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CONTACT SYSTEMS AND CONTACT INTEGRATORS

A Thesis Presented in Partial Fulfilment of the
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Seung-Hee Joo
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Abstract

This thesis is concerned with the study of contact systems, which are ordinary differential equations whose flow preserves a contact structure. We study contact systems from both an analytical and numerical point of view. The traditional point of view is to study the Reeb vector field of a contact form. However, if the contact Hamiltonian vanishes then its contact vector field is not the Reeb vector field of any contact form equivalent to the given one. In this thesis we study exactly this case, when the contact Hamiltonian vanishes on some submanifold of phase space. This submanifold is invariant under the flow and we study the flow on it, including its stability and fixed points.

The natural numerical method for a contact system is a ‘contact integrator’, a map that preserves the contact structure, which is suitable for exploring the long-time dynamics of contact systems. These have not been studied very much in geometric integration. In order to formulate our results and some consequences for contact integrators, we give a thorough development of the symplectification of a contact system and have found the integrable contact systems related to integrable homogeneous Hamiltonian systems via symplectification. We develop contact integrators by the splitting method, leading to an explicit contact integrator for any polynomial contact vector field. We also study how symplectic integrators for Hamiltonian systems and volume-preserving integrators for divergence-free systems are related to contact integrators for contact systems.

Acknowledgements

Life is filled with journeys. In my life, the longest journey was my PhD study. I commenced PhD study (Bezout's Theorem in Commutative Algebra and Algebraic Geometry) with Professor Wolfgang Vogel on June 1995. However, the sudden death of Professor Vogel during October 1996 changed my research topic. My research was suspended for 2 years, due to illness and to take care of my son. I was teaching at a NZ high School, Professor Robert McLachlan encouraged me to recommence my PhD study in Geometric Integration. I started work on this thesis in 2000. Before I could even organize this thesis, my father died, and I could not see him at the end due to the research.

I am happy to have achieved this milestone in my life. Even though it has brought me both joy and sorrow, it has become a part of my life.

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Nomenclature

We list below standard notation of Differential Geometry (see, for instance, [66, 70]) used in this thesis. Entries are generally arranged in order of occurrence in the thesis.

C^∞	the set of infinitely differentiable functions
\wedge	exterior product
T_x^*M	the set of 1-forms at the point x of a manifold M , dual to T_xM
T_xM	the set of tangent vectors at the point x of a manifold M
T^*M	the cotangent bundle of M
TM	the tangent bundle of M , dual to T^*M
d	exterior derivative
f_*	the push-forward map derived from f
f^*	the pull-back map derived from f
dx	the gradient of the x coordinate function
$L_X\omega$	the Lie derivative of ω by the vector field X , ω is a symplectic form
$i_X\omega$	the interior product of the differential form ω by the vector field X
$\mathcal{X}(M)$	the set of vector fields on M
S^n	the surface of the sphere in $(n + 1)$ -dimensional Euclidean space
π	a projection map
$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$	a basis for the tangent space of \mathbb{R}^3
$[X, Y]$	the Lie bracket of the vector fields X and Y
$\{F, G\}$	the Poisson bracket of the functions F and G
$[f, g]$	the Jacobi bracket of the functions f and g
div	the divergence operator, as in $\text{div}_\Omega X$

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