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Two Elliptic Generator Kleinian Groups

A thesis presented in partial fulfilment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

at Massey University, Albany, New Zealand

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2010

ABSTRACT

This thesis studies the discreteness of Kleinian groups and the geometry of their associated orbit spaces: hyperbolic 3-manifolds and 3-orbifolds.

Thurston's geometrization theorem states that the interior of every compact 3-manifold can be decomposed into pieces which have geometric structures. Most of these pieces have hyperbolic structures. Every hyperbolic 3-manifold can be described as \mathbb{H}^3/G , where \mathbb{H}^3 is the 3-dimensional hyperbolic space and G is a torsion free Kleinian group. The compact 3-manifolds other than hyperbolic 3-manifolds have been classified. Therefore the study of Kleinian groups holds the key to understanding 3-manifolds in general.

We investigate various conditions for determining the discreteness of a Kleinian group. Such a group is discrete if and only if all its two generator subgroups are discrete, thus we study the question of discreteness through two generator groups. Here we consider Kleinian groups generated by two elements of finite order. Once the order of these generators is fixed, these groups lie in a one complex-dimensional parameter space with a highly fractal boundary. Computational investigations of various types into the size and structure of this parameter space form an integral part of this thesis. In particular an analysis of Dehn surgery on 2-bridge knots and links gives us data indicating the boundary of the parameter space "internally" analogous to the boundary in the Riley slice (the parabolic case). We then investigate this boundary "externally" and numerically using the rational pleating rays and rational words, which were developed by Keen and Series to study the Riley slice boundary. We are then able to identify some boundary points via the interplay between algebraic and geometric convergence of sequences of Kleinian groups. We find an interesting connection between Conway's notation for 2-bridge knots and links and rational words.

We then apply these descriptions of parameter spaces to advance the identification problem for two generator arithmetic Kleinian groups, in fact we give a conjecturally complete list of certain families of such groups that were previously identified as being the most difficult cases.

ACKNOWLEDGMENTS

First of all, I would like to express my sincere gratitude to my chief supervisor, Distinguished Professor Gaven Martin. Without his encouragement, this thesis would not have eventuated. He has chosen a wonderful research subject and given me the necessary tools to explore it; and has always allowed me full freedom in my research. Above all, Gaven is a very warm and humorous person. It has been a real pleasure to work with him.

A few years ago, I gave a talk which turned out to be a disaster. I felt so low and doubted my ability to give any future talks. It was my co-supervisor Professor David Gauld who comforted me and assured me that with practice, I could succeed in giving talks. David encouraged me to give a talk at the post-graduate student conference that year and has always been happy to give me a chance to talk whenever possible; and he has always replied to my questions with great patience and wisdom. Aside from all of this, David is the person who has introduced me to the joys of tramping in the mountains of this beautiful country.

Staying in New York with a limited budget was not easy, Professor Linda Keen did so much to help me. She not only found a safe and affordable place for me, but also made sure that everything I needed for my research was there ready for me. I am especially grateful for her great effort at getting Mathematica installed in a computer in Lehman College. I have learned a lot from Linda. She has opened so many doors for me in studying Kleinian groups and what I have learnt from her has enhanced my research.

I am very grateful to Professor Chuck Livingston of Indiana University, Bloomington. First of all, my Dehn surgery data would not have existed without his website KnotInfo. Then when I visited him, he not only picked me up at the Indiana bus station but also let me stay in his house. Above all, he gave me a tip in my research: the convergence of the sequences of knots and links in my Dehn surgery data. With this tip, we have been able to find some boundary points in a $(2, m)$ commutator plane; and the beautiful connection between the 2-bridge knots and links and the rational words discussed in Chapter 4.

I am also grateful to Professor Paul Kirk of Indiana University for his time spent discussing my research.

I have a great supporting team at the Institute of Information and Mathematical Sciences (IIMS) of Massey University, Albany. IIMS's IT technicians, Anil Malhotra, Mike Yap, Yan Ou and Tony

Mao, have made my learning much easier by not letting me worry about any problem coming from my computers; Freda Mickisch, Michelle Campbell and Annette Warbrooke have made the IIMS Department a nice place in every possible way; the lecturers and professors whom I have learned from, Robert McKibbin, Mick Roberts, Graeme Wake, Alona Ben-Tal, Shaun Cooper, Yow-Tzong Yeh, Carlo Laing, Winston Sweatman, Tatiana (Tanya) Evans, all have contributed to this thesis in various ways.

I would like to give my thanks to Maarten Jordens, Haydn Cooper, Samuel Dillon, my fellow postgraduate students. Maarten has taught me a lot, especially his trick in using the Paint program has made it so much easier to obtain figures for this thesis; Haydn gave me tips in computer programming; Samuel is always happy to have a badminton game to freshen up my mind.

I am also grateful to various Institutes who have supported me in many ways. These are: the New Zealand Institute for Mathematics and its Applications, who have generously funded me for three years; The New Zealand Institute for Advanced Study, who have funded my visit to Linda Keen and have been funding me for the fourth year of my research; IIMS, who have generously provided me the space, the computers and funded the journal articles which were not available in Massey University's Library.

Finally, I am heavily indebted to my husband Brian for his endless support and encouragement and his great effort at proof reading this thesis. Without him, this thesis would not have had a conclusion.

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