# Some Shock Models in Reliability Theory.



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# ABSTRACT

This thesis is concerned with the lifetime distribution of a device subject to environmental shocks. The terms "device" and "shock" are used here in an abstract sense and although industrial interpretations are the most obvious, the models described in this thesis can also be applied in other fields, for example, in biology and finance.

Several models are presented and in each case the main question of interest is to determine the class of distributions to which the lifetime distribution associated with the model belongs. This approach to the study of shock models is taken since it is often difficult to derive an explicit expression for the lifetime distribution of a device, but if the class to which the distribution belongs can be identified it is usually possible to obtain a bound on the distribution.

Since classes of lifetime distribution have an important role to play in the study of shock models and in reliability theory generally, the first part of this thesis is devoted to a review of the classes which have proved useful in these areas. The classes are defined and some justification for their use in reliability theory is provided. In addition, alternative characterisations of the classes are given and it is shown that a function which is closely related to the Laplace transform can be used to characterise all the classes.

The discussion of shock models commences, in chapter two, with a survey of results pertaining to the standard shock model :-

$$\bar{H}(t) = \sum_{k=0}^{\infty} P(N(t)=k)\bar{P}_k,$$

where $\bar{H}(t) = 1 - H(t)$ is the lifetime distribution of a device subject to shocks whose arrival is governed by the stochastic counting process $\{N(t)\}$. The probability that the device survives K shocks is given by $\bar{P}_K$ where $k=0,1,2,\ldots\ldots$ This model has received a good deal of attention in the literature

(see, for example, Esary, Marshall and Proschan (1973), A-Hameed and Proschan (1975), Klefsjo (1981, 1985). The most striking feature of the model is that under appropriate conditions on {N(t)} the lifetime distribution inherits its class from the discrete class of the survival probabilities $(\bar{P}_k)_{k=0}^{\infty}$. Results are presented under a variety of assumptions on {N(t)} ranging from the assumption of a homogeneous Poisson process to the assumption that {N(t)} is a generalised renewal process. In addition, a model where {N(t)} is assumed to be a doubly stochastic Poisson process is introduced. For the more general models it is often the case that the life distribution H inherits its class not only from the survival probabilities but also from the class of the shock interarrival time distribution.

The final part of this thesis is concerned with shock models in which failure occurs according to some specified mechanism. In particular, two methods of failure are considered. Firstly, the case where failure occurs on the occurrence of a shock which exceeds some critical threshold is studied and, secondly, the case where failure occurs when the total accumulated damage due to shocks exceeds some critical threshold is considered. In both cases, the initial approach is to use the standard model with an appropriate structure imposed on the survival probabilities $(\bar{P}_k)_{k=0}^{\infty}$. A more general approach which allows for some dependency between the shock magnitudes and shock interarrival times and, in the case of the cumulative damage model, for wear or recovery between shocks, is then adopted. Such models have been studied by Shanthikumar and Sunmita (1983, 1984) and by Shanthikumar (1984). Their results are summarised and the importance of the class of the shock inter-arrival time distributions in determining the class to which the lifetime distribution of the model belongs is noted. In addition, some minor extensions to Shanthikumar's (1984) work on the cumulative damage model are made.

# INTRODUCTION

This thesis is concerned with the lifetime distribution of a device subject to shocks. The terms device and shock are used here in an abstract sense so, depending on the context, a device may be interpreted as: for example, a biological organism, a financial account, or a piece of industrial machinery. Similarly, a "shock" may be interpreted as a myocardial infarction, a demand or withdrawal, or, in the industrial context, as a blow which causes damage to the device or even as a repair causing negative damage to the device.

The study of shock models is approached from the point of view of establishing the class of distributions to which the lifetime distribution of the model belongs. The justification for taking this approach is that :-

a) in practice it is often difficult to obtain an explicit expression for the lifetime distribution of a device subject to shocks, and,

b) if the class to which a distribution belongs can be determined, it is usually possible to obtain bounds on the distribution.

Since the study of Shock Models is approached in this way, Chapter One of this thesis is devoted to a summary of the classes of life distribution useful in studying Shock Models and in Reliability Theory, generally. The classes are defined and some justifications for their use in Reliability Theory is presented. Alternative characterisations of the classes are provided using a function closely related to the Laplace Transorm and, where appropriate, the Total Time on Test transform. The Chapter concludes with a summary of the closure properties of the classes under the reliability operations of formation of coherent systems, mixture and convolution. Some bounds for distributions belonging to the classes are presented and the relationship between the classes are summarised.

Chapter Two includes a survey of the literature on the standard Shock Model,

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\ \bar{P}_k$$

proposed by Esary, Marshall and Proschan (1973). Here, it is assumed that shocks arrive at a device according to the Stochastic counting process $\{N(t)\}$ and that the probability of surviving k shocks is given by $\bar{P}_k$. $\bar{H}(t) = 1 - H(t)$ is the survivor function for the device in question, i.e. if T is a random variable denoting the lifetime of the device $\bar{H}(t) = P(T>t)$.

The main aim of the Chapter is to determine conditions on the survival probabilities $(\bar{P}_k)_{k=0}^{\infty}$ and on the process $\{N(t)\}$ which are sufficient for the lifetime distribution H(.) to belong to one of the classes of distribution introduced in Chapter one.

In Chapter three, some account is made of the actual mechanism by which devices fail. In particular, two modes of failure are considered :-

a) the maximum shock model where failure occurs when the magnitude of a shock exceeds some critical level, and
b) the cumulative damage model where shocks all cause damage which accumulates until the total accumulated damage exceeds some critical threshold and failure occurs.

Initially these models are studied by imposing an appropriate structure on the survival probabilities $(\bar{P}_k)_{k=0}^{\infty}$ of the standard model discussed in Chapter two. A more general approach which allows the possibility of correlation between the shock magnitudes and the intervals between shocks and, in the case of the cumulative damage model, the possibility of wear or recovery between shocks, is then considered.

Some remarks on the layout of this thesis are in order. Theorems, Lemmas and corallaries are numbered within subsections so that Theorem (3.1A.1) is the first theorem presented in subsection 1A of Chapter 3. Equations are numbered sequentially throughout sections, thus equation (2.1.24) is the 24th equation of section 1 of Chapter 2, regardless of whether it is located in subsection 2.1A or 2.1F. This should not cause too much inconvenience since, with few exceptions, numbered equations are only referred to locally.

Where a lemma is specific to a particular Theorem, it has usually been included within the proof of that Theorem so as not to interrupt the flow of the text.

Owing to some technical problems in the typing of this thesis some of the notation is not quite standard, for example 's' has been used where the Greek letter 'lambda' would normally be appropriate. Thus, the birth coefficients of a pure birth process are referred to as $s_1, s_2, \ldots$ and Poisson probabilities are given by $e^{-st}(st)^k/k!$

In section 2.1E a capital letter, B, has been used to denote a probability density function rather than a lower-case character as is more common.

In Chapter three the symbol $d^2/dxdy$ is used to denote partial derivative but it is clear from the context what is meant.

Finally, I would like to thank Dr C. D. Lai for his encouragement in the researching and writing of this Thesis, and Olaf Skarsholt et.al. for their efforts in making it legible for you to read.

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