Constructing Decimal Concepts in an Inquiry Classroom

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Abstract

This study examines the construction of decimal concepts of primary aged students in the classroom. It builds on previous work which has promoted the use of percentages as a means for rational number thinking and for the enhancement of such thinking through multiple modes of representation. In this study percentages provide a foundation for rational number understanding as represented through the decimal system.

The study is set within an inquiry classroom. In this classroom the pedagogical approach maps out an alternative to customary practice by shifting the traditional teacher-student relationship to one of partnership in knowledge construction. In this classroom both student engagement with well-designed learning activities, and mathematical discussion and debate are all deemed highly important to the production of decimal understandings.

The investigation revealed that students had a wealth of informal rational number knowledge. This informal knowledge created a useful context and springboard for the development of new conceptual understandings of decimal fractions. That development was not immediate—it traced out a lengthy, unpredictable and recursive path and required students to reflect on their thinking and allowed for subtle teacher and peer reconstruction of students' misconceptions. From those findings recommendations are made for a productive approach to the teaching of decimals in primary school classrooms.

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Chapter 1: Introduction

1.1 Background to the study

"To be numerate is to have the ability and inclination to use mathematics effectively—at home, at work and in the community" (Ministry of Education, 2002).

In research, current and past, there is an urgent call for change, to both the teaching and learning of mathematics (Anthony & Walshaw, 2002; Begg, 1999; Carpenter & Lehrer, 1999). The need for change is nowhere more evident than in the teaching of rational number and in particular decimal fractions (Ministry of Education, 1996, 1997, 2002). Internationally and in New Zealand, over a time span of more than six years, students study decimal fractions as part of the school curricula. However despite the regular teaching and reteaching of decimal concepts, research shows that more than thirty percent of students in their final year at school maintain erroneous decimal understandings (Irwin, 1996a; Moloney & Stacey, 1996; Stacey & Steinle, 1999; Steinle & Stacey, 2002). Explanation for this phenomenon is not simple. There is a complex interplay of many factors affecting the construction of decimal fraction concepts.

Many studies have documented the difficulties students and adults have with decimal fractions, in particular, a lack of conceptual understanding of decimal notation (Helme & Stacey, 2000; Hiebert & Wearne, 1986; Steinle & Stacey, 1998, 2002). The difficulties Steinle and Stacey (2002) maintain:

> ...lie both in the nature of the mathematical and psychological aspects of the task and in the teaching they receive. Understanding decimal notation is a complex challenge, which draws on previous learning and fundamental metaphors of number and direction, both to advantage and disadvantage (p. 633).

Decimal fractions are introduced as a Level 3 achievement objective to students aged nine to eleven years in New Zealand (Ministry of Education, 1992). Constructing conceptual understanding of decimal fraction knowledge is traditionally difficult for students within this age group (Moss & Case, 1999). Such understanding requires radical reconstruction of prior whole and fractional number concepts and integration of place value concepts using base 10 notation to represent the fractional quantities (Irwin, 1996a, 1999). The process is lengthy with regular recurring misconceptions and partial understandings occurring as students integrate their prior knowledge with new learning along the path to sense making (Condon & Hilton, 1999).

Studies suggest that school instruction, mathematics textbooks and classroom activity may cause many of the students' problems in constructing decimal concepts. Specifically, researchers contend that robust and deep understanding is unlikely for those students who do not construct richly connected concepts of decimal fractions as quantities before being introduced to the decimal notation system. Weak understanding of the notation system invariably leads to students applying formal algorithmic rule bound approaches (Hiebert, 1993; Post, Cramer, Behr, Lesh, & Harel, 1993; Wearne, 1990). Tenacious decimal misconceptions remain unchallenged because instruction has not promoted active student engagement in making the connections between decimal fractions and other mathematical concepts (Thompson & Walker, 1996).

1.2 Inquiry and reform type classrooms

In response to research findings of the past twenty years New Zealand mathematics curriculum documents have indicated a need for change in the teaching and learning of mathematics (Ministry of Education, 1992, 2002a). In these documents the vision is not one of passive transmission of knowledge and rules from teacher to student but rather a partnership of active learning in "a community of inquirers" where collaborative interaction occurs (Anthony & Walshaw, 2002, p. 4). In such classrooms student reasoning is foremost as students explain, explore alternative ideas, argue, justify and validate their thinking.

With respect to learning decimal fractions, students must engage in an active learning process in order to integrate their prior whole and fractional number thinking and build multi-levelled and multi-connected decimal fraction concepts. Translating across representations, that is, applying the equivalent fraction concept is a key understanding underpinning construction of robust concepts of fractions, decimals and percentages (Ministry of Education, 2002b; Vance, 1992).

In support of the need to make connections explicit between rational number concepts Moss and Case (1999) demonstrate the success of using percentages as the introductory representation in decimal fraction instruction with children. However, further studies are needed to explore what happens in the construction of decimal fraction understandings when students participate in learning activities that specifically build on their ability to connect and translate across rational number concepts.

The majority of the current studies which describe teaching and learning activities designed to support the construction of decimal knowledge (e.g., Hiebert & Wearne, 1986; Helme & Stacey, 2000; Irwin, 1996b; Moss & Case, 1999) focus on a range of teaching procedures—exploring the role of specific representations including concrete manipulatives, visualisation tools, computer games and contextualised problems. These studies describe the cognitive aspects of constructing decimal knowledge within a psychological perspective. However, to support reforms in mathematics teaching and learning further research is needed that focuses on the nature of learning decimal fractions in an inquiry classroom environment. Currently, there is limited research which links rational number development with the social perspective of learning. The focus of such research would not be the knowledge the teacher wants students to learn or whether students constructed decimal fraction knowledge but "the nature or quality of those constructions" and the mathematical community and environment that supports them (Cobb, Yackel & Wood, 1992b, p. 28).

1.3 Research objectives

The primary aim of this study is to explore how nine to eleven year old students construct decimal fraction concepts in the context of an inquiry classroom. The study also seeks to examine students' informal knowledge of decimal fractions and the ways in which these affect the construction of decimal fraction understandings. A related objective is to explore the classroom environment making links with the effect of specific classroom practices on individual students as they construct decimal fraction concepts.

In particular the following research questions have been addressed:

1. What informal knowledge of decimal fractions did the students hold before formal introduction in the classroom setting?
2. How did the students' informal decimal fraction knowledge support the construction of decimal fraction concepts?
3. How did an instructional model using a range of modes of representation affect the students' construction of decimal fraction concepts?
4. How did classroom practices and in particular the social and sociomathematical norms support conceptual learning of decimal fraction concepts?

1.4 Overview

Chapter 2 reviews the literature in the field and provides the background with which this study can be viewed. The context and framework for the current study is provided through summarising and linking appropriate and essential literature related to—active learning in an inquiry classroom, collaborative interaction and classroom discourse, social and sociomathematical norms, the construction of decimal knowledge and the effects of classroom instructional practices.

In Chapter 3 the methodology for the study is described. The research setting and sample, data collection and analysis are discussed, and a timeframe for the classroom teaching experiment project is outlined.

Chapter 4 provides examination of classroom practice in an inquiry classroom. The teacher's role is described and the effects of collaborative interaction and classroom discourse and social and sociomathematical norms on students' patterns of decimal fraction thinking are analysed. In Chapter 5, an analysis of the way the instructional model including informal knowledge and use of a range of modes of representation supported student construction of decimal fraction concepts is provided. Chapter 6 presents a summary for the four case study students of self-evaluation data and an analysis of the interview and classroom data.

In Chapter 7 the results are discussed and conclusions are drawn. Implications for the classroom are presented and suggestions for further research are described.

Chapter 2: Literature review

2.1 Introduction

Students in New Zealand schools are taught and re-taught decimal fraction concepts over more than six years. Despite the prolonged teaching and reteaching of decimals, tenacious misconceptions remain evident in many students' thinking patterns. A review of the literature will show that construction of decimal fraction concepts occur as a result of complex interplay between many factors.

Relevant literature falls into several categories. Research related to the way in which individual students construct decimal concepts, problems they encounter, and the use of students' informal knowledge, and a range of representations are of primary importance to this study. In addition, relevant research on the social construction of mathematical concepts in classrooms will also be summarised. This review outlines the growing body of literature that explores student mathematical activity and reasoning in inquiry or reform classrooms. Challenge to erroneous thinking patterns has been identified as a critical factor in student reorganisation of decimal thinking patterns. Relevant literature is reviewed which demonstrates how challenge to thinking patterns occurs in reform environments where during discussion and debate students are required to elaborate, argue and justify their current thinking. Since there is limited current literature on the teaching and learning of decimal concepts in inquiry classrooms the discussion focuses on more general mathematical learning within such classrooms.

2.2 Constructing mathematical knowledge in the classroom

Research over the past twenty years has signalled a need for change in the way mathematics is taught and learnt both in New Zealand and overseas (Begg, 1999; Carpenter & Lehrer, 1999; Ministry of Education, 1992; National Council of Teachers of Mathematics, 2000; Pirie & Kieren, 1994). Current mathematical learning theory emphasises the need for all students to acquire knowledge through active engagement in a constructive and interactive problem solving process. Through this process the learner is provided with rich opportunities to logically reason, reflect and communicate. Within this description mathematical teaching and learning is located within situated social contexts adopting a socio-constructivist framework (Boaler, 2000; Nickson, 2000). Cobb (2000b) maintains that the socio-constructivist framework links Piagetian and Vygotskian notions of cognitive development in a relationship of "reflexivity", that is, each perspective informing the other (p. 64).

2.2.1 The individual nature of learning

Piagetian cognitive theory maintains that learners are fundamentally active in constructing their own knowledge through purposeful activity. Activity can be seen primarily as problem finding and problem solving. Purposeful activity is filtered through the learner's cognitive lens, and used to construct knowledge through the adaptation and mediation of current perceptions and experiences and previous knowledge (Simon, 1995). Modification of existing conceptual structure depends on both challenge and a perception of it as a challenge to the viability of existing conceptual structures (Lerman, 1996). In this way mathematics learning can be characterised as a process of active individual reorganisation of conceptual schema (Cobb, 1995). Learners move back and forth through levelled but non-sequenced layers of knowing, in a series of recursive stages (Kieren, 1993; Storey, 2001). Each layer is embedded in subsequent layers and folds back to previous layers "to re-member and to re-construct new understanding" involving a non-unidirectional nature of coming to know mathematics (Pirie & Kieren, 1994, p. 84). The construction of mathematical concepts can be pictured as a chaotic process, not the neat tidy linking of ideas as presented in the past. This process is demonstrated by

Hiebert, Wearne and Taber (1991) in their study of fourth grader's gradual construction of decimal fractions "disconnecting, connecting and reorganising appear to be the rule rather than gradual addition to a stable structure" (p. 339).

2.2.2 The social nature of learning

Vygotskian sociocultural constructivist theory supports the view of the individual as the primary actor in the construction of knowledge. However this view is balanced with a social focus of describing learning as a "collective participatory process of active knowledge construction emphasising context, interaction and situatedness" (Salomon & Perkins, 1998, p. 2). Learning is characterised by a process of enculturation into established mathematical practices where mathematical symbols act as mediators linking the developing knowledge of the student to their cultural inheritance. In this way, constructing a decimal schema using the decimal system can be described as appropriation of a tool which is culturally specific (Cobb, 1995).

An implication of the Vygotskian sociocultural constructivist theory is the notion, that what is learnt is not able to be separated from how it is learnt, nor used. Vygotsky believed in the primacy of culture in shaping development (Begg, 1999). Brown, Collins and Duguid (1989) argue that recognition of the situated nature of cognition is essential in the construction of robust knowledge. Thus a Vygotskian view of instruction is often described metaphorically, as cognitive apprenticeship within which building mathematical conceptual knowledge is co-constructed within a community of learners through collaborative interaction.

Classroom practices which include modelling, scaffolding, discussing, explaining, arguing, reasoning, exploring and reflecting illustrate the apprenticeship model developed from Vygotskian principles. Collaborative interaction promotes the development of socially shared understandings, which are distributed across the social group and form a collective learning system. Collaboratively, mathematical understandings are developed with more expert others in a process of co-construction through active participation in learning situations (Salomon & Perkins, 1998). Collaborative interaction during mathematical activity creates what Vygotsky names as zone of proximal development. The zone of proximal development can be described as the "discrepancy between what a child is able to do at entry point into a problem situation and the level reached in solving the problems with assistance" (Nickson, 2000, p. 155). The successful use of older or more able students scaffolding construction of decimal knowledge with younger or less able students is illustrated in studies by Irwin (1996b) and Irwin, Lauaki, Jacobs and Marino (2000).

In this theorising frame, mathematical learning is situated; intricately connected to the socio-cultural context in which it is developed (Nickson, 2000). The key features of the socio-constructivist framework are that learners are purposefully active, engaging in problem finding and solving, and constructing knowledge which is individually and socially determined. The traditional mathematics classroom models a shift towards what is termed an 'inquiry mathematics classroom' or 'reform mathematics classroom' which supports:

- classrooms as mathematical communities
- logic and mathematical evidence as verification
- evaluation of own and others' mathematical thinking
- argumentation and mathematical reasoning
- conjecturing, inventing and problem-solving
- connecting mathematics, its ideas and its applications

(NCTM Professional Standards for Teaching Mathematics, 1991; 2000).

2.3 The inquiry classroom

A mathematical classroom identified as an "inquiry mathematics tradition" provides an environment in which students "experience understanding when they can create and manipulate mathematical objects in ways that they can explain and, when necessary, justify" (Cobb, Wood, Yackel & McNeal, 1992, p. 598). Cobb (2000b) argues that mathematical practice in this instance is characterised as "emergent phenomenon" in that the normative practices are constituted in the course of ongoing interaction (p. 66). Included within mathematical practice is classroom activity (otherwise termed mathematical tasks such as small collaborative groups, large group and whole class discussions, the use of tools including manipulatives and real world problems) and the social and sociomathematical norms. This model acknowledges the role of planned instructional programmes but also makes links to the classroom learning environment, classroom discourse, and classroom norms and sociomathematical norms (Cobb, 2000b).

Within this model of the inquiry classroom, it is contended that invisible and shared meanings are developed around norms and values that teachers and children bring to it, and these in turn control their actions and interactions within it (Nickson, 2000). This recognises that social and cultural influences are not only restricted to the process of learning and the development of mathematical knowledge but extend to its product—increasingly sophisticated mathematical ways of knowing, characterised by reflective discourse, patterns of mathematising and the development of a mathematical disposition (Boaler, 2000; Cobb, Boufi, McClain & Whitenack, 1997; Cobb, Gravemeijer, Yackel, McClain and Whitenack, 1997).

2.3.1 The autonomous learner in the inquiry classroom

A mathematical disposition is constructed within the class community as members within it actively negotiate, participate in, and contribute to the development of sociomathematical norms (McClain & Cobb, 2001). Yackel and Cobb (1996) describe a mathematical disposition as specifically referring to the mathematical beliefs and values students have that support the development of intellectual autonomy in mathematics. Cobb (2000b) contends that in the development of intellectual autonomy it is essential that students see themselves as a mathematical community. This implies that they are able to independently validate their own and others contribution to mathematical argumentation and reach consensus, maintaining confidence in their own validation or authorship, without need to refer to outside resources, such as the teacher or a text. In their studies, Kazemi (1998) and Kazemi and Stipek (2001) use the notion of 'press for learning' to describe the manner in which teachers support the growth of intellectual autonomy through reflective self-evaluation and personal responsibility. Intellectual autonomy is achieved through emphasising student efforts, pressing them to find multiple solutions, and to explain their reasoning, rather than only focus on the giving of correct answers. Cobb (2000b) contends that through analysis of sociomathematical norms, the degree of 'press for learning' can be measured, providing an explanation of the ways in which teachers are able to foster the development of intellectual autonomy, within a classroom community.

2.3.2 The role of the teacher in the inquiry classroom

As a proactive facilitator of interactive discussion, the teacher has a critical role in the inquiry classroom. A key goal is to establish reflective discourse, which involves both collaborative interaction and individual contribution. It is a balancing role, requiring appropriate timing, to allow students to struggle with ideas when to question further, probe deeper, and allow for reorganisation of thinking (Cobb et al., 1997; McClain & Cobb, 1998). The teacher as an active participant, inducts students into the mathematical community through appropriation of student responses, repeating, re-voicing, expanding or reformulating student explanations in such a way that other students are able to gain access to their peers' explanations to use as thinking tools (Cobb, Yackel & Wood, 1992; Foreman & Larreamendy-Joerns, 1998; Inagaki, Hatano & Morita, 1998). The teacher also acts as a regulating agent: since through the type of teacher response to aspects of mathematical activity, students are able to infer whether or not their responses and those of their peers are validated or sanctioned.

Teacher intervention acts as a model for the development of both social and sociomathematical norms, while also establishing paradigm cases with which students are able to reflectively access when involved with more conceptually advanced mathematical activity. Yackel and Cobb (1996) demonstrate this in a study which sought to interpret how classroom life supported the development of autonomous learners. When a student changed an answer in response to peer reaction, interpreting the social situation or pressure as more important than mathematical reasoning, the teacher intervened, presenting an inappropriate social scenario to clarify the inappropriate reason for changing the answer. Yackel and Cobb contend that "interventions of this type are powerful...as paradigm cases that students can refer to" (p. 11).

The teacher as a participant listener actively involved in making sense of explanations is an important model in the inquiry classroom. However, this is not sufficient; the teacher also requires sound pedagogical content knowledge. Close monitoring of student explanations and reasoning processes requires the teacher to not only listen to the student, but also to have knowledge of the common misconceptions held by students as they construct a decimal schema (Helme & Stacey, 2000; Ziukelis, 1988). In the study by Irwin et al. (2000), identified success factors in student's construction of decimal knowledge included teachers' understanding of place value as a multiplicative concept, the matching of learning to needs, and an awareness of common errors. The view of teachers having sound pedagogical knowledge not only supports them in making sense of partial understandings, but also means they are able to recognise when intervention is appropriate during classroom interaction and discourse (Ball & Bass, 2000).

2.4 Collaborative interaction and classroom discourse

The classroom participation structure of an inquiry classroom focuses on the central role of collaborative interaction and discourse. Classroom discourse has been identified as an important element in mathematical development, both in its role as supporting individual construction of mathematical knowledge and as a social act within the mathematical community. During collaborative interaction students construct mathematical understandings within a social context and at the same time learn to communicate mathematically through describing and justifying their solutions (Kazemi & Stipek, 2001) As a communal activity classroom discourse involves both individual and shared accountability (McClain & Cobb, 1998). Kazemi (1998) illustrated this in the study of "high press for conceptual thinking" (p.10). In her study, the most effective teacher emphasised individual accountability *and* the need for consensus in both whole class and small group discussions. Not only were all students required to contribute but they were also expected to make sense of each others' explanations.

But Cobb et al. (1997) caution, that participation in mathematical discourse only constitutes the *possibility* of mathematical development. Students may choose *not* to reorganise their thinking. Or they may create a commensurable paradigm where they appear to be collaboratively interacting but in reality are talking past each other; therefore they are unable to access each others' thinking (Cobb et al., 1992).

Intersubjectivity is described as "a reflexive phenomenon in that we have to assume it in order to attain it...that is, we each typically assume that what we are saying will be interpreted as we intend and, after the event, accept that this is the case unless we have direct evidence to the contrary" (Cobb et al., 1992a, p. 119). During collaborative interaction students may negotiate intersubjectivity. However, Cobb et al. and Hiebert (1993) argue that as they do not have direct knowledge of, nor access to their fellow students' prior mathematical experiences, or the current interpretations they are making, interpretations and conceptual construction is essentially individualised, and based on what they themselves know.

However it is through intersubjective engagements that individual interpretation is changed and modified. During collaborative interaction and task engagement, individual thinking is challenged when assumptions of intersubjectivity prove to be unviable, causing cognitive conflict as the students attempt to deal with incongruities (Hiebert et al., 1997). Using whole number concepts, Yackel, Cobb and Wood (1991) in a longitudinal teaching experiment, describe the learning opportunities gained through collaborative interaction and the resolution of the conflicting points of view held by individual students during small group problem solving and extended discourse. For example, the students used aspects of each others' solutions, re-conceptualised problems to analyse errors, and extended their own conceptual framework to make sense of alternative solutions and to reach consensus.

Hiebert and Wearne (1993) contend that extended discourse is linked to deeper reflection. Using teaching experiment methodology to make comparative observations of two sets of classrooms, when reassessed, one group of students, demonstrated higher levels of performance. They had been given fewer problems, but spent longer periods discussing and exploring alternative strategies, reflecting and reconceptualising the problem.

Reflective discourse is characterised as a series of repeated shifts in thinking which Cobb and colleagues (1997) link to the development of higher conceptual and more sophisticated mathematical thinking. In their study they demonstrated how the teacher during reflective discourse gradually translated a student's mathematical explanation into a symbolic record. This then became the explicit object of discourse available for reflective reviewing by class members and later used for conceptual manipulation as an experientially real mathematical object.

Wardekker (1998) argues for the significance of reflection for students in constructing "knowledge in action" through collaborative dialogue (p. 147). Wardekker contends that an important role of schools is that of giving students access to practices that are otherwise outside the reach of students. Moreover, that schools accelerate learning to participate in other practices by creating "virtual practices"—"this introduction is better when the virtual practice...retains the essential characteristics of the actual practice" (p.147). This links to the need for mathematics classrooms to bridge students' informal and intuitive knowledge with formal or scientific (school based) knowledge through contextualised problem solving and collaborative interaction. In a study of students constructing decimal knowledge, Irwin (2001), using collaborative dialogue and contextualised problems, demonstrated the bridging of students' scientific knowledge, with that of the informal knowledge.

McClain and Cobb (1998) suggest that "the development of mathematical understanding is a recursive, non-linear phenomenon" arguing that discourse that supports the growth of understanding shares these characteristics (p. 80). Cobb et al. (1992) describe learning, which occurs during, and as a result of collaborative interaction and classroom discourse, as "a circular, self-referential sequence of events" (p. 99). The view of mathematical development, intricately interwoven with classroom discourse in reform classrooms, contrasts with a traditional model "which consist of a series of teacher questions leading students to producing preferred procedures" (Wood & Turner-Vorbeck, 2001, p. 189).

Collaborative interaction in an inquiry classroom, supports both individual construction of mathematical concepts and the development of an autonomous learner as students participate in mathematical discourse modelled on the practice of mathematicians.

2.5 The social and sociomathematical norms

Cobb (2000b) describes classroom social norms as the ways in which students are obliged to make explanations and justify their solutions, question conflicting alternatives, make sense of others' explanations, and reach a point of consensus. In addition to the social norms, Yackel and Cobb (1996) discuss the idea of the sociomathematical norms: norms that are specific to the mathematics curriculum area and support higher level mathematical cognitive activity. It is the normative understandings of what counts as a mathematically different, sophisticated, efficient, or an elegant solution, or an acceptable explanation, argumentation, or justification, which are the focus of the sociomathematical norms. Explanations are not merely procedural or calculational, but have an expectation that students, will extend their explanation or justifications "to involve described actions on mathematical objects...and other students... are able to interpret the explanation in terms of actions on mathematical objects that are experientially real to them" (Yackel & Cobb, p. 462).

The sociomathematical norms shape classroom discourse and regulate learning and they are identified as critical elements enabling students to mathematise an activity or task and develop intellectual autonomy and a mathematical disposition (McClain & Cobb, 2001; Yackel & Cobb, 1996). During conceptual enculturation, as the students learn to make acceptable explanations, they appropriate the shared values and norms of the mathematics community. At the same time, the students are developing shared meaning of scientific (or school based) mathematical objects and participating in the discourse of the mathematics community (Foreman & Larreamendy-Joerns, 1998; Hiebert, 1993). Thus the social and sociomathematical norms are constituted collaboratively within the classroom community.

Explanations are tentative hypotheses that are reworked and reformed as they are interactively reconstituted. In classroom mathematical discourse, as students compare solutions, and make judgements about the similarities and differences, the role of the solution changes focus, and becomes the object of reflection (Cobb, 2000b). Hiebert and Wearne (1993) contend that through this process of argumentation, involving students listening to and making sense of each others' solutions, expressing and defending their opinions, students not only have to elaborate on their ideas but need to identify incongruities in their thinking. In this way they reorganise their thinking, and also engage in a more reflective process, which has depth beyond that of merely providing a solution or focusing on facts.

Currently there is limited research available on both student development of decimal knowledge in an inquiry classroom and on explanations of the effects of classroom norms and sociomathematical norms. However, studies by Lampert (1990; 2000) and Cobb (2000b, p. 69), using content from other mathematical strands, describe the "classroom participation structure" within which classroom norms and sociomathematical norms are established jointly by the teacher and students. Cobb (2000b) posits that analysis of sociomathematical norms has supported researchers in understanding ways in which teachers are able to foster development of intellectual autonomy. Kazemi and Stipek (2001), in a comparative study of teachers, describes the use of a "press for learning scale" to analyse the ways in which the sociomathematical norms were enacted effectively in classrooms. Differences among their 'high' and 'low press' samples identify the importance of the following sociomathematical norms:

- an explanation consists of a mathematical argument, not simply a procedural description;
- mathematical thinking involves understanding relations among multiple strategies;
- errors provide opportunities to reconceptualise a problem, explore contradictions, and pursue alternative strategies;
- collaborative work involves individual accountability and reaching consensus through mathematical argumentation.

2.6 The construction of decimal knowledge

Current research characterises rational number as a set of interconnected subconstructs, which at the same time are distinctly different from each other (Kieren, 1993; Marshall, 1993). Kieren's concept of the rational number domain, embeds it as "a significant window on the whole domain of mathematics" (p. 59). The interconnected subconstructs include notions of measure, quotient, operator, ratio, and part whole (Carpenter, Fennema & Romberg, 1993). This characterisation contrasts with the traditional school textbook notion of rational number as a linearly ordered, static and algorithmic extension of whole number.

Constructing conceptual understanding of decimal fractions is a lengthy and complex process of linking and interweaving new knowledge with prior knowledge (Post et al., 1993). The seminal study of Sackur-Grisvard and Leonard (1985) describes the construction process as one that uses a "succession of cognitive tools". Each tool is used "in a certain class of problems for which it will produce the correct answer...until it generates too many conflicts...new accommodation is required and a new tool is constructed" (p. 159). Within this process, Sackur-Grisvard and Leonard describe the use of intermediate cognitive organisations. These are tools which fall between the old and new concepts and are used to accommodate and modify knowledge. As such, mistakes are not viewed as random, rather they are rules established as stable intermediate organisers broadly based on whole number or fractional number thinking. The problem is, as Stacey and Steinle (1999) illustrate, these rules are tenacious, and while they may support learning of new concepts, they may also hinder it.

2.6.1 The complexities involved in constructing decimal knowledge

Behr, Lesh, Post and Silver (1983) contend that the rational number concepts encountered in the pre-secondary years are both the most deceptively complicated and significant mathematical ideas learnt. Given the complexity of the decimal system and the traditional instructional model used in texts the development of erroneous thinking is not unexpected (Hiebert, 1992; Post et al., 1993). The New Zealand results differ little from that from overseas. Irwin (1996a) describes students aged eleven and twelve as having a weak grasp of decimal knowledge and states that many adults are not able to operate with decimals despite the many years of schooling in which it has been part of the mathematics curriculum regularly taught to them.

Results from the National Education Monitoring Project (1997) and Third International Mathematics and Science Study (1996) highlight problems students are likely to develop with decimal numbers: those related to knowledge of relative sizes of decimal numbers, and the place value of the decimal system (Storey, 2001). Storey argues that the inability to quantify decimal numbers is linked to problems in being able to benchmark their fraction equivalent and implement and apply number sense when using them in operations such as multiplication and division. Findings from Steinle and Stacey (1998) suggest that only 50% of 13 year olds are able to order a set of five decimal numbers by relative size. Further studies describe the percentage of 'apparent expert' students as levelling off at 60% in Year 10 leaving 40% of the students devoid of conceptual knowledge of decimal numbers (Stacey & Steinle, 1999; Steinle & Stacey, 2002).

Decimal fractions appear to be an extension of the base ten number system and as such appear uncomplicated. In reality, the decimal system is deceptively complicated and difficult, with a complexity that conflicts with and is counter-intuitive to the whole number schema. The multiplicative decimal schema, with a place value system based on parts of a unit with specified sizes which relate to the unit either by building up or partitioning down, sharply contrasts the whole number notion of building repeatedly, groupings of ten (Hiebert, 1992, 1993; Irwin, 1996b). It is the continuous nature of decimals and the notion of partitioning down, which requires a major shift in thinking (Irwin, 1999; Thompson & Walker, 1996; Hiebert, 1992; Hiebert et al., 1991). To add to the complexity, the symbols used to record decimal fractions look like whole numbers however they represent quantities that are fractions.

Graeber and Tirosh (1990) argue that students need to learn that decimal notation may signify division, and they must also be able to interpret the division phase both as a partitive and measurement model, before they begin to work with decimal number operations. The part whole model of fractions is the dominant model used in school and within instructional texts, and therefore many students develop as a mental picture, parts of a whole rather than a continuous or measure model of comparison to a unit (Kieren, 1993; Post et al., 1993). Some researchers contend that in order to achieve this, it is essential that students construct a continuous rather than a discrete model of decimals initially (Moss & Case, 1999; Post et al., 1993). This contention was supported in the teaching experiment study of Moss and Case. They demonstrated that through the use of a continuous model the students constructed powerful mental referents for decimal symbols, which they were then able to use as experientially real thinking objects and not just unlinked symbols.

Hiebert and Wearne (1986) argue "that mathematical competence is characterised by connections between conceptual and procedural knowledge" (p. 199). They define conceptual knowledge as the "semantics of mathematics and procedural knowledge as the syntax" (p. 199). Conceptual knowledge can be defined as holding knowledge of the facts and properties and the rich relationships between them. Constructing conceptual knowledge of decimals is dependent upon an ability to make links between the various types of decimal knowledge, including the notation system, the quantities represented by the notation system, the application of rules for manipulating the quantities expressed in the notation, and their real world referents for the decimal fraction symbols including "when the quantities are moved, partitioned, combined or acted upon" (Hiebert, 1992, p. 291).

In contrast, procedural knowledge, the syntax of mathematics, is defined as "rich in rules and strategies for completing tasks but not rich in relationships" (Hiebert & Wearne, 1986, p. 201). The premature introduction of decimal notation and algorithmic operations is the cause of many problems. The students are unable to embed the decimal notation in experientially real relationships, and so the symbols are syntactically used within a set of rules or algorithms. In their studies, Bell, Swan and Taylor (1981) and Hiebert (1992) describe the procedural competence of students who were able to add and subtract with decimal symbols, while at the same time lacking ability to reason or make sense of the size of the numbers they were working with. Multiplying and dividing magnified the problem, with many students returning to 'whole number thinking' where multiplication makes bigger and division makes smaller, with little thought given to the reasonableness of answers. Hiebert maintains that the source of the problems is created for students in that they have limited experientially real opportunities to check their thinking against, and therefore are not able to see the need to suppress whole number thinking in decimal operations, particularly that of multiplication and division.

Sackur-Grisvard and Leonard (1985) describe how during the construction of decimal concepts, intermediate cognitive organisers coexist alongside more efficient tools, reappearing during cognitive overload as the learner attempts to integrate new ideas. Greer's research (1987), using a written test and a videotaped interview, found that application of conceptual understandings of operations on decimal numbers was strongly related to both the types of numbers used and the operations used. Whole number thinking was reverted to when the process of multiplication or division was required. Moss and Case (1999) using teaching experiment methodology describe similar outcomes when the control group was confronted with misleading cues. These students maintained their original erroneous thinking patterns which were primarily based on whole number thinking. However, Sackur-Grisvard and Leonard, and Condon and Hilton (1999) suggest that classroom activity may allow the rules to coexist through teacher and text adaptation of instructional tasks which avoid difficult problems and ensure correct responses in class based tasks. It is suggested that this lack of challenge to erroneous thinking may explain some of the reasons for the persistence of misconceptions. Kazemi (1998) argues that errors made by students provide an optimal teaching moment as they provide a context for the students to work within enabling students to reconceptualise the problems and contradictions and explore alternative strategies.

2.6.2 Classroom effects on the construction of partial understandings

The range of decimal misconceptions that student have hold significant pedagogical implications for the teaching of decimals (Resnick et al., 1989). Moloney and Stacey (1996, 1999) demonstrated the stability of thinking patterns in a study which explored the ordering of decimal numerals. Sixty percent of Year 6 and Year 8 students held misconceptions. Repeated a year later with the same Year 7 and 9 students, where there had been no intervention other than normal classroom instruction, the study showed fifty two percent of students had retained their prior misconceptions. The consistency of erroneous thinking patterns are confirmed by Stacey and Steinle (1999) in a four year longitudinal study in which students were regularly tested using a decimal comparison test. In tracking change in individual thinking patterns it was evident that those who did not develop expert knowledge tended to retain the same thinking patterns throughout the passage of time. These stable thought patterns signify persistent misconceptions and partial understandings constructed by the student as they attempt to integrate decimal knowledge into prior whole number or fractional number thinking.

It is of considerable concern that classroom instructional programmes seem to make little difference in addressing the problem. Thompson and Walker (1996) argue that the traditional methods of teaching decimal concepts have not supported the construction of rich connections between decimals and other mathematical contexts. Classroom observations and the analysis of classroom mathematics texts show that the premature teaching of rules and algorithms and resulting rote learning mitigates against the conceptual construction of a decimal schema (Hiebert & Wearne, 1985; Post et al., 1993). In such classrooms, many students develop procedural competence with decimals not as a sense-making activity, but rather within rule based and algorithmic behaviour. The duration of instruction makes little or no difference; consistent difficulties evident at thirteen are still present when the student is aged seventeen and leaving school (Carpenter, Corbett, Kepner, Lindquist & Reys, 1981; Helme & Stacey, 2000).

Research also indicates that within school instruction, the timing of the introduction of decimals and fractions may also influence patterns of misconceptions. Fractions introduced in Israel and America prior to decimal fractions resulted in a higher level of students with the fraction rule misconception when compared with that of France where decimals were introduced first (Resnick et al., 1989; Sackur-Grisvard & Leonard, 1985). This may also be explained by both practice and learning recency as causative factors (Hiebert & Wearne, 1985).

In exploring partial understandings of students, the age of students is also a factor that needs to be considered. Hiebert et al. (1991) in a study of middle school students, in which the instructional sequence was tracked and student explanation of tasks were analysed, describe a gradual and partial construction of understanding as normal. They noted that "students changed their reasoning to make sense of new information logically to deal with a particular context but sweeping across the board changes were rare" (p. 322). They also argued that when the movement to complete understanding was gradual, it was more usual for partial understandings to be constructed and retained. Similarly, Irwin (1996a) found that no 10-12 year old student in her three studies demonstrated complete theoretical understanding of the decimal fraction system. Irwin attributes this finding to the struggle inherent in adapting whole number knowledge to include decimal knowledge. The whole number rule, or 'longer is larger' decimal misconception is the predominant pattern of the younger students. This thinking pattern decreases from Year Five, to be replaced by the fractional number rule or 'shorter is larger' as the students move through the school system. In contrast, the fraction rule misconception decreases more slowly, remaining prominent in the thinking of twenty percent of students at Year 10 and retained as a stable thinking pattern into adulthood (Moloney & Stacey, 1996; Stacey & Steinle, 1999).

2.6.3 Decimal misconceptions as teaching tools

The error pattern in decimal misconceptions is both a powerful diagnostic and teaching tool which is often under utilised by teachers (Moloney & Stacey, 1996). A focus on the mathematical errors of the students in the past has been described as using a deficit model. In contrast, interpreting the errors as partial understandings supports a viewpoint which recognises that faulty rules are intelligent attempts made by students to integrate conflicting concepts within their current schema. As such, Resnick et al. (1989) argue that the errors themselves make an efficient diagnostic tool. In their study they extended previous research by exploring the rationale students use to explain their 'buggy rules' and argue that these should be used positively in the development of decimal knowledge.

As a teaching tool, the individual student's pattern of errors is of critical importance for further construction of decimal concepts. It is through the teacher listening to the student explanation and justification of their reasoning that specific misunderstandings can be addressed. Learning opportunities are optimised when the teacher understands the range of decimal misconceptions used by students in accommodating new learning with prior partial knowledge and appreciates how these partial understandings can be used to guide students to sound decimal understandings (Stacey & Steinle 1998; Ziukelis, 1988).

More than ten categories of erroneous rule use have been identified. These are consistent across studies and distributed by age in broadly similar ways (Stacey & Steinle, 1998). Three primary categories are commonly identified, using the classification system described by Resnick et al. (1989).

The first primary category is the 'Whole Number Rule' or 'Longer is Larger Rule'. Students who use this rule generally select the longer decimal as the larger. For example students using this category would select 4.177 as larger than 4.7—based on whole number thinking that 177 is larger than 7. The second primary category is the 'Fraction Rule' or 'Shorter is Larger Rule'. Students who use this rule would in contrast to the first rule, select the shorter decimal as larger. For example, students using this category would select 4.1 as larger than 4.177 using thinking based on knowledge of fractions—tenths are larger than hundredths and thousandths. The third primary category is the 'Zero rule'. Students who use this rule select the decimal, which has one or two zeros to the immediate right of the decimal point as the smaller. For example, students would select 4.007 as smaller than 4.7 correctly, but would not be able to explain the reason for the selection. This broad classification of rules has been described, explored, and extended in numerous studies in relation to decimal knowledge and strategies (e.g., Graeber & Tirosh, 1990; Moloney & Stacey, 1996; Resnick et al., 1989; Sackur-Grisvard & Leonard, 1985; Stacey, Chambers, Asp, Scott & Steinle, 2001; Steinle & Stacey, 1998, 2002; Ziukelis, 1988).

The three primary categories serve as a useful initial guide for teachers to assess how students are thinking. However, as an effective teaching tool, more detailed knowledge is required of the specific misconception a student might be using. Steinle and Stacey (1998) provide a useful detailed description of student thinking by extending the three primary categories of decimal misconceptions to include five additional categories under the 'Whole Number Rule' category and three additional categories under the 'Fraction Rule' category as follows:

The 'Whole Number Rule' is based on...

- String length thinking: Size is judged on length with the decimal number treated as a whole number. 4.03 > 4.3.
- Numerator focussed thinking: Similar to string length thinking but zeros after the decimal point are also ignored. 6.3 = 6.03.
- Reverse thinking: The numbers after the decimal point are seen as more whole numbers but written in the reverse order. .163 is read as one hundred and sixty three or 1 ten, six hundreds, 3 thousands.
- Zero makes smaller thinking: Similar to numerator focussed thinking but demonstrates knowledge that a decimal with a zero after the point is smaller. .08 > .8.
- Right hand overflow thinking: The numbers overflow from the right in such a way that all the numbers are squashed into one column. 0.12 is seen as twelve tenths.

The 'Fraction Rule' is based on...

- Denominator focussed thinking: Size is judged on the use of place value column names.

- Reciprocal thinking: Size is judged on an attempt to link decimals to fractions by the perception that the decimal number denotes how many parts. This is knowledge not based on decimals as ten equal parts of which a certain number are selected.
- Negative thinking: Confusion exists between notation for decimals and negative numbers.

(Steinle & Stacey, 1998, p. 549)

In addition, Irwin (1996a, 1999) describes a category of students who have problems caused by their whole number focused thinking, including the tight bond between counting numbers. For many students the notion of other numbers located between whole numbers is incomprehensible. For example, students in this category would say that there is no number between the numbers 1 and 2 nor between the decimal numbers 1.5 and 1.6 or 1.61 and 1.62.

Hiebert (1992) and Moss and Case (1999) identify the decimal point as another cause of difficulty. Swan (1993, cited in Irwin, 1996a, p. 246) uses the term 'decorative dot' to describe another category of misconceptions demonstrated by students when they treat the whole number and decimal fraction as separate units. For example, students in this category would state that the number which follows 2.1 is 3.2. An additional category linked to the notion of the 'decorative dot' is where the decimal point is ignored altogether and the number is treated as a one whole number rather than a whole number and fractional number part. For example, students in this category would state that the number 2.1 is the same as 21.

Moloney and Stacey (1997), Stacey and Steinle (1998, 1999) and Steinle and Stacey (1998) also note that when students are in transition, from one category of misconceptions to another they will frequently demonstrate inconsistent patterns. These studies repeatedly noted that when the students reconstruct their thinking patterns and develop more sophisticated ideas, they often maintain mixed and inconsistent patterns, or use dual strategies to assign meaning to decimal notation.

Many studies include one category in their list to describe students who demonstrate expert or apparent-expert behaviour. Within these studies, expert knowledge is described in various ways. Maloney and Stacey (1996) and Steinle and Stacey (1998) classify experts as those able to correctly compare pairs of decimals. Sackur-Grisvard and Leonard (1985) and Resnick, et al. (1989) use an 'expert' category which is applied to those students able to order pairs or sets of three decimals. However, in recognition of the difficulties in determining the quality of student conceptual knowledge in relation to the many strategies students use to compare decimal numbers, various other studies use the modifying terms of 'apparent' expert or 'task' expert.

Within the 'apparent expert' category students may apply procedural rules or rename decimal fractions without conceptual understanding or fit within a further category of those classified as using 'truncation thinking'. Students in this category use strategies based on their prior knowledge of money, metric measurement, or rounding to order decimal numbers to two places but have little meaning for the numbers beyond (Stacey & Steinle, 1999; Steinle & Stacey, 1998). However, when students become 'apparent experts' Stacey and Steinle (1999) note that they almost consistently remain in that category.

2.7 Linking instruction to the construction of decimal knowledge

The introduction of decimals in the Mathematics Curriculum in the middle school coincides with a change in expectation of outcomes during mathematical activity; many students and teachers no longer expect mathematical activity to directly link experientially to problems in the real world (Beswick, 2002; Wearne & Hiebert, 1988). More often instruction assumes that students within this age group are able to operate on numbers not tied to quantitative referents.

School mathematical activities based on procedures combined with texts that prematurely emphasise rule bound processes, support "students to think that decimal operational thinking originates in the nature of the symbols on the paper. Hence children look for concrete regularities in the symbols" (Kieren 1988, p. 177). For these students, symbols take on personalities of their own, and are used as tools representing ideas and procedural rules which may not relate to meaningful concrete representations nor realistic contexts (Wearne & Hiebert, 1988). According to Kieren this "premature formalism... leads to a person having technical knowledge which cannot be connected recursively to real situations...and without connections to the intuitive or ethnomathematical levels of knowing" (p. 178).

2.7.1 The need to construct quantitative concepts for decimal symbols

In order to build meaning for decimal symbols initial instruction, must intentionally focus on the connections between the written symbols and known concrete real world quantitative representations, and the language used to describe it (Graeber & Tirosh, 1990; Hiebert & Wearne, 1986; 1989; Hiebert, 1992). Through the use of a diagnostic test and individual interviews Padberg (2002) illustrated the tenuous hold 6^{th} Grade German students had of decimal concepts. Despite the students background of rich practical experience involving the metric system few students constructed conceptual understanding of the decimal place value system. Padberg maintains as essential, careful structuring of the connecting process between informal and formal knowledge of decimals. Furthermore, embedding construction of conceptual understanding of symbols in a concrete context provides students with thinking tools tied to concrete representations that link conceptual and procedural knowledge. To assist students to construct connections Hiebert maintains there are three locations where instruction needs to be focused:

- Site 1. Connecting individual symbols with meaningful referents
- Site 2. Connecting rules with actions on referents

• Site 3. Connecting answers with real world situations.

2.7.2 Cognitive conflict as a context for constructing decimal knowledge

Misconceptions in decimal knowledge have been demonstrated as based on prior knowledge—knowledge that is "robust to the point of requiring the very foundations to be tested even to the point of demolition" (Yates & Chandler, 1991, p. 148). Cognitive conflict as a part of building number sense and mathematical competence in decimal concepts has been identified as a critical determinant in the reconstruction of schema.

Studies by Bell, Swan and Taylor (1981) and Irwin (1996b) describe the use of instructional methods that utilise informal knowledge in contextualised problems to challenge or create cognitive conflict. Irwin (1999) describes students working in pairs to solve problems involving slightly unusual forms of decimals. An example of a problem used is: "The exchange rate between New Zealand dollar and Samoan tala is 1.5429. Thomas said that that means that you got 154 tala and 29 cents for every New Zealand dollar. Why did he say that? Do you agree?" (p. 3). In this instance the students' application of simplistic rules which they had developed in whole number thinking did not work in solving these problems, and coupled with knowledge of the real world context, caused cognitive conflict, which resulted in modification of their decimal schema. Furthermore, in a study to assess teaching effects, Stacey and Steinle (1999) describe the most positive results in classrooms were where cognitive conflict had occurred.

2.7.3 Formal and informal knowledge

Many researchers have identified the linking to, and building on, of informal and intuitive knowledge, as a key factor in development of conceptual understanding of rational number (Brown, 1993; Irwin, 1999; Mack, 1993, 2001; Moss &

Case, 1999; Resnick & Singer, 1993; Streefland, 1993). The knowledge students bring to formal instruction has been defined within various terms, including that of intuitive knowledge, situated knowledge and informal knowledge (Mack, 2001). Informal knowledge is characterised by Mack as "applied circumstantial knowledge constructed by individuals in response to their real life experiences" (p. 267). For example, Mack describes students as coming "to instruction with a rich store of informal knowledge related to partitioning" (p.271). Intuitive knowledge, is the term used by Kieren (1993) to describe the way in which students are able to use imagery to mentally manipulate experientially real images. For example "the act of partitioning can be thought of as an intuitive thinking tool" (p.52).

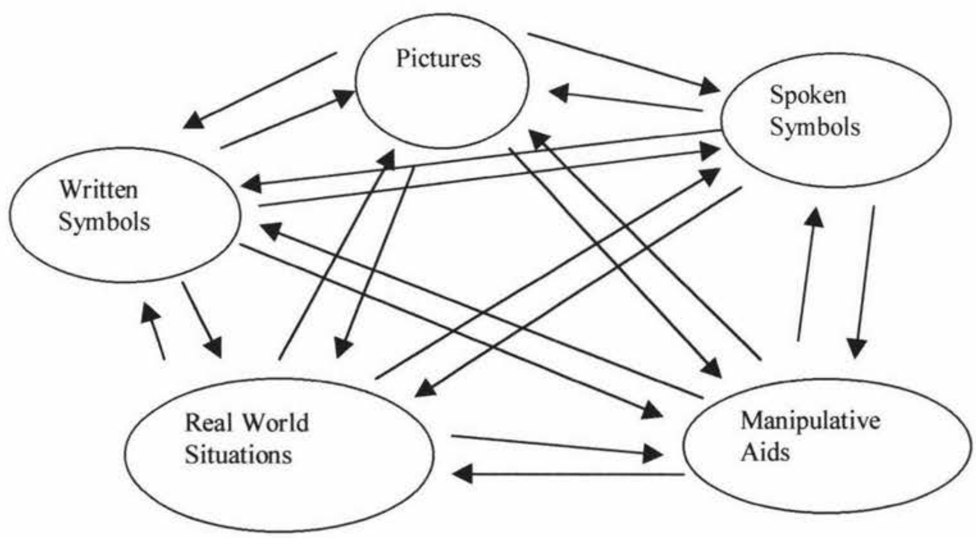
In contrast, formal knowledge concerns the "symbols, concepts and procedures that are taught in school" (Carpenter et al., 1993, p. 8). These scientific concepts are essentially school or academically based, and are situated; that is, tied to the socio-cultural context in which they are constructed. Scientific (school based) concepts, unlike informal or intuitive concepts are not implicitly understood, and therefore they require explicit linking to everyday experiences (Foreman & Larreamendy-Joerns, 1998; Mack, 2001). Within the domain of rational number the informal and intuitive knowledge of partitioning is used as a foundation for students to develop understandings of the conceptually complex domain of rational number. In a study of six, fifth grade students, Mack described how the informal partitioning knowledge of students, was used to support the construction of multiplication of fraction concepts.

Despite the purported importance of informal knowledge studies on students' understanding of decimal concepts have focused primarily on students' misconceptions with little regard to the role of the rich store of informal knowledge that students bring to the learning of decimals. An exception is a New Zealand based study by Irwin (1996a, 1999, 2001). Irwin interviewed 8-14 year old students to assess their informal knowledge of decimals. The 8-year-olds were able to describe a wide range of contexts in which they had encountered decimals informally, however the 10 to 12 year old students were more likely to refer to contexts commonly used in the school instructional setting. The range remained limited until students were aged 14, when decimals were no longer part of the school mathematics curriculum. Irwin (1996a) concluded that the quantities represented by notation used in the school setting demonstrated insufficient integration of school instructional experiences. Moreover, Irwin cautioned that teachers of students from low decile schools are less likely to access the informal experiences of their students through normal school texts and learning activities, noting the need to adapt activities to the informal knowledge of specific groups of children.

2.7.4 Representations and the construction of decimal knowledge

The construction of a robust decimal schema is dependent upon the use of representations as reasoning tools "to model and interpret physical, social and mathematical phenomena" (The National Council of Teachers of Mathematics, 2000, p. 70). Representations provide students with a set of tools that significantly expand their capacity to think mathematically. In this sense, the use of the term representation refers to both the process and to the product, including both concrete and mental embodiments. Within traditional classrooms representations have been used by teachers and learnt by students "as if they were ends in themselves" and not "essential elements...in understanding...communicating...and recognising connections" (NCTM, p. 67). In contrast, Ball (1993) maintains that the notion of classroom activity must extend beyond a specific instructional representation, to a wider meaning, where "fruitful representational contexts balance respect for the integrity and spirit of mathematics with an equal and serious respect for learners, serving as an anchor for the development of learners' mathematical ideas, tools and ways of reasoning" (p. 161).

Therefore, matching classroom activity with the needs of the learner is critical. During a five year study of reform Schwan Smith and Stein (1998) examined the use of small groups, tools (for example, manipulatives and calculators), and mathematical tasks. Their results demonstrated that the highest cognitive gains could be directly attributed to tasks which were organised and implemented in such a way that students were able to engage in high levels of cognitive reasoning and thinking. Furthermore, Stein and Schwan Smith (1998) argue that such tasks set a climate for students to complete fewer problems but engage for longer periods of time, discussing the mathematical ideas inherent in the tasks in depth. Hiebert and colleagues (1997) and Stein and Schwan Smith describe cognitively demanding tasks as 'problematic' in nature and worthy of making sense of, therefore leading to a different set of thinking processes. The tasks contain a sense of 'connectivity' and 'reflectivity', so that the activity provides opportunity for students to reflect on and develop important mathematical ideas. Initial instructional activities provide the opportunities for learning based on a highly situated and intuitive basis, from which students are able to construct a more sophisticated framework (Cobb, Yackel & Wood, 1992b). Lesh's model (cited in Post et al., 1993) show how representations connect within realistic contexts and then extend within a network of multi-levelled connectionist ways:



Lesh Translation Model (1979)

Within New Zealand schools, teachers use a wide range of concrete manipulatives to support students' construction of mathematical concepts, including those of rational number. Storey (2001) maintains that students need appropriate models with which to build strong visual images of the size of decimal fractions and suggests the use of multibase arithmetic blocks (MAB). Many classrooms use MAB as manipulatives to develop conceptual understanding of decimal concepts and these have been used in many studies (e.g., Hiebert et al., 1991; Vance, 1992; Wearne & Hiebert, 1989). However, because many students initially encounter MAB during the construction of whole number concepts Stacey, Helme, Archer and Condon (2002) argue that requiring the students to reinterpret the various block values to accommodate their decimal value creates a cognitive processing problem. Furthermore, Stacey and colleagues question the accessibility of MAB, noting that the transparency of the manipulative is dependent upon students having well developed notions of volume.

In contrast, Stacey and colleagues (2002) maintain that the use of linear arithmetic blocks (LAB) as concrete manipulatives are considerably more accessible for students. Findings from a classroom based research study claim that LAB proved more effective than MAB because LAB is modelled on length rather than volume. The fact that the mathematical concepts inherent in the LAB material did not conflict with students' previous use for whole number construction was also seen as positive. LAB engaged the students more actively and resulted in more in-depth discussion.

Although concrete manipulatives are used widely in New Zealand and overseas, the question of their effectiveness, or of which is most effective, is in contention (Behr et al., 1983; Irwin et al., 2000; Stacey et al., 2002; Pape & Tchoshanov, 2001). Pape and Tchoshanov and Nickson (2000) maintain a cautious attitude to the wholesale (uncritical) use of concrete material, warning that the mathematics inherent in some of the apparatus is not necessarily obvious to the student. In addition, Nickson describes the difficulties students encounter in making connections between their informal knowledge, the mathematics implicit in concrete apparatus and the transfer of it to problem solving situations.

Pape and Tchoshanov (2001) maintain that in the development of cognitive representational thinking the processes of internalisation and externalisation are "interrelated" (p. 126). With reference to decimal learning, Storey (2001) argues the need to advance representational thinking begun with images structured through the use of concrete manipulatives, for example, carrot cut to the size of the MAB cube then further cut is used to model the pattern of decimal fractions. In similar manner, Thompson and Walker (1996) suggest the use of individually wrapped cheese slices, which the students are able to slice in tenths, hundredths, and thousandths to provide a powerful visual pattern. New Zealand and overseas classroom teachers use a wide variety of procedures adapting their methods and models to meet perceived needs of students. The research of Irwin and colleagues (2000) involving the learning of decimal knowledge in a study of 14 classes describes the most effective methods as those which used one key visualisation tool repeatedly, bridging from the visualisation tool to numerical form.

Condon and Hilton (1999) in their research, describe the designing of activities including games and fictitious homework based on common misconceptions "which bring students into situations which conflict with their own constructs" (p. 21). They credit the success of these activities to active enjoyment and a "non-threatening opportunity...to hear their own misconceptions addressed through a fictitious person's mistake" (p. 28). Student misconceptions have also been the target of carefully designed computer games which "highlight the errors that students make and present them with conflict to be resolved" (McIntosh, Stacey, Tromp & Lightfoot, 2000, p. 410).

It is argued that decimals should be developed within an integrated network of rational number concepts (Case & Moss; Post et al., 1993; Thompson & Walker, 1996). Langford and Sarullo (1993) maintain that students need to develop "a mental map" to flexibly compare, order, and relate equivalence across the various symbols used to denote rational number (p. 242). Sowder and colleagues (1993) argue that the "process of making translations between and within modes of representation enhances students' flexibility of thought regarding the concepts being studied (p. 245). Case and Moss (1999) illustrated this in a study which began by developing a unidimensional representation of percents. This was followed by careful support to construct connections between rational number forms. Percentages were connected to their benchmark equivalent representations of decimals, and fractions. Case and Moss maintain that this process built on the notion of students' number sense, supporting the development of the ability to apply inventiveness and flexibility within a variety of representations in rational number.

However, while various studies using a range of representations have illustrated the development of conceptual and transferable understandings, other factors need to be considered. These include the complex interplay of language, the ability of individual students to make links with the emerging patterns, and the use of material which may represent adult rather than child based concepts. Furthermore, Lampert and Ball (1998) note that representations may make some aspects of the mathematical structure of a problem obvious or more accessible, but they also have the potential to complicate or obscure conceptual details. In addition, Cobb, Yackel and Wood (1992b) caution the use of an instructional strategy which explicitly makes links between the representation and notation, citing the potential it has to lose conceptual meaning and become algorithmatised.

2.8 Summary

An urgent need for change in the teaching and learning of mathematics has never been so evident as it is in the rational number domain where persistent misconceptions of decimal concepts abound. There is comprehensive description of decimal misconceptions in the literature and these provide both a useful diagnostic and teaching tool. However, the literature describes the way in which partial understandings of decimal concepts are persistent over many years of formal instruction and often maintained within traditional classroom practices.

Cognitive conflict has been identified as a critical factor in addressing decimal misconceptions. The literature has supported collaborative interaction and classroom discourse as a way in which student misconceptions are challenged. Collaborative interaction and classroom discourse are a feature of reform or inquiry classrooms. Mathematical learning in an inquiry classrooms is guided by the sociocultural constructivist theory of learning. Previous work has described how individual learning can be understood by studying the organisation of the social environment and the participation in social practices of the individual within it. While there is limited literature on the teaching and learning of decimal understandings in an inquiry classroom, many studies have shown how student mathematical activity and reasoning is enhanced when participating in a social environment where they explain, argue and justify their conjectures.

Students encounter rational number in the first instance with rich informal knowledge. The literature describes the way in which the informal knowledge can be used as a scaffold for formal school mathematics. Studies describe how contextualised problems, and students' informal knowledge are able to challenge their partial understandings. A range of representations used to support the construction of decimal concepts is described in the literature and support is given to the need for students to be able to translate flexibly across and between representations. Nevertheless, no one representation is able to capture all the features of decimal fractions and thus which representation best develops the differing features most effectively is subject to on-going debate and research.

Chapter 3: Methodology

3.1 Justification for methodology

After careful consideration of a variety of research methods, a qualitative approach was selected as most appropriate. Qualitative research is an umbrella term that encompasses a range of research methods covering a number of forms of inquiry, including interpretive research, case study, naturalistic inquiry, field study, and ethnography (Merriam, 1998).

The primary concern of a qualitative researcher is to understand reality, as it is constructed from the perspective of the participant. The characteristics of qualitative research include fieldwork, the researcher as the primary tool for data collection and analysis, an inductive strategy for theory building, and a richly descriptive product. Furthermore, an optimal design of a qualitative study is that it is emergent and flexible, and able to be responsive to conditions as they change during the study (Merriam, 1998).

The qualitative research paradigm has dominated mathematics education research in the past decade (Ernest, 1998). The decade has also seen a change in attitude towards the ways in which "the problems and issues of mathematics education have been framed and addressed" (Cobb, 2000a, p. 307). A psychological perspective views the constructive outcome of mathematical activity as resulting from individual student activity however, a social perspective holds with the view that the constructive outcome is socially situated. Cobb (2000a) argues for an "emergent perspective" in which the relationship between the psychological and social perspective is one of reflexivity. Each is balanced by the other. Furthermore, Cobb (2000a) links reflexivity to the relationship between the theory and practice, of learning and teaching mathematics. Cobb (2000a) argues for a cycle in which theory emerges from practice and in turn informs practice.

This study explored from the perspective of the individual learner, the effects of specific classroom practice on the construction of a decimal schema, within the naturalistic context of the classroom. The theoretical stance that underpinned the study was grounded in qualitative classroom teaching experiment methodology. The construction and reconstruction of decimal concepts were placed within an emergent perspective, which assumed reflexivity between individual student activity, and participation in classroom practices situated in the context of the mathematical classroom. Furthermore, the teaching and learning of mathematics situated within an inquiry classroom and coupled with classroom teaching experiment methods supported "educational innovation as a process of continual, iterative improvement" (Cobb, 2000b, p. 74).

A collaborative partnership between the researcher and the teacher supported the development of a hypothetical learning trajectory (Cobb, 2000b) and an instructional sequence, which through on-going discourse and data analysis was revised and modified as required.

Retrospective analysis of the entire data set provided the researcher with theoretical insight in a broader context. The projected learning trajectory, the instructional processes which were designed and subsequently modified, and the learning context and social practices of the classroom, provided a wider picture of the ways in which students construct decimal concepts in the naturalistic context of a inquiry classroom.

Multiple sources of data collection included participant observation, classroom observations, interviews, student case studies, and the collection of classroom artefacts. Multiple methods were used to support the triangulation of data.

3.2 Validity and Reliability

The common belief that qualitative research is "not scientific" and lacks ability to demonstrate validity and reliability is increasingly challenged (Merriam, 1998, p. 200). Reliability relates to the limit in which research findings can be repeated. However Tolich and Davidson (1999) argue that "reliability is not the goal" (p. 33). Reliability in qualitative research is not about repeating results instead, "the goal of reliability is to minimise the errors and biases in a study" to achieve dependable results (Yin, 1994, p. 36). Dependable results are based on well-documented procedures, a clear audit trail, triangulated results, and a clear statement of the researcher's position (Merriam, 1998; Yin). In this current study a clear statement of personal values has been articulated, the researcher position was clarified, multiple methods and sources of data were used, and procedures were clearly documented.

Validity, Hayes (2000) stated, "in its simplest form...was the question of whether a test, or test item, actually measures what it is supposed to measure" (p. 101). External validity is related to how generalisable the findings of a study are. Nickson (2000) claimed that "the swing to more descriptive, qualitative research that is interpretative rather than predictive is likely to be more accessible to teachers. Teachers may find it more relevant and identify with it and see themselves in it" (p. 176). Given that the current study was set within the naturalistic context of the classroom, modelled on the practice of the classroom teacher, involved the teacher as a collaborative peer in examining and critically reviewing all the data, rich descriptive data was provided, as far as possible therefore external validity was maintained.

Triangulation, referred to earlier in the chapter, is often cited as a means to enhance internal validity and reliability. Merriam (1998) described internal validity as concerned "with the question of how research findings match reality" (p. 201). Triangulation is described by Robson (1993, p. 290) as "a method of finding out where something is by getting a 'fix' on it from two or more places" and is a means to provide a partial explanation of a complex reality. However as Higgs (1998) cautioned, what constitutes reality has been a question which has "fascinated philosophers and researchers for centuries" (p. 137).

3.3 The research setting and sample

The research was conducted at a large Decile 8[1] inner city primary school. Students attending this school came from predominantly higher socio-economic levels and represented a range of ethic backgrounds.

The teacher in the current study chose to be involved in the classroom-based research on student learning. The teacher regarded her involvement as professional development and a means to reflectively inform instructional practice.

The classroom climate the teacher had developed was modelled on contemporary learning theory and can be described as a reform-oriented classroom with high demand for thinking, reasoning, and participation (Wood, 2002). The students were experienced at examining, discussing, and reflecting on their mathematical constructions. The classroom environment portrayed "a vision of mathematics learning... neither wholly individual nor wholly social" which enabled "connections to be made between the person, the cultural and the social". Student learning was supported within "a community of inquirers" (Anthony & Walshaw, 2002, p. 4).

Nineteen students from the Year Five and Six component of the class group, aged between nine and eleven years, were initially involved in the study. All

[1] Each state and integrated school is ranked into deciles, low to high on the basis of an indicator. The indicator used measures the extent to which schools draw from low socio-economic communities.

students were working within Level 3 of the Mathematics in the New Zealand Curriculum document (Ministry of Education, 1992) and prior to the study had no formal instruction in decimal concepts. Four students were selected as case studies from the nineteen students who had agreed to participate in the study.

The selection of four case study students was based on careful consideration of data collected in the initial interviews of the students. Tolich and Davidson (1999) support "theoretical sampling...drawn not according to probability theory (random selection) but upon essential and typical units" (p. 35). Given that an aim of this current study was to explore what effects prior knowledge had on the construction of decimal concepts, selection of students who represented a range of misconceptions common to children within this age group was appropriate.

The classroom teaching experiment project consisted of seven phases conducted over a 6-month period and involved 15 observed lessons.

In Phase One the researcher interviewed and audio recorded nineteen students. Items used in the interviews (Appendix A) were derived from other research and selected as appropriate to explore the partial understandings of decimal concepts children of this age group may have constructed. Items based on ordering decimals, reading and renaming decimals, exploring the denseness of decimals and sequences of decimals were derived from item banks produced by Hart, (1981, cited in Stacey et al., 2001), Carpenter et al., (1983, cited in Stacey et al., 2001) and Swann, (1983). Data was also derived from a class instructional activity, involving group brainstorms and the construction of a class concept map (Appendix B) of the informal decimal fraction knowledge the students had.

In Phase Two, data from the interviews and the concept map were analysed in order to identify the range of decimal misconceptions, and four case study students were selected. The teacher and researcher planned a teaching unit, which was mindful of current misconceptions and a hypothetical learning trajectory. The trajectory was comprised of the anticipated learning goals for the students, planned instructional activities, and a conjectured learning process, which anticipated how student thinking and learning might evolve in the context of mathematical activity in the classroom. However, the conjectured learning process and trajectory, was recognised as hypothetical and could not be applied to "each and every student's learning, for the straightforward reason that there are qualitative differences in their mathematical thinking at any point of time" (Cobb, 2000b, p. 62). The instructional sequences were modelled on a study by Moss and Case (1999) and designed to enable students to "integrate their existing understandings in a natural fashion and use the resulting cognitive structure as a basis for understanding the overall structure of the rational number system" (p. 125).

Data gathered in Phase Three consisted of video and audio recordings and researcher field observations of the lesson series.

Throughout the study the researcher and teacher worked in close collaboration, discussing and modifying the planned lessons based on on-going analysis of classroom events. "This daily cycle of planning, instruction, and analysis is highly consistent with the practices of skilled teachers whose overriding goal is to nurture their students' development of relatively deep mathematical understandings" (Cobb, 2000b, p. 45).

The first two lessons in the series were designed to enable the students to construct rich understandings of percentages as proportional amounts, which would then be linked to their decimal and fraction equivalent.

The next three lessons in the series introduced the students to two place decimals using percentages as an entry point. Long lengths of masking tape had been placed on the floor and marked in centimetres, with a longer line accentuating the marker for each set of ten centimetres. Numbers, exactly one metre apart, were placed on the line and individual students walked some part of a metre. The students were then required to problem solve the percentage of the distance walked and percentage of the distance to be walked to the next full metre. During these lessons the students' knowledge of percentages was used as a scaffold to build understandings of two place decimals both as a language, and a recorded notation system.

The following two lessons in the series included addition and subtraction of two place decimal 'real world' contextual problems. (Appendix C) These were designed to challenge misconceptions the students had constructed using whole and fractional number thinking and the role of zero. A further two lessons, based on 'real world' contextual problems, were used to confront misconceptions based on the denseness of counting numbers. (Appendix D) The final lesson in the series was a problem (Appendix E) which required the students to order any decimal number and make reasonable explanations of the ordering system they had used.

Phase Four followed the completion of the teaching sequence. The researcher interviewed the four case study students using the interview format as outlined in Appendix A

In Phase Five, based on analysis of data from prior lessons and the case study interviews, a sequence of five lessons was planned. These lessons, based on 'real world' contextualised problems (Appendix F) were designed to confront misconceptions concerning whole and fractional number thinking when decimal numbers are used as quantities in operations.

Phase Six began six weeks after the first series of lessons had been completed. Computations focused on addition and subtraction of any size decimal number for three lessons and in the final two lessons involved multiplying decimal numbers by multiples of ten or a hundred. Again all lessons in this phase were observed by the researcher and both video and audio recorded.

In Phase Seven the case study students were interviewed in the following week after the completion of Phase Six using applicable sections of the interview format (See Appendix A) and a further set of questions. (Appendix G) These required the students to provide estimated answers and make explanations based on reasons for their estimations. These questions were derived from those used by Irwin (2001).

Summary Time-Line

Week 1-2	Individual interviews of nineteen students
Week 3	Construction of concept map
Weeks 4-5	Collaborative planning of teaching unit.
Weeks 6-10	10 Lessons
Week 11	Interviews with case study students
Weeks16-17	Collaborative planning of teaching unit
Weeks18-21	5 Lessons
Week 22	Interviews with case study students

3.4 Data Collection

Consistent with an interpretive approach, data collection and analysis had complementary roles with one activity informing the other "as an iterative and reflexive process" (Tolich & Davidson, 1999, p. 108).

Observation is a key tool in educational ethnography (Scott & Usher, 1999). Cobb (2000b) argued that "the focus on the practices in which the students actually participate as they reorganise their mathematical reasoning brings context and meaning to the fore" (p. 75). Observations in the naturalistic context of the classroom maintained the ecological validity of the current study. However, as discussed later in the chapter, the researcher was required to consciously consider 'which hat was being worn' to avoid bias and maintain credibility of the current study (Tolich & Davidson, 1999; Yin, 1994).

The interview, as an investigative tool, was applicable within the constructivist paradigm of this study. Reliability and validity of interviews has been under dispute due to the flexibility and open-ended nature of the questioning techniques (Truran & Truran, 1998). However, this current study upheld the use of interviews as a viable tool given Truran and Truran's argument that reliability and validity "must be assessed in terms of the way the information is used and the nature of knowledge claims made" (p. 63). Furthermore, the use of the interview and the determining of the interview questions on the basis of those used by previous researchers is a well established practice in the study of the development of decimal understandings individual students. (e.g., Bell et al., 1981; Hiebert et al., 1991; Irwin, 2001; Moss & Case, 1999; Stacey & Steinle, 1999; Vance, 1992)

To assist analysis the audiotapes were wholly transcribed. The video-recorded observations were used as a means to corroborate field observations, audio tape recordings, and classroom artefacts. These artefacts included student written work, the reflective statements of the students written at the end of each session as well as recordings made in small collaborative problem solving groups and the charts recorded in the context of teacher facilitated large group discussions. Document analysis as an unobtrusive measure supported triangulation of data.

3.5 Data Analysis

Glesne and Peshkin (1992) describe data analysis as "the process of organising and storing data in light of your increasingly sophisticated judgements, that is, of the meaning-finding interpretations that you are learning to make about the shape of your study" (p. 129). It is a complex process of 'sense-making', which in the current study occurred simultaneously with the data collection process, each aspect informing the other. Categories, themes, and patterns were identified in order to bring meaning and understanding to how practices within the naturalistic context of a classroom may affect individual students as they constructed decimal concepts. Creswell (2002) maintains that qualitative data analysis "initially consists of developing a general sense of the data...then coding description and themes about the central phenomenon...it is primarily inductive in form...although the initial analysis consists of subdividing the data...the final goal is to generate a larger consolidated picture" (p. 257).

Audiotaped data was transcribed in its entirety and later mapped to contextualised information including interviews, written observations and classroom artefacts. The video recordings, supplemented written data, and were used solely to clarify details such as body language response, which student was speaking, recording, and what was recorded. It served as a means to cross-reference details.

The interview transcripts were read and reread to support the sorting of patterns and categories and led to refinement of questions asked in subsequent interviews with case study students. An ongoing process of data analysis of transcriptions and observations was undertaken by examination and re-examination of them, sorting and sifting, to identify codes, categories, themes and patterns. As patterns emerged, these were the basis of dialogue with the classroom teacher, refinement of the learning activities and, guided further observations and subsequent steps in the classroom teaching experiment.

When all major themes were identified, the categories were coded. Coding was used as a progressive tool to sort and define data (Glesne & Peshkin, 1992). Pattern coding was used to develop broad categories related to the research questions. Further coding was based on the reduction and refinement of the broad categories. As data analysis progressed, it was apparent that much of the data could be coded in a range of different ways and that there was considerable over-lap in the categories. This was not unexpected in this current study, and is typical in qualitative analysis, in that it highlights the complexity of this type of data, which was gathered in the complicated setting of a classroom.

3.6 Ethics

The current study, upheld the Massey University Code of Ethical Conduct for Teaching and Research (Massey University, 2001). The ethical standards including key principles of informed consent, confidentiality, minimising harm, truthfulness, and social sensitivity were upheld at all times for all participants (the school, the teacher involved, the students, and their parents).

However, the study included ethical dilemmas which are reportedly more problematic in qualitative case study research than in more traditional forms of research (Merriam, 1998). Anonymity within the classroom, and between participants was a difficult issue, as student participants and the teacher were known to each other. However, all practical steps, including the use of pseudonyms, was taken to ensure anonymity of all participants. No identifying information was recorded about any individual participants. In addition, potential harm to students was minimised through using classroom-based research methods consistent with everyday activities within the established culture of the classroom. Potential harm to the teacher was minimised through maintaining anonymity of the teacher and no evaluative data of the teaching and learning programme other than that which was grounded in the context of the current study was recorded. Potential harm to the school was minimised through maintaining anonymity of the school throughout the study and not reporting any identifiable features.

In addition, there existed a tension in role of the researcher, as a staff member (on study leave) and as a professional colleague to the teacher involved in the current study. Doerr and Tinto (2000) describe as inevitable, changes in the personal and professional relationship between the researcher and the classroom teacher. Given that the current study was grounded in practice, and sought to affirm the teacher's voice through maintaining an open and balanced dialogue between the researcher and teacher, and that the teacher considered participation in the research as professional development, a benefit for the teacher as participant Sowder (1998) argues can be that of knowledge.

3.7 The researcher's role

Although the perspective of individual students was central to the current study the researcher as the primary instrument for gathering and analysing data was able to maximise opportunities for collecting and yielding meaningful data (Merriam, 1998). Prior experience of teaching mathematics to students within this age group and within an inquiry classroom meant the researcher had realistic expectation of classroom practices and expected learning outcomes. However, this strength can also be seen as a weakness, limited by the very nature of the researcher, as human. Merriam describes how errors may be made, chances missed, and the ever-possible spectre of bias. As Creswell (2002) maintains "a qualitative stance is that all findings and all interpretations are subjective assessments by the researchers, and that individuals can never be "neutral" or remove themselves from the study to report objectively" (p. 278). Furthermore, Tolich and Davidson (1999) identify an additional problem for practitioners such as teachers as needing to face their "own over-familiarity

...dilemma" (p. 20). In the current study, this required the researcher to do what Tolich and Davidson describe as, metaphorically wearing two hats. In order to clearly define the role that was being assumed as researcher, the teacher role needed to be consciously cast aside so that the researcher was able to maintain an objective viewpoint.

3.8 Summary

A qualitative approach was selected as the most appropriate method of obtaining data which would provide answers for the research questions. Literature on student learning of mathematical concepts in the naturalistic context of the classroom provides support for such an approach. Three methods, interviews, classroom observations, and the collection of classroom artefacts, within the frame of a case study, obtained the data. The study was performed in a clearly documented and ethical manner. Data was analysed using a grounded approach of identifying codes, categories, patterns, and themes and these were used in conjunction with dialogue and quotes of all participants, in order to give voice to the students as they participated in classroom practices designed to support the construction of decimal concepts.

Chapter 4 Constructing mathematical concepts in an inquiry classroom

4.1 Introduction

This chapter provides a description of the ways in which the social and sociomathematical norms of the inquiry classroom support students as they construct decimal concepts. A picture of the four case study students solving realistic problems within collaborative groups is provided, and the ways in which patterns of dialogue including, explanation, argumentation, and justification may cause the reconstruction of decimal concepts are explored.

The effects of student sharing group strategies and solutions to a large sharing group where the strategies and solutions are re-presented and re-recorded as a notational scheme and used as a reflective tool are examined. I describe the use of both student recording of notational schemes and erroneous thinking as a reflective tool. The ways in which these were used by the students to analyse their thinking, and the thinking of other students, in order to find similarities and differences in the strategies and solutions is described.

I consider the stance of the teacher and how the students interpret this. The provision of on-going support and development of the social and sociomathematical norms—through the guidance of productive discourse, revoicing of student statements, and an expectation of active engagement in all aspects of mathematical activity as a sense-making process—is considered. Presented descriptions are based on researcher observations, samples of the student's work, audiotape and videotape evidence and teacher interviews.

4.2 The classroom context

The purpose of describing the classroom activity system is to establish a context within which interpretations and explanations may be made of the interactions and responses of the case studies. Moreover, the description is necessary and warranted, given that the learning and teaching programme had been specifically designed, and where necessary modified, in order to support the students in their construction of a decimal schema.

Although the focus of the study was that of student construction of mathematical concepts, the role of the teacher is also considered. The teacher figures, alongside that of the students, in a teaching and learning partnership in an inquiry classroom. It is the teacher stance, which is interpreted by the students and through which they are inducted into the mathematical norms of an inquiry classroom. However at no time during the study were any evaluative statements of the teaching practices made.

4.2.1. The structure of the learning sessions

In all learning sessions, the students began as a large group where mathematical tasks were briefly discussed. The students then worked in small collaborative problem solving groups for the following 20-30 minutes before returning to the larger group setting for 15-20 minutes. In this concluding session they discussed and shared their problem solving strategies and solutions and voiced any questions they had, or problems they had encountered. At the completion of every learning session each student recorded a reflection. The recording expressed a personal perspective of their learning, as well as commenting on their role as collaborative group members and the collective activity in which they had participated.

As the students worked in their small groups the teacher moved among them, listening to conjectures, advancing thinking through questioning, suggesting or modelling the use of concrete apparatus, and gaining a sense of the reasoning the students were using, as well as reinforcing appropriate classroom social and sociomathematical norms.

When the students returned to the large group context all members sat in a circle to discuss and listen to group explanations. The teacher began each session by asking a particular group to contribute. The selection of the first sharing group was based on teacher observation during the small group activity leading to careful consideration of a range of factors. Factors in selecting a group were described by the teacher. These included which group could provide other participants opportunity to examine their own reasoning, reflect on the strategies they had used, provide a more efficient, elegant, sophisticated or diverse approach to problem solving strategies and solutions. Alternatively, the teacher would select a group she had observed that needed support. At times, inability to reach consensus or errors and misconceptions formed the basis of the opening discussion. However, in this concluding session, all groups were given an opportunity to explain their strategies and solutions. The responsibility was theirs to do so if they had analysed that the solution method they had used differed from those shared.

The initial series of lessons and instructional tasks involved a high level of teacher involvement in structuring activity and facilitating links between the informal knowledge of the students and that of rational number concepts, and advancing understanding of the formal concepts of rational number.

However, as the students linked their informal knowledge of rational number to concrete representations involving percent of water in cups and proportions on a number line, they became more autonomous in the learning sessions. The measure of water and the number line served as visualised 'thinking tools' in 'taken as shared contexts' that as quantities could be represented as symbols in the decimal notation system. Increasingly they validated their own and other members of their collaborative group's mathematical thinking, questioning, justifying and proving through flexibly translating across representations. Within the case study group Eric took an active lead. For example, in finding the difference between .7 and .37 Eric recorded .7 as part of the notational scheme.

Brenda questions: *Why point seven?*

Eric: *Point seven that's seventy.*

Brenda: *Why? Can you explain more? What do you mean by seventy?*

Eric: *Like seven tenths is seventy.*

Brenda: *Oh you mean percent?*

Eric: *Yeah point seven is seventy hundredths and seventy percent and so we have .37 take away 70% of the chocolate bar.*

4.2.2. Elaborating the setting for a task and the importance of context

At the beginning of each learning session the context of the problem would be clarified in a brief teacher led discussion. Consequently, the students would engage in problems as 'experientially real' contexts (Kieren, 1993). In this way, exploration and argumentation of strategies in the small groups and explanation of solutions in the larger group were moved into 'taken as shared' reality. This was an important factor in the construction of decimal concepts. Decimal symbols represented quantities and operations represented action on quantities, not merely action on symbols as a consequence.

The importance of the context is illustrated in the following example. Fay and Jane are subtracting 1.13 metres from 2.41 metres. Both lengths represent wood to be cut for a shelf. Fay began her explanation stating: *I am going to start at 1.13 and add on .07.* She is interrupted by Jane: *No you can't start at 1.13 because he does not have 1.13 metres in wood, that's what he wants.* Brenda, the third member of the group maintains the contextual reality replying: *Yeah, but* *he may have worked that out before he does it so all he wants to know is how much wood he would have left as spare.*

Maintaining the context was a way in which all the target students checked their thinking and offered a framework for later explanation to the large group by Brenda: *Oh whoops he can't add on more wood! I was just thinking the logic so I was going to add on 20 centimetres of wood so can I disagree now I am going to take off .2 so that's 20 centimetres off instead.*

4.2.3. Active engagement in mathematical activity

The classroom norms included an expectation that all students would actively engage both mentally and physically in all mathematical activity. Active engagement also meant making sense of explanations in the small and larger sharing group.

The teacher regularly reinforced 'taken as shared' norms: *So as you listen to this group what are you going to be doing?*
Samuel: *Making sense of the strategies they use.*
Sara: *Listen as Eric talks and know what he is saying so I can predict what he is going to do next and if I need to I can ask more questions if I don't understand.*

Taken as shared norms included what the teacher referred to as 'sensing' which meant not only engaging, but questioning, clarifying and predicting.
Teacher: *Now while you are listening to the explanations I want you to turn your sensing on, ask questions at any time and search for answers. Listen carefully so that you can predict what might be said next.*

In addition, it was 'taken as shared' knowledge that problems might require sustained effort and involve a range of alternative strategies before resolution might be reached.

An example of this is given as the teacher states: *Don't forget to work as a group. Unpack what someone says make sure that you make sense of it. If you can't ask another question which you think will help. Don't look for quick solutions, prove it this way, then that way then check back to your benchmarks that you use. If you are having problems with the question or any part of it, go back to what you know so that you are able to go forward again. Keep checking that it really is making sense for you as you listen and rethink what you know.*

An episode involving Sara further illustrates the role the students took when unable to reach consensus. Close inspection of the dialogue shows that the three students were all talking past each other and making small numerical errors. When Sara begins to repeat a previous explanation she is stopped by Stefan: *Oh no don't say that again. You have to use a different way to explain because you are just going to tell us exactly what you told us the last time and that doesn't prove it because we can't really understand it and we keep coming out with all these answers.*

Sara: *Right, so we have to get an explanation together and prove that the answer is right together.*

4.3 Guiding productive discourse

The teacher maintained a fine balancing role in guiding productive discourse. Examples of this included, when to listen and when to ask questions, when to tell students something and when to allow them to struggle with ideas. Furthermore, during whole group discussions teacher balancing of student explanation with 'pause' time supported all listeners to ask questions and make sense of the explanations. Explanations coupled with the numerical recordings then became reflective tools.

Moreover, the students took for granted their right to challenge solutions. In doing so questioning extended the explanations to mathematical justification of the strategies used—or caused reconstruction of mathematical thinking for both the listeners and the group explaining. In the sharing group Fay and Eric actively lead group explanations and challenged solutions, while Sara and Jane preferred to listen. However reflective statements recorded consistently show that both Sara and Jane actively engaged in listening.

Jane: *Today I couldn't really get Tayler and Michelle's strategy at first but then when Anton got half-way through it I could predict what he was going to do next.*

Sara: *Today I lurnt when your talking about % = percentages its out of 100%. I also lurnt from Jane's mistake. I had Ideas about it and when Helen commented on it I thought back over my thinking sort of recaping and found out I was write. I think that this maths clinnic taught me a heap.*

A consistently high level of engagement was maintained as students individually and collectively recognised the teacher expectation of a shared role they all had in making mathematical sense of explanations. An example of the pivotal role of the teacher is described in the following extract.

Eric, Jane, and Anton are explaining to the sharing circle a solution to a problem[1] which required comparison of gymnastic score differences. They began by recording the numbers worked with then their estimation of the difference. Developing their explanation Eric recorded 6.967 at the start of a number line and .033 above the number line and 7 below as he explained: *First we plused .033 and that got us to seven.*

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[1] Helen went to watch four children compete in a team in a gymnastics competition. They had to compete in four different sections and each one was scored out of ten. These were the scores each child got

	Floor	Bar	Vault	Beam
Michelle	9.2031	6.967	8.895	9.03
Rosie	9.1	7.991	7.98	9.004
Emily	7.567	7.909	9	9.091
Bridget	9.705	7.99	8.005	9.039

Michelle was pleased with her Floor result but disappointed with her bar. What was the difference in the two results?

At this point the teacher asked Eric to pause and wait as she observed that many of the listeners were actively mentally engaged in adding .033 to 6.967: *Just stop and wait. I can see that some people need some thinking space and then we might have someone who wants you to explain something.*

Stopping the explanation at this point gave the watching students time to think, check, and question: *Can you explain where you got the .033 from? Oh but I see...*

Eric: *We're trying to make a tidy number. See three thousandths and seven thousandths is another hundredth then three hundredths and seven hundredths is another tenth. Nine tenths plus one tenth is another whole so that's seven.*

Adam: *Yeah that's an efficient step.* Eric and Adam have used 'taken as shared' understanding in the use of the words 'tidy' and 'efficient'. Both refer to the use of strategies that use minimal steps.

Teacher: *Can you see how it helps if you can use numbers like that? I wonder if anyone wants to ask why they chose this strategy to find the difference?* The teacher response validated the explanation; however it also asked for justification of the choice of mathematical strategy.

Eric: *I thought that some people got confused when they used the minusing way. We were going to well at first Anton said why don't we do it Will's way but they got confused yesterday so we decided not to do it that way because we wanted to use a strategy that we could all work with and that we could all make sense of.*

Eric then continues the explanation: *We plused two wholes so we got nine wholes.*

The teacher again stops the explanation allowing time for the other students to use the numerical recording as an object of reflection with which to make comparisons. After a long pause as the students look at their recordings, make comparisons, analyse similarities and differences, and justify these with other group members using their recorded steps the teacher asks: *Everybody happy* *with those steps? So now what? What do they have to do?* No answer is expected however the time allows the students to think and make a prediction.

Eric then continues the explanation: *We added .2031 so we had 9.2031. So to find the difference we added 2 to .2031. So that meant we had 2.2031.*

At this point a student comparing his recording comments: *That's one step more efficient* and a fellow student responds: *Their strategy is marginally different.* The teacher allows this interruption, recognising that these students are engaging in a reflective process which entails not only making sense of what is being explained but also making comparisons with their own strategies.

The students were then asked to predict what would be explained next and what the answer would be. After a long pause time Eric is directed to continue with the explanation.
Eric: *Then we added three hundredth and three thousandths so now there is three hundredth and six thousandth and so our answer is 2.2361 so that's why she was disappointed.*
At this point a student interrupts: *That's different from ours.* Fay responds: *Yeah that is where we are wrong look... oh yes see where we added the wrong tenth and hundredth to the wrong ones.* The teacher recognising this as a 'teachable moment' takes the recording sheet from the students who have identified the error. It is placed alongside the one that has just been completed. This action gives the watching students opportunity to engage in mathematical analysis. Responsibility is then passed back to the students who identified their error to compare and explain it to the listening group.

The teacher's use of the error indicates to the listening students the group's right to warrant their own explanation which is reinforced by the statement: *These children are checking their thinking, let's give them some thinking time and then they can explain what they did differently and justify their reasons.* Discussion of the importance of decimal place value then followed, which was student led and teacher supported. In doing this, the teacher indicated to the students a stance that they could interpret—errors are learning tools—and that not only is explanation required but also if necessary must be justified.

4.4 Patterns of collaborative discourse

Analysis of the classroom observations and transcription of the dialogues revealed regular patterns in the way that the students worked in small collaborative groups. Extensive discussion always preceded any recording. Initially the students discussed and made sense of what the problem required them to do. Then one of two distinct patterns would be used.

In the first situation a student would make a conjecture; its premise would be examined and justified through discussion in which all students contributed and built on each others' ideas. Then the students would collectively negotiate and construct a visual image of the notational scheme. Step-by-step within on-going discussion the notational scheme would be recorded as the group solution to the problem. Examination of classroom observations suggests this was the prevalent pattern for most groups.

An episode involving a case study student illustrates this pattern as they found the difference between 1.13 and 2.41. They collectively worked through the problem first verbally using decimals.

Georgina: *So you need to subtract the one whole so that leaves you 1.41?*

Brenda: *Then take off the .01 like the one hundredth cos that will tidy it up.*

Eric: *Now 10...*

Brenda: *.10 you mean well actually .13 so you have .27.*

Georgina: *So the answer is 1.27.*

Eric: *I am not sure that's right, oh yeah 1.28 actually remember the .01 we tidied up. So basically what we did was a number line in our head so let's write it now and then we will know how to share it like make an explanation.*

Georgina: *Yeah but can we do it another way too just to check?*

Eric: *Yeah you are right let's do it in fractions because you are basically doing the same thing except that you are putting them out of one hundred.*
Brenda: *Could do it in percentages because basically that's the same too.*
In this way collective thinking and discussion provided a scaffold for a solution, which was then recorded as a notational scheme.

The second pattern involved a less direct path. Verbally the problem would be discussed, conjectures proposed, disagreed with, and counter-arguments proposed until eventually justification and proof of one conjecture convinced the group. A notational scheme would be recorded when consensus was reached, although often group members remained only tentatively convinced asking for further discussion in the concluding sharing group.

A problem-solving[2] episode involving three case study students illustrates this pattern.
Jane: *Emily won.*
Fay: *Bridget won the first one.*
Jane and Fay continue to argue back and forth as Sara passively listens and occasionally asks one or the other to explain. Further time lapses and a later excerpt from the dialogue illustrates the on-going argument:
Jane: *And 8 hundredths so that is seventeen hundredths*
Fay: *Which means that you have to change that to one...oh no you can't.*

Further dialogue follows then Jane states: *That's plus .005.* In response Fay points at 7 in 34.0<u>7</u> and 5 in .00<u>5</u> : *Wait the seven plus the five will be .12.*
Jane interjects loudly: *What what what?*

[2] Helen went to watch four children compete in a team in a gymnastic competition. They had to compete in four different sections and each one was scored out of ten. These were the scores each child got

	Floor	Bar	Vault	Beam
Michelle	8.903	7.96	8.895	9.03
Rosie	9.1	7.991	7.98	9.004
Emily	7.567	7.909	9	9.091
Bridget	9.705	7.99	8.005	9.039

At the end of the competition all the scores are added up to get a winning total. What did Rosie get as her final total?

Sara has followed the 'to and fro' of the discussion contributing only occasional responses and now answers: *No that will be .075 because that's a hundredth and that's a tenth* as she points at the symbols.
Fay/Jane: *Oh yeah.*

Although Jane has agreed, her next question demonstrates only tentative acceptance of the answer: *So what did you do, can you show me again?*
Sara: *We added them together, 34.07 and .005 and that gave a score of 34.075.*
Jane still not convinced asks: *How?* Sara points at each decimal symbol using place value to justify the explanation: *No see those are the five thousandths and those are the seven hundredths. You need ten of those to make one of those but you haven't got ten of those so it's not one of those, yeah so you leave it like that.*
After recording the solution Jane states: *Yeah I still don't get how you got that .075 because plus .07 and .005 it just doesn't make sense.*

Analysis of the dialogue showed that this pattern occurred most often in the group Fay and Jane were in. The group had problems reaching consensus because both Fay and Jane had tenacious decimal misconceptions that they reconstructed only after much argument.

4.5 Mathematical explanations, justification and argumentation

Social and sociomathematical norms in the small and larger sharing groups were that students could explain and justify their solution strategy. Both the teacher and students maintained this expectation.
Teacher: *If you think you have an answer, then prove it. Ask yourself questions like what do I mean by that, how else can I explain it to prove it. It is also the job of your group to ask further questions if they don't totally understand, they have to push both your thinking and their own.*

Teacher: *Focus on the explanation that is given to you. Check in your head about whether you used the same method, ask yourself if the strategy being explained to you is efficient why and how it works and if you might use it.*

Fay, Jane and Eric consistently demanded full explanations from other members of their groups while Sara maintained a high level of active listening. However, the following extract illustrates that not only did she have an expectation that explanations should make sense, but also that she had a responsibility to ask questions in order to understand explanations.

Fay: *I don't get this.*

Sara: *Well just listen to him and see if he can make it clearer otherwise think of some other questions we need to ask him to make him explain it better.*

It was through Sara's insistence and her desire to understand other students' strategies and solutions that she was able to reconstruct many of her decimal concepts. The following episode illustrates the way in which she reinforced the need for clear explanation.

Sara: *So let's explain it again together using the number lines we have drawn so we know we can prove it.*

Georgina: *So what about another way? Let's use 3 metres and 37 centimetres so 3 minus .3 or 30 centimetres gives you 2.7 then .1 or 10 centimetres minus 7 centimetres means you have 2.63 or two metres and 63 centimetres. Do we all understand?*

Sara: *Yeah but I need to work this one again so let's prove it at the same time.*

Sara recognised that listening and making sense of other students' strategies was important for her learning as she recorded reflectively in response to the statement: <u>Activities that helped me achieve my goals were</u>: *Math clinics-other peoples strategies- working with a buddy.*

Furthermore, the students understood that collectively or individually they should be able to give a clear explanation of a group solution to the larger sharing group. The influence on their behaviour as they worked together is illustrated in the following episode.

Brenda: *What's like maybe an easier way of explaining it?*

Eric: *Oh yeah because remember that Helen (*the teacher) *said that we all have to understand how to do it and anyone might have to explain it so can you guys do that if you use my way? You could do a number line what about that?*

Brenda: *Yeah cos I go with number lines I find them easier to understand and so does everybody I reckon.*

In this example, they had recognised that they had to be able to explain and justify their strategies and solution in a way that was accessible to the wider audience of the sharing group.

Moreover, the students demonstrated confidence in their own ability and that of their peers to make mathematical decisions and warrant their own solutions. At no time did the students appeal to the teacher for help or authority. In contrast they would explore alternative strategies and validate the solutions using alternative representation.

4.5.1 Recording of student explanations

During large group sharing the students would use verbal explanation, coupled with drawings, diagrams, and the number line as concrete explanatory tools. In this way they re-recorded through visual representation, the notational schemes they had discussed in collaborative groups. These visual records allowed the students to make comparisons with their solutions and would be closely monitored as the other groups determined if their method constituted different ways of problem solving. Watching students of the other collaborative groups would track each step as it was explained and recorded. Members of other collaborative groups would quietly draw the attention of their own group members to the similarities and differences in the strategies being explained.

This is illustrated in the following episode as Eric records and explains: *Then we added .033 and 2 and .2031 so that meant we had 2.2361.*

Stefan comments quietly to another group member: *That's one step more efficient.* Acknowledging a more sophisticated step Fay states: *Yeah their strategy is marginally different, maybe less steps.*

In this way the recording of the notational scheme served as a thinking tool supporting students to reflect on their own and other students' mathematical activity.

In addition, the recording sheets were used as objects of reflection in situations where a conceptual error was illustrated that had not been questioned by the listening students. Teacher response supported the students to question the notation. In doing so the conflicting response caused restructuring of a cognitive concept. This is illustrated in the following example as Eric drew and then recorded on a number line and explained: *Well we started from 4.37 and then we took off .07 and that left 4.30. Then we took off 2 and that left 2.3 and then we had to take off .70 and that meant we had 1.60. Then we plused 7 so we had 1.67 and that is how much she really had left of the chocolate.*

The teacher attached the recording sheet to the whiteboard at this point as she re-voiced what Eric had said, and pointed at each step in the notational scheme.
Teacher: *So you say you took away .07 and that meant that when you had completed that subtraction you had 4.30. Then you subtracted what? Oh yes two wholes and that meant that you had 2.3 left and then you subtracted .7 and you had 1.60 left.*

After a long pause, the teacher continued asking: *Then you plused seven?*

Brenda: *Because of the...*
Eric: *Cos of the minus seven.*

Other students following the discussion have indicated by their body language that they want to question the students so the teacher passes the responsibility back: *Fay I can see that you have been tracking what they did every step and making sense of it. But you have a questioning look on your face. Is there something you want to ask them?*

Fay does this by challenging a statement in the explanation: *When you said plus seven at the end that meant seven wholes?*
Eric: *That's because of the seven there.*
Fay: *But I don't think that you mean seven wholes because look I'll show you.*

Fay picked up the pen and crossed out the +7 as she continued the explanation and justified the change which was made: *It will be plus point **zero** seven and that's not even anywhere near seven wholes.*

The importance of zero as a placeholder in decimals had been addressed in a way that supported the restructuring of prior knowledge. Eric recorded in his reflection at the end of that session: *Today I found that Louise thought that 0.07 was the same as the 0.70.* In contrast Fay recorded in her reflection: *We were talking about zeros and whether they need to go before numbers in decimals.*

Moreover, the recorded notational schemes became thinking tools with which students would 'fold back' to 'taken as shared knowledge' when working on subsequent problems (McClain & Cobb, 1998; Pirie & Kieren, 1994). Folding back to previous problems was also the means by which students were able to 'jump start' into strategies for solving new problems (Fraivillig, 2001). This is illustrated in the following episode of dialogue. Adam, Eric and Jane are required to find the difference between 7.991 and 7.909. In discussion preceding recording Adam suggests: *So it's the difference in the scores between Emily and Rosie. Oh we could add .008*
Eric: *Oh we could do the same as the strategy we used yesterday, that worked using a number line.*

Adam: *Yeah we could add and then we could take away...you know that strategy that Stefan used where he took away then he added it back on.*
Eric: *Yeah but that strategy confused me and we should stick with what we can explain.*
Jane: *Yeah I think stick with that too.*

Eric had realised the value of 'folding back' to a previous problem in order to go forward when he reflectively recorded: *Today it was the same problem as yesterday but with whole numbers. It was very easy because you could use the same strategy. Just it was a decimal not a whole number.*

Multiple strategies were often recorded by the small groups and then the most sophisticated would be selected to explain. The sharing of these notational schemes made the students aware of more conceptually advanced mathematical thinking. However, at the same time the students were certain in the knowledge that it was their choice to use them or not. This was reinforced in written reflections and dialogue.

For example, in an exchange between Eric and Adam, Adam challenges the length of the explanation: *If you just go minus 2 because she did eat 2 whole bars which gives you 2.37 so that's what she had left minus .7 which gives you 1.67 which is what she finally had so that is much quicker and gives you the answer quicker.* However Eric responds, justifying the length: *We have just got a lot more steps so if you look you will just see that we did extra steps but then we made sure that we all really understood by all the extra steps.* Eric's statement is reinforced by the teacher: *Adam is talking about a quicker way but the important thing to remember is what this group said, that their way meant that everybody in their group knew what they were doing and why they were doing it.*

Jane demonstrates confidence in her right to understand and be able to explain a strategy in a recorded reflection: *Today when we went off to do a problem we all* *argued over our strategy. In the end we all agreed. When we listened with Helen the other strategies were quicker but for me, ours was clearer.*

4.5.2 Revoicing of student explanation

Close examination of classroom observations and transcriptions of the dialogue show that the teacher frequently revoiced sections of the students' explanations. The teacher would capitalise on student explanations that demonstrated understanding at a higher conceptual level than other group members. Furthermore, teacher revoicing also served to legitimise what the students were explaining, and at the same time make the ideas potentially accessible to all other group members. In addition, some statements students made in their groups would be used for reflective discussion to advance conceptual understanding.

An example of this was demonstrated when the teacher recorded on a chart for discussion: **'Somebody the other day said in decimals the numbers go up in tenths'.** When the students had read and been given 'think time' she asked: *Do they actually go up like the whole numbers do in tens...or do they go down...get smaller and how?*
Jane: *They are going up in numbers like in total but they are going down...*

The teacher continued to explore the thinking by revoicing: *Going up in numbers?*
Fay: *In size. Yeah Jane said that the numbers are going up in the total but they are going down in size.*

Revoicing again: *Going down in size, so what do you know that means?*
Stefan: *Cos 1.9 is bigger than 1.30978 but people who don't know that think the one with the more numbers is bigger. They don't know about decimals.*

The teacher is aware of the active listening stance of the students so she pushes Stefan to explain: *They don't know about decimals?*

Stefan: *They are getting a smaller bit because it is like dividing, it's tenths, hundredths, thousandths, they are getting a smaller bit every time.*
Stefan has successfully explained both the partitive and continuous notion of decimals for the other students. Subsequently, Eric recorded in a self-assessment sheet: *I thought .63 meant divided into 63 bits now I know that the tenths are the key and after the decimal point the numbers get smaller and smaller.*

4.5.3 Mathematical difference

There was an expectation in this class that students would share a range of different mathematical solutions. What constituted a different mathematical solution was 'taken as shared' knowledge which the students took responsibility for. An example of this was demonstrated when Sara examined the differences and explained to the group: *Well we did the same as they did except that we took longer, like if you look, we did two steps and they did it in just one step in that bit.*

The students commented on efficient steps, or efficiency of a solution, both in justifying a step or their conjecture, as Eric illustrated in the following example: *We're trying to make a tidy number cos you see three thousandth and seven thousandth is another hundredth and then three hundredths and seven hundredths is another tenth and then nine tenth plus one tenth is another whole one so that means we have seven.*

Efficiency was a term used by the teacher and students to denote a more sophisticated strategy as Adam illustrates in the comment: *Yeah that's an efficient step.*

Sanctioning was also a feature of student response if a contribution was considered the same or similar to another previously shared. This is illustrated by Eric in his comment: *Well it is almost the same as the other one but you could have called that 7 hundredths instead of centimetres.*

4.6 Errors in strategies and solutions

Errors that were reinterpreted by students as failure to reach consensus were brought to the sharing group. Errors were seen as worthwhile opportunities to reconceptualise the problem, analyse or explore contradictions in solutions, and seek alternative strategies. An episode illustrates when three students could not reach consensus. Multiple solutions were given, yet at no time was there an appeal to the teacher or to students considered 'more knowledgeable others'.

Stefan: *You don't come with the same answer you said before, so how do you justify your answer? Prove it, make it make sense for us cos we are coming out with all these different answers.*

Furthermore, problems that explicitly addressed decimal misconceptions provided the students with an interpretation of the teacher's attitude to errors. *These problems you worked on today have all been based on people using erroneous thinking. That means that they had errors in the way they thought about the problems and that's okay as long as there are other people around who are able to ask questions, get them explaining their thinking and justifying their reasons. Having errors are fine because they give you lots of chances to rethink what you are doing.*

Errors were also used as reflective tools, when students requested support to reconceptualise the problem at the large group.

Fay: *Well how about we listen to both your explanations and then with our questions we ask, maybe you will be able to explain what you were thinking.*

Eric: *I can see where they made their error, see when they added 33.99 and .08 they got 34.09, but when we added .08 and .09 we got 17 hundredths so they have an adding error there.*

The students considered errors to be valuable learning opportunities and the teacher considered them valuable teaching opportunities.

4.7 Summary

Constructing robust decimal concepts is a lengthy and complex process, and one that requires on going discussion and exploration of students' partial understandings. In this chapter the social and sociomathematical norms of the inquiry classroom have been taken as units of analysis for investigating classroom processes. The investigation revealed how students elaborate their thinking, conjecturise, explain, argue, and justify their strategies and solutions through purposeful activity. These strategies lead to reflective reconstruction of decimal understandings.

The participation patterns differed in collaborative discourse for the case study students. Eric and Frances were more active in discussions, confident about making conjectures and challenging solutions, Jane and Sara while less verbal, remained active in listening and making sense of explanations. Explanations and justifications given by the case study students at various times demonstrate not only sound decimal concepts but also provide evidence of the growth of intellectual autonomy.

The teacher's role was shown to be facilitative, guiding effective discourse as the students engaged in collaborative interaction, and advancing conceptual thinking through careful listening to student explanations.

Chapter 5 Classroom Activity: Constructing decimal concepts

5.1 Introduction

This chapter describes the ways in which a group of students was supported to construct decimal concepts in a classroom.

I describe how informal knowledge of rational number is used as a scaffold to build understanding of formal decimal concepts, including the continuous and partitive nature of decimal numbers. I explain the rationale for introducing rational number through the use of percentages.

I discuss the use of a range of representations and describe the way in which these representations support quantitative understanding of decimal notation. I describe the importance of translation across representations, of cognitive conflict, of the number line as a concrete tool, and of rich contextualised problems as situated within the overarching goal of sense-making within the inquiry learning environment.

5.2 Informal rational number knowledge

It has long been recognised that students beginning formal instruction of rational number concepts already have a rich bank of informal rational number concepts (Mack, 1993, 2001; Streefland, 1993). Interviewing students prior to the start of the learning unit showed evidence of a wealth of informal decimal number concepts:

- *Oh well sometimes I ask my older sister what she's doing in her homework and she tells me about fractions and stuff like that.*
- *Well my Mum has taught me some tricks to do with decimals.*

- *See I learnt stuff when Helen (*classroom teacher) *was working with the kids in Year Six.*
- *Well you see I know about decimals because see I just use money and that works.*

5.3 Percentages and proportional reasoning.

The teaching and learning unit was structured on a teaching experiment of Moss and Case (1999). It began with percentages, enabling the students to make use of a visible representation they were already familiar with in their every day life.

In order to construct rational number concepts, students require whole number concepts of at least numbers one to one hundred and a "global structure for proportional evaluation and a numerical structure for splitting and doubling" (Moss & Case, 1999, p. 125). Using a context trialled by Moss and Case, the students were initially introduced to rational number using estimation of water in clear cups. Moss and Case found that individuals have little problem in seeing objects such as clear glasses in global proportional terms. Students in this study readily adopted percentage terminology to describe the relative 'fullness' of a cup.

Fay: *Hundred out of a hundred—a hundred percent full.*

Using percentages as an introduction to rational number delayed a need to engage with the complexities that are inherent in comparing ratios with different denominators. Also, every percentage has an easily seen decimal or fractional equivalent (Moss & Case, 1999).

A clear understanding of various numerical values was demonstrated in this study as the students worked with water. This was coupled with a spontaneous use of percentages and translation between percentage and fractional representation.

Jane: *If you have half you have 50% and half again is a quarter so that would be 25%.*

Sara: *...and if you add them both together you make 75% full, now our cup is three-quarters full again.*

Explanations of numerical splitting, decomposing or recomposing amounts using percentage values were also commonly matched with physical hand actions. For example, Fay's hand action: *In here I have twelve and a half percent because look...* indicated a total amount to be partitioned, the fingers were then moved to indicate a splitting and splitting again strategy as she continued to explain: *Because see I took one off and that left me twenty four and I split that into twelve and the one that's left over you split that into point five so that gives you twelve point five percent.*

Initially instruction remained closely aligned to the students' informal knowledge. However to advance student thinking, increasingly complex problems were introduced. Use of numerical problems requiring precise calculation provided information of how the students were applying their understanding of percentages.

In addition, the students' informal knowledge of other rational number concepts was evident as they used them in translation across percentages to their decimal and fraction equivalent when challenged in an explanation. For example, Eric explains an action on an amount:

Eric: *Then like fifty per cent of 1000mls is 500mls and then 250mls so that would be 750ml.*

Jane: *Why?*

Eric: *Because you know fifty percent, that is a half of 500mls is 250mls so they each get 250mls out of the point five and you just take away the one to get the point five so they each get 750mls.*

Brenda: *Why?*

Eric: *Because half of the point five is 250mls and half of one litre is 500mls and then you add the 250mls and the 500mls and you have 750mls.*
Brenda: *I am just not sure if point...*
Eric: *Point five is half!*

5.4 Proportional representation on a number line.

Running races of short distances and competing to jump the longest distance, measured arbitrarily by students is commonplace in the playground. To capitalise on this informal knowledge, problems were set within this context. Initially, the students were required to explain their reasoning using percentage terminology that was then used as a scaffold in order to introduce two-place decimals.

A number line marked in metre lengths was placed on the classroom floor (previously described in Chapter 3.3). Embedded within contextual problems, students were asked to walk some part of a metre along the number line between two adjacent numbers and then stopped. The students understood that the two adjacent numbers represented whole numbers, the first marking the number of metres which had been walked (for example 63 metres) and the second marking the end of the next complete metre which they were walking towards (64 metres). They were required to calculate the distance they had walked in complete metres (63 metres) and then calculate both, what percentage of the next whole metre they had walked and what percentage they needed to walk to complete the metre (e.g., 23% walked and 77% to walk).

Children were frequently observed to transfer strategies they used to estimate water quantity to estimate distance. The use of bench marking amounts through numerical splitting of percentages was applied—to compute the total. The strategy of numerical splitting is illustrated as Fay makes an explanation:

Fay: *Well she has walked 1 metre and see she is more than let me see half, sixty percent, seventy percent, eighty per cent, eighty one, eighty two percent of a metre.*

Eric: *So that means that she has 18% of a metre to go because if you go back from 2 metres like 100% 90% 89% 88% 87% 86% 85% 84% 83% 82%.*

Informal knowledge of fraction and decimal numbers, and their percentage equivalent, were flexibly applied. The students translated across representations of quantity to justify a description of a proportional amount of distance shown on the number line.

Eric: *Yeah 2 metres and 75 oh 2.75.*

Fay: *So it was 2 metres and 75%.*

5.5 Translating between equivalent representations.

Three of the case study students Fay, Jane, and Eric increasingly voiced translations between equivalent representations of fractions, percentages, and decimals. However, other students including Sara neither questioned the other students' use of equivalent fractions and decimals nor used them.

To encourage explicit linking of the equivalent percentage value and decimal value the teacher used the statement of Eric as a bridge to link the concept of equivalence of a percentage value with a decimal value.

Eric: *Fourteen percent or point one four.*

Explicit revoicing and emphasising the equivalent amounts by the teacher made what Eric had said accessible for other students to build on.

Teacher: *So you say that he has fourteen percent of the next metre still to walk, so what you are saying is that he still needs to walk point one four...yes* ***point one four of the next whole metre...that is the same as fourteen percent of the next whole metre.***

From this point on if a decimal equivalent was omitted from an explanation, the teacher and at times other students would explicitly request it.

Teacher: *So what did you say Stefan one metre and forty three percent of the next metre, so how else could we say that?*

Stefan: *Oh one point four three.*

Teacher: *Right Stefan* ***one metre and point four three of the next whole metre.***

In this way, the teacher established an expectation that distances described as a percentage value should be matched with a decimal equivalent. In addition the continuous nature of decimals and a need to state the unit referent was reinforced.

5.6 Decimal notation symbols, their referents and quantitative value.

Quantitative understanding of decimal notation is critical if students are to apply number sense when using decimal symbols in operations. Moreover, if decimal symbols are to be used as mental referents, they need to be tied to experientially real objects as quantities. Links between the students' informal knowledge of quantities represented as percentage proportions in cups and as proportions on the number line needed to be made. In the study these links were assisted by the teacher's explicit revoicing of explanations to construct links between the proportional amount as a percentage, its decimal equivalent and its recording as decimal notation: *So I just heard Jane say 3 metre and 47% of the next metre and then I heard Eric say that is 3.47.* As the teacher revoiced the explanation, the symbols were recorded as 3 metres and 47% on the whiteboard, and then along side recorded again as 3.47 metres.

The teacher continued: *So how can we record how much further she has to go to that next full metre?* When Jane replied: *53% of the next metre* the teacher recorded 53% of a metre, and then continued: *So she has 53% of the metre to walk. Who can show another way to record 53%?* The whiteboard marker was given to Eric who recorded in large writing .53 metres.

Until now all explanations had been verbally given and any recordings had used pictures, diagrams, and written words to represent the notation. Making explicit the connections between percentages and decimals represented through the decimal notational symbols was an important point of advancement in the study. Links from informal knowledge to the more formal school based decimal concepts had already been established; the students now had symbols as tools with which to visually represent an explanation of their problem strategies and solutions.

5.7 Understanding decimal numbers as referent units

Language conventions used in decimal number differ from that of whole number. In whole number, the numbers are counting numbers, whereas in decimals, the numbers are referent units used in particular contexts "as a basis for all measurements in that context" (Hiebert, 1992, p. 224). A fundamental concept which students must construct in decimal number understandings is that of the referent unit as one whole unit.

To reinforce the idea of one whole unit (in this instance a metre) as the referent unit the word percentage was used. The teacher recorded and underlined the words <u>per cent</u> as Jane described a distance walked along the number line:
Jane: *I walked five metres and thirty three per cent of a metre.*
Jane was then asked to clarify the meaning of 'per cent'.
Jane: *It means like...it means per hundred.*
Another student interjected quietly but sufficiently audibly to be heard by the teacher.
Eric: *Out of...*
Revoicing the interjection, the teacher built on the concept: *Out of what?*
Eric: *So 33% means you have 33 out of 100 of a metre.*

The teacher then re-voiced with an emphasis on **one**: *So 33% means you have 33 out of 100 of* ***one*** *whole metre.* The emphasis placed on the word 'one' focused the attention of the students on the notion of one whole unit and in this way reinforced the need to state the referent unit.

Links between the language of the referent and the symbols that represented the referent were explicitly maintained in subsequent lessons. Observing hesitation in the behaviour of Eric, while recording an answer to a problem for the large sharing group, the teacher probes for explanation: *I saw you hesitate, so I saw you write .71and then you looked like you wanted to write something else?*

Eric: *Yeah metres, I was going to write metres.*

Teacher: *Were you?*

Eric: *But I didn't cos it isn't a whole metre.*

Teacher: *Can you say .71 of a whole metre?*

Affirmative nods from peers, and the probing question, effectively prompted Eric to rethink and drew a positive response as Eric said: *Oh yeah* and the word **metre** was recorded next to .71 emphatically.

The explicit modelling by the teacher in this interaction formed the basis of behaviour of the students in successive small and large group situations. Students asked for specific clarification of the referents used in explanations of actions on quantities.

Brenda: *What do you mean by minus one hundred? Or do you mean point one zero zero, or one hundred metres or what?*

Sara: *I am talking about one hundred centimetres so one metre.*

Fay: *So if you add 3 and that makes 2.4metres and then you add 60 and that makes 3 metres so then you add them together and that means he has to cut off 63cm.*

Brenda: *Yeah but what are you talking about when you say 3 and 60, lets make it clearer on a number line and use decimals to record exactly what you are talking about.*

Furthermore, the students themselves explicitly modelled the specific referent in their own verbal or recorded explanations. For example as Eric reports the strategies for his group he describes: *Yeah and then from 2 and 41/100 you minus the 41/100 and that would equal just plain 2, that is two wholes.* Then making explicit the referent unit, underneath the number 2 the word **'wholes'** is written and underlined.

5.8 The number line as a concrete representation

Previous research has shown that students bring to the study of rational number a rich supply of informal knowledge related to partitioning. This knowledge more often than not, involves a strong part whole perspective (Mack, 2001). Instruction must extend this part whole understanding to include a perspective that not only conceptualises the continuous nature of decimals but also reconceptualises the unit, so that its fractional parts are viewed in relation to the unit whole. For example, .3 as 3/10 represents a quantity composed of three portions that are each one tenth the size of the unit whole. Each metre section of the number line served as a concrete representation of one hundred percent. It was also re-described by the students as a fraction of 100/100 and as a decimal referred to by the students as one point. This description reinforced the concept of the unit whole.

This was illustrated in a large sharing group when Sara became confused in an explanation Brenda suggested: *Maybe to make it look a bit easier, like to make it clearer you need to put a dot after the one so that we all know you are talking about one point.* The use of the term 'one point' became 'taken as shared' knowledge, and was used many times during the study by the students to denote the unit whole.

The use of a number line also provided a powerful visual concrete representation. Its use built on the continuous concept of percent of water and fitted within an informal context many students recognised: the downloading of computer games or the measurement marker denoting time left within which a computer game has to be played. These contexts were both described in the conceptual mind map of the students' informal knowledge of decimal numbers. (Appendix B)

The number line used in contextual problems highlighted two common misconceptions in decimal understandings. These are namely (i) The denseness between counting and decimal numbers and (ii) Zero as a placeholder in decimals (Case & Moss, 1999; Irwin, 1996a). All students including the four case study students had these misconceptions initially.

In order to cause cognitive conflict the teacher, using randomly selected whole numbers (to denote metres already completed) at the start of the number line, instructed a student to walk along it and then stop within a **very** short distance. The students were required to quantify the distance covered, and the distance to the next full metre.

In the first group with very little discussion Fay stated: *So I think that it is 761.4.* Jane agreed immediately confirming the misconception: *Yeah so .96 to go.*

The second group with Sara and Eric tentatively agreed but then in discussion became less sure. Eric indicated that he was making a conjecture but was unsure of the proof: *I reckon 761.4. Do you reckon that's it? You can argue if you don't agree. So how long do you think he's got to go to the next metre?*
Stefan confirms the misconception: *96.*
Eric takes Stefan's response as validation of his conjecture and records it: *Yeah yeah 96.*

At this point the teacher moves alongside the group asking: *So what have you written?* Eric states: *761.4 and then he had 96 percent to go.* The teacher responds in a questioning tone: *Did he?* The teacher's quizzical response effectively reinforces Eric and the group's unease with a definitive answer. They move to the number line and physically use it to discuss. Eric then states again tentatively: *Yeah it is 96 plus point 4 equals 100.* Stefan and Sara still unsure tentatively agree and then Stefan quietly states as an aside: *I don't know, I reckon it is wrong somehow.*

In the large sharing group the teacher questioning leads to a 'fold back' in thinking as Eric answers: *So we thought that he had walked 761.4 metres.* Re-presenting the distance as a percentage the teacher questions further causing reflection and activating cognitive conflict: *So are you saying that he had walked 40% of the way?*

Eric: *Oh no that's only 4 centimetres and...oh I can't explain it.*

Fay: *Well now I think it might be 761.04. See um the zero comes in because if you said 761.4 it's another way of saying 761.40 like 40% and so if you don't want to say 761.4 you have to put another zero in front of the 4 otherwise it will mean forty percent.*

However, misconceptions are known to be extremely robust. Responding to continuing puzzled faces the teacher reconstructs another context 'folding back' further by picking up a clear cup and asking: *If I walked 4% of the way Fay said that is the same as 40%. Now let's think of this cup if you fill it up 4% is that the same as 40%? If you want to fill it right up to the top how much do you have to put in if you have 4%?*

Sara: *96%*

Teacher: *If you have a cup that is filled to 40% how much do you have to put in to fill it to the top?*

Fay: *You have to fill it 60%.*

Teacher: *So have you walked 40% of the way or 4%?*

Fay: *Just 4%.*

Teacher: *So can you say .04 is just the same as .40?*

Fay: *No because .40 is 40% but .04 is just 4%.*

The teacher had responded by not just 'folding back' but also 'dropping back'; picking up the cup had constituted a new beginning in the discourse (McClain & Cobb, 1998; Pirie & Kieren, 1994). The drop back in context and translation across representations had served a useful purpose—causing sufficient cognitive conflict to potentially support the restructuring of concepts of fractional numbers (below point one). In recorded reflections at the end of the session the case study students wrote:

Jane: *Today I learnt that in a decimal if you put a zero after the point it does make a difference.*

Sara: *Today I learnt about tenths and hundredths, decimals, and I got stuck in the bit about .04. I thought it was .4. Tomorrow I want to learn more about decimals.*

Fay: *Today I thought about where you need to put zeros and why. I justified a lot about how and why I came up with an answer. It is difficult to justify so you have to be sure.*

The number line was used regularly as a concrete representation to 'fold back' or 'drop back' to (McClain & Cobb, 1998; Pirie & Kieren, 1994). It would be used during argumentation to prove or disprove a conjecture in an explanation. For example, Jane has added .3 to 2.37 metres and stated that it equals 2.4 metres. Brenda argues in response: *Wait with this if you add .3 you get 2.67 so you actually have to add .03 to get 2.40. Do you get it?* Jane maintains a perplexed stance as she asks for more explanation: *No I don't get it. Why?* In response, Brenda directs Jane and other members to the number line on the floor, using it as a model to reinforce her argument. Attention is directed to the segment marker for 10 centimetres and 1 centimetre as Brenda continues to explain: *Well it's like here, you are talking about these lines but you are only adding on 3 of these so that's the tenth and that's the hundredths.* Jane clarifies again by pointing at the 7 in 2.37 and asks: *So you are adding on to that one?* The 'fold back' to the concrete representation of the number line provided a tool for Brenda to explain her reasoning.

5.9 Translation between modes of representations

Connecting across and between the subconstructs of rational number supports deeper conceptual understanding. Mathematical thinking is enhanced, and becomes more flexible, when students are able to make translations within and between modes of rational number representations (Sowder et al., 1993). Flexibility in and between modes of rational number representations increased as the students made connections between problem contexts, their informal and formal rational number concepts, and the notation system. Eric and Jane illustrate this in a problem in which Eric explains: *Well he walked 6 metres and 14% of a metre so we wrote 6.14...*

Jane: *So he needs to walk another 86% of the metre.*

The teacher then asks for proof: *So how did you work that out?*

Jane justifies their explanation stating: *We started from 6.2 so that was 20% of a metre and then we went* (physically pointing at the number line) *to half way so that was 50% then, 60, 70, 80 and then we came back to 20% and we knew we had 6% to add on so then we went .81,.82, .83, .84, .85, so we wrote .86*

The added dimension of a flexible use of translations between representations provided a powerful thinking tool for the students. They frequently checked their reasoning, and the reasoning of others, by translating across and between other modes of rational number representation.

When a solution to a problem was explained and recorded as 2 metres and 75% of the next metre or 2.75 in the sharing group, Sara listened and then challenged the group: W*ouldn't you write that as 75 over 200?*

Jane who explained and recorded, repeats the explanation in a questioning voice, but renames the 2.75 as two and three quarters as she clarifies: *So you are saying she walked two and three quarter metres?*

The re-describing of .75 as three-quarters presented a conflicting representation for Sara. She responds by stating: *No wait I think maybe it is 75 out of 100 because if you use 75 out of 200 then you are not talking about three quarters of the way like then I think you are talking about much less.*

The flexible use of translation between representations had checked and clarified the proportion measured, causing the reconstruction of rational number concepts for Sara and confirmation for Jane as their reflection recordings demonstrate:

Sara: *Today I went to a mathema clinic. It was the hardest maths clinnics I've ever bin to. I recaped my nolege on desimal points. I think I'm realy getting the hang of it. I also clicked to some things like what it realy means to put 60 over 100 or something.*
Jane: *Today I learnt different ways of recording 5m and 54% what I also lurnt was that 54/600 wasn't the same as 54/100. I learnt a lot today.*

Translating between representations was a useful tool that the students continued to use for decimal numbers. An example is illustrated as Eric worked collaboratively on a problem which required exchanging NZ$1000 given that NZ$1 was equivalent to Australian $.88 or American $.470 and English £.232.
Stefan: *Geez look at that, what a rip off, you only get 232 pounds, what about for America?*
Eric: *Oh yeah it's about a half, no a little bit less than a half, so less than $500*
Brenda: *Yeah but with the English one you get about a quarter so I would go to Australia.*
Stefan: *So you get like three-quarters so like $750?*
Eric: *No more because it is .88 so you only need 2% more and you would have 90% so you actually get closer to $900 you can just estimate it in your head. It's closest to per cents.*
Brenda: *And fractions.*

As they explained their reasoning to the larger sharing circle Eric stated: *Yeah estimating you can use what you know about percentages and you don't need a* *calculator like when you go in the bank and it is in decimals you can just work it out as a percentage or fraction in your head.*

5.10 Mathematical tasks and cognitive conflict as tools to support the development of decimal concepts

Using activities based on other fictitious students' errors, provided non-threatening opportunities for the students to recognise and address misconceptions they had themselves. For example, Fay referred to the fictitious person and commented: *You know it's funny I used to think like that too* and Jane concurred with her: *Me too.*

Discussion of a misconception in a problem[1] enabled the students to reconstruct their own conceptual understandings of decimal numbers. For example, Fay referred to the erroneous thinking in the problem as she stated: *No she (*the fictitious person) *didn't really explain what she was thinking because she probably thought the 7 wasn't 70 but just a 7.*
Eric: *Yeah but...*
Fay continues to explain saying: *Because the zero...*

Sara has re-looked at the problem and applied a 'whole number' thinking misconception to the decimal number recorded as 2.7 and argues: *Maybe it is just a 7 cos see it isn't written like you did, it is just written 7.*
Eric has listened and then he points at the .7 (from 2.7) as he justifies the recording of it as .70: *If it was 7 it would be 7 hundredths but when I look at that I see it as 7 tenths because it doesn't have a zero in front of it. Do you get that?*
Sara: *Oh yes right... now I get it.*

[1] Louise' homework. Louise has handed in this homework and the teacher says that her answer is wrong. The problem she did was: *Mary was given heaps of chocolate bars at Easter and after she had eaten some she has 4.37 chocolate bars left. She eats 2.7 more then decides to save the rest. How much does she save?* Louise has written her answer as 2.30 and the teacher has told her that it is wrong. Can you work out what the correct answer should be and then work out a way you would explain to Louise why she got the answer wrong?

The number line as a representation of measure carried with it an inherent possibility that students could work with two place decimal numbers and still consider them two separate sets of numbers divided by a decimal point—a common misconception constructed through the use of money and measure (Irwin, 1996a). Moreover the tight bond (Discussed in Chapter 2 Section 6.3) between counting and decimal numbers which students of this age group commonly hold needed to be addressed. To achieve this, and advance student knowledge of quantities represented by **any** decimal number, a problem embedded in a realistic context of measure required the students to hypothesise two numbers between 1.36 and 1.37 and record the difference between them. (Appendix D) Initially, the students' behaviour paralleled that of those in other studies—they questioned the existence of other numbers between 1.36 and 1.37. (Irwin, 1996a).

Sara, Jane and their collaborative group had considerable difficulties. Sara stated: *Maybe we can't write because there aren't any.* Jane replied: *This is just stupid and I can't make sense of it.* After extended discussion Jane suggested: *1.36 and then .2 or something. It is hard I know it could be like 1 metre point and 36 centimetres and like 2 millimetres or .2 millimetres or .3 millimetres."*

Sara, relating the problem back to the number line and linear measurement states: *Basically we are working out of a hundred cos in a metre there are 100 centimetres so it could be any number between 1 and a 100.* Jane argues in response: *No it has to be any number between like after point like after 1 millimetre because it's like if Brook jumped more it can't be one yeah 1.36 centimetres and I millimetre because like if he jumped more then it wouldn't be exactly 1.36 it would be one millimetre more.*

Jane was tentatively persuaded through considerable argumentation to consider an enlarged section of the metric number line. She reconstructed her thinking by 'folding back' to measurement in order to accept more than two decimal place numbers: *Today what I thought was that there wasn't anything between 1.36 and*

1.37. But I relised that mm were inbetween cms. I didn't understand Brian's strategie but I understood when he explained that he was trying to enlage it.

However, Sara could not visualise decimal numbers extended beyond two places. The partial understanding she had constructed was of whole numbers and two decimal systems: *Today I got realy confused. I thought there is nothing in between 1.36 and 1.37. I also didn't understand how and why Brian wrote 1.361 Because to me that ses one point three hundred and sixty one. I think it makes more sence to put 1.36.1 its weird now Im still confused about that but I lurnt there is something in between 1.36 and 1.37 first I thought there were 100mm in a sentimetre but I lurnt there are only 10mm in a centimeture.*

In comparison Eric, Fay, and their collaborative group through extended and reflective discourse, generated many numbers, explored decimal place value concepts, and the role of zero in decimals. Eric began by stating: *I think it's like a way you can say 1.37 and a thousandth which is a tenth of a hundredth so the difference is one thousandth.* Brenda demanding clarification from Eric points at 1.37 and asks: *So they are trying to get to that?* Role playing the action Eric jumps his fingers across the page, demonstrating the jumps and states: *It's like Brook jumped further than Eugene but they both jumped 1.36 but um*...Brenda asks: *How much more?* Fay replies making a tentative conjecture: *Could be like 1 or 8.*

Eric building on the conjecture replies: *Yeah 1.368 and Eugene jumped 1.367.* Although the question is answered they provided no justification of their conjecture at this point.

Dialogue continued as they searched for proof. Eventually Eric recorded 1.368 and 1.367 and asked: *Well what was the difference?* Brenda replied: *One thousandth, you need to write that too.* Watched closely Eric records 0.001. Unconvinced Fay argues: *No it isn't.* Brenda points at the 8 in 1.368: *What's that one called again the hundredths?* Fay arguing in response, points at the 6 in 1.368 stating: *No that's the hundredths that's the tenths, no that, oh I don't know.*

Lengthy discussion of the place value system of the decimal numerals followed until they all voiced and pointed: *Wholes, tenths, hundredths, thousandths.*
Eric conjectures: *So they are the thousandths?* Fay providing proof says: *Yeah but look at the difference. You have no wholes, no tenths, no hundredth, hey why don't we write it?* Eric then records under each symbol the place value names.

The teacher, listening to the discussion, challenges: *So you are saying Brook jumped 1 metre and .368 of the next metre. Is that the only distance he could have jumped?* At this the students record all three place decimal numbers between 1.361 and 1.369. Challenging further, the teacher records 1.361: *What about if you write down under Brook 1.361 and under Eugene you write down 1.362? Now can you write something to show that Brook jumps further?*
Eric conjectures: *Could you put a zero on it?*
He adds zero twice then is stopped by Fay who argues: *If you add a one instead and that will make it bigger because one is like something but if you took the zero off then it would be no different like you could draw all these zeros and it wouldn't be any different.* A 'one' is added and the number now reads 1.361001.
To advance the thinking further the teacher asks: *Can you describe the distance between how far they jumped now?*
Fay: *One millionth.*
Brenda: *Yeah because all those zeros mean something now that the one is there at the end.*
Teacher: *So what is that?*

This question causes a 'fold back' to the place value system they had devised earlier and they point and say: *That's tenths, hundredths, thousandths.*
Then Fay and Brenda continue: *That's a millionths.*
Eric unconvinced asks: *A millionth, can we go and check on that?* They move to a whole number chart on the classroom wall as Eric questions: *That one is a trillionth?* Brenda points at the chart, and following the place value across argues: *Oh no look it's tenths of thousandths.* In response Fay excitedly states: *Oh I get it now, everything goes up in tens so tenth* as Brenda records under

1.361001as Fay continues: *Tenths then hundredths, then thousandths, then tenths of thousandths, hundredths of thousandths and millionth.*

In this way, extended discourse led to a 'folding back' to the visual tool of a whole number place value chart. It was used to construct place value of the decimal system and explore the way in which partitioning down is a feature of the decimal system (Hiebert, 1992, 1993; Irwin, 1996a). Furthermore, this example illustrated the way in which a rich task led to extended discourse, as the students actively engaged in making sense of their decimal understandings. Moreover, it reflected the way in which individual students when actively involved personally invest in strategies and solutions.

5.11 Operating on Decimal Quantities

Mathematical tasks that contain a sense of 'connectivity' and 'reflectivity' contribute to cognitive gains (Cobb et al., 1997; Hiebert & Wearne, 1993; Stein, 2001). In the final phase of the study the students continued to work with contextualised problems requiring addition, subtraction, and multiplying by units of 10 on varied sizes of decimal numbers. The problems proved sufficiently problematic prompting students to engage in extended discourse. However, at no time did any of the students use formal algorithms to solve them. Their preference to use a number line as a visual representation of their thinking emphasised decimal numbers representing quantities and supported justification of strategies using the place value system of decimals. Eric demonstrated this when he recorded .1 as part of a problem solving strategy. Jane challenged him: *But how do you know?* His proof is stated: *I know that nine hundredths is close to a tenth so we can round nine hundredths up to one tenth by adding .01.*

In another example Eric explains for Sara, Jane, and Fay to the sharing group *Well we added 33 and .17 and that gave us 33.17 then we added nine tenths* Samuel challenges the strategy: *You are working with the tenth then you went* *straight into the hundredth why?* Eric justifies the strategy: *We left that tenth which was one tenth and then nine tenths because we knew that would be one whole and we were working with tidy numbers so we added 33.17 to .9 and that equalled 34.07.* Jane then continued the explanation: *Then one thousandth and the .004 and we got .005 yeah because .001 and .004 equals five thousandths.*

Before operating on the decimal numbers, the students were asked to estimate the solution. Hiebert (1992) maintains that requiring students to estimate when operating with decimals provides clear information on how they are quantifying decimal numbers. Eric and Jane illustrated this when estimating the difference between 9.2031 and 6.967. Jane estimates 2.9 and Eric argues: *It's round about 2.55 because 6.967 is nearly to 7 and 9.2031 rounded to the nearest tenth is about 9.2* then rethinks: *No it will be a bit over 2.2.*

At the end of the session Jane wrote: *I think how estimating helps is you get clearer with numbers and how to deal with them. When we went off to solve our problem our group thought that it was better to add to the bigger number and our estimate was about rite.*

5.12 Summary

In this description students' informal knowledge provides a rich foundation for understanding percentages and later to robust decimal concepts and rational numbers. Knowledge of percentages became a valuable tool with which the students were able to check their reasoning, through translating across representations when working with decimal numbers.

The development of decimal understandings was shown to be a lengthy and complex process, where 'folding back' to prior knowledge formed the basis of moving forward to construct new partial understandings. Prior misconceptions of decimal concepts and partial understandings that were constructed in the study during mathematical activity were demonstrated to be robust. Extensive discourse including making conjectures, argumentation, and justification, around areas of cognitive conflict assisted reconstruction of the partial understandings.

Constructing quantitative knowledge of the decimal symbols so that they became experientially real mental referents was an extended process. At the conclusion of the study the students were solving contextualised addition and subtraction problems using decimals of varying lengths confidently and multiplying decimals by units of 10. Estimation of the answers for the problems showed that the students were able to quantify the decimal numbers in order to make reasonable conjectures.

Chapter 6 Case Studies

6.1 Introduction

Nineteen students were interviewed (See questions in Appendix A) at the commencement of the study; from these four students with different decimal misconceptions (e.g., whole number, fractional number, benchmarking, mixed patterns) were selected as case studies. Eric[1] was a ten-year-old Year Six student. Sara and Jane were nine-year-old Year Five students and Fay was an eight-year-old Year Five student.

In the description of decimal knowledge construction the students' prior whole and fractional number concepts are linked to the partial understandings of decimal concepts. The way in which partial understandings act as intermediate cognitive organisers is described as the students all moved through to the 'Apparent Expert' category. Apparent expert is a term used to describe students who can correctly compare and order sets of decimal numbers (Resnick et al., 1989; Sackur-Grisvard & Lincoln, 1985; Steinle & Stacey, 1998). In this section I map out the way in which the path travelled may best be described in the words of the students as: *Confusing, difficult, complicated, weird.*

Student self evaluation and concluding interview data provides a description of the decimal concepts the four students had constructed at the conclusion of the study. A measure of growth and self assessment in student's understanding over the 15 lessons focused on decimals during Terms 2 and 3 of their school year is provided.

[1] Pseudonyms were used to protect the identity of the case study students

6.2 Eric

6.2.1 Prior knowledge of decimal concepts and a summary of the first interview.

Eric had a mixed pattern of partial understandings of decimal concepts. When ordering decimals he selected the decimal number with only tenths as the largest—demonstrating that he knew tenths were the largest decimal placeholder. However, decimal numbers beyond tenths were classified as smaller if longer based on a 'Fractional Number' denominator focussed thinking pattern. Students using this pattern incorrectly generalise that tenths are bigger than hundredth therefore any number of tenths is greater than any number of hundredths (Steinle & Stacey, 1998). In ordering 0.19, 0.036, 0.195, 0.2 he selected correctly 0.2 as the largest explaining: *It's the biggest number because it has got no decimals in it.* Asked for further explanation he used the 5 in 0.195 explaining: *Like in that one that's like bits of the decimal, like a fraction of a decimal but 0.2 it's like the main part of the decimal.* He understood decimal place value notation, renaming 0.4 as four tenths: *Cos I'm saying it as one is ten tenths so a tenth is one part of ten of the pieces of one* and four tenths as forty hundredths explaining: *Well four tenths is four bits of one and a hundredth is a tenth of a tenth.* He provided a realistic context for 4.6 + 5.3 = 9.9 using measurement. He identified 1.6 as a number between one and two but could not identify a number between 8 and 8.1. He gave the next two numbers in sequence after 0.2, 0.4, and 0.6 correctly but could not sequence two decimal place numbers.

In summary, Eric had a range of partial understandings of decimal numbers. He knew that fractions linked to decimals and that there were numbers between one and two, but his conceptual understanding of decimals was limited to correct thinking at one decimal place. He demonstrated informal knowledge of partitioning when he explained the renaming of tenths for hundredths as: *Tenth of a tenth.*

In his self-evaluative summary recorded at the completion of the unit he described his initial decimal understandings:

- *I got confused with long decimals.*
- *I thought the zeros at the end were important.*
- *I thought that .63 ment divided into 63 bits.*
- *I forgot to write the zero before the hundredths eg 1.06 1.6.*
- *I didn't know what the dot meant.*

6.2.2 The construction of partial understandings

During the lesson series Eric constructed and reconstructed partial understandings to accommodate his developing concepts of rational number.

Recording decimals and adding a percentage symbol was his first challenge. Recording 2.63 metres as 2.63%, went unchallenged by his peers when they accepted it as part of his explanation. When the teacher asked him to draw a bottle of coke and mark on it 2.63% he indicated tentative understanding. However his continuing confusion was recorded in his written reflective statement: *Today I learnt that if you put the % sighn on the end of a decimal e.g. 2.37% it is really confuseing.*

The concept of fractional numbers below 0.1 was an on-going challenge for all the case study students. Restructuring to accommodate the concept took many lessons and resulted from a range of challenges causing cognitive conflict. A recorded comment Eric made indicated a tentative accommodation: *Oh it's 1.28 but let's just double check because I am not too sure, oh it's that .01 that's kind of weird.* The following lesson when giving an explanation he referred to

.07 as: *Plus seven.* This indicated the difficulties he had in constructing conceptual understanding of fractional numbers below 0.1.

Erroneous thinking based on a tight bond perceived between counting numbers and decimals was also the cause of ongoing conflict for all of the case study students. To assist in conceptual development activities were presented to challenge their thinking. In particular, Eric's thinking caused him to restructure his decimal place value concepts. He reflected on his growth in understanding: *Today I learnt about hundredths, tenths, thousenths, ones of thousendths and so on. I also know the difference between 1.160981 and 1.360981 the difference is in the 1.360981 there is 3 tenths but in the 1.160981 there is only 1 tenth.*

However, decimal numbers between decimal numbers remained a difficult concept and when asked to record 12 numbers between 1.36 and 1.37 he asked: *If you get to ten, no I mean nine, like you go 1.361 and then get to 1.369 what happens then?* Subsequently he drew a line across the sheet and divided it into twelve segments recording 1.36 at one end and 1.37 at the other and inserting the numbers in-between. In the first gap he recorded 1.361 and in the next he recorded 1.3612 (See Appendix H) indicating that the construction of partitioning of one whole unit within base ten was an on-going restructuring process. This activity caused a similar response from all the case studies.

6.2.3 Summary of the second and third interview

In the second interview at the completion of ten lessons Eric had moved into the 'Apparent Expert' category. Clear understanding of the quantities decimal numbers represent was shown in his explanation: *Well I start by looking at the tenths and if they are the same then I go to the hundredths and yeah then I look at the thousandths and so on. And sometimes I look at the ones that are closest to the next whole or the next whole tenths and so on.*

He clearly understood the place value structure of the decimal system renaming four tenths as forty hundredths: *Because in each tenth there is ten hundredths so in four tenths there must be forty hundredths.*

In the third interview at the completion of the study Eric applied his knowledge of the decimal symbols as mental referents for quantity to estimate the answers to the following problems:

- [12.5 - 5.75] *About six and three quarters* explaining: *5.75 is nearly 6 and then six from twelve and a half well then I guess it will be about 6.5 and then plus a quarter so the difference is six and three quarters so that's 6.75.*
- [5.07 - 1.3] *Around 3.8, cos 5.07 is 5.1 basically then minus 1 and that's 4 and then minus .3 is 3.8.*
- [10 x 0.5] *Um 5 cos point five is another name for a half, half of 10 is 5 that's sort of weird about multiplying by decimals it gets smaller.*

However, Eric's development of understanding is gradual and incomplete in some areas, estimating the answer for 0.12 divided by 10: *Oh I can't do divided by.*

His self-evaluative written summary included:

- *I can put decimals in order.*
- *I know that 4.24 is bigger than 4.23.*
- *I know that the tenths are the key, after the decimal point the numbers get smaller and smaller.*
- *I know ten tenths make a whole.*

6.3 Fay

6.3.1 Prior knowledge of decimal concepts and a summary of the first interview.

Fay used an informal range of strategies to make sense of decimal concepts with mixed success. She used money knowledge and benchmarking to correctly order decimal numbers to two places: *Zero point eight is bigger than 0.75 because eight is another way of saying eighty.* However, she reverted to a 'Whole Number' thinking pattern to order decimal numbers beyond hundredths: *The biggest is 0.195 then 0.036 because if you took away the zero after the point it would be just three six then 0.2 comes next because that's the same as twenty then smallest is 0.19.*

Fay used her limited knowledge of place value of decimals maintaining a benchmarking strategy to rename 0.4: *Four tenths cos well zero point five is one half and so zero point four it is kind of going down in tens.* This was repeated adding one tenth to 2.9: *Three because nine could be ninety and then another tenth makes a hundred hundredths,* and again in renaming four tenths as forty hundredths explaining: *Well it's like splitting the tens into twos or the fours times ten.* She provided a realistic context for 4.6 + 5.3 = 9.9 using shopping. She identified 1.6 as a number between one and two but could not identify a number between 8 and 8.1. She gave the next two numbers in sequence after 0.2, 0.4, 0.6 using whole number language stating: *Point eight, point ten.*

In summary, Fay had a range of decimal misconceptions. She used an informal strategy of benchmarking decimal numbers to hundredths and then used 'Whole Number' thinking. She had limited decimal place value knowledge and no understanding of zero as a placeholder. She demonstrated informal knowledge of partitioning as she explained how decimal numbers went up and down and could be split into tens.

In her self-evaluative summary recorded at the completion of the unit she describes her initial decimal understandings as:

- *Decimals got bigger not smaller.*
- *Percentages were different from decimals.*
- *I couldn't use decimals in problems.*
- *I thought .1 & .01 were the same.*

6.3.2 The construction of partial understandings

With the teaching unit Fay exhibited a misconception with percentages, lacking understanding of percentages as fractions with denominators of 100. She illustrated this in describing 200mls as 200%: *I think he has got 200%...200mls left because well if it's 800mls at the start and it goes down to 400mls because he drank 50% and you divide it by 2 again you get 200%.* This thinking pattern remained through four consecutive lessons until she drew one container and labelling it 5 litres 33ml and recording it as 5.33%. Group questioning challenged her thinking about what was represented by 5.33% and caused tentative restructuring as she reflectively recorded: *I concreted in a lot about Percentage Metre litre & Fractions. We really got into exact amounts. My strategies were challenged & made me come up with lots of different ways.*

A tenacious misconception caused problems linking percentages to decimals as she translated across representations. Each hundred percent or whole number was used as a denominator for the fraction of the whole and this thinking pattern was maintained through three lessons: *He's already walked 7.29 metres so let's see that's 7.29% or you could go 29 over 700.*

Later Fay used her benchmarking strategy of adding a zero as a scaffold to construct concepts of numbers below .1 explaining: *No because .4 is .40 and that's 40% but .04 is just 4%.* However, the tight bond between counting and decimal numbers continued to challenge Fay's thinking. She worked confidently within a group appearing to restructure her concepts. However, a misconception shared in the large group demonstrated how tenuous her understanding was when she reflectively recorded: *We were sharing our problems when Sara wrote down 1.63.1. I was feeling confident when I came to the clinic but now I'm not so sure. Decimals are very complicated.*

The following lesson when asked to record 12 numbers between 1.36 and 1.37 she drew a line marking each end 1.36 and 1.37 but drew 12 even segments. She then began recording at each segment, the first 1.361 then 1.362 until she reached 1.369 and she then recorded the next as 1.3691 (See Appendix H) However her recorded reflection showed a fold back to the previous day based on the use of a place value chart and her growing confidence: *We had to write 12 numbers between 1.36 & 1.37. I learnt more about smaller decimals.* She then recorded 1.36142 and named each symbol with whole, tenths, hundredths, thousandths, tenth of thousandths.

6.3.3 Summary of the second and third interview

In the second interview Fay had become an 'Apparent Expert'. She clearly understood quantities decimal numbers represented explaining: *Point two is biggest cos the first one says 0.19 and that's just one hundredths away from 0.2 so that wouldn't. The next one says 0.036 and since there are no tenths I'd forget about that and then 0.195 and the one tenth is smaller than the 0.2.* She understood the place value structure of the decimal system renaming four tenths as forty hundredths: *40 cos like one hundredth is .01 and 4 tenths is 40 so 5 hundredth is half of a tenth so 4 tenths makes 40 hundredths.*

In the third interview at the completion of the study Fay applied her knowledge of decimal symbols as mental referents for quantity to estimate the answers to the following problems:

- [12.5 - 5.75] *Around about 6.75* explaining: *12 - 5 = 7 and then minus .75 is 6.25 and then plus .5 gives an answer of 6.75.*
- [5.07 - 1.3] *Less than 4 because 5.07 - 1 = 4.07 and you still have more than .07 to take away so you would end up about 3.8 or nearly.*
- [10 x 0.5] *Well it will be around 5 cos if you swap the numbers around and then go point one times ten that's one and so point five times ten is 5.*
- [0.12 divided by 10] *Point one divided by ten is point zero one and point zero two divided by ten would be point zero zero two so the answer to that must be 0.012 so that's much smaller now.*

Her self-evaluative written summary included:

- *Zeros count.*
- *More figures doesn't always mean a bigger amount.*
- *How to add decimals together.*
- *About long decimals.*

6.4 Jane

6.4.1 Prior knowledge of decimal concepts and a summary of the first interview.

Jane's prior knowledge of decimal concepts was based on 'Fractional Number' thinking using a 'reciprocal thinking' pattern (Steinle & Stacey, 1998). She selected as biggest the decimal number which formed the biggest denominator of a fraction: *Point two is the biggest because when you say it as a fraction it would be one half. If you say 0.036 as a fraction it would be one by three six so that is next and if you say 0.19 as a fraction that will be one by nineteen and 0.195 is the smallest because when you say that as a fraction it is one by one* *hundred and ninety five.* She lacked decimal place value concepts, renaming 0.4 as one fourth. Using 'Whole Number' thinking in adding one tenth to 2.9 she answered 3.9 and likewise multiplying 5.13 by 10 her answer was 50.13. She provided a realistic context for 4.6 + 5.3 = 9.9 using measurement. She identified 1.6 on a number line between one and two using measurement as her reasoning: *I think it is 1.6 like six centimetres* but could not identify a number between 8 and 8.1. She gave the next two numbers in sequence after 0.2, 0.4, 0.6 using whole number language: *Zero point eight, zero point ten, no one whole.*

Her misconception was based on 'Fractional Thinking' with the decimal seen as the denominator of the fraction. She had limited decimal place value conceptualisation.

In her self-evaluative written summary recorded at the completion of the unit she described her initial decimal understandings:

- *I thought the longer the decimal was the bigger number.*
- *I thought that for example 1.4 was the same as 1.04.*
- *I couldn't justify my answers.*

6.4.2 The construction of partial understandings

Jane's 'Fractional Number' thinking pattern remained a tenacious misconception during the first five lessons. This thinking resurfaced when challenged by new concepts: *You could write it as 7.29 or 29 over 700* and *5.33 could be 33 out of a hundred or is it 533 out of 600?* When challenged about recording 54/600 alongside 6.54 she explained her reasoning: *Well instead of doing like a 100 there are 6 wholes like 6 of hundreds are 600 so instead of doing 6 wholes you do 6 times a 100 and have 600.* However, this showed that she was restructuring her thinking from a 'reciprocal thinking' pattern to a mixed pattern in which elements of a 'denominator focussed' thinking pattern were present. Her reasoning included some place value thinking interwoven with an erroneous image of an equivalent fraction.

She used measurement to scaffold her decimal concepts describing 5.54 as: *Five wholes and bits* when questioned further on the bits described them as: *54 centimetres.* However, Jane had the same difficulties as the other case study students recording decimal numbers between decimal numbers.

6.4.3 Summary of the second and third interview

In the second interview Jane had become an 'Apparent Expert'. She clearly understood quantities as represented by decimal numbers, explaining: *Point two is biggest because it has the biggest tenths then if you take .19 and .195 they both have the same up to the hundredth but .195 has 5 thousandth more than .19 so it is the next biggest.* She understood the place value of the decimal system renaming four tenths as forty hundredths: *40 cos I just know 4 tenths are 40 hundredths.*

In the third interview at the completion of the study Jane applied her knowledge of decimal symbols as mental referents for quantity to estimate the answers to the following problems:

- [12.5 - 5.75] *About 7.25* explaining: *12 -5 = 7 then 50 - 75 oh no it's a bit less than seven wholes.*
- [5.07 - 1.3] The statement: *It's around 4 wholes. No wait 4 point oh I don't know cos I don't know how to take off point three from point zero seven* indicated an understanding of what was required. However she was unable to manage subtraction of ragged decimals, that is decimals of differing lengths.
- [5.07 + 1.3] *Well I just know it's 6.37 but I could round that .07 to .1 cos that's a tidy number and then it would be about 6.4.*

- [10 x 0.5] *5 wholes because if you say like one whole times ten wholes it's 10 wholes but that .5 is only half of a whole so you half the answer and it is 5.* When asked to explain it using decimals she said: *I knew that point one times 10 is 1 so .5 times 10 is 5.*

However, Jane's development of understanding is gradual and incomplete in some areas, estimating the answer for 0.12 divided by 10: *I know that 0.1 divided by 10 is oh I don't know it is just too hard.*

Her self-evaluative written summary included:

- *That 1.4 isn't the same as 1.04 and that 1.4 is ten times bigger than 1.04.*
- *Now I know that the number after the decimal point just depends on the place value (tenths).*
- *Now I know how to justify in different ways so people can understand.*

6.5 Sara

6.5.1 Prior knowledge of decimal concepts and a summary of the first interview.

Sara had limited prior knowledge of decimal concepts. She had made a tenuous link between decimals and fractions related to one-place decimals using a 'Fractional Number' thinking pattern and selected as larger any number with only tenths explaining: *Zero point four because you divide something into four and one piece out of four would be bigger.* Decimal numbers were then ordered using 'Whole Number thinking'. She had no understanding of decimal place value concepts renaming 0.4 as one fourth. However, she provided a realistic context for 4.6 + 5.3 = 9.9 using money.

In her self-evaluative summary recorded at the completion of the unit she describes her initial decimal understandings:

- *I thought there was a set number untill the digit became a whole number.*
- *I knew some things but I couldn't explain it.*
- *I didn't know what the point meant.*
- *I thought the zeroes didn't count.*

6.5.2 The construction of partial understandings

Sara seldom voiced a conjecture when working in a collaborative group. This limited the collection of research evidence of the partial understandings she constructed in the learning activities.

The tight bond between counting and decimal numbers was a challenge to Sara's thinking. She listened to a peer explain a decimal number between 1.36 and 1.37 and stated: *Oh I get it.* However, describing 1.361 to the sharing group as: *Point three six and one millimetre* indicated that she erroneously considered decimal numbers had two decimal places and then began again. This misconception was confirmed when she reflectively recorded: *I also didn't understand how and why brook wrote 1.361 Because to me that ses one point three hundred and sixty one. I think it makes more sence to put 1.36.1*

Despite limited overt participation in group discussions Sara's active listening was evident in her recorded reflections. These reflections illustrate the challenges she faced in restructuring her thinking:

I learnt about tenths and hundreds, deimals and I got stuck in the bit about .04. I thought it was .4

When I read the problem I thought because I didn't have a buddy to explain to me and I thought it was way to hard, I lurnt before the decimal point it is <u>whole</u> numbers and they get bigger and bigger and on the other side the larger numbers are smaler actualy.

Today I went to a maths clinnic it was sort of easy and compaired to yesterday it was easy some parts I mixed up thou like I thought the more digits the bigger the number but I know it's the other way around desimals are realy confusing but say the number is 1.23000000000 and 1.2376 then even thou one number is a lot longer its still smaller.

6.5.3 Summary of the second and third interview

In the second interview following the first ten lessons Sara had constructed a partial understanding explaining the importance of tenths: *I looked at the tenths column and it's the highest number because it has the biggest tenth.* However, she then selected 0.19 as bigger than 0.195 basing her reasoning on: *Because it has one of the highest numbers in the tenths column but the other one has like a 5 as well so it goes into the thousandth and so that makes it smaller and the other one only goes to the hundredths.* Sara was demonstrating mixed patterns, partially correct and a 'Fractional Number' denominator focussed thinking pattern.

However in an interview at the end of five additional lessons Sara had become an 'Apparent Expert' explaining clearly: *0.2 is the biggest because the others only have one tenth, then 0.195 because it has 5 thousandths more than 0.19 and 0.036 is the smallest.* She renamed four tenths as: *Forty hundredths because I think four times ten tenths is forty hundredths.*

In the final interview at the completion of the study Sara applied her knowledge of decimal symbols as mental referents for quantity to estimate the answers to the following problems:

- [12.5 - 5.75] *About six point* explaining: *I would round the 5.75 to 5.8 and the 12.5 is like twelve and a half so it sort of would be a bit over six and a half.*
- [5.07 - 1.3] *It's around three and a half a bit more.*

- [5.07 + 1.3] *Well it's 6.37.*
- [10 x 0.5] *Well it will be 5 cos ten lots of one tenth is a whole and so five lots of one tenth is five wholes.*

However constructing decimal concepts is lengthy process and when asked to estimate the answer for 0.12 divided by 10: *Oh divided by, I don't know.*

Her self-evaluative written summary included:

- *There is a set number but it depends which column it's in.*
- *Now I can explain everything I know.*
- *I know what the point means.*
- *Now I know the zeroes do count.*

6.6 Summary of case studies

At the conclusion of the study all the students had moved to the 'Apparent Expert' category. Furthermore, they could all make reasonable estimates of answers to operations that used addition, subtraction, or multiplication by ten of decimals on decimal quantities. However, division of decimal numbers was perceived to be too difficult for three of the four students.

Each student followed a similar pattern of construction and reconstruction of partial understandings, restructuring initially from 'Whole Number' to 'Fractional Number' thinking patterns before reaching 'Apparent Expert'. Moreover, the 'Fractional Number' pattern was restructured from a reciprocal focus to a denominator focus. However, the length of time and reasons for reconstructing decimal concepts varied for each student.

Much of the student's rational number thinking including partitioning down and decimal numbers between decimal numbers was revealed as counter-intuitive. For all case studies, their misconceptions were tenacious and required repeated challenge to cause conflict and subsequent restructuring of thinking patterns. However the development of understanding of rational number is a gradual process and one that these students are only at the beginning of.

Chapter 7 Discussion and Conclusion

7.1 Introduction

The major goal of this study was to examine the construction of decimal fraction concepts of four case study students in the context of an inquiry classroom. A particular focus was on the way in which classroom activity and tasks in the mathematical classroom affected the construction of decimal understandings. A further focus was to examine the nature of collaborative discourse and the social and sociomathematical norms of an inquiry classroom as students engaged in mathematical activity and constructed formal knowledge of decimal fractions.

In this chapter I use the findings of the current study to illustrate what a complicated and protracted path Year Five and Six students take to construct robust decimal concepts. I describe the way in which rich classroom activity was used to challenge erroneous thinking and how this lead to reflective organisation and reorganisation of rational number concepts as students connected and translated across modes of rational number representations. I outline how collaborative discourse and the role of social and sociomathematical norms of an inquiry classroom supported the students as autonomous learners to actively participate in an intellectual community where reasoning was maintained as a core focus and consequently lead to on-going reconstruction of decimal fraction understandings.

Implications of this current study and suggestions for further research are outlined. The conclusion from this study is presented.

7.2 Constructing decimal fraction concepts

The complex, challenging and lengthy process that is involved in students constructing formal mathematical knowledge such as decimal concepts was illustrated by the findings of this study. All case study students at the completion of the study had progressed to an 'apparent expert' category and were able to correctly compare and order sets of decimal numbers. However, this study's findings were similar to those of other studies (e.g., Hiebert et al., 1991; Irwin, 1996b) in that no student had constructed completely correct decimal fractions concepts as evident in incomplete understanding shown in some decimal operations. For Year 5/6 students this was an expected outcome given the complexity of the decimal fraction concepts; the way in which reasoning with rational numbers is counter-intuitive to whole number thinking and the gradual partial construction process (Carpenter et al., 1993).

An analysis revealed that the students constructed decimal concepts gradually in small and often unpredictable steps, structuring and restructuring their thinking in response to conflicting ideas in a recursive non-linear process. As in earlier studies (e.g., Hiebert et al., 1991; Post et al., 1993; Sackur-Grisvard & Leonard, 1985) this study found that the students moved between layers of sophistication changing their reasoning to accommodate new information within a certain context, yet retaining prior partial understandings that they then applied to other contexts. The tenacious partial understandings came from many sources including prior experience with money and measure, as well as erroneous 'rules' devised by the students in an attempt to integrate new concepts with prior 'whole number' or 'fractional number' thinking.

Misconceptions based on partial understandings were a regular recurring feature of the study as the students progressed towards deeper conceptual understanding of the decimal system. However, a critical feature of the study was the teacher's knowledge of misconceptions (described in Chapter 2.6.3) commonly held by Year 5-6 students. The teacher frequently used this knowledge to ensure that erroneous thinking patterns were challenged and quantitative meaning for decimal fractions maintained through on-going flexible modification of learning activities.

Premature introduction and manipulation of decimal symbols is a reported contributing factor to student failure to apply meaningful understanding to decimal notation (Hiebert & Wearne, 1985; Post et al., 1993). In this study the teacher delayed the recording of decimal symbols until the students had made connections between their informal rational number concepts and concrete real world quantitative representations. The written symbols were then linked to proportional thinking embedded within a concrete continuous measure model of comparison to a unit. An examination of the findings showed that the students maintained a sense of quantity as they manipulated the symbols flexibly to solve real world problems using notational schemes as their explanatory tools. The teacher's careful linking had supported the students to develop conceptual understanding of the symbols as meaningful referents and provided them with a flexible thinking tool—one which would be expected to significantly expand their capacity to reason mathematically (NCTM, 2000).

Concluding interview results of the case study students showed significant growth in these students' ability to flexibly translate within and between modes of rational number representations. The students were able to accurately re-describe decimal understandings using equivalent rational number benchmarks. The teacher supported the students to construct a rapid and effective overview of rational numbers through building on prior knowledge of proportional thinking and the use of percentages as an introductory representation. An analysis revealed how important the development of in-depth rich understanding of a unidimensional representation was in enabling the students to connect percentages to their benchmark decimal and fraction equivalents. A case study student summarised the importance of connecting mathematical thinking in her reflective statement: *I've lurnt such a heap about percentages, fractions and decimals and this is some of the best lurning I've dun all year. People who are not used to decimals think that. Now I know a heap about them cos I've made sense of unfinished lurning overall I got gold.*

7.3 Classroom mathematical activity

Although the students began the study with different levels of partial understandings, they had similar informal knowledge of how decimal fractions might be used in everyday real world contexts. This knowledge gained through a class conceptual mind map (Appendix B) supported the use of contexts in instructional activities that were realistic for the students. More sophisticated conceptual thinking was advanced through embedding the concrete manipulatives in contextualised problems. In addition, the use of situated contexts provided a concrete or experientially real context for the students at varying times to do what McClain and Cobb (1998) and Pirie and Kieren (1994) describe as 'fold back to' or 'drop back to' for support for subsequent activity. As the student's understanding of decimal concepts developed contextualised problems replaced a need for concrete representations. These then became mental images used in explanations as experientially real 'taken as shared' objects within reflective discourse.

Connectivity and reflectivity in mathematical tasks are identified as key elements in students engaging in high levels of reasoning (Cobb et al., 1997; Hiebert et al., 1997; Schwan Smith & Stein, 1998; Stein & Smith, 1998). The findings in this study show that the use of contextualised problems provided a source for prolonged discussion as the students engaged in extended exploration and evaluation of possible solution strategies. Initially discussion took precedence over written recordings as the students actively examined mathematical ideas. A recorded summary using a notation scheme followed on from the discussion and became the explicit object of discourse providing a source for further reflective discussion of the mathematical ideas inherent in it.

The teacher chose not to introduce formal algorithms during the period of the study. The students' spontaneous use of their own informal strategies was expected and valued. Classroom data revealed that the students' use of informal strategies maintained a focus on proportional thinking of quantity which translated flexibly across rational number representations and supported the students to reflectively examine their erroneous thinking patterns.

Some studies show that classroom activity may allow misconceptions to co-exist with more sophisticated understandings (Condon & Hilton, 1999; Sackur-Grisvard & Leonard, 1985). Decimal misconceptions, in particular, are known to be robust, requiring deliberate challenge in order to cause reconstruction of prior thinking patterns (Bell et al., 1981; Irwin, 1996a; 1999; Stacey & Steinle, 1999; Yates & Chandler, 1991). Findings in this current study suggest that for students in Years Five and Six misconceptions are most effectively addressed as they arise. The teacher's use of problems that explicitly focused on common misconceptions of fictitious individuals (See Appendix C) were useful tools and supported the students to safely explore, identify and examine and reconstruct their own erroneous thinking patterns.

Within the current study, both the way in which the teacher elaborated the setting of problems and the use of known contexts ensured that the students made explanations describing action on mathematical objects that were experientially real to both themselves and their peers. Incongruity between the student's informal knowledge of a problem context, individual partial understandings, and mathematical explanation always resulted in inability of the group to reach consensus. In such instances, the students would then engage in extended discourse that usually lead to re-conceptualisation of the problem, cognitive conflict and eventual restructuring of their thinking patterns.

7.4 Classroom practice: Collaborative discourse and the social and sociomathematical norms.

Collaborative discourse and the social and sociomathematical norms of the classroom significantly supported the construction of decimal concepts for the four case study students who participated in this study. The teacher proactively maintained interactive discourse, facilitating an inquiry classroom environment where mathematical learning was both an active individual and a social process of constructing decimal concepts. In this environment the case study students, each in their own way, took ownership of their mathematical activity demonstrating both the growth of intellectual autonomy and a mathematical disposition.

The teacher skilfully ascertained student knowledge of decimal understandings through active listening and careful questioning of student explanation and justification. Errors in explanations became learning opportunities used in discussion to explicitly address erroneous thinking. Discussion supported the exploration of contradictions students encountered between prior whole number thinking and decimal fraction concepts resulting in reconstruction of concepts. Furthermore, valuing errors as learning tools inducted students into a mathematical environment where reasoned explanation was more important than a solution.

Observations confirmed that the teacher regularly halted explanations to provide a 'wait time' in order to support students giving explanations and to encourage questions from students in the larger group. During the 'wait time' the listening students would analyse their strategies, reflectively compare and contrast these and predict the next step in the explanation. These explanations recorded as notational schemes became the basis of further mathematical activity—used by the teacher as reflective tools ensuring that mathematical reasoning was accessible to all students. The student's conceptual understanding of decimal fractions was advanced through teacher questioning and revoicing of student explanation. The teacher sought alternative strategies and solutions and the students confidently analysed ways in which the strategies differed, identifying those considered the most efficient. However, the students were also certain in their right to use a less sophisticated strategy recognising that understanding and explaining a strategy and the reasoning behind it meaningfully was of critical importance.

Using Fraivillig, Murphy and Fuson's (1999) framework the analysis showed that the teacher shared the role of intellectual validator of mathematical reasoning with the students. The intellectual community established was one in which all class members had rights matched to corresponding responsibilities. These included a high level of active engagement in collaborative discourse with a focus on making sense of their own and others' explanations, working for sustained lengths of time and reaching group consensus. Group consensus was achieved through peer scaffolding, extended discourse, and the exploration of a range of alternative strategies and solutions. Students spending extended time discussing conjectures, arguing, exploring and justifying alternative strategies and solutions, was a key factor in their reconstruction of decimal concepts as they identified incongruities between their thinking and that of others.

When working collaboratively the students began with a conjecture or possible solution. It would collectively be discussed and explored and alternative or more efficient strategies would be considered. Scaffolding supported any group member being able to share the group strategy to the larger group. Alternatively, conjectures would be followed by intense argumentation before consensus was reached. Argumentation occurred most often when a student had a tenacious misconception that could only be addressed through cognitive conflict to cause reconstruction of decimal fraction concepts.

The sociomathematical norms shaped the classroom discourse and regulated the learning of decimal concepts although the ways in which individual case study students engaged in mathematical discourse differed. Two students took leading roles—asked and answered questions, made conjectures, challenged and extended their explanations to justification of strategies and solutions. However, the other two students took less active roles using group members' explanations as scaffolds for new conceptual thinking. Although the higher level of participation in classroom discourse appeared to cause a difference in the results initially, ultimately all four students constructed sound decimal fraction concepts.

7.5 Implications for the classroom

When interpreting the results of this study, the complex nature of classrooms, the complicated interaction of many features of classroom practice, and the nature of individual construction of concepts must be considered. Interpretation of the results can only provide an emerging understanding of the ways in which students may be supported to construct decimal concepts in an inquiry classroom given that the number of participants was limited.

Decimal fractions are introduced in the mathematics curriculum in New Zealand as a Level 3 achievement objective for students aged 9-11 (Ministry of Education, 1992). The emphasis at this level is on ordering and explaining meaning for three place decimal fractions. Appropriately the focus is on constructing rich conceptual knowledge of decimal fractions as quantities. The research results confirm that the construction of this foundation knowledge is a complex and lengthy process and Year 5/6 students must be carefully supported in an unhurried manner in recognition that this knowledge is a critical scaffold for future meaningfully manipulation of decimal fractions. It is possible that recording decimal fraction symbols is best delayed until students have developed robust mental referents for decimal fractions as proportional amounts. Results suggest that students should be encouraged to use informal recordings of notational schemes in preference to formal algorithmic to maintain focus on proportional thinking of quantity.

Percentages are the final rational number concept introduced in Mathematics in the New Zealand Curriculum (Ministry of Education, 1992) and in the current New Zealand Numeracy Project (Ministry of Education, 2002b). The results of this research suggest that such a placement should not be considered definitive. An earlier introduction to percentages has the potential to provide a meaningful scaffold to deeper richer rational number concepts. Percentages are visible in students' everyday life and every percentage has an easily seen decimal or fractional equivalent number. The earlier introduction of rational number through percentages as a rich connective base could provide a serviceable structure for students to translate across representations of rational number. Moreover, beginning with percentages and immediate linking to decimals and fractions integrates rational number in an interconnected manner and supports student use of flexible translation across representations to check reasoning.

In order for students to construct decimal fraction concepts, New Zealand studies (e.g., Irwin, 2000; Storey, 2001) and the New Zealand Numeracy project (Ministry of Education, 2002b) promote an initial use of concrete manipulatives gradually replaced by visualisation. The results of this current study reveal that the number line was a significant tool that represented a concrete embodiment of key aspects of decimal concepts—the continuous nature of decimal fractions and the notion of the referent unit as one whole unit. Furthermore the number line embedded within real life contextual problems supported students to manipulate the decimal fraction symbols as experientially real objects in meaningful mathematical activity.

Many Year 5/6 students have available a rich bank of informal rational number knowledge. Results indicate the importance of teachers taking heed of this knowledge in order to provide an authentic context for problems to engage student attention and enhance thinking. McClain and Cobb (1998) and Pirie and Kieren (1994) suggest that for students to maintain meaningful mathematical activity they need richly connected models as mental referents available to 'fold back' or 'drop back' to when needed in order to check their reasoning. Informal knowledge may also include partitioning concepts—a valuable scaffolding tool for formal decimal number concepts. However, there is no assurance that students' prior knowledge is not based on erroneous thinking.

Teachers need knowledge of the misconceptions students within this age group commonly hold as partial understandings. Misconceptions are both powerful diagnostic and teaching tools (Moloney & Stacey, 1996). Teachers need to carefully consider partial understandings recognising them not as errors but intelligent attempts to integrate new learning with prior whole or fractional number thinking (Resnick et al., 1989). As such, they are a significant indication of current thinking and can potentially scaffold new and deeper conceptual learning. Teachers' knowledge should also include richly connected rational number concepts and the learning progression students take in developing robust decimal. Furthermore, sound pedagogical knowledge is essential if teachers are to be able to listen to and make sense of students' explanations as they describe their current understanding concepts (Anthony & Walshaw, 2002; Cobb, 2000b).

The description of the classroom in this study reinforced that it is the teacher who makes possible a classroom environment where all members have interactively constituted the social and sociomathematical norms. In order to establish a mathematical environment that is conducive to active student engagement and sense making teachers of a more traditional mode will need to review their own role in the mathematical classroom (Cobb et al., 1992; Cobb, 2000b; Lampert, 1990, 2000). Moreover, if students are to construct rich decimal fraction concepts then classroom environments must move towards the inquiry model within which students' recurring erroneous decimal thinking patterns will be challenged—and through discussion and debate lead to the reconstruction of robust decimal fraction concepts.

7.6 Opportunities for further research

The following issues identified from the results and implications of this study warrant further research.

1. Students in this study constructed sound conceptual understanding of decimal fractions and were able to apply this knowledge within problem contexts that required addition, subtraction, or multiplication by 10 of any decimal quantity. To solve problems which included decimal fraction quantities they used a range of informal strategies. It would be timely to explore the types of informal strategies students use to solve problems which involve decimal fractions and the ways in which these maintain understanding of decimal symbols as quantitative representation.
2. New Zealand teachers use a variety of concrete manipulatives in their instructional practice. However, which manipulative best suits the mathematical requirements of students at a particular age or stage of decimal fraction conceptual development needs further exploration.
3. Students in this study were from a high decile inner city school. They had a rich knowledge of informal rational number contexts which supported learning activities within authentic contexts. Further research would be appropriate to compare and examine the informal knowledge of students in other areas of New Zealand and within other decile level schools. In addition, studies of how teachers use the informal knowledge of their

students as a scaffold for complex formal knowledge of rational number is needed.

4. Students in this study constructed powerful mental referents for decimal fractions as proportional quantity through building on their prior rational number concepts. Percentages were used as a rich connective base for other representation of rational number. The current Advanced Numeracy Project (Ministry of Education, 2002b) teaching and learning activities focus on fractions initially followed by decimals and finally percentages. A further study is needed to compare which order of rational number representation is most effective in supporting students to develop richly connected rational number concepts. Furthermore, introducing decimal concepts through the use of percentages supported flexible translation across representations. Comparisons need to be made of students' ability to translate across representations in relationship to which form of rational number is introduced first.
5. In this study the teacher had strong mathematical knowledge and was able to listen to student explanation, build on their current mathematical reasoning and advance conceptual thinking. Further studies are needed to examine the critical decimal fraction knowledge Year 5/6 teachers require in order to plan learning activities, listen to student explanation, ask appropriate challenging questions to advance student thinking and challenge the validity of student statements.
6. A related area to consider is the role of collaborative interaction. The ways students interacted in this study influenced their construction of decimal fraction concepts. Further investigation is warranted to establish how patterns of discourse best support mathematical learning.
7. Analysis of the social and sociomathematical norms in this study indicated their key role in student construction and reconstruction of decimal fraction thinking. Many questions related to these norms merit further investigation including:
    - The role of active listening and making sense of explanations

- Age, ability and gender related differences in making explanations and justification of conjectures
- Characteristics of individual explanations compared to collaboratively constructed explanations.
- How collaborative groups reach consensus. The effect on individual learning resulting from all students having to explain group strategies and solutions.
- Strategies students use to analysis the similarities and differences between their explanation and others and how they define efficient strategies.
- The language teachers use to develop and support classroom sociomathematical norms.
- Exploration of teacher behaviour used to advance students' conceptual thinking when giving explanations.
- Teacher questions in an inquiry classroom.

7.7 Concluding thoughts: The point of it all.

This research adds to an aggregation of knowledge about the teaching and learning of decimal fraction concepts. The design of the study was modelled on good practice of classroom teachers using classroom teaching experiment methods (Cobb, 2000a). A flexible learning progression was designed and implemented which incorporated the informal rational number knowledge of the students within classroom activity that adapted responsively to emerging mathematical understanding of individual students. However, another dimension was added and one which needs careful consideration when examining student construction and reconstruction of decimal fraction concepts—collaborative interaction and the social and sociomathematical norms of an inquiry classroom. Evidence from this research would suggest that mathematical reasoning is located at the core of classroom activity in an inquiry classroom. Student participation in meaningful activity that included explaining, justifying, and critically reflecting maintained learning as a sense-making activity leading to rich and robust conceptual understanding of decimal fractions.

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Appendices

Appendix A: First interview questions.

1. Say the number 0.29
2. Which number is the bigger of these two? 0.75 or 0.8
3. Which number is the biggest of the three numbers? 0.62   0.236   0.4
4. Which number is the biggest of the four numbers? 0.19   0.036   0.195   0.2
5. 0.4 is the same as… four…four tenths…four hundredths…one fourth?
6. Can you tell a story about…4.6+5.3=9.9
7. If you multiplied…5.13 by ten… What would be your answer?
8. If you added…one tenth to 2.9…What would you have?
9. Four tenths is the same as how many hundredths?
10. What decimal number is shown by the arrow on the number lines? (1.6 was shown on a number line)
11. Look at the scale carefully and then say what decimal number you think each box shows. (8.05, 2.03 were shown on a number line)
12. What are the next two numbers after 0.2   0.4   0.6
13. What are the next two numbers after 0.3   0.6   0.9
14. What are the next two numbers after 0.94   0.96   0.98
15. What are the next two numbers after 1.13   1.12   1.11

Appendix B: Class concept map of the informal knowledge of decimal concepts

Where are decimals seen when we are not at school?

Calculators
Answers to division
Money

Drivers
Work out their speed and distance on the motorways (4.5 km to the next exit)
Buying petrol (24.6 litres)

Shopping
1.5 litres of coke
2.25 litres of coke
1.5 kg of sugar/flour
Price tags

Computers
Hard disk space (24.55 gigabytes)
Down loading programmes off net (5.3 mega bytes)
System numbers (8.6 system)
Computer game scores and time to go (3.5)

Sports
Cricket batting average (39.4)
Times at swimming club (14.861 milli-seconds)
Running races (10.2 seconds)
Depth of the pool 1.2 metres
Ballet and gymnastics results (6.987 out of a score of 10)
Marching scores
Length of fishing line and weight of fish
A position on an athletic track and horse racing track

Sewing
Material for a net ball skirt (1.2 metres)
Curtains for the lounge 6.3 metres wide

Builders/Engineers/Carpenters
Lengths of rooms exact measurement of wood and windows (4.342 metres needed for floorboards)
How much cement. 8.2 kg
Paint to buy (1.3 litres)
Height of walls for exact wall paper measurement
Carpet for floor

Body measurements
Height (1.89 metres)
Weight (65.5 kilogram)
Waist (19.4 cm for a ballet costume)
Arm span (1.2 metres)
Blood pressure
Temperature

Real estate agents
Rich houses (1.2 million)
Size of land (2.8 acres)

Money and Banks
Pay (6.80 an hour)
Money exchange (NZ $1 = A$ .812
Interest

Library
A system for finding books
Decimals in books, magazines, newspapers

Measurement
Volume of bottles and cans
Amount of water in a container or in a spa or swimming pool
Knowing how big cargo is.
Depth of submarines
Hospital temperature charts

Appendix C: Contextualised two-place decimal problems.

1. Problem: Shelves for the play-shed.

At school we need to build a new shelf in the play-shed to store some more sports equipment and Wendy wants John to do it as soon as possible. So instead of going out and buying a pre-cut shelf he decides to use up some of the pieces of timber the builders doing the hall left over.

The shelf needs to be exactly 1.13 metres but the only spare piece of timber John can find measures 2.41 metres.

How much will he have to trim off to make the shelf fit in the play-shed?

2. Problem: Louise's homework.

Louise has handed in this homework and the teacher says that her answer is wrong.

The problem she did was:

Mary was given heaps of chocolate bars at Easter and after she had eaten some she had 4.37 chocolate bars left. She eats 2.7 more then she decides to save the rest. How much does she save?

Louise has written her answer as 2.30 and the teacher has told her that it is wrong.

Can you explain what she was thinking and why she got the answer wrong? Can you work out what the correct answer should be and then work out a way you would explain to Louise why she got the answer wrong?

Appendix D: Contextualised problems—decimal numbers between 1.36 and 1.37

1 Problem

Evan and Ben were having a jump off in the sandpit to see where they could jump to if they stood with their toes just before the edge. Nick measured each jump and he said that Ben won because although they both jumped 1.36 metres it wasn't exactly 1.36 metres for either of them but neither of them reached 1.37 metres.

So if Ben jumped a tiny bit further what different distances could you record for their jumps which show that Ben did jump further than Evan?

Be ready to explain and justify your answers using equipment, diagrams, drawings, percentages, fractions and decimals.

2 Problem

Gillian and Rachel have been having an argument because Gillian says she can write more than 10 numbers between 1.36 and 1.37.

Rachel says she not only has to prove it by writing 12 numbers between 1.36 and 1.37 but then she has to show them on a number line.

You write the twelve numbers for her and then show them on a number line.

Appendix E: Contextualised problems—ordering decimal fractions.

1 Problem

Louise had to do some more homework. She had to put some decimal fractions in order from largest to smallest and this is what she did:

A.	.90146	.9115	.97	.9
B.	.4500000	.451001	.5104	.54

You put each row in the right order to help her out and then choose one of the rows to explain to her why you needed to change the order she had them in.

In your group discuss the explanation you would give. You could use equipment to support your explanation. When you have all agreed on a clear explanation write what you would say.

2 Problem

Helen went to watch four children compete in a team in a gymnastics competition. They had to compete in four different sections and each one was scored out of ten.

These were the scores each child got:

	Floor	Bar	Vault	Beam
Michelle	8.903	7.96	8.895	9.03
Rosie	9.1	7.991	7.98	9.004
Emily	7.567	7.909	9	9.091
Bridget	9.705	7.99	8.005	9.039

Prizes were awarded for the three highest scores in each category. Who got the first, second and third prize for the Floor? Bar? Vault? Beam?

Appendix F: Contextualised problems involving addition, subtraction and multiplication.

1 Problem

Helen went to watch four children compete in a gymnastics competition. They had to compete in four different sections and each one was scored out of ten.

These were the scores each child got

	Floor	Bar	Vault	Beam
Michelle	8.903	7.96	8.895	9.03
Rosie	9.1	7.991	7.98	9.004
Emily	7.567	7.909	9	9.091
Bridget	9.705	7.99	8.005	9.039

At the end of the competition all the scores were added up to get a winning total. What did Rosie get as her final total?

2 Problem

Helen went to watch four children compete in a team in a gymnastics competition. They had to compete in four different sections and each one was scored out of ten.

These were the scores each child got

	Floor	Bar	Vault	Beam
Michelle	9.2031	6.967	8.895	9.03
Rosie	9.1	7.991	7.98	9.004
Emily	7.567	7.909	9	9.091
Bridget	9.705	7.99	8.005	9.039

Michelle was pleased with her Floor result but disappointed with her Bar. What was the difference in the two results? First make a quick estimate of the difference and then work it out exactly.

3 Problem

If \$1 New Zealand exchanges for .8544 Australian, how much will you get for \$10 New Zealand?

4 Problem

The Herald advertises everyday the rate you can exchange New Zealand dollars for. If you were going to England and wanted to exchange your \$1000 New Zealand dollars to get the most pounds:

Which day would you exchange your money and why?

Which day would you really not want to exchange your money and why?

Monday	\$1 = £.27
Tuesday	\$1 = £.2699
Wednesday	\$1 = £.27003
Thursday	\$1 = £.269
Friday	\$1 = £.26799

5 Problem

The Herald advertises everyday the rate you can exchange New Zealand dollars for. If you were going to Samoa and wanted to exchange your \$1000 New Zealand dollars to get the most number of tala.

Which day would you exchange your money and why?

Which day would you really not want to exchange your money and why?

Monday	\$1 = 2.1936
Tuesday	\$1 = 2.190
Wednesday	\$1 = 2.19904
Thursday	\$1 = 2.109
Friday	\$1 = 2.10095
Saturday	\$1 = 2.19
Sunday	\$1 = 2.19361

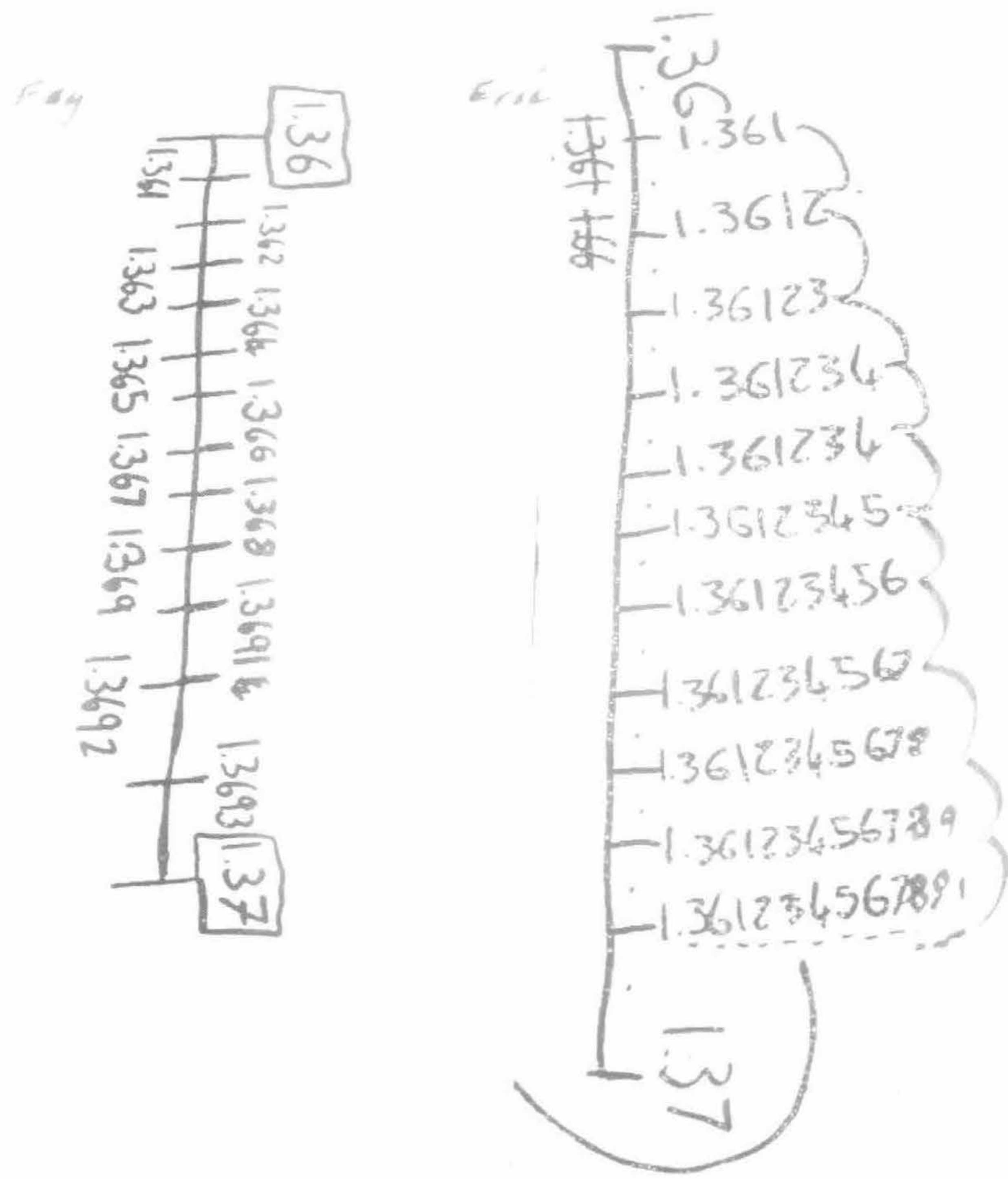
Appendix G: Additional interview questions

Estimate what the answers might reasonably be.

Explain what you think the answer is and how you worked it out.

- 12.5 - 5.75
- 5.07 - 1.3
- 5.07 + 1.3
- 10 x 0.5
- 0.12 divided by 10

Appendix H: The number lines drawn by Eric and Fay.



Appendix I: Information Sheet for Board of Trustees

Dear

As you know I have been a teacher at *(School name)* for the past 8 years and am to be on study leave for the next two terms to complete a thesis for a Master of Education at Massey University. My thesis is a qualitative study examining the knowledge and strategies children use in constructing decimal concepts.

(Teacher's name) has tentatively agreed to participate in a collaborative research role for teaching decimals. She will be formally approached pending B.O.T. acceptance. Permission to participate in the study will be sought from both the parents of the children and the children within the class. The consent will be twofold: consent for individual interviews, and consent if randomly selected as one of four students participating in a case study which tracks more closely how individual children construct decimal concepts as they engage in mathematical activity in the classroom.

Individual interviews will explore the child's current knowledge of decimals in much the same way as we conducted interviews in the classroom for the 'Advanced Numeracy Project' last year. Using the data from the interviews the teacher and I will plan a unit of six lessons and I will observe these as *(Teacher's name)* teaches them. I will focus on the case study children and the observations will involve the use of audio recording. From the first lesson series the data gathered will be analysed and will lead to a new teaching and learning cycle.

The time involved in the complete study for the teacher will be no more than fifty hours, over a period of one and half school terms. The time involved for each child's interview will be no more than 40 minutes. The teacher, the children, and their parents/caregivers will be given full information and consent will be requested in due course.

Information gathered from interviews and observations of case study children engaging in learning activities will be used to formulate conceptual frameworks that attempt to structure children's construction of decimal concepts. Data will be stored in a secure location, and used only for this research and any related publications. After the completion of the thesis, the information will be destroyed.

All efforts will be taken to maximise confidentiality and anonymity for all participants. Names of participants will not be used once information has been gathered and only non-identifying information will be used in reporting.

If you have any questions about this study you are welcome to phone on [redacted] or email me at bobbiehunter@clear.net.nz or contact either of my supervisors.

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Please note:

The Board of Trustees has the following rights:

- To decline the right of participation of a staff member or children from *(School name)* in the study
- To withdraw consent for this study at any time
- To ask questions about the study at any time
- To allow access on the understanding that the school will not be identified at any time
- To be provided with a summary of the findings at completion

Thank you very much for your support so far in supporting my application for study leave.

Bobbie Hunter.

Appendix J: Information Sheet for the Teacher.

Dear (*Teacher's name)*,

As you know I am to be on study leave for the next two terms to complete a thesis for a Master of Education at Massey University. My thesis is a qualitative study examining the knowledge and strategies children use in constructing decimal concepts.

Together we have discussed the problems inherent in children learning decimal concepts and the need for teachers to have cognisance of the conceptual framework individual children are using as part of the process of their construction of decimal concepts. Now I am formally inviting you to be a part of a collaborative research process in which we look at some of the ways children construct decimal concepts as they participate in mathematical activity in classroom.

Permission to participate in the study will be sought from both the parents of the children in your class and the children themselves. The consent will be twofold: consent for individual interviews, and consent if randomly selected as one of four students participating in a case study which tracks more closely how individual children construct decimal concepts as they engage in mathematical activity in the classroom.

Individual interviews will explore the child's current knowledge of decimals in much the same way as we conducted interviews in the classroom for the 'Advanced Numeracy Project' last year. Using the data from the interviews, we will plan a unit of six lessons, which you will teach and I will observe. I will focus on the case study child and the observations will involve the use of audio recording. From the first lesson series, the data will be analysed and will lead to a new teaching and learning cycle.

The time involved in the complete study for you will be no more than fifty hours, over a period of one and half school terms. During each cycle of the teaching and learning phase you will be asked to keep a diary to provide a retrospective account of classroom instruction but no evaluation of the instructional programme will occur other than that which is grounded in the context of the study. The time involved for each child's interview will be no more than 40 minutes. The children and their parents/caregivers will be given full information and consent will be requested in due course.

Information gathered from interviews and observations of case study children engaging in learning activities will be used to formulate conceptual frameworks that attempt to structure children's construction of a decimal schema. Data will be stored in a secure location, and used only for this research and any related publications. After the completion of the thesis, the information will be destroyed.

All efforts will be taken to maximise confidentiality and anonymity for all participants. Names of participants will not be used once information has been gathered and only non-identifying information will be used in reporting.

If you have any questions about this study you are welcome to discuss it with me personally, or phone me on 09 8460721 or 025 988 204, or email me at bobbiehunter@clear.net.nz, or contact either of my supervisors:

Dr Glenda Anthony, Massey University, Private Bag 11222, Palmerston North. Telephone: (06) 356 9099 extn 8600 Email: G.J.Anthony@massey.ac.nz

Dr Margaret Walshaw, Massey University, Private Bag 11222, Palmerston North. Telephone: (06) 356 9099 extn 8782 Email: m.a.walshaw@massey.ac.nz

Please note you have the following rights:

- To decline the right of participation in the study
- To withdraw consent for this study at any time

- To ask questions about the study at any time
- To allow access on the understanding that the school and yourself will not be identified at any time
- To be provided with a summary of the findings at completion
- To ask that the audio tape be turned off at any time

Thank you for your support for this study. I know that professionally we will both benefit through the knowledge we will gain about the conceptual framework children use in constructing a decimal fraction concepts and I am very grateful to you for your own desire to be involved as part of your professional development.

Bobbie Hunter

Appendix K: Information Sheet for Parents of Students

I have been a teacher at *(School name)* for the past 8 years and have been granted study leave for the next two terms to complete a thesis for a Master of Education at Massey University. My thesis is a qualitative study examining the knowledge and strategies children use in constructing decimal concepts.

The Board of Trustees has agreed that I may undertake this study. *(Teacher's name)* has agreed as a classroom teacher to participate in this study. Consent is now requested from both you and your child. The consent is twofold: consent for individual interviews, and consent to participate in a case study which tracks more closely how individual children construct decimal concepts as they engage in mathematical activity in the classroom. Four children will be randomly selected from those who provide consent for the case studies. All children (parents/caregivers) who consent to participate in the case studies will be informed of the outcome of this selection process in writing.

An individual interview will explore your child's current knowledge of decimals in much the same way as we conducted interviews in the classroom for the 'Advanced Numeracy Project' last year. Using the data from the interviews *(Teacher's name)* and I will plan a unit of six lessons. I will observe these as *(Teacher's name)* teaches them. I will focus on the case study child and the observations will involve the use of audio recording. From the first lesson series, the data gathered will be analysed and will lead to a new teaching and learning cycle.

The time involved for your child for the interview will be no more than 40 minutes. The interview with your child will be audio recorded and at any time your child can request that the tape recorder be turned off. All teaching and learning activities which involve case study participants will be audio recorded and at any time your child as either a case study or classroom participant can ask that the audio recorder be turned off and their comments deleted from the transcript.

Information gathered from interviews and observations of case study children engaging in learning activities will be used to formulate conceptual frameworks that attempt to structure children's construction of decimal concepts. Data will be stored in a secure location and used only for this research and related publications. After the completion of the thesis, the information will be destroyed.

All efforts will be taken to maximise your child's confidentiality and anonymity. Their name will not be used in this study, and only non-identifying information will be used in reporting.

If you have any questions about this study you are welcome to discuss it with me personally, or phone me on 09 8460721 or 025 988 204, or email me at bobbiehunter@clear.net.nz, or contact either of my supervisors:

Dr Glenda Anthony, Massey University, Private Bag 11222, Palmerston North. Telephone: (06) 356 9099 extn 8600 Email: G.J.Anthony@massey.ac.nz

Dr Margaret Walshaw, Massey University, Private Bag 11222, Palmerston North. Telephone: (06) 356 9099 extn 8782 Email: m.a.walshaw@massey.ac.nz

Your child will also be given full information and I ask that you discuss it fully with them before they give their consent to participate.

Should you agree to your child taking part in this study, you have the following rights:

- To decline to allow your child to participate in the study
- To withdraw your child from the study at any time
- To ask questions about the study at any time
- To allow your child to participate on the understanding that the school will not be identified at any time
- To be provided with a summary of the findings at completion

Appendix L: Information Sheet for Students

I am currently undertaking my thesis research for a Master of Education at Massey University. My research is a study of the ways in which children construct decimal concepts. It will also examine the strategies children use to solve decimal problems as they build decimal knowledge.

I would like to invite you with your parent's permission to be involved in this study. (*Teacher's name)* has also agreed to participate in the study. The Board of Trustees has also given their approval for me to invite you to participate, and for me to undertake this research.

Your involvement in the study will include being interviewed individually about your current decimal knowledge much the same as the interviews you participated in last year in the 'Advanced Numeracy Project.' (*Teacher's name)* and I will plan a unit of mathematics based on what we have learnt about how you think about decimals. (*Teacher's name)* will teach the mathematics lessons and I will observe and these lessons will be audio recorded. Four children will be randomly selected to be observed more closely and you will be informed in writing if this involves you or not.

The interview will be tape-recorded and you may at any time ask that the tape recorder be turned off and that any comments you have made deleted. During classroom mathematics activities you may at any time ask that the audio recorder be turned off and any comments you have made deleted. If you are one of the four children you will also be asked for copies of your mathematics reflections, written work and charts you make to support your explanations to the group during the decimal unit. You have the right to refuse to allow the copies to be taken.

Taking part in this research will not in any way affect your learning, but rather may help you clarify what you know about decimals and what you need to know next. The interview and observations will take place in the classroom and be part of the normal mathematics programme.

All the information gathered will be stored in a secure location and used for this research and any related publications. After the completion of the research the information will be destroyed. All efforts will be taken to maximise your confidentiality and anonymity which means that your name will not be used in this study and only non-identifying information will be used in reporting.

If you have any questions about this study you are welcome to discuss it with me personally, or phone me on 09 8460721 or 025 988 204, or email me at bobbiehunter@clear.net.nz, or contact either of my supervisors:
Dr Glenda Anthony, Massey University, Private Bag 11222, Palmerston North. Telephone: (06) 356 9099 extn 8600 Email: G.J.Anthony@massey.ac.nz
Dr Margaret Walshaw, Massey University, Private Bag 11222, Palmerston North. Telephone: (06) 356 9099 extn 8782 Email: m.a.walshaw@massey.ac.nz

I ask that you discuss all the information in this letter fully with your parents before you give your consent to participate.

Please note that you have the following rights:

- To say you do not want to participate in the study
- To withdraw from the study at any time
- To ask for the audio recorder to be turned off and any comments you have made be deleted
- To refuse to allow copies of your written work to be taken
- To ask questions about the study at any time
- To participate knowing that you will not be identified at any time
- To be given a summary of what is found at the end of the study