COMBINATORIAL MAPS AND THE FOUNDATIONS OF TOPOLOGICAL GRAPH THEORY

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ABSTRACT

This work develops the foundations of topological graph theory with a unified approach using combinatorial maps. (A combinatorial map is an n-regular graph endowed with proper edge colouring in n colours.) We establish some new results and some generalisations of important theorems in topological graph theory. The classification of surfaces, the imbedding distribution of a graph, the maximum genus of a graph, and MacLane's test for graph planarity are given new treatments in terms of cubic combinatorial maps. Among our new results, we give combinatorial versions of the classical theorem of topology which states that the first Betti number of a surface is the maximum number of closed curves along which one can cut without dividing the surface up into two or more components. To conclude this thesis, we provide an introduction to the algebraic properties of combinatorial maps. The homology spaces and Euler characteristic are defined, and we show how they are related.

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PREFACE

Topological graph theory is concerned with the study of graphs imbedded in surfaces. During the past two decades, graph imbeddings in surfaces have received considerable analysis by combinatorial methods. The concept that motivated this thesis is the use of a special kind of edge coloured graph, called by Lins [11] a gem, to provide such a method to model graph imbeddings. For the most part, we shall push Lins' model further by using more general edge coloured graphs, called cubic combinatorial maps, to establish some new results and some generalisations of important theorems in topological graph theory. (A cubic combinatorial map is defined as a cubic graph endowed with a proper edge colouring in three colours.) It is the use of combinatorial maps that is the unifying feature in this thesis and its development of the foundations of topological graph theory.

An advantage of this approach over previous attempts to combinatorialise topological graph theory is that the theorems can be easily visualized, encouraging geometric intuition. We demonstrate how this axiomatic non-topological definition of a graph imbedding means that no topological apparatus needs to be brought into play when proving theorems in topological graph theory.

Following a chapter of introductory material, Chapter II gives a simple graph theoretic proof of the classification of surfaces in terms of cubic combinatorial maps. This provides our first example of the naturalness of cubic combinatorial maps as a variation on the simplicial complex approach to topology. As in topology, we can now assign an orientability character and genus or cross cap number to a given cubic combinatorial map. This chapter also serves as an introduction to the special operation or "move" on combinatorial maps that permeates this thesis.

In [24], Stahl presents a purely combinatorial form of the Jordan curve theorem from which graph theoretical versions (for example [26]) follow as corollaries. This was later (in [13]) presented in terms of cubic combinatorial maps. Generalisations of the Jordan curve theorem abound in topology, and therefore we make progress along these lines in Chapter III by presenting a generalisation of Stahl's work, motivated by the work in [13]. We give a combinatorial version of the theorem of topology which states that the first Betti number of a surface is the maximum number of closed curves along which one can cut without dividing the surface up into two or more components.

No text on the foundations of topological graph theory would be complete without some study of the set of surfaces a given graph can be imbedded on, or more precisely the imbedding distribution of a graph. The principal objective of topological graph theory is to determine the surface of smallest genus such that a given graph imbeds in that surface. In general, this surface is difficult to find. By way of contrast, the surface of largest genus such that a given graph imbeds in that surface can be found. We define a special partition of the set of all cubic combinatorial maps, and we say that two cubic combinatorial maps that belong to the same cell are congruent. In particular, two congruent gems correspond to two possible imbeddings for a given graph. We analyse the distribution of the genus or cross cap numbers associated with the cubic combinatorial maps congruent to a given one in Chapter IV. In Chapter V, we calculate the maximum value in this distribution. This work generalises results of Khomenko [9, 10] and Xuong [32].

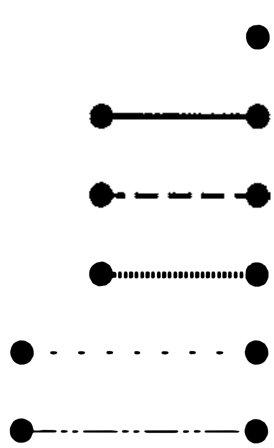
In [30], short proofs of three graph theoretic versions of the Jordan curve theorem are given. In the spirit of Chapter III, we generalise the version, expressed in terms of a double cover for a graph, in Chapter VI. (A double cover is a family of circuits such that each edge belongs to exactly two.). Furthermore, we show how this work is related to our work on cubic combinatorial maps in Chapter III, and hence we proceed in the direction of Little and Vince in [14].

In an attempt to make a partial separation between graph theory and topology, MacLane proved that a given graph would be imbeddable on the sphere if and only if it had a certain combinatorial property. However, his characterization was proved by topological arguments. The tools introduced in Chapter VI which relate cubic combinatorial maps to double covered graphs are further applied in Chapter VII. Here we classify which cubic combinatorial maps are congruent to planar ones, where planarity is defined in terms of orientability and Euler characteristic. The classification given is a combinatorial generalisation of MacLane's test for planarity.

A more general version of the cubic combinatorial map is found by dropping the restriction of cubic graphs so as to include n-regular graphs. Of course we increase the number of colours for the edge colouring to n. To conclude this thesis, we provide an introduction to the algebraic properties of such maps. The homology spaces and Euler characteristic are defined, and we show how they are related. Furthermore, a general form of the "move" that permeates this thesis is presented, and we show how this move affects the Euler characteristic.

NOTES ON FIGURES

This thesis is mainly concerned with edge coloured graphs. Unfortunately, colour was not achieveable on laser printers at our disposal. It is possible, using the postscript language, to dash curved lines and to vary the width of a line. Therefore we represent the various colours by dashing edges according to the following figure.



vertex

a blue edge

a red edge

a yellow edge

a path of red and yellow edges

a path of blue and yellow edges

For labellings, we will usually use a, b or c, together with a subscript or a prime, to label *red*, *blue* and *yellow* edges respectively. A vertex will always be labelled with u, v, w, x, y or z, together with a subscript or a prime.

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To the University Grants Committee, for financial assistance.

DEDICATION

To

my wife, Karyn,

my Mum and Dad,

and my brother, Alex.

THIS THESIS WAS TYPESET ON APPLE LASERWRITER IINT PRINTERS IN TIMES 12/24, WITH HEADINGS IN COPPERPLATE AND PALATINO. DOCUMENT PREPARATION WAS ON APPLE MACINTOSH.COMPUTERS, USING MICROSOFT WORD, DESIGN SCIENCE MATH TYPE, ALDUS FREEHAND, AND CLARIS CAD. CROSS REFERENCING AND BIBLIOGRAPHY PREPARATION WERE ACHIEVED WITH CLARIS FILEMAKER PRO AND WORD REF.