On Two Problems of Arithmetic Degree Theory

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Alistair Allan Compton

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To the memory

of

Wolfgang Vogel

whose guidance was invaluable.

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ABSTRACT

The reader of this thesis should already have a basic understanding of ideal theory. For this reason it is recommended that a good introduction to this subject would be gained from reading D. G. Northcott's book "Ideal Theory", paying special attention to chapters one and three. This thesis consists of three chapters, with chapter one providing the definitions and theorems which will be used throughout. Then I will be considering two problems on the arithmetic degree of an ideal, one posed by Sturmfels, Trung and Vogel and the other by Renschuch. These problems will be described in the introductions to chapters two and three.

CHAPTER 1

PRELIMINARY RESULTS

Let I be a homogeneous ideal of the polynomial ring $S = F[x_0, \ldots, x_n]$ where F is any field.

Let P be a prime ideal belonging to I.

If P is isolated, we know from the corollary of theorem 3 of Northcott's book [6, p.19], that the primary component corresponding to P is the same for all normal decompositions of I.

However, if P is embedded, then this is not true, as the following example [6, p.30] shows.

Consider the ideal (x^2, xy) in the ring $F[x, y], F$ any field.

It is shown in Northcott's book [6, p. 30] that

$(x) \cap (y + ax, x^2)$ (where a is any element of F),

$(x) \cap (y + bx, x^2)$ (where $b \in F,\ b \neq a$),

and $(x) \cap (x^2, xy, y^2)$

are all normal decompositions of (x^2, xy) with $(y + ax, x^2)$, $(y + bx, x^2)$, (x^2, xy, y^2) all (x, y) – primary.

So the primary component corresponding to an embedded prime ideal need not be unique.

Therefore, if we have two normal primary decompositions of I, one having a primary component Q_1 corresponding to an embedded prime P, and the other having a primary component Q_2 corresponding to P, $Q_1 \neq Q_2$, then in general, the classical length multiplicity of Q_1 does not equal the classical length multiplicity of Q_2.

However, in arithmetic degree theory, we do have a way of defining the length multiplicity of an embedded component of an ideal which is well-defined.

The definitions that are needed to do this are given later in this chapter.

We will also need some basic facts about the Hilbert function from the classical degree theory.

Let $V(n+1,t)$ denote the F–vector space consisting of all forms of degree t in $x_0, \ldots, x_n$.

Then $\dim_F V(n+1,t) = \binom{t+n}{n}, \quad t \geq 0,\ n \geq 0.$

Let $V(I,t)$ be the F–vector space consisting of all forms in $V(n+1,t)$ which are in I.

Definition 1. The function $H(I,-) : Z^+ \rightarrow Z^+$ [10, p.43] defined by $H(I,t) = \dim_F V(n+1,t) - \dim_F V(I,t)$ is called the Hilbert function of I.

For large enough t, the Hilbert function is a polynomial $P(I,t)$ in t with coefficients in Z. The degree $d\,(0 \leq d \leq n)$ of this polynomial is called the dimension of I and is denoted by $\dim(I)$.

The polynomial $P(I,t)$ can be written in the following form:

$P(I,t) = h_0(I)\binom{t}{d} + h_1\binom{t}{d-1} + \ldots + h_d$ [10, p.45] where $h_0(I)$ is a positive integer.

The leading coefficient of $P(I,t)$, namely $h_0(I)$, is called the degree of I.

There is of course a great deal of theory on the Hilbert polynomial, but for our purposes the following definition and theorem will suffice.

Let $I = (f_1, \ldots, f_t)$.

Definition 2. I is said to be a complete intersection if $(f_1, \ldots, f_{i-1}) : f_i = (f_1, \ldots, f_{i-1})$ for all $i = 1, \ldots, t$.

Theorem 1 [10, p.46]. Let the generators $f_1, \ldots f_t$ of I be forms of degrees $s_1, \ldots, s_t$ respectively. If I is a complete intersection then $h_0(I) = s_1 \ldots s_t$.

We will now state the other definitions, theorems and propositions that will be used in chapters two and three.

Definition 3 [5, p.1]. Given any homogeneous ideal I and prime ideal P in S, we define J to be the intersection of the primary components of I with associated primes strictly contained in P. We let $J = S$ if there are no primes p belonging to I with $p \subsetneq P$.

Let Q be a P–primary ideal belonging to I.

Definition 4 [3]. We define the length–multiplicity of Q, denoted by $\mathrm{mult_I(P)}$, as the length of a maximal strictly increasing chain of ideals, $I \subseteq J_\ell \subset J_{\ell-1} \subset \ldots \subset J_2 \subset J_1 \subset J$ where each J_k equals $q \cap J$ for some P–primary ideal q.

As we will be making repeated use of an algorithmic approach to calculate $\mathrm{mult_I(P)}$ it is convenient to state it here, followed by a theorem.

Step 1. Take a maximal strictly increasing chain of primary ideals from Q to P.

$$Q \subset \ldots \subset Q_{i-1} \subset Q_i \subset \ldots P. \tag{1}$$

Step 2. Intersect each primary ideal in (1) with J.

$$Q \cap J \subseteq \ldots \subseteq Q_{i-1} \cap J \subseteq Q_i \cap J \subseteq \ldots \subseteq P \cap J = J. \tag{2}$$

Step 3. Eliminate duplicates in (2) in order to get a strictly increasing chain of ideals in the sense of definition 4.

$$Q \cap J =: J_\ell \subset J_{\ell-1} \subset \ldots \subset J_1 \subset J. \tag{3}$$

Note: If P is an isolated prime ideal of I, then $\mathrm{mult_I(P)}$ gives the classical length multiplicity of Q.

Theorem 2 [5, p.2]. Using the above notation we have $\ell = \mathrm{mult_I}\,(\mathrm{P})$.

Definition 5 [2, p.1]. A polynomial of the form $a_{(i)}x_1^{i1}\,x_2^{i2}\ldots x_n^{in}$, where $i_1, i_2, \ldots i_n$ are any non-negative integers and $a_{(i)}$ is any element of F, is a monomial.

Definition 6 [2, p.1]. If A is an ideal of S then A is a monomial ideal of S if and only if A is generated by monomials. That is, $A = (m_1, \ldots, m_s)$, where m_ℓ are monomials for $\ell = 1, \ldots, s$.

Proposition 1 [2, p.2]. Let P_1 be a monomial ideal of $S = F\,[x_0, \ldots, x_n]$; P_1 is a prime ideal if and only if $P_1 = (x_{i_0}, \ldots, x_{i_r})$, $i_j \in \{0, \ldots, n\}$ for $j = 0, \ldots, r$.

Proposition 2 [2, p.2]. Let P_1, Q_1 be monomial ideals of $S = F\,[x_o, \ldots, x_n]$ where P_1 is prime and, say $P_1 = (x_{i_0}, \ldots, x_{i_r})$, $i_j \in \{0, \ldots, n\}$ for $j = 0, \ldots, r$. Q_1 is P–primary if and only if $Q_1 = (x_{i_0}^{t_0}, \ldots, x_{i_r}^{t_r}, m_0, \ldots, m_s)$ where $t_j \geq 1$ for $j = 0, \ldots, r$, and m_ℓ are monomials in $x_{i_0}, \ldots, x_{i_r}$ for $\ell = 0, \ldots, s$.

Definition 7. Consider a primary decomposition of $I = Q_1 \cap \ldots \cap Q_k$ where Q_i is P_i–primary. The arithmetic degree of I, denoted by arith-deg (I), is given by
arith-deg $(I) := \sum_{i=1}^{k} \mathrm{mult}_I\,(P_i)$ degree (P_i).

Let $I = (f_1, \ldots, f_t)$.

Definition 8. $M\,(I) := \max\limits_{i=1 \text{ to } t} \left\{ \text{degree } (f_i) \right\}$.

Theorem 3 (criterion of $\mathrm{mult_I(P)} = 1$) [1, p.2].

Let R be a Noetherian ring.

Let A and B be ideals in R such that $B \subsetneqq A$.

Let P be a prime ideal such that all primes belonging to A and B are contained in P.

Necessary and sufficient conditions, that there exists no ideal, say C, with $B \subsetneqq C \subsetneqq A$, and all primes that belong to C are also contained in P, are the following:

(i) there exists an element x in A such that $A = B + R \cdot x$.

(ii) $PA \subseteqq B$.

Definition 9 [2, p.3]. Two monomials λ and τ are said to be relatively prime if, when

$$\lambda = x_{i_0}^{n_{i_0}} \dots x_{i_j}^{n_{i_j}} \text{ and } \tau = x_{k_0}^{m_{k_0}} \dots x_{k_r}^{m_{k_r}},$$

$$\text{then } \{x_{i_0}, \dots, x_{i_j}\} \cap \{x_{k_0}, \dots, x_{k_r}\} = \phi.$$

Theorem 4 [2, p.3]. Let $S = F[x_0, \dots, x_n]$ be a ring of polynomials in $n+1$ indeterminates. Let $\lambda, \tau, m_0, \dots, m_r$ be monomials in F. If λ and τ are relatively prime, then:

$$(\lambda \cdot \tau, m_0, \dots, m_r) = (\lambda, m_0, \dots, m_r) \cap (\tau, m_0, \dots, m_r).$$