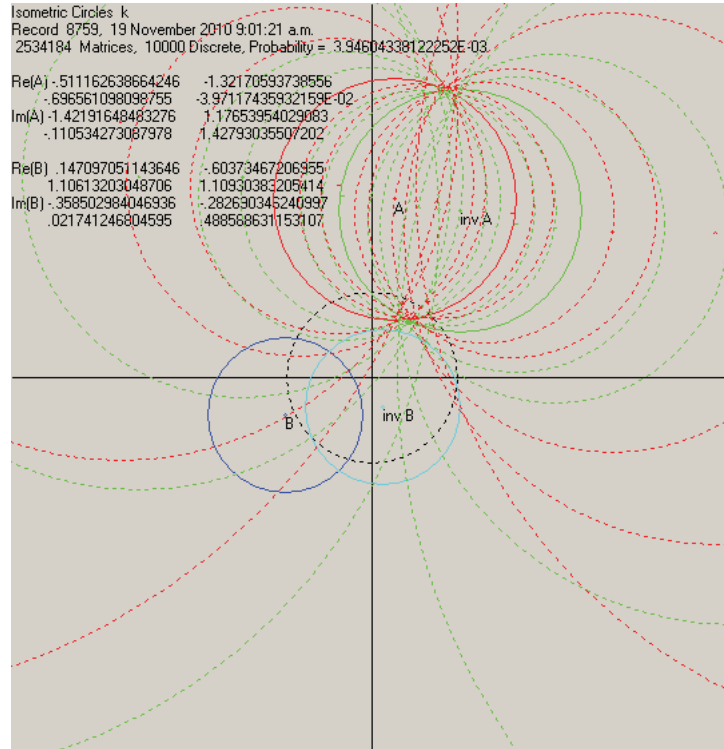


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# Random Discrete Groups in the Space of Möbius Transformations



A Thesis presented in partial fulfillment of the requirements for the degree of  
Master of Science  
in  
Mathematics  
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Graeme K O'Brien  
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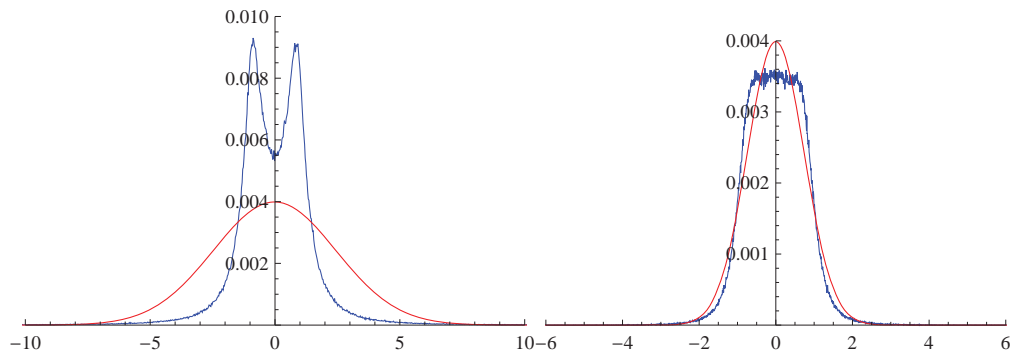
"I do not quite understand you," I said, with an uneasy foreboding as to what she meant...

"Surely a man must do a day's work first!"

I gazed in the white face of the woman, and my heart fluttered. She returned my gaze in silence.

"Let me first go home," I resumed, "and come again after I have found or made, invented, or at least discovered something!"

- George MacDonald

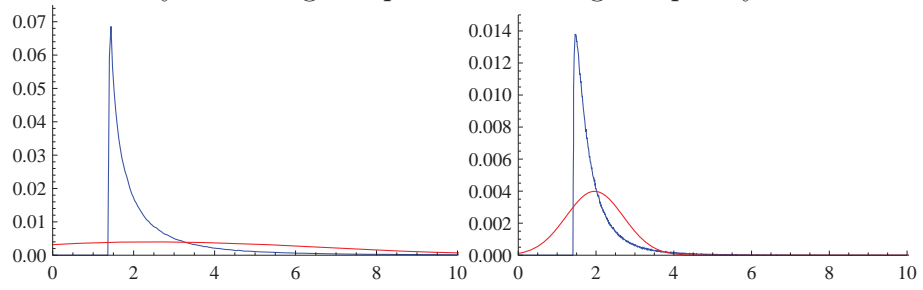


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Distinguished Professor Gaven Martin.

“If you don’t give up then I won’t give up on you”.



## ABSTRACT

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Graeme K O'Brien

Discrete subgroups of random Möbius transformations are investigated using computational methods together with collateral mathematical analysis. The main results include quantification of the likelihood of occurrence of two generator discrete groups and studies of the sharpness of the Hadamard inequality for random matrices and of the scale invariance for the domain of definition for matrix entry distributions derived by normalisation of matrices in  $GL(2, \mathbb{C})$  to  $SL(2, \mathbb{C})$ .

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