Developing Early Algebraic Understanding in an Inquiry Classroom

A thesis presented in partial fulfilment of the
requirements for the degree of
Master of Education
at Massey University, Palmerston North, New Zealand

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2007

ABSTRACT

This study explores Year 5 and 6 students' construction of early algebraic concepts within an inquiry classroom context. Also under consideration are the tools—the instructional tasks and models, the forms of notation and symbolisation, the discourse and interaction, and the teacher's pedagogical actions—which mediate student development of early algebraic reasoning.

An emergent theoretical perspective which brings together social and constructivist theories of learning underpins the focus of the study. Relevant literature is drawn on to illustrate the need for student focus to shift from a procedural perspective of number operations and relations to understanding their structural aspects. Comprehensive evidence in the literature is provided of the significant role of the teacher in developing the students' early algebraic reasoning through facilitating their participation in making conjectures, generalising, justifying and formalising.

A classroom-based qualitative research approach—teaching experiment—matched the emergent theoretical frame taken in the study. The teaching experiment approach supported a collaborative teacher-researcher partnership. Student interviews, participant and video recorded observations, and classroom artefacts formed the data collection. On-going and retrospective data analysis was used to develop the findings as one classroom case study.

Important changes in student reasoning were revealed in the findings as the teacher guided development of productive discourse and facilitated extended time and space for student discussion and exploration within an inquiry context. Students were provided with many rich opportunities to engage with tasks and models which explicitly focused on developing relational thinking, understanding of algebraic notation, the exploration of the properties and relationships of numbers, and functional patterns. Evidence is provided that through engaging with the tasks and models, the students learnt to make conjectures, represent, justify, generalise and formalise their observations. Of significance in deepening student understanding of early algebraic concepts were the repeated challenges to their partial understandings.

The research findings provide insights into ways teachers can assist students to use their implicit understanding of number relations and properties as a foundation for the construction of early algebraic reasoning. The results of this study suggest that student participation in mathematical activity which included explanation, argumentation and justification supported their development of rich algebraic reasoning.

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CHAPTER ONE

INTRODUCTION

1.1   BACKGROUND TO THE STUDY

Over the past decade, there has been increased focus in both national and international research and curricula reforms on the teaching and learning of algebraic reasoning (e.g., Irwin & Britt, 2005; 2006; 2007; Ministry of Education (MOE), 2006; National Council of Teachers of Mathematics (NCTM), 2000). Such emphasis has arisen from growing acknowledgment of the insufficient algebraic understandings students develop during schooling and the way in which this denies them access to potential educational and employment prospects (Knuth, Stephens, McNeil, & Alibabi, 2006). As a result, teachers are charged with providing their students with opportunities to learn algebra with rich understanding. Teachers need to find ways to make algebra accessible to all students and create an environment in which students learn with conceptual understanding (Chazan, 1996; Kaput, 1999; Stacey & Chick, 2004).

Historically within primary school classrooms there has been a strong emphasis on arithmetic and computation with less focus on the connections between arithmetic and algebraic foundations. Many studies illustrate primary students' lack of understanding of important algebraic concepts (Hunter, 2007; Knuth, Alibabi, McNeil, Weinberg, & Stephens, 2005; MacGregor & Stacey, 1997). These studies suggest that school instruction and classroom activity may cause many of the students' difficulties in constructing algebraic concepts.

The shift from arithmetic to algebra is described as an important transition in mathematical reasoning. However it is very often a major hurdle for many students (Chazan, 1996; Stacey & Chick, 2004). Developing algebraic reasoning with primary age children acts as a bridge for conceptual understanding of algebra in later years (Blanton & Kaput, 2005; Carpenter, Franke, & Levi, 2003; Lins & Kaput, 2004). Studies which have investigated the teaching and learning of early algebraic reasoning demonstrate that students' intuitive

knowledge of patterns and numerical reasoning can provide a useful foundation for learning algebra (Carpenter, et al.; Kaput & Blanton, 2005). This is particularly so because algebraic reasoning develops over a lengthy period of time, involves two way connections with other mathematical topics, and involves generalisation, justification and formalisation (Kaput & Blanton). Additionally, further studies (e.g., Carpenter, et al.; Schliemann, Carraher, & Brizuela, 2007a) have shown that scaffolding student use of algebraic tools such as notation and diagrams allows younger students opportunities to operate at a higher level of generality.

Many researchers maintain that inquiry classroom practices and culture provide optimal algebraic reasoning opportunities (e.g., Carpenter, et al., 2003; Kaput & Blanton, 2005). Construction of rich conceptual algebraic reasoning requires that students build understanding of key mathematical ideas and develop mathematical ways of thinking. This is achieved when students are provided with opportunities to engage in the process of "generating [mathematical] ideas, deciding how to express them …justifying that they are true, and in using them to justify the mathematical procedures they are learning" (Carpenter et al., p. 6). Moreover, as Kaput and Blanton argue frequent, viable algebraic reasoning opportunities occur within classroom contexts where students are supported to make purposeful conjectures and construct mathematical arguments.

Whilst the *Mathematics in the New Zealand Curriculum* document (MOE, 1992) organises algebra as a separate strand from number, the most recent draft of the curriculum document (MOE, 2007) advocates teaching number and algebra in an integrated manner. However, within the New Zealand context there is little research completed with primary students which examines how this can be successfully accomplished. There also seems to be limited research within a New Zealand context which focuses on students' construction of early algebraic understanding within inquiry classrooms. For these reasons it is important to examine the development of primary students' understanding of early algebraic concepts within an inquiry setting. Successful international research has demonstrated that algebraic reasoning can be integrated into primary grades. This study aims to build on this research within a New Zealand context.

1.2 RESEARCH OBJECTIVES

The purpose of this study is to explore 9-11 year old students' construction of early algebraic concepts within the context of an inquiry classroom. The study will investigate how specific instructional tasks and models can be used to develop early algebra concepts. The study also seeks to explore how the classroom environment can support students to engage in algebraic reasoning and develop mathematical ways of thinking.

In particular the study addresses the following research questions:

1) What types of instructional tasks and models can support primary students' development of early algebra concepts?

2) How can notation and symbolisation be effectively used to support primary student's development of early algebra concepts?

3) How do classroom practices, and in particular the social norms and socio-mathematical norms, support the development of early algebraic reasoning?

1.3 OVERVIEW

Chapter 2 reviews the literature from both a New Zealand and an international perspective providing a background with which this study can be viewed. It summarises and connects literature related to the establishment of effective participation structures and students' mathematical activity and development in inquiry classrooms. It also outlines relevant findings on the construction of early algebraic reasoning in the classroom.

Chapter 3 describes the methodology used in this research. Data collection methods and data analysis are discussed and the research setting, sample and timeframe for the teaching experiment are outlined.

The following three chapters present and discuss the results of the study. Chapter 4 examines the classroom practice and culture illustrating the important role of social and socio-mathematical norms in guiding productive discourse and collaborative interaction.

An analysis is also provided of the pedagogical actions which the teacher took to shift students from a numerical to an algebraic focus. Chapter 5 provides a description of the ways in which primary students can be supported to construct early algebraic reasoning through exploration of number. Chapter 6 documents students' construction of early algebraic concepts through exploration of patterns and functional relationships.

In Chapter 7, the conclusions from the study are discussed. The recommendations and implications for classroom practice are presented and suggestions are made for areas of further research.

CHAPTER TWO
LITERATURE REVIEW

2.1   INTRODUCTION

The earlier chapter outlined the growing recognition of the inadequate algebraic understandings many students develop during their schooling and the role this has in denying them access to prospective educational and employment opportunities. Although *The New Zealand Curriculum draft for consultation 2006* (MoE, 2007) promotes teaching algebra and number in an integrated manner, there has been little research which suggests how to successfully achieve this within New Zealand primary schools. This review of the literature examines both the nature of early algebraic reasoning and international research studies that investigate possible classroom activity and practice that can usefully support the development of young children's algebraic understandings.

Section 2.2 examines relevant literature on both the way individual students construct early algebraic concepts and the social construction of mathematical concepts in classrooms. Section 2.3 reviews the literature on students' mathematical activity and development of algebraic reasoning in inquiry classrooms. It also outlines the important role of the teacher in an inquiry classroom. Section 2.4 considers the critical role teachers take in establishing effective participation structures which facilitate students' growing algebraic understandings. Section 2.5 examines literature on the construction of early algebraic reasoning in the classroom. Relevant literature is reviewed in section 2.6 which identifies constructing notions of variables, the equal sign, and operational laws as critical factors in students' development of early algebraic reasoning. Section 2.7 considers the role of functional thinking in developing early algebraic understanding and section 2.8 highlights the importance of algebraic processes (for example, making conjectures, generalising, justifying and formalising).

2.2 CONSTRUCTING MATHEMATICAL KNOWLEDGE IN THE CLASSROOM

Historically, school mathematics teaching has emphasised accurate results and infallible procedures which equate 'knowing' mathematics with identification of basic concepts and a competent procedural performance (Boaler & Greeno, 2000). However, in recent years, mathematics education reform has advocated a shift in classroom practices to focus on communication, interaction and understanding of deeper mathematical ideas (Anthony & Hunter, 2005; Kazemi & Stipek, 2001; McCrone, 2005). A learning theory which supports this shift is the emergent perspective taken by Cobb (1995). This socio-constructivist learning perspective links Piagetian and Vygotskian notions of cognitive development connecting the person, cultural, and social factors. Thus a view of mathematical learning is presented which is neither wholly individual, nor wholly social (Cobb, 1994; 1995; 2000a).

2.2.1 SOCIO-CONSTRUCTIVIST LEARNING THEORY

According to Cobb, Boufi, McClain, and Whitenack (1997) the emergent theory brings together social and constructivist theories of learning and provides a way to account for the differences in students' thinking as they participate in collective activities. Piagetian cognitive learning theory views an individual's knowledge of the world as constructed from their perceptions and experiences filtered through their previous knowledge (Cobb, 1995; Simon, 1995). Cognitivist theories view individuals as actively responding to the environment and other people by "building structure of meanings that are constantly adapted and developed under the influence of events outside themselves" (Morgan, Watson, & Tikly, 2004, p. 74). In this frame, learning is viewed as a process of conceptual re-organisation and active individual construction (Cobb, 1994; 1995). New learning is constructed through relating new ideas to those already understood and layers of understanding are created in a non-unidirectional manner (Pirie & Kieren, 1994). Students would then construct algebraic understandings from their numerical knowledge and individually create their own unique and different understandings, regulating their own knowledge construction process.

Vygotskian cognitive learning theory emphasizes the importance of the social aspect of learning. Learning is viewed as a process of enculturation into established mathematical practices with cultural tools operating as mediators. Learners internalise such cultural tools as language or conventional mathematical symbols and these become psychological tools which can be used to regulate and control activity (Cobb, 1995; Cobb et al., 1997). In this way, mastering algebraic thinking and notation may be viewed as an appropriation of a culture specific tool. Socio-cultural theory places emphasis on context, the situated nature of learning, and interaction (Nickson, 2000). Knowledge is viewed as embedded within the context it is learnt—situated, "being in part a product of the activity, context, and culture in which it is developed and used" (Brown, Collins, & Duguid, 1987, p. 32).

An integral factor of socio-cultural theory is that the processes of learning and teaching are interactive, both involving implicit and explicit negotiation of mathematical meaning. Collaborative interaction creates zones of proximal development. This zone first described by Vygotsky is explained by Nickson (2000) as the "discrepancy between what a child is able to do at the point of entry into a problem situation and the level reached in solving problems with assistance" (p. 155). Through negotiation of meaning, teachers and students elaborate taken-as-shared knowledge and this forms the foundation for ongoing communication (Boaler, 2000; Cobb, 2000a). Negotiation is a process of mutual appropriation with teachers and students utilising each others' contributions while the teacher mediates between students' personal meanings and culturally established mathematical meanings. Individual students and the teacher actively contribute to the development of the practices of the mathematics classroom (Cobb, 1994; Cobb, Yackel, & Wood, 1992).

In this theorising frame, early algebraic learning is recognised as both an individual constructive process and as social negotiation of meaning. The central feature of this perspective is that the learner is actively engaged in pattern-seeking and finding. Accounts of students' learning and developing conceptions of mathematics within classroom communities need to take into account the classroom environment, instructional tasks, tools, discourse, interaction, and other social factors. Also individual students' beliefs about

school and mathematics are an additional factor in the knowledge constructed (Cobb, 2000a; Cobb, et al., 1992; McCrone, 2005).

2.3 INQUIRY CLASSROOMS

Inquiry classrooms are those in which students are expected to publicly "express their thinking and engage in conjecture, argument and justification" (Zack, 1999, p. 134). Practices in such classrooms may include small group collaborative work on problems or practical activities, whole class discussions, sharing back and discussing problem solutions, and student evaluation of their own work and the work of others (Anthony & Hunter, 2005; Yackel, 1995; Lampert, 2001). Such classrooms support student questioning, increased feedback and student engagement, and create an environment where error is welcomed and explored (Hattie, 2002; Kazemi, & Stipek, 2001). Inquiry classrooms aim to develop connections between different aspects of mathematics, representations and students' methods as well as linking student's prior knowledge to new learning (Manoucheri & St John, 2006, Pape, Bell, & Yetkin, 2003). Within inquiry classrooms, both understanding mathematics and socialisation into practices of making and using mathematics are regarded as of similar importance (Lampert).

2.3.1 DEVELOPING STUDENTS' MATHEMATICAL DISPOSITIONS IN AN INQUIRY CLASSROOM

In inquiry classrooms, the locus of responsibility is shared by all participants within the classroom community. Both students and teachers have responsibility to develop a social community of learners who are able to share their own thinking, listen and learn from others and reflect both on their own and others' thinking (McCrone, 2005). Student autonomy is fostered through classroom activities which allow students to make sense of mathematics in a way that is personally meaningful. Many studies (e.g., Cobb, et al., 1997; R. Hunter & Anthony 2003; McCrone, 2005; Yackel, Cobb, & Wood, 1991) have illustrated that when students are provided with opportunities to take a reflective stance, critique solution methods, justify and self-evaluate, students' sense of identity is extended and positive dispositions towards mathematics are fostered.

Students and teachers co-construct the social and socio-mathematical norms of the classroom and this ensures students learn about valued ways of working (Yackel & Cobb, 1996). Through participating in the classroom communities, students learn the classroom expectation and obligations for how to manage mathematical activity. As students participate in the negotiation of socio-mathematical norms they develop mathematical beliefs and values. These constitute students' mathematical dispositions and frame their sense of autonomy (Gorgorio & Planas, 2005; Khisty & Chval, 2002; Lampert & Cobb, 2003).

2.3.2   ROLE OF THE TEACHER IN AN INQUIRY CLASSROOM

The role of the teacher in the inquiry classroom is a significant one. It is the teacher who guides student engagement in the mathematical discourse. Through their actions, a range of social norms are developed. These norms are important in that they shape how the students participate and communicate in collaborative dialogue and explain and justify their reasoning. The teacher:

> initiates and guides the development of norms that are specific to mathematical aspects of the children's activity. These socio-mathematical norms are critical to the children's understanding of what constitutes mathematical difference, mathematical sophistication and mathematical elegance. (Yackel, 1995, p. 134)

Establishment of socio-mathematical norms are an important factor to maintain the environment of inquiry and to guide the quality of discourse within a classroom (Anthony & Walshaw, 2002a; Kazemi, 1998). The socio-mathematical norms regulate mathematical argumentation within the classroom and influence learning opportunities for both teachers and students (Wood, 2002). Research by Kazemi found key socio-mathematical norms which were linked to a high press for conceptual thinking. These included:

> explanations consisted of mathematical arguments, not simply procedural summaries of the steps taken to solve the problem, errors offered opportunities to reconceptualise a problem and explore contradictions and alternative strategies, mathematical thinking involved understanding relations among multiple strategies, collaborative work involved individual accountability and reaching consensus through mathematical argumentation. (p. 411)

The teacher plays a central role in orchestrating and facilitating discourse (Khisty & Chval, 2002; McCrone, 2005; Walshaw & Anthony, 2007). As a facilitator, the teacher leads shifts in the discourse to ensure that it is conceptually focused and reflective. In this way, students' thinking can be advanced when the rationale for specific actions becomes the explicit topic of conversation (Cobb et al., 1997; Lampert & Cobb, 2003). Studies by Kazemi (1998) and Kazemi and Stipek (2001) illustrated how discourse promoting conceptual thinking can be achieved through specific teacher actions. These included pressing students to give conceptually focused reasons for mathematical actions, questioning in sustained mathematical exchanges and facilitating student examination of similarities and differences among multiple strategies. In a study involving third grade students, Blanton and Kaput (2005) also illustrated how a teacher created algebraic 'conversations' by pressing students to engage in generalising, formalising, and reasoning with generalisations.

Teachers also take a key role in the questions and prompts used. The study by Wood and McNeal (2003) illustrated the significant role the teacher played in shifting students from explaining strategies to justifying and defending solution strategies. This was achieved by the use of teacher-led questions and prompts. Specific evidence of the importance of questions and prompts to promote algebraic reasoning is provided in Blanton and Kaput's (2003) study involving students from elementary level classrooms.

Within the classroom, the teacher not only facilitates student participation, they are also required to position students to take a specific stance. A key strategy used by teachers to position students is described by O'Connor and Michaels (1996) as revoicing. Revoicing is used to clarify reasoning, highlight specific aspects, extend, rephrase and further develop it (R. Hunter, 2005; Lampert & Cobb, 2003). R. Hunter (2002) reported on a teaching experiment which focused on developing decimal understanding. In this study, the teacher used revoicing of student ideas and positioning to progress the students' rational number reasoning.

Thus, we see from the research studies that within inquiry classrooms teachers have a dual role—to both promote the development of conceptual knowledge in students and to facilitate shared knowledge in the classroom community (Lampert & Cobb, 2003).

2.4   COLLABORATIVE INTERACTION AND CLASSROOM MATHEMATICAL DISCOURSE

Collaborative interaction and classroom mathematical discourse are inter-linked. Discourse fosters a learning community and at the same time learning communities have the possibility of generating useful dialogue between learners (Manoucheri & St John, 2006). A study by Zack (1999) illustrated how fifth grade students used the norms of the learning community to work together to prove or refute arguments and counter-arguments when solving an algebraic problem.

Effective collaborative interaction requires a shift in the role of students from passive member of the classroom to critical participant and listener. Whilst not easy to achieve, we do have some insights from research studies about effective pedagogical practices that support such a shift. Research by McCrone (2005) illustrates how a teacher shifted fifth grade students' participation in discourse from parallel conversations characterised by a lack of active listening to that of critical active participants. The teacher used specific actions such as modelling active listening and reflecting on the ideas of others. She also initiated explicit discussions to emphasise the importance of active reflection and participation in mathematical discussions. Later in the year, the teacher gradually modified her role to become a facilitator during discussions; "interpreting students' solutions and encouraging them to respond to each other (redirecting and suggesting)" (p. 130).

Other researchers (e.g., Mercer, 1995; 2000; Wegerif, Mercer, & Dawes, 1999) examined the forms of mathematical talk used by students when working in small groups. Separate studies were carried out in English and in Mexican schools. It was found that students used three different forms of talk which the researchers termed exploratory, disputational and cumulative talk. These types of talk involved different levels of constructive engagement with the reasoning of others. Disputational and cumulative talk are both characterised as

unproductive. Students engaged in disputational talk focused on self-defence and holding control rather than trying to reach joint agreement. Cumulative talk avoided questions and argument and lacked evaluative examination of reasoning. In contrast, exploratory talk was productive. Using this type of talk, students explored and critically examined their shared reasoning. Importantly for this study, Mercer and his colleagues found that exploratory talk required specific teacher attention, intervention and scaffolding.

Whole class discussions that make substantive mathematical ideas an explicit area of conversations are another way that teachers can advance the instructional agenda (Cobb, 2000a; Yackel et al., 1991). Cobb's study illustrated the role that classroom discussions take in highlighting the need for justification to participants. Interaction between students during discussion supported students to develop their understanding of what constitutes an acceptable mathematical argument.

During small group work learning opportunities also arise from collaborative dialogue and the resolution of differing points of view. Teachers can support students to learn appropriate ways to disagree which are mathematically productive and socially acceptable (Lampert & Cobb, 2003). Further to this, as students work together they develop a knowledge base where both their own constructions and other students make sense (Wood & Yackel, 1990).

Formation of a shared knowledge base and the established social norms enable students to autonomously communicate and validate knowledge claims. This requires the development of a shared focus of understanding and purpose, or intersubjectivity. Intersubjectivity is a process in which participants in the discussion construct personal interpretations which are taken-as-shared (Cobb et al., 1992; Smith, 1996). Successful communication occurs if individual interpretations are compatible. However if the shared interpretations are not viable, "aspects of the taken-as-shared basis for communication… then become the object of explicit negotiation" (Cobb et al, p. 17-18). In this process, the individual participants may modify their perspective in an attempt to achieve intersubjectivity.

Engaging in discourse and argumentation demands that both students and teachers can use common mathematical language and symbols to communicate their thinking. Meanings of symbols and language are shaped by the way they are used in common activities (Lampert, 1991). In the early stages of learning new concepts, students may use definitions and terms in ways which closely relate to the ideas they aim to signify but challenge the conventional meanings. In this case, teachers need to assume the role of mediator between students' informal uses of terms and the standard mathematical use (Lampert & Cobb, 2003). Through teacher modelling of symbolic records of students' contributions, conventional representations and terms can be appropriated by the students and used to represent their thinking (Cobb et al., 1997). This is illustrated in the study by Carraher, Schliemann, Brizuela, and Earnest (2006) with third grade students. Following classroom work involving number lines, the students were able to represent a problem involving changes in unknown amounts using multiple representations and linking these to a number line.

2.5 THE CONSTRUCTION OF EARLY ALGEBRAIC UNDERSTANDING IN THE CLASSROOM

Historically within school settings, algebra has been presented in a highly abstract manner as a form of symbol manipulation (Chazan, 1996; Stacey & Chick, 2004). However algebra takes many forms. Thomas and Tall (2001) categorise algebra into four types; generalised arithmetic, manipulation algebra, evaluation algebra and axiomatic algebra. Other researchers describe school based algebra as encompassing two broad conceptual areas— the representation of numbers, ideas about numbers and number systems and secondly symbolic representation and theory of equations (Warren, 2000; Woodbury, 2000).

Advocates of mathematics curriculum reform have called for a move to re-characterise the nature of algebra and algebraic thinking to include a wider definition (Blanton & Kaput, 2005; Carpenter, Levi, Berman & Pligge, 2005a; Warren & Cooper, 2001a). Blanton and Kaput define algebraic reasoning as a process whereby "students generalise mathematical ideas from a set of particular instances, establish those generalisations through the discourse of argumentation, and express them in increasingly formal and age appropriate ways" (p. 413). Within this definition, algebraic reasoning takes varying forms including:

(a) the use of arithmetic as a domain for expressing and formalising generalisations (generalised arithmetic); (b) generalising numerical patterns to describe functional relationships (functional thinking); (c) modelling as a domain for expressing and formalising generalisations; and (d) generalising about mathematical systems abstracted from computations and relations. (p. 413)

Development of algebraic reasoning occurs over a long period of time and has inter-related connections with all other types of mathematics, particularly arithmetic (Kaput & Blanton, 2005). However, algebraic reasoning can be differentiated from arithmetical thinking through the shift from a procedural perspective of operations and relations to a structural perspective (Carpenter et al., 2005a).

2.6   EARLY ALGEBRAIC UNDERSTANDING AS GENERALISED ARITHMETIC

Algebra in the form of generalised arithmetic requires students understand the structural aspects of mathematics. Students need to develop understanding of variables and algebraic notation, the equal sign as a relation, and the properties of numbers. Students' familiarity with structure is understood to be a consequence of their experiences with arithmetic through a process of inductive generalisation (Hannah, 2006; Warren, 2001a; Warren, 2001b; Warren & Cooper, 2001a). Benefits for younger students focusing on mathematical structure have been found by Mulligan, Mitchelmore, and Prescott (2005). These researchers linked early school mathematics achievement with the student's development and perception of mathematical structure. Another study by Irwin and Britt (2005) with older students found that students who were encouraged to use a "flexible array of skills for manipulating arithmetical relations in ways that exhibit number sense as well as operational sense" (p. 182) were provided with a foundation for algebra in secondary school. Their other New Zealand based studies (2006; 2007) with Year 7 to Year 10 students found that students who had been exposed to part-whole thinking and had participated in instruction where numbers were used as quasi-variables displayed higher levels of algebraic reasoning.

Researchers claim that limited classroom experiences in exploring the properties of numbers in combination with extensive experiences with arithmetic as a procedural process work as a cognitive obstacle for students later on when there is a need to abstract the properties of numbers and operations (Warren 2001a; 2001b; Warren & Pierce, 2004).

2.6.1 UNDERSTANDING OF VARIABLES AND ALGEBRAIC NOTATION

Developing conceptual understanding of variables and algebraic notation is central to the transition from arithmetic to algebra. Using instruction which utilises quasi-variables is a way of providing a gateway to the concept of variables in the primary years of schooling (Fujii, 2003). Quasi-variables are described by Fujii as "a number sentence or group of number sentences that indicate an underlying mathematical relationship which remains true whatever the numbers used are" (p. 59). They provide opportunities for students to understand general properties of arithmetic within the context of number. Irwin and Britt (2007) argue that when students have had classroom experiences where numbers are used as quasi-variables they can move "without too much difficulty to the use of letters as variables" (p. 33). Research by Warren and Cooper (2002) in Year Two classes examined how the use of quasi-variables assisted young students in expressing generalisations. In this case, the students were able to generalise a pattern through the use of quasi-variables to illustrate the underlying mathematical structure of the pattern. Other studies (e.g., Blanton & Kaput, 2003; 2005) illustrated how quasi-variables can be used to facilitate students to examine the properties of odd and even numbers. Through extending the numbers, students investigated the structure of the numbers and solved the task algebraically rather than using arithmetic.

Historically, developmental theory suggested that the use of letters as algebraic notation was beyond younger children's cognitive capabilities (Carraher et al., 2006). However, a study by MacGregor and Stacey (1997) suggested that students' difficulties with notation could be both attributed to lack of opportunities to explore notation and their classroom experiences. These researchers found that inappropriate teaching methods such as the use of letters to represent an object led to students viewing letters as abbreviated words. Also a number of the students in the study based their interpretation of symbolic letters on

intuition, guessing or false analogies. Other research studies (e.g., Fujii, 2003; Weinberg, Stephens, McNeil, Krill, Knuth, & Alibabi, 2004: A. Stephens, 2005) have highlighted common student misconceptions linked to notation including the notions that a single letter variable can only stand for a single number and that variables represented by different letters could not be the same number.

Students need opportunities to both use algebraic notation and explore the concept of variables (MacGregor & Stacey, 1997; A. Stephens, 2005). To support algebraic reasoning, students need to develop understanding of variables as specific unknowns or generalised numbers (Knuth et al., 2005; Subramaniam & Banerjee, 2004). A. Stephens asserts that students should be encouraged to view algebraic notation as a shorthand tool for expressing recognised ideas about quantities.

We know that students' algebraic understandings develop with experience (Flockton, Crooks, Smith, & Smith, 2006). Studies involving middle school students have shown that students' performance in interpreting algebraic notation and understanding of variables as representing multiple values increases over year levels (Knuth et al., 2005; Weinberg et al., 2004). However, recent research with primary age students advocates that younger students are able to understand and work with algebraic notation (Ameron, 2003; Carraher et al., 2006; Carpenter & Levi, 2000b; Carpenter et al., 2003; Lee, 2006; Schliemann et al., 2007a). Results of an interview based study carried out with third grade students by Schliemann, Carraher, Brizuela, and Jones (2007b) found that the students could develop consistent notations such as circles or shapes to "represent elements and relationships in problems involving known and unknown quantities" (p. 59). Two classroom intervention studies involving third grade students, the first by Carraher and his colleagues and the second by Lee, found that the students were able to use formulas to represent functions and treated the symbolic letter in the additive situation as having multiple possible solutions. Another study by A. Stephens (2005) with Year 7 and 8 students illustrated how a mathematical problem could be used to confront common misconceptions students held about notation. Teachers in the study noted that concentrating on symbolic representations allowed them to address misconceptions.

2.6.2   UNDERSTANDING EQUIVALENCE

An understanding of the equal sign relationally as an equivalence symbol meaning the "same as" is crucial for students to abstract their structural numerical reasoning across to algebraic reasoning (Carpenter & Levi, 2000a; Carpenter et al., 2005a; Falkner, Levi & Carpenter, 1999; Knuth et al., 2006; McNeil & Alibabi, 2005). Inadequate understanding of the equal sign creates difficulties for students solving symbolic expressions and equations and makes the transition to algebra difficult for them (Jacobs, Franke, Carpenter, Levi, & Battey, 2007; Kieran, 1981; Knuth et al.; McNeil & Alibabi). Some researchers have demonstrated the relationship between students' understanding of the equal sign and their success in solving both traditional algebraic equations and equivalence problems. Knuth and his colleagues (2005) found students with a relational view of the equal sign were more likely to be able to solve equivalence problems and use the more efficient strategy of recognising equivalence to solve the problem. A second study by Knuth et al. (2006) demonstrated that students with a relational view of the equal sign were more likely to use an algebraic strategy to solve the problems. These researchers argue that understanding of the equal sign is a crucial feature of success in solving both equivalence and algebraic equations.

Many primary and middle school students show limited understanding of equivalence or the meaning of the equal sign. The studies by Kieran (1981), and more recently Knuth and his colleagues (2006), demonstrated that student understanding of the equal sign did not necessarily improve as students advanced in year levels. According to Knuth et al., students commonly believe the equal sign is an operational sign rather than a symbol of a mathematically equivalent operation. Students who hold this operational view equate the equal sign with a need to find a 'sum' or 'answer' or a left to right action of adding all the numbers to the left of the equal sign (Carpenter et al., 2005a; Knuth et al.; McNeil & Alibabi, 2005). Other studies by Freiman and Lee (2004) and J. Hunter (2007) illustrated the common errors students made when solving equivalence problems due to their misconceptions about the equal sign.

Researchers have suggested that classroom mathematics experiences in the early years of schooling are the basis for many difficulties that children experience with developing initial understanding of algebra. Within classrooms, equations are often presented with only one number after the equal sign. Focus on computation means that students are presented with the equal sign as an indication to carry out a calculation (Carpenter et al., 2003; Knuth et al., 2006; Warren, 2001a; Warren & Pierce, 2004). McNeil and her colleague's (2006) study of the presentation of the equal sign in textbooks found very few textbooks showed operations on both sides of the equal sign. This study also found a link between the context the equal sign was presented to students and whether they gave a relational definition indicating that context can influence students' interpretation of the equal sign.

Developing understanding of equivalence is recognised as a complex and difficult task and one which requires considerable time and explicit teacher attention (Falkner et al., 1999; Knuth et al., 2006). Teachers need to provide students with learning situations that will enrich and expand their understanding of symbols and challenge students' conceptions (Carpenter & Levi, 2000a; Carpenter et al., 2003; Falkner et al.; Rivera, 2006). Various studies have investigated ways in which this can be accomplished. Warren and Cooper (2005) suggest the use of a balance model to explore arithmetic equations with students. Carpenter and his colleagues (2005a) investigated the use of true and false number sentences and open number sentences to challenge student's conceptions of equality. In contrast to traditional classrooms where student understanding of the equal sign was under-developed, Carpenter et al.'s study showed 84% of Grade 6 students developed relational understanding of the equal sign following classroom work with true and false number sentences. However, when working with younger Grade 1 and 2 children, Falkner and her colleagues noted that understanding of the notion of equality took considerable time to develop. Students began to develop notions of equality but "the concept was not easily or quickly understood" (p.235) and required re-visiting over the year.

Students' understanding of the equal sign takes multiple forms. Some students view the numbers on each side of the equal sign as representing separate calculations. These students use computation to solve equivalence problems. In contrast, students with relational

thinking are able to look at the "expressions in their entirety, noticing number relations among and within these expressions and equations" (Jacobs et al., p. 260). This enables them to use number relations to simplify calculations (Carpenter, Levi, Franke, & Zeringue, 2005b). However, studies by J. Hunter (2007) and M. Stephens (2006) with primary and middle school students demonstrated that unless students encounter explicit teaching of relational approaches, many continue to use computational approaches.

2.6.3 UNDERSTANDING ARITHMETIC OPERATIONS AND OPERATIONAL LAWS

Students draw implicitly on the properties of arithmetic operations and operational laws to solve a range of computational problems. For example, students may solve word problems by using different operations and utilising inverse relationships. Likewise, number properties which incorporate the commutative or distributive property are used implicitly. However students often still lack deep understanding of these properties (Carpenter et al., 2005a; Schifter, 1999). To develop students' algebraic thinking, the underlying properties of arithmetic operations need to be made explicit. Carpenter and his colleagues contend that if students "understand arithmetic at a level that they can explain and justify the properties that they are using as they carry out calculations, they have learned some critical foundations of algebra" (p. 82).

Studies investigating students' understanding of commutativity have illustrated that many students lack understanding of the operational laws. Anthony and Walshaw's (2002b) study of Year 4 and Year 8 students demonstrated that many students failed to reach correct generalisations regarding commutativity. Similar results were gathered from Warren's (2001a; 2001b) studies involving Year 3, Year 7 and Year 8 students. In all of these studies, students recognised the commutative nature of addition and multiplication; but also thought that subtraction and division were commutative. Anthony and Walshaw found that although students were able to offer some explanation of the commutative property none offered generalised statements. These researchers argued that most of the students in their study did not have deep understanding of arithmetic which would support structural understanding.

2.7   EARLY ALGEBRAIC UNDERSTANDING AS FUNCTIONAL THINKING

Functional thinking is a form of algebraic reasoning which can be described as representational thinking that concentrates on the relationship between two differing quantities. Functions are the representational system used to show a generalisation of a relationship among quantities (Blanton & Kaput, 2004). Arithmetic operations can be viewed as functions and functions offer an opportunity to integrate algebra into the existing mathematics curriculum (Carraher et al., 2006; Blanton & Kaput, 2005). Recent research has indicated that young children are capable of functional thinking (e.g., Blanton & Kaput, 2004; Carraher et al.; Lee, 2006; McNab, 2006). Blanton and Kaput's study using a task involving a functional relationship between the number of dogs and total number of eyes or eyes and tails illustrated that students in Grade 1 and 2 were able to engage in functional thinking and to recognise the relationship between quantities. When comparing younger and older students, these researchers found that students in senior grades were able to formulate functional relationships from smaller data sets. In another study involving a classroom intervention with Grade 2 students, McNab (2006) also found that all students were able to recognise and describe functional rules for patterns.

2.7.1   GENERALISATION STRATEGIES TO SUPPORT FUNCTIONAL THINKING

Strategies that students use to generalise numeric situations emerge through different types of reasoning (Lannin, Barker, & Townsend, 2006; Swafford & Langrall, 2000). The framework developed by Lannin and his colleagues outlines a continuum of algebraic generalisation strategies that students utilise when solving problems. At the less sophisticated end of their continuum, they describe the commonly used recursive generalisation strategy whereupon students "describe a relationship that occurs in the situation between consecutive values of the independent variable" (p. 6). Further along their continuum, they describe chunking when the student "builds on a recursive pattern by building a unit onto known values of the desired attribute" (p. 6) and whole-object strategies when a portion is used as a unit "to construct a larger unit using multiples of the unit" (p. 6). Both chunking and whole-object strategies are often used by students to

quickly calculate particular values. Finally, the most effective strategy described on their continuum is the use of explicit generalisation whereupon a rule is constructed by the student which "allows for immediate calculation of any output value given a particular input value" (p. 6).

Students' use of generalisation strategies are influenced by a range of factors (Becker & Rivera, 2006; Lannin et al., 2006; Swafford & Langrall, 2000). In a teaching experiment involving two Grade 5 students, Lannin and his colleagues investigated the broad factors which influenced students' use of generalisation strategies. These researchers found that when input values were close, students tended to use recursive strategies. When input values were multiples or doubles of previous values, students would often use a whole-object strategy. However, by setting tasks whereupon students considered increasingly large input values, students' use of explicit generalisation strategies was promoted. Use of explicit strategies was also influenced by visual images. For example, when students were able to link their rules to the problem situation, they were more successful at creating correct explicit rules. Students also attempted to find explicit rules in an effort to find a more efficient strategy.

2.7.2   TOOLS AND REPRESENTATIONS TO SUPPORT FUNCTIONAL THINKING

Tools that support algebraic reasoning include symbolic notation, t-charts[1], and diagrams. These tools allow students to understand and express functional relationships across a variety of problem contexts (Blanton & Kaput, 2005; Carraher et al., 2006). In the year-long teaching intervention with third grade students, Schliemann and her colleagues (2007a) demonstrated that algebraic notation offered students a means to succinctly form descriptions of functional patterns and express generalisations. Blanton and Kaput's (2004) study investigated types of representations students used to represent functional relationships. These researchers found the introduction of t-charts built an early representational infra-structure that supported algebraic reasoning. Older students were able to fluently use t-charts and symbolise varying quantities and relationships with letters.

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[1] Function tables with a column of data for the independent variable and a column of data for the dependent variable

Some researchers (e.g., McNab, 2006; Swafford & Langrall, 2000) argue that sequential examples on t-charts position students to focus on recursive patterns. In contrast, the use of non-sequential examples ensure students focus on the "across" or functional rule rather than what comes next.

Making connections across representations including visual and numeric patterns allows students to identify, communicate, and justify functional rules (Beatty & Moss, 2006; Becker & Rivera, 2006). McNab's (2006) classroom teaching intervention highlights the importance of using multiple representations. Integrating the use of ordinal numbers, geometric constructions and the number of tiles in the construction promoted students' ability to move fluently across representations. Beatty and Moss also argue that students need opportunities to build awareness of functions in multiple representations—specifically to enable them to move beyond pattern spotting to generalisation of specific cases. Their design research-based study with fifth graders found that students who used the representational context to solve problems, rather than numeric strategies (ie., tables of data) had more robust understanding of functions and were able to fluently use representations, including symbolic notation, for problem solving.

2.8   DEVELOPING THE PROCESSES OF EARLY ALGEBRAIC REASONING

Current research on early algebra has investigated how instruction can be targeted to build on students' numerical reasoning (Lins & Kaput, 2004). This includes making conjectures and generalisations, justifying and proving, and formalising the generalisations (Carpenter & Levi, 2000b; Carpenter et al., 2003).

2.8.1   MAKING CONJECTURES

Making conjectures, generalising, and formalising are fundamental processes of algebraic reasoning (Carraher et al., 2006; Cooper, 2003; Kaput, 1999). However, a New Zealand study by Anthony and Walshaw (2002a) involving Year 4 and Year 8 students noted that students were generally unable to model conjectures concerning arithmetic properties using material or provide quality responses which referred to generalisations of number

properties. It appeared that few students were able to draw upon learning experiences which bridged number and algebra. Research by Carpenter and Levi (2000b) reports on the use of true and false and open number sentences to scaffold students to generate conjectures. Whilst the first and second grade students in this study were successful at generating conjectures the researchers noted that they exhibited limited understanding of what was required to justify their conjectures.

2.8.2   MAKING GENERALISATIONS

In order to develop early algebraic thinking, students need experiences in making and proving generalisations. Both generalisations and the mathematical reasoning which supports the generalisation should be expressed explicitly (Anthony & Walshaw, 2002a; Carpenter & Levi, 2000a; Carpenter et al., 2005a). Kaput (1999) states that:

> generalisation involves deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases, or lifting the reasoning or communication to a level where the focus is no longer on the cases or situations themselves but rather on the patterns, procedures, structures and relations across and among them. (p. 136).

Young children are capable of thinking in general terms and can learn to construct and justify generalisations about the fundamental structure and properties of arithmetic (Carpenter & Levi, 2000a; Warren & Pierce, 2004). Whilst spontaneous opportunities for numerical generalisation may occur within the classroom Carpenter and his colleagues (2005a) maintain that a coherent framework is needed for tasks that provide a context to focus on generalisation. Carpenter and Levi (2000b) describe how number sentences can be selected and used to "highlight and provide a context for discussion of basic properties of numerical operations and relations" (p. 3). Their study in a first and second grade class illustrated how the use of number sentences containing zero assisted the students to make generalisations about the properties of zero when adding and subtracting. A further study by Carpenter and Levi (2000a) across elementary grades demonstrated that the discussion of number sentences supported students to generalise about number properties.

2.8.3   JUSTIFYING CONJECTURES AND GENERALISATIONS

Justification is a critical element of the generalisation process. Simon (1995) argues that students' competence in providing justification develops from a consistent expectation that justification will be supplied. However, determining the legitimacy of a general statement is a demanding task for students (Lannin, 2005). Studies into the forms of arguments that elementary students used to justify generalisations have found that students use two types of justification: empirical justification and generic examples (Carpenter et al., 2005a; Lannin). Initially, most students viewed specific examples or trying a number of cases as valid justification. But Carpenter and colleagues classroom studies (Carpenter & Levi, 2000b; Carpenter et al.) demonstrated that younger students could use counter examples to challenge other students' claims or generalisations.

Supporting students to justify generalisations using arguments which are precursors of mathematical proof is challenging. Carpenter et al.'s (2005a) study of sixth grade students reported on the students' use of justification following a year-long teaching intervention. In the final interview although only 37% of students could produce a valid general justification, 79% of the students acknowledged examples of general justifications as superior to individual number examples. Kazemi (1998) contends that in classrooms where socio-mathematical norms have been established, students learn they can justify through triangulation of verbal, numerical and graphical strategies. Carpenter and his colleagues and Kaput (1999) have demonstrated that using materials can support students in making concrete justification which then can be formalised through the introduction of algebraic notation.

Within mathematics classrooms, students establish and negotiate norms for articulating and justifying generalisations (Carpenter et al., 2005a; Simon, 1995). Lannin (2005) proposes that justification is satisfactory when it meets the norms that have been established by the community. That is, proof is a matter of social acceptability at a particular time. Argumentation allows exploration of conjectures and assists students to develop understanding of what comprises a suitable explanation or justification. With regards to

algebra, Blanton and Kaput (2005) argue that argumentation positions students to be better prepared to make explicit links between numerical and algebraic reasoning.

2.8.4 FORMALISING CONJECTURES AND GENERALISATIONS

Formalising knowledge into abstract systems and constructing notations for representing procedures and generalisations abstractly are also important elements of the generalising process (Carpenter & Levi, 2000b; Krebs, 2003). In Carpenter and Levi's study, Grade 1 and 2 students first provided generalisations in natural language; however at times this proved awkward and imprecise. Use of number sentence offered the students a notational system to express generalisations precisely. Carpenter and his colleagues (2005a) worked with students across the primary grades and found that students who had worked with number sentences were able to easily adapt these to represent generalisations. At the conclusion of the study, 80% of 4th and 6th Graders were able to use algebraic notation to express generalisations.

2.9 SUMMARY

Recent mathematic curriculum reform has called for changes in classroom practices to include a focus on both the individual and the social factors in mathematics classrooms. Students construct understanding as they negotiate mathematical meanings and actively engage in pattern-seeking and finding. The literature describes how student learning is influenced by instructional tasks, tools, discourse, interaction and other social factors.

Inquiry classrooms are recognised as those in which students express their thinking and engage in making conjectures, justifying and argumentation. The teacher takes a significant role in guiding the construction of social and socio-mathematical norms which shape how students communicate and participate within the classroom. The literature also highlights the important role the teacher takes in leading shifts in students' roles to active listeners and critical participants ensuring productive mathematical talk and collaborative interaction occurs. Many studies demonstrate how students' mathematical reasoning and early algebraic understanding is enhanced through participating in the practices of an inquiry classroom.

Students' development of early algebraic reasoning has been the focus of many reform programmes and advocates have called for a re-definition of the nature of algebra and algebraic reasoning. The need for change is highlighted by the many studies which show persistent student misconceptions about equality, algebraic notation and the properties of numbers. The literature describes how student understanding can be enriched and expanded by providing experiences that challenge their misconceptions through use of number sentences or contextual problems. Previous work also describes how students can be encouraged to develop robust understanding of functions through integrating different representations including visual and numeric patterns. There is comprehensive evidence within the literature which illustrates how students' early algebraic reasoning develops from participating in the processes of algebraic reasoning—making conjectures, generalising, justifying and formalising.

CHAPTER THREE
RESEARCH DESIGN

3.1   INTRODUCTION

This chapter outlines the design and methods used in the study. Section 3.2 provides justification for the qualitative approach which was selected for this study and describes the use of teaching experiment methodology. Section 3.3 outlines the role of the researcher. Section 3.4 outlines data collection methods used in the study, including observation, interviews and classroom artefacts. Section 3.5 details the setting of the study, the participants and the research schedule. Section 3.6 and section 3.7 describe the data analysis used in the study and the measures taken to ensure the validity and reliability of the findings. Section 3.8 summarises the methods used to ensure ethical standards were upheld at all times.

3.2   JUSTIFICATION FOR METHODOLOGY

This study aims to investigate how students between the ages of nine to eleven years old construct early algebraic understanding within an inquiry classroom. A case study of one classroom is used to describe the teacher's actions and the students' learning of early algebraic concepts. After thorough reflection on a range of research methods, a teaching experiment which used qualitative methods was selected as the most appropriate approach for the following reasons. Firstly, the study explores the construction of early algebraic understanding within the naturalistic setting of a classroom. Secondly, the purpose of this study is to investigate the types of tasks and teacher practices which support students' development of algebraic understanding. Thirdly, teaching experiments are aligned with the theoretical perspective that underpins this study.

Qualitative research is described by Bogdan and Biklen (2003) as an umbrella term for research which is naturalistic and contextual, collects and uses descriptive data, focuses on processes rather than outcomes and products, uses inductive analysis for theory building and focuses on meaning and participant perspectives. It is conducted within a natural

setting and it uses multiple methods to collect descriptive data to give a rich description of the social world (Denzin & Lincoln, 2003).

Qualitative research aims to encapsulate individual points of view and perspectives while acknowledging that there are multiple realities and many ways of understanding a situation (Denzin & Lincoln, 2003). It is the researchers who make interpretations therefore there is no claim of truth instead interpretations are viewed as "plausible given the data" (Bogdan & Biklen, 2003, p. 24). The use of methods related to qualitative research and case study are well-matched to the socio-constructivist framework taken in this study.

Case study research includes detailed examination of one site and one group in a natural environment. It is a holistic form of research which uses multiple sources and methods of data collection (Denscombe, 2003). The data provides a richly descriptive product of words and pictures which is used to support theoretical conclusions. Case study research aims to develop understanding and expand the range of interpretations (Scott & Usher, 1999). Use of a case study design is appropriate as this study investigates the development of a group of students' early algebraic understanding within a classroom. This study includes an in-depth examination of one site and one group within a natural environment and utilises multiple sources of methods and data collection.

Teaching experiments have been used widely within mathematics education research (e.g., Cobb, 2000a; Cobb, Yackel, & Wood, 1995; Falkner et al., 1999; Warren & Cooper, 2003). They allow researchers to comprehend mathematical learning and reasoning and to understand "the progress students make over extended periods" (Steffe and Thompson, 2000, p. 274). The design of teaching experiments focuses on using instructional strategies and tools to develop, enact, and maintain innovative learning environments (The Design-Based Research Collective, 2003).

Within mathematics education, notable users of the teaching experiment (e.g., Cobb and colleagues) reference the emergent perspective as their theoretical foundation. From this perspective, learning is viewed as a process of active construction and influenced by the

social and cultural norms within the context of the classroom. Through the use of multiple methods of data collection, the researcher develops understanding of the social interactions and individual students' constructions (Cobb, 2000b; Steffe & Thompson, 2000).

Teaching experiments include a recursive cycle of "hypothesis formulation, experimental testing and reconstruction of the hypothesis" (Steffe & Thompson, 2000, p. 300). In this study, a hypothetical learning trajectory was formulated collaboratively by the researcher and teacher participant. Data collection involved multiple sources including interviews, observations and classroom artefacts. Data was then used to inform ongoing modification of the hypothetical learning trajectory which took place during the teaching experiment.

Collaboration is an important element of teaching experiment research as the practitioner and the researcher work together to "produce meaningful change in contexts of practice" (The Design-Based Research Collective, 2003, p. 6). In this study, daily discussion and reflection involving the researcher and the teacher participant ensured the development of 'taken-as-shared' interpretations of the classroom events and this facilitated modification of learning goals and instructional activities (Cobb, 2000b).

3.3 RESEARCHER ROLE

Within any qualitative research study the researcher holds the role of "the primary instrument for data collection and analysis" (Merriam, 1998, p. 7). In my role as researcher in this study, I took the role of a participant observer which is defined by Clarke (1996) as a "participant in the actions, events and contexts being studied" (p. 4).

I am an experienced primary school teacher who has taught mathematics to students within this age range and within an inquiry classroom. This familiarity means that I have understanding of expected learning outcomes and classroom practices. However, this prior experience also creates new challenges for me as a researcher—it is acknowledged that all findings and interpretations are influenced and shaped by my identity and viewpoint (Bogdan & Biklen, 2003). As a former teacher at the school where the study took place I had an "insider" role, and therefore needed to consciously establish a new role as a

researcher within the classroom. Detailed field-notes and reflections were recorded to prevent errors, bias and missed opportunities (Bogdan & Biklen; Merriam, 1998).

3.4 DATA COLLECTION

Data collection during this study was a cyclic process with "continuous cycles of design, enactment, analysis and redesign" (The Design-Based Research Collective, 2003, p. 5). Conjectures were formulated, tested and re-defined in an iterative process and a relationship of reflexivity was maintained between data collection and ongoing data analysis as each informed the other (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003).

Consistent with a teaching experiment design, the study used multiple methods of data collection. Methods included interviews, observations, and classroom artefacts. Triangulation of data collected through the multiple sources of data assisted in establishing the credibility and validity of the study (Merriam, 1998).

3.4.1 OBSERVATION

Observation is a key tool within a teaching experiment design. This study involved a sequence of video-recorded teaching episodes (Steffe & Thompson, 2000). One group was followed closely throughout the whole lesson in order to investigate conceptual development which occurred within the group. The students included in this group varied in different lessons. Video-taping of the large group discussion offered opportunities to document the diversity of methods between the groups.

Video-taping lessons supported retrospective analysis, a central component of teaching experiments. It provided the researcher and teacher participant with opportunities to rethink explanations and validate interpretations of student responses made during instructional activities (Lesh & Lehrer, 2000; Steffe & Thompson, 2000).

Video-taped lessons are a record of events open to multiple analyses however they cannot be considered an objective representation because some aspects maybe highlighted while others are suppressed. Detailed observational field-notes to support the use of video-taping

enabled a broad range of factors to be observed within the naturalistic setting of the classroom (Clarke, 1996; Lesh & Lehrer, 2000).

3.4.2 INTERVIEWS

Exploratory clinical interviews were used in the initial phase to gather in-depth information about students' early algebraic understanding. The interviews used pre-set questions but included an exploratory form as the researcher explored the responses asking "new questions in order to clarify and extend the investigation" (Clement, 2000, p. 547). At the end of the teaching experiment classroom study, final clinical interviews were used to trace the changes in students' early algebraic understanding. I was interested in exploring how understandings had developed over time in response to the conceptual challenges and classroom activities. The interviews were audio-recorded to allow a less intrusive method of recording data. Field notes taken during the interviews supplemented the recordings.

Use of clinical interviews allowed the researcher to document naturalistic forms of thinking and collect data which reflected the student participants' understanding of the algebraic processes under investigation. Data gathered from the clinical interviews was triangulated with data from field-notes and the classroom observations. This strengthened the viability of models of students' thinking generated from the interviews (Clement, 2000).

3.4.3 CLASSROOM ARTEFACTS

Data collection in the study involved the collection of artefacts from the classroom. This included samples of the students' written work and reflections. This collection of artefacts supported and complemented other methods of data collection.

A reflective field log was also maintained by the researcher to capture analytic thoughts (Glesne & Peshkin, 1992). Informant diaries completed by the researcher and teacher provided a source of data which included factual data, significant events and personal interpretations (Denscombe, 2003). These were secondary to the focus of the study and were used as a retrospective account of classroom practices and a further means of triangulating data.

3.5   THE RESEARCH STUDY: SETTINGS, SAMPLE AND SCHEDULE

This section provides the setting of the study, the details of the participants and outlines the phases of the study.

3.5.1   THE SETTING AND THE SAMPLE

The research was conducted at a large inner city full primary school during term one of the 2007 school year. Bellview School has a decile[2] rating of eight. Students attending this school represented a range of ethnicities and came from pre-dominantly high socio-economic backgrounds.

The study was conducted within one classroom in the school. Following an invitation from the researcher, the teacher in the study agreed to be involved in the research, viewing it as an opportunity to reflect on her own practice and develop awareness of effective teaching practices in the area of early algebra.

Twenty-five Year Five and Six students aged between nine and eleven years old were involved in the study. Initially, all the students were included in the lesson observations in terms of data collected. However as the study progressed, specific groups of students working at more advanced levels of algebra were targeted for observations. These students included a 'middle' group of ten students who were working within Level Three/Four of *Mathematics in the New Zealand Curriculum* (Ministry of Education, 1992) and a 'hard' group of nine students who were consistently working within Level Four of the New Zealand Curriculum document. Further discussion of the groups of students participating in activities is provided in the following section.

3.5.2   THE RESEARCH STUDY SCHEDULE

The classroom teaching experiment project consisted of four phases conducted over a 3 month period involving 18 lessons.

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[2] Each state and integrated school is ranked into deciles, low to high, on the basis of an indicator. The decile indicator measures the extent to which schools draw from low or high socio-economic communities.

Phase One

In Phase One, the researcher individually interviewed all the students. Interview questions (Appendix A) were derived from previous research studies including items from item banks produced by Knuth et al. (2005) and Carpenter (2003). Items investigated students' understanding of the equal sign, operational laws, variables, and their use of relational and functional thinking. The interviews were audio-recorded and wholly transcribed. Transcriptions and data from the interviews were analysed by the researcher to identify the range of understanding of early algebraic concepts. The results were discussed with the teacher participant.

Phase Two

Phase Two involved the researcher and teacher collaboratively planning a teaching unit which included a hypothetical learning trajectory. This trajectory was composed of three components:

> learning goals for students, planned learning or instructional activities, and a conjectured learning process in which the teacher anticipates how students' thinking and understanding might evolve when the learning activities are enacted in the classroom. (Cobb, 2000b, p. 316)

The instructional learning sequences were modelled on work by Kaput and Blanton (2005) and Carpenter and his colleagues (2003). They were designed to support students in "developing ways of thinking about arithmetic that are more consistent with the ways that students have to think to learn algebra successfully" (Carpenter et al., 2003, p. 1).

Phase Three

In Phase Three the teacher taught a sequence of 18 lessons as part of the study. Some of the lessons involved all the class members and others were focused on specific groups of students. The research data collected in this phase included video recordings, written work samples from the students, field notes, and researcher recorded observations of the lessons.

Additional data included audio recordings of the collaborative planning meetings and reflective discussions between the teacher and researcher.

Although lessons varied in nature, there was a standard format that was typically followed for each lesson. Lessons began with a 20 to 30 minute whole-class session in which all students were involved and worked in pairs. Throughout the teaching experiment study, these sessions focused on the properties of number or were used to maintain and consolidate concepts covered in earlier lessons. Within any single lesson format, the whole-class session was followed by a small group lesson. There were three mathematics groups in the class classified as 'easy', 'middle' and 'hard'. With teacher support students self-selected their participation in these group activities.

The summary time-line (see Table 3.1) provides an overview of the lessons which took place during the study and the students from the mathematics groups who attended the lessons. Further details of the activities and problems used in each session are provided in Appendix C. Students involved in the small group lesson worked in collaborative problem-solving groups consisting of four students. Students who did not attend the small group lesson worked independently on mathematical problems consolidating material from earlier lessons. A richer description of the structure of the learning sessions is provided in Chapter Four.

Phase Four

Phase Four occurred immediately after the instructional sequence associated with the algebra teaching experiment finished. All students in the class were individually interviewed using a set of tasks (Appendix B) modified to match the initial interview tasks. The interviews were audio-taped and wholly transcribed. Students also completed written reflections in which they recorded their personal perspective of their mathematical learning over the term.

Table 3.1   Summary time-line of data collection

Week 1	Individual interviews of twenty-five students		
Week 2	Collaborative planning of teaching unit		
Week 3-8	*Whole class warm-up*	*Focus of group lesson*	*Focus mathematics group/s in the lesson*
Week 3	N/A	Understanding the equal sign	Whole class
	Commutative Law	Relational thinking	Easy and middle group
	Commutative Law	Relational thinking	Middle group
Week 4	Commutative Law	Relational thinking	Hard group
	Commutative Law	Representing unknowns algebraically	Middle and hard group
	Relational thinking	Representing unknowns algebraically	Middle and hard group
Week 5	Relational thinking	Geometrical patterning problem	Hard group
	Commutative Law	Geometrical patterning problem	Easy and middle group
	Commutative Law	Geometrical patterning problem	Hard group
Week 6	Odd / even numbers	Geometrical patterning problem	Easy and middle group
	Odd / even numbers	Recording all possibilities	Easy and middle group
	Odd / even numbers	Functional relationship problem	Hard group
Week 7	Odd / even numbers	Functional relationship problem	Middle group
	Relational thinking	Understanding the equal sign as balance	Hard group
	Algebraic number sentences	Understanding the equal sign as balance	Middle group
Week 8	Properties of zero	Understanding the equal sign as balance	Hard group
	Properties of zero	Inverse relationship of multiplication and division	Hard group
	Properties of zero	Inverse relationship of addition and subtraction	Middle group
Week 9	Individual interviews of twenty-five students		

3.6   DATA ANALYSIS

Bogdan and Biklen (2003) describe data analysis as a process of "working with the data, organising them, breaking them into manageable units, coding them, synthesising them and searching for patterns" (p. 147). As appropriate to the use of teaching experiment methodology, two phases of data analysis were used in the study. Ongoing data analysis shaped the study and involved the researcher and teacher in collaborative examination of

classroom practices, modification of the instructional sequence and amendment of the hypothetical learning trajectory.

Retrospective data analysis occurred beyond the classroom and included the identification of categories and themes (Cobb et al., 2003). Video and audio recordings were wholly transcribed and transcripts were read and reread to assist the development of broad themes which formed the initial categories. These broad themes form the headings of the findings chapters. Further coding reduced and refined the initial themes and the development of sub-categories narrowed the coding and analysis (Glesne & Peshkin, 1992). Often the data collected was coded in a variety of different ways and this resulted in some over-lap between the categories. Themes that developed during the coding were explored and verified. Each theme was examined against the wide array of data collected during the teaching experiment. This included the transcriptions, field-notes, written work samples and reflections.

3.7   VALIDITY AND RELIABILITY

Within a qualitative framework, key aspects of validity are honesty, depth, richness and the scope of the data. Validity is gained through provision of rich, detailed descriptions and comprehensive measurements (Babbie, 2007; Cohen, Manion & Morrison, 2003). Internal validity refers to the credibility of the findings and whether the explanation of an event can be sustained by the data. Prolonged engagement in the field, persistent observation and triangulation of data establishes the credibility of a study (Guba & Lincoln, 2006). In the current study, there were lengthy periods of observation within the classroom and comprehensive field-notes were recorded. This was coupled with multiple methods and triangulation of the sources of data to maintain the credibility of the study.

Transferability of the findings or external validity is the second element of validity. This refers to the degree to which results can be generalised to a wider population (Scott & Usher, 1999). Provision of thick descriptions and rich data enables readers and users of the research to determine whether transferability is possible and to what extent the results can be generalised to other contexts (Cohen et al., 2003; Guba & Lincoln, 2006). In this study,

set within the naturalistic context of the classroom, this process is supported through the provision of rich descriptive data.

Reliability refers to the accuracy and comprehensiveness of the research. Multiple realities evident in qualitative accounts challenge traditional notions of reliability (Seale, 1999). Within qualitative research, reliability is viewed as the fit between what is recorded as data and what occurs in the setting under study (Bogdan & Biklen, 2003; Cohen et al., 2003). To maintain reliability there is a need to clearly define the researcher's position and identity. Using multiple sources of data and the provision of a comprehensive audit trail also allows the data to be confirmed (Seale). In this study, the researcher's position and assumptions are clearly defined and detailed documentation of the procedures and descriptions of the multiple methods and sources of data are provided.

Trustworthiness can be established in qualitative studies through the use of triangulation of the data (Cohen et al., 2003). Using multiple sources of data collection provides a wider and deeper view and helps to eliminate bias that results from only relying on one research method. The corresponding outcomes of the data collected from different methods provide greater confidence in the findings of this study

3.8   ETHICAL CONSIDERATIONS

This study was designed and conducted in accord with the Massey University code of ethical conduct for research, teaching and evaluations involving human participants (Massey University, 2004). The key principles of this code include respect for persons, minimisation of harm, informed and voluntary consent, confidentiality, truthfulness and social and cultural sensitivity. Ethical approval was sought and obtained prior to data collection. All participants involved in the research were provided with the relevant information required to gain their informed consent (Appendix D, E & F). As the research involved children under the age of fifteen years old consent from their parents or guardians was also obtained.

As a practising teacher and a member of the New Zealand Educational Institute, the researcher also upheld the ethical guidelines of the New Zealand Educational Institute's Code of Ethics (New Zealand Educational Institute, 2001). In accordance with the core principles of this code, the researcher conducted the research and reported in a way that upheld what is best for the education and welfare of children and for the profession of teaching.

Ethical dilemmas particular to this research pertain to the setting of the study within a school and the position of the researcher as a teacher (on study leave). Particular ethical dilemmas due to the researcher's position as a teacher and colleague of the teacher participant were anticipated. With shifts in roles, changes in the relationship between the teacher and researcher are inevitable (Doerr & Tinto, 2000). However, this was not a negative change as the study was grounded in practice and the teacher's voice was made visible and valued throughout the process through clear communication and interaction with the researcher. No evaluations of the teaching and learning programmes were made other than those grounded in the context of the study. Harm to the teacher participant was minimised through open and honest discussion. Collaborative analysis of data, planning and reflective meetings was undertaken in times and settings of the teacher participant's choice. The anonymity of the teacher was maintained at all times and care was taken to make no evaluative judgments of the teacher or instructional programmes in the classroom.

Potential harm to students was minimised as the research was undertaken using normal classroom practices. Anonymity within the classroom and between participants was difficult as the participants were known to each other therefore confidentiality of the participants could not be wholly guaranteed. Steps taken to maintain anonymity of the teacher and students involved in the research included the assignment of pseudonyms and absence of identifying information within any written reports. Potential harm to the school was minimised by the use of non-identifying information in reporting.

3.9　SUMMARY

A qualitative research design was selected as the most appropriate research method for this study. In order to examine students' development of early algebraic understanding in a naturalistic classroom environment a teaching experiment design was used. Data was obtained through three key methods: observation, interviews and classroom artefacts. Validity and reliability of the results was ensured through clear documentation of data. At all times, ethical principles were upheld and the voice of both the teacher participant and student participants was considered. The findings of this study are documented in Chapter Four, Five and Six.

CHAPTER FOUR

DEVELOPING EARLY ALGEBRAIC UNDERSTANDING IN AN INQUIRY CLASSROOM

4.1   INTRODUCTION

This chapter describes how the social and socio-mathematical norms of the classroom contributed to students' construction of early algebraic understanding. Section 4.2 establishes the classroom context in which this study took place and summarises the structure of the learning sessions. Section 4.3 outlines the patterns of interaction and discourse in the classroom and the specific pedagogical actions which led to collaborative interaction and productive discourse. Section 4.4 describes the pedagogical actions which were used to shift students' thinking from numerical to algebraic focus.

4.2   THE CLASSROOM CONTEXT

The following section describes the classroom context and how it was specifically designed as an inquiry environment and adapted to support student construction of early algebraic understanding. The discussion highlights the role the teacher takes in establishing the social and mathematical norms of an inquiry classroom.

4.2.1   STRUCTURE OF THE LEARNING SESSIONS

All learning sessions began with whole-class activity in which the students were paired to work for 10-15 minutes. In their pairs, students discussed a range of early algebraic concepts (see Chapter Three for unit overview). These included discussion of true and false number sentences and position statements constructed from previous student conjectures. As the students worked, the teacher moved among them listening, questioning, and selecting specific students to contribute to the later whole-class discussion. The whole-class activity concluded with a 10-15 minute whole-class discussion where students shared solution strategies, made further conjectures or generalisations and justified their reasoning.

After the whole-class activity, there was a small group lesson involving 12 to 16 students (see Section 3.5). Those students who did not participate in the group lesson worked

independently on mathematical problems consolidating material from earlier lessons. The small group lesson typically began with an introduction to a mathematical task. Students in this group then worked in collaborative problem-solving groups consisting of four students for 20-25 minutes. During this time, the teacher moved among the small groups of students. She listened to solution strategies, used questioning to advance their discussion and, modelled and scaffolded desired mathematical behaviour to establish appropriate social interactions. The group lesson concluded with a 15-20 minute whole group discussion based on the focus problem and associated solution strategies. At the end of every maths session all students recorded a reflection in which they presented their personal perspective of their mathematical learning.

4.3   DEVELOPING COLLABORATIVE INTERACTION

Collaborative interaction in productive classroom mathematical discourse requires students to take the role of active listeners and participants. However, when the study began many students viewed their role as passive listeners. For example, when Ella (the teacher) asked the students what their role was during class or whole group discussions, a typical response was: *Sit quietly and listen to what they are saying and don't interrupt.* Class or whole group discussions at this point were characterised by unproductive silence and question asking was limited to teacher modelled questions which did not relate to sense-making of an explanation. For example:

> Mike: *Could you explain that in a different way?*
> Hamish: *Could you have done it any other way…?*

At this stage, many students engaged in disputational talk focusing on self-defence and holding control rather than reaching joint agreement (Mercer, 2000). For example, during small group work Peter explained his choice of solution strategy as: *Because I felt like it.* In another instance when a small group examining true and false number sentences failed to reach a group consensus they began to use disputational talk:

> Rani: *If you plus 3 to equal that.*
> Matthew: *No you can't do that.*
> Rachel / Rani: *Why?*

Matthew: *Because if you do then it's changing the whole thing.*

Zhou: *I'm getting even more confused.*

Rachel: *Guys can you please be quiet for one sec.*

Rani: *Be quiet.*

As Wegerif and his colleagues (1999) described, this type of talk does not support collaborative examination of the different reasoning used by members of the small group.

Through our discussion and reflections on the existing interaction patterns, Ella realised that she needed to refine the social norms so that all students engaged in collaborative interaction. As McCrone (2005) argues specific teacher actions are required to lead shifts in students' participation in productive discourse so they become critical active participants. The specific teacher actions that were either introduced or strengthened within the research project phase are detailed as follows.

Scaffolding small group norms

In order to directly address how the students worked in groups, Ella introduced a code of working together that involved only one pen and one piece of paper for the group to use. She also required that *all* group members be able to share back a group selected strategy solution.

Throughout the term Ella maintained discussions about the responsibilities when working in small groups. She reminded the students of their responsibility to each other but she also indicated their need to actively engage, question and individually sense-make:

Ella: *You have to help and you have to understand, everyone in your group needs to understand the strategy. It is not a good enough if it is only one person you need to try and help the rest of your group understand it.*

Ruby: *You have to ask questions if you don't understand.*

Ella: *Exactly you don't just sit there and hope that others will explain it to you. You need to ask questions yourself.*

In this way Ella attended to the group processes. Other researchers (e.g., Manoucheri & St John, 2006; Wood & Yackel, 1990; Yackel et al., 1991) have described similar pedagogical actions to develop collaborative interaction.

Modelling active listening and questioning

Ella used class and whole group discussions to develop productive mathematical talk. Active listening and questioning were emphasised:

> Ella: *Your job in maths is to actually think about what other people are saying and whether or not you are agreeing. Think about is there a question I need to ask as she goes along.*

Student attention was drawn to students who modelled appropriate behaviour. For example, when Zhou made a recording error, Ella asked if anyone wanted to comment. When Josie, a member of Zhou's group, stepped in to progress his explanation, Ella affirmed Josie's participation:

> Ella: *Thank you Josie for helping to clarify there. Can you see what she did then, that's what I mean get the help, the support from your group. That was a really good example of that.*

As Cobb (2000a) illustrated, class and whole group discussions offer opportunities for teachers to advance the instructional agenda. They also provide opportunities for teachers to model and encourage desired mathematical behaviour.

Learning to agree and disagree

The students in this classroom worked in groups where the emphasis was placed on developing collaborative agreement. Our review of the initial set of videoed lessons enhanced Ella's awareness that the students interpreted this as always needing to agree. Early in the project (Lesson Four) Ella directly discussed with the class how to disagree within discussions:

> Ella: *What if you don't agree?*
>
> Mike: *If you don't agree ask them why… why did you do that?*
>
> Ella: *You can say I'm not sure about that, I'm not convinced by that part there. Can you convince me? …it's not just sitting there and tuning out while the other people*

are talking, you actually need to be involved and engaged in what is happening. So if you disagree and are not sure of that part you actually need to say that.

In following lessons, Ella sought other opportunities to affirm students' right to disagree:

Ella: *Good on you Bridget, because that actually takes someone brave to say they are not entirely convinced.*

Valuing student errors

Student errors are useful as reflective tools to provide students with opportunities to recognise errors and re-conceptualise their reasoning (R. Hunter, 2002; Kazemi & Stipek, 2001). The following two vignettes illustrate how when students presented erroneous explanations Ella intervened and asked all students to think carefully about what was being explained. At the same time she gave the student explaining, time to re-conceptualise their misconceptions.

Using student errors as reflective tools

During a whole group discussion, Heath explained his solution strategy to a problem[3].

Heath writes $Z - 9 = X$.

Heath	The first one is how much we needed to save, the next one is how much is in the bank account, that's the nine and the last one is the cost of the T-shirt.
Ella	Talk to the person next you about a question you might need to ask Heath.
Students talk to the person next to them.	
Ella	I'm just going to ask Heath now that you've had a little opportunity to rethink that…you said the amount that needs to be saved, takeaway nine equals the cost of the T-shirt. Can you rethink that a little bit…
Heath	I actually meant that that [points to the z] is the cost of the T-shirt then we minus nine off that to get how much we need.
	(Lesson 5)

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[3] If you had $9 in your bank and wanted to buy a t-shirt for $17, how much do you need to save?
What about if the t-shirt cost $20 or $26 or $40?
Have a go at solving the problem and see what changes and what stays the same.
See if you can find a way to write a number sentence algebraically so someone could use your number sentence to work out how they need to save no matter what the cost of the t-shirt.

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During a whole class discussion about the properties of zero Gareth made a conjecture.

Gareth   H times zero plus Z equals X.

Ella   Talk to the person next to you. Do you agree with this statement H times zero plus Z equals X? You need to convince us why you agree or disagree.

Following the discussion time, Ella asked Gareth to re-evaluate his conjecture.

Gareth   We worked out there had to be two Z's, one after the equal sign because H times zero equals zero… the Z's should be on both sides of the equal sign. H times zero plus Z equals Z.

(Lesson 17)

Comparing mathematical solution strategies

Reflective space allowed students to explore and analyse the similarities between explanations. The following vignette illustrates how when a student offered a similar algebraic number sentence to one already given, Ella asked the students to examine and compare the algebraic number sentences.

Comparing algebraic number sentences

During a whole group discussion, Rachael had shared the following algebraic number sentence as a solution strategy.

Rachael writes $9 + \blacksquare = A$ …

Ella   Does anyone else have a different way of representing that problem? Okay Gareth.

Gareth writes $9 + \bullet = A$.

Ella   I want everyone to look at that and I want you to think has Gareth shown us a different way or is it similar to a way that is already there?

She then provided space for the students to think and discuss each sentence with others seated near them.

Ella   Gareth do you think that it is similar or different to one that is already there?

Gareth   Similar.

Ella   Why is it similar?

Gareth   Because that [points to $9 + \bullet = A$] is just another way of doing that (points to $9 + \blacksquare = A$].

(Lesson 5)

Questioning

Questioning was a tool Ella used to develop richer discussions. She also listened closely to how students questioned and then engaged the students in direct discussion:

    Ella: *What I have noticed often is that people are asking the question can you*

           explain it in a different way? Now that isn't always helpful and sometimes we just

use it because we don't know what else to ask so what are some other questions that we might have to ask during this session?...

Bridget: *We had to convince people that it would work for any number including zero.*

Ella: *Great so you can use words like convince us that it would work for any number?...What are some other questions we might ask?*

Georgia: *Why did you do that?*

The teacher modelling of questions was observed to impact on students' use of questioning. In the later part of the study, students were frequently observed using questions that focused on justification. It appeared that the requirement that students provide conceptually focused reasons for their mathematical actions facilitated discourse which involved conceptual reasoning and justification (Kazemi & Stipek, 2001).

Revoicing and positioning students

The development of discourse premised on inquiry and argument is dependent on specific teacher actions. These include revoicing and the positioning of students (Wood & McNeal, 2003). Ella increasingly used student positioning in conjunction with revoicing to ensure that all students in the group took a stance. As an example, when discussing the commutative law a student stated:

Rachel: *It didn't work with everything.*

Ella: *So what did it work with?*

Rachel: *Pluses.*

Ella: *So you're saying it only worked with the plus or it worked with plus?*

Rachel: *It worked with plus.*

Through her revoicing, Ella positioned Rachel to further explain and justify her stance.

Ella guided the development of norms including the requirement that explanations consist of mathematical arguments and that consensus be reached through mathematical argument. The following vignette illustrates Ella's facilitation of the development of these norms. The vignette shows how she positioned students to take a stance to justify their reasoning and she also stepped in as a participant to model how to justify the reasoning.

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Developing the norms for mathematical argumentation

During a whole group discussion Mike has stated an incorrect generalisation for the task[4].

Mike   You would times it by five then you would minus one because of the six…it would be one over so you would have to minus this to make it fair.

Ella   Mike said the number of houses times five minus one because of the six you get at the start. Does everyone agree? Could someone show us why or why not you agree?

Students discuss the generalisation.

Ruby   [draws a house] That is one house and if you added another one, that is always going to be a six when you times it by five you would actually add one because you have timesed that by five and it's still a six so you would add it on.

Ella steps in to provide a counter argument using equipment.

Ella   [builds representation of two houses] I could show you another way why it doesn't work. Now I have to times by five and two times five is ten, now if I take it away I am going to have an incomplete house. I have to add one so that is my two times five, to make it complete I need to add one.

(Lesson 8)

4.4   THE EMERGENCE OF INQUIRY AND JUSTIFICATION

Evident in the data is how the students' mathematical talk during small group work changed after social norms which facilitated collaborative mathematical discourse were established. As the project progressed, students were frequently observed to be engaging in exploratory talk as they investigated and critically examined their shared reasoning. For

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[4] Jasmine and Cameron are playing "Happy houses". They have to build a house and add onto it. The first one looks like this…..

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 / \
|_|
```

The second building project looks like this….

```
 / \/ \
|_|_|
```

How many sticks would you need to build four houses?
How many sticks would you need to build eight houses?
Can you find a pattern and a rule?

example, as a group examined a functional relationship problem[5] during Lesson Nine, they collaboratively interacted and explored their ideas. Initially, Josie made an error when solving the output value for Cross 6:

Josie: *It's fourteen.*

Steve: *No it is thirteen because you are adding two each time. It doesn't work because if you are adding two on each time and it is odd numbers it can't be fourteen because it's an even number.*

Josie listened carefully to Steve's argument and subsequently used his reasoning to explain how her explicit generalisation was linked to the geometric model.

Josie: [points to the vertical line] *There is always one in the middle. It is always an uneven number because there is always one in the middle for that line there.*

Teacher modelling that focused on the requirement for justification of claims and a press for collaborative interaction led to shifts in students questioning. The following three vignettes illustrate different aspects of how the students appropriated the models Ella had provided and began to independently press their peers for justification.

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a) Draw in the next two crosses in this pattern.
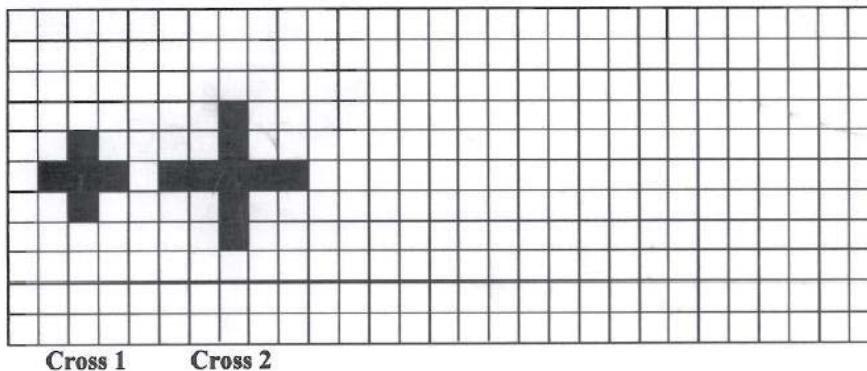
Cross 1   Cross 2

Complete the table

	Cross 1	Cross 2	Cross 3	Cross 6	Cross 20
Number of squares across					
Total number of squares					

What is the word rule for the pattern?
Can you write it as an algebraic number sentence?
Use the rule to find the number of squares across and the total number of squares for Cross 101.

Shifting to justification

During small group work Ella asked a group of four students to identify what changed and stayed the same in an equation.

Bridget   Five
Heath   So you think five. Why do you think five is the main number?
Then when Bridget could not provide justification, Heath stepped in.
Heath   I think it's because that's how much you get paid.

(Lesson 6)

During a whole group discussion Heath explained his solution strategy for a problem[6] and another student asked a question focused on eliciting justification.

Heath   You add three to each table then the plusing two bit…
Josie   Why isn't it two fives added together?
Matthew   [points to the model] Because you couldn't put one there…
Josie   But each table is meant to have five.
Heath   Yeah but on one table its five, it starts off with five but then you…
Hayden   You can't sit someone right in the middle of the table. They can't sit here [points to the middle of the desks] because they'd be on the table.

(Lesson 7)

A student pressed for a contextual link and justification of an algebraic number sentence.

Josie   A times two plus one equals Q.
Gareth   Is the times two and plus one always the same?...What does A represent?
Josie   A represents the number of the cross because this is cross three…that's the number of the cross and you've got it there.

(Lesson 9)

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[6] At the table 5 people can sit like this ……

When another table is joined this many people can sit around it…

Can you find a pattern? How many people could sit at 3 tables or 5 tables or 10 tables? See if your group can come up with a rule and make sure you can explain why your rule works.

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4.5   SHIFTING FOCUS TO ALGEBRAIC REASONING

Algebraic reasoning requires a shift in students' thinking beyond the numerical nature of a task to investigate the more general properties (Blanton & Kaput, 2003; Carpenter et al., 2005a; Kaput & Blanton, 2005). As will be discussed in more detail in Chapter Five, students initially attended to the numerical aspect of the tasks rather than their algebraic nature. Building on their earlier experiences with number computations many students focused on the use of a range of different solution strategies as the goal of the activity. As they constructed various strategy solutions they engaged in cumulative talk avoiding questions and argument. This meant each strategy remained unexplored or unexamined. The following vignette illustrates students' initial focus on the numerical when engaging in an algebraic task.

Constructing numerical strategy solutions

Students were working on a problem[7] in a small group.	
Heath	Because it's nine you could just add that up to ten and then you'll have to plus seven and then you'll minus the one again and plus on to the seven to make the eight...
Sangeeta	Another way you could do it is…you could take one of the seven and add on to the nine and that will make ten a tidy number and then
Ruby	See how much it is to get to the seventeen…
Hamish	If it was twenty-six and you have nine dollars…you would minus six off the twenty-six so you're got twenty dollars and then three left to minus, seventeen, so all you need to save is seventeen dollars to get the T-shirt.
	(Lesson 5)

One of the powerful ways that enabled Ella to help shift the mathematical focus from constructing many strategy solutions to closer examination and analysis of each strategy was to provide students with space to reflect on their reasoning and the reasoning of others. She did this through the introduction of a 'think time' whereupon she interceded, asked for questions, and then paused during explanations:

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[7] If you had $9 in your bank and wanted to buy a t-shirt for $17, how much do you need to save?
What about if the t-shirt cost $20 or $26 or $40?
Have a go at solving the problem and see what changes and what stays the same. See if you can find a way to write a number sentence algebraically so someone could use your number sentence to work out how they need to save no matter what the cost of the t-shirt.

Ella: *Does anyone need to ask a question?*

Or she asked the students to use the space to reflect on their own reasoning:

Ella: *I just want everyone to reflect on their own learning and I want you to think are you convinced by what Susan said there? That four hundred and seventy-one take away three hundred and eighty-two equals four hundred and seventy-four take away three hundred and eighty-five because you add three to both those numbers four hundred and seventy-one and three hundred and eighty-two so you are going to end up with the same number. If you're not convinced that's fine you can just tell us. Anyone not convinced?*

Active engagement in discussing and examining reasoning was required to shift students from numerical to algebraic focus. Other researchers (e.g., R. Hunter & Anthony, 2003; Kazemi & Stipek, 2001) also illustrated the importance of providing students with a space to think, particularly when developing understanding of difficult concepts. To move the students beyond focusing on numerical strategy sharing, Ella had to explicitly intervene and refocus their attention on the algebraic nature of the tasks. Ella prompted students during small group work by saying: *Remember it's not so much about the number strategies to get to thirty-five what I'm wanting to know is how to represent this algebraically.* She also provided further guidance through specific prompts and questions: *Have a look to see what they've done there and up here as well. What stays the same and what changes?*

Maintaining emphasis on the algebraic nature of the task in whole class discussions also required ongoing teacher attention. Ella used prompts: *Remember how we said we want to do it algebraically rather than numerically.* As a result, many of the students shifted their focus to the algebraic nature of tasks. This is illustrated in the following vignette.

Shifting focus to the algebraic nature of a task

During small group work four students were working together to solve a problem[8].

Caitlin	I had an idea that we could say if you had two numbers and added them together.
Peter	Do you have any specific numbers?
Caitlin	No because we have got to do something that will work for everything we can't just do something that will only work for one number sentence…
Peter	So are you plusing or minusing?...
Caitlin	Plus so if you got two numbers and added them together and you got the answer…when you took away that number there, you would have to take-away the same number.

(Lesson 18)

4.5 SUMMARY

This chapter has illustrated how the social and socio-mathematical norms of the classroom influence students' construction of early algebraic understanding. Productive mathematical discourse and collaborative interaction requires specific teacher actions and attention. The teacher in this study took a key role in facilitating the establishment of norms which guided student interaction within her classroom. Through pedagogical actions she guided effective discourse and supported student engagements in collaborative interaction.

The chapter outlined the changing participation patterns. It demonstrated the initial unproductive patterns which were characterised by students taking a passive role or engaging in disputational or non-evaluative cumulative talk. It highlighted the shifts in participation and interaction following specific pedagogical actions which aimed to develop productive discourse, collaborative interaction and shift focus from explanation to justification. The investigation revealed how the pedagogical actions effectively shifted students in this study towards greater use and facility with the mathematical practices of argumentation and justification.

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[8] Ella had 5 grams of salt for her science experiment but the instructions said she needed 11 grams. How much more salt does Anna need?
Solve this problem using addition.
Solve this problem using subtraction.
Can you always switch addition and subtraction like you have done to solve the problem?
Can you write a number sentence to show this using **If….and then** which would work for any number?

CHAPTER FIVE

EXPLORING EARLY ALGEBRAIC UNDERSTANDING THROUGH NUMBER

5.1   INTRODUCTION

The previous chapter drew attention to the way the context of the inquiry classroom supported students as they constructed early algebraic concepts. Evidence was provided of the important role the teacher had in establishing norms which supported collaborative interaction and productive mathematical discourse.

This chapter describes the ways in which students were supported to construct early algebraic understanding through exploration of number. The findings are reported in three broad sections. Section 5.2 describes the role of the equal sign in developing students' early algebraic understanding. Section 5.3 describes the link between algebraic notation and early algebraic understanding and Section 5.4 examines how exploration of the properties of numbers can lead to early algebraic reasoning. Each section begins by outlining students' initial understandings followed by a description of the classroom activities and outcomes of the activities. Finally a summary of student understanding following the teaching intervention is provided.

5.2   UNDERSTANDING THE EQUAL SIGN
5.2.1   STUDENTS' INITIAL UNDERSTANDING OF THE EQUAL SIGN.

In the initial interview when students were asked what the equal sign meant, 68% of the students presented an operational view as illustrated in the following responses: *"the answer"; "equal means what the answer is".* According to Carpenter and his colleagues (2003) and McNeil and Alibabi (2005) this operational view of the equal sign is a typical expectation of instruction that focuses on conventional number operation tasks (e.g., 6 + 4 =).

Open number sentence equivalence problems [9] were used to further probe student understanding of the equal sign. These questions proved to be very challenging, with only 8 students able to solve question A and B and 6 students able to solve question C. Erroneous responses were grouped into categories identified by Freiman and Lee (2004): the direct sum error, when the equivalence equation was treated as a direct sum; sum of all terms, when all the numbers in the equation were added; or complete the sum, when the blank was filled in to complete the equation. Table 5.1 shows the percentage of all students making errors for each equation.

Table 5.1   Percentage of all students' errors (n=25) when solving equivalence problems

	Direct sum error e.g., 27 + 16= 43 + 14 or 84 – 18 = 86 – 68	Sum of all terms error e.g., 27 + 16= 57 + 14 or 54 + 147 = 57 + 36	Complete the sum error e.g., 54 + 3 = 57 + 36
27 + 16 = __ + 14	52%	4%	N/A
54 + __ = 57 + 36	N/A	4%	60%
84 – 18 = 86 - __	16%	N/A	N/A

Although the errors made by the students were consistent with those identified in other research studies (e.g., Freiman & Lee; J. Hunter, 2007) the level of erroneous responses was significantly higher than reported in the previous New Zealand study by J. Hunter.

The open number sentence equivalence problems also highlighted students' use or non-use of relational strategies. Table 5.2 illustrates the percentage of students using relational or computational strategies to solve each problem.

Table 5.2   Percentage of students (n=25) using relational or computational strategies

	Relational strategy	Computational strategy	Error or no response
27 + 16 = __ + 14	20%	12%	68%
54 + __ = 57 + 36	24%	8%	68%
84 – 18 = 86 - __	20%	16%	64%

The percentage of students at this age group using relational strategies is consistent with previous studies of J. Hunter (2007) and M. Stephens (2006).

---

[9] A) 27 + 16 = __ + 14
  B) 54 + __ = 57 + 36
  C) 84 – 18 = 86 – __

An additional question[10] offered to students from the 'middle' and 'hard' group (n=19) proved difficult. Only 3 students were able to identify that the same number would be in both spaces and provide reasonable justification for their response. For example:

> Steve: *Same number...there's an extra minus nine and minus nine added on but if they both get minused by nine it just takes them both down by nine so it doesn't make them uneven or unbalanced.*

Interestingly, 15 students stated that the numbers would be different and attributed the difference to the subtraction of nine in the second equation:

> Peter: *Different number because that one's got minus nine and that one doesn't.*

> Sabrina: *It would be a different number because that one equals thirty-one and that one equals thirty-one minus nine.*

5.2.2 CLASSROOM ACTIVITY TO DEVELOP STUDENT UNDERSTANDING OF THE EQUAL SIGN

For students to abstract their structural numerical reasoning across to algebraic reasoning it is necessary they understand the equal sign relationally as an equivalence symbol (McNeil & Alibabi, 2005). To support the development of a relational view of the equal sign students were required to examine true and false number sentences[11], an activity appropriated from Carpenter and his colleagues (2003). The tasks setting employed small groups of students with varying levels of understanding of the equal sign in order to foster opportunities for students to hear relational explanations and reflect on their own thinking.

Following this activity, there was evidence in the whole class discussion that some students had successfully made the transition to understanding the equal sign as relational equivalence. For example, Mike explained his shift from considering the equal sign indicated an answer: *I thought it was false at first...we kind of thought it was seven plus five equals three, I didn't get the plus nine. Then I found out it was seven plus five equals twelve*

---

[10] Is the number that goes in the ___ the same number in both of these equations?

$2 \text{ x } \underline{} + 15 = 31$ $\qquad\qquad$ $2 \text{ x } \underline{} + 15 - 9 = 31 - 9$

[11]

$8 = 3 + 5$	$8 = 8$	$3 + 5 = 5 + 3$	$19 = 1 + 8 + 10$	$57 + 12 = 69 + 10$
$9 = 6 + 3$	$14 = 14$	$54 + 10 + 1 = 64 + 1$	$7 + 5 = 3 + 9$	$16 - 14 = 14 - 12$
$22 - 9 = 13$	$16 = 19$	$25 - 6 = 19 - 7$	$27 - 8 = 25 - 10$	$57 + 12 = 59 + 10$

and then equals three plus nine and its like what Heath said, equals is the same as. However, the shift took more time for some students. In a following lesson, the misconception of the equal sign as an operator surfaced in a group discussion of the problem 11 – 4 = 10 – 3:

> Peter: *I think it's false because eleven minus four doesn't equal ten.*
>
> Mike: *No it doesn't have to equal ten…is eleven minus four the same as ten minus three? Does eleven minus four equal the same answer as ten minus three? Equals means the same as.*

These findings are consistent with those of Carpenter and his colleagues (2000a; 2005b) who maintain that students need multiple opportunities and experiences of the relational use of the equal sign presented in a variety of contexts.

5.2.3 CLASSROOM ACTIVITY TO DEVELOP RELATIONAL THINKING

During the development of student understanding of the equal sign as relational equivalence, the students engaged in two major representations. Some students considered balance equations as two equations separated by the equal sign (Carpenter et al., 2005b). For example, Mike reflectively wrote: *My thinking was challenged…instead of thinking 7+5 ≠ 3+9 [seven plus five does not equal three plus nine]… and thinking 3 was the answer but seeing that there were two problems 7 + 5 and 3 + 9.* Students with this view use computation to solve open number sentence problems (Carpenter et al., 2003; M. Stephens, 2006). For example, Peter justified 8 + 6 = 9 + 5: *True…because eight plus six equals fourteen and nine plus five equals fourteen.* In a more sophisticated model, students are able to solve the problems by using the relation between both expressions without carrying out a calculation (Jacobs et al., 2007; M. Stephens). This relational type of thinking is recognised as a precursor to formal algebraic thinking (M. Stephens). Consequently, a focus of the study was to develop student use of this form of thinking.

True and false number sentences and open number sentence equations proved useful to develop relational strategies. Initially, instruction tasks which focused on number sentences which used closely related numbers proved effective. During small group work Hannah voiced: *Eleven minus four is the same as ten minus three…because you're just taking away*

one more away from the eleven than the ten. Rani later in a whole group discussion used the difference of one as a justification: *If you do eleven minus four…both of these two numbers they are just one number higher than these ones. If you have a number like twelve minus five and then there was thirteen take away six, they would both be the same and you don't have to subtract the numbers to find out if it's true.*

Specific teacher actions were also important to support the students towards developing confidence with relational strategies. During the whole group discussion, Ella selected a student to model the use of a relational strategy to prove the number sentence was true:

> Steve: *Seven plus five equals three plus nine… equals is like a set of scales… on one side there's seven plus five and the other it's three plus nine and they have to weigh the same thing. If you changed the actual numbers that were there,* [draws scales] *if you minused two off the five and then you plused the two you took off the five on to the seven you'd get nine and three so it would be the same because if you've nine and three on one side and nine and three on the other side it would be the same.*

Ella consistently pressed students beyond calculating the answer to justify using relational strategies. Before commencing an activity she indicated: *I want you to think whether there is a way to prove without actually adding up the numbers.* When noting the use of computation she prompted the students: *I don't want you to have to add the numbers up so think about how can you look at the statement and think is it true or false.* When a computational strategy was used in a whole group discussion she probed for a relational strategy: *is there a way that you can show seven plus five is the same as three plus nine without actually adding it up? Just have some thinking time about how you can prove that without actually adding up those numbers.* Ella highlighted the efficiency of the relational approaches: *So you're telling us that you didn't have to subtract the numbers on both sides? You just looked at the number? Right I want everyone looking at Rani because this is really important.* To make explanations more accessible, Ella modelled how relational strategies could be recorded using arrows (see Figure 5.1).
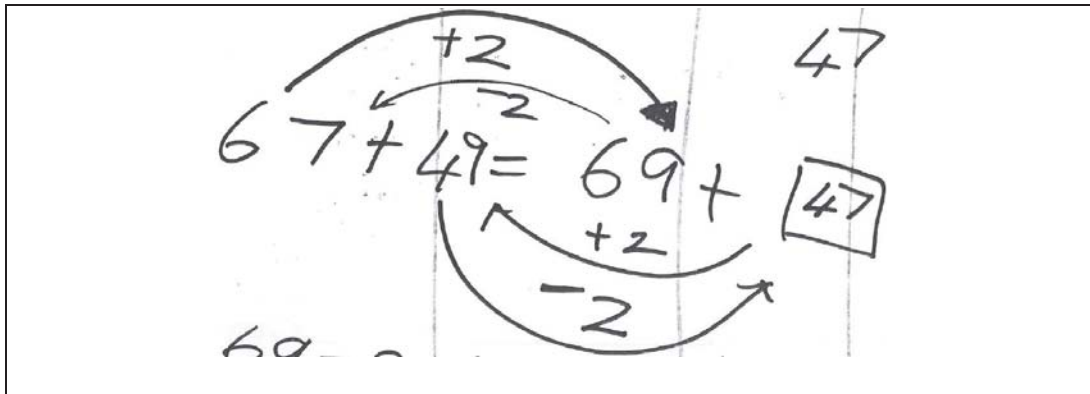
Figure 5.1   Recording relational strategies using arrows

She encouraged students to be explicit in their explanations. For example, Rani explained her solution strategy for 256 + 3 = 246 + 13:

> Rani: *From the two hundred and forty-six to the two hundred and fifty-six there is ten there and from the three to the thirteen there is ten there as well.*
>
> Ella: *Are you adding or subtracting that ten?... Talk to the person next to you about whether or not it is adding or subtracting the ten?...*
>
> Rani: *Subtracting ten and that's adding ten.*

Relational strategies, however, require time to develop and the students fluctuated between their use of calculational and relational strategies. In an early lesson, Matthew gave a relational explanation for 498 + 12 = 488 + 22: *Four hundred and ninety-eight plus twelve is the same as four hundred and eighty-eight plus twenty-two because twelve and twenty-two are pretty much the same except that has ten more* [points to 22] *and that has ten less than that* [points to 12] *…take the ten off that* [points to 498] *to make it four hundred and eighty-eight and then adding that ten onto twelve so it will be the same.* However, in a later lesson he preferred to use computation to solve the number sentence 256 + 3 = 246 + 13: *I think it is true because if you added a three onto two hundred and fifty-six that would equal two hundred and fifty-nine and if you added a thirteen onto two hundred and forty-six that would equal two hundred and fifty-nine as well.* The students also drew on computational strategies when they encountered difficulties convincing their peers through relational explanation. In one example, Rachel began using a relational strategy to solve 583 − 529 = 83 - 29: *You'd take away the five hundreds…if you look at it carefully…you go five hundred*

and eighty-three minus five hundred and twenty-nine is the same as eighty-three minus twenty-nine…take away the five hundreds on those and then it will be eighty-three take away twenty-nine is the same as eighty-three take away twenty-nine. When Rani disagreed her group began to calculate the answers:

> Rani: *If you take away five hundred and eighty-three from five hundred and twenty-nine it will be a higher number than that…maybe if we took away this first.*
> Rachel: *So you take that away.*
> Rani: *Eighty-three round that and the closest number to that is eighty.*
> Matthew: …*The closest number to that is thirty…*
> Rani: *Eighty minus thirty is fifty.*

5.2.4 CLASSROOM ACTIVITY TO ENRICH UNDERSTANDING OF RELATIONAL EQUIVALENCE

In order to further students' understanding of the equal sign as relational equivalence specifically constructed tasks were used. These aimed to develop understanding of the equal sign as balance and resulted in spontaneous use by the students of an analogy of the equal sign as scales. For example, as Zhou explained a solution strategy for a problem[12] he drew a scale. Ella revoiced to allow all students to access the idea of balance. Subsequently, another student appropriated the analogy of a balance scale embedding it in a real-life context: *If you're minusing from one side on a scale, say there is seven pounds of butter on one side and seven pounds of butter on the other side and you are minusing seven pounds of butter then that does matter because one side will just go down and other side will go up… but if you're minusing from both sides there is not really a point because if you take it both off it will still be equal.* Ella revoiced extending this explanation to a more general context: *If they take it only off one side, the other side is going to tilt one way so it won't be balanced, it won't be equal. So you're taking seven off one side then you have to do exactly the same to the other side to make it equal.* In later lessons, the students continued to use the balance scale as a means to sense-make equivalence problems. Heath used his hands to

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[12] The solution to the equation $3n + 15 = 39$ is $n = 8$
What is the solution to the equation $3n + 15 - 7 = 39 - 7$?
(adapted from Knuth et al., 2005)

indicate a scale when discussing a problem[13]: *If it said eighteen plus fifteen minus nine that will keep one side up and the other side down if you were weighing it and if it said thirty-three minus nine that will keep it even.* In the same lesson, Ruby justified using the model of a scale: *You would have to take-away the same number otherwise it wouldn't be right because one side… using the scale thing it would go down* [uses her hands to indicate a balance scale going up and down] *because you took away too much from one side and you took away not the right amount from both sides.*

The use of the balance tasks provided the foundation to press students towards making conjectures, generalising and formalising. Specific tasks[14] facilitated students to construct generalisations. Generalisations were voiced during small group work and whole group discussion and also recorded in reflective statements following the lessons. For example, Zhou recorded in a reflection: *If you have a number sentence that is true…to keep it true you have to add, subtract, multiply, divide the same numbers on each side of the equal sign.* Student generated generalisations were formalised using 'if and then' algebraic number sentences. During small group work, Steve and Sabrina symbolised the groups' generalisation: *If C plus D equals star then C plus D minus Z equals star minus Z. Minus or plus.*

5.2.5   STUDENTS' PROGRESS IN UNDERSTANDING THE EQUAL SIGN

Carpenter et al. (2003) argue that learning situations which challenge students' notions of equality and encourage them to think about relations support the transition from

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[13] Ruby took a quick look at these true/false number sentence problems. Without working out any bit of it she said both problems were true.
The first part said $18 + 15 = 33$ is true.
She looked at the second part which said $18 + 15 - 9 = 33 - 9$
How did she know that the problem was true?
How did she know that the problem was true?

[14] Ruby took a quick look at these true/false number sentence problems. Without working out any bit of it she said both problems were true.
The first part said $11 + 115 = 233$ is true.
She looked at the second part which said $118 + 115 - 9 = 233 - 9$
How did she know the problem was true?
Then she looked at the second problem which said $254 - 89 = 165$ is true.
She looked at the second part which said $254 - 89 + 11 = 165 + 11$.
How did she know that the problem was true? Can you make a generalisation from the problem? Can you think of an algebraic number sentence using if and then that would show your generalisation always works?

computational to relational thinking. The students in this study provided evidence that the learning situations they had experienced supported the development of relational understanding of the equal sign. In the final interview, all students provided relational explanations for the equal sign. Their explanations drew on the many classroom discussions they had participated in:

> Charlotte: *The same as…eight equals eight means eight is the same as eight.*
>
> Hannah: *The same as…six plus two equals two plus six.*

Likewise, a shift in understanding was also evident in response to the open number sentence equivalence problems[15]. Twenty-two students gave correct responses for question A and B, and 14 students responded correctly to question C. There was a significant increase in students using relational strategies (see Table 5.3). In accord with previous research (e.g, Carpenter & Levi, 2000a; Carpenter et al., 2005a; Carpenter et al., 2005b; M. Stephens, 2006) the use of true and false and open number sentences and a focus on relational strategies proved an effective tool to develop student understanding of the equal sign as a relation.

Table 5.3   Percentage of students (n=25) using relational or computational strategies

	Relational strategy	Computational strategy	Error or no response
23 + 15 = _ + 17	68% *(20%)*	28% *(12%)*	4% *(68%)*
81 +_ = 83 + 26	84% *(24%)*	12% *(8%)*	4% *(68%)*
76 – 27 = 78 - _	76% *(20%)*	12% *(16%)*	12% *(64%)*

Note. Initial interview results are in brackets.

Using a question[16] requiring transformation, in the final interview 15 of the 19 students from the 'middle' and 'hard' group were able to recognise the similarities between the equations and justify their response:

> Sabrina: *It would equal the same answer because you are just minusing six from both sides so it wouldn't really matter.*

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[15] A) 23 + 15 = _ + 17
  B) 81 +_ = 83 + 26
  C) 76 – 27 = 78 - _
[16] Is the number that goes in the ___ the same number in both of these equations?
     3 x __ + 12 = 27   3 x __ + 12 – 6 = 27 – 6

Gareth: *It will be the same because that has got minus six and that has also got minus six from the twenty-seven.*

The 4 students from the 'middle' and 'hard' group who continued to demonstrate difficulties with recognising the similarities between the equations still required ongoing development in the understanding of the equal sign as relational equivalence.

5.3   ALGEBRAIC NOTATION
5.3.1   STUDENTS' INITIAL UNDERSTANDING OF ALGEBRAIC NOTATION

Many of the students in this research project had little prior experience with algebraic notation. An initial interview item[17] investigated student use of notation to represent an unknown quantity. Students predominantly used a specific number to represent the unknown quantity (see Table 5.4). These results demonstrate students' unfamiliarity or reluctance to symbolically represent an unknown quantity.

Table 5.4   Percentage of students (n=25) using forms of notation for an unknown quantity

	Correct notation e.g., ■+ 3 N + 3 + 2	Non-standard or incorrect notation e.g., B + 3 = A A + A = C	Number as notation e.g., 2 + 3 = 5	No response
Situation A	24%	4%	60%	12%
Situation B	28%		60%	12%
Situation C	16%	12%	60%	12%

Students from the 'middle' and 'hard' group (n= 19) were asked additional questions[18] to elicit their understanding of what a letter meant in a mathematical context. In response to Question B, 6 students responded that the symbol could stand for four and correctly justified this response. However, 7 students' constructed their interpretation of symbolic

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[17]   What is a mathematical statement or sentence to represent each of the following situations:
  A) I have some pencils and then get three more.
  B) I have some pencils, then I get three more and then I get two more.
  C) I have some pencils then I get three more and then I double the number of pencils I have.

[18] 2f + 3
  a)   What does the symbol stand for?
  b)   Could the symbol stand for 4?
  c)   Could the symbol stand for 37?

letters by guessing or false analogies. Most frequently, these students viewed the letter as an abbreviated word.

> Sangeeta: *There is sort of a clue in that letter because the reason I said four is because four starts with an f.*

In response to Question C, 3 students stated that the symbol could stand for thirty-seven and justified this response. Two students reported that a single letter could only stand for a single digit number. For example Rachel stated: *It would have to be ff. It has to be one number.* Ten students stated that the symbol could not stand for thirty-seven. These results correspond with those described in previous studies (e.g., Knuth et al., 2005; MacGregor & Stacey, 1997; Weinberg et al., 2004).

A further item[19] was used to probe the 'middle' and 'hard' group of students' understanding. Only 4 students stated that the sentence was sometimes true and justified their response.

> Zhou: *These two letters could stand for either different or the same numbers.*

Five students stated the number sentence could never be true and indicated that they considered different letters in an equation could not represent the same number.

> Josie: *M and the P couldn't stand for the same thing.*

5.3.2 CLASSROOM ACTIVITY TO DEVELOP STUDENT UNDERSTANDING OF ALGEBRAIC NOTATION

Understanding of algebraic notation is necessary for students to make the transition from arithmetic to algebra. Development of this understanding requires rich classroom opportunities (MacGregor & Stacey, 1997). Algebrafied arithmetic problems were used as a context for engaging the students in dialogue about algebraic notation (Blanton & Kaput, 2003). Students used both shapes and letters to represent unknowns.

To solve the word problem[20] Ella supported students to record in a systematic way to emphasise patterns in the equations. For example, during the whole group discussion, Ella

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[19] Is h + m + n = h + p + n always, sometimes or never true?

revoiced Hannah's explanation scaffolding her to record in a logical manner: *Nine plus something equals seventeen…what if the T-shirt was twenty dollars, what would the equation look like if was twenty dollars?* In this way, she emphasised the patterns in the equations. She focused the students on what was constant in the equations: *Talk to the person next to you about what has stayed the same in all of those equations?* Then she facilitated the students to identify the unknowns: *What changes in each of those equations?* To introduce students to using algebraic notation to represent a generalised situation Ella asked them to: *Come up with a rule or a statement to tell them what you would do no matter what the cost of the T-shirt.* Students used informal algebraic notation to represent the situation.

Rachel: [writes □ + 9 = a] *We drew a box plus nine equals a.*

Heath also shared his group's alternative strategy z − 9 = x. In subsequent lessons, the students modelled their recording (see Figure 5.2) on Hannah's explanation which Ella had earlier scaffolded. This enabled them to quickly identify the unknowns and construct algebraic notation to represent the situation. For example, Heath constructed a solution strategy for a problem.

Heath: [writes equation] *Triangle divided by five equals spiral.*

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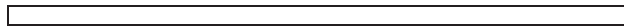
[20] If you had $9 in your bank and wanted to buy a t-shirt for $17, how much do you need to save?
What about if the t-shirt cost $20 or $26 or $40?
Have a go at solving the problem and see what changes and what stays the same.
See if you can find a way to write a number sentence algebraically so someone could use your number sentence to work out how they need to save no matter what the cost of the t-shirt
(adapted from Blanton & Kaput, 2003).
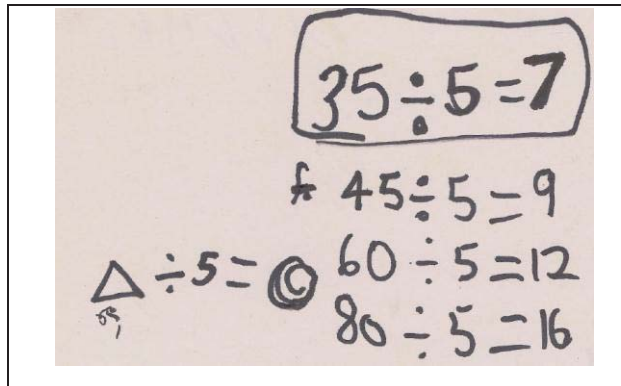
Figure 5.2   Solution strategy for CD player problem[21]

Algebrafied word problems provided a context for students to give conceptual explanations. For example, Zhou identified the constant in the equation: *The nine because that's how much you have in the bank.* Their explanations of the unknowns also linked to the context of the problem. Gareth said: *The amount that you have to save,* Sabrina added to the explanation: *The cost of the T-shirt.* Consistently the students began to link the context of the problem to their explanation. If procedural explanations were used, the students were pressed to provide a more conceptual response. The following vignette illustrates how Ella facilitated the students to collaboratively construct a conceptually focused explanation of their algebraic number sentence and use of notation.

Construction of a conceptually focused explanation of an algebraic number sentence

Students were working in a small group of four to solve the CD player problem.

Bridget	He went like this, five.
Ella	What's the five?
Bridget	How much he gets paid.
Ella	How much he gets paid?
Caitlin/Ruby	How much he gets paid every hour.
Bridget	And then he did times the triangle.

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[21] You would like to buy a CD player that costs $35. You earn $5 an hour at your job. How many hours do you need to work?
What about if the CD player costs $45 or $60 or $80?
Have a go at solving the problem and see what changes and what stays the same.
See if you can find a way to write a number sentence algebraically so someone could use your number sentence to work out how many hours they need to work no matter what the cost of the CD player.
(adapted from Blanton & Kaput, 2003)

Ella	What's the triangle?
Heath	It's how much the CD player costs, oh no that's how much you need to earn an hour.

Ella questions this response and the students discuss what the triangle represents.

Ruby	It's the hours, isn't it?
Caitlin	No because that's the end.
Ruby	No I think hours isn't the end, I think it's after.
Heath	Five hours…for example you need to times seven…you always times the answer which means that's how much hours that you need to do it and then the answer is how much the CD player costs.

Ella asks the students to have some reflective thinking time and re-voices Heath's explanation.

Bridget	I agree because that's how much per hour, and how many hours and that's how much the CD player is [points at specific parts of the problem and the equation the group has written].
Caitlin	But if you do the divideds, the hours, how much the thing is, is at the start, how much the CD player, and then how much you get per hour and then how much you need, how many hours you need to work before you get that much money, to get the CD player.

(Lesson 6)

5.3.3   CONFRONTING A MISCONCEPTION ABOUT ALGEBRAIC NOTATION

Previous research (e.g., A. Stephens, 2005; Weinberg et al., 2004) identified a common misconception held by students is that different letters in an equation could not be the same numbers. Midway through the study, an examination of an algebraic number sentence revealed that many students had this misconception. For example, Josie argued that J + T = T + L could never be true because:

Josie: *L and J can't equal the same number…two letters can't represent the same number in the same equation.*

Ella: *So you are saying that J and L can't represent the same number?*

Josie: *Yeah but T and T have to.*

Other students agreed with Josie's argument.

Sabrina: *I think they can if they are in different equations.*

Josie: *They can if they are in completely different equations…these are two equations which are joined so that means that they can't represent the same number.*

Specifically designed algebraic number sentences [22] were used to confront this misconception. The first number sentence reinforced student understanding that the same letter had to represent the same number. Then the second number sentence positioned the students to engage in argumentation in order to confront the misconception. After lengthy discussion Ella recorded fifteen and fifteen as a possible solution and challenged the students with the question: *Can J equal fifteen and B equal fifteen?* Zhou responded: *Even if they are not the same letters they can still equal the same value… if it is a different letter it still could.* Ella revoiced and emphasised the statement: *In the equation there, that J can equal fifteen and B can equal fifteen there.* Reflective statements recorded by the students following this lesson identified a change in their thinking:

> Ruby: *If it was the same letter it had to be the same number…I thought then the algebraic letters couldn't represent the same thing.*

5.3.4 STUDENTS' PROGRESS IN UNDERSTANDING ALGEBRAIC NOTATION

The final interview responses indicated that there had been a considerable shift in student use of notation to represent varying quantities. Table 5.5 illustrates the percentage of students correctly using notation in response to the questions[23].

Table 5.5 Percentage of students (n=25) using forms of notation for an unknown quantity

	Correct notation e.g., (A + 5) x 2	Non-standard or incorrect notation e.g., A + 5 = B B + B =	Number as notation e.g., 5 + 5 = 10	No response
Situation A	92% *(24%)*	0% *(4%)*	4% *(60%)*	4% *(12%)*
Situation B	92% *(28%)*		4% *(60%)*	4% *(12%)*
Situation C	44% *(16%)*	44% *(12%)*	8% *(60%)*	4% *(12%)*

Note. Initial interview results are in brackets.

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[22] H + H = 30   J + B = 30
What could H be? What could J be? What could B be?
(adapted from Carpenter et al., 2003).

[23] What is a mathematical statement or sentence to represent each of the following situations:
 A) I have some lollies and then get five more.
 B) I have some lollies, then I get five more and then I get three more.
 C) I have some lollies then I get five more and then I double the number of lollies I have.

Their correct statements drew on their classroom experiences:

> Charlotte: *Box plus five.*

> Peter: *A plus five plus three.*

Student use of non-standard notation included for example two equations to indicate the results of the first computation:

> Rachel: *A plus five equals B…and then you'll plus B onto B.*

Additional questions[24] were used with students from the 'middle' and 'hard' groups (n=19) which examined their understanding of what letters represented in a mathematical context. Significant improvement was demonstrated in their responses. In response to Question A, all of these students stated that the symbol in the equation could stand for six and justified this response.

> Sangetta: *M can stand for any number…no matter how big the number, it can stand for it.*

In response to Question B, 17 students stated that the symbol could stand for forty-five and justified this response. However, 1 student maintained the misconception that the variable could only represent a single digit.

> Mike: *No it's only one letter so it would have to be mf.*

A following question[25] was used with the students from the 'middle' and 'hard' groups. This question probed their understanding of the concept that a different letter could represent the same number. Fifteen students identified that the number sentence was sometimes true and provided justification for their assertion.

> Josie: *Sometimes…because e and f could be the same numbers but then they could also be different numbers…b and b have to be the same number and n and n have to be the same number.*

However, 3 students of these students maintained that the same number could not be represented by two different letters in an equation.

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[24] $2m + 5$
  a)   What does the symbol stand for?
  b)   Could the symbol stand for 6?
  c)   Could the symbol stand for 45?
[25] Is $b + f + n = b + e + n$ always, sometimes or never true?

Matthew: *I don't think it's ever true because the first and the last letter are the same but the e and the f aren't the same and one number can't represent two letters.*

The results of the final interview showed that student understanding of algebraic notation had improved. The intervention provided opportunities to explore algebraic notation through a range of contextual tasks. In these tasks the students used letters or shapes as notation to represent a range of values and engaged in extensive discussions about notation. However, there remained some students who demonstrated misconceptions about algebraic notation in the final interview. The need for provision of numerous opportunities to explore symbolic representations is reinforced in the research literature (e.g., Carpenter et al., 2003; Kaput, 1999; A. Stephens, 2005; Swafford & Langrall, 2000).

5.4 UNDERSTANDING PROPERTIES OF NUMBERS
5.4.1 STUDENTS' INITIAL UNDERSTANDING OF THE PROPERTIES OF NUMBERS

True and false number sentences[26] were used to explore student understanding of the commutative principle. In the initial interview, 5 students provided a correct application of the commutative principle to addition and multiplication. These results are consistent with those described by Anthony and Walshaw (2002a; 2002b) and Warren (2001a).

Students' initial knowledge of number properties and operational laws was further probed[27]. Only one student provided a number sentence that was true for any number.

Rachel: *f plus f and then the same as sign and then f plus f.*

These results concur with Anthony and Walshaw (2002a) and Schifter (1999) and suggest that the students had had limited opportunities to explore conjectures and generalisations about number properties and operational laws.

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[26] $15 + 6 = 6 + 15$ $15 - 6 = 6 - 15$
   $15 \times 6 = 6 \times 15$ $15 \div 6 = 6 \div 15$
[27] What is a number sentence that is true for any number?

5.4.2 CLASSROOM ACTIVITIES TO DEVELOP STUDENT UNDERSTANDING OF THE PROPERTIES OF NUMBERS.

Previous research has recognised that students possess a wealth of implicit knowledge about mathematical properties (Carpenter et al., 2003). But when they draw implicitly on these properties they may lack deep understanding (Schifter, 1999). The development of algebraic reasoning requires that the underlying properties of numbers and arithmetic become explicit.

In this section of the study, the activities trialled by Carpenter and his colleagues (2003) were drawn on. This included using true and false number sentences as a context to begin exploring conjectures about number properties and operational laws. Number sentences were also used to develop student use of notation to represent the identities and properties of numbers.

5.4.3 DEVELOPING UNDERSTANDING OF THE COMMUTATIVE PRINCIPLE

From the initial introduction of these activities, an immediate focus was placed on developing the commutative principle through use of true and false number sentences generated by the students. The students readily recognised that addition number sentences were true (e.g., $15 + 3 = 3 + 15$ and $5 + 6 = 6 + 5$).

> Hamish: *It's just the same equation spelt backwards.*
>
> Matthew: *Three plus fifteen is just fifteen plus three twisted around so it is exactly the same.*

However the commutative principle of multiplication posed more challenges. In an early lesson (Lesson Three), one group of students concluded that the commutative law only applied to addition. For example, Ruby stated: *Six times five equals more than five times [six] so it wouldn't work in that way.* Hamish supported her argument: *One times zero is zero and zero times one is one.*

In the initial interviews, it was evident that students did not have access to representations on which they could base their understanding of the commutative law. Therefore an explicit focus was placed on the use of equipment to justify their conjectures and shift their

arguments into general terms. Ella selected students to model how they represented and justified conjectures using equipment. For example, a student demonstrated her explanation using popsicle sticks. Hannah swapped the pile of three sticks with the pile of fifteen sticks: *Three plus fifteen equals eighteen but you could just swap the other ones like the fifteen with the three and the three with the fifteen so it does equal eighteen.* Ella revoiced Hannah's explanation saying: *So you are saying that if you just swap them around it will still be exactly the same amount?* Then she shifted the discussion into general terms and encouraged the students to make a general statement: *Would that work for any set of numbers then when you are adding?* Other forms of equipment were also used to justify conjectures about the commutative nature of multiplication. This included the use of animal arrays and counters. For example, Sabrina used counters to explain: [builds an array with counters] *We put four down there and then we did five across…we thought that if you turn it around and put them down here, it is the same five rows of four.*

Many researchers (e.g., Carpenter et al., 2003; Kaput & Blanton, 2005) maintain that students need to formalise their generalisations. After lengthy discussion of the commutative principle with concrete representation of specific examples, Ella prompted the students and asked them to use an algebraic number sentence to represent the conjectures about the commutative law:

> Ella: *Can you write this as a number sentence that would be true for any number?*
> As a result the students readily provided a range of algebraic number sentences:
> Susan: *Z times Y equals Y times Z.*
> Steve: *A B equals B A*
> Gareth: *We did rectangle plus B equals B plus rectangle*

The combination of time with equipment and press for generalisation appeared to support some students to offer generalisations:

> Josie: *If you get two numbers and you times them by each other…and then if you times them by each other the other way around it will always be the same answer.*

Ella facilitated discussion of symbolically represented conjectures. For example, she recorded symbolised conjectures[28] on the white-board then asked students to discuss these: *Can you look for the ones which are always true…think about why it is always true as well?* During further discussion, the students illustrated their knowledge that addition was always commutative:

Heath: *They always will be true... because they are just a reflection…*

Ruby: *It's just swapped… it is just the same numbers the opposite way.*

Sangeeta also used the symbolised conjecture to make a general statement about the commutative law: *If you use two sets of numbers which are the same the statement will always be true… with addition or multiplication.*

Specific teacher prompts encouraged student exploration of the commutative principle with other operations. For example, Ella probed: *Does it work with other things like division or subtraction?* Lengthy discussion led to students' recognition of the non-commutative nature of subtraction and division. They justified their responses using counter-examples:

Gareth: *Seven minus four doesn't equal four minus seven… because seven minus four equals three and four minus seven equals minus three.*

5.4.4 DEVELOPING UNDERSTANDING OF THE PROPERTIES OF ODD AND EVEN NUMBERS

Previous research (e.g., Blanton & Kaput, 2003; Carpenter et al., 2003) illustrated that odd and even numbers can be used to investigate the properties of numbers. To introduce odd and even numbers Ella used a dice game[29]. Students used their recorded patterns to list the results and then began to formulate conjectures from these:

Rani: *That's odd, even, odd and then there is even, even, even, and even, odd, odd.*

Matthew: *There is no odd, odd, odd.*

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[28] $B + \square = \square + B$ $\qquad$ $J + T = T + L$ $\qquad$ $Q + R = R + Q$

[29] Odd and even dice game. Students rolled the dice twice and recorded the numbers in a table, noting whether they were odd or even. The resultant numbers were added and whether the number was odd or even was recorded

Following discussion of these results Matthew generalised: *An odd plus an even or an even plus an odd always equals an odd.* In the whole group discussion, students shared a range of conjectures they had formulated through examination of the patterns:

Rachel: *If you add two odds they equal an even.*

Hamish: *An odd plus an even equals an odd.*

In the next lesson, Ella introduced the use of popsicle sticks and grid paper to test the students' conjectures. Many students initially responded by attempting to justify their conjectures by making numbers and symbols with the equipment. Ella intervened and redirected the students: *I am not wanting to see plus signs and equal signs and E's made out of pop sticks to say E plus O equals O. That's not proving to me that an odd number plus an odd number equals an even number. You need to think about how you are going to prove it.* She directly modelled the pattern of odd and even numbers using popsicle sticks (see Figure 5.3).
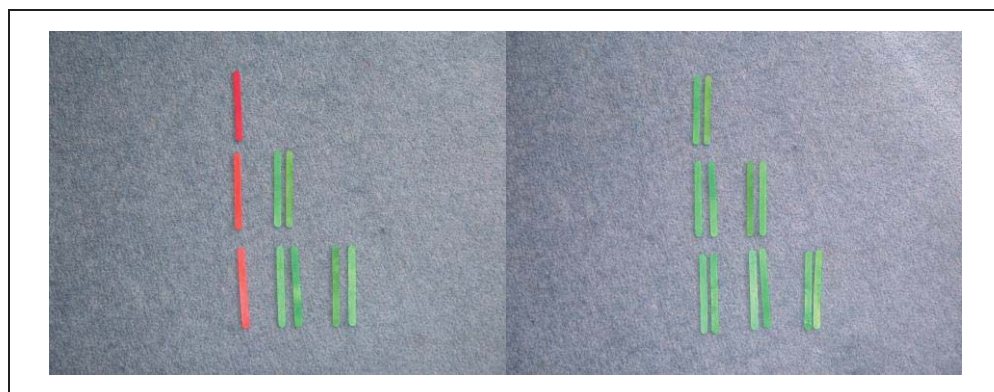
Figure 5.3   Patterns of odd and even numbers

Subsequently, the students built on this model. Hannah justified the conjecture that adding two odd numbers made an even number: *You could move the three* [moves single sticks to make pair] *or you could move the five* [moves single sticks to make pair]. Again Rani used the teacher's model to justify: *For an even plus even equals even and for every number you could have pairs like twos, two and two and two and two.*

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To deepen student understanding of odd and even numbers, Kaput and Blanton's (2005) work was drawn on. They demonstrated that quasi-variables can be used to focus student examination of the structural features of odd and even numbers. Equations[30] with large numbers as quasi-variables were introduced. The students' initial discussions of the equations focused on the structure of the odd and even numbers:

> Rani: *In the six one it goes even number, odd number, odd number and even number.*
>
> Matthew: *No on the first one they are both even…*
>
> Rani: *But the five and the six aren't even.*
>
> Matthew: *But it is the number at the end that matters.*

Again the structural aspects of these numbers were discussed during whole class discussion. As a result the discussion addressed other misconceptions. For example, Michelle looked at the two first numerals in the numbers (e.g., 6398 + 5296) and stated that the first equation was odd: *Because six plus five equals eleven and eleven is odd.* Gareth challenged this: *How do you know that that works to make that question odd because it is just using the six thousand and the five thousand?* Ella noted the misconception and stepped in to redirect the focus: *When you add odd numbers and even numbers you don't necessarily look at the first number. Which part do you think we should look at?* Caitlin responded: *The numbers at the end.* This episode was noted in several student reflections. For example, Rani noted: *At first I thought you look at the front numbers and now I think you look at the back.*

The students were also required to examine the structural features of numbers to determine whether the sum would be odd or even, a practice frequently prompted by the teacher:

> Ella: *You don't have to add the numbers though. What do you notice about five and seven Hannah?*
>
> Hannah: *They are both odd.*
>
> Ella: *They are both odd. What do two odd numbers make together?*
>
> Hannah: *An even number.*

Following this episode several students were noted to be using similar justification:

---

[30] 1) 6398 + 5296   2) 189 197 + 36 455   3) 192 197 + 124 364

Matthew: *So eight and six. Even plus even equals even right? So it would be even…*
for the second one the end numbers are seven and five and they are both odd so you
would go odd plus odd equals even.

Tasks that used larger numbers as quasi-variables and prolonged discussion appeared to assist the development of students' understanding of the properties of odd and even numbers. These results matched those of Kaput and Blanton (2005). They showed that by using sufficiently large numbers, students "had to examine the structural features to reason whether a sum would be even or odd, and were led to focus on the properties of evenness and oddness and (implicitly) to treat the numbers as abstract placeholders" (p. 114).

5.4.5   DEVELOPING UNDERSTANDING OF THE PROPERTIES OF ZERO

Anthony and Walshaw (2002a) described how students in their study over-generalised the properties of numbers. Their challenge to explicitly discuss identity elements and zero within the classroom was taken up in this study. Building on a student's conjecture[31] Ella initiated discussion of the properties of zero. The students recognised the conjecture was true. They justified it using numerical examples and their understanding of algebraic notation:

Rani: *True because the H and H can stand for any number so it could be eight and*
eight or nine and nine.
Hayden: *They are the same letter so they have to be the same number.*

The use of a true and false number sentence (e.g., 256 + 72 = 256) further facilitated student exploration of the properties of zero. After discussion of the false number sentence, Hayden suggested a solution to make the number sentence true: *Two hundred and fifty-six plus seventy-two minus seventy-two equals two hundred and fifty-six.* Gareth built on Hayden's suggestion with a conjecture: *If you add a number and then minus it, it equals zero.* Following this discussion, Sangeeta formalised the conjecture and offered the generalisation: *Q plus H take away H equals Q.* Steve offered a warrant to back up the

---

[31] Matthew made a statement, he said H – H = 0.

claim: *It's true because if you add something to a number and take it away you get the original number.*

Teacher facilitated discussions of the properties of zero supported students to co-construct mathematical arguments. For example, Georgia stated: *P times zero equals zero*. Cameron challenged this: *I actually think P times zero equals P.* In the following discussion other students co-constructed a mathematical argument supported by concrete justification to convince Cameron the conjecture did work.

Rani: *Because if you times any number by zero it will equal zero.*

Sabrina: [draws a diagram on the white-board] *Four times zero. It would be four lots of zero which is nothing. There is nothing in each of the groups which is zero.*

5.4.6 DEVELOPING UNDERSTANDING OF THE RELATIONSHIP BETWEEN ADDITION AND SUBTRACTION AND MULTIPLICATION AND DIVISION.

Specific tasks drawn from the work of Carpenter and his colleagues (2003) were used to facilitate students to explicitly examine inverse relationships. In solving the tasks students recorded the equations and identified the numerical link. For example, Ruby recognised the relationship in the problem[32] using recorded equations: *I am going to write down the eleven take-away five equals six and then the five plus six equals eleven* [writes $11 - 5 = 6$ and $5 + 6 = 11$]...*you can see you are just switching them around.* Similarly in another lesson, Rani recorded the equations $6 \times \_\_ = 120$ and $120 \div 6 = \_\_$ to solve the problem[33]. Hayden

---

[32] Ella had 5 grams of salt for her science experiment but the instructions said she needed 11 grams. How much more salt does Ella need?
Solve this problem using addition.
Solve this problem using subtraction.
Can you always switch addition and subtraction like you have done to solve the problem?
Can you write a number sentence to show this using **If….and then** which would work for any number?
(adapted from Carpenter et al., 2003)
[33] Ella had 6 science groups and she had 120 grams of salt to share amongst the groups for their science investigation. Solve this problem using multiplication. Solve this problem using division.
Can you always switch multiplication and division like you have done to solve the problem?
Can you write a number sentence to show this using **If….and then** which would work for any number?
(adapted from Carpenter et al., 2003)

pointed to the equations asking: *Why don't we just find an answer for this* [points to 6 x _ = 120] *and put it in down there?* [points to 120 ÷ 6 =].

In one example during small group work, Rani constructed a 'if and then' number sentence that used specific numbers: *If six times twenty equals one hundred and twenty then one hundred and twenty divided by six equals twenty.* Ella listened and used questioning to deepen the reasoning and encourage generalisation of their ideas: *How could we represent that say if the numbers weren't six and twenty if they were any numbers?* This resulted in the students constructing algebraic notation to illustrate the relationship.

> Peter: [writes if Z x X = A then A ÷ Z = X] *Z times X equals A… then A divided by Z equals X.*

Further attention was required to direct student attention to the general link between addition and subtraction and multiplication and division. For example, Ella probed students with questions such as: *Why is A there and there? Why is B there and there?* She supported the students suggesting they look at: *The link between the multiplication and the division.* She revoiced student explanations and modelled the use of arrows to emphasise the links (see Figure 5.4): [draws arrows linking the algebraic number sentences] *That E there links to the E in the division problem, that A there links there.*
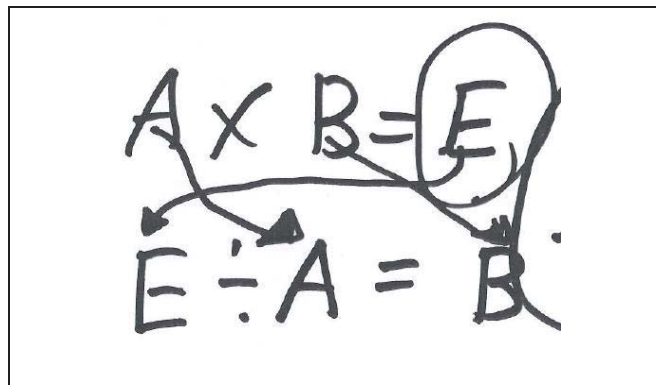
Figure 5.4   The link between multiplication and division

These teacher actions led to students' reflectively recording generalised statements and diagrams about inverse relationships. Following a lesson on the inverse relationship of addition and subtraction, Hamish drew a diagram in his reflection (see Figure 5.5).
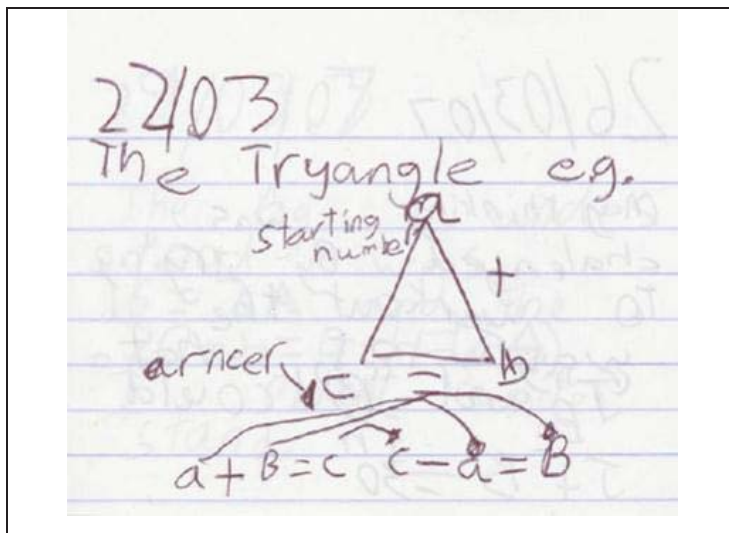
Figure 5.5   Hamish's reflection

After another lesson Josie recorded reflectively: *Key ideas about inverse were that if you say if A x B = Q then you can't say then A ÷ B = Q but you can say Q ÷ A = B.*

5.4.7   STUDENTS' PROGRESS IN UNDERSTANDING THE PROPERTIES OF NUMBERS

In the final interview, all students were able to recognise the commutative nature of addition and multiplication with some offering sophisticated arguments and justification using concrete examples. For example, Josie indicating an array with her hands offered an explanation: *Because if you swap it around it is like 12 groups of 4 or 4 groups of 12 which is the same.* Josie followed this explanation by drawing an array of six times five (see Figure 5.6) and stating: *Because you can put it into groups that way or that way and it always works.*

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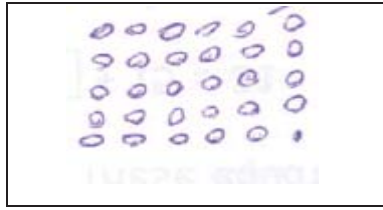
Figure 5.6   Josie's array

Some students also constructed number sentences which drew on the commutative nature of addition or multiplication. For example, Steve remarked: *J plus C equals C plus J...because you can always reverse stuff in adding.* However, 6 of the 25 students still over-generalised the commutative principle to include subtraction and division. Although all 6 students participated in the whole class activities involving exploration of the commutative principle, 5 students were from the group of students who were classified as working at a lower level within the classroom. Whilst these students were observed on several occasions during collaborative group work activities to be able to recognise the non-commutative nature of subtraction and division, without such scaffolding in the interview process they reverted to over-generalising the commutative properties. These results suggest the need for further opportunities for students' to develop deep understanding of operational laws.

Carpenter and his colleagues (2005a) contend that number sentences provide a context whereby students' implicit knowledge becomes explicit. They also promote using number sentences to provide students with access to notation for expressing generalisations. In the final interview, 21 students constructed a number sentence that would be true for any number and were able to justify their reasoning. Most commonly students provided number sentences which were based on the properties of zero.

> Hannah: *H plus 0 equals H...if there was a number and you add or subtract zero you always get the same number.*
>
> Gareth: *If you use zero when you multiply a number it will always equal zero...zero times B equals zero.*

These results support the findings of Carpenter and his colleagues (2005a) that classroom activity using the true and false number sentences provides students with the basis to generalise number properties and then formalise the generalisations.

In this study, students examined numerical properties and operational laws through the use of conjectures. An expectation was established that conjectures would be supported by justification and this led to students engaging in argumentation. Students were pressed to shift beyond justifying by examples to engage in more general forms of justification using equipment. Anthony and Walshaw (2002a) emphasised the need for students to be provided with opportunities to explore conjectures with equipment. In this research, the teacher requirement and scaffolding of students to use equipment to justify their claims supported students' developing understanding. Evidence is provided in the episodes from the lessons and the final interviews that many students developed an understanding of what constituted a valid general justification of a generalisation. Of key importance was the need for both time and explicit attention in developing students' rich constructions of number properties and operational laws.

5.5   SUMMARY

The discussion of the student activities has demonstrated how students' understanding of number can provide a rich foundation for developing early algebraic reasoning. However, the findings also illustrate the robustness of prior misconceptions and the challenging nature of students' construction of early algebraic concepts.

Within the intervention, it was evident that considerable time and attention was required to support students' development of early algebraic reasoning. Students were provided with many opportunities to engage in the processes of early algebraic reasoning including making conjectures, generalising, justifying and formalising. Classroom episodes also highlight the important role of the teacher in using questioning and prompts to extend children's thinking to algebraic reasoning. These prompts alongside the provision of time and space to reflect were important factors in developing student understanding. Specifically developed tasks shifted students' discussions and arguments into general terms. The opportunities for rich discussion coupled with a requirement for reasoned explanations and a press for justification led to students developing deep understanding of early

algebraic concepts. The following chapter describes how algebraic patterning activities were used to further develop students' understanding of early algebraic concepts.

CHAPTER SIX

EXPLORING EARLY ALGEBRAIC UNDERSTANDING THROUGH PATTERNING

6.1   INTRODUCTION

The previous chapter examined how students' early algebraic understanding was developed through exploration of number. It outlined the important roles of the equal sign, algebraic notation and exploration of number properties in developing early algebraic understanding.

This chapter describes the ways in which students were supported to construct early algebraic understanding through exploration of patterns and functional relationships. Section 6.2 outlines the students' initial understandings of functional relationships. Section 6.3 examines how tools and representations can be used to develop students' understanding of functional relationships. It also describes how students' early algebraic understanding was developed through generalising, justifying, and formalising functional patterns and relationships. Section 6.4 summarises students' understanding of functional patterns and relationships at the end of the teaching intervention.

6.2   STUDENTS' INITIAL UNDERSTANDING OF FUNCTIONAL RELATIONSHIPS

Initial interview questions investigated student understanding of functional relationships. Table 6.1 shows student responses to part A and B of the first question[34].

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[34] 1) To make copies of a CD, a store charges a set-up fee and an additional amount per CD. Use the information to answer the questions.
To make copies of a CD, a store charges $2 as a set-up fee and an additional $3 for each copy.
A) What is the cost to make 10 copies of a CD?
B) What is the cost to make 21 copies of a CD?
C) What is a mathematical equation that you could use to find the cost to make copies of a CD if you know the number of copies you want?

Table 6.1   Percentage of students (n=25) correctly using the functional relationship

	Correct use of functional relationship	Incorrect response - directly modelled the equation erroneously adding two and three.	Other incorrect response	No response
Part A	40%	32%	20%	8%
Part B	28%	32%	20%	20%

These findings illustrate that 40% of the students were able to correctly use the functional relationship to calculate the costs of CDs in the problem. Thirty-two percent of the students erroneously directly modelled the equation adding both numbers in order without considering the relationship.

Students from the "middle" and "hard" group (n=19) were asked to respond to question C and provide an algebraic number sentence to represent the functional relationship. Most of this group of students (n=12) did not provide a response. However, 3 students correctly provided an algebraic number sentence, such as: *A times three plus two.* An additional 4 students were able to identify and express the pattern or the rule verbally:

>Mike: *The two stays the same but every time you make a copy plus on three.*
>Hamish: *You'd always times it by three and then add two on.*

A second functional relationship problem[35] was used in the initial interview with students from the "middle" and "hard" groups. Similarly, few students provided responses. Three students were able to identify the cost of the video returned ten days late, and only 2 students correctly identified the cost after 21 days:

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[35]A video store charges a certain fee per video rental and an additional amount each day if the video is returned late. Use the information below to answer the following questions.
It cost $3 to rent the video.
If the video is 6 days late, it costs $15 to rent (including the rental fee)
If the video is 7 days late, it costs $17 to rent (including the rental fee)
If the video is 13 days late, it costs $29 to rent (including the rental fee)
A) What is the cost if the video is returned 10 days late?
B) What is the cost if the video is returned 21 days late?
C) What is a mathematical equation that you could use to find the cost to rent the video game if you know the number of days it is late.

Josie: *Twenty-six dollars divided by thirteen equals two for each day so if it's ten days late then it will be twenty dollars and altogether twenty-three dollars…I timesed ten by two which equals twenty and then I plused three.*

Mike: *I know each day is plus two so thirteen days is twenty-nine so because you said ten I just minused six dollars off because two per day.*

Two of the students provided an algebraic number sentence which represented the functional relationship.

6.3 CLASSROOM ACTIVITIES TO DEVELOP STUDENT UNDERSTANDING OF FUNCTIONAL RELATIONSHIPS

Functional tasks provide opportunities to develop early algebraic reasoning. As students connect algebraic symbols to quantitative referents they establish meaning for them (Blanton & Kaput, 2004; Lannin et al., 2006). Geometric problems involving multiple representations—numeric and visual patterns—allow students to identify, communicate and justify functional rules (Beatty & Moss, 2006; McNab, 2006). In this study, a range of linear functional problems including tasks with geometric contexts were used. These were structured to promote use of a wide range of generalisation strategies and formalisation of functional rules.

6.3.1 USING T-CHARTS TO SUPPORT FUNCTIONAL THINKING

Researchers (e.g., Blanton & Kaput, 2005; Carraher et al., 2006) have shown that t-charts can be used to allow students to understand and express functional rules across a range of problem contexts. Building on their work, Ella introduced the t-chart as a way of recording data (see Figure 6.1).

Figure 6.1   T-chart

Her introduction scaffolded the students to link the data they were recording to the context of the problem. Throughout the series of lessons, Ella consistently cued students to use t-charts to record data: *Can we remember how we can record data?*

As was noted in previous research by Carraher and his colleagues (2006) and Blanton and Kaput (2005), t-charts were appropriated by the students as representational tools that aided their thinking. The vignette below illustrates students independently choosing to use a t-chart during small group work.

Using a t-chart to solve a problem and aid thinking

Students were discussing how to solve a problem[36]. They began by verbally listing possibilities.

Heath	One in one cage, seven in the other. Two in one cage and six in the other.
Bridget	Or four in one cage and four in the other.
Heath	Yeah and then three in one cage and five in the other.
Ruby	We could write a table.
Hayden begins drawing a t-chart [See Figure 6.2].	

---

[36] Rachael has two mice cages which are joined together. She has 8 mice in the cages. Can you show all the different ways the 8 mice could be in the cages? What about if Rachael had 9 mice or 20 mice? Can you write a number sentence so Rachael can find out how many different ways the mice can be in the cages no matter how many mice there are? (Adapted from Carpenter et al., 2003).

Heath	So cage one mice and cage two mice. So in cage one there could be one mouse [points to cage two column] and seven mice.
Hayden	No it would be like this zero, eight, one, seven.

Hayden fills in the t-chart writing the numbers zero to eight in the cage one column and the corresponding numbers eight to zero in the cage two column. The students continued to use the t-chart to record the possibilities for different amounts of mice and justify that they had found all possible combinations.

(Lesson 11)

Figure 6.2   T-chart for the mouse-cage problem

In the whole group discussion that followed, other groups also shared their solution strategies using a t-chart to record data:

> Cameron: [draws a t-chart] *We just kept on going on with all the numbers going down like that and we didn't really have to add because it will go down on this side and up on this side so it makes it easier.*

6.3.2   CLASSROOM ACTIVITIES TO DEVELOP STUDENTS' GENERALISATION STRATEGIES FOR ALGEBRAIC PATTERNS

Initially, many students applied additive recursive strategies when using closely related input values. These strategies involved listing successive values until the desired output number was reached. For example, during small group work while solving a functional

relationship problem[37] Ruby noted the recursive pattern: *Look there's five people here but there's three added on.* Throughout the lesson, the students continued to use the recursive pattern:

> Heath: *We are plusing three, so on one table there is five, on two tables which makes eight.*
>
> Matthew: *So then four tables will be fourteen.*

These results conformed to the findings of many researchers (e.g., Becker & Rivera, 2006; Lannin et al., 2006; Swafford & Langrall, 2000) who found that students commonly used additive recursive strategies when input values were relatively close.

Ella required the students to find more efficient generalisation strategies. She used questioning to promote students to use more effective strategies than recursive generalisation. During one lesson, Ella's prompt led to a student developing a chunking strategy. Chunking enables the student to build on the recursive pattern using the known values (Lannin et al., 2006). This strategy is on a continuum between recursive reasoning and explicit reasoning and is identified by Lannin and his colleagues as more effective than recursive strategies:

> Ella: *What would be a quicker way than going plus three?...*
>
> Ruby: [points to model] *The first table is five so you could ignore that and just go nine times three... you could just ignore that because you know it is five, so nine times, because that's table one, nine times three then add the five on.*

However, shifting students beyond using recursive additive strategies was challenging. For example, although Ruby developed a chunking strategy, Rani continued to use the recursive pattern. She filled in the t-chart additively (see Figure 6.3): *You have to keep adding three*

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[37] At the table 5 people can sit like this ……

When another table is joined this many people can sit around it…

Can you find a pattern? How many people could sit at 3 tables or 5 tables or 10 tables? See if your group can come up with a rule and make sure you can explain why your rule works.

all the time and if you do it this way twenty-seven won't come here, nine would be twenty-nine and ten would be thirty-two.

No. of tables	No. of People
1	5
2	8
3	11
4	14
5	17
6	20
7	23
8	26
9	29
10	32
11	35

Fig

Lannin and his colleagues (2006) suggest that input values which are multiples or doubles of previous values encourage students' use of whole object generalisation strategies. Often this strategy leads to over-counting or under-counting. This was evident when students were asked to calculate how many people could sit around ten tables and then one hundred tables. This prompted a group to use the whole object generalisation strategy erroneously:

> Heath: [points to 10 in the table of data] *If it is a hundred we will just plus a zero to that.*
>
> Matthew: [points to 10 and 32] *You can add a zero to that and a zero to that.*

Observations of students' engaged in over-counting strategies led Ella to provide tasks which facilitated further examination of the whole object strategy. Tasks structured so the input values doubled appeared to prompt students to examine and discuss the whole object

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generalisation strategy in-depth. For example, one problem[38] involved input values that doubled from four to eight. Gareth's explanation over-counted the values: *So if four is twenty-one so it is twenty-one plus twenty-one.* His explanation was challenged:

> Ruby: *Instead of just doing twenty-one plus twenty-one, you don't because you wouldn't just build another four separate and there is not going to be another six one so it's not really adding twenty-one...*
>
> Ella: *So you're saying you can't just double it because there's not going to be another six one like at the start.*
>
> Ruby: S*o you just do twenty.*

Gareth responded to the reasoned argument by correctly using a whole object generalisation strategy to find the output value: *So it's only twenty because you take a one away at the start, you add on twenty from here... like Ruby said you can't add on six that would mean there would be two of those sides* [points to middle stick] *so it can't be twenty plus twenty-one so it's twenty-one plus twenty.*

Development of efficient generalisation strategies was a lengthy process and required ongoing teacher attention. Ella consistently required that the students find quicker, more efficient generalisation strategies: *Is there another way you can do it without adding from twenty?... Can you think of an equation or a rule that would help you get from four to twenty-one?* Such pedagogical actions supported many students to shift from using additive to multiplicative approaches and facilitated a range of different generalisation strategies.

Lannin and his colleagues (2006) argue that development of explicit generalisation strategies are linked to students' attempts to find more efficient strategies. This was evident

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[38] Jasmine and Cameron are playing "Happy houses". They have to build a house and add onto it. The first one looks like this…..

```
 /\
|_|
```

The second building project looks like this….

```
 /\/\
|_|_|
```

How many sticks would you need to build four houses?
How many sticks would you need to build eight houses?
Can you find a pattern and a rule?

in the study. The students shifted from recursive to explicit generalisation as they sought easier ways to calculate output values. In doing so, they shifted from additive to multiplicative strategies. For example, in the following episode Ruby's challenge facilitated a shift in the group's additive strategy:

> Ruby: *If you are doing four houses instead of going five plus five plus five you can just go… four times five…plusing five isn't good because you want a quicker way.*
>
> Gareth: *You could count it but that would take ages if you wanted to get it to a hundred or something it would take too long.*

6.3.3 CLASSROOM ACTIVITIES TO DEVELOP STUDENT JUSTIFICATION OF FUNCTIONAL RULES

Integrating visual and numeric schema and making connections between visual and numeric patterns enables students to find, express and justify functional rules (Beatty & Moss, 2006; Becker & Rivera, 2006; McNab, 2006). Many studies (e.g., Beatty & Moss; Lannin et al., 2006; Swafford & Langrall, 2000) demonstrate that when students link algebraic rules to the contextual base of the problem they are more successful at creating correct rules and developing robust understanding of functions. Justification is an important element of developing early algebraic understanding (Carpenter et al., 2003). Initially, in this study, many students maintained a view that using lots of different numbers justified their reasoning. For example, when Ella asked to be convinced about a rule the response she drew was:

> Hayden: *We tested it with this number, it works, this number, it works and we are just finishing this one and so far it is working.*

To shift students beyond this view required that Ella press them to integrate visual and numeric schema. Students were encouraged to connect their explicit rules with the geometric problem representation. For example, Hamish explained his group's explicit generalisation in response to the problem of how many people could sit at ten tables: *Thirty-two people sit at the table…you get the ten and times it by three and the two people who are sitting on those ends, one of them stays there and the other one gets moved to the end of the new table.* Ella then asked him to connect his contextual explanation with the geometric pattern: *Hamish can you show…the times three part of your model there and the*

plus two part? This request was accompanied by Ella demonstrating the link between the functional rule and the geometric representation through modelling (see Figure 6.4): *This is the first table here so I'm going to use blue for the times three and the plus two here* [uses pink counters on each end]*...so two tables, two groups of three do you see that?* [points to blue counters] *Plus two* [points to pink counters].
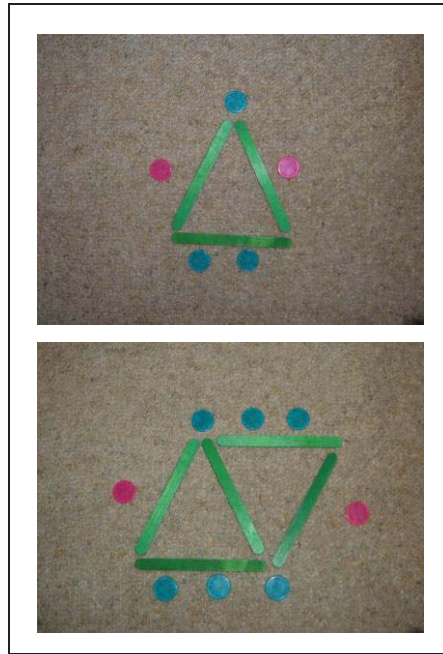
Figure 6.4   Using a geometric representation to show a functional rule

In another lesson, Ruby and Bridget challenged the view that using lots of different examples was sufficient justification and pressed others in the group to connect the visual pattern and functional rule:

> Ruby: *It's not really saying anything...do you think it would be more convincing if we used equipment* [for our explanation]*?*
>
> Bridget: *Yeah.*

Through modelling and sustained press to explain ones' thinking many students began to recognise the need for justification to extend beyond examples. This was demonstrated by Susan when her group recognised a pattern: *We have got to make proof that it actually works because if you think it is eleven but you don't know that it is eleven.*

Ella also required that the students link their use of a t-chart with visual representations to justify their explanations: *You need to be able to show and prove that by either drawing a diagram or using equipment.* She emphasised the importance of constructing links between the numeric pattern in the t-chart and the geometric models: *This is really important this part, what he is going to show you will help you make the links between that and the table of data.* This resulted in student explanations which used the t-chart and geometric model as tools to assist their justification to peers. For example, Matthew provided a solution strategy for the table problem:

> Matthew: [draws t-chart and uses equipment to build a geometric model] *We used the table…for one table there would be five people and then you put another one on…and that was eight…whenever you added another table on you would plus three more people.*

Using multiple representations appeared to provide students with opportunities to construct deep understanding of functions. Requiring students to link numeric and geometric patterns encouraged students to construct reasoned explanations which incorporated multiple representations.

The linear functional problems involving geometric patterns coupled with the press from the teacher prompted students to increasingly use equipment to support their justifying processes. For example, during a large group discussion, Ruby justified her explicit generalisation of the house problem using the geometric pattern: [builds model] *The first one is six but then when you add another house it is only five because you don't need another wall…if you wanted to see how many for eight you could just go eight times five and then plus the one, you are plusing the one because you have to still understand that that is six* [points to first house].

In the following vignette from another lesson, Josie and Matthew repeatedly referred to the geometric context of the problem to convince their other group members of their explicit generalisation. When the other group members remained unconvinced, Josie and Matthew pointed to the already drawn model, re-drew the model with different colours and used visualised representations which they indicated with hand actions.

	Using a geometric context to justify an explicit generalisation

Students were working in a small group of four on a geometric function problem[39]. After examining the model, Josie begins by pointing to the model with her hands.

Josie   This is cross one. There is one on each side plus one in the middle. This is cross two, so two here and two here and one in the middle so that makes five. So you double it and then add one to get the number across.

Matthew builds on her generalisation explaining how to find the number of square across for cross twenty.

Matthew   [Indicates a vertical line with his hands] It would be twenty and twenty plus one so that would be forty-one.

The other group members Steve and Rani continue to press for justification.

Steve   So why does it equal forty-one?

Josie and Matthew use the model to build justification for their explicit generalisation.

Matthew   [points to each side] Because like cross one has one there and one there.

Josie   Plus one in the middle.

Matthew   Two has two there and two there and one in the middle and three has three there and three there plus one.

As the other group members remained unconvinced, Matthew draws two models.

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a)   Draw in the next two crosses in this pattern.
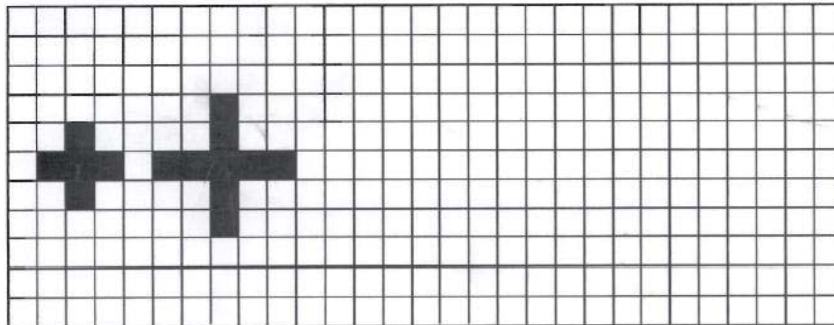
Cross 1   Cross 2

Complete the table

	Cross 1	Cross 2	Cross 3	Cross 6	Cross 20
Number of squares across					
Total number of squares					

What is the word rule for the pattern?
Can you write it as an algebraic number sentence?
Use the rule to find the number of squares across and the total number of squares for Cross 101.
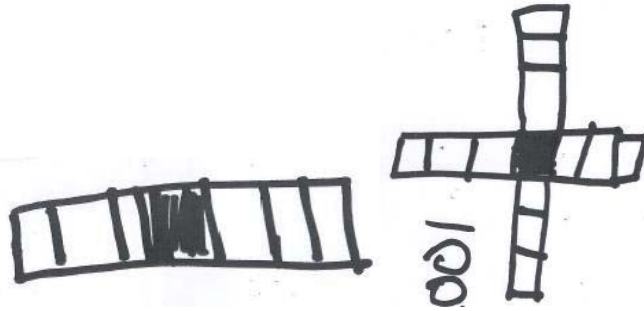
Both Matthew and Josie continue to construct an argument for their generalisation.

Matthew	One, two, three and the one in the middle is black. So this is for cross three right? Have you noticed a pattern yet?
Steve	No I haven't.
Matthew	Cross three has got three, cross two has got two on the outside.
Josie	Each part that sticks out has the amount, the number of the cross then there is one in the middle to join it up…it is the number of the cross doubled and then you add one in the middle to get that amount there [covers the model so only the horizontal row is showing].

After some individual thinking time Steve and Rani continue to question until they are convinced.

Steve	So when you double it, what are you actually trying to get to by doubling it?...
Josie	[covers the vertical row so only the horizontal row is visible] The number of squares in that line there.

Following this, Rani revoices the generalisation referring to the model of the cross.

Rani	[points to one arm of the cross] so you double that and add one.
Josie	You double the number of the cross and add one.
Rani	[points to the arms of the cross] So you double these?...
Josie:	[points to right horizontal arm] That little bit here. [points to upwards vertical arm] This little bit here is also three squares wide and this is three squares wide [points to left horizontal arm] and that is three squares wide [points to downwards vertical arm] so to get the bit across here in the middle [points to the horizontal row] you do times two plus one.

(Lesson 9)

6.3.4 CLASSROOM ACTIVITIES TO DEVELOP STUDENT FORMALISATION OF FUNCTIONAL RULES

Many researchers (e.g., Blanton & Kaput, 2004; Carpenter & Levi, 2000b; Krebs, 2003) highlight formalisation as an important aspect of the generalisation process. In this study, constructing notations was an important focus of the students' formalisation of functional rules. Students used notation to represent the rules and generalisations their groups had constructed for the functional relationships. For example, after Hamish had shared his

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group's explicit generalisation, Rachel developed the explanation using the written rule: A x 3 + 2 = Q: *We did A times three plus two because you always times three and then you add two. We did A for the number of tables.* Ella further extended students' use of formal notation introducing an algebraic convention: [writes 3A + 2 = Q] *I can write it like this three A plus two equals Q because that's like putting brackets around here and plusing two because three A is the same as saying three times A.*

Formalising explicit generalisations into algebraic rules provided opportunities for students to develop their understanding of notation. For example, when Ruby recorded a generalisation as ▲ x 5 = ▲, Gareth disagreed: [pointing to both triangles] *But these aren't the same numbers you need to change it from the triangle.* In another lesson (Lesson 13), the activity[40] provided students the opportunity to extend their understanding of algebraic notation as a quantitative referent. During small group work, the students in one group discussed how their notation linked to the contextual basis of the functional relationship. Building on their notation S x 10 + 5 to represent the functional relationship, their discussion moved their understanding towards a generalised rule:

> Tim: *So S is meaning three.*
>
> Ruby: *No it is not meaning only three, it is just meaning the number of minutes you have had on the phone.*
>
> Tim: *So at the moment it is meaning three minutes?*
>
> Ruby: *No it is just meaning any number of minutes.*

During the whole group discussion which followed small group work on this problem, students also extended their understanding of formal notation. Sarah notated the generalisation as P x 10 + 5 = N. Rani suggested the notation could be shortened: *Instead of going P times ten couldn't you just go 10P?* Several other students initially disagreed with this:

> Bridget: *No because it is a number and a letter.*

---

[40] Vodafone is currently offering a calling plan that charges 5 cents per call and 10 cents per minute..
  1) How much would a 3 minute phone call cost? 6 minutes? 15 minutes?
  2) Write a number sentence to show how much a phone call will cost no matter how long you talk for.
  3) Can you use your number sentence to show how much a 101 minute phone call would cost?

Susan: *You need to know if you are timesing it or plusing it.*

Sensing an opportunity, Ella facilitated a discussion of using this notation:

Ella: *What does it mean if you see a letter with a number in front of it? What does it mean in mathematics?*

Steve: *It's like ten times P because it is an algebraic short-cut.*

6.4 STUDENTS' PROGRESS IN UNDERSTANDING FUNCTIONAL RELATIONSHIPS

Evidence from the final interview showed that many students in this study were beginning to develop rich understanding of functions. Table 6.2 shows student responses to part A and B of the first question[41], with corresponding responses for the initial interview in brackets.

Table 6.2   Percentage of students (n=25) correctly using the functional relationship

	Correct use of functional relationship	Incorrect response - directly modelled the equation erroneously adding two and three.	Other incorrect response	No response
Part A	88% *(40%)*	8% *(32%)*	4% *(20%)*	0% *(8%)*
Part B	84% *(28%)*	4% *(32%)*	8% *(20%)*	4% *(20%)*

Note. Initial interview results are in brackets

These results show a significant increase in the number of students able to use the functional relationship to find out the cost of copies of posters.

Students from the "middle" and "hard" groups (n=19) were also asked to provide an algebraic number sentence which represented the functional relationship. Seventeen students were able to provide a correct algebraic number sentence such as:

Sangetta: *A times two plus three.*

Matthew: *Two J plus three.*

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[41] To make copies of a poster a store charges a set-up fee and an additional amount per poster. Use the information to answer the questions.

To make copies of a poster, a store charges $3 as a set-up fee and an additional $2 for each copy.
  A) What is the cost to make 10 copies of a poster?
  B) What is the cost to make 21 copies of a poster?
  C) What is a mathematical equation that you could use to find the cost to make copies of a poster if you know the number of copies you want?

A second functional relationship problem [42] was also asked of the students from the "middle" and "hard" group. This proved more difficult for students to solve. However, 11 students were able to identify the cost of returning a video after ten days and 10 students identified the cost of returning a video after twenty one days. For example, Caitlin reasoned: *Twenty-four because it says on here if the video is five days late it costs fourteen dollars including the rental fee and I figured out that not including that right now that it will be ten dollars and then…I think it costs two dollars a day.* Seven students were able to construct an algebraic number sentences which represented the functional situation.

Overall, these results indicate that many of the students gained understanding of functional patterns and relationships through their engagement with a range of targeted activities that involved identifying the functional relationships of geometric models and connecting these to numeric patterns.

6.5   SUMMARY

This chapter has offered strong evidence that students' early algebraic understanding can be developed through tasks which involve functional patterns. However, development of efficient generalisation strategies was shown to be a lengthy and complex process requiring multiple opportunities to engage in functional patterning tasks.

The discussion noted the various ways the teacher was able to facilitate students' development of more efficient generalisation strategies and make connections across numeric and geometric patterns. Investigation of geometric functional relationships

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[42] A video store charges a certain fee per video rental and an additional amount each day if the video is returned late. Use the information below to answer the following questions.
It cost $4 to rent the video.
If the video is 5 days late, it costs $14 to rent (including the rental fee)
If the video is 6 days late, it costs $16 to rent (including the rental fee)
If the video is 13 days late, it costs $30 to rent (including the rental fee)
    A) What is the cost if the video is returned 10 days late?
    B) What is the cost if the video is returned 21 days late?
    C) What is a mathematical equation that you could use to find the cost to rent the video game if you know the number of days it is late?

provided a context for students to develop and justify algebraic generalisations. Teacher introduced representations such as t-charts were appropriated by students as representational tools which further supported their reasoned explanations. Students initially approached justification through use of examples. However, the specific teacher attention to justification using equipment and connection of geometric models with functional rules led to students justifying through integration of a range of representations. Formalising generalisations allowed opportunities for students to further their understanding of algebraic notation.

CHAPTER SEVEN
CONCLUSION

7.1 INTRODUCTION

An important aim of this study was to investigate primary students' construction of early algebraic concepts within an inquiry classroom. A particular focus was placed on examination of how instructional tasks and models and notation and symbolisation supported students' construction of early algebraic concepts. A further focus was to analyse how classroom practices and the social and socio-mathematical norms of an inquiry classroom supported development of early algebraic reasoning. Drawing together these findings serves to illustrate the complex nature of the teaching and learning process, the importance of selecting and implementing instructional tasks, and the crucial role of classroom practices in students' construction of early algebraic reasoning.

Drawing on the conclusions of the study, implications for current classroom practice and suggestions for further research are outlined.

7.2 THE COMPLEX NATURE OF TEACHING AND LEARNING

This study was situated within the complexities of teaching and learning within a real classroom. Interpretation of the results requires consideration of the complex interaction of many features of classroom practice and the social and individual nature of students' construction of understanding—interactions that are specific to the classroom under investigation. This uniqueness, combined with a relatively small number of participants in the study, necessarily means that the results can only offer an emerging insight into the ways in which students can be supported to construct early algebraic reasoning within an inquiry classroom.

Importantly, the results of this study support the contention made by other researchers (e.g., Blanton & Kaput, 2005; Carpenter et al., 2003, Schliemann et al., 2007a) that primary age students' mathematics learning can benefit from sustained exposure to algebraic reasoning

opportunities. It was evident that student understanding of key algebraic concepts (e.g., the equal sign, variables, number properties) evolved gradually with students moving between levels of understanding. To foster rich algebraic understanding and shift students beyond misconceptions and partial understanding, the students in this study needed repeated challenges to their thinking.

7.2.1 INSTRUCTIONAL TASKS AND MODELS

In this study, particular instructional tasks and models acted as key tools which supported student construction of early algebraic reasoning. In the first instance, the understanding of the equal sign as relational equivalence was perceived as an important step towards student development of early algebraic reasoning. The specifically designed tasks involving true and false number sentences engaged students in rich examination of the equal sign. Similarly, the number sentences with operations on both sides of the equal sign caused cognitive conflict and confronted student expectation that the answer would be presented after the equal sign. The repeated use of such number sentence activities over an extended time enabled the majority of students to successfully re-construct the equal sign as a symbol of relational equivalence.

The study successfully trialled a set of instructional tasks to mediate student understanding of relational thinking. These included open number sentence problems and word problems which scaffolded an image of the equal sign as scales. Although many students fluctuated between calculational and relational strategies during the classroom activities, in their final interview the majority of students displayed the ability to use relational strategies to solve the open number sentence problems. Carpenter and his colleagues (2003) described similar findings. In this study, true/false and open number sentences also provided a context to engage students in discussions of the use of symbols and notation.

The provision of opportunities for students to both use algebraic notation and explore the concept of variables was important in students' construction of rich algebraic understanding. Algebrafied word problems were used to introduce notation and provided the students with a context in which they engaged in rich discussion about algebraic notation and

symbolisation. As in earlier studies (e.g., Carpenter et al., 2003, Carraher et al., 2006; Lee, 2006) the Year 5 and 6 students in this study developed increased understanding of algebraic notation and were able to make use of notation to represent important mathematical ideas about number properties.

Schifter (1999) noted that many primary age students have a wealth of implicit understanding of number properties and draw on this knowledge to solve computational problems. This was evident in this research. As happened in Carpenter et al.'s (2003) study, the students' early algebraic understanding was developed through direct discussion of underlying number properties. True and false number sentences provided a context in which students explored and extended conjectures about the properties of numbers. Number sentences were used to generate conjectures centred on the properties of zero. Many students quickly formalised the conjectures and offered justification to convince others of their validity. Similarly, true and false number sentences were used to examine the commutative principle. Initially, the series of number sentences focused student attention on examination of when the commutative principle was valid. Subsequently, the students generalised the commutative nature of addition and multiplication. However, the findings illustrated that understanding of the commutative principle is complex and difficult and requires on-going attention. This was demonstrated by a small group of students in the final interview who continued to over-generalise and include subtraction and division as commutative.

During the observed lessons, whole class discussion provided an opportunity for explicit attention to be given to the properties of odd and even numbers. The introduction of quasi-variables enabled students to eventually construct understanding that the last digit in number could be used to determine whether a particular number was odd or even. In accord with classroom studies by Blanton and Kaput (2003) and Fujii (2003), the use of quasi-variables offered many possibilities for students to construct understanding of the general properties of arithmetic.

The study introduced specifically designed tasks to facilitate student examination of the inverse relationship between addition/subtraction and multiplication/division. The requirement to systematically record equations and establish a pattern successfully scaffolded student identification of the numerical link between equations. The introduction of "if and then" number sentences directly mediated students' generalisation of the link and their construction of algebraic notation to represent the situation.

Tasks which involved functional patterns and relationships were used to support student development of functional reasoning. These tasks designed with specifically selected input values led to student use of different generalisation strategies. The use of multiple or double input values resulted in student examination of whole number generalisation strategies. Students were prompted to use more efficient explicit generalisation strategies through the extension to large input values. Further to this, the use of deliberately designed functional tasks which included numeric and geometric patterns offered possibilities for students to integrate their visual and numeric schema. The geometric patterns and use of multiple representations provided further opportunities for students to justify using equipment and to construct deep understanding of functions.

7.2.2   NOTATION AND SYMBOLISATION

The transition from arithmetic to algebraic reasoning requires that students have opportunities to construct conceptual understanding of variables and algebraic notation (Knuth et al., 2005; MacGregor & Stacey, 1997). At the beginning of the study, most of the students had limited experience in the use or interpretation of algebraic notation. This was highlighted by their attempts to use specific numbers to represent unknown quantities and to sense-make symbolic letters as abbreviated words.

Algebrafied word problems were used to engage students in discussion about algebraic notation. The students recorded a series of equations to solve the algebrafied word problems and engaged in discussions where they linked their equations back to the context of the problem. Teacher scaffolding of the use of notation to represent the generalised situation supported students to construct conceptually focused explanations of their

notation and algebraic number sentences. The use of true and false number sentences further developed student use of notation.

Carpenter and his colleagues (2003) describe symbolic notation as a way of providing students with "a concise and precise means of representing basic properties about number operations and relations" (p. 5). In this study, true and false number sentences provided students with a context from which they constructed conjectures and articulated their ideas about basic mathematical principles—for example, conjectures related to the commutative principle and the properties of zero. Students completing these tasks were encouraged to extend their use of algebraic notation as a precise way to express mathematical ideas rather than articulating a verbal conjecture.

In this study, functional relationship tasks offered further opportunities for students to investigate and use algebraic notation. The students generalised and formalised the functional rules. The teacher built on student use of notation to introduce algebraic conventions—for example, how to represent multiples algebraically. Discussion of the formalised rules provided opportunities to deepen understanding of algebraic notation as a quantitative referent.

Many studies (e.g., Fujii, 2003; MacGregor & Stacey, 1997; Weinberg et al., 2004) have illustrated students' difficulties with algebraic notation and variables. In this study, students examined the conventions of algebraic notation through the use of specifically structured algebraic number sentences. These reinforced student understanding that the same symbol represented the same number. They also addressed a common misconception that two different symbols in an equation can not be the same number. The use of algebraic number sentences mediated a deeper understanding of algebraic notation. In support of the findings of previous research studies (e.g, Carraher et al., 2006; Carpenter et al., 2003; Schliemann et al., 2007a), the study illustrated the significant shift students had made in the use of algebraic notation and interpretation of symbolic notation.

7.2.3 INQUIRY CLASSROOM PRACTICE: SOCIAL AND SOCIO-MATHEMATICAL NORMS

It is the contention of this study that such significant shifts in students' algebraic thinking and understanding were dependent on more than the intervention tasks. Throughout the study, it was evident that the development of inquiry classroom practice and culture mediated student engagement in algebraic reasoning. An integral aspect of the inquiry classroom culture was the social and socio-mathematical norms established by the teacher. Initially, classroom discussions were hindered by silence or unproductive disputational or cumulative talk. However, through the explicit focus on productive group processes, the teacher led shifts towards more collaborative interaction practices.

Specific pedagogical actions shifted students to become active listeners and critical participants. The teacher skilfully facilitated class discussions to encourage appropriate mathematical talk and advance the instructional agenda. Student responsibility to actively engage, question and sense-make was emphasised and attention was drawn to students who modelled appropriate mathematical behaviour. The introduction of reflective 'space' and time for thinking and questioning during explanations allowed students opportunities to reflect on errors and misconceptions. The reflective space engaged students in examining and analysing mathematical thinking and thus students were inducted into a classroom culture which emphasised the importance of examining reasoning rather than sharing multiple solution strategies.

Evidence from the final interview questions completed by students clearly indicated that for many students the underlying properties of arithmetic had become explicit objects of their understanding. It appears that through participation in mathematical inquiry and argumentation, the multiple opportunities to develop mathematical reasoning supported their ability to provide sophisticated justification statements related to the commutative nature of addition and multiplication. These mathematical argumentation practices were also supported through teacher revoicing that modelled and highlighted appropriate forms of justification.

7.3   OPPORTUNITIES FOR FURTHER RESEARCH

This study illustrated that rich algebraic understandings of the properties and relationships of number can be developed with Year 5 and 6 students. The students participating in this study had strong conceptual understanding of number strategies and were achieving within the higher stages of the New Zealand Numeracy project. It would be timely to examine how early algebraic reasoning can be fostered with students who have differing levels of numerical understanding within the same age group. It would also be appropriate to investigate how particular areas of early algebraic reasoning (e.g., understanding of the equal sign and relational strategies, understanding of number properties) can be developed with younger students.

This study investigated student construction of early algebraic reasoning over one school term. For some students this understanding was relatively fragile. Given the complexity and ongoing development of algebraic understanding a longitudinal study which investigated student construction of early algebraic understanding over a longer period of schooling within the primary years would be warranted.

The results of this study indicated the value of mathematical tasks and activities which engaged students in algebraic reasoning. These included providing opportunities to model conjectures and justify using equipment. Initially, the students accepted using multiple numerical cases as justification. They also demonstrated many difficulties in using equipment to justify conjectures about number properties. An investigation into artefacts and instructional practice that support students to make and justify conjectures is warranted.

Within this study, the need for students to be provided with appropriate classroom experiences and multiple opportunities to explore the use of notation was highlighted. The findings of the study illustrated that students required both space and time to develop rich understanding of algebraic notation and variables. An investigation of how to build on students' informal use of notation to develop more formal notational forms is an important area of research.

Blanton and Kaput (2005) identified a need for teachers to develop their 'algebra ears and eyes'. In this study, the teacher participant's sound mathematical knowledge enabled her to provide an effective range of algebraic reasoning opportunities within arithmetically-based tasks. Examining how other teachers can be supported to utilise algebraic reasoning opportunities across a range of content areas within their everyday classroom practice would be beneficial.

In this study, the attention given to the development of an inquiry culture and its corresponding social and sociomathematical norms was a significant factor in shifting the students towards algebraic reasoning. Whilst the establishment of these norms are documented within studies on discourse and inquiry classrooms (e.g., R Hunter, 2006; 2007) in general, studies which focus on the development of specific mathematical understanding in relation to engagement in productive mathematical discourse (e.g., Wood, Williams, & McNeal, 2006) are relatively rare. This study makes some contribution to this research agenda, but more are needed.

7.4   CONCLUDING THOUGHTS

The intention of this research was to examine students' construction of early algebraic reasoning within an inquiry classroom. The study adds to the wealth of knowledge about the teaching and learning of early algebraic reasoning. In this study, a flexible set of learning tasks were designed which incorporated learning sequences modelled on previous research studies (e.g., Carpenter et al., 2003; Kaput & Blanton, 2005). As the study progressed, the tasks and activities were adapted to meet the emerging needs of the individual students. The results highlighted how particular instructional tasks and models could be successfully used to facilitate the development of early algebraic reasoning. However, a second element within the study which needs careful consideration was the role of the classroom practices, in particular the social and socio-mathematical norms. Evidence from this study suggests that student participation in mathematical activity which included explanation, argumentation and justification supported their development of rich algebraic reasoning.

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APPENDIX A: Interview Questions (Pre-Unit)

1) What does the '=' mean?
   Can it mean anything else?

2) Are these number sentences true or false?

   $15 + 6 = 6 + 15$

   $15 - 6 = 6 - 15$

   $15 \times 6 = 6 \times 15$

   $15 \div 6 = 6 \div 15$

3) What is a mathematical statement or sentence to represent each of the following situations:
   a) I have some pencils and then get three more.
   b) I have some pencils, then I get three more, and then I get two more.
   c) I have some pencils, then I get three more, and then I double the total number of pencils I have.

4) What numbers do you think should go in these boxes?

   a) $27 + 16 = \boxed{} + 14$

   b) $54 + \boxed{} = 57 + 36$

   c) $84 - 18 = 86 - \boxed{}$

5) Is the number that goes in the $\boxed{}$ the same number in both of these equations?

   $2 \times \boxed{} + 15 = 31$ $\qquad$ $2 \times \boxed{} + 15 - 9 = 31 - 9$

6) $2f + 3$
   a. What does the symbol stand for?

   b. Could the symbol stand for the number 4?

   c. Could the symbol stand for the number 37?

7) Is $h + m + n = h + p + n$ always, sometimes or never true?

8) What is a number sentence that is true for any number?

9) To make copies of a CD, a store charges a set-up fee and an additional amount per CD. Use the information to answer the questions.

> To make copies of a CD, a store charges $2 as a set-up fee and an additional $3 for each copy.

    a. What is the cost to make 10 copies of a CD?

    b. What is the cost to make 21 copies of a CD?

    c. What is a mathematical equation that you could use to find the cost to make copies of a CD if you know the number of copies you want?

10) A video store charges a certain fee per video rental and an additional amount each day if the video is returned late. Use the information below to answer the following questions.

> It cost $3 to rent the video.
> If the video is 6 days late, it costs $15 to rent (including the rental fee)
> If the video is 7 days late, it costs $17 to rent (including the rental fee)
> If the video is 13 days late, it costs $29 to rent (including the rental fee)

    a. What is the cost if the video is returned 10 days late?

    b. What is the cost if the video is returned 21 days late?

    c. What is a mathematical equation that you could use to find the cost to rent the video game if you know the number of days it is late?

APPENDIX B: Interview Questions (Post-Unit)

8) What does the '=' mean? Can it mean anything else?

9) Are these number sentences true or false?

    $12 + 4 = 4 + 12$
    $12 - 4 = 4 - 12$
    $12 \times 4 = 4 \times 12$
    $12 \div 4 = 4 \div 12$

10) What is a mathematical statement or sentence to represent each of the following situations:
    a) I have some lollies and then get five more.
    b) I have some lollies, then I get five more and then I get three more.
    c) I have some lollies then I get five more and then I double the number of lollies I have.

11) What numbers do you think should go in these boxes?

    a) $23 + 15 = \boxed{} + 17$

    b) $81 + \boxed{} = 83 + 26$

    c) $76 - 27 = 78 - \boxed{}$

12) Is the number that goes in the $\boxed{}$ the same number in both of these equations?

    $3 \times \boxed{} + 12 = 27$ $3 \times \boxed{} + 12 - 6 = 27 - 6$

6) $2m + 5$
    a) What does the symbol stand for?
    b) Could the symbol stand for 6?
    c) Could the symbol stand for 45?

7) Is $b + f + n = b + e + n$ always, sometimes or never true?

8) What is a number sentence that is true for any number?

9) To make copies of a poster, a store charges a set-up fee and an additional amount per CD. Use the information to answer the questions.

> To make copies of a poster, a store charges $3 as a set-up fee and an additional $2 for each copy.

a. What is the cost to make 10 copies of a poster?

b. What is the cost to make 21 copies of a poster?

c. What is a mathematical equation that you could use to find the cost to make copies of a poster if you know the number of copies you want?

10) A video store charges a certain fee per video rental and an additional amount each day if the video is returned late. Use the information below to answer the following questions.

> It cost $4 to rent the video.
> If the video is 5 days late, it costs $14 to rent (including the rental fee)
> If the video is 6 days late, it costs $16 to rent (including the rental fee)
> If the video is 13 days late, it costs $30 to rent (including the rental fee)

a. What is the cost if the video is returned 10 days late?

b. What is the cost if the video is returned 21 days late?

c) What is a mathematical equation that you could use to find the cost to rent the video game if you know the number of days it is late?

APPENDIX C: Tasks and problems

1) $8 = 3 + 5$ $8 = 8$ $3 + 5 = 5 + 3$

       $19 = 1 + 8 + 10$ $9 = 6 + 3$ $16 - 14 = 14 - 12$ 7

  $+ 5 = 3 + 9$ $22 - 9 = 13$ $16 = 19$ $25 - 6 =$

  $19 - 7$ $57 + 12 = 59 + 10$ $14 = 14$

2) $8 + 6 = 9 + 5$ $27 = 19 + 8$ $11 - 4 = 10 - 3$

3) $54 + 10 + 1 = 64 + 1$ $27 - 8 = 25 - 10$ $57 + 12 = 69 + 10$

       $75 + 68 = 77 + 66$

4) $583 - 529 = 83 - 29$ $498 + 12 = 488 + 22$

       $471 - 382 = 474 - 385$ $186 + 277 = 188 + 279$

5)   If you had \$9 in your bank and wanted to buy a t-shirt for \$17 how much do you need to save?
   What about if the t-shirt cost \$20 or \$26 or \$40?
   Have a go at solving the problem and see what changes and what stays the same.
   See if you can find a way to write a number sentence algebraically so someone could use your number sentence to work out how they need to save no matter what the cost of the t-shirt.

6)   You would like to buy a CD player that costs \$35. You earn \$5 an hour at your job. How many hours do you need to work?
   What about if the CD player costs \$45 or \$60 or \$80?
   Have a go at solving the problem and see what changes and what stays the same.
   See if you can find a way to write a number sentence algebraically so someone could use your number sentence to work out how many hours they need to work no matter what the cost of the CD player.

7)   At the table 5 people can sit like this ……

When another table is joined this many people can sit around it…

```
         o o o
         ___
       o/ \ /o
         o o o
```

Can you find a pattern?
How many people could sit at 3 tables or 5 tables or 10 tables?
See if your group can come up with a rule and make sure you can explain why your rule works.

8)   Jasmine and Cameron are playing "Happy houses". They have to build a house and add onto it. The first one looks like this…..

```
       / \
       |_|
```

The second building project looks like this….

```
     / \ / \
     |_|_|
```

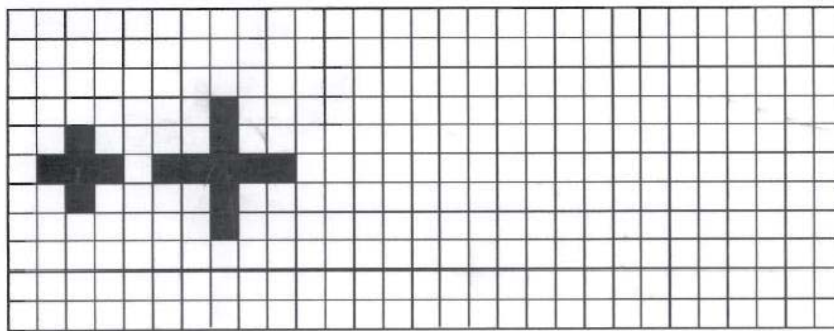How many sticks would you need to build four houses?
How many sticks would you need to build eight houses?
Can you find a pattern and a rule?

9)

a) Draw in the next two crosses in this pattern.

Cross 1   Cross 2

Complete the table

	Cross 1	Cross 2	Cross 3	Cross 6	Cross 20
Number of squares across					
Total number of squares					

123

What is the word rule for the pattern?
Can you write it as an algebraic number sentence?
Use the rule to find the number of squares across and the total number of squares for Cross 101.

10) Tim is making pyramid towers.
The first one looks like this….. □
□□□

The second one looks like this….. □
□□□
□□□□□

The third one looks like this….. □
□□□
□□□□□
□□□□□□□

How many blocks are in each bottom layer?
How many blocks would be in the bottom layer for the fourth tower?
How many blocks would be in the bottom layer for the eighth tower?
Tim wanted to build the 100th pyramid tower – how many blocks would he need for the bottom layer?
Can you find a rule and describe it in words?

11) Rachael has two mice cages which are joined together. She has 8 mice in the cages. Can you show all the different ways the 8 mice could be in the cages? What about if Rachael had 9 mice or 20 mice?
Can you write a number sentence so Rachael can find out how many different ways the mice can be in the cages no matter how many mice there are?

12) Ella is thinking of joining Video Ezy offers two rental plans.
Plan A costs $18 annual membership fee plus $2 per video rented.
Plan B costs $9 membership fee but costs $3 per video rented.
Can you show both these plans as algebraic number sentences?

Use your number sentence to work out which plan would be better if you rented 6 videos? What about if you rented 12 videos?
Can you work out how many videos you would need to rent for the plans to become equal?

13) Vodafone is currently offering a calling plan that charges 5 cents per call and 10 cents per minute..
1) How much would a 3 minute phone call cost? 6 minutes? 15 minutes?
2) Write a number sentence to show how much a phone call will cost no matter how long you talk for.
3) Can you use your number sentence to show how much a 101 minute phone call would cost?

14) The solution to the equation $3n + 15 = 39$ is $n = 8$
What is the solution to the equation $3n + 15 - 7 = 39 - 7$?

15) Ruby took a quick look at these true/false number sentence problems. Without working out any bit of it she said both problems were true.
The first part said $18 + 15 = 33$ is true.
She looked at the second part which said $18 + 15 - 9 = 33 - 9$
How did she know that the problem was true?
How did she know that the problem was true?

16) Ruby took a quick look at these true/false number sentence problems. Without working out any bit of it she said both problems were true.
The first part said $11 + 115 = 233$ is true.
She looked at the second part which said $118 + 115 - 9 = 233 - 9$
How did she know the problem was true?
Then she looked at the second problem which said $254 - 89 = 165$ is true.
She looked at the second part which said $254 - 89 + 11 = 165 + 11$.
How did she know that the problem was true? Can you make a generalisation from the problem? Can you think of an algebraic number sentence using if and then that would show your generalisation always works?

17) Ella had 6 science groups and she had 120 grams of salt to share amongst the groups for their science investigation.
Solve this problem using multiplication.
Solve this problem using division.
Can you always switch multiplication and division like you have done to solve the problem?
Can you write a number sentence to show this using **If….and then** which would work for any number?

18) Ella had 5 grams of salt for her science experiment but the instructions said she needed 11 grams. How much more salt does Ella need?

Solve this problem using addition.

Solve this problem using subtraction.

Can you always switch addition and subtraction like you have done to solve the problem?

Can you write a number sentence to show this using **If….and then** which would work for any number?

APPENDIX D: Teacher information sheet and consent form

Dear

As you know I am to be on study leave for the next two terms to complete a thesis for a Master of Education at Massey University. My thesis is a qualitative study examining how classroom environments and instructional tasks can support children in developing early algebraic reasoning.

Together we have discussed the need for arithmetic and algebra to be taught in an integrate way in order to develop children's early algebraic understandings. Now I am formally inviting you to be a part of a collaborative teaching design experiment research process in which we look at some of the ways children construct algebraic understanding as they participate in mathematical activity in classroom. We will also examine how the instructional environment and tasks support children to develop early algebraic understandings.

Permission to participate in the study will be sought from both the parents of the children in your class and the children themselves. The children and their parents/caregivers will be given full information and consent will be requested in due course.

Two individual interviews will explore the children's early algebraic understanding. One interview will take place at the beginning of the project and the second during the project. These interviews will be conducted in much the same way as the interviews in the classroom for the 'Numeracy Project' a year ago. The time involved for each child for each interview will be no more than 20 minutes. The interview with each child will be audio recorded.

Using the data from the interviews, we will plan a unit of lessons which will be taught. The observations will involve the use of audio and video recording. During classroom mathematics activities you may at any time ask that the audio or video recorder be turned off and any comments you have made deleted.

The time involved in the complete study for you will be no more than thirty-five hours, over a period of one school term. During the data collection phase you will be asked to keep a diary to provide a retrospective account of classroom instruction but no evaluation of the instructional programme will occur other than that which is grounded in the context of the study.

All project data will be stored in a secure location, with no public access and used only for this research and any publications arising from this research. After completion of five years, all data pertaining to this study will be destroyed in a secure manner. All efforts will be taken to maximize confidentiality and anonymity for participants. The school name and names of all participants will be assigned pseudonyms to maintain their anonymity. However total anonymity cannot be guaranteed due to my position as both a researcher and current member of staff. Near the end of the study a summary will be presented to you to verify accuracy, and following any necessary adjustments, a final summary will be provided to the school and teachers involved.

Please note you have the following rights in response to my request for you to participate in this study.
- decline to answer any particular question;
- withdraw from the study after four weeks;
- ask any questions about the study at any time during participation;
- provide information on the understanding that your name will not be used unless you give permission to the researcher;
- To ask for the audio or video recorder to be turned off and any comments you have made be deleted
- be given access to a summary of the project findings when it is concluded.

If you have further questions about this project you are welcome to discuss them with me personally:

Jodie Hunter. Phone: (09) 846 0721 0r 021 022 52204. Email. jodiehunter@slingshot.co.nz

or contact either of my supervisors at Massey University (Palmerston North)

- Associate Professor Glenda Anthony: Department of Technology, Science and Mathematics Education. Phone: (06) 350 5799 Extension 8600. Email. G.J.Anthony@massey.ac.nz

This project has been reviewed and approved by the Massey University Human Ethics Committee: Southern B, Application 06/57. If you have any concerns about the conduct of

this research, please contact Dr Karl Pajo, Chair, Massey University Human Ethics Committee: Southern B, telephone 04 801 5799 x 6929, email *humanethicsouthb@massey.ac.nz*

CONSENT FORM: TEACHER PARTICIPANT

THIS CONSENT FORM WILL BE HELD FOR A PERIOD OF FIVE (5) YEARS

I have read the Information Sheet and have had the details of the study explained. My questions have been answered to my satisfaction, and I understand that I may ask further questions at any time.

I agree/do not agree to_____ being audio-taped

I agree/do not agree to_____ being videotaped

I agree to _____ participating in this study under the conditions set out in the Information Sheet.

Signature: .... **Date:** ....

Full Name - printed ....

APPENDIX E: Student and parent information sheet and consent form

Dear

I am doing a research project for a Master of Education at Massey University. I am going to look at how the mathematical activities children work on in the classroom help them develop their number and algebra understandings.

I would like to invite you with your parent's permission to be involved in this study. Ella, your teacher, has also agreed to participate in the study. The Board of Trustees has also given their approval for me to invite you to participate, and for me to do this research.

If you agree to be involved, I will interview you about your number and algebra understandings. There will be two interviews: one at the beginning of the year and one at the end of the first term. They will be like the interviews we do for the Numeracy Project. The time involved in the interview will be no more than 20 minutes. The interview will be tape-recorded and you may at ask that the tape recorder be turned off and that any comments you have made be deleted if you change your mind or are not happy about what you said.

Ella and I will plan a unit of mathematics based on what we have learnt about how you think about number and algebra. The lessons will be taught in your classroom and will be audio and video recorded. During classroom mathematics activities you may at any time ask that the audio or video recorder be turned off and any comments you have made deleted. With your permission I might sometime collect copies of your mathematics reflections, written work and charts you make to support your explanations to the group during the unit. You have the right to refuse to allow the copies to be taken.

The mathematics activities you do in class will be the same whether you agree to be in the study or not. The interview and observations will take place in the classroom and be part of the normal mathematics programme. It is possible that talking about your learning may help you clarify what you know about number and algebra and what you need to know next.

All the information gathered will be stored in a secure location and used only for this research. After the completion of the research the information will be destroyed. All efforts will be taken to maximise your confidentiality and anonymity which means that your name will not be used in this study and only non-identifying information will be used in reporting. However total anonymity cannot be guaranteed due to my position as both a researcher and current member of staff.

I ask that you discuss all the information in this letter fully with your parents before you give your consent to participate.

Please note that you have the following rights:
- To say you do not want to participate in the study
- To withdraw from the study at any time
- To ask for the audio or video recorder to be turned off and any comments you have made be deleted
- To refuse to allow copies of your written work to be taken
- To ask questions about the study at any time
- To participate knowing that you will not be identified at any time
- To be given a summary of what is found at the end of the study

If you have further questions about this project you are welcome to discuss them with me personally:

Jodie Hunter. Phone: (09) 846 0721. Email. jodiehunter@slingshot.co.nz

or contact my chief supervisor at Massey University (Palmerston North)

- Associate Professor Glenda Anthony: School of Curriculum and Pedagogy. Phone: (06) 350 5799 Extension 8600. Email. G.J.Anthony@massey.ac.nz

This project has been reviewed and approved by the Massey University Human Ethics Committee: Southern B, Application 06/57. If you have any concerns about the conduct of this research, please contact Dr Karl Pajo, Chair, Massey University Human Ethics Committee: Southern B, telephone 04 801 5799 x 6929, email humanethicsouthb@massey.ac.nz.

CONSENT FORM: STUDENT PARTICIPANTS

THIS CONSENT FORM WILL BE HELD FOR A PERIOD OF FIVE (5) YEARS

I have read the Information Sheet and have had the details of the study explained. My questions have been

answered to my satisfaction, and I understand that I may ask further questions at any time.

I agree/do not agree to being audio-taped

 I agree/do not agree to being videotaped

I agree/do not agree to participating in this study under the conditions set out in the Information Sheet.

Child's Signature: .... **Date:** ....

Full Name - printed ....

CONSENT FORM: PARENTS OF STUDENT PARTICIPANTS

THIS CONSENT FORM WILL BE HELD FOR A PERIOD OF FIVE (5) YEARS

I have read the Information Sheet and have had the details of the study explained. My questions have been

answered to my satisfaction, and I understand that I may ask further questions at any time.

I agree/do not agree to_____ being audio-taped

I agree/do not agree to_____ being videotaped

I agree to _____ participating in this study under

the conditions set out in the Information Sheet.

Parent's Signature: .... **Date:** ....

Full Name - printed ....

APPENDIX E: Board of Trustees information sheet and consent form

Dear Sir/Madam

As you know I have been a teacher at Bellview School for the past 2 years and am to be on study leave for the next two terms to complete a thesis for a Master of Education at Massey University. My thesis is a qualitative study examining how classroom environments and instructional tasks can support children in developing early algebraic reasoning.

Ella Belmain has tentatively agreed to participate in a teaching design experiment for teaching number and algebra. She will be formally approached following B.O.T. approval of the study. The parents and children will be informed of the nature of the study through information sheets and a discussion in class.

Two individual interviews will be conducted during the study to explore the children's early algebraic understanding. One interview will take place at the beginning of the project and the second during the project. These interviews will be conducted in much the same way as the interviews in the classroom for the 'Numeracy Project' a year ago. The time involved for the children for each interview will be no more than 20 minutes. The interview with the children will be audio recorded.

Using the data from the interviews, Ella and I will plan a unit of lessons. During the teaching of the unit, several of the lessons will be video and audio-taped. Work samples will also be collected and photo-copied. Ella and I will review the tapes to provide feedback for planning of the next series of lessons.

The time involved in the complete study for Ella will be no more than thirty-five hours, over a period of one school term. The teacher, the children, and their parents/caregivers will be given full information and consent will be requested in due course. Specifically, permission to allow the children to be included in video footage of classroom lessons and to participate in individual interviews will be sought from both the parents of the children and the children within Ella's class.

All project data will be stored in a secure location, with no public access and used only for this research and any publications arising from this research. After completion of five years, all data pertaining to this study will be destroyed in a secure manner. All efforts will be taken to maximize confidentiality and anonymity for participants. The school name and names of all participants will be assigned pseudonyms to maintain their anonymity. However total anonymity cannot be guaranteed due to my position as both a researcher and current member of staff. Near the end of the study a summary will be presented to the teacher to verify accuracy, and following any necessary adjustments, a final summary will be provided to the school and teacher involved.

Please note you have the following rights in response to my request for your school to participate in this study.
- withdraw from the study after ten weeks;
- ask any questions about the study at any time during participation;
- provide information on the understanding that the participants' names will not be used unless you give permission to the researcher;
- be given access to a summary of the project findings when it is concluded.

If you have further questions about this project you are welcome to discuss them with me personally:

Jodie Hunter. Phone: (09) 846 0721 0r 021 022 52204. Email. jodiehunter@slingshot.co.nz

or contact my chief supervisor at Massey University (Palmerston North)

- Associate Professor Glenda Anthony: School of Curriculum and Pedagogy. Phone: (06) 350 5799 Extension 8600. Email. G.J.Anthony@massey.ac.nz

This project has been reviewed and approved by the Massey University Human Ethics Committee: Southern B, Application 06/57. If you have any concerns about the conduct of this research, please contact Dr Karl Pajo, Chair, Massey University Human Ethics

Committee: Southern B, telephone 04 801 5799 x 6929, email humanethicsouthb@massey.ac.nz

CONSENT FORM: BOARD OF TRUSTEES

THIS CONSENT FORM WILL BE HELD FOR A PERIOD OF FIVE (5) YEARS

We have read the Information Sheet and have had the details of the study explained. Our questions have been

answered to our satisfaction, and we understand that we may ask further questions at any time.

We agree to _____ participating in this study under

the conditions set out in the Information Sheet.

Signature: .... **Date:** ....

Full Name - printed ....