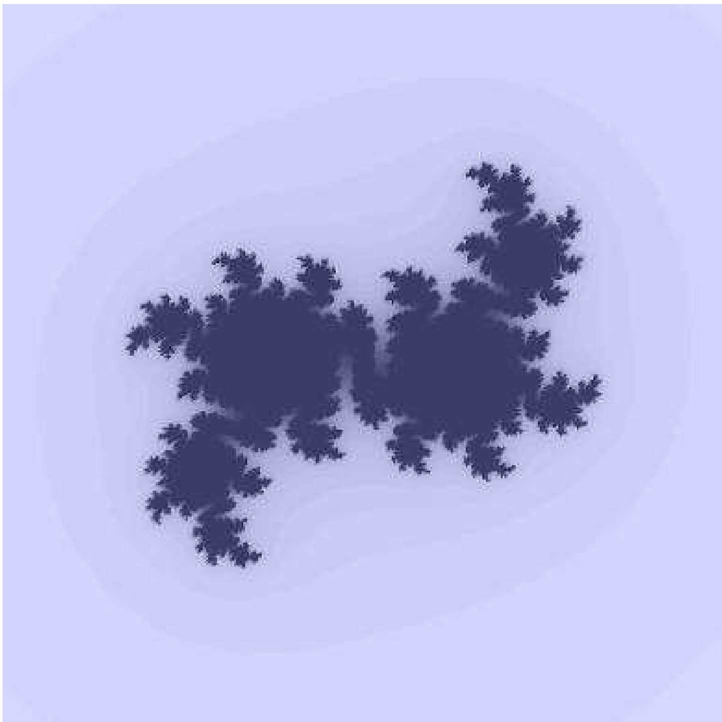


Figure 1.1.3: Julia set of a quadratic polynomial.

Here the map defined on the exterior is conjugate to z^2 by the Riemann Mapping Theorem (see Theorem 2.3), and the dynamics yield an identification pattern between pairs of points on the circle - joined by closed hyperbolic lines (subarcs of orthogonal circles) - and a model for this Julia set is the image of the plane via the quotient map of the decomposition associated with these lines. A similar situation arises when shrinking a curve on a Riemann surface: the lift of a closed geodesic to the hyperbolic disk gives a family of lines which, when pinched, give either a surface at a boundary point of Teichmüller space or a model of the limit set (which one depends on the choice of point of view of the map). See [37] for a discussion of Kleinian groups and their degenerate limits. Other examples are discussed with regard to the theory of convergence groups in [36].

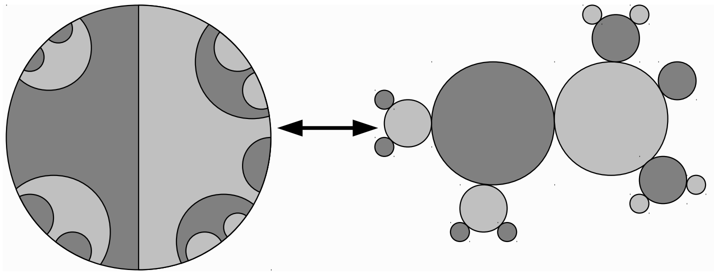


Figure 1.1.4: Quotient map from left to right. Blowing up points from right to left.

There is no map $\mathbb{C} \setminus E \to \mathbb{C} \setminus \{0\}$, where E is a non-degenerate continuum, by a mapping whose distortion is in any reasonable integrability class (i.e. L^1 or better); see 3.18 for a proof of this result. However, we can blow up points with distortion in a good class, and thus opening up the possibility of finding an analytic inverse to these quotient maps. Although we could choose to resolve these singularities by blowing up bigger sets, for certain applications (and to stay in sufficiently nice Sobolev classes) we require the singular set to be as small as possible.

By *resolving a decomposition* we mean taking a planar curve separating the plane into a countable collection of disjoint domains Ω_i, say with a subcollection touching at the origin, $\bigcap_{i\in I} \overline{\Omega_i} = \{0\}$ (where I is the (possibly countable) subcollection's indexing set). We then seek a mapping f of the plane which is of finite distortion and a homeomorphism away from the origin so that $\operatorname{int}\left(f\left(\bigcup_{i\in I}\overline{\Omega_i}\right)\right)$ is a domain, thereby reducing the (possibly infinite) number of components. Resolutions of decompositions can be achieved by blowing up points and in Chapter 4 we investigate what can be done in regards to mappings whose distortion is in an L^p class.

1.2 Quasiconformal Extensions

The definition of quasiconformal mappings (which we shall come to in Chapter 2) requires them to be defined on open sets; however, it is often useful to have a similar property on more arbitrary planar subsets. Quasisymmetry serves this purpose; it was introduced by Ahlfors and Beurling on the real line [5], and for general metric spaces by Tukia and Väisälä [52].

Although we will give more precise definitions later, we shall introduce the two variants of quasisymmetric mappings. A self-mapping f of the real line is M-quasisymmetric (where $M \geq 1$) if for all $x \in \mathbb{R}$ and all $h \in \mathbb{R} \setminus \{0\}$.

$$\frac{1}{M} \leq \frac{f(x+h) - f(x)}{f(x) - f(x-h)} \leq M.$$

A mapping f defined on any subset Ω of the complex plane $\mathbb{C}$ is η-*quasisymmetric* (where η an increasing homeomorphism of $[0,\infty)$) if for each triple $z_0, z_1, z_2 \in \Omega$ we have that

$$\frac{|f(z_0)-f(z_1)|}{|f(z_0)-f(z_2)|} \leq \eta\left(\frac{|z_0-z_1|}{|z_0-z_2|}\right), \tag{1.2.1}$$

The two classes of functions (quasisymmetric and quasiconformal) are closely related: quasisymmetric mappings are, in fact, quasiconformal, and although quasiconformal mappings are not necessarily quasisymmetric, those whose domain is the entire complex plane are, and there are certain results for dealing with local equivalence. These can be found in the work of Astala, Iwaniec and Martin[6].

One of the desired properties of such a function would be to allow us to construct quasiconformal mappings by extending these quasisymmetric mappings onto open domains; in particular, if we have a quasisymmetric mapping defined on the real line (respectively, the unit circle), if we can extend this function into the upper half-plane (respectively, the unit disk) so that this new function is quasiconformal, then this will be very useful: for example, the fact that we can do this has important applications to conformal welding and Teichmüller theory. For more details on why this is so, see [6].

The Beurling-Ahlfors Extension

The first proof that we are in fact able to perform this extension process is due to Beurling and Ahlfors [5]. They first define the *dilatation* of a differentiable topological mapping $f(x,y) = u(x,y) + iv(x,y)$ by $D \geq 1$ where

$$D + \frac{1}{D} = \frac{u_x^2 + u_y^2 + v_x^2 + v_y^2}{|u_x v_y - u_y v_x|};$$

that is geometrically, the ratio between the major and minor axis of the infinitesimal ellipse obtained by mapping an infinitesimal circle of centre (x,y). Given the mapping $\tilde{f} : \mathbb{R} \to \mathbb{R}$, we define the Beurling-Ahlfors extension $f : \mathbb{H} \to \mathbb{H}$ by $f(x,y) = u(x,y) + iv(x,y)$ where

$$\begin{aligned} u(x,y) &= \frac{1}{2}\int_0^1 \tilde{f}(x+ty) + \tilde{f}(x-ty)\,dt, \\ v(x,y) &= \frac{1}{2}\int_0^1 \tilde{f}(x+ty) - \tilde{f}(x-ty)\,dt. \end{aligned}$$

We have the following theorem from [5] with refined estimates from [9] (see also [49] for a summary of the evolution of estimates):

Theorem 1.1. *There exists a quasiconformal mapping f of the upper half-planes with the boundary correspondence $x \mapsto \tilde{f}(x)$ if and only if $\tilde{f}$ is M-quasisymmetric* *for some constant $M \geq 1$. More precisely, there exists a mapping f whose dilatation $K(z, f)$ satisfies*

$$\frac{3}{4}\ln 2 \leq \liminf_{y\to 0^+} \frac{K(x+iy, f)}{\varrho(x,y)\,\varrho_m(x,y)} \leq \limsup_{y\to 0^+} \frac{K(x+iy, f)}{\varrho(x,y)\,\varrho_m(x,y)} \leq 2,$$

where

$$\varrho(x,y) = \max\left\{\frac{\tilde{f}(x+y)-\tilde{f}(x)}{\tilde{f}(x)-\tilde{f}(x-y)}, \frac{\tilde{f}(x)-\tilde{f}(x-y)}{\tilde{f}(x+y)-\tilde{f}(x)}\right\} \leq M$$

and

$$\varrho_m(x,y) = \min\left\{\varrho\left(x+\frac{y}{2},\frac{y}{2}\right), \varrho\left(x-\frac{y}{2},\frac{y}{2}\right)\right\}.$$

There are equivariant versions of this due to Tukia [51] and Douady-Earle, which we shall now discuss in more depth.

The Douady-Earle Extensions

The Douady-Earle extension [11] provides us with an extension with a particularly useful property: it has conformal naturality. That is, if the quasisymmetric function commutes with a Fuchsian group (a discrete subgroup of Möbius transformations that leaves a disk invariant), the Douady-Earle extension will also commute with this group.

Let G be the group of conformal automorphisms of $\mathbb{D}$. If G operates on X and Y, and $T : X \to Y$ is a mapping, then T is *conformally natural* if, for all $g \in G$ and $a \in X$

$$T(g \cdot a) = g \cdot T(a), \tag{1.2.2}$$

while if $G \times G$ operates on X and Y, then T is conformally natural if, for all $g_1, g_2 \in G$ and $a \in X$,

$$T((g_1, g_2) \cdot a) = (g_1, g_2) \cdot T(a). \tag{1.2.3}$$

In particular, when X is the the space $C(\mathbb{S}, \mathbb{S})$ where $\mathbb{S}$ is the unit circle

$$\mathbb{S} = \{z : \ |z| = 1\},$$

and Y is the space $C(\overline{\mathbb{D}}, \overline{\mathbb{D}})$

$$T(g_1 \circ a \circ g_2^{-1}) = g_1 \circ T(a) \circ g_2^{-1}.$$

We will assume henceforth that probability measures have no atoms to avoid going into the details of specifics on the matter.

Theorem 1.2. (Douady-Earle Extension Theorem) *There is a conformally natural extension of any quasisymmetric homeomorphism $\varphi : \mathbb{S} \to \mathbb{S}$ to a quasiconformal homeomorphism $f = E(\varphi)$ where $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, such that $E(\varphi)$ maps the identity on $\mathbb{S}$ to the identity on $\overline{\mathbb{D}}$, and if*

$$\int_{\mathbb{S}} \varphi(z)dz = 0,$$

then $f(0) = 0$.

The construction, as stated before, is in [11].

The Schoen Conjecture

The Douady-Earle extension is of particular interest because it has the property of being conformally natural. The automorphisms of the Riemann sphere (and thus the half-plane or unit disk) are Möbius transformations; these also happen to be the isometries of the Poincaré half-plane and Poincaré disk models of the hyperbolic plane. A natural question is then: is there a quasiconformal extension which is also harmonic with respect to the hyperbolic metric?

The Schoen conjecture [45] suggests that the answer to this question is "yes"; in particular, that every quasisymmetric self mapping of the unit circle extends to a quasiconformal homeomorphism of the disk that is harmonic with respect to the hyperbolic metric. Such extensions are unique by Li and Tam [30], and also, since isometries preserve the property of being harmonic, the extension would commute with the group of hyperbolic isometries (and thus be an alternative to the Douady-Earle extension operator). It would also have the nice property of producing homeomorphic harmonic maps between Riemann surfaces in a given homotopy class.

Although the conjecture has not been proven, there have been some steps made in that direction.

A Theorem Of Hardt And Wolf

In [19], Hardt and Wolf proved the following theorem (we restate it here for the two-dimensional case only, although the result was proved for higher dimensions as well):

Theorem 1.3. *The set of quasisymmetric maps $\tilde{f}:\mathbb{R}\to\mathbb{R}$ which admit a quasi-isometric harmonic extension $f:\mathbb{H}\to\mathbb{H}$ is open in the set of quasisymmetric self-maps of* $\mathbb{R}$.

Here the topology is the one derived from the distance function $d(\cdot,\cdot)$ on $\mathbb{H}$. The outline of the proof goes as follows: extend the quasisymmetric map to a quasi-isometry F; that is, show that there exists $A\geq 1, B\geq 0$ such that for all $z,w\in\mathbb{H}$,

$$\frac{1}{A}\rho_{\mathbb{H}}(z,w)-B\leq\rho_{\mathbb{H}}(F(z),F(w))\leq A\rho_{\mathbb{H}}(z,w)+B$$

(for example, by using either the Beurling-Ahlfors or Douady-Earle extension), chosen so that unit tangent vectors v have close-to-unit images:

$$|\|dF(v)\|-1|<\epsilon_1$$

for small $\epsilon_1>0$. We then divide $\mathbb{H}$ into compact isometric two-dimensional blocks B, by dyadic decomposition, and on each construct a hyperbolic isometry G_B that is close to F, namely for small $\epsilon_2>0$ we construct a hyperbolic isometry G_B on each block B so that

$$d(F(b),G_B(b))+\|(dF)_b-(dG_B)_b\|<\epsilon_2$$

for all $b\in B$, and construct the rest of G from these blocks by interpolating in neighborhoods of the dyadic 1-skeleton, and then the dyadic 0-skeleton. (By dyadic n-skeleton, we mean the subspace formed by the union of the cells, arising from the dyadic decomposition, of dimensions $m\leq n$.) Thus the tension field $\tau(G)$ has small norm, and G has the same asymptotic boundary values as F. We then construct a compact exhaustion of $\mathbb{H}$, and construct the sequence of the unique harmonic maps that agree with G on the boundaries (see [18]).

The harmonic maps are estimated by their distance from G by observing

$$\rho_{\mathbb{H}}(x+iy,u+iv)=\cosh^{-1}\left(1+\frac{(u-x)^2+(v-y)^2}{2vy}\right),$$

(also see later as (2.2.5)) and calculating the Laplacian of the hyperbolic cosine of the distance, creating a bound where this Laplacian is at most zero. The energy density of the harmonic maps is then bounded, and by the Arzelà-Ascoli theorem (see Theorem A.1) we can construct a subsequence of these harmonic maps that converge uniformly on compacta (metrizable compact spaces) to a map of small bounded distance from G, which must have the same asymptotic boundary values as G, and therefore as F.

The Harmonic Map Equation

The equation for a harmonic map u between Riemann surfaces with given conformal structures is a nonlinear second order equation called the *harmonic map equation*:

$$u_{z\bar{z}} + \phi\left(u\right) u_z u_{\bar{z}} = 0,$$

where $\phi\left(u\right) = \left(\log\, \rho\right)_u \left(u\right)$ and $\rho\left(u\right)$ is the metric density of the range Riemann surface. We will come back to this equation later (see (2.3.1)).

One wishes to solve the associated boundary value problem and this has been done in a very limited setting. In Chapter 5 we show that this existence question can be related to a nonlinear inhomogeneous Beltrami equation and discuss some of the consequences. The results hold in more generality for other conformal metrics as well.

Finally, in Chapter 6, we shall summarise the results we have obtained, as well as discuss some prospects for future research.

Chapter 2

Preliminary Topics

This chapter contains the definitions and foundation theorems used in this thesis. Since we wish to consider quasisymmetric mappings, their extension to quasiconformal mappings, and to the closely related class of mappings of finite distortion, this chapter will generally be dedicated to providing the relevant background to explain these mapping classes. We then define a few classes of functions that we will be making use of in this thesis in several places, and provide some properties of them that we will be using.

We shall first define the complex differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

where $z = x + iy \in \mathbb{C}$ and x and y are the real and imaginary parts respectively. We will often use the shorthand f_z for $\frac{\partial f}{\partial z}$ and similarly for the $\overline{z}$-derivative. Also, Df is the differential matrix of the function $f(x, y) = u(x, y) + iv(x, y)$:

$$Df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}.$$

The most fundamental concept that we are concerned with is the distortion caused by a mapping. Distortion measures the deformation to a space caused by a given mapping: for example, the mapping $f(x + iy) = 2x + iy$ on $\mathbb{C}$ stretches the plane in the horizontal direction by a factor of two. What we would like to be able to do would be to describe the distortion for an arbitrary mapping, in a way that ignores the direction of the deformation - as this may vary from region to region - but still tell us of the magnitude of this deformation.

For our purposes, we want a mathematical definition of distortion that works at the pointwise level: since a function can distort the image in different localised regions, we need a local definition, and having it be pointwise makes it simple to express when we try to find solutions which minimise this distortion. We do this by considering the distortion of circles of small radii around the point, then taking the limit as the radius goes to zero:

Definition 2.1. The *linear distortion function* is defined to be

$$H(z, f) = \limsup_{r \to 0} \frac{\max_{|h|=r} |f(z+h) - f(z)|}{\min_{|h|=r} |f(z+h) - f(z)|}.$$

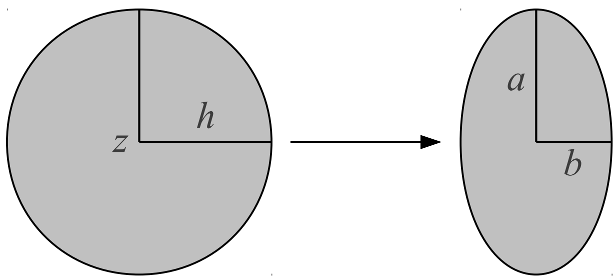


Figure 2.0.1: Diagram illustrating the effect of linear distortion.

For example, in Figure 2.0.1 on page 12, the circle of radius h centred at z has been transformed by f to the ellipse with semi-major axis a and semi-minor axis b. The linear distortion $H(z, f)$ is then a/b.

Conformal mappings are another core concept: quasiconformal mappings can be seen as a generalised form of conformal mappings, and because mappings of finite distortion are another relaxation of these same ideas, it is important to understand them as well, especially as some of our results we will be using will require that certain mappings be conformal.

Definition 2.2. A *conformal mapping* of a domain $\Omega \subset \mathbb{C}$ is a holomorphic homeomorphism. Two Riemannian metrics g and h are *conformally equivalent* if there is a positive function α^2 such that $g = \alpha^2 h$. A mapping $f : (M, g) \to (N, h)$ between two Riemannian manifolds (M, g) and (N, h) is a *conformal mapping* if the pull-back metric f^*h is conformally equivalent to g.

From the perspective of distortion, conformal mappings map infinitesimal circles to infinitesimal circles, and so for a conformal mapping f, $H(z, f) = 1$.

By *domain* we mean a connected, open subset of the complex plane. Two domains are *conformally equivalent* if there is a holomorphic homeomorphism between them. For simply connected domains, the answer as to whether two such domains are conformally equivalent is answered by the Riemann Mapping Theorem.

Theorem 2.3. (Riemann Mapping Theorem) *Let Ω be a simply connected subset of the complex plane, not equal to $\mathbb{C}$. Let $a \in \Omega$ be arbitrary. Then there exists a (unique) conformal map f from Ω onto the unit disk $\mathbb{D}$ such that $f(a) = 0$ and $f'(a) > 0$.*

An extension of this theorem is the uniformisation theorem:

Theorem 2.4. (Uniformisation Theorem) *Any simply connected surface is biholomorphically equivalent to the Riemann sphere, the complex plane or the unit disk. No two of these surfaces are conformally equivalent.*

Proofs for the Riemann Mapping Theorem and uniformisation theorem can be found in [15], [27] and [21].

Another core definition in this area is that of the symmetric map. Because we are primarily interested in these mappings on the real line and the unit circle, we will define it explicitly for both for convenience.

Definition 2.5. A mapping $f : \mathbb{R} \to \mathbb{R}$ is *symmetric* if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{f(x) - f(x-h)} = 1, \tag{2.0.1}$$

and a mapping $f : \mathbb{S} \to \mathbb{S}$ is *symmetric* if

$$\lim_{h \to 0} \frac{f\left(e^{i(\theta+h)}\right) - f\left(e^{i\theta}\right)}{f\left(e^{i\theta}\right) - f\left(e^{i(\theta-h)}\right)} = 1. \tag{2.0.2}$$

2.1 Möbius Transformations

Möbius transformations are mappings of the form

$$z \mapsto \frac{az+b}{cz+d}, \tag{2.1.1}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$, (or $ad - bc = 1$ as we may cancel out any common factors of the coefficients without changing the mapping). These mappings form a group which we can identify with the projective linear group $PSL(2, \mathbb{C})$. The groups of Möbius transformations that we discussed earlier are subgroups of this group whose elements fix the real line (in the case of the hyperbolic half-plane) or the unit circle (in the case of the disk).

A *Kleinian group* is a discrete subgroup of Möbius transformations; a Kleinian group is *Fuchsian* if it it leaves a disk invariant.

2.2 The Hyperbolic Plane

This thesis will focus on two-dimensional Riemannian manifolds. While we will be talking about certain results with regards to general metrics, we are particularly interested in the hyperbolic metric. A Riemannian manifold is hyperbolic if it is of constant negative sectional curvature; and the hyperbolic plane is thus a hyperbolic Riemannian manifold of two dimensions. There are several models of the hyperbolic plane: the hyperboloid model, the Klein model, the Poincaré disk and half-plane models. We will be making use of the latter two, so we shall go into further detail for them. We will also briefly mention the hyperbolic metric on the punctured unit disk.

2.2.1 Poincaré Disk

The Poincaré disk model uses the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ equipped with the Riemannian metric

$$ds_{hyp\mathbb{D}}(z) = \frac{2\,|dz|}{1 - |z|^2}, \tag{2.2.1}$$

and thus the metric

$$\rho_{\mathbb{D}}(z, w) = \inf_{\gamma} \int_{\gamma} ds_{hyp\mathbb{D}} = \ln \frac{|1 - \bar{z}w| + |z - w|}{|1 - \bar{z}w| - |z - w|}, \tag{2.2.2}$$

where the infimum is taken of all rectifiable curves γ joining z to w in $\mathbb{D}$. Lines are in fact represented by diameters of the boundary circle as well as the arcs of circles orthogonal to the boundary.

The group of all isometries is the group of Möbius transformations

$$z \mapsto e^{i\theta} \frac{z - a}{1 - \bar{a}z}, \quad a \in \mathbb{D}, \theta \in [0, 2\pi) \tag{2.2.3}$$

and their complex conjugates. We denote $G_{\mathbb{D}}$ as the group of these isometries, and $G_{\mathbb{D}}^{+}$ as the group of Möbius transformations.

2.2.2 Poincaré Half-Plane

The Poincaré half-plane model uses the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} : \ \Im(z) > 0\}$$

equipped with the Riemannian metric

$$ds_{hyp\mathbb{H}} = \frac{|dz|}{\Im(z)} \tag{2.2.4}$$

and metric

$$\rho_{\mathbb{H}}(x+iy, u+iv) = \cosh^{-1}\left(1 + \frac{(u-x)^2 + (v-y)^2}{2vy}\right). \tag{2.2.5}$$

The isometries here are similarly the Möbius transformations

$$z \mapsto \frac{az+b}{cz+d}, \tag{2.2.6}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc \neq 0$. We shall denote this group by $G_{\mathbb{H}}$, which we note is isomorphic to $PGL(2,\mathbb{R})$, and the group of orientation-preserving isometries by $G_{\mathbb{H}}^{+}$, which is isomorphic to $PSL(2,\mathbb{R})$.

The mapping

$$z \mapsto i\frac{1-z}{1+z}$$

is an isometry from the hyperbolic metric of the disk to the hyperbolic metric of the upper half-plane. The inverse mapping is

$$z \mapsto \frac{i-z}{i+z}.$$

We will primarily be using the Poincaré disk model, although we will make use of the isometry and its inverse to choose a model which suits the particular problem at hand better.

2.2.3 Hyperbolic Punctured Disk

As a brief aside, we will mention the hyperbolic metric on the punctured disk later as an illustration, so we shall formulate it here although it is not used as frequently as the other two in this thesis. The hyperbolic metric on the punctured unit disk $\mathbb{D}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ is

$$ds = \frac{|dz|}{|z| \ln |z|}. \tag{2.2.7}$$

2.3 Harmonic Mappings

Harmonic mappings are critical points of the Dirichlet energy functional

$$E(f) = \frac{1}{2} \int_M \|df\|^2 (\det\, g)^{1/2}\, dz$$

on functions $f : (M, g) \to (N, h)$ between Riemannian manifolds (M, g) and (N, h). Often we are interested in these critical points; in particular, there are numerous physical models that require some system to be in lowest energy state possible. Harmonic maps have some other nice properties, though we shall leave that for other sources to go into, for example, [12] and [25].

When examining mappings between Riemannian manifolds with various metrics, we often want the properties that we examine to be independent of the coordinate system. The Dirichlet energy functional of such a mapping is one geometrically important invariant of these mappings. Given compact oriented Riemannian surfaces M and N without boundary, with metrics $\lambda(z)\,|dz|^2$ and $\rho(u)\,|du|^2$ respectively, we can represent the energy density by

$$|du|^2 = 2\frac{\rho(u(z))}{\lambda(z)} \left(|u_z(z)|^2 + |u_{\bar{z}}(z)|^2\right)$$

and so the Dirichlet energy functional as

$$E(u) = \int_M |du|^2\, \lambda(z)\ |dz|^2 = 2\int_M \rho(u(z)) \left(|u_z(z)|^2 + |u_{\bar{z}}(z)|^2\right)\ |dz|^2 .$$

From this we can derive (see [46]) the harmonic map equation as the Euler-Lagrange equation for the minimiser of $E(f)$.

$$u_{z\bar{z}} + (\ln\, \rho)_u\,(u)\, u_z u_{\bar{z}} = 0. \tag{2.3.1}$$

Hence for mappings between copies of the hyperbolic disk, the associated harmonic map equation is

$$u_{z\bar{z}} + \frac{2u}{1-|u|^2} u_z u_{\bar{z}} = 0, \tag{2.3.2}$$

while for the hyperbolic half-plane, it is

$$u_{z\bar{z}} + \frac{i}{\Im(u)} u_z u_{\bar{z}} = 0. \tag{2.3.3}$$

We will also note that for Euclidean metrics this gives us the expected definition for the complex plane in complex analysis (see for example [8]). In this case we have that $\rho(z) = 1$ and so $(\log\, \rho)_z = 0$.

2.4 L^p-Spaces And Sobolev Spaces

The definitions of the core concepts of bounded and finite distortion make use of L^p and Sobolev spaces, amongst others. We will provide a brief introduction to them here.

For $1 \leq p \leq \infty$, we denote the standard L^p-norm for mappings of $\Omega \subset \mathbb{C} \to \mathbb{C}$ as usual:

$$\|f\|_p = \begin{cases} \left(\int_\Omega |f|^p \rho(z) \, |dz|^2\right)^{1/p} & p < \infty, \\ \inf\{C \geq 0 : \, |f(z)| \leq C \text{ for almost every } x\} & p = \infty, \end{cases}$$

and say that $f \in L^p(\Omega)$ if $\|f\|_p < \infty$. Note that$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$ if $f \in L^\infty(\Omega) \cap L^q(\Omega)$ for some $q < \infty$. More general discussion of the L^p-norms can be found in [38].

The Sobolev norm $\|f\|_{k,p}$ we define by

$$\|f\|_{k,p} = \begin{cases} \left(\sum_{|\alpha|\leq k} \|D^\alpha f\|_p^p\right)^{1/p} & p < \infty, \\ \sum_{|\alpha|\leq k} \|D^\alpha f\|_\infty & p = \infty, \end{cases}$$

where $D^\alpha f$ denotes the weak partial derivative associated with the multi-index α; by *weak partial derivative* of $f \in L^1_{loc}(\Omega)$, we mean any $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega f D^\alpha \varphi \, |dz|^2 = (-1)^{|\alpha|} \int_\Omega g\varphi \, |dz|^2$$

for all infinitely differentiable functions φ with compact support in Ω.

The Sobolev space $W^{k,p}(\Omega)$ is the space of functions $f : \Omega \to \mathbb{C}$ where $\|f\|_{k,p} < \infty$.

2.4.1 Local L^p-Spaces and Local Sobolev Spaces

The definition of quasiconformality will depend on local versions of these: $f \in L^p_{loc}(\Omega)$ (resp. $W^{k,p}_{loc}(\Omega)$) if $f \in L^p(\Omega')$ (resp. $W^{k,p}(\Omega')$) for every relatively compact subdomain Ω' with $\overline{\Omega'} \subset \Omega$. More information on Sobolev spaces can be found in [41].

2.5 Distortion And Quasiconformality

The major results of this thesis centre around finding families of functions, in particular minimizing functions, where there is some integral limit applied to (some simple function of) some measurement of distortion: the mappings of bounded distortion (quasiconformal functions) and the mappings of finite distortion. To explain these concepts, and to introduce related concepts that will appear in this thesis, we begin with a brief introduction to the Beltrami equation, which we will be using to help define different measures of distortion.

2.5.1 The Beltrami Equation And Complex Dilatation

Let D be a domain in the complex plane $\mathbb{C}$, and $\mu : D \to \mathbb{C}$ be a measurable function. The *Beltrami equation* is then given by

$$f_{\bar{z}} = \mu(z) f_z \quad \text{for almost every } z \in \mathbb{D}. \tag{2.5.1}$$

Because we are interested with this in terms of quasiconformal mappings, we note that if $\|\mu\|_\infty = k < 1$ then the solution to the Beltrami equation is a quasiconformal mapping with *complex dilatation* μ, also known as the *Beltrami coefficient.* If we are given a function f, we may write the associated Beltrami coefficient as μ_f.

We note some results associated to this from [24]. First is the transformation for the Beltrami coefficient of a composition:

$$\mu_{f\circ g^{-1}}(g(z)) = \frac{\mu_f(z) - \mu_g(z)}{1 - \mu_f(z)\overline{\mu_g(z)}} \left(\frac{g_z(z)}{|g_z(z)|} \right)^2. \tag{2.5.2}$$

We also note the relation derived from the observations from the chain rule Theorem A.2 between the Beltrami coefficient of f and that of its inverse g:

$$\mu_f(g(z)) = \frac{f_{\bar{w}}(g(z))}{f_w(g(z))} = -\frac{g_{\bar{z}}(z)}{\overline{g_z(z)}} = -\mu_g(z) \frac{g_z(z)}{\overline{g_z(z)}}. \tag{2.5.3}$$

2.5.1.1 The Hopf Differential

The *Hopf differential* is defined on mappings $f : M \to N$ between Riemannian manifolds M and N, respectively equipped with the metrics

$$ds_M^2 = \lambda(z)\,|dz|^2 \qquad \text{and } ds_N^2 = \rho(f(z))\,|df|^2$$

by

$$\Phi_f(z)\,dz^2 = \rho(f(z)) f_z(z) \overline{f_{\bar{z}}(z)} dz^2. \tag{2.5.4}$$

Note that f is harmonic if and only if the Hopf differential Φ_f is holomorphic [46].

2.5.2 Quasiconformal Mappings

Quasiconformal mappings are a generalization of conformal mappings. Where conformal mappings are mappings that are distortion-free, quasiconformal mappings are principally mappings of bounded distortion.

A mapping $f : \Omega \to \Omega'$ has bounded distortion if, for all $z \in \Omega$, we have that

$$H(z, f) \leq K < \infty.$$

However, because this definition can be difficult to work with, we shall instead use an alternate form as our definition for quasiconformal mappings, which will be formulated here in terms of the Beltrami coefficient.

Definition 2.6. (Quasiconformality)The distortion function $K(z, f)$ for a mapping f at a point z is defined by

$$K(z, f) = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}. \tag{2.5.5}$$

A mapping $f : \Omega \to \Omega'$ is *quasiconformal* if f is orientation preserving, $f \in W^{1,2}_{loc}(\Omega)$ and if

$$\|K(z, f)\|_\infty < \infty.$$

In particular, we say that a mapping f is *K-quasiconformal* if $K \geq \|K(z, f)\|_\infty$.

In Chapter 5 and any other time we discuss quasiconformal mappings, this will be the formulation for distortion that we mean. A different definition will be required when we discuss mappings of finite distortion; we will introduce this definition later.

2.5.3 Some Established Results Of Quasiconformal Mappings

Here we shall state some important previously-established results of quasiconformal mappings.

Theorem 2.7. (Existence Theorem)

Let $\mu : \mathbb{D} \to \mathbb{D}$ be a measurable function with $\|\mu\|_\infty = k < 1$. Then there is a quasiconformal homeomorphism $f : \mathbb{D} \to \mathbb{D}$ with complex dilatation equal to μ,

$$\mu_f(z) = \mu(z), \qquad a.e. \quad z \in \mathbb{D}.$$

See [1] for details. In fact we have the stronger result of the measurable Riemann mapping theorem from [4] that this correspondence is holomorphic: if μ depends holomorphically on a complex variable, then so does f.

Theorem 2.8. (Composition and Inversion of Quasiconformal Mappings) *Let $f : \Omega \to \Omega'$ be a K-quasiconformal mapping from the domain $\Omega \subset \mathbb{C}$ onto $\Omega' \subset \mathbb{C}$, and let $g : \Omega' \to \mathbb{C}$ be a K'-quasiconformal mapping. Then*

- *$f^{-1} : \Omega' \to \Omega$ is K-quasiconformal,*
- *$g \circ f : \Omega \to \mathbb{C}$ is KK'-quasiconformal, and*
- *for all measurable sets $E \subset \Omega$, $|E| = 0$ if and only if $|f(E)| = 0$.*

For the proof of these results, see [6].

Lemma 2.9. (Quasiconformality, Harmonicity and the Hopf differential) *The complex dilatation of a harmonic quasiconformal mapping is the product of a real-valued function and an anti-holomorphic function (that is, the complex conjugate of a holomorphic function).*

Proof. Because f is quasiconformal, we can rewrite Φ_f as

$$\Phi_f(z) = \rho(f(z))\,|f_z(z)|^2\,\overline{\mu_f(z)},$$

hence μ_f can be factored in the manner described, after rearrangement and the fact that Φ_f is holomorphic.

□

2.5.4 Quasisymmetric Mappings

The definition of quasiconformality requires that the function is defined on an open set of $\mathbb{C}$. However, there are certain places where this will not be the case: for example, we consider quasiconformal extensions of functions from the unit circle $\mathbb{S}$ to the unit disk $\mathbb{D}$; however $\mathbb{S}$ is not open in $\mathbb{C}$. Therefore we would like to have a more general form of quasiconformality. The concept of quasisymmetry, which was introduced by Ahlfors and Beurling[5] for the real line and for general metric spaces by Tukia and Väisälä[52], gives us a satisfactory alternative.

Definition 2.10. Let $\eta : [0, \infty) \to [0, \infty)$ be an increasing homeomorphism, $\Omega \subset \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ a mapping, and orientation preserving if Ω is open. Then f is *η-quasisymmetric* if for each triple $z_0, z_1, z_2 \in \Omega$ we have that

$$\frac{|f(z_0) - f(z_1)|}{|f(z_0) - f(z_2)|} \leq \eta \left(\frac{|z_0 - z_1|}{|z_0 - z_2|} \right), \tag{2.5.6}$$

and f is *quasisymmetric* if some such η exists.

We may also speak of M-quasisymmetric mappings, where M is a constant, when f is a self-mapping of the real line $\mathbb{R}$.

Definition 2.11. Let $M \geq 1$. Then $f : \mathbb{R} \to \mathbb{R}$ is *M-quasisymmetric* if for all $x \in \mathbb{R}$ and all $h \in \mathbb{R} \setminus \{0\}$.

$$\frac{1}{M} \leq \frac{f(x+h) - f(x)}{f(x) - f(x-h)} \leq M.$$

Quasisymmetric mappings defined on open subsets are mappings of bounded distortion, as can be seen in [6]. From here as well we have that the inverse of quasisymmetric mappings are quasisymmetric.

2.5.5 Mappings Of Finite Distortion

Although quasiconformal mappings are an important class of functions, there are some limitations: some problems which can be modelled by the Beltrami equation (2.5.1) where again $|\mu| < 1$ almost everywhere, but it might be the case that

$$\|\mu\|_\infty = 1.$$

Such problems arise in several areas, such as in two-dimensional hydrodynamics when the flow approaches a critical value, and in holomorphic dynamics when examining the flow of quasicircles of the Julia set of $\lambda z + z^2$ as $|\lambda| \to 1$. Obviously we would like to have solutions to these problems as well, so we consider a class of functions derived from the relationship between the Beltrami equation and the distortion inequality (2.5.5).

We refer to [6] for further details on this class of mappings and their role in modern geometric function theory and analysis.

Definition 2.12. A mapping $f : \Omega \to \mathbb{C}$ is a *mapping of finite distortion* if $f \in W^{1,1}_{loc}(\Omega)$, $J(z,f) \in L^1_{loc}(\Omega)$, and there is a measurable function $\mathbb{K}(z,f)$, finite almost everywhere, such that f satisfies the distortion inequality

$$\|Df\|^2 \leq \mathbb{K}(z,f) J(z,f).$$

Principally in Chapter 3 and Chapter 4, where we are primarily interested in mappings of finite distortion (rather than quasiconformal mappings), when we refer to distortion we will be using it to refer to $\mathbb{K}(z, f)$.

Note the similarity to the definition of mappings of bounded distortion, or quasiconformal mappings. The differences are that f is in the class of $W^{1,1}_{loc}$ rather than $W^{1,2}_{loc}$, the addition of the integrability condition of the Jacobian, and the use of the *Hilbert-Schmidt norm*

$$\|A\|^2 = \frac{1}{2}\operatorname{tr}\left(A^t A\right)$$

rather than the *operator norm*

$$|A|^2 = \max_{|\zeta|=1} |A\zeta|.$$

(Note these are not the most general of definitions, but are sufficient for our applications).

We note that $\mathbb{K}(z, f)$ can also be written as

$$\mathbb{K}(z,f) = \frac{1+|\mu_f(z)|^2}{1-|\mu_f(z)|^2}.$$

OFrom this formula, it is clear that $\mathbb{K}(z,f) \leq K(z,f)$ so all quasiconformal mappings are mappings of finite distortion. Also note that, as we are dealing with two-dimensional spaces, a linear mapping can be written as a linear matrix $A(z)$; if $A^t(z)A(z)$ has eigenvalues $\lambda_1(z), \lambda_2(z)$ then

$$K(z, A) = \max\left\{\frac{\lambda_1(z)}{\lambda_2(z)}, \frac{\lambda_2(z)}{\lambda_1(z)}\right\},$$

whereas

$$\mathbb{K}(z, A) = \frac{1}{2}\left(\frac{\lambda_1(z)}{\lambda_2(z)} + \frac{\lambda_2(z)}{\lambda_1(z)}\right),$$

and so, where the eigenvalues cross, $\mathbb{K}$ is differentiable whereas K is not. We also have

$$\mathbb{K}(z, f) = \frac{1}{2}\left(K(z,f) + \frac{1}{K(z,f)}\right).$$

As this is a convex function of $K(z,f)$, the L^∞ minimisers will be the same (see [6]).

We shall list some additional properties of this distortion similar to that used for quasiconformality. Suppose that φ is conformal; then $\varphi_{\overline{z}} = 0$ and so

$$\begin{aligned}
\mu_{f\circ\varphi}(z) &= \frac{(f\circ\varphi)_{\overline{z}}(z)}{(f\circ\varphi)_{z}(z)} \\
&= \frac{f_{\overline{z}}(\varphi(z))\,\overline{\varphi_z(z)}}{f_z(\varphi(z))\,\varphi_z(z)}.
\end{aligned}$$

Thus $|\mu_{f\circ\varphi}(z)| = |\mu_f(\varphi(z))|$ and, as a consequence,

$$\mathbb{K}(z, f\circ\varphi) = \mathbb{K}(\varphi(z), f). \tag{2.5.7}$$

Similarly

$$\begin{aligned}
\mu_{\varphi\circ f}(z) &= \frac{(\varphi\circ f)_{\overline{z}}(z)}{(\varphi\circ f)_{z}(z)} \\
&= \frac{\varphi_z(f(z))\,f_{\overline{z}}(z)}{\varphi_z(f(z))\,f_z(z)} \\
&= \frac{f_{\overline{z}}(z)}{f_z(z)} = \mu_f(z),
\end{aligned} \tag{2.5.8}$$

so

$$\mathbb{K}(z, \varphi\circ f) = \mathbb{K}(z, f). \tag{2.5.9}$$

We also have the following proof due to Hencl, Koskela and Onninen [20]

Theorem 2.13. *Let $f \in W^{1,1}_{loc}(\Omega, \Omega')$, where Ω, Ω' are bounded domains in $\mathbb{C}$, be a homeomorphism of finite distortion with*

$$\int_{\Omega} \mathbb{K}(x, f)\, dx < \infty.$$

Then the inverse map $h : \Omega' \to \Omega$ belongs to $W^{1,2}(\Omega', \Omega)$ and

$$\int_{\Omega'} |Dh(y)|^2\, dy = \int_{\Omega} \mathbb{K}(x, f)\, dx.$$

2.6 Separable Functions

We say that a function f of $n \geq 2$ variables $\{x_i : i = 1, \ldots, n\}$ is *separable* if it can be written in the form

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_i(x_i).$$

In this thesis we will often make use of certain classes of separable functions, as they are often useful classes for simplifying problems. In particular, we will see that a class of such functions form the minimisers we seek for Theorem 3.13.

2.6.1 Radial Stretchings And Radially Symmetric Mappings

For a few reasons, not the least of which is the frequency with which we will be doing calculations in polar coordinates, we will introduce a class of functions where the argument is kept fixed. They also form very useful classes when examining problems on (subsets of) the unit disk due to being the class of mappings the minimal solutions to Theorem 3.13 fall into when transformed into the Nitsche version of that problem.

Let $\rho : [0,\infty] \to [0,\infty]$ be a piecewise differentiable, strictly increasing function. We then say $f : \mathbb{C} \to \mathbb{C}$ is a *radial stretching* if it is of the form

$$f(z) = \frac{z}{|z|}\rho(|z|).$$

For functions with domains as subsets of $\mathbb{C}$, in particular for mappings from the unit disk $\mathbb{D}$ to itself, we will say that such functions are radial stretchings if they are restrictions of a radial stretching to the appropriate domain, and if the mapping preserves boundaries.

We note some results from [24] that will be convenient. First the complex partial derivatives of a harmonic mapping are given by

$$f_z = \frac{1}{2}\left[\dot{\rho}(|z|) + \frac{\rho(|z|)}{|z|}\right],$$

and

$$f_{\bar{z}} = \frac{1}{2}\left[\dot{\rho}(|z|) - \frac{\rho(|z|)}{|z|}\right]\frac{z}{\bar{z}}.$$

The Jacobian is then

$$J(z,f) = |f_z|^2 - |f_{\bar{z}}|^2 = \frac{\dot{\rho}(|z|)\,\rho(|z|)}{|z|} \tag{2.6.1}$$

and Beltrami coefficient is then

$$\mu_f = \frac{|z|\,\dot{\rho}(|z|) - \rho(|z|)}{|z|\,\dot{\rho}(|z|) + \rho(|z|)}\frac{z}{\bar{z}},$$

so

$$K(z,f) = \max\left\{\frac{|z|\,\dot{\rho}(|z|)}{\rho(|z|)}, \frac{\rho(|z|)}{|z|\,\dot{\rho}(|z|)}\right\}$$

and

$$\mathbb{K}(z,f) = \frac{1}{2}\left(\frac{|z|\,\dot{\rho}(|z|)}{\rho(|z|)} + \frac{\rho(|z|)}{|z|\,\dot{\rho}(|z|)}\right). \tag{2.6.2}$$

Note that the inverse of a radial stretching is a radial stretching. We will also use the term *anti-radial stretching* to refer to a mapping that is the complex conjugate of a radial stretching, that is a function of the form

$$f(z) = \frac{\bar{z}}{|z|}\rho\left(|z|\right).$$

A similarly useful class is that of *radially symmetric functions*: the only difference is that ρ need not be strictly increasing. We may similarly define *anti-radially symmetric functions* in the same manner.

2.6.2 Radially Fixed Mappings

For radially symmetric mappings, we fix the argument; we shall now propose to fix the modulus. Let $\tau : [0, 2\pi) \to [0, 2\pi)$ be a piecewise differentiable, injective function that is 2π-periodic (that is, if we extend the domain of τ to $\mathbb{R}$ the extension is 2π-periodic). We then say $f : \mathbb{C} \to \mathbb{C}$ is *radially fixed* if it is of the form

$$f\left(re^{i\theta}\right) = re^{i\tau(\theta)},$$

and we extend this definition to subdomains of $\mathbb{C}$ in a similar fashion to radial stretchings.

In this case, we have that

$$f_z\left(re^{i\theta}\right) = \frac{1}{2}\left[1 + \tau'\left(\theta\right)\right] e^{i(\tau(\theta)-\theta)}$$

and

$$f_{\bar{z}}\left(re^{i\theta}\right) = \frac{1}{2}\left[1 - \tau'\left(\theta\right)\right] e^{i(\tau(\theta)+\theta)}.$$

The Beltrami coefficient is therefore given by

$$\mu_f\left(re^{i\theta}\right) = \frac{1 - \tau'\left(\theta\right)}{1 + \tau'\left(\theta\right)} e^{i2\theta}.$$

Note that when f is orientation preserving, τ is increasing, and τ is decreasing when f is orientation reversing.

2.6.3 Polar Independent Mappings

We can combine these two concepts to create a more general class of functions: in this case, we have a function $f : re^{i\theta} \to \rho e^{i\tau}$ where ρ and τ are functions of r and θ respectively:

$$f\left(re^{i\theta}\right) = \rho\left(r\right) e^{i\tau(\theta)}.$$

The derivatives are

$$f_z\left(re^{i\theta}\right)=\frac{1}{2}\left[\rho'\left(r\right)+\frac{\rho\left(r\right)}{r}\tau'\left(\theta\right)\right]e^{i(\tau(\theta)-\theta)}$$

and

$$f_{\bar{z}}\left(re^{i\theta}\right)=\frac{1}{2}\left[\rho'\left(r\right)-\frac{\rho\left(r\right)}{r}\tau'\left(\theta\right)\right]e^{i(\tau(\theta)+\theta)}.$$

In this case the Jacobian is

$$J\left(re^{i\theta},f\right)=\frac{\rho'\left(r\right)\rho\left(r\right)\tau'\left(\theta\right)}{r}$$

and Beltrami coefficient is

$$\mu_f\left(re^{i\theta}\right)=\frac{r\rho'\left(r\right)-\rho\left(r\right)\tau'\left(\theta\right)}{r\rho'\left(r\right)+\rho\left(r\right)\tau'\left(\theta\right)}e^{i2\theta}.$$

Chapter 3

Blowing Up Points

We now define some common terms for this chapter and the following one: we define $\mathbb{D}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ to be the punctured unit disk, and also the family of annuli of inner radius 1 as $\mathbb{A}_R = \{z \in \mathbb{C} : 1 < |z| < R\}$.

Definition 3.1. We say f *blows up a point* if $f : \mathbb{D}^* \to \mathbb{A}_R$ is an orientation preserving homeomorphism with

$$\lim_{z \to 0} |f(z)| = 1, \qquad \lim_{|z| \to 1} |f(z)| = R.$$

3.1 Extremal Problems

3.1.1 Modulus Of Annuli

In this thesis we shall be making use of two similar and related concepts: that of the extremal length and modulus of a collection of (locally rectifiable) curves. Because one is the reciprocal of the other, we do not cause too much confusing by using modulus to refer to extremal length for the most part; many other results pertaining to the classic modulus we can import (with perhaps some slight adjustment where necessary) because of this reciprocity. Although the concept can be generalized, here we shall only define it in the two-dimensional case.

Definition 3.2. Let Λ be a collection of curves in $\mathbb{C}$. A non-negative Borel measure $\rho : \mathbb{C} \to \mathbb{R}$ is *admissible* if

$$\int_\lambda \rho ds \geq 1$$

for every locally rectifiable curve $\lambda \in \Lambda$. The collection of all admissible Borel measures with respect to Λ is denoted $\operatorname{adm}(\Lambda)$. We define the *modulus* to be

$$\operatorname{mod}(\Lambda) = \left[\inf_{\rho \in \operatorname{adm}(\Lambda)} \int_{\mathbb{C}} \rho^2 dm \right]^{-1}. \tag{3.1.1}$$

Where the curves are given in, or are understood to lie within, some $\Omega \subset \mathbb{C}$, then we may define

$$\operatorname{mod}(\Omega) = \left[\inf_{\rho \in \operatorname{adm}(\Lambda)} \int_{\Omega} \rho^2 dm \right]^{-1}.$$

We note some useful properties of the modulus, which we take from [34]. First, if $\Lambda_1 \subset \Lambda_2$, then $\operatorname{mod}(\Lambda_2) \leq \operatorname{mod}(\Lambda_1)$. Second, if Λ_1 is *minorised* by Λ_2 (that is, each curve $\lambda_1 \in \Lambda_1$ has a subcurve $\lambda_2 \in \Lambda_2$) then $\operatorname{mod}(\Lambda_2) \leq \operatorname{mod}(\Lambda_1)$ as well. Finally, we have the important theorem which ties this modulus in with conformal mappings.

Theorem 3.3. *The modulus of a family of curves is a conformal invariant.*

The proof for this theorem can be found in [34].

As an example, let $\mathbb{A}(r, R)$ be the annulus with inner radius r and outer radius R. If we consider the set of paths to be those lying between the inner and outer boundary components, then

$$\operatorname{mod}(\mathbb{A}(r, R)) = \frac{1}{2\pi} \ln \left(\frac{R}{r} \right).$$

A proof of this can be found in [54]. In the first half of this thesis we shall be examining mappings to and from annular regions, and as the mappings will generally be within the realm of conformal mappings and those of bounded distortion, we shall use this measure for reasons we will explain in a moment.

The Riemann mapping and uniformisation theorems (Theorem 2.3 and Theorem 2.4 respectively) classify all simply connected spaces as being biholomorphically equivalent to exactly one of $\hat{\mathbb{C}}$, $\mathbb{C}$ and $\mathbb{D}$; the next simplest case will be the doubly connected surfaces: the punctured plane $\mathbb{C} \setminus \{0\}$, the punctured disk $\mathbb{D} \setminus \{0\}$ and the annuli $\mathbb{A}(r, R)$.

We shall now mention a few pertinent results in this area. The first result is due to Schottky [47]:

Theorem 3.4 (Schottky Theorem)**.** *The annuli $\mathbb{A}_{R_1} = \mathbb{A}(1, R_1)$ and $\mathbb{A}_{R_2} = \mathbb{A}(1, R_2)$ are conformally equivalent if and only if $R_1 = R_2$. In general, there exists a conformal homeomorphism h from $\mathbb{A}(r, R)$ onto $\mathbb{A}(r', R')$ if and only if the annuli have the same modulus; thus,*

$$\frac{R}{r} = \frac{R'}{r'}.$$

Moreover, up to the rotation of the annuli, every such map takes the form

$$h(z) = \begin{cases} \frac{r'}{r} z, & \textit{if preserving the order of the boundary components, and} \\ \frac{rR'}{z} & \textit{if reversing the order of boundary components.} \end{cases}$$

We refer to [23] for the proof. We gain flexibility by considering this mapping problem for harmonic homeomorphisms (univalent complex-valued harmonic functions).

Another pertinent theorem is the following.

Theorem 3.5. *Every doubly connected region* Ω *in the complex plane is conformally equivalent to the round annulus* $\mathbb{A}(r, R)$ *with the same modulus.*

This is a special case of a result found in [2, 255-256].

3.1.2 Nitsche-Type Extremal Problems

In [42], Nitsche demonstrated that a harmonic homeomorphism between annuli does not exist if the image annulus is too conformally thin in comparison, which led him to conjecture the following:

Theorem 3.6. (Nitsche Conjecture) *A harmonic homeomorphism* $h : \mathbb{A}(r, R) \to \mathbb{A}(r', R')$ *onto exists if and only if*

$$\frac{R'}{r'} \geq \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right).$$

This inequality is known as the *Nitsche bound.* In [22] this was proven by Iwaniec, Kovalev and Onninen when the domain annulus satisfies $\log \frac{R}{r} \leq \frac{3}{2}$, and in general under the assumption that h or its normal derivative has vanishing average on the inner boundary circle. It was then proved in full in [23].

In our situation, we wish to do something similar; but instead of minimizing some functional of the energy, we wish to minimise some functional of the distortion; so in our case we are interested in the inverse mapping. In [35], Martin and McKubre-Jordens generalised this approach to solve a class of similar problems by post-transformation of the distortion $\mathbb{K}$ by a convex function Φ; for our purposes, we will be particularly interested in the cases where $\Phi : x \mapsto x^p$, for they provide us with L^p-bounds and related results.

Definition 3.7. Consider $f : \mathbb{A}_R \to \mathbb{A}_S$ a homeomorphism which maps the boundary components of $\mathbb{A}_R = \mathbb{A}(1, R)$ to the appropriate ones of $\mathbb{A}_S = \mathbb{A}(1, S)$,

$$f(\{|z| = 1\}) = \{|z| = 1\} \quad \text{and} \quad f(\{|z| = R\}) = \{|z| = S\}.$$

Let $\Phi : [1, \infty) \to [0, \infty)$ be a convex function. A *Nitsche-type extremal problem* is to find the existence or otherwise of a minimiser or stationary point of the functional

$$f \mapsto \iint_{\mathbb{A}_R} \Phi(\mathbb{K}(z, f))\, \eta^2(z)\, |dz|^2. \tag{3.1.2}$$

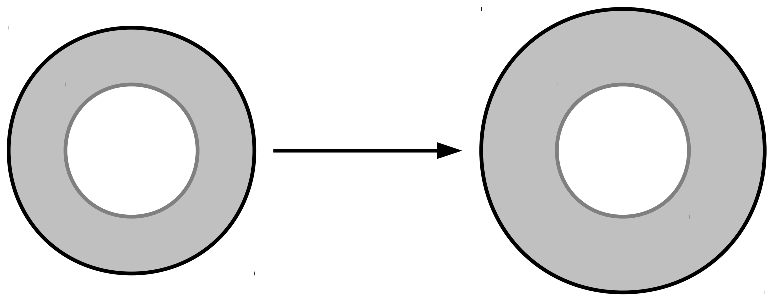


Figure 3.1.1: A Nitsche-type extremal problem.

When posed in this manner, 3.6 may be written in the following way.

Corollary 3.8. *An extremal mapping* $f : \mathbb{A}(r, R) \to \mathbb{A}(r', R')$ *for the functional* (3.1.2) *exists if and only if*

$$\frac{R}{r} \geq \frac{1}{2}\left(\frac{R'}{r'} + \frac{r'}{R'}\right).$$

3.1.3 Grötzsch-Type Extremal Problems

Classically, the Grötzsch problem is the identification of the homeomorphism of least maximal distortion f that maps the rectangle $\mathbb{Q}_1 = [0, \ell] \times [0, 1]$ to the rectangle $\mathbb{Q}_2 = [0, L] \times [0, 1]$ where the mapping is orientation-preserving and maps the edges to the edges as follows:

- $\Re(f(0, y)) = 0$,
- $\Re(f(\ell, y)) = L$,
- $\Im(f(x, 0)) = 0$ and
- $\Im(f(x, 1)) = 1$.

The solution of this is, in fact the linear mapping. The formulation of the generalization this problem is very similar to that of Nitsche-type extremal problems.

Definition 3.9. Consider $f : \mathbb{Q}_1 \to \mathbb{Q}_2$ a homeomorphism of finite distortion which maps edges to edges in the manner described before. Let $\Phi : [1, \infty) \to [0, \infty)$ be a convex function and $\lambda(z)$ a positive weight function. A *Grötzsch-type extremal problem* is to establish the existence or otherwise of a minimiser satisfying the boundary conditions from the edge-mapping of the functional

$$f \mapsto \iint_{\mathbb{Q}_1} \Phi(\mathbb{K}(z, f))\, \lambda(z)\, |dz|^2.$$

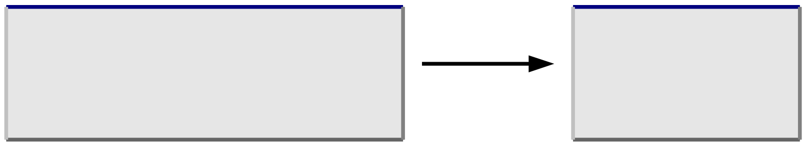


Figure 3.1.2: A Grötzsch-type extremal problem.

3.2 The Condition And Bounding Theorems

We have two main theorems for this chapter that we will prove together. The first states the conditions under which we have a mapping whose distortion lies in L^p for a given radially symmetric metric.

Theorem 3.10. *Let $p \geq 1$ and let $\eta^2(z)$ be a radially symmetric metric of finite area defined on the punctured unit disk $\mathbb{D}^*$. Set*

$$\alpha = \frac{2}{p+1} \leq 1. \tag{3.2.1}$$

Then there is a mapping of finite distortion blowing up the origin $f : \mathbb{D}^* \to \mathbb{A}_R$ for each $R > 1$, with

$$\mathcal{K}_p = \iint_{\mathbb{D}^*} \mathbb{K}(z,f)^p \eta^2(z)\ |dz|^2 < \infty \tag{3.2.2}$$

if and only if

$$\mathcal{I}_p = \iint_{\mathbb{D}^*} \eta^\alpha(z)\, |z|^{\alpha-2}\ |dz|^2 < \infty. \tag{3.2.3}$$

The second provides a lower bound on $\mathcal{K}_p$, which we use to prove Theorem 3.10, in terms of $\mathcal{I}_p$.

Theorem 3.11. *If* (3.2.3) *holds, then there is a minimiser, unique up to rotation, with $\mathbb{K}(z,f) \in L^p\left((\mathbb{D}^*, \eta^2)\right)$ and the estimate ,*

$$\left(\frac{c^2}{2^{2p+1}p}\right)^{p/(p+1)} \frac{\mathcal{I}_p}{(2\pi)^{2-2/(p+1)}} \leq \mathcal{K}_p, \tag{3.2.4}$$

where c is chosen so that

$$\iint_{\mathbb{D}^*} \frac{\eta^\alpha(z)\, |z|^{\alpha-2}}{c^\alpha + p^{1/(p+1)} \left(2\pi\, |z|\, \eta(z)\right)^\alpha}\ |dz|^2 = \frac{2\,(2\pi)^{1-\alpha}}{p^{1/(p+1)}} \ln R; \tag{3.2.5}$$

thus $c \nearrow \infty$ as $R \searrow 1$ and for fixed R, $\mathcal{K}_p \nearrow \infty$ as $p \to \infty$.

In the estimate (3.2.4) it is the term $c^{\alpha p}$ which is large and grows rapidly; we will give explicit estimates later in this chapter. We also have the following corollary to Theorem 3.10 for the flat metric:

Corollary 3.12. *Suppose $\eta \equiv 1$. Then for all finite $p \geq 1$ there exists an $f : \mathbb{D}^* \to \mathbb{A}_R$ with $\mathbb{K}(z, f) \in L^p(\mathbb{D}^*)$. However, $p = \infty$ is not allowed.*

Proof. From Theorem 3.10 we know that we have such an f whenever

$$\iint_{\mathbb{D}^*} |z|^{\alpha-2} |dz|^2 < \infty.$$

Since

$$\alpha - 2 = \frac{2}{p+1} - 2 = \frac{-2p}{p+1},$$

if we convert to polar coordinates we get

$$2\pi \int_0^1 \frac{r\, dr}{r^{2p/(p+1)}} = 2\pi \int_0^1 \frac{dr}{r^{(p-1)/(p+1)}};$$

this last integral is integrable when $\frac{p-1}{p+1} < 1$. This is always true for finite $p \geq 1$. When $p = \infty$ then $\alpha = 0$ and so the similar conversion gives the integral

$$2\pi \int_0^1 \frac{dr}{r},$$

which means we do not have integrability. $\square$

3.3 Preliminary Results

In order to prove Theorems 3.11 and 3.10, we need some preliminary results: first, how to reduce the Nitsche-type problem to a Grötzsch-type problem, so that we may easily find the distortion minimisers. We shall them perform some preliminary calculations on the $\mathcal{I}_p$ term that arises as a result of this reduction. We then prove the result explicitly for the case where $p = 1$, before proceding to the full proof of Theorem 3.11; this is then used to prove Theorem 3.10.

3.3.1 Reductions To A Grötzsch Problem

Following the analysis in [35], we have that if the minimisers exist, then they are radially symmetric. However, the radially symmetric form of the Euler-Lagrange equations are quite difficult to deal with. However, we note that if we transform the problem into a Grötzsch problem, then we find not only that the Euler-Lagrange equations are much easier to deal with, but they are in fact not really differential equations at all. We shall make heavy use of this fact in this proof to provide explicit estimates, so we will go into this in more detail.

Figure 3.3.1 on page 33 illustrates the general idea from [35]: the mappings $\sigma : z \mapsto \frac{1}{2\pi} \ln z$ and $\tau : w \mapsto e^{2\pi w}$ are both conformal, so they are easy to deal with

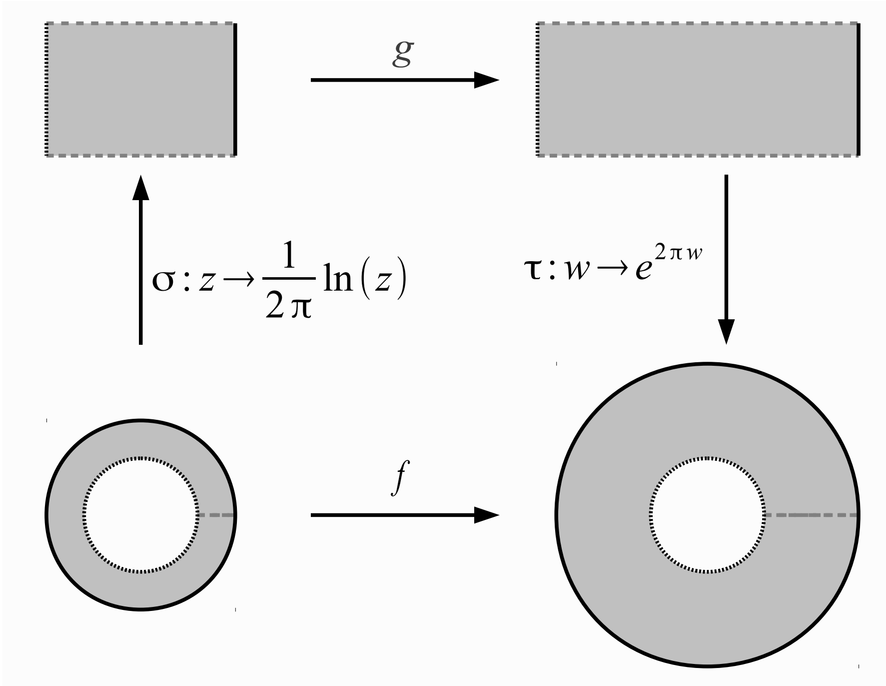


Figure 3.3.1: Diagram demonstrating the conversion of a Nitsche problem to a Grötzsch problem; dashing and colour of the boundaries indicates correspondence between the boundary lines and/or circles between the mappings.

in the analysis. For our purposes we will essentially be doing the same; the only change being that instead of f mapping from an annulus we are mapping from a punctured disk (which we can consider as a "limit annulus" informally).

Let $\mathbb{Q}_1 = [-\infty, 0] \times [0, 1]$ and $\mathbb{Q}_2 = [0, L] \times [0, 1]$, and define $\sigma_1 : \mathbb{D}^* \to \mathbb{Q}_1$ and $\tau_2 : \mathbb{Q}_2 \to \mathbb{A}_R$ where $R = \exp(2\pi L)$ to be the equivalent to the conformal mappings σ and τ mentioned earlier, that is, $\sigma_1 : z \mapsto \frac{1}{2\pi} \ln z$ and $\tau_2 : w \mapsto e^{2\pi w}$, choosing a branch of the logarithm (although which branch is immaterial for our analysis). Given f we define g by

$$f(z) = \tau_2 \circ g \circ \sigma_1(z) = \exp\left(2\pi g\left(\frac{1}{2\pi}\ln z\right)\right),$$

or more explicitly, where $w = \frac{1}{2\pi}\ln z$,

$$g : w \mapsto \frac{1}{2\pi}\ln\left(f\left(e^{2\pi w}\right)\right).$$

Since τ_2 is conformal, we have that

$$\begin{aligned}\mathbb{K}(z, f) &= \mathbb{K}\left(z, \exp\left(2\pi g\left(\frac{1}{2\pi}\ln z\right)\right)\right) \\ &= \mathbb{K}\left(z, g\left(\frac{1}{2\pi}\ln z\right)\right),\end{aligned}$$

and by our rules for composition by conformal mappings (2.5.7) and (2.5.9),

$$\mathbb{K}(z,f) = \mathbb{K}(\sigma_1(z), g) = \mathbb{K}(w,g).$$

So by the change of variables $w = \sigma_1(z)$, we want

$$\begin{aligned}
\iint_{\mathbb{D}^*} \Phi(\mathbb{K}(z,f))\,\eta^2(z)\,|dz|^2 &= \iint_{\mathbb{D}^*} \Phi(\mathbb{K}(w,g))\,\eta^2\left(e^{2\pi w}\right) 4\pi^2 \left|e^{2\pi w}\right|^2 |dw|^2 \\
&= \iint_{\mathbb{D}^*} \Phi(\mathbb{K}(w,g)) \left(4\pi^2 \left|e^{2\pi w}\right|^2 \eta^2\left(e^{2\pi w}\right)\right) |dw|^2 \\
&= \iint_{\mathbb{D}^*} \Phi(\mathbb{K}(w,g))\,\lambda^2(w)\,|dw|^2 .
\end{aligned}$$

We obtain this with the choice

$$\lambda(w) = 2\pi \left|e^{2\pi w}\right| \eta\left(e^{2\pi w}\right) = 2\pi e^{2\pi \Re(w)} \eta\left(e^{2\pi w}\right). \tag{3.3.1}$$

We have that the original Nitsche problem and the Grötzsch problem are equivalent for our problem: certainly Nitsche problems can be converted readily into Grötzsch problems; and because we are fortunate that, for our problem concerning the minimisers of the distortion functional, the solutions have the nice feature of having the horizontal boundary values match (see the grey lines in Figure 3.3.1 on page 33), which is a requirement for the conversion from Grötzsch problems to Nitsche problems.

We slightly rephrase the result from [35] which identifies the extremal solutions to the Grötzsch problem, which become radial when lifted to the annuli. Note that we want η^2 to be radially symmetric; under the relationship defined by (3.3.1), this means λ^2 is constant on the imaginary axis, so we can write $\lambda^2(x)$ instead.

Theorem 3.13. *Let $\lambda^2(x) > 0$ be a positive weight and $\Phi : [1,\infty) \to [0,\infty)$ be convex. Let the function $u : [-\infty, 0] \to [0, L]$ with*

$$\lim_{t\to-\infty} u(t) = 0, \qquad u(0) = L \tag{3.3.2}$$

be a solution to the ordinary differential equation

$$\lambda^2(x)\left(1 - \frac{1}{u_x^2(x)}\right)\Phi'\left(u_x(x) + \frac{1}{u_x(x)}\right) = C, \tag{3.3.3}$$

where C is a nonzero constant. Set

$$f_0(z) = u(x) + iy, \qquad f_0 : \mathbb{Q}_1 \to \mathbb{Q}_2. \tag{3.3.4}$$

Let $f : \mathbb{Q}_1 \to \mathbb{Q}_2$ be a homeomorphism of finite distortion with

$$\lim_{t\to-\infty} \Re\left(f\left(t,y\right)\right) = 0, \quad \Re\left(f\left(0,y\right)\right) = L, \quad \Im\left(f\left(x,0\right)\right) = 0, \quad \Im\left(f\left(x,1\right)\right) = 1.$$

Then

$$\iint_{\mathbb{Q}_1} \Phi\left(\mathbb{K}(z,f)\right) \lambda^2(x)\, |dz|^2 \geq \iint_{\mathbb{Q}_1} \Phi\left(\mathbb{K}(z,f_0)\right) \lambda^2(x)\, |dz|^2, \tag{3.3.5}$$

with equality if and only if $f = f_0$ up to rotation. (This means any other map for which equality holds at (3.3.5) is of the form $z \mapsto \zeta f_0(z)$ and $|\zeta| = 1$.)

The question then becomes: can we choose C so that there is such a function u? Furthermore, in [35] we have a construction of minimizing sequences when these equations cannot be solved, so no solution implies no (homeomorphic) minimiser.

3.3.2 Converting $\mathcal{I}_p$

We shall now show that the results for the Grötzsch problem we obtain later are in fact the same as those we need to prove Theorem 3.10. We begin by considering the natural change of coordinates from $\mathbb{D}^*$ to $\mathbb{Q}_1$:

$$w \mapsto z = e^{2\pi w}.$$

Under this transformation,

$$\begin{aligned}
\mathcal{I}_p &= \iint_{\mathbb{D}^*} \eta^\alpha(z)\, |z|^{\alpha-2}\, |dz|^2 \\
&= \iint_{\mathbb{Q}_1} \eta^\alpha\left(e^{2\pi w}\right) \left|e^{2\pi w}\right|^{\alpha-2} 4\pi^2 \left|e^{2\pi w}\right|^2\, |dw|^2 \\
&= 4\pi^2 \iint_{\mathbb{Q}_1} \left|e^{2\pi w}\right|^\alpha \eta^\alpha\left(e^{2\pi w}\right)\, |dw|^2,
\end{aligned}$$

and because $\lambda(w) = 2\pi\, |e^{2\pi w}|\, \eta\left(e^{2\pi w}\right)$ from (3.3.1)

$$\begin{aligned}
\mathcal{I}_p &= \frac{4\pi^2}{(2\pi)^\alpha} \iint_{\mathbb{Q}_1} (2\pi)^\alpha \left|e^{2\pi w}\right|^\alpha \eta^\alpha\left(e^{2\pi w}\right) |dw|^2 \\
&= (2\pi)^{2-\alpha} \iint_{\mathbb{Q}_1} \lambda^\alpha(w)\, |dw|^2.
\end{aligned}$$

So $\mathcal{I}_p < \infty$ if and only if $\iint_{\mathbb{Q}_1} \lambda^\alpha(w)\, |dw|^2 < \infty$.

3.4 The L^1 Problem: Mean Distortion

We first examine the case of $\mathbb{D}^*$ equipped with a radially symmetric metric $ds = \eta(z)\,|dz|$ that is smooth away from the origin, and the calculation of the mean distortion of mappings of finite distortion $\mathbb{D}^* \to \mathbb{A}_R$. We start with the conversion into a Grötzsch problem. From Theorem 3.13, with λ and η associated as in (3.3.1), we must solve the equation

$$\lambda^2(x)\left(1 - \frac{1}{u_x^2(x)}\right) = C. \tag{3.4.1}$$

At some point we must have $u_x(x) < 1$, because we are compressing the infinite rectangle $(-\infty, 1] \times [0, 1]$, and as $\lambda^2(x) > 0$, we require $C < 0$. Writing $C = -c^2$ gives

$$u_x(x) = \frac{1}{\sqrt{1 + \frac{c^2}{\lambda^2(x)}}}, \tag{3.4.2}$$

so the solution u will be defined by

$$u(x) = \int_{-\infty}^{x} \frac{dt}{\sqrt{1 + \frac{c^2}{\lambda^2(t)}}} = \int_{-\infty}^{x} \frac{\lambda(t)dt}{\sqrt{\lambda^2(t) + c^2}}. \tag{3.4.3}$$

As $c \neq 0$ is constant, this integral converges if and only if

$$\int_{-\infty}^{x} \lambda(t)dt < \infty. \tag{3.4.4}$$

To show this, we note

$$0 < \frac{1}{\sqrt{\lambda^2(t) + c^2}} \leq \frac{1}{c},$$

and so we have

$$\lim_{t \to -\infty} \frac{1}{\sqrt{\lambda^2(t) + c^2}} = 0$$

if and only if we have $\lim_{t \to -\infty} \lambda(t) = \infty$. Therefore by the limit comparison test we have that the integrals of both $\lambda(t)$ and $\lambda(t)\left(\sqrt{\lambda^2(t) + c^2}\right)^{-1}$ are either both convergent or both divergent.

Since

$$\int_{-\infty}^{x} \lambda(t)dt = \int_{-\infty}^{x} 2\pi\eta\left(e^{2\pi t}\right)e^{2\pi t}dt = 2\pi \int_{0}^{r} \eta\,(s)\,ds,$$

we have, from symmetry, that this is proportional to the length of the geodesic terminating at the boundary point 0. That it is finite means the metric is incomplete at the origin.

Corollary 3.14. *Let $\mathbb{D}^*$ be endowed with the radially symmetric metric $\eta^2(z)\,|dz|^2$*

of finite area. Then there is a minimiser of the mean distortion problem

$$\min_f \iint_{\mathbb{D}^*} \mathbb{K}(z,f)\eta^2(z)\,|dz|^2$$

if and only if the metric is incomplete at the origin.

As an aside, from the hyperbolic metric on the punctured unit disk (2.2.7), we have that $\eta(z) = |z|^{-1} \ln |z|^{-1}$ and so, by (3.3.1), that $\lambda(x) = x^{-1}$; then

$$u(x) = \int_{-\infty}^{x} \frac{dt}{\sqrt{1+t^2c^2}} = \infty.$$

Therefore, in the case of the hyperbolic metric we have no L^1-minimisers. This is a similar result to that obtained in Corollary 3.14, as the hyperbolic metric on this space is complete.

3.5 Proof Of The Bounding Theorem

Proof. (See Theorem 3.11) Most of the details we shall leave for the next section, but we shall provide an outline here. First, we construct a minimal solution for the problem; we know that minimal solutions of these problems are radial stretchings. We transform this problem into a Grötzsch problem, creating some estimates above and below the real parts of the Grötzsch mappings (since the minimal solutions of the original problem are radial stretchings, the imaginary components remain fixed by the mapping, so don't impact the analysis much). These create bounds on $\mathcal{K}_p$, that establish the requirement (3.2.4), and with further calculations we obtain the theorem. □

3.6 The L^p Problem

Following the same argument as above, we start by noting that

$$\iint_{\mathbb{D}^*} \mathbb{K}(z,f)^p\eta^2(z)\ |dz|^2 = \iint_{\mathbb{Q}_1} \mathbb{K}(w,g)^p\,\lambda^2(w)\ |dw|^2 . \tag{3.6.1}$$

As there must be some point with $u_x(x) < 1$, we have that $C < 0$ from (3.6.2). Again we set $C = -c^2$. We thus wish to solve

$$p\lambda^2(x)\left(\frac{1}{u_x^2(x)} - 1\right)\left(u_x(x) + \frac{1}{u_x(x)}\right)^{p-1} = c^2. \tag{3.6.2}$$

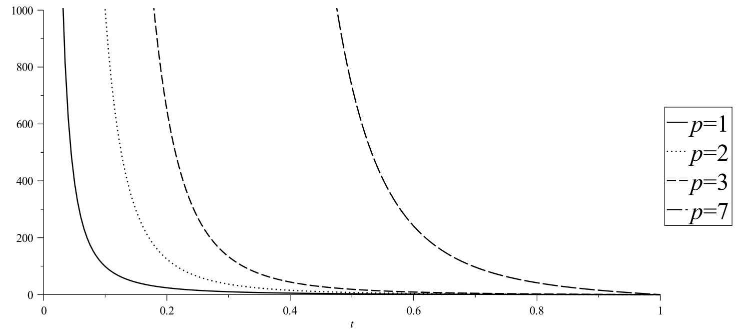


Figure 3.6.1: Graphs of $P(t)$ for $p = 1, 2, 3$ and 7.

3.6.1 Determining Bounding Functions

For even small p and nice λ, a closed form will be difficult to calculate, so instead we try to find some bounding functions for which we can solve explicitly. Define

$$P(t) = \left(\frac{1}{t^2} - 1\right)\left(t + \frac{1}{t}\right)^{p-1}, \tag{3.6.3}$$

and as illustrated by Figure 3.6.1 on page 38, we can see some general trends that we should consider when constructing our bounding functions for various values of p and t. First, that it is decreasing for all values of p. The second thing to note is the values as t gets close to 0 and 1: as $t \to 0$, $P(t) \to \infty$, and in particular as $t \to 1$, we have that $P(t) \to 0$.

Let us set

$$A(t) = \left(\frac{1}{t} - 1\right)^{p+1} \tag{3.6.4}$$

and

$$B(t) = \left(1 + \frac{1}{t}\right)^{p+1} - 2^{p+1}. \tag{3.6.5}$$

Lemma 3.15. *For $t \in [0, 1]$ we have functions A and B strictly decreasing with $A(1) = B(1) = 0$ and*

$$A(t) \le P(t) \le B(t). \tag{3.6.6}$$

Proof. The fact that $A(1) = P(1) = B(1) = 0$ and A and B are strictly decreasing can be seen by simple substitution. Noting

$$P(t) = \left(\frac{1}{t} - 1\right)\left(\frac{1}{t} + 1\right)\left(\frac{1}{t} + t\right)^{p-1}$$

and

$$\frac{1}{t} + 1 \ge \frac{1}{t} + t \ge \frac{1}{t} - 1,$$

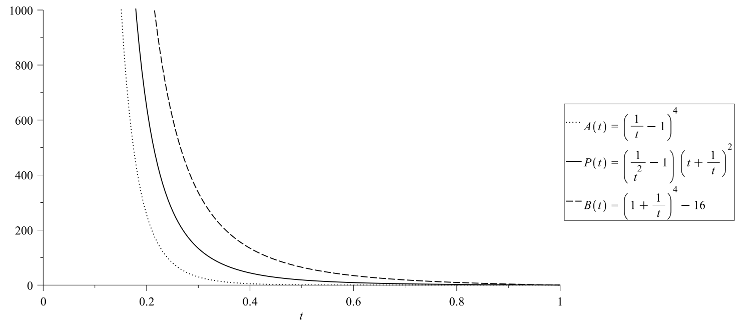


Figure 3.6.2: Graph demonstrating $A(t) \leq P(t) \leq B(t)$ for $p = 3$

we have that

$$P(t) \geq \left(\frac{1}{t} - 1\right)^{p+1} = A(t).$$

For the second inequality, we note

$$\begin{aligned} P'(t) &= -\frac{(t^2+1)^{p-2}}{t^{p+2}}\left(1 + 4t^2 - t^4 + p\left(1 - t^2\right)^2\right) \quad \text{and} \\ B'(t) &= -\frac{(p+1)(t+1)^p}{t^{p+2}}. \end{aligned}$$

The second inequality follows if we establish that $B'(t) \leq P'(t)$. Note that this follows from

$$(p+1)(t+1)^2 \geq 1 + 4t^2 - t^4 + p\left(1 - t^2\right)^2,$$

reducing to

$$3pt^2 + 2(p+1)t \geq 3t^2 + (p-1)t^4,$$

which follows as $p \geq 1$ and $t \in [0, 1]$ in the domain in question.

$\square$

These estimates are good if t is small, but for specific values or ranges of p they can be improved. Some examples are given in Appendix B.

3.6.2 Bounding Solutions And Bounding Distortion

We now fix x temporarily. Then, as $A(t)$, $P(t)$ and $B(t)$ are strictly decreasing with range $[0, \infty)$, by the intermediate value theorem we can find unique numbers t_-, t, t_+ such that

$$\begin{aligned} p\lambda^2(x) A(t_-) &= c^2, \\ p\lambda^2(x) P(t) &= c^2, \quad \text{and} \\ p\lambda^2(x) B(t_+) &= c^2; \end{aligned}$$

and furthermore that these numbers vary continuously with c and x. As the graph of $P(t)$ lies above $A(t)$ and below $B(t)$, and all are decreasing, we have that

$$t_- \le t \le t_+. \tag{3.6.7}$$

We then define for each x

$$u_x^-(x) = t_-, \qquad u^-(s) = \int_{-\infty}^{s} u_x^-(x)\, dx, \tag{3.6.8}$$

$$u_x(x) = t, \qquad u(s) = \int_{-\infty}^{s} u_x(x)\, dx, \tag{3.6.9}$$

$$u_x^+(x) = t_+, \quad \text{and} \quad u^+(s) = \int_{-\infty}^{s} u_x^+(x)\, dx; \tag{3.6.10}$$

and define, for $z = x + iy$,

$$f^-(z) = u^-(x) + iy, \quad f(z) = u(x) + iy, \quad \text{and} \quad f^+(z) = u^+(x) + iy.$$

Then for each p and each choice of constant c at (3.6.2), f is the unique minimiser onto its range. Although we have used them to provide bounds, neither f^+or f^- can be extremal: f^+ has a better distortion integral than f, but its image is a larger rectangle (fatter annulus in terms of the original problem), while f^- has a worse distortion integral.

We have, for the bound-derived functions, explicit formulae: first, for A we have that

$$c^2 = p\lambda^2 \left(\frac{1}{u_x^-} - 1 \right)^{p+1},$$

which we rearrange to give

$$\frac{1}{u_x^-} = 1 + \left(\frac{c^2}{p\lambda^2} \right)^{1/(p+1)}$$

or

$$u_x^- = \frac{(p\lambda^2)^{1/(p+1)}}{c^{2/(p+1)} + (p\lambda^2)^{1/(p+1)}}.$$

Integration gives

$$u^-(x) = \int_{-\infty}^{x} \frac{p^{1/(p+1)} \lambda^{2/(p+1)}(s)\, ds}{c^{2/(p+1)} + p^{1/(p+1)} \lambda^{2/(p+1)}(s)}. \tag{3.6.11}$$

Similarly,

$$c^2 = p\lambda^2 \left[\left(1 + \frac{1}{u_x^+}\right)^{p+1} - 2^{p+1} \right]$$

becomes

$$\frac{1}{u_x^+} = \left(\frac{c^2}{p\lambda^2} + 2^{p+1}\right)^{1/(p+1)} - 1$$

or

$$u_x^+ = \frac{1}{\left(\frac{c^2}{p\lambda^2} + 2^{p+1}\right)^{1/(p+1)} - 1},$$

which, after integrating, gives us that

$$u^+(x) = \int_{-\infty}^{x} \frac{\lambda^{2/(p+1)}(s)\, ds}{\left(\frac{c^2}{p} + 2^{p+1}\lambda^2(s)\right)^{1/(p+1)} - \lambda^{2/(p+1)}(s)}. \tag{3.6.12}$$

Noting that the first of these is in a similar form to that which we used earlier for the L^1 case (see Section 3.4), and the second is bounded above and below by similar integrals by using Lemma A.4, we have that if $\mathcal{I}_p$ converges, then both of these solutions converge after the substitution $\lambda \mapsto \lambda^\alpha$ into (3.4.3) and (3.4.4). Therefore u exists and $\mathcal{K}_p$ converges for the associated $f(x+iy) = u(x) + iy$.

3.6.3 Bounding c

The equation (3.6.12) and (3.6.11) define $u^-(0)$ and $u^+(0)$ as strictly decreasing continuous functions of c with infinite limit as $c \searrow 0$ and limit zero as $c \nearrow \infty$. From (3.6.7) we have that

$$u^-(x) \le u(x) \le u^+(x), \tag{3.6.13}$$

and by the squeeze theorem and continuity, we have for each $T > 0$ that there exists a c so that $u(0) = T$. In fact, noting that $u^-(0) \le u(0)$, we have that

$$p^{1/(p+1)} \int_{-\infty}^{0} \frac{\lambda^{2/(p+1)}(s)\, ds}{c^{2/(p+1)} + p^{1/(p+1)}\lambda^{2/(p+1)}(s)} \le T, \tag{3.6.14}$$

which places a lower bound on c.

Integrating by y over $[0,1]$ then gives

$$p^{1/(p+1)} \iint_{\mathbb{Q}_1} \frac{\lambda^{2/(p+1)}(s)\, ds\, dy}{c^{2/(p+1)} + p^{1/(p+1)} \lambda^{2/(p+1)}(s)} \leq T,$$

and as $ds\, dy = \frac{1}{2}\left|dw\right|^2$, by converting back into a Nitsche problem we have that

$$\frac{p^{1/(p+1)}}{2} \iint_{\mathbb{Q}_1} \frac{\left(2\pi \left|e^{2\pi w}\right| \eta\left(e^{2\pi w}\right)\right)^{\alpha} \left|dw\right|^2}{c^{\alpha} + p^{1/(p+1)}\left(2\pi \left|e^{2\pi w}\right| \eta\left(e^{2\pi w}\right)\right)^{\alpha}} \leq T;$$

the change of coordinates $w \mapsto z = e^{2\pi w}$ gives

$$\frac{(2\pi)^{\alpha-2} p^{1/(p+1)}}{2} \iint_{\mathbb{D}^*} \frac{\eta^{\alpha}(z) \left|z\right|^{\alpha-2}}{c^{\alpha} + p^{1/(p+1)}\left(2\pi \left|z\right| \eta(z)\right)^{\alpha}} \left|dz\right|^2 \leq T.$$

For our problem we want $u(0) = L = \frac{1}{2\pi} \ln R$; substituting this for T and rearranging for the equality gives (3.2.5).

As $R \to 1$, the right hand side of (3.2.5) goes to zero; the only term on the left that varies with R is c, so that must tend to infinity.

3.6.4 Bounding $\mathcal{K}_p$

Since (3.6.13) and as $t \mapsto t + 1/t$ is convex decreasing for $t \leq 1$ then

$$\mathbb{K}^+ = \mathbb{K}\left(z, f^+\right) \leq \mathbb{K} = \mathbb{K}(z, f) \leq \mathbb{K}^- = \mathbb{K}\left(z, f^-\right),$$

thus providing our bounds on $\mathbb{K}$.

From (3.6.12) and (3.6.11) we can see that

$$\lambda(x) \in L^{2/(p+1)}\left((-\infty, 0]\right). \tag{3.6.15}$$

We also want a lower bound on $\mathcal{K}_p$. So

$$\mathbb{K}^p \geq \left(\mathbb{K}^+\right)^p = 2^{-p}\left(u_x^+ + \left(u_x^+\right)^{-1}\right)^p;$$

as $0 \leq u_x^+ \leq 1$, we have that

$$\left(u_x^+ + \frac{1}{u_x^+}\right)^p \geq \frac{1}{(u_x^+)^p} = \left(\left(\frac{c^2}{p\lambda^2} + 2^{p+1}\right)^{1/(p+1)} - 1\right)^p,$$

and thus

$$\mathcal{K}_p \geq 2^{-p} \int_{-\infty}^{0} \left(\left(\frac{c^2}{p\lambda^2(x)} + 2^{p+1}\right)^{1/(p+1)} - 1\right)^p \lambda^2(x)\, dx.$$

We shall now use Lemma A.4 here two times. First note that

$$\left(\frac{c^2}{2^p p\lambda^2}\right)^{1/(p+1)} + 2^{1/(p+1)} \leq \left(\frac{c^2}{p\lambda^2} + 2^{p+1}\right)^{1/(p+1)}.$$

As $2^{1/(p+1)} \geq 1$ we then get the inequality

$$\left(\frac{c^2}{2^p p\lambda^2}\right)^{1/(p+1)} + 2^{1/(p+1)} - 1 \geq \left(\frac{c^2}{2^p p\lambda^2}\right)^{1/(p+1)},$$

and so our lower bound is

$$\mathcal{K}_p \geq 2^{-p}\left(\frac{c^2}{2^p p}\right)^{p/(p+1)} \int_{-\infty}^{0} \lambda^{2/(p+1)}(x)\, dx. \tag{3.6.16}$$

This is equivalent to

$$\mathcal{K}_p \geq 2^{-p}\left(\frac{c^2}{2^p p}\right)^{p/(p+1)} \iint_{\mathbb{Q}_1} \lambda^{\frac{2}{p+1}}(w)\ |dw|^2,$$

which we can rewrite in terms of our initial problem: first note that

$$\iint_{\mathbb{Q}_1} \lambda^{\frac{2}{p+1}}(w)\ |dw|^2 = (2\pi)^{\alpha-2}\mathcal{I}_p.$$

This gives us (3.2.4):

$$\mathcal{K}_p \geq \left(\frac{c^2}{2^{2p+1}p}\right)^{p/(p+1)} \frac{\mathcal{I}_p}{(2\pi)^{2-\alpha}}.$$

3.6.5 Near Extremals

Through our previous arguments, we have a method for finding near extremal mappings. Even though we have chosen them to be nice for calculating distortion integral bounds, the functions $f^-$and f^+ will not be easy to express in closed form for even nice metrics. However, we can still examine the asymptotics of these functions.

Let us suppose that λ is small. For fixed c, (3.6.2) implies that

$$P(t) = c^2 p^{-1}\lambda^{-2}(x)$$

will be solved for small values of t (as $P(t)$ is decreasing, and $\lambda^{-2}(x)$ is large), and so $P(t) \sim t^{-(p+1)}$. Therefore solving

$$t^{-(p+1)} = \frac{c^2}{p\lambda^2(x)}$$

should give us examples with correct asymptotic behaviour near $-\infty$. To do this we should really have λ finite; then we can normalise so that

$$\sup_{x\in(-\infty,1]} \lambda(x) = 1. \tag{3.6.17}$$

We must have $c^2 \geq p\lambda^2(x)$ for all x, so from our normalization we have the bound $c^2 \geq p$. This provides us with the candidate map

$$f_c(z) = \left(\frac{p}{c^2}\right)^{1/(p+1)} \int_{-\infty}^{x} \lambda^{2/(p+1)}(s)\, ds + iy, \tag{3.6.18}$$

and as

$$0 \leq f_c(0) \leq \int_{-\infty}^{0} \lambda^{2/(p+1)}(s)\, ds,$$

we cannot map arbitrarily far with this map. However, we can calculate

$$\begin{aligned} 2^p\mathbb{K}^p(z, f_c) &= \left(\left(\frac{c^2}{p\lambda^2(x)}\right)^{1/(p+1)} + \left(\frac{p\lambda^2(x)}{c^2}\right)^{1/(p+1)}\right)^p \\ &\leq 2^p \left(\frac{c^2}{p\lambda^2(x)}\right)^{p/(p+1)}, \end{aligned}$$

so if $f_c(0) = T \geq 0$ then

$$T = \left(\frac{p}{c^2}\right)^{1/(p+1)} \int_{-\infty}^{0} \lambda^{2/(p+1)}(s),\ ds$$

and we then obtain the inequality

$$\begin{aligned} \int_{-\infty}^{0} \mathbb{K}^p(z, f_c)\lambda^2(x)\, dx &\leq \left(\frac{c^2}{p}\right)^{p/(p+1)} \int_{-\infty}^{0} \lambda^{2/(p+1)}(x)\, dx \\ &\leq \frac{1}{T^P}\left(\int_{-\infty}^{0} \lambda^{2/(p+1)}(x)\, dx\right)^{p+1}. \end{aligned}$$

Thus

$$\|\mathbb{K}(z, f_c)\|_{L^p(\lambda^2)} \leq \frac{1}{T}\left(\int_{-\infty}^{0} \lambda^{2/(p+1)}(x)\, dx\right)^{1+1/p}. \tag{3.6.19}$$

3.6.6 Limit Case: The L^∞ Problem

Rearranging (3.6.2) we have that

$$\left(\frac{1}{u_x^2(x)} - 1\right)\left(u_x(x) + \frac{1}{u_x(x)}\right)^{p-1} = \frac{c^2}{p\lambda^2(x)}.$$

We shall now examine what happens when $p \to \infty$. We can intuitively see what will happen from this equation: since c is 'constant' and $\lambda(x) > 0$, on the right hand side the term tends to zero as $p \to \infty$: $c^2 p^{-1} \lambda^{-2}(x) \to 0$. On the left, as $(u_x + u_x^{-1}) \geq 2$, we must then have that$1 - u_x^2(x) \to 0$, which implies that $u_x(x) \to 1$. This also means that $c \to 0$ and $u(x) \to \infty$. This intuitive idea has one problem: c depends on p, so we could potentially have $c^2 p^{-1} \lambda^{-2}$ not going to zero. However, as we shall soon see, we still get no minimisers here by fixing R.

For fixed R, from (3.6.16) we have that as $p \to \infty$, $\alpha = \frac{2}{p-1} \to 0$ and so $\lambda^\alpha \to 1$. If the constant term does not go to zero as $p \to \infty$, then the integral will tend to infinity and so will $\mathcal{K}_p$. Suppose that the constant coefficient does tend to zero; then for sufficiently large p we have the bound

$$c^2 < \epsilon 2^{2p+1} p$$

for some $\epsilon \in (0, 1)$. Substituting this into (3.6.14) and using the fact we chose $\epsilon < 1$, we have that

$$\int_{-\infty}^{0} \frac{\lambda^\alpha(s)\, ds}{2^{(2p+1)/(p+1)} + \lambda^\alpha(s)} < \frac{1}{2\pi} \ln R,$$

but while the right side is fixed as $p \to \infty$,

$$\frac{\lambda^\alpha(s)}{2^{(2p+1)/(p+1)} + \lambda^\alpha(s)} \to \frac{1}{5},$$

as $p \to \infty$ and so the left side tends to infinity. Therefore, $\mathcal{K}_p \nearrow \infty$ as $p \to \infty$.

3.7 Proof Of The Condition Theorem

Proof. (See Theorem 3.10) Theorem 3.10 follows from Theorem 3.11 with the following observations.

First, in the previous section, we proved the theorem (as well as Theorem 3.11) explicitly for the case $p = 1$, that is for mappings in $L^1(\mathbb{D}^*, \eta^2)$.

Next suppose $\mathcal{I}_q$ diverges at $q > 1$. Set

$$p_0 = \sup_p \{p : \mathcal{I}_q < \infty\} \leq q.$$

If there is a mapping of finite distortion $f : \mathbb{D}^* \to \mathbb{A}_R$ for which $\mathcal{K}_q < \infty$ then $\mathcal{K}_p \leq \mathcal{K}_q$ for all $p \leq p_0$. In particular, $\mathcal{K}_q$ is an upper bound for the extremal problem for each $p < p_0$. However, the lower bound on the value of the extremal map (3.2.4) from Theorem 3.11, which does exist, increases to ∞ as $p \to p_0$ as both c and $\mathcal{I}_p$ necessarily tend to infinity (as R is fixed). So if $\mathcal{I}_q$ diverges, then there is no such f.

The other direction follows from our construction of bounds on u and a similar argument to that appearing in the L^1 case. $\square$

3.8 Minimisers On The Euclidean Metric

The case of the flat metric is of particular interest, as we shall need to make use of it in the following chapter when we apply the example mapping we construct here to the problem of resolving decompositions. For the Euclidean metric on the punctured disc, $\eta(z) \equiv 1$ so $\lambda(x) = 2\pi e^{2\pi x}$. In this case, we see a corollary to Theorem 3.11.

Corollary 3.16. *Let $p \geq 1$. Then, for each $R > 1$ there is an extremal mapping of finite distortion blowing up a point $f : \mathbb{D}^* \to \mathbb{A}_R$, unique up to rotation, with $\mathcal{K}_p < \infty$ and the lower estimate*

$$\frac{\pi^{1/p}\left(p+1\right)^{1/p}}{2^{(2p+1)/(p+1)}} \frac{1}{R^{2/(p+1)}-1} \leq \left\|\mathbb{K}\left(z,f\right)\right\|_p < \infty. \tag{3.8.1}$$

This estimate is sharp for each R.

We shall prove this result in the following sections. However, we shall take a brief look at the asymptotics of this mapping: for large p the lower bound is

$$\frac{\pi^{1/p}\left(p+1\right)^{1/p}}{2^{(2p+1)/(p+1)}} \frac{1}{R^{2/(p+1)}-1} \approx \frac{1}{4}\frac{1}{R^{2/p}-1},$$

so tends to ∞ with p. This fits in with what we observed for 3.12 for the behaviour of the L^p norm of $\mathbb{K}\left(z,f\right)$ at $p = \infty$.

3.8.1 Example Mapping

Our primary question is to determine what is the optimal degree of integrability of the distortion functions of mappings which blow up points. By way of an example, the simplest obvious candidate is the map $f : \mathbb{D}^* \to \mathbb{A}_R$ for $R > 1$ defined as follows. Let $\beta > 0$ and $\rho\left(t\right) = R^{t^\beta}$; then the radial stretching

$$f\left(z\right) = \frac{z}{|z|} R^{|z|^\beta} \tag{3.8.2}$$

blows up the origin, and the derivatives are

$$f_z\left(z\right) = \frac{\left(|z|^\beta \beta \ln R + 1\right)}{2\left|z\right|} R^{|z|^\beta}$$

and

$$f_{\bar{z}}(z) = \frac{\left(|z|^{\beta}\beta\ln R - 1\right)}{2|z|}\frac{z}{\bar{z}}R^{|z|^{\beta}}.$$

Let Ω be a compact subset of $\mathbb{D}^*$; then Ω is closed and thus cannot be arbitrarily close to zero. Let z_0 be a point of Ω closest to zero and define $\delta := |z_0|$.

$$\begin{aligned}
\|f\|_{1,p,\Omega}^{p} &= \iint_{\Omega}\left(|f|^{p} + \left(|f_z|^2 + |f_{\bar{z}}|^2\right)^{p/2}\right)|dz|^2 \\
&= \iint_{\Omega} R^{p|z|^{\beta}}|dz|^2 \\
&+ \iint_{\Omega}\frac{R^{p|z|^{\beta}}}{2^{p}|z|^{p}}\left(\left(|z|^{\beta}\beta\ln R + 1\right)^2 + \left(|z|^{\beta}\beta\ln R - 1\right)^2\right)^{p/2}|dz|^2,
\end{aligned}$$

which we can rearrange to give

$$\|f\|_{1,p,\Omega}^{p} = \iint_{\Omega} R^{p|z|^{\beta}}\left(1 + \frac{\left(|z|^{2\beta}\beta^2\ln^2 R + 1\right)^{p/2}}{2^{p/2}|z|^{p}}\right)|dz|^2.$$

Then using the fact that $|z| < 1$, we have the upper bound

$$\|f\|_{1,p,\Omega}^{p} \leq \pi R^{p}\left(1 + \frac{(\beta^2\ln^2 R + 1)^{p/2}}{2^{p/2}\delta^{p}}\right) < \infty,$$

the latter following as $\delta > 0$. So f is certainly in the locally Sobolev class $W^{1,1}_{loc}(\mathbb{D}^*)$ (such mappings need not be continuous); and by (2.6.1), the Jacobian

$$J(z,f) = \beta\ln R\,|z|^{\beta-2}R^{2|z|^{\beta}}$$

is locally integrable (that is, $J(z,f) \in L^1_{loc}(\mathbb{D}^*)$) by [24, page 7] as f preserves orientation, and by (2.6.2) we have that

$$\iint_{\Omega}\mathbb{K}(z,f)^{p}|dz|^2 = \frac{1}{2^{p}}\iint_{\Omega}\left(|z|^{\beta}\beta\ln R + \frac{1}{|z|^{\beta}\beta\ln R}\right)^{p}|dz|^2. \tag{3.8.3}$$

From changing to polar coordinates ($|dz|^2 = r\,drd\theta$), the fact that $\Omega \subset \mathbb{D}^*$ and Lemma A.4, we have the inequality

$$\iint_{\Omega}\mathbb{K}(z,f)^{p}|dz|^2 \leq \pi\left(\beta^{p}\ln^{p} R\int_0^1 r^{p\beta+1}dr + \frac{1}{\beta^{p}\ln^{p} R}\int_0^1\frac{dr}{r^{p\beta-1}}\right)$$

after integrating through by $d\theta$, and so

$$\iint_{\Omega} \mathbb{K}(z,f)^p \, |dz|^2 \quad \leq \quad \pi \left(\beta^p \ln^p R + \frac{1}{\beta^p \ln^p R} \int_0^1 \frac{dr}{r^{p\beta-1}} \right).$$

For convergence we require that $1 - p\beta > -1$, or $p\beta < 2$. The integral $\mathcal{K}_p$ for this f has a closed expression in terms of the hypergeometric function ${}_2F_1$ which might give slightly better asymptotics, but we will simply use the above estimate. The choice $\beta := \alpha = 2/(p+1)$ gives us the estimate

$$\begin{aligned} \|\mathbb{K}(z,f)\|_p &\leq \left(\pi \left(\beta^p \ln^p R + \frac{1}{\beta^p \ln^p R} \frac{1}{2 - p\beta} \right) \right)^{1/p} \\ &= \pi^{1/p} \left(\left(\frac{2}{p+1} \right)^p \ln^p R + \frac{(p+1)^{p+1}}{2^{p+1} \ln^p R} \right)^{1/p} \\ &\leq \pi^{1/p} \left(\left(\frac{2}{p+1} \right) \ln R + \frac{(p+1)^{1+1/p}}{2^{1+1/p} \ln R} \right) \\ &\sim \frac{(p+1)}{2 \ln R} = \frac{1}{\ln R^{2/(p+1)}} \end{aligned}$$

once p is sufficiently large. Although this is not the extremal mapping we should compare with (3.8.1) to see that this is close to a minimum as soon as p is sufficiently large:

$$\frac{\pi^{1/p}}{2^{(2p+1)/(p+1)}} (p+1)^{1/p} \frac{1}{R^{2/(p+1)} - 1} \sim \frac{1}{R^{2/(p+1)} - 1}$$

and $\ln x \sim x - 1$ when (for small ϵ) $1 - \epsilon < x \leq 1$, which we certainly have for large p.

Next, we shall make some observations of the inverse map,

$$g(z) = \frac{z}{|z|} \left(\frac{\ln |z|}{\ln R} \right)^{1/\beta}. \tag{3.8.4}$$

First, it shrinks a disk. The complex derivatives of this are

$$g_z(z) = \left(\frac{\ln |z|}{\ln R} \right)^{1/\beta} \frac{(\beta \ln |z| + 1)}{2\beta |z| \ln |z|}$$

and

$$g_{\bar{z}}(z) = \left(\frac{\ln |z|}{\ln R} \right)^{1/\beta} \frac{(\beta \ln |z| - 1)}{2\beta |z| \ln |z|} \frac{z}{\bar{z}}.$$

Therefore because

$$\|g\|_{1,p,\Omega}^p \quad = \quad \iint_{\Omega} \left(|g|^p + \left(|g_z|^2 + |g_{\bar{z}}|^2 \right)^{p/2} \right) |dz|^2,$$

we have that

$$\begin{aligned}\|g\|_{1,p,\Omega}^{p} &= \iint_{\Omega}\left(\frac{\ln|z|}{\ln R}\right)^{p/\beta}|dz|^{2}\\ &+ \iint_{\Omega}\left(\frac{\ln|z|}{\ln R}\right)^{p/\beta}\frac{\left((\beta\ln|z|+1)^{2}+(\beta\ln|z|-1)^{2}\right)^{p/2}}{2^{p}\beta^{p}|z|^{p}\ln^{p}|z|}|dz|^{2};\end{aligned}$$

rearrangement gives

$$\|g\|_{1,p,\Omega}^{p} = \iint_{\Omega}\left(\frac{\ln|z|}{\ln R}\right)^{p/\beta}\left(1+\frac{\left(\beta^{2}\ln^{2}|z|+1\right)^{p/2}}{2^{p/2}\beta^{p}|z|^{p}\ln^{p}|z|}\right)|dz|^{2}.$$

Converting to polar coordinates gives

$$\|g\|_{1,p,\Omega}^{p} = \iint_{\Omega}\left(\frac{\ln r}{\ln R}\right)^{p/\beta}\left(1+\frac{\left(\beta^{2}\ln^{2}r+1\right)^{p/2}}{2^{p/2}\beta^{p}r^{p}\ln^{p}r}\right)r\,dr\,d\theta.$$

Then, by the change of variables $t=\ln r$,

$$\|g\|_{1,p,\Omega}^{p} = \iint_{\Omega}\left(\frac{t}{\ln R}\right)^{p/\beta}\left(1+\frac{\left(\beta^{2}t^{2}+1\right)^{p/2}}{2^{p/2}\beta^{p}e^{tp}t^{p}}\right)e^{2t}\,dt\,d\theta.$$

By using the fact that $t<\ln R$, we have the upper bound

$$\|g\|_{1,p,\Omega}^{p} \leq 2\pi R^{2}\ln R+\frac{2\pi R^{2}}{\ln^{p/\beta}R}\frac{\left(\beta^{2}\ln^{2}R+1\right)^{p/2}}{2^{p/2}\beta^{p}}\int_{0}^{\ln R}t^{p/\beta-p}\,dt,$$

the latter being integrable when $\frac{p}{\beta}-p>-1$ or $p<\frac{\beta}{\beta-1}$. In particular, it is integrable if $\beta=1$. Therefore, $g\in W_{loc}^{1,p}\left(\mathbb{A}\left(1,R\right)\right)$ whenever $p<\frac{\beta}{\beta-1}$.

Also

$$\mathbb{K}\left(z,g\right)=\frac{1}{2}\left(\beta\ln|z|+\frac{1}{\beta\ln|z|}\right).$$

This distortion is in $L^{p}\left(\mathbb{A}_{R}\right)$ for all $p<1$, but it is not in $L^{1}\left(\mathbb{A}_{R}\right)$: first we note

$$\iint_{\mathbb{A}_{R}}\mathbb{K}(z,g)^{p}|dz|^{2} = \frac{1}{2^{p}}\iint_{\mathbb{A}_{R}}\left(\beta\ln|z|+\frac{1}{\beta\ln|z|}\right)^{p}|dz|^{2};$$

since $\beta\ln|z|\geq 0$, we have that

$$\iint_{\mathbb{A}_{R}}\mathbb{K}(z,g)^{p}|dz|^{2} \geq \frac{2\pi}{2^{p}\beta^{p}}\int_{1}^{R}\frac{r\,dr}{\ln^{p}r},$$

so by a change of coordinates, we have that

$$\iint_{\mathbb{A}_R} \mathbb{K}(z,g)^p\,|dz|^2 \;\geq\; \frac{2\pi}{2^p\beta^p}\int_0^{\ln R}\frac{ds}{s^p}.$$

Similarly

$$\begin{aligned}\iint_{\mathbb{A}_R} \mathbb{K}(z,g)^p\,|dz|^2 &\leq \frac{2\pi\left(\beta+\frac{1}{\beta}\right)^p}{2^p}\int_1^R \frac{r\,dr}{\ln^p r}\\ &\leq \frac{2\pi}{2^p\beta^p}\int_0^{\ln R}\frac{ds}{s^p}.\end{aligned}$$

Nor is it in the Lorentz-Zygmund space $L^1/\ln(L)$; however it does lie in the space $L^1/\ln^p(L)$ for all $p>1$:

$$\begin{aligned}\iint_{\mathbb{A}_R}\frac{\mathbb{K}(z,g)}{(\ln\mathbb{K}(z,g))^p}\,|dz|^2 &= \iint_{\mathbb{A}_R}\frac{\frac{1}{2}\left(\beta\ln|z|+\frac{1}{\beta\ln|z|}\right)}{\ln^p\left(\frac{1}{2}\left(\beta\ln|z|+\frac{1}{\beta\ln|z|}\right)\right)}\,|dz|^2\\ &= \pi\int_1^R \frac{r\left(\beta\ln r+\frac{1}{\beta\ln r}\right)\,dr}{\ln^p\left(\frac{1}{2}\left(\beta\ln|z|+\frac{1}{\beta\ln|z|}\right)\right)}\\ &= 2\pi\int_0^{\beta\ln R}\frac{e^{2s/\beta}}{\beta}\frac{s^2+1\,ds}{2s\ln^p\left(\frac{s^2+1}{2s}\right)}.\end{aligned}$$

Since $1\leq e^{2s/\beta}\leq R^2$, and $\beta>0$ and $1\leq s^2+1\leq\beta^2\ln^2 R+1$, we need only worry about the integrability of

$$\int_0^S \frac{C ds}{s\ln^p\left(\frac{C}{s}\right)}$$

for some non-zero constant C and $S=\beta\ln R$. The change of variables $t=\ln\left(\frac{C}{s}\right)$ gives $\int_T^\infty t^{-p}dt$ where $T=\ln\left(\frac{C}{S}\right)$ and so we have proved our earlier claim about which Lorentz-Zygmund spaces these minimal solutions lie in.

It is a nontrivial theorem that a map shrinking a disk cannot have distortion in L^1, though this is easy to see for radially symmetric mappings.

Theorem 3.17. *There is no radially symmetric mapping $f:\mathbb{A}(1,R)\to\mathbb{D}^*$ such that $\mathbb{K}(z,f)\in L^1(\mathbb{A}(1,R),\mathbb{D}^*)$.*

Proof. The result follows as a result of the proof given in [23] for the proof of the Nitsche conjecture. Suppose there exists a mapping; then

$$\iint_{\mathbb{A}(1,R)}\mathbb{K}(z,f)\,dz<\infty.$$

Let $\delta>0$ be small; then because f is homeomorphic, the image $f(\mathbb{S}(1+\delta))$ of the circle of radius $1+\delta$ is a simple closed curve within $\mathbb{D}^*$, so the region between this and $\mathbb{S} = f(\mathbb{S}(R))$ is doubly connected, and thus conformally equivalent to a round annulus. Let

$$\tilde{f} : \mathbb{A}(1+\delta, R) \to \mathbb{A}(\epsilon(\delta), 1)$$

be defined by

$$\tilde{f}(z) = \varphi \circ f(z),$$

where φ is a conformal map mapping the image $f(\mathbb{A}(1+\delta, R))$ to the conformally equivalent annulus. Then

$$\iint_{\mathbb{A}(1+\delta,R)} \mathbb{K}\left(z, \tilde{f}\right) dz = \iint_{\mathbb{A}(1+\delta,R)} \mathbb{K}(z, f)\, dz < \infty.$$

So

$$\mathbb{K}\left(z, \tilde{f}\right) \in L^1(\mathbb{A}(1+\delta, R), \mathbb{A}(\epsilon(\delta), 1)),$$

and from the Nitsche conjecture,

$$\frac{1+\delta}{2}\left(\epsilon(\delta) + \frac{1}{\epsilon(\delta)}\right) \leq R;$$

by choosing δ small enough (as $\epsilon(\delta) \to 0$ as $\delta \to 0$) we must obtain a contradiction when R is finite. □

Corollary 3.18. *There is no map $f : \mathbb{C} \setminus E \to \mathbb{C} \setminus \{0\}$, where E is a non-degenerate continuum, by a radially symmetric mapping with distortion $\mathbb{K}(z, f) \in L^1$.*

Proof. Choose an open disk $D = \mathbb{D}(0, R)$ such that $E \subset D$. Let $f : \mathbb{C} \setminus E \to \mathbb{C} \setminus \{0\}$ be a mapping with $\mathbb{K}(z, f) \in L^1$. From Theorems 3.17, and from 3.5 we know that $D \setminus E$ is conformally equivalent (by, say, the mapping ψ) to an annulus, and therefore the image $f \circ \psi(D \setminus E)$ cannot be conformally equivalent to $\mathbb{D}^*$. This is a contradiction. □

3.8.2 L^1-minimisers In The Euclidean Metric

From (3.4.3) we have, noting that $\lambda(x) = 2\pi e^{2\pi x}$ for the Euclidean metric, that

$$u(x) = \int_{-\infty}^{x} \frac{e^{2\pi y} dy}{\sqrt{c^2 + e^{4\pi y}}} = \frac{1}{2\pi} \int_0^S \frac{ds}{\sqrt{c^2 + s^2}} = \frac{1}{2\pi} \log \frac{1}{c}\left(S + \sqrt{c^2 + S^2}\right),$$

where $S = e^{2\pi x}$. Hence the extremal radial mapping is defined by

$$\rho(r) = \frac{1}{c}\left(r + \sqrt{r^2 + c^2}\right),$$

with $\rho(0) = 1$ and
$$\rho(1) = R = \frac{1+\sqrt{1+c^2}}{c}.$$
Checking that the mean distortion is in fact finite, we observe that

$$\begin{aligned}\iint_{\mathbb{D}^*} \mathbb{K}(z,f)\,|dz|^2 &= \pi \int_0^1 \left(\frac{t}{\sqrt{t^2+c^2}} + \frac{\sqrt{t^2+c^2}}{t} \right) t\, dt \\ &= \pi\sqrt{1+c^2} = \pi \frac{R^2+1}{R^2-1} = \pi \coth 2\pi\sigma,\end{aligned}$$

where $\sigma = \frac{1}{2\pi} \ln R$ is the modulus of the ring $\mathbb{A}_R$.

From this, and the invariance under postcomposition by a conformal mapping, we have the estimate (which is the Nitsche result of Astala, Iwaniec and Martin [6]):

Theorem 3.19. *Let $\mathbb{D}^*_\epsilon = \{z : 0 < |z| < \epsilon\}$. Let f be a mapping of finite distortion $f : \mathbb{D}^*_\epsilon \to \mathbb{C}$. Then*
$$\coth 2\pi\sigma \le \frac{1}{\pi\epsilon^2} \iint_{\mathbb{D}^*_\epsilon} \mathbb{K}(z,f)\ |dz|^2, \tag{3.8.5}$$
where $\sigma = \operatorname{mod}(f(\mathbb{D}^*_\epsilon))$ is the modulus of the image.

This result tells us that the modulus cannot be too small unless the L^1 norm of the distortion is large.

3.8.3 L^p-minimisers In The Euclidean Metric, $1 < p < \infty$

Note that $\lambda(x) = 2\pi e^{2\pi x} \le 2\pi$ for $x \in (-\infty, 0]$. The lower bound on c from (3.6.14) reads as
$$\int_{-\infty}^0 \frac{p^{1/(p+1)}\left(2\pi e^{2\pi s}\right)^{2/(p+1)} ds}{c^{2/(p+1)} + p^{1/(p+1)}\left(2\pi e^{2\pi s}\right)^{2/(p+1)}} \le T.$$
Writing $c = 2\pi p^{1/2} C$ and integrating the left hand side gives

$$\frac{p+1}{4\pi} \ln\left(1 + C^{-2/(p+1)}\right) \le T,$$

so we have the estimate for c^2

$$c^2 \ge \frac{4\pi^2 p}{\left(e^{4\pi T/(p+1)} - 1\right)^{p+1}},$$

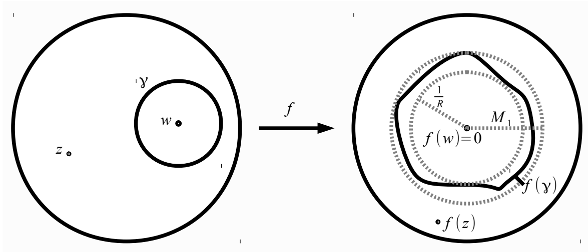


Figure 3.8.1: Illustration of the definitions of $\frac{1}{R}$ and M_1 from a given f.

Returning to the estimate (3.6.16), as

$$\int_{-\infty}^{0} e^{4\pi x/(p+1)}\, dx = \frac{p+1}{4\pi},$$

we then obtain the bound on the original problem on the punctured disk:

$$\begin{aligned}\mathcal{K}_p &\geq 2^{-p}\left(\frac{c^2}{2^p p}\right)^{p/(p+1)} (2\pi)^{2/(p+1)} \frac{p+1}{4\pi} \\ &\geq \frac{\pi}{\left(e^{4\pi T/(p+1)}-1\right)^p} \frac{p+1}{2^{p(2p+1)/(p+1)}},\end{aligned}$$

which, upon taking the pth root, becomes

$$\|\mathbb{K}(z,f)\|_p \geq \frac{\pi^{1/p}}{\left(e^{4\pi T/(p+1)}-1\right)} \frac{(p+1)^{1/p}}{2^{(2p+1)/(p+1)}}.$$

Taking the choice $T = \frac{1}{2\pi}\ln R$ gives us (3.8.1).

As a consequence of the proofs of Theorem 3.10 and Theorem 3.11, we obtain the following 'large scale injectivity' estimate.

Corollary 3.20. *Let $p \geq 1$ and $0 < \epsilon < \infty$. Suppose f is a mapping of finite distortion $f : \mathbb{D} \to \mathbb{D}$ with $\mathcal{K}_p < \infty$. Then there is an $M = M(\mathcal{K}_p, \epsilon)$ such that*

$$\rho_{\mathbb{D}}(f(z), f(w)) \geq M \Longrightarrow \rho_{\mathbb{D}}(z,w) \geq \epsilon.$$

Proof. Because postcomposition by a Möbius transformation does not change $\mathbb{K}(z,f)$ and so $\mathcal{K}_p$, we may assume $f(w) = 0$. We then consider the hyperbolic ϵ-ball about w; we set γ to be its boundary, and then set $\frac{1}{R}$ to be the minimum and M_1 to be the maximum Euclidean distances from $f(\gamma)$ to the origin.

From (3.8.1), we have the lower bound of $\|\mathbb{K}(z,f)\|_p$ for fixed p, and so we have the constant

$$C_p = \frac{\pi^{1/p}(p+1)^{1/p}}{2^{(2p+1)/(p+1)}}.$$

Using the Euclidean metric, we construct a mapping of finite distortion $\tilde{g} : \mathbb{D}^* \to \mathbb{A}_R$

for which $\mathcal{K}_p < \infty$ and

$$\mathcal{K}_p^{1/p} \geq \frac{C_p}{R^{2/(p+1)} - 1},$$

for the R we established earlier, noting R is finite. A simple conformal dilation allows us to change the range to $\mathbb{A}\left(\frac{1}{R}, 1\right)$ without affecting $\mathcal{K}_p$. Since we have $M_1 \geq \frac{1}{R}$, and noting

$$\left(\frac{\mathcal{K}_p^{1/p}}{C_p + \mathcal{K}_p^{1/p}}\right)^{(p+1)/2} \geq \frac{1}{R},$$

we choose M_2 to be the maximum of M_1 and $\left(\frac{C_p+\mathcal{K}_p^{1/p}}{\mathcal{K}_p^{1/p}}\right)^{-(p+1)/2}$.

From the definition of $\rho_{\mathbb{D}}$ we know that for $z \in \mathbb{D}$ we have $f(z) \in \mathbb{D}$ and so

$$\rho_{\mathbb{D}}(f(z), 0) = \ln \frac{1 + |f(z)|}{1 - |f(z)|}.$$

As the mapping

$$\mathcal{M} : t \mapsto \ln \frac{1+t}{1-t}$$

is strictly increasing, letting $M = \mathcal{M}(M_2)$ gives us $\rho_{\mathbb{D}}(f(z), f(w)) \leq M$ and this gives us our result. $\square$

Chapter 4

Resolving A Decomposition

4.1 Decompositions

A subset E of the complex plane $\mathbb{C}$ is a *continuum* if E is compact and connected, while E is *cellular* if E is compact connected and $\hat{\mathbb{C}} \setminus E$ is simply connected. A *decomposition* of the complex plane $\mathbb{C}$ is a collection $\mathcal{E}$ of disjoint cellular continua such that $\cup_{E \in \mathcal{E}} E = \mathbb{C}$. Moreover we will assume in this thesis that only countably many of these continua are non-degenerate; that is, does not consist of a single point. In fact we shall mostly restrict our study to the case that the non-degenerate elements of the decomposition are generated by a countable family of closed geodesic lines $\{\gamma_i\}_{i \in \mathbb{N}}$ in the disk; all the results apply equally when the decomposition is ambiently quasiconformally equivalent to such a decomposition. We shall also make some geometric assumptions that imply upper-semicontinuity and regularity on the continua: we shall explain what each of these mean now.

The definition for upper-semicontinuity comes from Moore [39]. We say that the distance $d(x,E)$ between a point x and a point set E to be defined as

$$d\left(x, E\right) = \inf_{y \in E} d\left(x, y\right).$$

Given two point sets E and E', we define the *lower distance*

$$l\left(E, E'\right) = \inf_{x \in E} d\left(x, E'\right),$$

and E is said to be the lower distance $l\left(E, E'\right)$ from E'. We also have the *upper distance*

$$u\left(E, E'\right) = \sup_{x \in E} d\left(x, E'\right),$$

provided the supremum exists, and E is said to be the upper distance $u(E, E')$ from E'. Note that the order of E and E' matters here, as the definition for upper distance is not symmetric. *Upper-semicontinuity* on a collection $\mathcal{E}$ of continua means that for each continuum $E \in \mathcal{E}$ and for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $E' \in \mathcal{E}$ at a lower distance from E less than δ, then the upper distance of E' from E is less than ϵ.

The definition of regularity comes from Chinen [10]. The *mesh* of an open cover $\mathcal{U}$ of a continuum E is the supremum of the diameter of the elements of $\mathcal{U}$, which is denoted by mesh$\mathcal{U}$. For an open cover $\mathcal{U}$ of E we set $\mathrm{Bd}\,(\mathcal{U}) = \bigcup \{\mathrm{Bd}\,(U) : U \in \mathcal{U}\}$, where $\mathrm{Bd}\,(U)$ denotes the boundary of U in E. A continuum E is said to be *regular* if for each $\epsilon > 0$, there exists a finite open cover $\mathcal{U}$ of E with mesh$\mathcal{U} < \epsilon$ such that $\mathrm{Bd}\,(\mathcal{U})$ is finite.

Given such a decomposition, we can define an equivalence relation $\sim$ on $\mathbb{C}$ by using the elements of $\mathcal{E}$ as the equivalence classes: that is $z \sim w$ if and only if there is an $E \in \mathcal{E}$ such that $z, w \in E$. From the assumptions of upper-semicontinuity, we have from Moore [39] that the quotient space is homeomorphic to the complex plane, or in the words of the article:

Theorem 4.1. (Theorem 25 of *[39]*) *If, in a plane S, M is a closed and bounded point set no subset of which separates S, and every maximal connected subset of M is considered as an element, and every point which does not belong to M is considered as an element, then the set of all such elements is topologically equivalent to the set of all points in a plane.*

We shall restate what we mean by *resolving a decomposition*: taking a planar curve separating the plane into a countable collection of disjoint domains Ω_i, say with a subcollection touching at the origin, $\bigcap_{i\in I} \overline{\Omega_i} = \{0\}$ (where I is the (possibly countable) subcollection's indexing set), then seeking a mapping f of the plane which is of finite distortion and a homeomorphism away from the origin so that $\mathrm{int}\left(f\left(\bigcup_{i\in I} \overline{\Omega_i}\right)\right)$ is a domain.

An example of what we mean by this resolution is given in Section 4.2 and illustrated by Figure 4.2.2 on page 62, where we illustrate the decomposition of a finite collection of disjoint domains. However, we first need to make a few definitions that we shall make use of later in this chapter in proving some useful results.

4.1.1 Separation In Modulus And Distance

Definition 4.2. A decomposition is *separated in moduli* if there is a positive number δ such that for each non-degenerate component $E \in \mathcal{E}$ there is an open neighborhood U of E meeting no other non-degenerate components of $\mathcal{E}$ and such that the modulus of $U \setminus E$ is greater than δ.

A more general situation which arises naturally - for instance, in the lifts of simple geodesics - is when there is a uniform lower bound on the hyperbolic distance between these lines. Then non-degenerate continua can accumulate, but only at the endpoints.

Definition 4.3. Let $\mathcal{E}$ be a decomposition of $\mathbb{C}$ whose non-degenerate components form a family $\{\gamma_i\}_{i\in\mathbb{N}}$ of closed geodesic lines in the hyperbolic Poincaré disk. We say that $\mathcal{E}$ is *separated in distance* if there is a constant $M > 0$ such that the hyperbolic distance $d_{hyp}(\gamma_i, \gamma_j) \geq M$ whenever $i \neq j$.

Lemma 4.4. *If the non-degenerate components of a decomposition that is separated in moduli are geodesic arcs γ_i, then for each i there is an annulus A_i of definite modulus (natural logarithmic ratio of inner and outer radii greater than ϵ) which has γ_i as its central curve and meets no other ($A_j \cap A_i = \emptyset$ if $i \neq j$).*

Proof. For each arc γ_i, choose a Möbius transformation φ_i such that $\varphi_i(\gamma_i) = [-1, 1]$. Set $V_i = \varphi_i(U_i)$ where U_i is given by the definition of separation in modulus (so $U_i \cap \gamma_j = \emptyset$ when $j \neq i$, and the modulus of $U_i \setminus \gamma_i > \delta$ for some given $\delta > 0$).

There is a $\delta' > 0$ so that for any open neighborhood X of $[-1, 1]$, if $l(\partial X, [-1, 1]) < \delta'$ then $\operatorname{mod}(X \setminus [-1, 1]) < \delta$. Set $W' = \{z : d(z, [-1, 1]) < \alpha\}$ where $\alpha = \alpha(\delta')$ is chosen such that $\operatorname{mod}(W' \setminus [-1, 1]) = \frac{\delta}{8}$; we also define $W_i = \varphi_i^{-1}(W')$. Clearly $W_i \subset U_i$, and $\operatorname{mod}(W_i \setminus \gamma_i) > \epsilon = \epsilon(\delta')$ for $\epsilon(\delta') > 0$ as φ_i are conformal and W' has definite modulus.

If $W_i \cap W_j \neq \emptyset$ for $j \neq i$, then let $z \in W_i \cap W_j$. We define the following collections of locally rectifiable curves (that is, for these definitions we will assume the curves are locally rectifiable):

- Λ is defined to be the set of all curves from γ_i to γ_j.
- Λ' is the set of all curves from γ_i to the boundary ∂U_i of U_i. Note that Λ is minorised by Λ' because $\gamma_i \subset U_i$ while $U_i \cap \gamma_j = \emptyset$, so any such curve must cross the boundary of U_i at some point. Therefore, from the definition of U_i and the properties of moduli, we have that

$$\delta < \operatorname{mod}(\Lambda') \leq \operatorname{mod}(\Lambda).$$

- Λ_z is the set of all curves from γ_i to γ_j which pass through z. Note that $\Lambda_z \subset \Lambda$, so

$$\delta < \operatorname{mod}(\Lambda) \leq \operatorname{mod}(\Lambda_z).$$

- Λ_i is the set of all curves from γ_i to z. If Λ_i' is the set of all curves from γ_i to ∂W_i then note Λ_i' is minorised by Λ_i, since $z \in W_i$; therefore $\operatorname{mod}(\Lambda_i) \leq \frac{\delta}{8}$.

- $\circ$ Λ_j is the set of all curves from z to γ_j. If Λ_j' is the set of all curves from ∂W_j to γ_j then note Λ_j' is minorised by Λ_j, since $z \in W_j$; therefore $\operatorname{mod}(\Lambda_j) \leq \frac{\delta}{8}$.

Suppose $\rho \in \operatorname{adm}(\Lambda_z)$; then for every $\lambda : [a,b] \to \mathbb{C} \in \Lambda_z$

$$\int_\lambda \rho ds = \int_a^b \rho(\lambda(t))\, dt \geq 1.$$

Note that λ can be represented by

$$\lambda(t) = \begin{cases} \lambda_i\left(a + (b-a)\frac{t-a}{c-a}\right) & a \leq t \leq c, \\ \lambda_j\left(a + (b-a)\frac{t-c}{b-c}\right) & c \leq t \leq b, \end{cases}$$

where $\lambda(c) = z$, $\lambda_i : [a,b] \to \mathbb{C} \in \Lambda_i$ and $\lambda_j : [a,b] \to \mathbb{C} \in \Lambda_j$. So

$$\begin{aligned} \int_a^b \rho(\lambda(t))\, dt &= \int_a^c \rho(\lambda(t))\, dt + \int_c^b \rho(\lambda(t))\, dt \\ &= \int_a^c \rho\left(\lambda_i\left(a + (b-a)\frac{t-a}{c-a}\right)\right) dt \\ &\quad + \int_c^b \rho\left(\lambda_j\left(a + (b-a)\frac{t-c}{b-c}\right)\right) dt \\ &= \int_a^b \rho(\lambda_i(t))\, dt + \int_a^b \rho(\lambda_j(t))\, dt. \end{aligned}$$

This means

$$\int_a^b \rho(\lambda_i(t))\, dt + \int_a^b \rho(\lambda_j(t))\, dt \geq 1,$$

which gives us

$$\int_a^b \rho(\lambda_i(t))\, dt = \alpha \quad \text{and} \quad \int_a^b \rho(\lambda_j(t))\, dt = \beta$$

with $\alpha + \beta \geq 1$.

If $\alpha \geq 1$ then define $\rho_i := \rho$ and $\rho_j \in \operatorname{adm}(\Lambda_{\mathrm{j}})$ be arbitrary; and define $\sigma_i = 1$ and $\sigma_j = 0$. Similarly, if $\beta \geq 1$ then $\rho_j := \rho$ and $\rho_i \in \operatorname{adm}(\Lambda_{\mathrm{i}})$ be arbitrary; and define $\sigma_i = 0$ and $\sigma_j = 1$. Otherwise $0 < \alpha, \beta < 1$, and we define $\rho_i := \frac{\rho}{\alpha}$ and $\rho_j := \frac{\rho}{\beta}$ with $\sigma_i = \frac{\alpha}{2}$ and $\sigma_j = \frac{\beta}{2}$. In all cases, $\rho_i \in \operatorname{adm}(\Lambda_i)$ and $\rho_j \in \operatorname{adm}(\Lambda_j)$ and $\rho = \sigma_i\rho_i + \sigma_j\rho_j$ with $\sigma_i, \sigma_j \in [0,1]$.

Therefore, for each $\rho \in \operatorname{adm}(\Lambda_z)$ we have that

$$\int_\Omega \rho^2 dm = \int_\Omega (\sigma_i\rho_i + \sigma_j\rho_j)^2\, dm,$$

so

$$\int_\Omega \rho^2 dm \geq \sigma_i^2 \int_\Omega \rho_i^2 dm + \sigma_j^2 \int \rho_j^2 dm.$$

If we consider the infima for each integral, we have that

$$\frac{1}{\operatorname{mod}(\Lambda_z)} \geq \frac{\sigma_i^2}{\operatorname{mod}(\Lambda_i)} + \frac{\sigma_j^2}{\operatorname{mod}(\Lambda_j)} = \frac{\sigma_i^2 \operatorname{mod}(\Lambda_j) + \sigma_j^2 \operatorname{mod}(\Lambda_i)}{\operatorname{mod}(\Lambda_i)\operatorname{mod}(\Lambda_j)},$$

or

$$\operatorname{mod}(\Lambda_z) \leq \frac{\operatorname{mod}(\Lambda_i)\operatorname{mod}(\Lambda_j)}{\sigma_i^2 \operatorname{mod}(\Lambda_j) + \sigma_j^2 \operatorname{mod}(\Lambda_i)},$$

and from the moduli of Λ_i and Λ_j we have the bound

$$\operatorname{mod}(\Lambda_z) \leq \frac{\delta}{8} \frac{1}{\sigma_i^2 + \sigma_j^2}.$$

Finally, from the definitions of σ_i and σ_j, we know

$$\sigma_i^2 + \sigma_j^2 = \begin{cases} 1 & \alpha \geq 1 \text{ or } \beta \geq 1, \\ \frac{\alpha^2+\beta^2}{4} & 0 < \alpha, \beta < 1, \end{cases}$$

and in the latter case we have $1 \leq \alpha + \beta < 2$, so by Lemma A.4,

$$\frac{1}{8} \leq \frac{(\alpha+\beta)^2}{8} \leq \frac{\alpha^2+\beta^2}{4} \leq \frac{(\alpha+\beta)^2}{4} < 1.$$

Hence $\operatorname{mod}(\Lambda_z) \leq \delta$, which is a contradiction, as we have already shown $\delta < \operatorname{mod}(\Lambda_z)$.

Finally, we construct A_i by extending W' to cover the whole of the real line:

$$A' = \{z : d(z, \mathbb{R}) < \alpha\} = \{z : |\Im(z)| < \alpha\};$$

and define the annulus $A_i = \varphi_i^{-1}(A')$. For convenience, for $j \neq i$ we let $S_{j,i} = \varphi_i(S_j)$ where S_j is the circle or diameter for which $S_j \cap \overline{\mathbb{D}} = \gamma_j$; similarly $A_{j,i} = \varphi_i(A_j)$. The tangent lines l_1, l_2 of $S_{j,i}$ where it intersects $\mathbb{S}$ pass through zero, and divide the plane into four regions. Moreover, $S_{j,i}$ belongs to exactly one of the two regions from these four which only intersect the real line at the origin, as otherwise γ_i and γ_j would intersect. We observe that in either of these two regions, the points of $A_{j,i}$ nearest the origin must lie within $\mathbb{D}$, and so within W'; however, this is a contradiction.

□

This result leads us to the following theorem.

Theorem 4.5. (Separation in Modulus Decomposition Resolution Theorem) *Let $\mathcal{E}$ be a decomposition of $\mathbb{C}$ whose non-degenerate components are closed geodesic lines* *in the disk, and which is separated in modulus. Then there is a mapping* $g : \mathbb{C} \backslash \mathcal{E} \to \mathbb{C}$ *which acts as a quotient map of* $\mathcal{E}$ *whose inverse mapping* $f = g^{-1} : \mathbb{C} \to \mathbb{C}$ *gives a resolution of this decomposition such that*

- g *is a continuous monotone mapping in* $W^{1,2}_{loc}(\mathbb{C})$,

- *the image of the non-degenerate continua under* g *form a finite set or a countable set* K *of Hausdorff dimension* 0,

- f *is a mapping of finite distortion with* $\mathbb{K}(z, f) \in L^1_{loc}(\mathbb{C})$, *and*

- *each component of* $\mathbb{C} \setminus g(\mathbb{S})$ *is conformally equivalent to a round disk.*

See Figure 1.1.4 on page 4 for an illustration of this theorem. We shall prove this theorem later in 4.2.1. However, we shall be making use of an intermediary result, Theorem 4.7, which we need to establish first.

Notice that the property of a decomposition being separated in moduli is preserved (with variation of constants) by quasiconformal mappings. The condition of separation in modulus does not allow the accumulation of non-degenerate continua at a point of another non-degenerate continua (they can of course accumulate elsewhere). We have the following result.

Lemma 4.6. *If* $\mathcal{E}$ *is separated in modulus, then* $\mathcal{E}$ *is upper semicontinuous.*

Proof. Choose an arbitrary $E \in \mathcal{E}$ and $\epsilon > 0$. Choose $E' \in \mathcal{E}$ such that $u(E', E) \geq \epsilon$. Then $E' \neq E$ (the upper distance between a set and itself is easily shown to be zero by the definition). As $\mathcal{E}$ is separated in modulus we have an open neighborhood of E, U_E, such that $U_E \cap E' = \emptyset$ and the modulus of $U_E \setminus E$ is greater than some $\delta_0 > 0$. The lower distance of ∂U_E from E is therefore greater than some $\delta > 0$ where δ depends on δ_0 and E; the lower distance of E' from E' must also be at least this distance. $\square$

4.2 Resolution Of Decompositions

Before we proceed, let us make what we want clear, as there are many ways that we could resolve the singularities which arise from decompositions. Let us first discuss briefly what we don't want: that is, resolving the singularity without separating the "point of contact". Consider the cusp defined by the equation

$$\mathcal{C} = \left\{ z = x + iy : y = \pm |x|^\beta, -1 \leq x \leq 1 \right\}.$$

The angle at the origin will be 0 as soon as $\beta < 1$. One obvious way to resolve such a cusp is consider the mapping for $a, b > 0$,

$$h : (x, y) \to \left(\operatorname{sgn}(x) |x|^a, \operatorname{sgn}(y) |y|^b\right), \qquad \beta b = a.$$

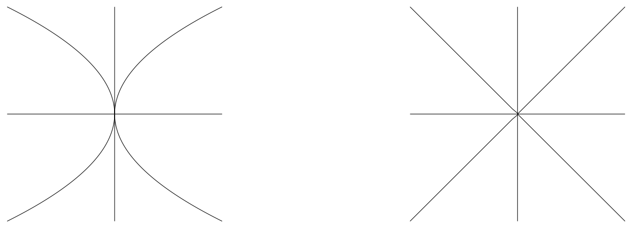


Figure 4.2.1: Resolving a cusp without separation.

The image of $\mathcal{C}$ under this mapping is the (noncusped) curve $\{y = |x|\}$. We calculate that (in the positive quadrant)

$$Dh = \begin{pmatrix} ax^{a-1} & 0 \\ 0 & by^{b-1} \end{pmatrix} \qquad \mathbb{K}(z, f) = \frac{1}{2}\left(\frac{ax^{a-1}}{by^{b-1}} + \frac{by^{b-1}}{ax^{a-1}}\right),$$

so that

$$\begin{aligned} \iint_{Q_1} \mathbb{K}^p(z, f) |dz|^2 &= \frac{1}{2^{p-1}} \int_0^1 \int_0^1 \left(\frac{ax^{a-1}}{by^{b-1}} + \frac{by^{b-1}}{ax^{a-1}}\right)^p dx\, dy \\ &\leq \int_0^1 \int_0^1 \frac{a^p x^{p(a-1)}}{b^p y^{p(b-1)}} + \frac{b^p y^{p(b-1)}}{a^p x^{p(a-1)}} dx\, dy \\ &= \frac{a^p}{b^p (1 + p(a-1))(1 - p(b-1))} \\ &\quad + \frac{b^p}{a^p (1 - p(a-1))(1 + p(b-1))} \end{aligned}$$

is finite when

$$1 - \frac{1}{p} < \beta b \leq b < 1 + \frac{1}{p},$$

but this resolution does not separate the "point of contact" as zero is fixed by this family of functions. Since this is what we are primarily interested in, we need to find another technique.

Let us now consider resolving a multi-cusped object as illustrated by Figure 4.2.2 on page 62. The parametric equation of a $2q$-lobed curve is

$$z(\theta) = |\sin(q\theta)| e^{i\theta}, \quad \theta \in [0, 2\pi]. \tag{4.2.1}$$

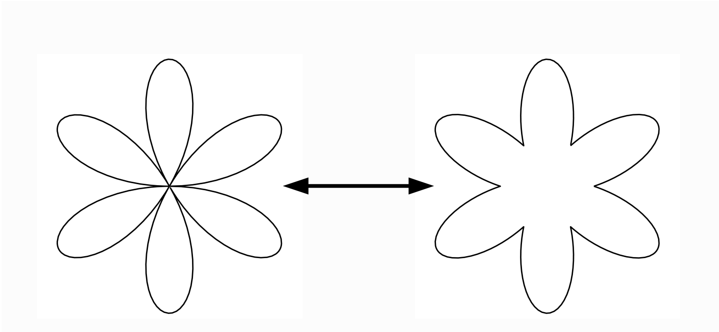


Figure 4.2.2: Resolution of a 6-lobed curve to a quasidisk.

Let us consider what happens near $\theta = \pi/q$. We have the derivatives

$$z'(\theta) = \begin{cases} (q\cos(q\theta) + i\sin(q\theta))\, e^{i\theta}, & \theta < \pi/q \\ -\left(q\cos(q\theta) + i\sin(q\theta)\right) e^{i\theta}, & \theta > \pi/q \end{cases}$$

and so the tangent vectors $z'_+(\pi/q) = -qe^{i\pi/q}$ and $z'_-(\pi/q) = qe^{i\pi/q}$ turn an angle π at π/q and we have a cusp. Let us examine the image under the map

$$z \mapsto \frac{z}{|z|} e^{|z|^\alpha}$$

that we explored earlier. The parametric equation of the image is

$$w(\theta) = e^{|\sin(q\theta)|^\alpha} e^{i\theta}, \quad \theta \in [0, 2\pi]$$

and

$$w'(\theta) = \begin{cases} e^{|\sin(q\theta)|^\alpha} \left(\alpha q\, |\sin(q\theta)|^{\alpha-1} \cos(q\theta) + i\right) e^{i\theta}, & \theta < \pi/q \\ e^{|\sin(q\theta)|^\alpha} \left(-\alpha q\, |\sin(q\theta)|^{\alpha-1} \cos(q\theta) + i\right) e^{i\theta}, & \theta > \pi/q \end{cases}.$$

If $\alpha < 1$, then as $\theta \to \pi/q$, the tangent turns through π and the image is again a cusp. However, if $\alpha = 1$ then

$$\begin{aligned} w'_+(\pi/q) &= (-q + i)\, e^{i\pi/q}, \\ w'_-(\pi/q) &= (q + i)\, e^{i\pi/q}. \end{aligned}$$

The angle turned is the argument of $w'_-\overline{w'_+}$,

$$\arg\left(w'_-\overline{w'_+}\right) = \arg\left((q + i)(-q - i)\right),$$

which shows that the angle formed by the image curve at $\theta = \pi/q$ is

$$\arg\left(-(i + q)^2\right) = 2\arctan\left(\frac{1}{q}\right) + \pi \bmod 2\pi,$$

and so the image does not have a cusp for finite q.

Consider the mapping $F : \mathbb{C} \to \mathbb{C}$ defined by

$$F\left(re^{i\theta}\right) = re^{|\sin(q\theta)|}e^{i\theta}.$$

This function is invertible (as it is multiplication by a non-zero value for all θ), and we calculate $|\mu_F|$:

$$\left|\mu_F\left(re^{i\theta}\right)\right| = \frac{q\,|\cos(q\theta)|}{\sqrt{4 + q^2\,|\cos(q\theta)|^2}},$$

which attain a maximum when $q\theta = 0 \mod \pi$. Then

$$|\mu_F| \leq \frac{q}{\sqrt{4+q^2}},$$

and so

$$\mathbb{K} \leq 1 + \frac{q^2}{2};$$

therefore the image is a quasicircle (the image under the unit circle of a K-quasiconformal homeomorphism of $\mathbb{C}$). Important for us is the case $q = 1$ which shows we can resolve the tangency of two round disks.

Theorem 4.7. *There is a mapping $f : \mathbb{D} \to \mathbb{C}$ of finite distortion with $\mathbb{K}(z, f) \in L^1(\mathbb{D})$ with the following property: the boundary of the two tangent disks $D_1 = \left\{\left|z - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$ and $D_2 = \left\{\left|z + \frac{1}{2}\right| \leq \frac{1}{2}\right\}$ is mapped to the unit circle and the image $f\left((D_1 \cup D_2) \setminus \{0\}\right)$ is the disk minus a line segment, $\mathbb{D} \setminus i\,[-1, 1]$.*

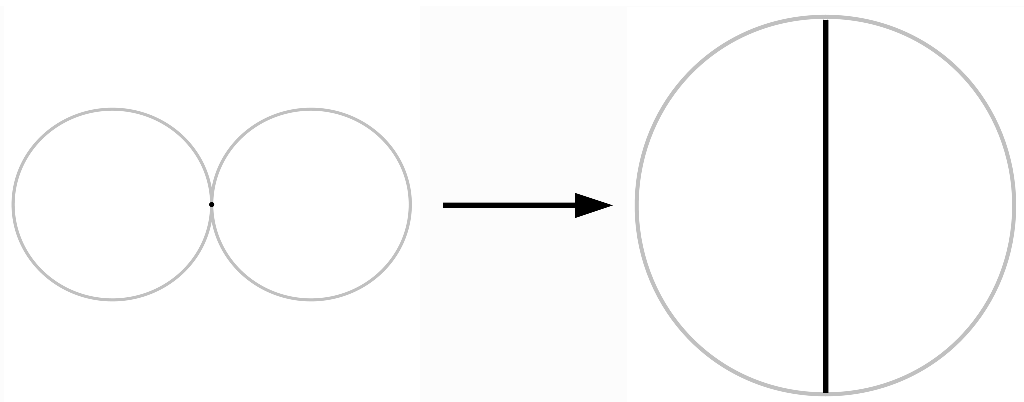


Figure 4.2.3: Resolution of a cusp

Proof. The boundary of two disks is smoothly parametrised by the equation $z = |\cos\theta|\,e^{i\theta}$: we show this is the case by first noting that for $\theta \in [-\pi/2, \pi/2)$ we require that

$$\begin{aligned} r\cos\theta &= \frac{1}{2}(1+\cos\alpha) \quad \text{and} \\ r\sin\theta &= \frac{1}{2}\sin\alpha \end{aligned}$$

for some α. Squaring and adding both sides gives

$$\begin{aligned} r^2 &= \frac{1}{4}\left[(1+\cos\alpha)^2+\sin^2\alpha\right] \\ &= \frac{1}{4}\left[1+2\cos\alpha+\cos^2\alpha+\sin^2\alpha\right], \end{aligned}$$

and so

$$r^2 = \frac{1}{2}(1+\cos\alpha) = r\cos\theta.$$

Thus $r = \cos\theta$ in this region; note also that $\cos\theta \geq 0$ in this region as well. For $\theta \in [\pi/2, 3\pi/2)$ we have that $\cos\theta \leq 0$, and for some α

$$\begin{aligned} r\cos\theta &= -\frac{1}{2}(1-\cos\alpha) \quad \text{and} \\ r\sin\theta &= \frac{1}{2}\sin\alpha. \end{aligned}$$

Squaring and adding both as before gives us, after a similar simplification

$$r^2 = \frac{1}{2}(1-\cos\alpha) = -r\cos\theta,$$

so $r = -\cos\theta$. From this we determine $r = |\cos\theta|$.

We first blow up the origin via the map $f_1 : z \mapsto \frac{z}{|z|}e^{|z|}$; the image now omits the unit disk. We calculated and examined the integrability properties of the distortion of this mapping in 3.8.1. We follow by the conformal mapping $f_2 : z \mapsto \frac{1}{2}\left(z - \frac{1}{z}\right)$ defined on the exterior of the disk. (Note that this is similar to the case above, just rotated so the lobes lie on the real axis.) The conformal map takes $\mathbb{C} \setminus \mathbb{D}$ to $\mathbb{C} \setminus i[-1,1]$, and as this last map is conformal, it does not change the integrability properties of $\mathbb{K}$.

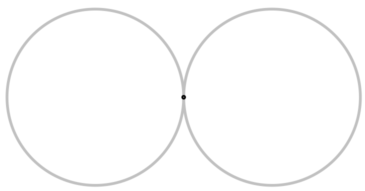
The image of the boundary of the two disks is smoothly parametrised by the equation

$$\{z(\theta) = \sinh(|\cos(\theta)| + i\theta)\}. \tag{4.2.2}$$

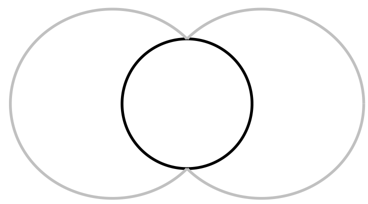
The composition of the two maps given earlier is

$$z \mapsto \frac{1}{2}\left(\frac{z}{|z|}e^{|z|} - \frac{|z|}{z}e^{-|z|}\right) = \sinh\left(|z| + \ln\frac{z}{|z|}\right)$$

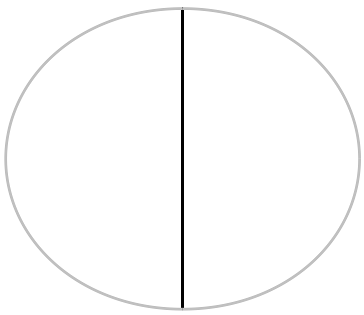
and when substituting in the boundary value parametrization $z = |\cos\theta|\, e^{i\theta}$ we obtain this fact, noting that $\ln \frac{z}{|z|} = i\theta$. We must now finally map this boundary to the unit circle, which can be done by a minor postcomposition with a conformal mapping f_3 using the Riemann mapping theorem (we can assume that $f_3(\pm i) = \pm i$ as we can simply transform the circle by a Möbius transformation). Again, as this map is conformal, it does not change the integrability properties of $\mathbb{K}$. $\square$



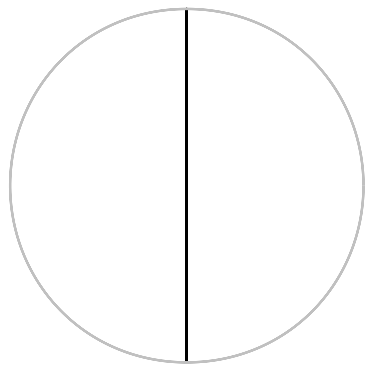
(a) The two tangent disks.



(b) Blowing up the origin to the unit disk.



(c) Squashing the unit disk to $i[-1,1]$



(d) Mapping the image to the unit circle.

Figure 4.2.4: Illustration of Theorem 4.7

Notice that the inverse of the map we have constructed has a continuous extension to the "missing arc" by defining it to be zero there. Furthermore, it is a mapping of finite distortion, and although the distortion is not in any reasonable integrability class (as we noted earlier when we examined this example function) we note that the mapping itself is in the better Sobolev space $W^{1,2}_{loc}$ (which we can note by setting $p = 2$ into the appropriate working in 3.8.1). This inverse mapping shrinks a diameter of the unit disk in the plane. The image of the unit disk is two tangent disks.

Clearly we can further modify the map of Theorem 4.7. Any postcomposition by a quasiconformal map will not effect the integrability properties of the distortion, which will be multiplied by an L^∞ function, or the Sobolev $W^{1,2}$ regularity of the inverse, and so on.

Thus, using the quasiconformal Schoenflies Theorem (see for example [17], [32] and [53] for results pertaining to this theorem) we can construct such a mapping f blowing up the origin for which:

- the domain of f is $\mathbb{C}$,
- outside $\mathbb{D}(0,\epsilon)$ the map is a dilation $z \mapsto \lambda z$, $\lambda \in \mathbb{R}^+$,
- for properties we want in its capacity as an inverse, $\mathbb{K} - 1 \in L^1(\mathbb{C})$, and
- given any infinite collection of disks $\{D_i = \mathbb{D}(z_i, r_i)\} \subset \mathbb{D}(0,\epsilon) \setminus \{0\}$ which is separated, and separated away from 0, in the sense that there is a $\delta > 0$ such that

$$\mathbb{D}(z_i,(1+\delta)r_i) \cap \mathbb{D}(z_j,(1+\delta)r_j) = \emptyset, \quad i \neq j \text{ and}$$

$$\mathbb{D}(0,\delta) \cap \mathbb{D}(z_j,(1+\delta)r_j) = \emptyset, \quad \text{for all } j,$$

  the map $f|D_j$ is a similarity.

We first make some observations. First, we can change the domain $\mathbb{D}$ to $\mathbb{D}(0,\epsilon)$ by simply scaling the variable. Although this is conformal, it affects the integral of the distortion by the action of the Jacobian. Set $g(z) = f(z/\epsilon)$. Then $\mathbb{K}(z,g) = \mathbb{K}(z/\epsilon, f)$ and

$$\iint_{\mathbb{D}(0,\epsilon)} \mathbb{K}(z,g)\ |dz|^2 = \iint_{\mathbb{D}(0,\epsilon)} \mathbb{K}(z/\epsilon,f)\ |dz|^2 = \epsilon^2 \iint_{\mathbb{D}} \mathbb{K}(w,f)\ |dw|^2,$$

so the change in the integral of the distortion is just the change in areas of the domains. Of course our initial map is radial and therefore can be extended by a similarity, with a little attention where we had to change the smooth curve to the circle.

Second, our mapping f is a $\mathbb{K}_\delta$-quasiconformal diffeomorphism outside $\mathbb{D}(0,\delta)$ and the separation of these disks by annuli of a definite modulus tells us - via the quasiconformal Schoenflies Theorem - that the modified mapping has had its distortion increased to a number which depends only on δ (and $\mathbb{K}_\delta$). The construction is to modify the map inductively on $\mathbb{D}(z_j, 2r_j)$ keeping the boundary values the same near $\mathbb{S}(z_j, 2r_j)$. Of course, we do not modify the map in $\mathbb{D}(0,\delta)$ and so the integral of the distortion has gone up by a factor which depends only on δ.

We do not seem to be able to allow these disks to accumulate at the origin. One can see that a basic requirement for the techniques we have used so far is that the images of the disks in question are all K-quasicircles, for some bounded K. The eccentricities of any family of disks accumulating at the origin must tend to ∞ under this mapping.

Another problem we face is that if we move things around into a geometrically nice configuration by a quasiconformal map before we blow up some points, then when we compute the L^p class of the distortion of the mapping, there is a change of variable involved in the integral (it has a term involving the Jacobian of this quasiconformal mapping). Post composition is fine, since the distortion of a quasiconformal map is in L^∞ and can be pulled out of an integral estimate. There are ways to resolve these problems, for example Muckenhoupt weight estimates (see [40]), etc., but the easiest way is to simply ensure that when we move important things around (sets which contain points to be blown up) that we do so by similarity transformations. Then the (constant) Jacobian term comes out and is proportional to the change in area, and therefore easily controlled.

4.2.1 Proof of Separation in Modulus Decomposition Resolution Theorem

Proof. (Theorem 4.5) We first prove this result where $\mathcal{E}$ consists of one closed geodesic line. We first map this closed geodesic line to the imaginary interval $i\,[-1,1]$ by the use of an appropriate Möbius transformation η. We define f as the inverse of g, where $g = g_1 \circ g_2 \circ g_3 \circ \eta$ and where $g_i = f_i^{-1}$ are the inverses of the appropriate mappings from Theorem 4.7: in particular, for g_1 and g_2 these are

$$g_1(z) = \frac{z \ln |z|}{|z|}$$

and

$$g_2(z) = \begin{cases} z + \sqrt{z^2+1} & \Re(z) > 0, \\ z - \sqrt{z^2+1} & \text{otherwise.} \end{cases}$$

Each of the component functions is continuous and monotone, so the function itself is continuous and monotone. Let us write h for the composition $g_2 \circ g_3 \circ \eta$. We have on the compact subset $\Omega \subset \mathbb{C} \setminus (i\,[-1,1])$that

$$\|g\|_{1,2,\Omega} = \iint_\Omega |g_1 \circ h|^2 + |(g_1 \circ h)_z|^2 + |(g_1 \circ h)_{\overline{z}}|^2 \, |dz|^2 .$$

As h is conformal,

$$\|g\|_{1,2,\Omega} = \iint_\Omega |g_1(h(z))|^2 + \left(|(g_1)_w (h(z))|^2 + |(g_1)_{\overline{w}} (h(z))|^2\right) |h'|^2 \, |dz|^2 .$$

This means we have that g is in $W^{1,2}_{loc}\left(\mathbb{C}\setminus\left(i\left[-1,1\right]\right)\right)$: by letting $\Omega' = h\left(\Omega\right)$, we may write the previous equation as

$$\left\|g\right\|_{1,2,\Omega} = \iint_{\Omega} \left|g\left(z\right)\right|^2 \left|dz\right|^2 + \iint_{\Omega'} \left(\left|\left(g_1\right)_w\left(w\right)\right|^2 + \left|\left(g_1\right)_{\overline{w}}\left(w\right)\right|^2\right) \left|dw\right|^2.$$

The first integral is finite, as $|g|$ is bounded on Ω. The second is finite from (3.8.4); as g_1 is in $W^{1,2}_{loc}(\mathbb{A}_R)$ for the annulus $\mathbb{A}_R = \{z : 1 < |z| < R\}$, and the domain of g_1 we know to be a compact (and hence bounded) subset of $\mathbb{C}\setminus\overline{\mathbb{D}}$. This satisfies our first requirement.

The closed geodesic line is first mapped by η to $i\left[-1,1\right]$. This imaginary interval is kept constant by g_3. Under g_2 this interval gets mapped to the unit disk, and finally under g_1 the unit disk is mapped to the origin. A set consisting of one point is certainly Cantor, therefore this mapping satisfies our second requirement.

The inverse f we have already discussed somewhat: in Theorem 4.7 we proved that $\mathbb{K}\left(z,f\right) \in L^1\left(\mathbb{D}\right)$. We shall use this case to prove the result to say that $\mathbb{K}\left(z,f\right) \in L^1_{loc}\left(\mathbb{C}\right)$: the mapping f here may be written

$$\eta^{-1} \circ f_3 \circ f_2 \circ f_1,$$

where η^{-1}, f_3 and f_2 are all conformal, so do not affect the integrability properties of $\mathbb{K}$. Let Ω be any compact subset of $\mathbb{C}$; then $\Omega \subset \mathbb{D}\left(0,S\right)$ for some S, since Ω must be closed and bounded, and hence

$$\iint_{\Omega} \mathbb{K}(z,f_1)\left|dz\right|^2 \le \iint_{\mathbb{D}(0,S)} \mathbb{K}(z,f_1)\left|dz\right|^2.$$

From (3.8.3) we have, given $\beta = 1$ and $p = 1$ by our assumptions and a change of coordinates, that

$$\iint_{\mathbb{D}(0,S)} \mathbb{K}(z,f_1)\left|dz\right|^2 = \frac{1}{2}\iint_{\mathbb{D}(0,S)} \left(r\ln R + \frac{1}{r\ln R}\right) r\,drd\theta,$$

where R comes from the definition $f_1 : \mathbb{D}^* \to \mathbb{A}\left(1,R\right)$. Since

$$\begin{aligned}\iint_{\mathbb{D}(0,S)} \mathbb{K}(z,f_1)\left|dz\right|^2 &= \frac{1}{2}\int_0^{2\pi}\int_0^S r^2\ln R + \frac{1}{\ln R}\,drd\theta \\ &= \pi\left(\frac{S^3}{3}\ln R + \frac{S}{\ln R}\right) < \infty,\end{aligned}$$

this mapping satisfies our third requirement.

Both components are also round disks, satisfying the last requirement.

Before we continue we make the following observation. Certain problems may

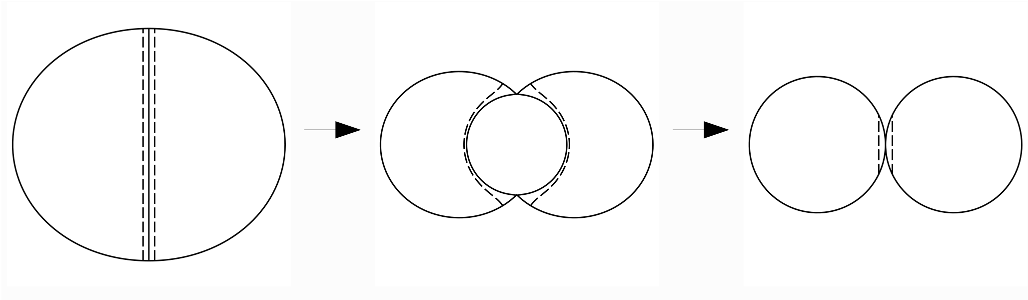


Figure 4.2.5: Illustration of mappings g_2 and g_1 shrinking a line to a point.

arise, however, when we come to iterate the process (say for a countable number of geodesic arcs). First, g_1 is not conformal, so there is not necessarily a conformal mapping which will map the next geodesic arc to the imaginary interval in order to shrink it, and though we may be able to handle this for the case of a finite number of arcs, there is no assurance that we can continue in such a way. Second, we should have some control over the images of the earlier contracted arcs. We therefore wish to amend our function for each of the geodesic arcs γ_i in such a way that no other geodesic arc is affected. Since the γ_i are separated in moduli, this is possible.

From Lemma 4.4, for each geodesic arc γ_i in the decomposition $\mathcal{E}$, let A_i be the open annular neighborhood of γ_i as described in the lemma. Let $U_i \subset A_i$ be an open subset chosen such that U_i is simply connected, and that under the Möbius transformation η_i which maps γ_i to the interval $i\,[-1,1])$, U_i is mapped to the open rectangle

$$\mathbb{R}_i\,(1, 1+\epsilon) = \{x + iy : -1 < x < 1, -(1+\epsilon) < y < 1+\epsilon\}, \qquad \epsilon > 0.$$

By the Riemann mapping theorem, there is a conformal mapping τ_i that maps $R_i\,(1, 1+\epsilon)$ to $\mathbb{D}\,(0, 1+2\delta)$ for some $\delta > 0$ such that τ_i maps $i\,[-1,1]$ to itself: we can see this by using the inverse of the Schwarz-Christoffel transformation (see [8, p. 333]) that maps the upper half-plane $\mathbb{H}$ to $\mathbb{R}_i\,(1, 1+\epsilon)$, outside of a scaling factor, we have that

$$\int_0^{iy} \frac{ds}{\sqrt{(1-s^2)\,(1-k^2s^2)}} = i \int_0^{y} \frac{dt}{\sqrt{(1+t^2)\,(1+k^2t^2)}},$$

so any imaginary values remain imaginary under this mapping, and similarly for the inverse for values within the rectangle. The conformal mapping $z \mapsto -i\,(1+2\delta)\,\frac{i-z}{i+z}$ similarly maps imaginary values to imaginary values. Define $\omega_i = \tau_i \circ \eta_i$ on U_i, and define W_i and V_i by

$$W_i = \omega_i^{-1}\,(\mathbb{D}) \quad \text{and} \quad V_i = \omega_i^{-1}\,(\mathbb{D}\,(0, 1+\delta))\,.$$

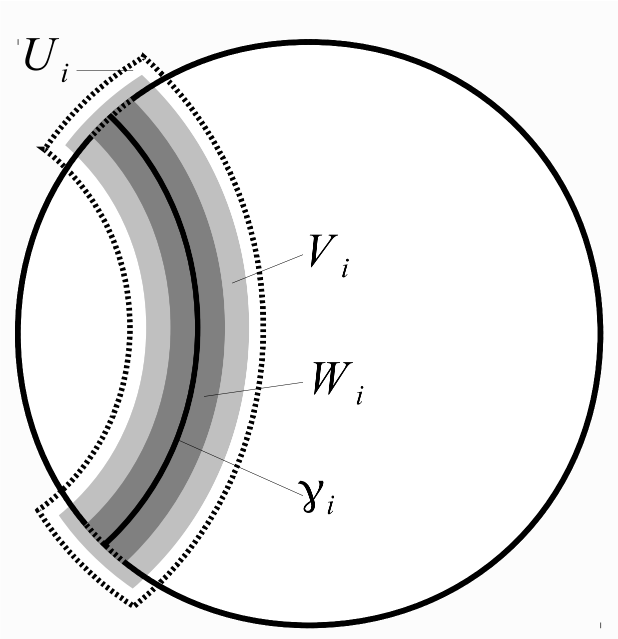


Figure 4.2.6: Example of neighborhoods U_i, V_i and W_i of geodesic arc γ_i.

For each geodesic arc γ_i, we construct a mapping h_i in the following manner: we first focus on U_i, which we map to the disk $\mathbb{D}\left(0, 1+2\delta\right)$ via the conformal mapping ω_i described above.

We then focus on $\omega_i\left(V_i\right) = \mathbb{D}\left(0, 1+\delta\right)$. Let $\tilde{f}$ be the mapping $f_3 \circ f_2 \circ f_1$from Theorem 4.7 restricted to $\mathbb{D}\left(0, 1+\delta\right)$ and $\tilde{g} = \tilde{f}^{-1}$. We now need a quasiconformal mapping which maps $\mathbb{D}\left(0, 1+\delta\right)$ to $D = \tilde{f}\left(\mathbb{D}\left(0, 1+\delta\right)\right)$ and fixes $i\left[-1, 1\right]$. Away from the origin (say in $\mathbb{D}\left(0, 1+\delta\right) \setminus \overline{\mathbb{D}\left(0, 1+\frac{\delta}{2}\right)}$), $\tilde{f}$ is quasiconformal: as f_2 and f_3 are conformal, we only need worry about f_1. We note that, as f_1 is a radial stretching, that

$$K\left(z, f_1\right) = \max\left\{\left|z\right|, \frac{1}{\left|z\right|}\right\}.$$

Since $0 < 1+\frac{\delta}{2} < \left|z\right| < 1+\delta$, K is finite over this region. Therefore, by using the quasiconformal Schoenflies theorem (at least, the version mentioned as Theorem 17c of [32]), we may construct a κ-quasiconformal $\chi : \mathbb{D}\left(0, 1+\delta\right) \to \tilde{f}\left(\mathbb{D}\left(0, 1+\delta\right)\right)$ which fixes the imaginary axis (since we can ensure f does, and because we may rotate conformally the result from [32]) and agrees with $\tilde{f}$ near the boundary, and where κ is a function of K and δ.

We then define

$$h_i : z \mapsto \begin{cases} \omega_i^{-1} \circ \tilde{g} \circ \chi \circ \omega_i\left(z\right) & z \in V_i \\ z & z \notin V_i \end{cases}.$$

We define g to be the concatenation of all $\left\{h_i\right\}_{i\in I}$ where I is a countable index of the geodesic arcs in the decomposition $\mathcal{E}$. Note that then, on each U_i, $g|_{U_i} = h_i$.

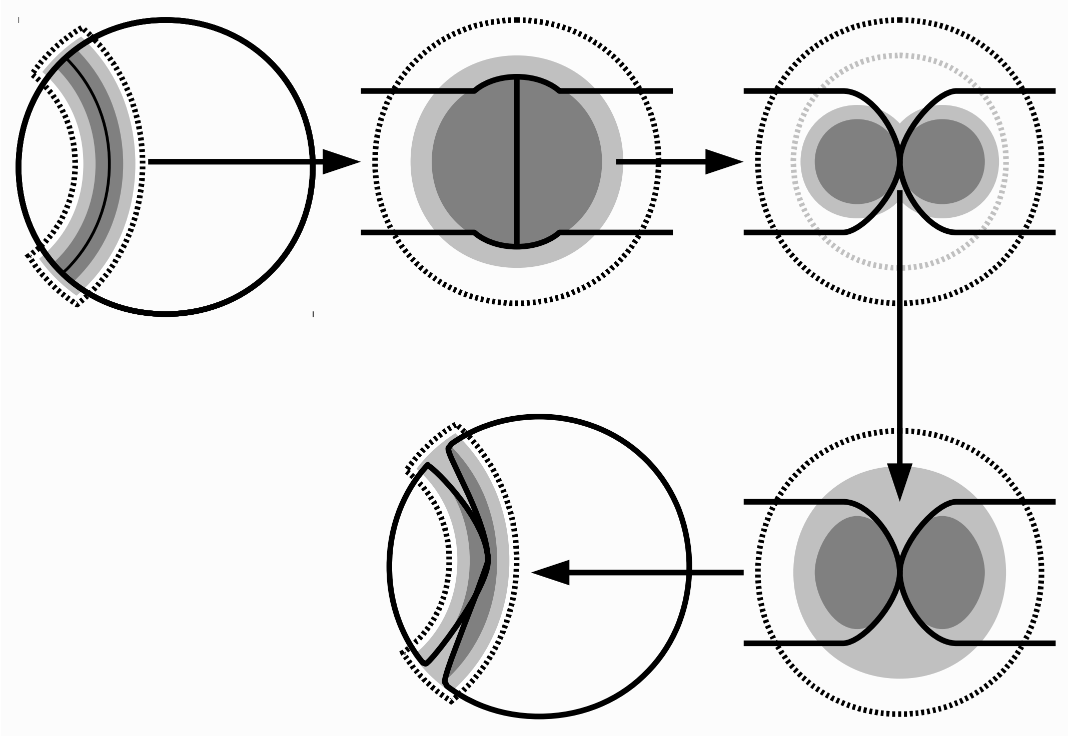


Figure 4.2.7: Illustration of the construction of h_i inside U_i.

- From the definition of h_i, we have that g is continuous and monotonic on each V_i. Outside of the V_i, g is the identity, and is therefore continuous and monotonic there as well. Because we ensured that the boundaries ∂V_i match up in a sufficiently nice way, we have continuity and monotonicity of g on the entire domain. For every compact $\Omega \subset \mathbb{C}$,

$$\begin{aligned}
\|g\|_{1,2,\Omega} &= \iint_{\Omega} |g(z)|^2 + |g_z(z)|^2 + |g_{\overline{z}}(z)|^2 \, |dz|^2 \\
&= \sum_{i \in I} \left[\iint_{\Omega \cap V_i} |h_i(z)|^2 + \left|(h_i)_z(z)\right|^2 + \left|(h_i)_{\overline{z}}(z)\right|^2 |dz|^2 \right] \\
&\quad + \iint_{\Omega \setminus \cup V_i} |z|^2 + 1 \ |dz|^2 .
\end{aligned}$$

Because Ω is compact, it is bounded (say, $|z| \leq R$ for $z \in \Omega$) and so $|z|^2 + 1 \leq R^2 + 1$; therefore

$$\iint_{\Omega \setminus \cup V_i} |z|^2 + 1 \ |dz|^2 \leq \left(R^2 + 1\right) \iint_{\Omega \setminus \cup V_i} |dz|^2 ,$$

this last integral being the area of $\Omega \setminus \cup V_i$. This is bounded, and as a result we may safely ignore it when considering whether $g \in W^{1,2}(\Omega)$. Instead, we focus on whether the restriction of g to $\Omega \cap V_i$ lies in $W^{1,2}(\Omega \cap V_i)$ for each i;

if this is the case, then we have that $g \in W^{1,2}(\Omega)$. For each $i \in I$,

$$\begin{aligned}
\|g\|_{1,2,\Omega\cap V_i} &= \iint_{\Omega\cap V_i} \left|\omega_i^{-1}\circ\tilde{g}\circ\chi\circ\omega_i(z)\right|^2 |dz|^2 \\
&\quad + \iint_{\Omega\cap V_i} \left|\left(\omega_i^{-1}\circ\tilde{g}\circ\chi\circ\omega_i\right)_z(z)\right|^2 |dz|^2 \\
&\quad + \iint_{\Omega\cap V_i} \left|\left(\omega_i^{-1}\circ\tilde{g}\circ\chi\circ\omega_i\right)_{\overline{z}}(z)\right|^2 |dz|^2 .
\end{aligned}$$

Because ω_i is conformal and χ is quasiconformal, by the chain rule we have

$$\begin{aligned}
\left(\omega_i^{-1}\circ\tilde{g}\circ\chi\circ\omega_i\right)_z &= \left(\omega_i^{-1}\right)_u \cdot \left[(\tilde{g})_v\cdot\chi_w + (\tilde{g})_{\overline{v}}\cdot\overline{\chi_{\overline{w}}}\right]\cdot(\omega_i)_z, \quad \text{and} \\
\left(\omega_i^{-1}\circ\tilde{g}\circ\chi\circ\omega_i\right)_{\overline{z}} &= \left(\omega_i^{-1}\right)_u \cdot \left[(\tilde{g})_v\cdot\chi_{\overline{w}} + (\tilde{g})_{\overline{v}}\cdot\overline{\chi_w}\right]\cdot\overline{(\omega_i)_z}.
\end{aligned}$$

Therefore, by substituting $\chi_{\overline{w}} = \mu_\chi\chi_w$ where possible, we get that

$$\begin{aligned}
\|g\|_{1,2,\Omega\cap V_i} &= \iint_{\Omega\cap V_i} \left|\omega_i^{-1}\circ\tilde{g}\circ\chi\circ\omega_i(z)\right|^2 |dz|^2 \\
&\quad + \iint_{\Omega\cap V_i} \left|\left(\omega_i^{-1}\right)_u\right|^2 \left|(\omega_i)_z\right|^2 |\chi_w|^2 \left|(\tilde{g})_v\cdot\frac{\chi_w}{\overline{\chi_w}} + (\tilde{g})_{\overline{v}}\cdot\overline{\mu_\chi}\right|^2 |dz|^2 \\
&\quad + \iint_{\Omega\cap V_i} \left|\left(\omega_i^{-1}\right)_u\right|^2 \left|(\omega_i)_z\right|^2 |\chi_w|^2 \left|(\tilde{g})_v\cdot\mu_\chi + (\tilde{g})_{\overline{v}}\cdot\frac{\overline{\chi_w}}{\chi_w}\right|^2 |dz|^2 .
\end{aligned}$$

By the triangle inequality we have

$$\begin{aligned}
\|g\|_{1,2,\Omega\cap V_i} &\leq \iint_{\Omega\cap V_i} \left|\omega_i^{-1}\circ\tilde{g}\circ\chi\circ\omega_i(z)\right|^2 |dz|^2 \\
&\quad +2 \iint_{\Omega\cap V_i} \left|\left(\omega_i^{-1}\right)_u\right|^2 \left|(\omega_i)_z\right|^2 |\chi_w|^2 \left(\left|(\tilde{g})_v\right|^2 + |\mu_\chi|^2\left|(\tilde{g})_{\overline{v}}\right|^2\right) |dz|^2 \\
&\quad +2 \iint_{\Omega\cap V_i} \left|\left(\omega_i^{-1}\right)_u\right|^2 \left|(\omega_i)_z\right|^2 |\chi_w|^2 \left(|\mu_\chi|^2\left|(\tilde{g})_v\right|^2 + \left|(\tilde{g})_{\overline{v}}\right|^2\right) |dz|^2 .
\end{aligned}$$

Since χ is a quasiconformal diffeomorphism, $|\mu_\chi| < 1$, and so

$$\begin{aligned}
\|g\|_{1,2,\Omega\cap V_i} &\leq \iint_{\Omega\cap V_i} \left|\omega_i^{-1}\circ\tilde{g}\circ\chi\circ\omega_i(z)\right|^2 |dz|^2 \\
&\quad +4 \iint_{\Omega\cap V_i} \left|\left(\omega_i^{-1}\right)_u\right|^2 \left|(\omega_i)_z\right|^2 |\chi_w|^2 \left(\left|(\tilde{g})_v\right|^2 + \left|(\tilde{g})_{\overline{v}}\right|^2\right) |dz|^2 .
\end{aligned}$$

Note $\left|\omega_i^{-1}\circ\tilde{g}\circ\chi\circ\omega_i\right|^2$ is bounded on $\Omega\cap V_i$ (as it maps $\Omega\cap V_i$ to itself, and Ω is bounded). So

$$\sum_{i\in I}\iint_{\Omega\cap V_i} \left|\omega_i^{-1}\circ\tilde{g}\circ\chi\circ\omega_i(z)\right|^2 |dz|^2 \leq R^2\sum_{i\in I}\iint_{\Omega\cap V_i} |dz|^2$$

which again is simply the area of $\cup_i (\Omega \cap V_i)$. As this is finite, we can focus on the second integral. By the conformal change of variables by the mapping ω_i, we have that this integral becomes

$$\iint_{\omega_i(\Omega \cap V_i)} \left|\left(\omega_i^{-1}\right)_u\right|^2 \left(\left|(\tilde{g})_v\right|^2 + \left|(\tilde{g})_{\overline{v}}\right|^2\right) |\chi_w|^2 |dw|^2 .$$

Then, as this is bounded above by

$$\iint_{\omega_i(\Omega \cap V_i)} \left|\left(\omega_i^{-1}\right)_u\right|^2 \left(\left|(\tilde{g})_v\right|^2 + \left|(\tilde{g})_{\overline{v}}\right|^2\right) \left(|\chi_w|^2 + |\chi_{\overline{w}}|^2\right) |dw|^2 ,$$

we can then change variables on this bound by using the quasiconformal diffeomorphism χ to get

$$\iint_{\chi \circ \omega_i(\Omega \cap V_i)} \left|\left(\omega_i^{-1}\right)_u\right|^2 \left(\left|(\tilde{g})_v\right|^2 + \left|(\tilde{g})_{\overline{v}}\right|^2\right) |dv|^2 .$$

Then as $\tilde{g} \in W^{1,2}_{loc}(\mathbb{D})$, and as ω_i is conformal, we obtain a bound on this depending on the area of $\chi \circ \omega_i (\Omega \cap V_i)$ similar to . The summation of all of these terms will then be bounded above by a term proportional to the area of Ω, which is finite. This implies that$g \in W^{1,2}_{loc}(\mathbb{C})$.

- The image of the non-degenerate continua under g form a finite set or a countable set K: if $\mathcal{E}$ consists of a finite number of continua, then K is certainly finite; otherwise as it is separated in modulus it must be countable. We can see this by first noting that $\mathbb{S}$ with the usual topology is second countable (the topology has a countable basis), so any collection of disjoint open sets must be countable. If we then note that each U_i overlaps with $\mathbb{S}$ twice (one at either end of the geodesic arc γ_i), and that the U_i must be disjoint we can form a collection of disjoint open sets by considering the collection $\{U_i \cap \mathbb{S}\}$. Such a collection must be countable, which means the collection $\{U_i\}$ is countable; hence the number of geodesic arcs, and so points under the image mapping, must be countable. K has Hausdorff dimension 0, because it is the countable (possibly finite) union of a set of points, each of which has Hausdorff dimension 0.

- For the inverse f of g, for a compact $\Omega \subset \mathbb{C}$, the $L^1(\Omega)$ norm of the distortion is

$$\begin{aligned}\|\mathbb{K}(z,f)\|_1 &= \iint_{\Omega} \mathbb{K}(z,f)\,|dz|^2 \\ &= \sum_{i \in I} \iint_{\Omega \cap V_i} \mathbb{K}\left(z, h_i^{-1}\right) |dz|^2 + \iint_{\Omega \setminus \cup V_i} |dz|^2 .\end{aligned}$$

On each $\Omega \cap V_i$, we have $h_i^{-1} = \omega_i^{-1} \circ \chi^{-1} \circ \tilde{f} \circ \omega_i$. Since $\tilde{f} = f_3 \circ f_2 \circ f_1$ with f_3 and f_2 conformal, and since χ^{-1} is quasiconformal because χ is, we may write $h_i^{-1} = \tilde{\chi}_i \circ f_1 \circ \omega_i$ with $\tilde{\chi}_i$ quasiconformal. Since postcomposition with quasiconformal mappings and precomposition with conformal mappings will not affect the integrability of $\mathbb{K}$, we only need worry about $\mathbb{K}(w, f_1)$. From (3.8.3), and the fact that $R = e$ and $p = \beta = 1$, we obtain that

$$\iint_{\omega_i(\Omega \cap V_i)} \mathbb{K}(z, f_1)\, |dw|^2 \leq \frac{e^2+1}{2e} \iint_{\omega_i(\Omega \cap V_i)} |dw|^2,$$

so

$$\sum_{i \in I} \iint_{\Omega \cap V_i} \mathbb{K}\left(z, h_i^{-1}\right) |dw|^2 \leq \frac{e^2+1}{2e} \tilde{\kappa} \sum_{i \in I} \iint_{\omega_i(\Omega \cap V_i)} |dw|^2$$

where $\tilde{\kappa}$ is the factor of distortion caused by the κ-quasiconformalχ, which will be the same for each of the $\tilde{\chi}_i$. We then have $\|\mathbb{K}(z, f)\|_1$ bounded above by some factor depending on the area of Ω and some constant. Since Ω is compact, this is finite. Therefore $f : \mathbb{C} \to \mathbb{C}$ is a mapping of finite distortion with $\mathbb{K}(z, f) \in L^1_{loc}(\mathbb{C})$.

- Finally, each component of $\mathbb{C} \setminus g(\mathbb{S})$ is conformally equivalent to a round disk by the Riemann mapping theorem, as the components are simply connected open sets.

□

4.3 Shrinking The Line Segment

Here we revisit the sort of maps we need to resolve decompositions. Our map needs to shrink the interval $[-1, 1]$ to a point, but be a homeomorphism away from this interval; therefore although the composition of $\frac{1}{2}\left(z + \frac{1}{z}\right)$ and $\frac{1}{2}\left(z - \frac{1}{z}\right)$ would shrink the disk to the interval, and then the interval to a point, and is conformal, it is not a homeomorphism on the unit disk. Here we explore an interesting difference between radial maps and more general mappings for this problem.

We know from 3.16 that there exists $h : \mathbb{D}^* \to \mathbb{A}_R$ with $\mathbb{K}(z, h) \in L^p$ for any $1 \leq p < \infty$; let $\varphi := z \mapsto \frac{1}{2}\left(z + \frac{1}{z}\right)$ be the conformal mapping which maps $\mathbb{A}_R$ to $\mathbb{D} \setminus [-1, 1]$; therefore the composition $\varphi \circ h : \mathbb{D}^* \to \mathbb{D} \setminus [-1, 1]$ has distortion $\mathbb{K}(z, \varphi \circ h) \in L^p$ as well, for $\mathbb{K}(z, \varphi \circ h) = \mathbb{K}(z, h)$. However, we shall now show that, if we assume that $f : \mathbb{D}^* \to \mathbb{D} \setminus [-1, 1]$ has some radially symmetric properties, there is an upper limit on the p for which we can have $\mathbb{K}(z, f) \in L^p$, in stark contrast to our earlier result.

We shall begin the analysis by considering a class of functions that act on a slightly larger disk: on $\mathbb{D}$ we take some nice homeomorphic function that fixes the interval (say, the identity) and multiply by argument-depended weight $h(\theta)$ that shrinks the interval; outside of $\mathbb{D}$ we construct it so that it is homeomorphic, matches the common boundary with the function defined on $\mathbb{D}$ and the outer boundary is a quasicircle.

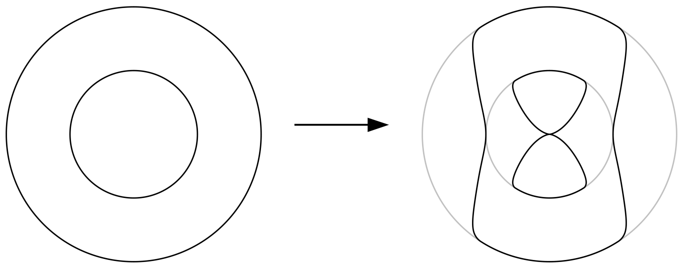


Figure 4.3.1: Shrinking the line segment, showing the circles of radius 1 and 2, and their images under a sample mapping. The grey lines represent the preimage.

Let us consider the simplest of such functions. The map f is defined on $\mathbb{C}$ by the following formula:

$$re^{i\theta} \mapsto \begin{cases} rh(\theta)\, e^{i\theta} & r \leq 1 \\ (r - 1 + h(\theta))\, e^{i\theta} & r > 1 \end{cases}, \tag{4.3.1}$$

where $h(\theta)$ has the following properties:

- h is C^∞ smooth and 2π periodic,
- $0 \leq h(\theta) \leq 1$,
- for a given $\epsilon > 0$, $h(\theta) = 1$ if $\theta \notin \bigcup_{k\in\mathbb{Z}} [k\pi - \epsilon, k\pi + \epsilon]$, and
- $h(\theta) = 0$ if and only if $\theta = k\pi$, $k \in \mathbb{Z}$.

Away from $[-1, 1]$, the mapping f is a homeomorphism: outside of the unit disk it is invertible by the mapping $se^{i\phi} \mapsto (s + 1 - h(\phi))\, e^{i\phi}$, and inside and away from the interval it is $se^{i\phi} \mapsto \frac{s}{h(\phi)} e^{i\phi}$ as $h(\phi) > 0$ away from the origin. Also, f shrinks this segment to a point as the points are multiplied by $h(\theta) = 0$. The differential of this map is

$$
\begin{aligned}
\|Df\|^2 &= |f_r|^2 + \frac{|f_\theta|^2}{r^2} \\
&= \begin{cases} \left|h\left(\theta\right)e^{i\theta}\right|^2 + \frac{\left|\left(rh'(\theta)+irh(\theta)\right)e^{i\theta}\right|^2}{r^2} & r<1, \\ \left|e^{i\theta}\right|^2 + \frac{\left|\left(h'(\theta)+i(r-1+h(\theta))\right)e^{i\theta}\right|^2}{r^2} & r>1 \end{cases} \\
&= \begin{cases} 2h^2(\theta) + h'(\theta)^2, & r<1, \\ 1 + \frac{(r-1+h(\theta))^2}{r^2} + \frac{h'(\theta)^2}{r^2} & r>1. \end{cases}
\end{aligned}
$$

Therefore, any reasonable function h will put the map f in $W^{1,2}$.

We calculate the distortion of the inverse map (away from the origin). The inverse is defined by g,

$$
se^{i\phi} \mapsto \begin{cases} \frac{s}{h(\phi)}e^{i\phi} & s \le h(\phi) \ne 0, \\ 0 & h(\phi)=0, \\ \left(s+1-h\left(\phi\right)\right)e^{i\phi} & s>h(\phi) \end{cases} \tag{4.3.2}
$$

and since we have that $h(\phi)=0$ only at $\phi=0$ and $\phi=\pi$, the problematic definition of the inverse g of f which shrinks $[-1,1]$ to $\{0\}$ does not affect us. Since

$$
\begin{aligned}
\|Dg\|^2 &= |g_s|^2 + \frac{|g_\phi|^2}{s^2} \\
&= \begin{cases} \left|\frac{e^{i\phi}}{h(\phi)}\right|^2 + \left|\left(-\frac{h'(\phi)}{h^2(\phi)} + \frac{i}{h(\phi)}\right)e^{i\phi}\right|^2 & s<h\left(\phi\right), \\ \left|e^{i\phi}\right|^2 + \frac{\left|\left(-h'(\phi)+i(s+1-h(\phi))\right)e^{i\phi}\right|^2}{s^2} & s>h\left(\phi\right), \end{cases} \\
&= \begin{cases} \frac{2}{h^2(\phi)} + \frac{h'(\phi)^2}{h^4(\phi)} & s<h\left(\phi\right), \\ 1 + \frac{h'(\phi)^2+(s+1-h(\phi))^2}{s^2} & s>h\left(\phi\right), \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
J(se^{i\phi},g) &= \frac{2}{s}\Im\left(g_\phi\overline{g_s}\right) \\
&= \begin{cases} 2\Im\left(\left(-\frac{h'(\phi)}{h^2(\phi)} + \frac{i}{h(\phi)}\right)e^{i\phi}\frac{e^{-i\phi}}{h(\phi)}\right) & s<h\left(\phi\right), \\ \frac{2}{s}\Im\left(\left(-h'\left(\phi\right) + i\left(s+1-h\left(\phi\right)\right)\right)e^{i\phi}e^{-i\phi}\right) & s>h\left(\phi\right), \end{cases} \\
&= \begin{cases} \frac{2}{h^2(\phi)} & s<h\left(\phi\right), \\ \frac{2(s+1-h(\phi))}{s} & s>h\left(\phi\right), \end{cases}
\end{aligned}
$$

the distortion is

$$\mathbb{K}(w,g) = \begin{cases} 1 + \frac{h'(\phi)^2}{2h^2(\phi)} & s < h(\phi), \\ 1 + \frac{(1-h(\phi))^2}{2s(s+1-h(\phi))} + \frac{h'(\phi)^2}{2s(s+1-h(\phi))} & s > h(\phi). \end{cases} \tag{4.3.3}$$

and note that for no function h can this term be bounded: for if so, $h'(\phi)\,[h(\phi)]^{-1}$ must be bounded, and as $h(\phi) \to 0$ as $|\phi| \to 0$, then $h'(\phi) \to 0$; therefore by definition

$$\lim_{\phi \to 0} \frac{h(\phi)}{\phi} < \infty.$$

Write $a(t) = \lim_{\phi \to t} \phi^{-1} h(\phi)$; then

$$h(t) = \lim_{\phi \to t} h(\phi) = \lim_{\phi \to t} \phi \lim_{\phi \to t} \frac{h(\phi)}{\phi} = ta(t),$$

and so

$$\lim_{t \to 0} \frac{h'(t)^2}{h(t)^2} = \lim_{t \to 0} \frac{1}{t^2} + \lim_{t \to 0} \frac{a'(t)^2}{a(t)^2} = \infty.$$

Let us now calculate the L^p norm of $\mathbb{K} - 1$ of the map on the image $f(\mathbb{D}(0,2)) = \Omega$. Note that $f(e^{i\theta}) = h(\theta)\, e^{i\theta}$ and $f(2e^{i\theta}) = (1 + h(\theta))\, e^{i\theta}$. We shall use the norm of $\mathbb{K} - 1$ and not $\mathbb{K}$ because from Lemma A.4 they are more or less the same problem as Ω has finite area under the flat metric, and it makes our calculations a little cleaner.

$$\begin{aligned} \|\mathbb{K}(w,g) - 1\|^p_{L^p(\Omega)} &= \iint_\Omega (\mathbb{K}(w,g) - 1)^p \; |dz|^2 \\ &= \frac{1}{2^{p+1}} \int_0^{2\pi} \frac{h'(\phi)^{2p}}{h^{2p-2}(\phi)} \, d\phi \\ &\quad + \frac{1}{2^p} \int_0^{2\pi} \int_{h(\phi)}^{1+h(\phi)} \frac{((1-h(\phi))^2 + h'(\phi)^2)^p}{s^{p-1}\,(s - h(\phi) + 1)^p} \, ds\, d\phi. \end{aligned}$$

The behaviour of the second integral changes at various values of p: we shall investigate them in turn, and assume $\|\cdot\|$ to be the appropriate L^p norm for each value of p.

4.3.1 L^1 Norm Of $\mathbb{K}-1$

If $p=1$ then we can calculate the norm precisely:

$$\begin{aligned}
\|\mathbb{K}(w,g)-1\| &= \frac{1}{4}\int_0^{2\pi} h'(\phi)^2\,d\phi \\
&\quad+\frac{1}{2}\int_0^{2\pi}\int_{h(\phi)}^{1+h(\phi)}\frac{(1-h(\phi))^2+h'(\phi)^2}{(s-h(\phi)+1)}\,ds\,d\phi \\
&= \int_0^{2\pi}\left(\frac{1}{4}+\frac{\ln 2}{2}\right)h'(\phi)^2+\frac{\ln 2}{2}\left(1-h\left(\phi\right)\right)^2\,d\phi,
\end{aligned}$$

and from our conditions on h we have that

$$\left(\frac{1}{4}+\frac{\ln 2}{2}\right)\int_0^{2\pi} h'(\phi)^2\,d\phi\le\|\mathbb{K}(w,g)-1\|\le\frac{1+2\ln 2}{4}\int_0^{2\pi} h'(\phi)^2\,d\phi+\pi\ln 2.$$

If we take as "sufficiently nice" h the condition that $\int_0^{2\pi} h'\left(\phi\right)\,d\phi$ is bounded, this ensures that the L^1 norm of $\mathbb{K}-1$ is finite.

4.3.2 L^p Norm Of $\mathbb{K}-1$, $p>1$

Let

$$\mathcal{J}(\phi)=\int_{h(\phi)}^{1+h(\phi)}\frac{ds}{s^{p-1}\left(s-h(\phi)+1\right)^p}. \tag{4.3.4}$$

If $p>1$ we have that

$$\begin{aligned}
\|\mathbb{K}(w,g)-1\|^p_{L^p(\mathbb{D}(0,2))} &\ge \frac{1}{2^p}\int_0^{2\pi}\frac{h'(\phi)^{2p}}{h^{2p-2}(\phi)}\,d\phi+\frac{1}{2^p}\int_0^{2\pi}h'(\phi)^{2p}\mathcal{J}(\phi)\,d\phi \\
&\quad+\frac{1}{2^p}\int_0^{2\pi}(1-h(\phi))^{2p}\mathcal{J}(\phi)\,d\phi,
\end{aligned}$$

and

$$\begin{aligned}
\|\mathbb{K}(w,g)-1\|^p_{L^p(\mathbb{D}(0,2))} &\le \frac{1}{2^p}\int_0^{2\pi}\frac{h'(\phi)^{2p}}{h^{2p-2}(\phi)}\,d\phi+\frac{1}{2}\int_0^{2\pi}h'(\phi)^{2p}\mathcal{J}(\phi)\,d\phi \\
&\quad+\frac{1}{2}\int_0^{2\pi}(1-h(\phi))^{2p}\mathcal{J}(\phi)\,d\phi.
\end{aligned}$$

Since $1\le s-h\left(\phi\right)+1\le 2$ on this region we have the bounds

$$2^{-p}\mathcal{J}_0\left(\phi\right)\le\mathcal{J}\left(\phi\right)\le\mathcal{J}_0\left(\phi\right),$$

where

$$\mathcal{J}_0(\phi) = \int_{h(\phi)}^{1+h(\phi)} \frac{ds}{s^{p-1}} = \begin{cases} \frac{(1+h(\phi))^{2-p} - h^{2-p}(\phi)}{2-p} & 1 < p \neq 2, \\ \ln(1+h(\phi)) - \ln h(\phi) & p = 2. \end{cases}$$

Also note that $(1-h(\phi))^p$ is bounded above and below, so we may ignore it for the purposes of integrability. Therefore we find the integrability properties of this mapping by considering the integrability of

$$\frac{h'(\phi)^{2p}}{h^{2p-2}(\phi)}, \mathcal{J}_0(\phi) \quad \text{and} \quad h'(\phi)^{2p}\mathcal{J}_0(\phi).$$

The behaviour of $\mathcal{J}_0(\phi)$ changes depending on where p is in relation to 2. We shall consider the three cases momentarily, after making an observation that will simplify this problem. Since we already have specified behaviour for $h(\phi)$ outside of some ϵ-balls about $\phi = 0$ and $\phi = \pi$ (namely that it is constantly 1, and thus the mapping f and its inverse g become the identity mapping outside of these ϵ-balls), where it certainly is integrable we can instead focus on the behaviour of $h(\phi)$ within these ϵ-balls. Observe that near π we can make similar arguments as we can near zero, and that we only really need to concern ourselves with one side of the interval because of smoothness and the fact that similar arguments can be made for either side, even if the function is not symmetric about these points, and through some appropriate smoothing with, say the identity mapping, we can determine the rest of $h(\phi)$.

L^p Norm Of $\mathbb{K} - 1$, $1 < p < 2$

If $1 < p < 2$ then the integrand of $\mathcal{J}_0(\phi)$ is finite everywhere, so integrability really only depends on $h'(\phi)^{2p}h^{2-2p}(\phi)$ as long as $h'(\phi)^2$ is integrable. As $h(0) = 0$, and it is a minimum of a smooth function, we must also have $h'(0) = 0$. Let us assume that near zero $h(\phi) \approx \phi^q$. Then $q > 1$ and we have that

$$h'(\phi)^{2p}h^{2-2p}(\phi) \approx q^p\phi^{2q-2p}, \tag{4.3.5}$$

so for integrability (locally near zero) we require $p < q + \frac{1}{2}$. Certainly the choice of $q = 2$ satisfies this for all such p. We shall see later why this particular choice of q was made here.

L^2 Norm Of $\mathbb{K} - 1$

When $p = 2$ we have that

$$\mathcal{J}_0(\phi) = \ln(1 + h(\phi)) - \ln(h(\phi)).$$

As before, let us suppose that near zero $h(\phi) \approx \phi^2$; we have already established integrability for one term (see (4.3.5)), we just need to calculate it for the other two. We can then explicitly evaluate the integral for this in some small interval about zero. First note

$$\int \mathcal{J}_0(\phi)\, d\phi = \phi\left(\ln\left(1+\phi^2\right) - \ln\left(\phi^2\right)\right) + 2\arctan(\phi);$$

which is finite when integrated on sufficiently small intervals:

$$\int_0^{\epsilon} \mathcal{J}_0(\phi)\, d\phi = \epsilon\left(\ln\left(1+\epsilon^2\right) - \ln\left(\epsilon^2\right)\right) + 2\arctan(\epsilon).$$

As $h'(\phi)^4 \approx 16\phi^4 \leq 1$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ we have the integrability of $\mathcal{J}_0 h'(\phi)^4$ as well.

L^p Norm Of $\mathbb{K}-1$, $2 < p < \frac{5}{2}$

When $2 < p < \frac{5}{2}$ we evaluate

$$\mathcal{J}_0(\phi) = \frac{h^{2-p}(\phi) - (1+h(\phi))^{2-p}}{p-2},$$

as $p-2$ is some fixed constant and $1 \leq (1+h(\phi)) \leq 2$ we are principally interested in whether $h^{2-p}(\phi)$ is integrable.

Again, we consider the case $h(\phi) \approx \phi^2$; since this choice ensures integrability of $h'(\phi)^{2p}h^{2-2p}(\phi)$ from (4.3.5), and for sufficiently small intervals around the origin $h'(\phi)^{2p} < 1$ we need only concern ourselves with the integrability of $h^{2-p}(\phi) \approx \phi^{4-2p}$. Since $4-2p > -1$, this is integrable as well.

L^p Norm Of $\mathbb{K}-1$, $p \geq \frac{5}{2}$

Our example where $h(\phi) \approx \phi^2$ near the origin breaks down once $p \geq \frac{5}{2}$; we shall expand on this in more detail later, but for now let us consider the more general $h(\phi) \approx \phi^q$. If $h'(\phi)^{2p}h^{2-2p}(\phi)$ is to be integrable near zero, we must have $p - \frac{1}{2} < q$ from (4.3.5); meanwhile, just as in the previous case, we also require the integrability of $h^{2-p}(\phi) \approx \phi^{(2-p)q}$. In this case,

$$(2-p)\,q > -1 \Leftrightarrow q < \frac{1}{p-2}.$$

So $p - \frac{1}{2} < \frac{1}{p-2}$, or $p < \frac{5}{2}$.

We can now see why the choice $q = 2$ may be considered ‘best’ for this problem: let $p = \frac{5}{2} - \epsilon$ for some small positive ϵ. Then the range from which we may pick q is $\left(2-\epsilon, \frac{2}{1-2\epsilon}\right)$ and as both $2-\epsilon \to 2$ and $\frac{2}{1-2\epsilon} \to 2$ as $\epsilon \to 0$ this is the one choice of q that works for any $p < \frac{5}{2}$.

We shall now prove this result in more generality. We want $h(0) = 0$, so let us suppose that

$$h(t) = t^{2q} a(t), \tag{4.3.6}$$

where $2q > 0$, $a(0) > 0$ (as $h(t) \geq 0$) and, from the definition of h, $a(t)$ is C^∞ smooth (which implies $a(t) > 0$ for $t \in (-\delta, \delta)$ for some $\delta > 0$) and bounded (we use $2q$ instead of q to try to ensure a common definition for both positive and negative values of t). Then

$$h'(t) = t^{2q-1} \left(2qa(t) + ta'(t)\right), \tag{4.3.7}$$

and so

$$\frac{h'(t)^{2p}}{h^{2p-2}(t)} = t^{2(2q-p)} \frac{\left(2qa(t) + ta'(t)\right)^{2p}}{a^{2p-2}(t)}. \tag{4.3.8}$$

As stated earlier, we need the integrability of $h^{2-p}(t)$, which we shall handle first. We have that

$$h^{2-p}(t) = \frac{t^{2q(2-p)}}{a^{p-2}(t)},$$

and if we choose our interval of integration as being within $(-\delta, \delta)$, we only need to focus on the integrability of $t^{q(2-p)}$, as then $a(t)$ has nonzero upper and lower bounds. This happens when

$$(2-p)\, 2q > -1 \qquad \text{or} \qquad 2q < \frac{1}{p-2}.$$

Returning to (4.3.8), we see that our bound on $2q$ implies

$$\frac{h'(t)^{2p}}{h^{2p-2}(t)} \geq t^{2\left(\frac{1}{p-2} - p\right)} \frac{\left(2qa(t) + ta'(t)\right)^{2p}}{a^{2p-2}(t)}.$$

Let us examine the term $2qa(t) + ta'(t)$ for a moment. Note that

$$\lim_{t \to 0} 2qa(t) + ta'(t) \neq 0,$$

as otherwise

$$\lim_{t \to 0} ta'(t) = -2qa(0) \neq 0;$$

this would mean $a'(t) \approx -2qa(0)t^{-1}$ near zero, and would make $a(t)$ not smooth at zero. As it is also continuous, by choosing δ sufficiently small then we also can bound $2qa(t) + ta'(t)$ away from zero, and so can bound the term $\left(2qa(t) + ta'(t)\right)^{2p} a^{2-2p}(t)$ from below by a nonzero amount. Hence for integrability we require $2\left(\frac{1}{p-2} - p\right) > -1$ or $p < \frac{5}{2}$. Therefore we have the following result.

Theorem 4.8. *When $p \geq \frac{5}{2}$ there is no choice of analytic function h with the properties listed above for* (4.3.1)*, such that*

$$0 < \lim_{t \to 0} \frac{h(t)}{t^q} < \infty$$

for any $q > 0$ satisfying (4.3.1) *such that the inverse g of the mapping $f \in W^{1,2}_{loc}(\mathbb{C})$ has distortion in $L^p_{loc}(\mathbb{C})$.*

Although we have eliminated all analytic choices, there may be a choice of $h(\theta)$ that is smooth but non-analytic. However, as we shall soon see, we do not even get this when $p \geq \frac{5}{2}$.

Corollary 4.9. *Theorem 4.8 also holds when*

$$\lim_{t \to 0} t^{-q} h(t) = 0 \tag{4.3.9}$$

for all $q > 0$.

Proof. Since (4.3.9) holds for all $q > 0$, we must certainly have it true for $q = 2$, so

$$\lim_{t \to 0} t^{-2} h(t) = 0.$$

Thus there exists some $\delta > 0$ such that for any $0 < t < \delta$ we have that $t^{-2}h(t) < \epsilon$ where $0 < \epsilon < 1$ and ϵ depends on δ. As we need integrability of $h^{2-p}(t)$, as $\epsilon^{2-p} t^{4-2p} \leq h(t)^{2-p}$ when $|t| < \delta$ and if h is integrable near zero then t^{4-2p} must be as well. This happens only if $4 - 2p > -1$ or $p < \frac{5}{2}$. $\square$

4.3.3 Generalization

Following roughly the same argument, we can extend this result to a more general class of functions than those of the form given in the previous theorem. Since we know solutions exist for $p < \frac{5}{2}$, we shall assume that $p \geq \frac{5}{2}$.

Theorem 4.10. *Theorem 4.8 also holds where f has the form*

$$f : re^{i\theta} \mapsto \begin{cases} R(r)\, h(\theta)\, e^{i\theta} & r \leq 1, \\ (R(r) - 1 + h(\theta))\, e^{i\theta} & r > 1, \end{cases} \tag{4.3.10}$$

where $h(\theta)$ has the same properties as given above, and $R : [0, \infty) \to [0, \infty)$ is a strictly increasing C^1 diffeomorphism whose inverse is also in C^1 and $R(1) = 1$.

Before we prove this, we note that the conditions on R follow from how we want it to behave:

- ∘ We first want to ensure that it fixes $[-1,1]$, as then our definition of $h(t)$ can remain the same, therefore $R(1)=1$.

- ∘ In order to perform the analysis, we will need to take derivatives; R must therefore be differentiable; similarly, its inverse S must also be differentiable. We would also like it to be as smooth as possible to avoid issues; however, we shall require at least that the function and its inverse are in C^1.

- ∘ It has to be invertible, as we require f to be invertible (at least sufficiently far away from the interval to be shrunk), and we also want $R(0)=0$ and $\lim_{r\to\infty} R(r)=\infty$ so we need R to be increasing; and since we want it to be invertible and for there to be a finite derivative at every point for both R and its inverse, we will assume that R is strictly increasing diffeomorphism.

Proof. (4.10) We start by squaring the Hilbert-Schmidt norm of the differential of f: this is

$$\|Df\|^2 \quad = \quad \begin{cases} \left(R'(r)^2+\frac{R(r)^2}{r^2}\right)h(\theta)^2+\frac{R(r)^2h'(\theta)^2}{r^2} & r<1, \\ R'(r)^2+\frac{h'(\theta)^2+(R(r)-1+h(\theta))^2}{r^2} & r>1, \end{cases}$$

and thus we have that, as in the case we have just established, the choice of sufficiently nice R and h will put the map f in $W^{1,2}$. The inverse g is then given by:

$$g: se^{i\phi} \mapsto \begin{cases} S\left(\frac{s}{h(\phi)}\right)e^{i\phi} & s\leq h(\phi)\neq 0, \\ 0 & h(\phi)=0, \\ S\left(s+1-h\left(\phi\right)\right)e^{i\phi} & h\left(\phi\right)<s, \end{cases} \tag{4.3.11}$$

which we shall further explain. The mapping f maps the unit circle $\mathbb{S}$ to a two lobed structure parametrised by $h\left(\theta\right)e^{i\theta}$: h evaluates to zero precisely at two points in $[0,2\pi)$, namely 0 and π, and $R\left(1\right)=1$. Outside of $f\left(\overline{\mathbb{D}}\right)$ the inverse is simple and obtained by rearrangement; and if $se^{i\phi}\in\mathbb{C}\setminus f\left(\overline{\mathbb{D}}\right)$, we have that $s>h\left(\phi\right)$. A similar argument holds when $h\left(\phi\right)\neq 0$ inside $f\left(\overline{\mathbb{D}}\right)$; here if $se^{i\phi}\in f\left(\overline{\mathbb{D}}\right)$, then $s\leq h\left(\phi\right)$. Otherwise we lie in $f\left(\overline{\mathbb{D}}\right)$ and $h\left(\phi\right)=0$; let be such a point. Then if $se^{i\phi}$ is such a point, $s=0$ (because $0\leq s\leq h\left(\phi\right)$) and so $se^{i\phi}=0$. As $f\left(0\right)=0$ we want $g\left(0\right)=0$.

As before, we do not need to worry about the problematic definition of g. Here

$$\|Dg\|^2 = \begin{cases} \frac{1}{h^2(\phi)}S'\left(\frac{s}{h(\phi)}\right)^2\left(1+\frac{h'^2(\phi)}{h^2(\phi)}\right)+\frac{1}{s^2}S^2\left(\frac{s}{h(\phi)}\right) & s<h\left(\phi\right)\neq 0, \\ S'\left(s+1-h\left(\phi\right)\right)^2\left(1+\frac{h'(\phi)^2}{s^2}\right)+\frac{S^2(s+1-h(\phi))}{s^2} & s>h\left(\phi\right), \end{cases}$$

and

$$J\left(se^{i\phi},g\right)=\begin{cases}\frac{2}{sh(\phi)}S'\left(\frac{s}{h(\phi)}\right)S\left(\frac{s}{h(\phi)}\right) & s<h\left(\phi\right)\neq 0,\\ 2\frac{S(s+1-h(\phi))}{s}S'\left(s+1-h\left(\phi\right)\right) & s>h\left(\phi\right),\end{cases}$$

so the distortion is

$$\mathbb{K}\left(se^{i\phi},g\right) = \begin{cases}\dfrac{\frac{1}{h^2(\phi)}S'\left(\frac{s}{h(\phi)}\right)^2\left(1+\frac{h'^2(\phi)}{h^2(\phi)}\right)+\frac{1}{s^2}S^2\left(\frac{s}{h(\phi)}\right)}{\frac{2}{sh(\phi)}S'\left(\frac{s}{h(\phi)}\right)S\left(\frac{s}{h(\phi)}\right)} & s<h\left(\phi\right)\neq 0,\\ \dfrac{S'(s+1-h(\phi))^2\left(1+\frac{h'(\phi)^2}{s^2}\right)+\frac{S^2(s+1-h(\phi))}{s^2}}{2\frac{S(s+1-h(\phi))}{s}S'(s+1-h(\phi))} & s>h\left(\phi\right),\end{cases}$$

or, after some rearranging,

$$\mathbb{K}\left(se^{i\phi},g\right) = \begin{cases}\frac{1}{2}\left(\frac{s}{h(\phi)}\frac{S'\left(\frac{s}{h(\phi)}\right)}{S\left(\frac{s}{h(\phi)}\right)}\left(1+\frac{h'^2(\phi)}{h^2(\phi)}\right)+\frac{h(\phi)}{s}\frac{S\left(\frac{s}{h(\phi)}\right)}{S'\left(\frac{s}{h(\phi)}\right)}\right) & s<h\left(\phi\right)\neq 0,\\ \frac{1}{2s}\left(\frac{S'(s+1-h(\phi))\left(s^2+h'(\phi)^2\right)}{S(s+1-h(\phi))}+\frac{S(s+1-h(\phi))}{S'(s+1-h(\phi))}\right) & s>h\left(\phi\right).\end{cases}$$

We then calculate the L^p norm of the distortion (from here we shall assume that the norm is the $L^p\left(\Omega\right)$ norm). We shall suppress the arguments of the various functions here for clarity.

$$\begin{aligned}\|\mathbb{K}(w,g)\|^p &= \iint_\Omega \left(\mathbb{K}(w,g)\right)^2\ |dw|^2\\ &= \frac{1}{2^p}\int_0^{2\pi}\int_0^h\left(\frac{s}{h}\frac{S'}{S}\left(1+\frac{h'^2}{h^2}\right)+\frac{h}{s}\frac{S}{S'}\right)^p s\,ds\,d\phi\\ &\quad+\frac{1}{2^p}\int_0^{2\pi}\int_h^{1+h}\frac{1}{s^{p-1}}\left(\frac{S'\left(s^2+h'^2\right)}{S}+\frac{S}{S'}\right)^p ds\,d\phi.\end{aligned}$$

Let us construct some bounds on these integrals. From Lemma A.4 we have that

$$\mathcal{A}\leq\left(\frac{s}{h}\frac{S'}{S}\left(1+\frac{h'^2}{h^2}\right)+\frac{h}{s}\frac{S}{S'}\right)^p s\leq 2^{p-1}\mathcal{A}$$

and

$$\mathcal{B}\leq\frac{1}{s^{p-1}}\left(\frac{S'\left(s^2+h'^2\right)}{S}+\frac{S}{S'}\right)^p\leq 2^{p-1}\mathcal{B},$$

where

$$\mathcal{A}=\frac{s^{p+1}}{h^p}\frac{S'^p}{S^p}\left(1+\frac{h'^2}{h^2}\right)^p+\frac{h^p}{s^{p-1}}\frac{S^p}{S'^p} \tag{4.3.12}$$

and

$$\mathcal{B}=\frac{1}{s^{p-1}}\left(\frac{S'^p\left(s^2+h'^2\right)^p}{S^p}+\frac{S^p}{S'^p}\right); \tag{4.3.13}$$

we thus only have to worry about the integrability of $\mathcal{A}$ and $\mathcal{B}$.

Let us begin with $\mathcal{B}$. The change of variables $s\mapsto t=s+1-h$ transforms the domain of integration to the annulus $\mathbb{A}_2$. As $S'(t)$ is continuous on the closed interval $[1,2]$, we have, by the extreme value theorem, that there are some $a,b \in [1,2]$ such that $S'(a) \leq S'(t) \leq S'(b)$; and as $S'(t) > 0$ (since R is increasing, so S is), then $S'(a) > 0$; moreover $S'(b) < \infty$. We also have $1 \leq S(t) \leq S(2) < \infty$ on this interval as well, which means

$$\mathcal{B} \geq \frac{S'(a)^p}{S^p(2)} s^{p+1} + \frac{S'(a)^p}{S^p(2)} s^{1-p} h'^{2p} + \frac{s^{1-p}}{S'(b)^p},$$

and as

$$\int_h^{1+h} s^{1-p} ds = \frac{h^{2-p}}{p-2} - \frac{(1+h)^{2-p}}{p-2},$$

we see that integrability of $\mathcal{B}$ requires integrability of $h'^{2p} h^{2-p}$ and h^{2-p}.

Returning to $\mathcal{A}$, we perform the change of variables $s \mapsto t = \frac{s}{h}$;

$$\mathcal{A} = h t^{p+1} \frac{S'^p}{S^p} \left(1 + \frac{h'^2}{h^2}\right)^p + \frac{h}{t^{p-1}} \frac{S^p}{S'^p},$$

and as $h \geq 0$ and the second term has no other terms dependent on ϕ we have the bound

$$\mathcal{A} \geq h t^{p+1} \frac{S'^p}{S^p} \left(1 + \frac{h'^2}{h^2}\right)^p.$$

We may do this as $h \neq 0$ almost everywhere. Since $ds = h\, dt$, we have the integral

$$\int_0^{2\pi} \int_0^1 \mathcal{A} h \, dt \, d\phi \geq \int_0^{2\pi} \frac{h'^{2p} \, d\phi}{h^{2p-2}} \int_0^1 t^{p+1} \frac{S'^p}{S^p} \, dt.$$

When $t > 0$, $S' > 0$ and $S > 0$, then $\int_0^1 t^{p+1} \frac{S'^p}{S^p} \, dt > 0$, so integrability of $\mathcal{A}h$ on this region requires the integrability of $h'^{2p} h^{2-2p}$, however, these are the same two requirements as in Theorem 4.8 and there we proved there were no solutions where $\lim_{t \to 0} h(t) t^{-q} < \infty$. □

4.3.4 Distortion Properties Of f

We know that there is no extremal mapping of the functional (3.1.2) that maps the annulus to the punctured disk; however, because of the change of variables induced in the mapping $\frac{1}{2}\left(z + \frac{1}{z}\right)$, it may be possible that there is such a mapping from $\mathbb{D} \setminus [-1,1]$ to the punctured disk. Let us examine that problem now for functions of this form:

The Jacobian is

$$
\begin{aligned}
J(re^{i\theta}, f) &= \frac{2}{r}\Im\left(f_\theta \overline{f_r}\right) \\
&= \begin{cases} 2h^2(\theta) & r < 1, \\ \frac{2}{r}(r-1+h(\theta)) & r > 1, \end{cases}
\end{aligned}
$$

and so, the distortion is

$$
\mathbb{K}\left(re^{i\theta}, f\right) = \begin{cases} 1 + \frac{h'(\theta)^2}{2h^2(\theta)} & r < 1, \\ 1 + \frac{(1-h(\theta))^2}{2r(r-1+h(\theta))} + \frac{h'(\theta)^2}{2r(r-1+h(\theta))} & r > 1. \end{cases} \tag{4.3.14}
$$

We shall calculate the L^p norm of $\mathbb{K}-1$ (again, for the same reason as before) of the map on the disk of radius 2.

$$
\begin{aligned}
\|\mathbb{K}(z,f)-1\|^p_{L^p(\mathbb{D}(0,2))} &= \iint_{\mathbb{D}(0,2)} (\mathbb{K}(z,f)-1)^p \; |dz|^2 \\
&= \int_0^{2\pi}\int_0^1 \frac{h'(\theta)^{2p}}{2^p h^{2p}(\theta)} r \, dr \, d\theta \\
&\quad + \int_0^{2\pi}\int_1^2 \frac{\left((1-h(\theta))^2 + h'(\theta)^2\right)^p}{2^p r^{p-1}(r-1+h(\theta))^p} \, dr \, d\theta \\
&= \frac{1}{2^{p+1}} \int_0^{2\pi} \frac{h'(\theta)^{2p}}{h^{2p}(\theta)} \, d\theta \\
&\quad + \frac{1}{2^p}\int_0^{2\pi}\int_1^2 \frac{((1-h(\theta))^2 + h'(\theta)^2)^p}{r^{p-1}(r-1+h(\phi))^p} \, dr \, d\theta.
\end{aligned}
$$

For $p = 1$ we then have

$$
\|\mathbb{K}(z,f)-1\| = \int_0^{2\pi} \frac{h'(\theta)^2}{4h^2(\theta)} + \frac{[\ln(1+h(\theta)) - \ln(h(\theta))]\,[(1-h(\theta))^2 + h'(\theta)^2]}{2} \, d\theta.
$$

Let us now examine what happens near $\theta = 0$, and use the same argument as used earlier. We want $h(0) = 0$, so approximating $\ln(1+h(\theta)) \approx 0$ and $(1-h(\theta))^2 \approx 1$ (we may treat these as bounded by constants that do not affect the analysis, and these will be the approximate values of such constants) gives

$$
\|\mathbb{K}(z,f)-1\| \gtrsim \int_0^{\epsilon} \frac{h'(\theta)^2}{4}\left(h^{-2}(\theta) + 2\ln\left(\frac{1}{h(\theta)}\right)\right) + \frac{1}{2}\ln\left(\frac{1}{h(\theta)}\right) \, d\theta,
$$

so we certainly need this integral to be finite. Near zero, h will be small and therefore $\ln\left([h(\theta)]^{-1}\right)$ will be positive. Let us examine the term

$$
\left(\frac{h'(\theta)}{h(\theta)}\right)^2
$$

for a moment; as in our earlier proof, let us focus on t small and positive and assume that $h(t) = t^q a(t)$ for some $q > 0$ where $a(0) \geq 0$, $a'(t) \geq 0$ and $a(t)$ is C^1 and bounded; then

$$\frac{h'(t)}{h(t)} \geq \frac{q}{t},$$

so the integral is not convergent.

Thus we have established the following result.

Lemma 4.11. *There is no extremal mappings, in the form given at* (4.3.1)*, of a functional analogous to* (3.1.2) *for maps from* $\mathbb{D} \setminus [-1, 1]$ *to the punctured disk.*

Chapter 5

Quasiconformal Harmonic Extensions

5.1 Properties Of The Inverse

If we put $z = g(w)$, then from the property of the Jacobian we have the identity

$$J(w,g)J(z,f) = 1.$$

Multiplying through by $\rho(f) = \rho(w)$ and noting that from (2.5.3) we have that $|\mu_f(g(w))| = |\mu_g(w)|$, and so

$$\begin{aligned}
\rho(w) &= \rho(w)\left(|f_z|^2 - |f_{\bar{z}}|^2\right) J(w,g) \\
&= \rho(w)\,|f_z|\,|f_{\bar{z}}|\left(\frac{1}{|\mu_f(g)|} - |\mu_f(g)|\right) J(w,g) \\
&= |\Psi_f(g)|\left(\frac{1}{|\mu_g|} - |\mu_g|\right) J(w,g),
\end{aligned}$$

We wish to write this in terms of the dilatation of g. Note that

$$K - \frac{1}{K} = \frac{1+|\mu|}{1-|\mu|} - \frac{1-|\mu|}{1+|\mu|} = \frac{4\,|\mu|}{1-|\mu|^2}.$$

We now recollect from (2.3.2) and (2.3.3) the definition of a harmonic mapping in the hyperbolic metric for the disk and half-plane models. With this, we have the following lemma.

Lemma 5.1. *Let $f : \mathbb{D} \to (\mathbb{D}, \rho)$ be a harmonic diffeomorphism, $g = f^{-1}$ and Ψ_f the Hopf differential. Then*

$$\rho(w)\left(K(w,g) - \frac{1}{K(w,g)}\right) = 4\,|\Psi_f(g)|\,J(w,g) \tag{5.1.1}$$

and

$$\rho(w)\left(K^2(w,g) - 1\right) = 4\,|\Psi_f(g)|\,|Dg(w)|^2,$$

where $K(\cdot, g)$ is the distortion function of g.

Let us first note that $t \mapsto t - 1/t$ is increasing on $[1, \infty)$ and therefore if f (and hence g) is K-quasiconformal, then

$$K(w,g) - \frac{1}{K(w,g)} \leq K - \frac{1}{K},$$

and then by integrating both sides of (5.1.1) we have the following estimate on the growth of the Hopf differential:

Corollary 5.2. *Let $f : \mathbb{D} \to (\mathbb{D}, \rho)$ be a harmonic diffeomorphism that is K-quasiconformal and Ψ_f the Hopf differential* (2.5.4). *Let $E \subset \mathbb{D}$ be a measurable set. Then*

$$\int_{f^{-1}(E)} |\Psi_f(z)| \, |dz|^2 \leq \frac{1}{4}\left(K - \frac{1}{K}\right) |E|_\rho ,$$

where $|E|_\rho$ is used to denote the area of E in the ρ metric.

Let

$$\rho(w) = \frac{1}{\left(1 - |w|^2\right)^2};$$

if we have that$|\rho(w)| < M |\rho(g)|$, then

$$\frac{\rho(w)}{\rho(g)}\left(K(w,g) - \frac{1}{K(w,g)}\right) = \frac{4 |\Psi_f(f)| \, J(w,g)}{\rho(g)}$$

and integrating and estimating gives

$$4 \int_{f^{-1}(E)} \frac{|\Psi_f(z)|}{\rho(z)} |dz|^2 \leq KM.$$

Letting $E = \mathbb{D}$ gives

$$\int_{\mathbb{D}} \frac{|\Psi_f(z)|}{\rho(z)} |dz|^2 \leq \frac{1}{4} KM.$$

The term $\frac{|\Psi_f(z)|}{\rho(z)}$ is related to the Bloch norm of the Hopf differential: in general, this norm is given by

$$f \mapsto \|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right) |f'(z)| .$$

5.2 A First-Order Nonlinear PDE For A Beltrami Coefficient

From Theorem 2.8 we have that the inverse of a quasiconformal mapping is quasiconformal. Since the expression of (2.3.1) contains the term $(\log\, \rho(u))_u$, the expression is somewhat complicated by its dependence on derivatives from both spaces. However, we may rewrite (2.3.1) in such a way that this complication is removed.

Theorem 5.3. (Inverse Complex Dilatation Condition)*A differentiable function* $\mu : \mathbb{D} \to \mathbb{C}$*,* $\|\mu\|_\infty = k < 1$*, is the complex dilatation of the inverse of a harmonic mapping if and only if* μ *satisfies the first order PDE*

$$\mu_z + \mu\phi = \overline{\mu}\left(\mu_{\bar{z}} - \mu\overline{\phi}\right). \tag{5.2.1}$$

*Proof. (Theorem 5.3)*Let $g = f^{-1}$ and we use coordinates w and z for f and g respectively, so $z = f(w)$ and $w = g(z)$. The Beltrami coefficient $\mu = \mu_g$ then satisfies (2.5.1):

$$g_{\bar{z}} = \mu g_z.$$

From the chain rule (Theorem A.2), we have that

$$-f_{\bar{w}} = \mu\overline{f_w},$$

and taking the w derivative then gives

$$-f_{w\bar{w}} = \mu\overline{f_{w\bar{w}}} + \mu_w\overline{f_w}.$$

Next, we expand the μ_w term by using the chain rule:

$$\mu_w = \mu_z f_w + \mu_{\overline{z}}\overline{f_{\overline{w}}},$$

thereby giving us

$$-f_{w\bar{w}} = \mu\overline{f_{w\bar{w}}} + \mu_z f_w\overline{f_w} + \mu_{\overline{z}}\overline{f_{\overline{w}}f_w}.$$

We then substitute for the $f_{w\overline{w}}$ term by using the equation for harmonic maps (2.3.1). Writing $\phi(z)$ for $(\ln \rho(z))_z$ and rearranging (2.3.1) gives

$$f_{w\bar{w}} = -\phi(z)\, f_w f_{\bar{w}}.$$

And so the substitution gives

$$\phi(z) f_w f_{\bar{w}} = -\mu \overline{\phi(z) f_w f_{\bar{w}}} + \mu_z f_w \overline{f_w} + \mu_{\overline{z}} \overline{f_{\overline{w}}} f_w.$$

Our next goal is to rearrange and substitute so there is no mention of f. Dividing through by $|f_w|^2$ gives, after some rearrangement,

$$\phi(z) \frac{f_{\bar{w}}}{\overline{f_w}} = \mu_z + \frac{\overline{f_{\overline{w}}}}{f_w} \left(\mu_{\overline{z}} - \mu \overline{\phi(z)} \right).$$

From (2.5.3), we obtain

$$-\phi(z) \mu = \mu_z - \overline{\mu} \left(\mu_{\overline{z}} - \mu \overline{\phi(z)} \right).$$

After some rearrangement, this argument shows that the complex dilatation of the inverse of a harmonic map satisfies (5.2.1). The converse implication follows by the reverse argument, where we obtain the harmonic mapping from the existence theorem, Theorem 2.7. □

Remark 5.4. We first investigate some general results for all metrics. Later, we will provide an alternate proof using Möbius transformations later for the case of the hyperbolic metric by substituting into the case of the Euclidean metric.

From Theorem 5.3 we can take any smooth function μ and using (5.2.1) define a function ϕ. We can solve the equation $(\ln\rho)_{\overline{z}} = \phi$ using the Cauchy transform (see [6]). However, we have already made the observation that the complex dilatation of an inverse harmonic map has a special form (real positive multiples of anti-holomorphic functions). The reconciliation is that not every Cauchy transform yields a real valued function (in this case $\ln\rho$).

5.3 Observations From Inverse Complex Dilatation Condition

We shall now make some well known observations on the complex dilatation which follow from this theorem.

We begin with two lemmas based on (2.3.1).

Lemma 5.5. *Let $f : \mathbb{D} \to \mathbb{D}$ be harmonic and define $h : \mathbb{D} \to \mathbb{D}$ by $h(z) = \overline{f(\overline{z})}$. Then h is harmonic.*

Proof. (Lemma 5.5) We first note that if $a(w) = \overline{b(\overline{w})}$, then

$$a_w(w) = \overline{b_w(\overline{w})}, \quad \text{and} \quad a_{\overline{w}}(w) = \overline{b_{\overline{w}}(\overline{w})}. \tag{5.3.1}$$

Then

$$\begin{aligned} h_{z\overline{z}}(z) + \phi(h(z))\, h_z(z)\, h_{\overline{z}}(z) &= \overline{f_{z\overline{z}}(\overline{z})} + \phi\left(\overline{f(\overline{z})}\right) \overline{f_z(\overline{z}) f_{\overline{z}}(\overline{z})} \\ &= \overline{f_{z\overline{z}}(\overline{z})} + \overline{\phi(f(\overline{z})) f_z(\overline{z}) f_{\overline{z}}(\overline{z})}, \end{aligned}$$

by applying (5.3.1) to $\phi(w) = (\ln \rho(w))_w$. Since this is the conjugate of (2.3.1), since f is a harmonic mapping then this equals zero. □

Lemma 5.6. *Let $f : \mathbb{D} \to \mathbb{D}$ be a homeomorphism, $g = f^{-1}$ and $h(z) = \overline{f(\overline{z})}$. Then*

$$h^{-1}(z) = \overline{g(\overline{z})}.$$

Proof. We compute

$$h\left(\overline{g(\overline{z})}\right) = \overline{f(g(\overline{z}))} = z.$$

□

From these two lemmas, we have the following corollary to Theorem 5.3.

Corollary 5.7. *Suppose that $\mu : \mathbb{D} \to \mathbb{D}$ is the complex dilatation of the inverse of a harmonic mapping. Then so is*

$$\sigma(z) = \overline{\mu(\overline{z})}.$$

Proof. Suppose that g is the inverse of a harmonic mapping. The previous two lemmas imply that $h(z) = \overline{g(\overline{z})}$ is also the inverse of a harmonic mapping, and that

$$\mu_h(z) = \frac{h_{\overline{z}}(z)}{h_z(z)} = \frac{\overline{f_{\overline{z}}(\overline{z})}}{\overline{f_z(\overline{z})}} = \overline{\mu_g(\overline{z})}.$$

It now follows that if μ satisfies (5.2.1) then so does $\sigma(z) = \overline{\mu(\overline{z})}$: we have that

$$\mu_z(\overline{z}) + \mu(\overline{z})\, \phi(\overline{z}) = \overline{\mu(\overline{z})} \left(\mu_{\overline{z}}(\overline{z}) - \mu(\overline{z})\, \overline{\phi(\overline{z})}\right).$$

Substitution gives

$$\overline{\sigma_z(z)} + \overline{\sigma(z)} \phi(z) = \sigma(z) \left(\overline{\sigma_{\overline{z}}(z)} - \overline{\sigma(z)} \phi(z)\right),$$

which is the conjugate of (5.2.1). □

5.3.1 Ellipticity

From the definition of μ_g we have that (5.2.1) is equivalent to

$$g_{z\overline{z}} = \frac{g_z\overline{g_{\overline{z}}}}{|g_z|^2 + |g_{\overline{z}}|^2} g_{\overline{z}\overline{z}} + \frac{g_{\overline{z}}\overline{g_z}}{|g_z|^2 + |g_{\overline{z}}|^2} g_{zz} - \frac{g_z g_{\overline{z}} \left(\overline{g_{\overline{z}}}\overline{\phi} - \overline{g_z}\phi\right)}{|g_z|^2 + |g_{\overline{z}}|^2}.$$

If we put

$$a = a(z) = \frac{g_{\overline{z}}\overline{g_z}}{|g_z|^2 + |g_{\overline{z}}|^2},$$

then the equation reads as

$$g_{z\overline{z}} = a g_{zz} + \overline{a} g_{\overline{z}\overline{z}} - \left(g_{\overline{z}}\overline{a}\overline{\phi} - g_z a \phi\right). \tag{5.3.2}$$

In [7] it is shown that for such systems, the condition for ellipticity takes the form

$$\left|1 + a\lambda + \overline{a}\overline{\lambda}\right| > 0$$

for all $\lambda \in \mathbb{C}$ with modulus 1. The choice $\lambda = -\frac{g_z\overline{g_{\overline{z}}}}{|g_z||g_{\overline{z}}|}$ gives

$$\left||g_z|^2 + |g_{\overline{z}}|^2 - 2|g_z||g_{\overline{z}}|\right| > 0,$$

which is equivalent to the condition that $J(z,g) > 0$. Since

$$J(z,g)J(g(z),f) = 1,$$

we have the corollary to Theorem 5.3.

Corollary 5.8. *The equation defined at* (5.3.2) *for g is elliptic, and uniformly elliptic if and only if $\|J(z,f)\|_\infty < \infty$. A homeomorphic solution $g : \mathbb{D} \to \mathbb{D}$ to the equation* (5.3.2) *will have as its inverse a harmonic map $f : \mathbb{D} \to (\mathbb{D}, \rho)$.*

5.3.2 Gradient And Laplacian Of $|\mu|^2$

Note that $\nabla|\mu|^2$ vanishes if and only if $|\mu|^2_{\overline{z}} = 0$ as $|\mu|^2_z = \overline{|\mu|^2_{\overline{z}}}$. At these critical points we have that

$$\mu_{\overline{z}}\overline{\mu} + \mu\overline{\mu_z} = 0,$$

and so from (5.2.1) we have, after rearrangement, that

$$\mu_z = -\mu\left(\overline{\mu_z} + \left(\phi + \overline{\mu}\overline{\phi}\right)\right).$$

Substituting the complex conjugate into itself gives

$$\mu_z = -\mu\left(-\overline{\mu}\left(\mu_z + \left(\overline{\phi} + \mu\phi\right)\right) + \left(\phi + \overline{\mu}\overline{\phi}\right)\right),$$

or

$$\left(1 - |\mu|^2\right)\mu_z = -\mu\phi\left(1 - |\mu|^2\right);$$

therefore $\mu_z = -\mu\phi$ (as $\|\mu\|_\infty \leq k < 1$) and also $\mu_{\overline{z}} = \mu\overline{\phi}$ at points where the gradient vanishes. Among these are the points where the dilatation of a harmonic mapping has its maximum.

For the Laplacian of $|\mu|^2$, we have that

$$\begin{aligned} |\mu|^2_{z\overline{z}} &= \mu_{z\overline{z}}\overline{\mu} + |\mu_{\overline{z}}|^2 + |\mu_z|^2 + \mu\overline{\mu_{z\overline{z}}} \\ &= 2\Re\left(\mu_{z\overline{z}}\overline{\mu}\right) + |\mu_{\overline{z}}|^2 + |\mu_z|^2, \end{aligned}$$

so at critical points we have that

$$\frac{1}{4}\Delta|\mu|^2 = 2\Re\left(\mu_{z\overline{z}}\overline{\mu}\right) + |\mu|^2|\phi|^2.$$

5.4 Solutions For Separable Families

We shall now take a short look into cases where either f or μ have a specific form. We first note that the case of the Euclidean and hyperbolic metric they are radially symmetric: their metrics can be represented as a function of $|z|$, that is $\rho(z) = \lambda(|z|)$. We will consider metrics of this form on the disk. Note

$$\phi(z) = (\log\rho(z))_z = \frac{\lambda'(|z|)}{2\lambda(|z|)}\frac{\bar{z}}{|z|},$$

and that it is separable. We shall now investigate when we have solutions belonging to certain families of separable functions.

5.4.1 Solutions With f Separable

We know that the identity is the only radial stretching solution from the following argument. From [30], we have that any quasiconformal harmonic extension of a quasisymmetric mapping is unique. Any radial stretchings mapping the unit disc to itself must be the identity on the restriction the boundary unit circle. Since the identity mapping of the unit disk is quasiconformal and harmonic, this must be the unique extension of any mapping that fixes the boundary. A similar argument shows an analogous result holds for anti-radial stretchings.

Before we examine the case for radially fixed solutions $re^{i\theta} \mapsto re^{i\tau(\theta)}$ and similar, we shall note first that when we say τ is linear, etc., that it is linear on $[0, 2\pi)$ after some rotation.

Lemma 5.9. *The identity is the only radially fixed solution.*

Proof. (Lemma 5.9) We have, since the inverse of a radially fixed mapping is radially fixed, when $g\left(re^{i\theta}\right) = f^{-1}\left(re^{i\theta}\right) = re^{i\tau(\theta)}$ where τ is not a simple rotation (that is, $\tau'(\theta) \not\equiv 1$), that

$$\mu\left(re^{i\theta}\right) = \frac{1-\tau'(\theta)}{1+\tau'(\theta)} e^{i2\theta}.$$

The θ-derivative is

$$\mu_\theta\left(re^{i\theta}\right) = 2i\left(\frac{1-\tau'(\theta)}{1+\tau'(\theta)} + i\frac{\tau''(\theta)}{\left(1+\tau'(\theta)\right)^2}\right) e^{i2\theta}.$$

Therefore

$$\mu_z = \frac{1}{r}\left(\frac{1-\tau'(\theta)}{1+\tau'(\theta)} + i\frac{\tau''(\theta)}{\left(1+\tau'(\theta)\right)^2}\right) e^{i\theta},$$

and

$$\mu_{\overline{z}} = -\frac{1}{r}\left(\frac{1-\tau'(\theta)}{1+\tau'(\theta)} + i\frac{\tau''(\theta)}{\left(1+\tau'(\theta)\right)^2}\right) e^{i3\theta}.$$

After rearranging, (5.2.1) becomes

$$\frac{2}{1+\tau'}\left[\frac{1}{r}\left(\frac{1-\tau'}{1+\tau'} + i\frac{\tau''}{\left(1+\tau'\right)^2}\right) + \frac{1-\tau'}{1+\tau'}\frac{\lambda'}{2\lambda}\right] = 0,$$

so

$$1 + i\frac{\tau''}{\left(1-\tau'^2\right)} = -r\frac{\lambda'}{2\lambda}.$$

The right-hand side is real, which means we must have $\tau(\theta)$ linear (in the same sense as given before), and also $\lambda(r) = Cr^{-2}$ for some constant C. By Picard-Lindelöf (Theorem A.6) this solution is unique for fixed C away from 0, thus there is no solution of this form as $\lim_{r\to 0}\lambda(r) = \infty$. □

5.4.2 Solutions With μ Separable

Lemma 5.10. *Suppose $\mathbb{D}$ is equipped with a radially symmetric metric. If μ satisfies (5.2.1) and μ is radially symmetric or anti-radially symmetric, then μ is identically zero or $\mu(z)$ is a constant multiple of $\frac{z}{|z|}$ and the metric density $\rho(z)$ is a constant multiple of $|z|$.*

Proof. (Lemma 5.10) Suppose $\mu(z) = \frac{z}{|z|}\alpha(|z|)$, we then have from (5.2.1), after some rearranging,

$$\alpha'(|z|)+\frac{\alpha(|z|)}{|z|}+\alpha(|z|)\frac{\lambda'(|z|)}{\lambda(|z|)}=\frac{z}{|z|}\alpha(|z|)\left(\alpha'(|z|)-\frac{\alpha(|z|)}{|z|}-\alpha(|z|)\frac{\lambda'(|z|)}{\lambda(|z|)}\right).$$

Note that the left side of this equation is real for all z; therefore the right must be as well. This can only happen if both sides equal zero, otherwise for any choice of z not real the equality will be incorrect; so either $\alpha(t)=0$ for all t, or both

$$\alpha'(|z|)+\frac{\alpha(|z|)}{|z|}+\alpha(|z|)\frac{\lambda'(|z|)}{\lambda(|z|)}=0$$

and

$$\alpha'(|z|)-\frac{\alpha(|z|)}{|z|}-\alpha(|z|)\frac{\lambda'(|z|)}{\lambda(|z|)}=0.$$

We add and subtract these last two equations from one another, then rearrange the results. Upon cancellation of a common $\alpha(|z|)$ term, which we have assumed is nonzero, we obtain that

$$\alpha'(|z|)=0 \quad \text{and that} \quad \frac{1}{|z|}=-\frac{\lambda'(|z|)}{\lambda(|z|)}.$$

In this case $\lambda(t)=Ct^{-1}$ for some constant C, and $\alpha(t)$ is constant. So for the Euclidean and hyperbolic metrics, the only radially symmetric solution has $\mu(z)=0$ everywhere.

Similar analysis for anti-radially symmetric case gives (we suppress the arguments for clarity)

$$\left[\alpha'-\frac{\alpha}{|z|}+\alpha\frac{\lambda'}{\lambda}\right]\frac{\bar{z}}{z}=\frac{z}{|z|}\alpha\left(\left[\alpha'+\frac{\alpha}{|z|}\right]-\alpha\frac{\lambda'}{\lambda}\right)$$

requires either $\alpha(t)=0$ everywhere or $\alpha(t)$ constant and $\lambda(t)=Ct$ for C constant.

□

Since $\|\mu\|_\infty=k<1$, we cannot have μ radially fixed. However, we can scale the family by some constant $k\in[0,1)$.

Lemma 5.11. *Suppose $\mathbb{D}$ is equipped with a radially symmetric metric. If μ satisfies* (5.2.1) *and $\mu\left(re^{i\theta}\right)=kre^{i\tau(\theta)}$ for some constant $k\in[0,1)$, then $k=0$.*

Proof. We will assume that $k\neq 0$, as we know the result for $\mu(z)\equiv 0$. Suppose $\mu\left(re^{i\theta}\right)=kre^{i\tau(\theta)}$; (5.2.1) becomes

$$\left(1+\tau'(\theta)+\frac{r\lambda'(r)}{\lambda(r)}\right)e^{i(\tau(\theta)-2\theta)}=kr\left(1-\tau'(\theta)-\frac{r\lambda'(r)}{\lambda(r)}\right).$$

Note that the right side of this equation is real for all $z = re^{i\theta}$; therefore the left must be as well. This can only happen if $\tau(\theta) = 2\theta$, $\tau(\theta) = 2\theta + \pi$ or both sides are zero.

In the first and second cases, we then have

$$3 + \frac{r\lambda'(r)}{\lambda(r)} = \mp kr\left(1 + \frac{r\lambda'(r)}{\lambda(r)}\right),$$

so, for some constant c,

$$\lambda(r) = c\frac{(rk \pm 1)^2}{r^3},$$

and by Picard-Lindelöf (Theorem A.6) this solution is unique for fixed c away from 0. Since λ must be finite on $\mathbb{D}$, this is impossible as this formula goes to infinity as $r \to 0$.

Therefore we must have the third case, where

$$\left(1 + \tau'(\theta) + \frac{r\lambda'(r)}{\lambda(r)}\right) = rk\left(1 - \tau'(\theta) - \frac{r\lambda'(r)}{\lambda(r)}\right) = 0,$$

but this too is impossible as $r \not\equiv 0$ and $k \neq 0$, as otherwise

$$1 + \tau'(\theta) + \frac{r\lambda'(r)}{\lambda(r)} = 0 = 1 - \tau'(\theta) - \frac{r\lambda'(r)}{\lambda(r)},$$

and adding both sides together would have $2 = 0$. □

Lemma 5.12. *Suppose $\mathbb{D}$ is equipped with a radially symmetric metric λ. If μ satisfies* (5.2.1) *then μ is a polar independent function if and only if*

- *$\mu \equiv 0$, or*
- *$\lambda(r) = Kr^c$ for some constants $K, c > 0$ and $\mu = Le^{-i(c\theta+\alpha)}$ for $L \geq 0$ and α constant.*

In particular, the only polar independent solution for the Euclidean and hyperbolic metrics is $\mu(z) = 0$.

Proof. (Lemma 5.12) Suppose $\mu\left(re^{i\theta}\right) = \rho(r)\, e^{i\tau(\theta)}$ where $\rho(r) \not\equiv 0$; (5.2.1) becomes

$$\left(\rho' + \frac{\rho}{r}\tau' + \rho\frac{\lambda'}{\lambda}\right)e^{i(\tau-2\theta)} = \rho\left(\rho' - \frac{\rho}{r}\tau' - \rho\frac{\lambda'}{\lambda}\right).$$

If $\tau(\theta) = 2\theta + n\pi$ where $n \in \mathbb{Z}$, we have that

$$\rho'(r) = -\rho(r)\frac{1+\rho(r)}{1-\rho(r)}\left(\frac{2}{r} + \frac{\lambda'(r)}{\lambda(r)}\right),$$

so, for some constant C, we have that

$$\rho(r) = \frac{2}{r\left(Cr\lambda(r) \pm \sqrt{C^2r^2\lambda(r)^2 - 4C\lambda(r)}\right) - 2},$$

however, as $r \to 0$, $\rho(r) \to -1$, which is impossible as $\rho \geq 0$.

Otherwise, as before, the absolute value of both sides must equal zero; since $\rho \not\equiv 0$ we require

$$\rho' + \frac{\rho}{r}\tau' + \rho\frac{\lambda'}{\lambda} = 0 = \rho' - \frac{\rho}{r}\tau' - \rho\frac{\lambda'}{\lambda},$$

so ρ is constant, and $\tau' = -r\lambda'\lambda^{-1}$; therefore $\tau = c\theta + L$ and $\lambda = Kr^{-c}$. For orientation preserving maps, no λ work and so no solutions exist. Therefore, μ must be orientation-reversing and be of the form stated in the statement of the theorem. □

5.5 Investigation Of The Euclidean Metric Case

In the case of the Euclidean metric, $\phi \equiv 0$ and (5.2.1) reduces to

$$\mu_z = \overline{\mu}\mu_{\overline{z}}. \tag{5.5.1}$$

Although we are obviously interested in these results primarily in the hyperbolic case, we shall discuss the Euclidean case briefly, as using only this version of Theorem 5.3 we may derive the result for the hyperbolic case as well. We formulate the result in the following corollary.

Corollary 5.13. *We may derive* (5.2.1) *for the hyperbolic metric from* (5.5.1)*.*

Proof. (Corollary 5.13) First note that at the origin the hyperbolic metric is flat, so harmonic functions with respect to either metric are indistinguishable at this point. Therefore

$$\mu_z(0) = \mu_{\overline{z}}(0)\,\overline{\mu}(0)$$

for both metrics.

Let g be the inverse of a harmonic mapping and $\eta(z)$ be a Möbius transformation:

$$\eta(z) = \lambda\frac{z-a}{1-\overline{a}z}, \qquad |\lambda| = 1, a \in \mathbb{D}.$$

Then $g \circ \eta$ is also the inverse of a harmonic mapping as composition with isometries preserves harmonicity. We compute $\mu_{g\circ\eta}$:

$$\begin{aligned}
(g\circ\eta)_z &= g_z(\eta)\,\eta' \\
(g\circ\eta)_{\overline{z}} &= g_{\overline{z}}(\eta)\,\overline{\eta'} \\
\eta'(z) &= \lambda\frac{1-|a|^2}{(1-\overline{a}z)^2} \\
\left(\frac{\overline{\eta'}}{\eta'}\right)(z) &= \lambda^{-2}\frac{(1-\overline{a}z)^2}{(1-a\overline{z})^2}
\end{aligned}$$

so

$$\mu_{g\circ\eta}(z) = \mu_g(\eta(z))\,\lambda^{-2}\frac{(1-\overline{a}z)^2}{(1-a\overline{z})^2}. \tag{5.5.2}$$

Calculating the first-order derivatives, we obtain

$$\begin{aligned}
(\mu_{g\circ\eta})_z(z) &= [\mu_g(\eta(z))]_z\,\lambda^{-2}\frac{(1-\overline{a}z)^2}{(1-a\overline{z})^2} + \mu_g(\eta(z))\,\lambda^{-2}\left[\frac{(1-\overline{a}z)^2}{(1-a\overline{z})^2}\right]_z \\
&= (\mu_g)_z(\eta(z))\,\eta'(z)\,\lambda^{-2}\frac{(1-\overline{a}z)^2}{(1-a\overline{z})^2} - \mu_g(\eta(z))\,\lambda^{-2}2\overline{a}\frac{(1-\overline{a}z)}{(1-a\overline{z})^2},
\end{aligned}$$

and

$$\begin{aligned}
(\mu_{g\circ\eta})_{\overline{z}}(z) &= [\mu_g(\eta(z))]_{\overline{z}}\,\lambda^{-2}\frac{(1-\overline{a}z)^2}{(1-a\overline{z})^2} + \mu_g(\eta(z))\,\lambda^{-2}\left[\frac{(1-\overline{a}z)^2}{(1-a\overline{z})^2}\right]_{\overline{z}} \\
&= (\mu_g)_{\overline{z}}(\eta(z))\,\overline{\eta'(z)}\lambda^{-2}\frac{(1-\overline{a}z)^2}{(1-a\overline{z})^2} + \mu_g(\eta(z))\,\lambda^{-2}2a\frac{(1-\overline{a}z)^2}{(1-a\overline{z})^3},
\end{aligned}$$

so at zero, we have that

$$\begin{aligned}
(\mu_{g\circ\eta})_z(0) &= (\mu_g)_z(\eta(0))\,\eta'(0)\,\lambda^{-2} - 2\overline{a}\mu_g(\eta(0))\,\lambda^{-2} \\
&= (\mu_g)_z(-\lambda a)\left(1-|a|^2\right)\lambda^{-1} - 2\overline{a}\mu_g(-\lambda a)\,\lambda^{-2},
\end{aligned}$$

and that

$$\begin{aligned}
(\mu_{g\circ\eta})_{\overline{z}}(0) &= (\mu_g)_{\overline{z}}(\eta(0))\,\overline{\eta'(0)}\lambda^{-2} + 2a\mu_g(\eta(0))\,\lambda^{-2} \\
&= (\mu_g)_{\overline{z}}(-\lambda a)\left(1-|a|^2\right)\lambda^{-3} + 2a\mu_g(-\lambda a)\,\lambda^{-2}.
\end{aligned}$$

Substituting this into our earlier equation gives, with $w=-\lambda a$

$$\left(\left(1-|w|^2\right)(\mu_g)_w + 2\overline{w}\mu_g\right)\lambda^{-1} = \left(\left(1-|w|^2\right)(\mu_g)_{\overline{w}} - 2w\mu_g\right)\overline{\mu_g}\lambda^{-1},$$

which upon simplification gives

$$\left(\mu_g\right)_w(w)+\frac{2\overline{w}}{1-|w|^2}\mu_g(w)=\left(\left(\mu_g\right)_{\overline{w}}(w)-\frac{2w}{1-|w|^2}\mu_g(w)\right)\overline{\mu_g}(w), \tag{5.5.3}$$

which equals (5.2.1) for the hyperbolic metric (as can be seen by the substitution $\phi(z)=2\overline{z}\left(1-|z|^2\right)$); since w can be any value in $\mathbb{D}$ we have Theorem 5.3 for that case. □

Remark 5.14. Let us return to the first-order derivatives of $\mu_{g\circ\eta}$: the choice of $\lambda=-1$ gives

$$\begin{aligned}
\left(\mu_{g\circ\eta}\right)_z(0) &= -\left[\left(\mu_g\right)_z(a)\left(1-|a|^2\right)+2\overline{a}\mu_g(a)\right]\\
\left(\mu_{g\circ\eta}\right)_{\overline{z}}(0) &= -\left[\left(\mu_g\right)_{\overline{z}}(a)\left(1-|a|^2\right)-2a\mu_g(a)\right],
\end{aligned}$$

so

$$\begin{aligned}
-a\left(\mu_{g\circ\eta}\right)_z(0) &= a\left(1-|a|^2\right)\left(\mu_g\right)_z(a)+2|a|^2\mu_g(a)\\
-\overline{a}\left(\mu_{g\circ\eta}\right)_{\overline{z}}(0) &= \overline{a}\left(1-|a|^2\right)\left(\mu_g\right)_{\overline{z}}(a)-2|a|^2\mu_g(a),
\end{aligned}$$

which upon adding gives

$$a\left(\mu_{g\circ\eta}\right)_z(0)+\overline{a}\left(\mu_{g\circ\eta}\right)_{\overline{z}}(0)=-\left(1-|a|^2\right)\left(a\left(\mu_g\right)_z(a)+\overline{a}\left(\mu_g\right)_{\overline{z}}(a)\right). \tag{5.5.4}$$

Therefore we obtain

$$\left(\mu_{g\circ\eta}\right)_z(0)=\overline{\left(\mu_{g\circ\eta}\right)(0)}\left(\mu_{g\circ\eta}\right)_{\overline{z}}(0)=\overline{\mu_g(a)}\left(\mu_{g\circ\eta}\right)_{\overline{z}}(0).$$

5.6 The Hyperbolic Metric Case

We already derived this formula earlier, but we will restate it here: we are interested in finding solutions to

$$\mu_z+\frac{2\overline{z}}{1-|z|^2}\mu=\overline{\mu}\left(\mu_{\overline{z}}-\frac{2z}{1-|z|^2}\mu\right). \tag{5.6.1}$$

Suppose that $\mu(z_0)=0$. From (5.5.2) we have, when $\lambda=1$ and $a=-z_0$, that

$$\mu_{g\circ\eta}(0)=\mu_g(z_0);$$

therefore we may assume that $z_0=0$. Hence either $\mu\equiv 0$ or there is some other value at, say, w with $\mu(w)\neq 0$. In the first case, $g_{\overline{z}}\equiv 0$ and so g (and thus f) must be holomorphic; we shall assume the second case is true.

Let us suppose $\mu(z) = z^m \overline{z}^n \alpha(z)$ where $m+n > 0$, α has finite (at least) first-order partial derivatives, $\alpha(0) \neq 0$ and $|\alpha|$ is bounded. Then

$$\begin{aligned}
\mu_z(z) &= m z^{m-1} \overline{z}^n \alpha(z) + z^m \overline{z}^n \alpha_z(z), \\
\mu_{\overline{z}}(z) &= n z^m \overline{z}^{n-1} \alpha(z) + z^m \overline{z}^n \alpha_{\overline{z}}(z),
\end{aligned}$$

and

$$z^{m-1}\overline{z}^n \left(z\alpha_z + \left(m + \frac{2|z|^2}{1-|z|^2} \right) \alpha \right) = z^{m+n} \overline{z}^{m+n-1} \overline{\alpha} \left(\overline{z}\alpha_{\overline{z}} + \left(n - \frac{2|z|^2}{1-|z|^2} \right) \alpha \right),$$

so when $z \neq 0$ we have that

$$z\alpha_z + \left(m + \frac{2|z|^2}{1-|z|^2} \right) \alpha = z^{n+1} \overline{z}^{m-1} \overline{\alpha} \left(\overline{z}\alpha_{\overline{z}} + \left(n - \frac{2|z|^2}{1-|z|^2} \right) \alpha \right).$$

Choose $\epsilon > 0$ and evaluate this at $\epsilon e^{i\theta}$:

$$\epsilon e^{i\theta} \alpha_z + \left(m + \frac{2\epsilon^2}{1-\epsilon^2} \right) \alpha = \epsilon^{m+n} e^{i\theta(n-m+2)} \left(\epsilon e^{-i\theta} \alpha_{\overline{z}} \overline{\alpha} + \left(n - \frac{2\epsilon^2}{1-\epsilon^2} \right) |\alpha|^2 \right).$$

This means

$$\left| \epsilon e^{i\theta} \alpha_z + \left(m + \frac{2\epsilon^2}{1-\epsilon^2} \right) \alpha \right| \leq \epsilon^{m+n} |\alpha| \left(\epsilon |\alpha_{\overline{z}}| + \left(n - \frac{2\epsilon^2}{1-\epsilon^2} \right) |\alpha| \right).$$

If we take the limit as $\epsilon \to 0$, as α_z and $\alpha_{\overline{z}}$ are finite then the right hand goes to zero while the left goes to $m|\alpha(0)|$; as $|\alpha(0)| \neq 0$ we must have $m = 0$. So $\mu(z) = \overline{z}^n \alpha(z)$.

5.7 Extending Quasisymmetric Maps

We shall now briefly investigate the use of the equation (5.2.1) to the problem of the Schoen conjecture. First, suppose we are given a quasiconformal mapping $f : \mathbb{D} \to \mathbb{D}$ with smooth complex dilatation. We wish to know when the boundary values of f are also those of a harmonic mapping. Equivalently, that there is a harmonic mapping $h : \mathbb{D} \to \mathbb{D}$ such that $(h^{-1} \circ f)(z) = z$, $z \in \mathbb{S}$.

It is a difficult question to decide when two mappings have the same boundary values from their complex dilatation. However, because of stability results of Wolf, Markovic, etc., (see [19] and [33], for example) , we need only show that $g \circ h$ is quasisymmetric with small constant (depending on h). To remove this dependence we in fact need that $h^{-1} \circ f$ is symmetric. This can be formulated as a condition on the dilatation of the composition.

Lemma 5.15. *The mapping $f : \mathbb{D} \to \mathbb{D}$ has symmetric boundary values if $|\mu_f| \to 0$ appropriately where μ_f is the complex dilatation of f.*

As we have stated earlier, a proof for this can be found in [33].

Note that if f has symmetric boundary values, then so does $g = f^{-1}$: we know that $|\mu_g| = |\mu_f|$ by (2.5.3).

From 2.5.2, we have that

$$\mu_{h^{-1}\circ f}(z) = \mu_{f\circ g^{-1}}(g(z)) = \frac{\mu_{h^{-1}}(z) - \mu_{f^{-1}}(z)}{1 - \mu_{h^{-1}}(z)\overline{\mu_{f^{-1}}(z)}} \left(\frac{\left(f^{-1}\right)_z(z)}{\left|\left(f^{-1}\right)_z(z)\right|} \right)^2,$$

and therefore we set $\mu = \mu_{f^{-1}}$ and define (for an as yet undetermined harmonic function h)

$$\nu(z) = \frac{\mu_{h^{-1}}(z) - \mu(z)}{1 - \mu_{h^{-1}}(z)\overline{\mu(z)}},$$

where we want $|\nu(z)| \to 0$ as appropriate. Solving this for $\mu_{h^{-1}}$ gives

$$\begin{aligned}
\nu - \nu\mu_{h^{-1}}\overline{\mu} &= \mu_{h^{-1}} - \mu \\
\nu + \mu &= \mu_{h^{-1}}\left(1 + \nu\overline{\mu}\right) \\
\mu_{h^{-1}}(z) &= \frac{\nu(z) + \mu(z)}{1 + \nu(z)\overline{\mu(z)}},
\end{aligned}$$

and as the left-hand side of this equation is supposed to be the complex dilatation of a harmonic mapping, it must satisfy (5.2.1). Noting

$$\begin{aligned}
\left(\frac{\nu + \mu}{1 + \nu\overline{\mu}}\right)_z &= \frac{\left(\nu_z + \mu_z\right)\left(1 + \nu\overline{\mu}\right) - \left(\nu + \mu\right)\left(\nu_z\overline{\mu} + \nu\overline{\mu_{\overline{z}}}\right)}{\left(1 + \nu\overline{\mu}\right)^2} \\
&= \frac{\nu_z\left(1 - |\mu|^2\right) + \left(1 + \nu\overline{\mu}\right)\mu_z - \left(\nu^2 + \mu\nu\right)\overline{\mu_{\overline{z}}}}{\left(1 + \nu\overline{\mu}\right)^2} \\
\left(\frac{\nu + \mu}{1 + \nu\overline{\mu}}\right)_{\overline{z}} &= \frac{\nu_{\overline{z}}\left(1 - |\mu|^2\right) + \left(1 + \nu\overline{\mu}\right)\mu_{\overline{z}} - \left(\nu^2 + \mu\nu\right)\overline{\mu_z}}{\left(1 + \nu\overline{\mu}\right)^2},
\end{aligned}$$

then (5.2.1) becomes

$$\begin{aligned}
&\frac{\nu_z\left(1 - |\mu|^2\right) + \left(1 + \nu\overline{\mu}\right)\mu_z - \left(\nu^2 + \mu\nu\right)\overline{\mu_{\overline{z}}}}{\left(1 + \nu\overline{\mu}\right)^2} + \frac{\nu + \mu}{1 + \nu\overline{\mu}}\phi \\
&\quad = \frac{\overline{\nu} + \overline{\mu}}{1 + \overline{\nu}\mu}\left(\frac{\nu_{\overline{z}}\left(1 - |\mu|^2\right) + \left(1 + \nu\overline{\mu}\right)\mu_{\overline{z}} - \left(\nu^2 + \mu\nu\right)\overline{\mu_z}}{\left(1 + \nu\overline{\mu}\right)^2} - \frac{\nu + \mu}{1 + \nu\overline{\mu}}\overline{\phi}\right),
\end{aligned}$$

so

$$\nu_z = \frac{\overline{\nu}+\overline{\mu}}{1+\overline{\nu}\mu}\left(\nu_{\overline{z}} + \frac{(1+\nu\overline{\mu})}{\left(1-|\mu|^2\right)}\mu_{\overline{z}} - \nu\frac{(\nu+\mu)}{\left(1-|\mu|^2\right)}\overline{\mu_z} - \frac{(1+\nu\overline{\mu})(\nu+\mu)}{\left(1-|\mu|^2\right)}\overline{\phi}\right)$$
$$- \frac{(1+\nu\overline{\mu})}{\left(1-|\mu|^2\right)}\mu_z + \frac{\nu(\nu+\mu)}{\left(1-|\mu|^2\right)}\overline{\mu_{\overline{z}}} - \frac{(1+\nu\overline{\mu})(\nu+\mu)}{\left(1-|\mu|^2\right)}\phi.$$

Chapter 6

Conclusions And Future Work

The most interesting results of this thesis are Theorem 3.10 and Theorem 3.11. It provides us with a condition on a radially symmetric metric which says whether there exists any mappings of finite distortion blowing up the origin whose L^p-norm of their distortion is finite. Using this theorem, we may then attempt to resolve decompositions that arise in the studies of Kleinian groups and the iterations of rational maps. This generalises some results previously established by Martin and McKubre-Jordens that ties into many problems in the physical sciences.

Theorem 4.10 was quite a surprising result: in the equivalent problem of the disk, we may find mappings that blow up the point to a disk in any L^p space for any $p \geq 1$; $\mathbb{D} \setminus [-1, 1]$ is quasiconformally equivalent, yet we can only blow up a point to this interval in the method listed for spaces where $1 \leq p < \frac{5}{2}$ for functions of the given form from Section 4.3.

6.1 Blowing Up Points And Resolving Decompositions

In Chapter 3 we focused primarily on establishing some results that we would be using in Chapter 4 to resolve singularities arising from decompositions in certain situations. Theorems 3.10 and 3.11 provide us with a good test for the existence of functions of finite distortion whose distortions lie in L^p spaces for particular values of p, as well as bounds on the L^p norm and the constant that appears in Theorem 3.13. One obvious thing that could be done would be to investigate this for other metrics; this would probably depend on the application of interest.

For Theorem 4.10, there are a few more extensions that could be investigated: for example, an arbitrary function that mapped the disk to itself that preserved the interval multiplied the same type of $h(t)$ listed in Chapter 4. However, it seems unlikely, as near $\phi = 0, \pi$ the function would have to act quite similarly to the form given in Theorem 4.10, and since that is the area where the problems generally arise, it seems reasonable to conjecture that the same difficulties crop up once again. Other potential functions would have to have a very different form to this.

Let us consider another problem. Let $t \in \mathbb{R}$ and $r < t \leq 1$. For an interval about t of radius r, $I = [t-r, t+r]$ we define the mapping

$$\phi_I(z) = z - \frac{r^2}{z - t + \sqrt{(z-t)^2 - r^2}} \text{``} = \text{''} t + \sqrt{(z-t)^2 - r^2}$$

where " $=$ " is used here because of some issues around the choice of branches; it is illustrated below for $t = 0$, $r = 1$. A way to confirm these observations is via composition $\alpha_- \circ \alpha_+^{-1}$ where

$$\overline{\mathbb{C}} \setminus [-1,1] \xleftarrow{\alpha_+} \mathbb{C} \setminus \mathbb{D} \xrightarrow{\alpha_-} \overline{\mathbb{C}} \setminus [-i, i]$$

and $\alpha_\pm(z) = \frac{1}{2}\left(z \pm \frac{1}{z}\right)$: first, the inverse of α_+ is

$$\alpha_+^{-1}(z) \text{``} = \text{''} z + \sqrt{z^2 - 1},$$

where again we use " $=$ " to be mindful of the branch choice. The composition is then

$$\alpha_- \circ \alpha_+^{-1}(z) = \frac{1}{2}\left(z + \sqrt{z^2-1} + \frac{1}{z + \sqrt{z^2 - 1}}\right) = z - \frac{1}{z + \sqrt{z^2-1}}.$$

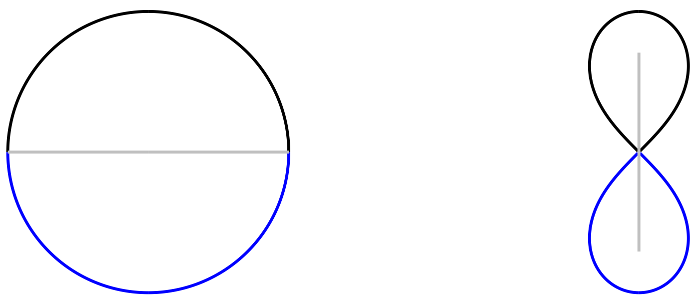


Figure 6.1.1: The map $z \mapsto z - \frac{1}{z + \sqrt{z^2-1}}$ maps the unit circle (left) to the lemniscate (right)

For our purposes, a key property will be that the image of the unit circle is the figure eight curve, with four angles of $\pi/2$ formed at the origin. The map has the additional properties that it is symmetric in the real axis and imaginary axis, as each of the composites is, which makes the map odd: a fact not obvious from the simplification to $\sqrt{z^2-1}$. Given an interval I we write $\phi_I(2i) = a + ib$; clearly $b > 0$, and set

$$\varphi_I(z) = \frac{2}{b}\left(\phi_I(z) - a\right).$$

This map conformally takes $\overline{\mathbb{C}} \setminus [t-r, t+r] \to \overline{\mathbb{C}} \setminus [t-is, t+is]$ for some real s and the exterior of the disk $\mathbb{D}(t,r)$ to the exterior of a lemniscate with vertex at t. In particular,

- $\varphi_I(2i) = 2i$,
- $\varphi_I(\mathbb{R} \setminus [t-r, t+r]) = \mathbb{R} \setminus \{t\}$, and
- φ_I has a continuous surjective extension to the upper half-space $\overline{\mathbb{H}^+} \to \overline{\mathbb{H}^+}$, and also to the lower half-space.

Let $\mathcal{I}_0 = \bigcup I_k^0$ be a possibly infinite collection of disjoint closed intervals $I_k^0 = [t_k - r_k, t_k + r_k]$. Set

$$\varphi_1(z) = \varphi_{I_1^0}(z), \quad \mathcal{I}_1 = \bigcup_{k=2}^{\infty} I_k^1$$

where

$$I_k^1 = \varphi_1([t_k - r_k, t_k + r_k]) = \left[\tilde{t_k} - \tilde{r_k}, \tilde{t_k} + \tilde{r_k}\right].$$

Then we may inductively define

$$\varphi_k(z) = \varphi_{I_k^{k-1}}(\varphi_{k-1}(z)), \quad \mathcal{I}_k = \bigcup_{j=k+1}^{\infty} I_j^k.$$

This gives us a sequence of conformal mappings $\overline{\mathbb{C}} \setminus \{\mathcal{I}_0\} \to \mathbb{C}$. We would like to prove some result such as:

Conjecture 6.1. *Suppose $\overline{\mathbb{C}} \setminus \{\mathcal{I}_0\}$ is a domain. Then the sequence $\{\varphi_k\}_{k=1}^{\infty}$ converges locally uniformly on $\overline{\mathbb{H}^+}$, and hence $\overline{\mathbb{H}^-}$, to a mapping $\varphi : \mathbb{C} \to \mathbb{C}$ which is*

- *continuous on $\overline{\mathbb{C}} \setminus \bigcup_k I_k^0$,*
- *conformal on $\overline{\mathbb{C}} \setminus \overline{\bigcup_k I_k^0}$, and*
- *$\varphi(\mathbb{D}(t_k, r_k))$ consists of two lobes (forming a lemniscate) meeting the real line with internal angle $\pi/2$ and external angles $\pi/4$.*

Remark. Note that $\overline{\mathbb{C}} \setminus \{\mathcal{I}_0\}$ may not be a domain: for example, setting $t_k = \frac{1}{k}$ and $r_k = \frac{1}{3k(k+1)}$ gives an $\mathcal{I}_0$ that is not closed.

From Theorem A.3, we know that there is a subsequence which converges locally uniformly to a conformal mapping on $\overline{\mathbb{C}} \setminus \overline{\bigcup_k I_k^0}$ since we have a family of normalised (by $\varphi_k(\pm 2i) = \pm 2i$) conformal mappings. Consider such a convergent subsequence $\{\varphi_k\}$. Also, by the inductive construction, $\varphi_k \circ \varphi_{k-1}^{-1}$ maps the kth disk $\mathbb{D}(t_k, r_k)$ to a lemniscate whose lobes meet at 0. The problem arises for $n \neq k$ $\varphi_n \circ \varphi_{n-1}^{-1}$: the disks

$\mathbb{D}(t_k, r_k)$ do not remain circular under these mappings; and certain obvious fixes such as 'filling in' the area in some bounded quasi-annulus around the appropriate disk in a simple way to ensure the mapping on the boundary is the identity, or using the technique used in the proof of Theorem 4.5 via the quasiconformal Schoenflies extension do not permit the given function to remain conformal.

If we could solve this problem, the inductive construction shows that the sequence $\varphi_k \circ \varphi_1^{-1}$ converges locally uniformly on $\overline{\mathbb{C}} \backslash \overline{\bigcup_k I_k^1}$. The properties then follow from the properties of local uniform convergence (continuity), Theorem A.3 (conformality) and of the inductive construction (lemniscates).

6.2 Quasiconformal Harmonic Extensions

In Chapter 5, we tied together the two properties of quasiconformal and harmonic mappings to construct Theorem 5.3. This is a very tempting result: it is a first order partial differential equation (albeit a nonlinear one), and we already know how to turn the solution of this problem into the final form that we want: a quasiconformal harmonic extension of a quasisymmetric mapping on the boundary of some model of the hyperbolic plane. However, the big difficulty here is the lack of boundary conditions for this new equation: we have the quasisymmetric mapping, but outside of certain classes of such mappings the ability to relate the boundary conditions of one problem to another is still unknown.

Other analysis for certain more general classes of functions were done, but in general the ability to analyze them was rather limited, due to the corresponding forms of (5.2.1) being very complex. Perhaps there is a currently unknown class of functions that provide solutions to this PDE: if there are any, they would have to be somewhat asymmetric, for as we have noted in Chapter 5 there are no radially symmetric solutions.

Aside from this, it is a rather elegant equation that captures this problem: it is structurally similar to the simpler case for the Euclidean metric if we consider (say) the operators $D_1 : \mu \mapsto \mu_z + \phi\mu$ and $D_2 : \mu \mapsto \mu_{\overline{z}} - \overline{\phi}\mu$ as some variants of the usual derivative operators that have been altered to depend on the metric.

Future work that could be done in this area would be to perhaps use this, alongside some other techniques that were not covered during this thesis, to try to work on, for example, situations where $|\mu| \not\to 0$ at the boundary. For example, if we examine (5.2.1), for the hyperbolic disk case

$$\left(1 - |z|^2\right)\mu_z + 2\overline{z}\mu = \overline{\mu}\left(\left(1 - |z|^2\right)\mu_{\overline{z}} - 2z\mu\right).$$

If we consider the case where $|z| \to 1$, if μ_z and $\mu_{\overline{z}}$ are finite going towards the boundary, we get (if $\mu\left(e^{i\theta}\right) \neq 0$),

$$e^{-2i\theta} = -\overline{\mu\left(e^{i\theta}\right)}$$

and as $\|\mu\|_\infty < 1$ we need $|\mu| \to 0$. However, there are certainly ways for us to construct a bounded function μ over $\mathbb{D}$ (or $\mathbb{H}$, as appropriate) where both first derivatives go to infinity at the boundary, which if it decreased at a sufficient rate would balance out the factor $\left(1-|z|^2\right)$.

Appendix A

Miscellaneous Theorems And Results

In this section we will list theorems mentioned and arguments used in the text in multiple areas whose details would break the flow of the argument.

Theorem A.1 (Arzelà-Ascoli Theorem)**.** *A family $\mathcal{F}$ of continuous functions from a separable metric space to a compact metric space is normal if and only it is equicontinuous.*

A proof appears in [43].

Lemma A.2 (Chain Rule for Partial Derivatives (Two-Dimensional))**.** *If g is a function real differentiable at z and f is a function real differentiable at $w = g(z)$ then*

$$(f \circ g)_z(z) = f_w(w)\, g_z(z) + f_{\bar{w}}(w)\, \overline{g_{\bar{z}}(z)},$$

and

$$(f \circ g)_{\bar{z}}(z) = f_w(w)\, g_{\bar{z}}(z) + f_{\bar{w}}(w)\, \overline{g_z(z)}.$$

We highlight it here to make the following observation which we will use multiple times: suppose that g is the inverse of f. Then, writing $w = g(z)$ we have that

$$f_w(w) = \frac{\overline{g_z(z)}}{J(z, g)},$$

and

$$f_{\bar{w}}(w) = -\frac{g_{\bar{z}}(z)}{J(z, g)}.$$

Theorem A.3. *Let $\{f_k\}_{k=0}^{\infty}$ be a family of K-quasiconformal mappings $f_k : \mathbb{C} \to \mathbb{C}$ normalised by the conditions $f_n(0) = 0$ and $f_n(1) = 1$. Then $\{f_k\}_{k=0}^{\infty}$ is a normal family. Moreover every limit mapping is a nonconstant K-quasiconformal homeomorphism of $\hat{\mathbb{C}}$.*

The proof can be found in [6, p.134]. Note that we can change these normalization conditions easily as there are many suitable conformal mappings.

Lemma A.4 (Application of Jensen's Inequality)**.** *If $p \geq 1$ and $a, b \geq 0$ then*

$$a^p + b^p \leq (a+b)^p \leq 2^{p-1} (a^p + b^p) . \quad \text{(A.0.1)}$$

Proof. The second inequality follows from the convexity of $x \mapsto x^p$ on $[0, \infty)$ and Jensen's inequality. The first follows from the Minkowski inequality. □

For the next theorem we require a specific case of the *Lipschitz condition.*

Definition A.5. Let $f : \mathbb{R} \to \mathbb{R}$. We say that f satisfies a *Lipschitz condition if for all $x, y \in \mathbb{R}$,*

$$|f(x) - f(y)| \leq M |x - y|$$

for some constant M.

We quote the theorem from [28]:

Theorem A.6 (Picard-Lindelöf Theorem)**.** *Let Ω be an open subset of $\mathbb{R}^2$ and a continuous function $f(x, y)$ defined as $f : \Omega \to \mathbb{R}$. If $(x_0, y_0) \in \Omega$ and f satisfies a* Lipschitz condition *in the variable y in Ω. Then the ordinary differential equation defined as $\frac{dy}{dx} = f(x, y)$ with the initial condition $y(x_0) = y_0$ has a unique solution $y(x)$ on some interval $|x - x_0| \leq \delta$.*

A proof for this can be found in [26].

Appendix B

Alternative Bounds For P

Lemma B.1. *For $1 \le p \le 2$ and $t \in [0,1]$ we have that*

$$P(t) \le P_2(t) = \frac{1}{t^{p+1}} - t^{p+1}, \tag{B.0.1}$$

whilst for $2 \le p$ we have that

$$P(t) \ge P_1(t) = \frac{1}{t^{p+1}} - 1. \tag{B.0.2}$$

Proof. For the first result, we want

$$\left(1 - t^2\right)\left(1 + t^2\right)^{p-1} \le 1 - t^{2p+2}.$$

Let $s = t^2$ and write the difference

$$h(p) = 1 - s^{p+1} - (1 - s)(1 + s)^{p-1}$$

as a function of p. Then $h(1) =s - s^2$ and $h(2) = s^2 - s^3$, both of which are positive in the domain so the result is true for $p = 1, 2$ (noting for later that $h(2) \le h(1)$). Differentiating with respect to p, we obtain

$$h'(p) = s^{p+1} \ln\left(\frac{1}{s}\right) - (1 - s)(1 + s)^{p-1} \ln(1 + s).$$

We wish to find the minimum value of this function. Setting $h'(p) = 0$ gives

$$s^{p+1} \ln\left(\frac{1}{s}\right) = (1 - s)(1 + s)^{p-1} \ln(1 + s).$$

The left hand side is a decreasing function of p as $s \le 1$ while the right hand side is increasing. So there is a unique p value for which this is true for a given s; therefore, as a function of p, there is at most a single maximum or minimum. As $h(1) > 0$ and $h(2) > 0$ and

$$h'(2) = s^3 \ln\left(\frac{1}{s}\right) - \left(1 - s^2\right) \ln\left(1 + s\right) \leq 0$$

then the minimum occurs at $p = 1$ or $p = 2$, and as we observed earlier, it is in fact at $p = 2$.

For the second, choose N such that $N \leq p < N + 1$; from our assumptions, $N \geq 2$. We want to show

$$1 - t^{p+1} \leq \left(1 - t^2\right)\left(t^2 + 1\right)^{p-1},$$

which follows from

$$1 - t^{N+2} \leq \left(1 - t^2\right)\left(t^2 + 1\right)^{N-1}.$$

This is clearly true for $t = 1$, so assume $t < 1$. We can then rewrite this as

$$\sum_{k=0}^{\left\lfloor \frac{N+1}{2} \right\rfloor} t^{2k} + \sum_{k=0}^{\left\lfloor \frac{N}{2} \right\rfloor} t^{2k+1} \leq \sum_{k=0}^{N-1} \binom{N-1}{k} t^{2k} + \sum_{k=0}^{N-1} \binom{N-1}{k} t^{2k+1}.$$

and as $\left\lfloor \frac{N}{2} \right\rfloor \leq \left\lfloor \frac{N+1}{2} \right\rfloor \leq N - 1$ and $1 \leq \binom{N-1}{k}$ this result holds.

$\square$

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Nomenclature

$\mathbb{A}_R$	The annulus with inner radius 1 and outer radius R.
$\mathbb{A}(r, R)$	The annulus with inner radius r and outer radius R.
$\mathbb{C}$	The complex plane.
C^∞	The class of all infinitely differentiable functions.
$\mathbb{D}$	The unit disk of the complex plane.
$\mathbb{D}^*$	The unit disk with the origin removed.
$G_\mathbb{D}$	The group of isometries of the Poincaré disk.
$G_\mathbb{D}^+$	The group of Möbius transformations of the Poincaré disk.
$G_\mathbb{H}$	The group of isometries of the Poincaré half-plane.
$G_\mathbb{H}^+$	The group of Möbius transformations of the Poincaré half-plane.
$\mathbb{H}$	The upper half-plane.
$\mathcal{I}_p$	A condition integral of a radially symmetric metric that, if finite, is necessary and sufficient to show that there is a mapping of finite distortion that blows up the origin with distortion in the specified Lebesgue space.
$\mathcal{K}_p$	Given a mapping of finite distortion, an integral specifying the associated Lebesgue norm of the distortion.
$K(z, f)$	The quasiconformal distortion function.
$\mathbb{K}(z, f)$	The finite distortion function.
μ, μ_f	The Beltrami coefficient or complex dilatation.
$PGL(2, \mathbb{R})$	The projective general linear group of 2 by 2 real matrices.
$PSL(2, \mathbb{R})$	The projective special linear group of 2 by 2 real matrices.

$\Phi_f(z)$	The Hopf differential of the function.
$\mathbb{S}$	The unit circle.

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