Nonholonomic Dynamical Systems

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Abstract

The dynamics of mechanical systems subject to nonholonomic (i.e. non-integrable velocity) constraints is only poorly understood. It is known that (i) they preserve energy and, (ii) they are reversible. In this thesis I explore the conjecture that (i) and (ii) are the *only* general features of the entire class. The discovery of dissipative orbits, ones that behave differently as $t \to +\infty$ and $t \to -\infty$, would strongly support this conjecture.

This dissipation can appear in various forms, e.g. sinks (attractors) or sources (repellers) in the phase space, but in every form the dynamics have the property that the forwards time orbit occupies a different region of the phase space than the reverse time orbit.

In nonholonomic dynamical systems that are reversible and possess an integral, theory predicts that near the fixed set of a reversing symmetry, e.g. $R : p \mapsto -p$ with fixed set $\mathrm{Fix}(R) = \{(q,p) : p = 0\}$, no dissipation can occur. If the system can be integrated analytically, then all the orbits are quasi-periodic and even away from the fixed set of any reversing symmetries, dissipation cannot occur. But, if the system cannot be integrated analytically, then away from the fixed set of any reversing symmetries, dissipative orbits can exist.

The minimum dimension needed for a nonholonomic system is 6. So, in this thesis I study the simplest class of nonholonomic dynamical systems that are reversible with an integral, namely the contact particle in $\mathbb{R}^3$. I search for evidence of dissipative behaviour in this class of systems by taking a known contact particle system that can be integrated analytically, such as the harmonic oscillator, where no dissipation can occur and calculating (numerically and analytically) the dynamics of its orbits. Then I perturb the system so that it cannot be integrated analytically and search for orbits that exhibit the dissipative behaviour described above away from the fixed set of the reversing symmetries of the system.

To achieve this I implemented a semi-explicit reversible integrator in C to integrate the system forwards (or backwards when desired) in time from an initial point. The C code interacts with MATLAB via the "mex" interface to make use of MATLAB's graphing facilities, which I used to plot the forwards and backwards orbits in blue and red respectively. This allows the orbits to be observed and any dissipative behaviour should become immediately apparent as the orbits will cover different portions of the phase space if dissipation occurs. The phase space of the system is actually $\mathbb{R}^6$, which is beyond my capabilities to visualise, but it can be reduced to $\mathbb{R}^3$, as I have done, through the use of the integral, the nonholonomic constraint and a Poincaré section.

Acknowledgements

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Chapter 1

Introduction

1.1 Overview

In this thesis I am studying a class of simple mechanical systems subject to a single nonholonomic constraint.

A simple mechanical system is a dynamical system that has a Hamiltonian

$$H(q,p) = \frac{1}{2}p^T\mu(q)^{-1}p + V(q)$$

where $\mu(q)$ is a mass matrix and $V(q)$ is a potential. The equations of motion are given by

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

A nonholonomic dynamical system is a dynamical system which has a non-integrable constraint, $A(q)p = 0$, on the velocities. A simple mechanical system subject to a single nonholonomic constraint, with $\mu(q) = Id$, can be written as

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -\nabla V(q) + A^T(q)\lambda \\ A(q)p &= 0. \end{aligned} \tag{1.1}$$

One property of simple mechanical systems is that the energy, $H(q,p)$, is preserved. This means that for a given initial point, $(q(0), p(0))$, in the system, with energy $H(q(0), p(0)) = h_0$, any point in the phase space that can be reached by integrating forwards, or backwards, in time from that initial point, must have the same energy, $H(q,p) = h_0$.

Another feature of simple mechanical systems is that they are reversible with respect to the reversing symmetry $R : (q,p) \mapsto (q,-p)$. So, if we apply R to the unconstrained system above, we get

$$\begin{aligned} \dot{q} &= -p \\ -\dot{p} &= -\nabla V(q) + A^T(q)\lambda, \end{aligned}$$

then, applying $t \mapsto -t$ returns us to the original system.

The dynamics of mechanical systems subject to nonholonomic constraints is poorly understood. In particular, it is not known whether or not these nonholonomic systems are Hamiltonian, or volume preserving. In this thesis I explore the conjecture that in the class of simple mechanical systems subject to nonholonomic constraints, the *only* general features of the dynamics is that they preserve the energy and are reversible.

The theory of nonholonomic dynamical systems predicts that for a system as described above, near the fixed set of a reversing symmetry the path of almost any point in the phase space is such that it will return to a neighbourhood of the initial point infinitely many times [32]. If the system can be integrated analytically, then all the orbits are quasi-periodic and even away from the fixed set of any reversing symmetries, the above holds. But, if the system cannot be integrated analytically, then away from the fixed set of any symmetries, this condition no longer applies and dissipation can occur. That is, the path of a point in phase space as we integrate forwards in time can occupy one region of the phase space while the path of the same point can occupy a different region of the phase space when we integrate backwards in time.

The discovery of dissipative behaviour in a nonholonomic dynamical system would strongly support the above conjecture. It can be shown that the lowest dimension needed for a nonholonomic dynamical system is 6. Thus the simplest class of nonholonomic dynamical systems is the "contact particle" in $\mathbb{R}^3$, defined by the Hamiltonian, $H(q,p) = \frac{1}{2}\|p\|^2 + V(q)$, and the constraint $p_1 + q_2 p_3 = 0$. It is in this class of systems that I am searching for evidence of dissipative behaviour.

1.2 History and Background

The word 'holonomic' is due to Hertz and means 'universal' or 'integrable', literally it translates as '$\acute{o}\lambda o\varsigma$' = 'whole', '$\nu o\mu o\varsigma$' = 'law' [36]. Thus the term 'nonholonomic' is synonymous with 'non-integrable'. The theory of nonholonomic systems is the subject of many papers, some of which date back to the turning of the last century, but the term 'nonholonomic' is scarcely even mentioned in most texts.

Nonholonomic variational problems have much in common with optimal control problems and occur in thermodynamics and quantum theory. They are also closely connected with the general theory of partial differential equations.

The beginnings of nonholonomic theory can be traced back to Lagrange in 1788, with his *Méchanique Analytique* [22], in which the equations of unconstrained motion are written in Euler-Lagrange form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{\partial L}{\partial q^i},$$

in section V. He used the notation $Z = T - V$ for his Lagrangian and concluded that a coordinate invariant expression for *mass* $\times$ *acceleration* is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial v}\right) - \frac{\partial T}{\partial q}.$$

Curiously [26], Lagrange does not recognise the equations of motion as being equivalent to the variational principle

$$\delta \int L\mathrm{d}t = 0,$$

despite knowing the general form of the differential equations for variational problems; he had actually commented on Euler's proof of them. This was only recognised a few decades later by Hamilton.

Although Euler had already treated nonholonomic constraints [17], they were not clearly understood until the turn of last century. Foremost in clarifying the features of nonholonomic mechanical systems was Hertz with his *Prinzipien der Mechanik* in 1894. Gibbs and Carathéodory also dealt with contact structures in the formulation of thermodynamics. As for pure mathematics, the study of nonholonomic systems began with the theory of Pfaffian systems and the subsequent work on the general theory of differential equations. E. Cartan introduced the powerful tools of differential forms and codistributions but, unfortunately, these were not widely used in nonholonomic problems.

In the 1920's Levi-Civita and H. Weyl defined the notions of Riemannian and affine connections and discovered deep relations between mechanics and geometry. Nonholonomic mechanics served as a source of new geometrical structures which provided mechanics and physics with a convenient and concise language. This mutual interaction was started in the pre-war years by Vranceanu and Synge. In 1931, Vranceanu [37] gave the first precise definition of a nonholonomic structure on a Riemannian manifold in two short notes and an article, outlining its relation to dynamics of nonholonomic systems. Synge [34, 35] studied the stability of free motion of nonholonomic systems and anticipated the notion of curvature of a manifold.

According to Vershik and Gershkovich [36], in the post-war years the research on nonholonomic systems waned due to the vagueness of how the papers were written, giving vast differences in the notations and coordinates used, resulting in large, mostly incomprehensible texts. This was emphasised in 1948 when V. Vagner wrote: "The lack of rigour which is typical for differential geometry is reflected also in the absence of precise definitions of such notions as spaces, multi-dimensional surfaces, etc. Differential geometry is certainly dropping behind and this became even more dangerous when it lost its direct contact with theoretical physics" [36]. In the 1950's and 1960's nonholonomic theory was almost completely left alone.

In 1975, Vershik and Faddeev produced a paper in which nonholonomic mechanics was exposed systematically in terms of differential geometry. With the introduction of a more consistent notation, there came a renewed interest in nonholonomic theory.

More recently, the amount of interest in nonholonomic theory has increased significantly, with several papers in the late 1990's on symmetries, reduction and the application of nonholonomic theory to molecular dynamics and control theory, as I now briefly survey.

In a paper in 1995, Sarlet, Cantrijn and Saunders [33] discuss the concepts of symmetries and adjoint symmetries for Lagrangian systems with nonholonomic constraints. The following year Bloch, Krishnaprasad, Marsden and Murray [7] developed the geometry and dynamics of mechanical systems with nonholonomic constraints and symmetries from the perspective of Lagrangian mechanics with a view to control-theoretical applications. In particular they derived the evolution equation for momentum and distinguished geometrically and mechanically between the cases where it is conserved and where it is not. Also in 1996, Cardin and Favretti [14] derived the vakonomic equations (equations of motion of nonholonomic systems, originally derived by Arnold, Kozlov and Neishtadt [2] in 1988) from a nonholonomic variational problem and gave a geometrical interpretation of the terms.

Koon and Marsden [20] wrote a paper, in 1998, comparing the Hamiltonian and Lagrangian formulations of nonholonomic systems. They further developed the theory of Poisson reduction, which is important in stability theory for nonholonomic systems and tied it to other work in the area. In another paper in the same journal, Cantrijn, de Leon, Marrero and de Diego [13] presented a geometric reduction procedure for Lagrangian systems. In 1999, Cortes and de Leon [15] developed a reduction scheme in terms of the nonholonomic momentum mapping.

Earlier in 1999, Cantrijn, de Leon and de Diego [12] unified several approaches to the 'almost-Poisson' bracket for mechanical systems with nonholonomic constraints and used the almost-Poisson structure to describe phase-space dynamics of a nonholonomic system. In a paper on molecular dynamics simulations, Kutteh [21] described three algorithms for imposing nonholonomic constraints with any number of additional holonomic constraints.

The theory of nonholonomic systems continues to develop, with strong links to other fields, such as molecular dynamics, thermodynamics, control theory and quantum mechanics, and is an increasingly popular field of research.

1.3 Definitions

Throughout this thesis I will be discussing simple mechanical dynamical systems subject to nonholonomic constraints, so a few definitions are in order.

Definition 1 (Dynamical System) *A* continuous time dynamical system *is a system of ODEs*

$$\dot{x} = F(x)$$

where $F(x)$ is a vector field on a manifold M, x is a point of the manifold and $\dot{x}$ denotes the time derivative of x.

For a finite dimensional manifold the system is representable by a set of first order ODEs, giving us a finite dimensional dynamical system. The solutions to the dynamical system are integral curves of F, which constitute a flow on the underlying manifold.

Definition 2 (Mapping) *A* mapping *is a diffeomorphism $f : M \mapsto M$ and is the discrete time analog of the flow of a vector field. Typically mappings are written as*

$$x_{i+1} = f(x_i) \qquad or \qquad x' = f(x).$$

Many ODEs can be reduced to a mapping via the *Poincaré* (or *first return*) *map*, which can be defined as follows:

Definition 3 (Poincaré Map) *Consider a dynamical system, $\dot{x} = F(x)$, in a manifold, M. Then a* Poincaré section, Σ, *can be defined as follows:*

Let $\Sigma \subset M$ be of co-dimension 1 such that F is transverse to Σ, i.e. $F(x) \notin T_x\Sigma$. Then Σ is a Poincaré section if, for all $x \in \Sigma$, there exists a minimum $t = \tau > 0$ such that $\phi_\tau(x) \in \Sigma$ and $\phi_\tau(x)$ crosses Σ in the correct sense.

From the Poincaré section we can define the Poincaré map, $\phi : \Sigma \mapsto \Sigma$, *as*

$$\phi(x) := \phi_\tau(x).$$

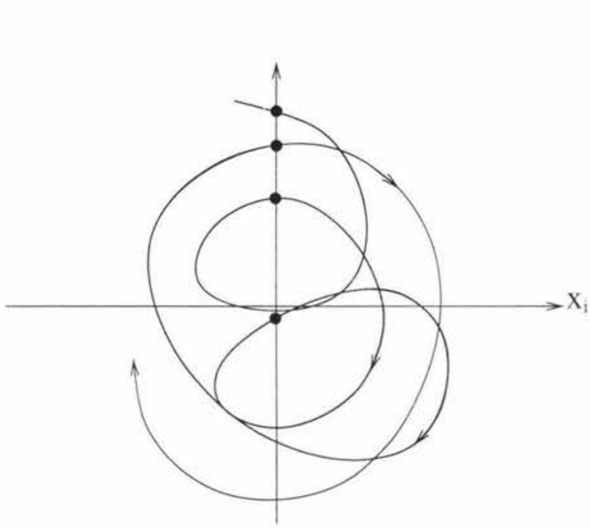

Figure 1.1: An example of a Poincaré map. The dots indicate where the trajectory crosses $x_i = 0$ in the positive direction.

For example, on $M = \mathbb{R}^n$,

$$\Sigma = \{x : x_i = 0, \dot{x}_i > 0\}$$

is a Poincaré section if all points in Σ eventually return to Σ.

Definition 4 (Orbit) *For the system $\dot{x} = F(x)$ with flow $x(t) = \phi_t(x(0))$, an* orbit *(or* trajectory*) $\mathcal{O}(x)$ passing through a point x is the set of points in the phase space reachable from x by flowing (or mapping in the case of maps) forwards or backwards in time.*

i.e.

$$\mathcal{O}(x) = \{\phi_t(x) : t \in \mathbb{R}\} \qquad or \qquad \mathcal{O}(x) = \{f_i(x) : i \in \mathbb{Z}\}.$$

If an orbit passing through a point x^* returns to that point after a time t^*, then the orbit is called *periodic* of period t^*. If the orbit can be written as

$$\mathcal{O}(x) = g(\omega_1 t, \ldots, \omega_k t) \text{ with } \frac{\omega_i}{\omega_j} \text{ irrational for } i \neq j$$

and g being 2π-periodic in each argument, then $\mathcal{O}(x)$ is said to be *quasi-periodic* and the phase space covered by the orbit, $g : \mathbb{T}^k \mapsto M$, is a *k-torus.*

Definition 5 (Fixed Point) *A* fixed point x^* *of a system is an orbit which remains at* x^* *for all time* t.

In the case of maps, if a point x^* in an orbit returns to x^* after q steps, where $q \geq 1$ is the smallest such integer, then the point x^* is called a *periodic point* of *period* q and the orbit $\mathcal{O}(x) = \{x^*, f(x^*), \ldots, f_{q-1}(x^*)\}$ is called a *q-cycle*. So a fixed point in a map is really a periodic point of period 1.

Definition 6 (Fixed Set) *Let* R *be a map* $R : M \mapsto M$. *Then the* fixed set *of* R *is*

$$\text{Fix}(R) = \{x : R(x) = x\}.$$

That is, the set of points in the phase space that remain where they are when R *is applied.*

Definition 7 (Hamiltonian) *A* Hamiltonian *system is one in which there exists a twice continuously differentiable function* $H(q, p)$, *such that the system can be written as*

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, \ldots, m$$

Definition 8 (non-Hamiltonian) *Any system that is not Hamiltonian.*

Definition 9 (Simple Mechanical) *A* simple mechanical *system is a Hamiltonian system in which the Hamiltonian is of the form*

$$H(q, p) = \frac{1}{2} p^T \mu(q)^{-1} p + V(q)$$

where $\mu(q)$ *is a symmetric mass matrix and* $V(q)$ *is a potential.*

Definition 10 (Volume Preserving) *A system is said to be* volume preserving *if there exists a volume element* $m(x)\mathrm{d}^n x$ *such that*

$$\det(\mathrm{D}\phi(x)) = \frac{m(x)}{m(x')}$$

for maps, or

$$\nabla \cdot (mF) = 0$$

for flows.

For a fixed point, $x' = x$, of a map the eigenvalues of $\mathrm{D}\phi(x)$ have product 1.

Definition 11 (Reversible) *A* reversible *system is a dynamical system with a reversing symmetry $R : M \mapsto M$ that reverses the direction of time. That is, if*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x) \qquad \textit{for all } x \in M$$

then

$$\frac{\mathrm{d}(R(x))}{\mathrm{d}t} = -F(R(x)),$$

i.e.

$$\mathrm{T}R.F(x) = -F(R(x)).$$

For example, consider the Hamiltonian

$$H(q,p) = \frac{1}{2}\|p\|^2 + V(q)$$

with the reversing symmetry

$$R : (q,p) \mapsto (q, -p).$$

The Hamiltonian describes the system

$$\begin{aligned} \dot{q} &= p, \\ \dot{p} &= -\nabla V(q). \end{aligned}$$

Applying R we get $p \mapsto -p$ giving the system

$$\begin{aligned} \dot{q} &= -p, \\ \dot{p} &= \nabla V(q). \end{aligned}$$

Then letting $t \mapsto -t$ we regain the original system, as can be seen in Figure 1.2. The reversing symmetry together with the reversal of time leaves the equations invariant.

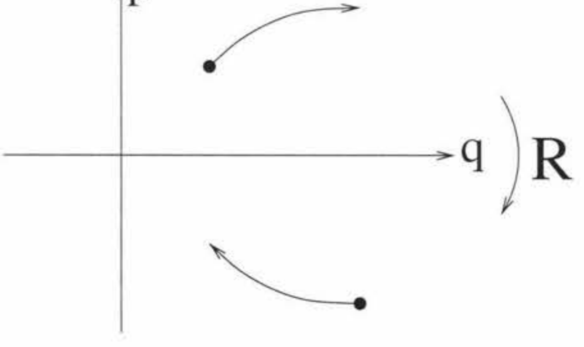

Figure 1.2: A point (q, p) being mapped forwards in time is equivalent to R being applied, that point being mapped backwards in time and then R being applied again.

Definition 12 (Nonholonomic) *[27] A* nonholonomic *system is a dynamical system with non-integrable constraints on the velocities. That is, for an n-dimensional nonholonomic system with $n-k$ constraints, the constraints*

$$\sum_{j=1}^{n} A_{ij}(q)p_j = 0, \qquad i = 1, \ldots, n-k$$

cannot be put into the form

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial q_j} p_j = 0$$

for some functions $f_1, \ldots, f_{n-k}$.

Examples of nonholonomic systems are the rolling penny, and the rattleback (see section 3.4), which both have a no-slip rolling constraint.

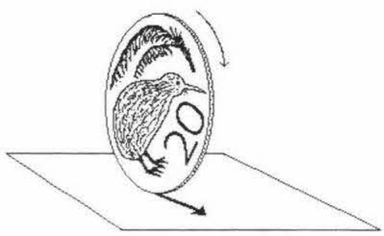

Figure 1.3: Rolling penny: the coin can only move in the direction it is rolling.

Figure 1.4 shows how dynamical systems can be classified into reversible mechanical systems, unconstrained mechanical systems and constrained mechanical systems. The constrained, reversible mechanical systems with nonholonomic constraints that I am discussing in this thesis are indicated.

1.4 Outline

As mentioned before, the discovery of dissipative behaviour in a non-integrable nonholonomic dynamical system would strongly support the conjecture that the only general features of the dynamics of the class of mechanical systems subject to nonholonomic constraints are that they preserve the energy and are reversible.

To find evidence of dissipative behaviour I am beginning with an unperturbed, simple Hamiltonian, which gives rise to an integrable, simple mechanical system and can be solved analytically. Then I compute the orbits of the system and find the period 2 points, as the amount of dissipation is easier to measure at fixed points, although no dissipation can occur yet. The code that I have written is a semi-explicit reversible integrator in C that interfaces with MATLAB via the "mex" interface to use MATLAB's graphing facilities. My code also

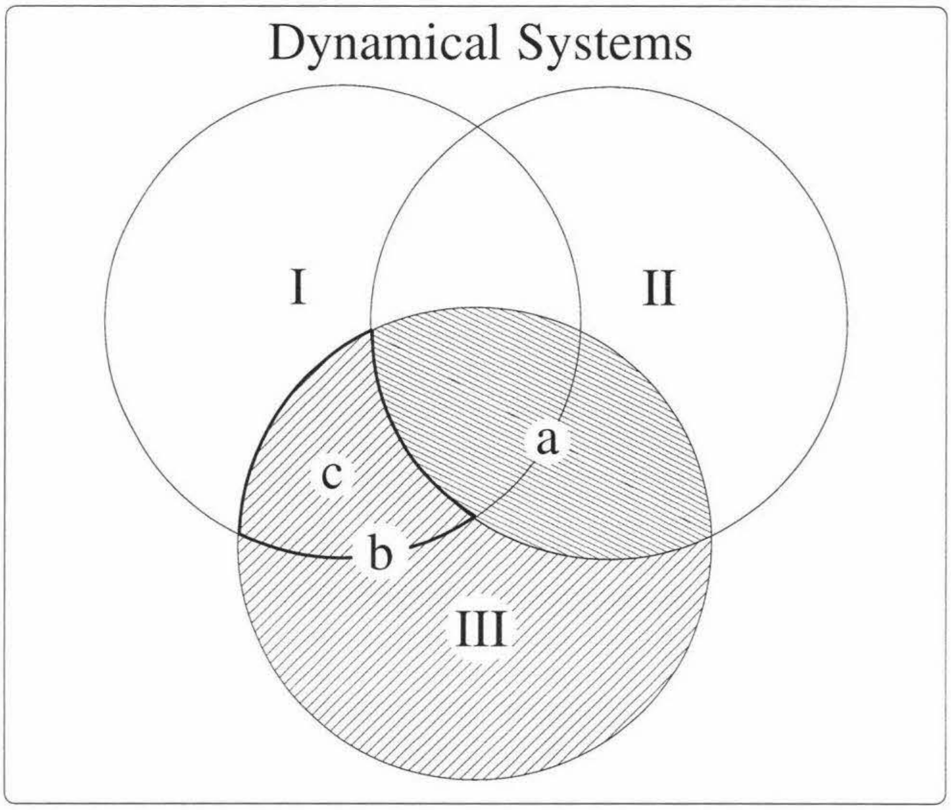

Figure 1.4: Dynamical systems.

I Reversible systems, $\frac{\mathrm{d}(R(x))}{\mathrm{d}t} = -F(R(x))$.

II Unconstrained mechanical systems. These are the systems defined purely by Euler-Lagrange / Hamiltonian equations of motion.

III Constrained mechanical systems.

a) The constraints are integrable, the system is said to be holonomic. It can be shown that the dynamics is equivalent to an unconstrained system on a lower dimensional manifold.

b) The constraints are non-integrable, the system is said to be nonholonomic.

c) The constrained, reversible mechanical systems with nonholonomic constraints that I am discussing in this thesis are contained in this region.

reduces the six dimensional system via the Hamiltonian integral, the nonholonomic constraint and a Poincaré section to three dimensions allowing the orbits to be plotted and visualised in MATLAB. The forwards time orbits are plotted in different colours to the reverse time orbits to make it immediately obvious when an orbit differs in forwards and backwards time.

Next, I perturb the system to make it non-integrable allowing for dissipation and I search for orbits that differ in forwards and backwards time. I also look for the period 2 points that survive the perturbation and measure the eigenvalues of the linearised coefficient matrix, which gives the amount of expansion in each direction at the periodic point and describes the amount of dissipation. Fixed points are not the only places where dissipation can occur, orbits such as invariant tori and some chaotic orbits can show dissipation but measuring it involves finding the eigenvalues of the Jacobian matrix of the map, which has to be calculated at each step and is more expensive in processor time.

The content of the remainder of this thesis is as follows. In chapter 2 I will discuss the history and properties of reversible dynamical systems, including Hamiltonian systems, maps and the linearisation and stability of fixed points.

Chapter 3 deals with the principle of nonholonomic mechanics, which is based on the Lagrange-d'Alembert principle and the associated Lagrange-d'Alembert equations. I will also cover some examples of nonholonomic dynamical systems, namely the simple, integrable, rolling penny, the complicated, high dimensional, rattleback and the contact particle with a spherically symmetric potential.

I will explain, in detail, how I went about searching for dissipative behaviour in the contact particle class of nonholonomic systems in chapter 4. This will involve the methods that I used to integrate the systems and my strategies for locating dissipative orbits, as well as the systems themselves and their perturbations. Also covered is the multitude of problems I encountered with the symmetries of the first system and the first Poincaré section and the resolution of these problems.

Finally, my results and conclusions will be discussed and summarised in chapter 5. The code that I have written is included in the appendices.