The Development of The Elliptic Functions According To Ramanujan

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Abstract

Srinivasa Ramanujan (1887-1920) was one of the world's greatest mathematical geniuses. He made substantial contributions to elliptic functions, continued fractions, infinite series, and the theory of numbers. For many years people have studied Ramanujan's work and tried to obtain a better understanding of his work.

The main purpose of my thesis will be to consider some important classical results on elliptic functions and give proofs of these results using the methods which could have been used by Ramanujan. This will give an insight into how Ramanujan may have proved many of his results since his own proofs are often unknown.

This thesis contains five chapters. Chapter 1 is the introduction and this is related to Chapter 2 up to Chapter 4. The goal for Chapter 2 is to write the transformation of $S_{2n+1}(q)$, $\phi_{r,s}(q)$, $U_{2n}(q)$, and $V_{2n}(q)$ in terms of $P(p)$, $Q(p)$, and $R(p)$. Chapter 3 discusses Ramanujan's congruence for partitions and we give a proof for Ramanujan's modulus 5 partition congruence. In Chapter 4, we investigate a method of determining the number of representations of an integer n as the sum of two, four, six, and eight squares and triangular numbers. Then we present two computer programs which are for the sums of squares and triangles. Finally, some interesting relations between the sums of squares and the sums of triangles are shown.

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Contents

Introduction

Srinivasa Ramanujan (1887-1920) was an Indian mathematician who had to contend with a lack of education and resources. In 1913, Ramanujan wrote his first letter to G. H. Hardy. Hardy ([19], p. 9) said, "...they defeated me completely; I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them. Finally... the writer must be completely honest, because great mathematicians are commoner than thieves or humbugs of such incredible skill."

Ramanujan lived a short life. He died on April 26 1920, and left behind three notebooks, a "lost notebook" and other manuscripts, and published papers [21].

The goal of this thesis is to gain a better understanding of Ramanujan's work. All my work, methods, and ideas are based on those in K. Venkatachaliengar's [36] monograph. Venkatachaliengar's work is significant to this thesis.

"We have no idea how he did the marvelous things he did, what led him to them, or anything else," said mathematician Richard Askey ([26], p. 280), a Ramanujan scholar at the University of Wisconsin in Madison. Bruce Berndt ([26], p. 280) said "I still don't understand it all. I may be able to prove it, but I don't know where it comes from and where it fits into the rest of mathematics," after years of working through Ramanujan's notebooks. He also said, "The enigma of Ramanujan's creative process is still covered by a curtain that has barely been drawn." So I think it is worthwhile to study his work and to have a better understanding of Ramanujan's work and of himself.

This thesis consists of five Chapters. Chapter 1 is an introduction to Chapter 2, 3, and 4. We will introduce some of Ramanujan's identities and use some theorems and results due to Venkatachaliengar to prove them. Then we introduce Ramanujan's ${}_1\psi_1$ summation formula, which is related to many different identities. The Jacobi triple product identity and the Jordan-Kronecker function are special cases of Ramanujan's summation formula. Then we introduce the fundamental multiplicative identity, the Weierstrass function and three special cases of $F(a,t)$. Next we give series expansions for $\phi_1(a)$, $\phi_2(a)$, and $\wp(a)$ and introduce the series P, Q, and R.

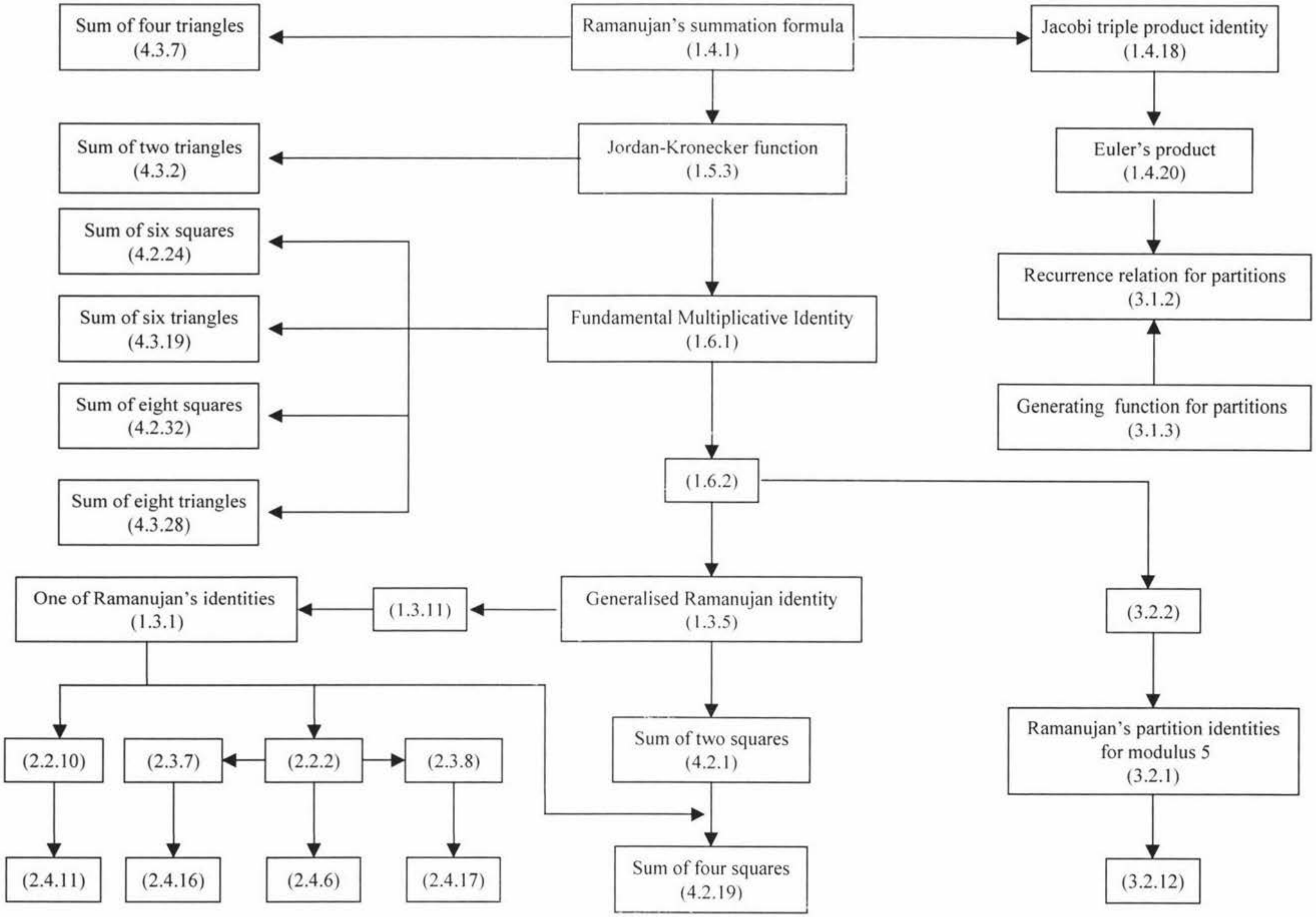
In 1916, Ramanujan published a paper called "On certain arithmetical functions" ([21], pp.136-143) which contain the formulas for S_n, $\phi_{r,s}$, and Ramanujan differential equations. According to Venkatachaliengar, the transformation formulas for S_{2n+1}, $\phi_{r,s}$, U_{2n}, and V_{2n} can be calculated. The aim of Chapter 2 is to write the transformation of $S_{2n+1}(q)$, $\phi_{r,s}(q)$, $U_{2n}(q)$, and $V_{2n}(q)$ in terms of $P(p)$, $Q(p)$, and $R(p)$.

Ramanujan discovered properties of $p(n)$, the number of partitions of n, by studying MacMahon's table of values of $p(n)$. Ramanujan observed that the number of partitions of numbers $5m+4$, $7m+5$, and $11m+6$ are divisible by 5, 7, and 11 respectively. In Chapter 3, we will give a proof of Ramanujan's modulus 5 partition congruence and the idea is based on Venkatachaliengar's work.

The problem of the representation of an integer n as the sum of a given number k integral squares and triangles is one of the most interesting in the theory of numbers.

Chapter 4 will present proofs of formulas for the sums of two, four, six, and eight squares and triangles, all based on Ramanujan's summation formula. For instance, Jacobi proved an identity for the sum of six triangles (which is not widely known) but I found a proof using Ramanujan's result. Ramanujan has given a formula for the sum of twenty-four squares. I wrote two computer programs which can find the number of ways of writing an integer as a sums of k squares and triangles, how many groups there are, and their nature. The last section will indicate some interesting relationships between sums of squares and sums of triangles.

A flow chart in the next page represents the main results of this thesis and indicates how we connect some of our results to Ramanujan's works. Each box contains either the equation and the equation number or just the equation number. The equation numbering is standard so that for example (1.4.22) can be found in Chapter 1, section 4, equation 22.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Chapter 1
Basic definition

1.1 Introduction

In this Chapter, I summarise some results given by K. Venkatachaliengar [36] on "Development of the elliptic functions according to Ramanujan", and S. Cooper [13] on "The development of the elliptic functions according to Ramanujan and Venkatachaliengar", either to be used later in the thesis or provide a better understanding of Ramanujan's works. Notice that S. Cooper's [13] is based on the monograph by K. Venkatachaliengar [36].

1.2 Notation

Let τ be a fixed complex number satisfying $\operatorname{Im}\tau > 0$ and let $q = e^{i\pi\tau}$, so that $|q| < 1$. We will make use of the following notation for products. Let

$$\begin{aligned}(a;q)_n &= \prod_{j=0}^{n-1}\left(1-aq^j\right), \qquad n = 1,2,3,...,\\ (a;q)_\infty &= \prod_{j=0}^{\infty}\left(1-aq^j\right).\end{aligned}$$

Then

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

and we take this as a definition for $n \in \mathbb{R}$. We define

$$(a_1,a_2,...,a_n;q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_n;q)_\infty .$$

Note that Venkatachaliengar and some others take $q = e^{2i\pi\tau}$ instead of $q = e^{i\pi\tau}$.

1.3 The generalised Ramanujan identity

Ramanujan gave many different identities. One of his most famous identities is

$$\begin{aligned} &\left(\frac{1}{4}\cot\frac{\theta}{2} + \sum_{n=1}^{\infty}\frac{x^n}{1-x^n}\sin n\theta\right)^2 \\ = &\left(\frac{1}{4}\cot\frac{\theta}{2}\right)^2 + \sum_{n=1}^{\infty}\frac{x^n}{(1-x^n)^2}\cos n\theta + \frac{1}{2}\sum_{n=1}^{\infty}\frac{nx^n}{1-x^n}(1-\cos n\theta). \qquad (1.3.1) \end{aligned}$$

In 1916, Ramanujan ([21], p. 139) published a paper containing (1.3.1). G. H. Hardy has also mentioned (1.3.1) in ([20], p. 134) and ([19], p. 312). We will state the following theorem and then use it to prove the identity (1.3.1).

Theorem 1.3.1 (Theorem 1.1, [36]**)** *Let*

$$\begin{aligned} \rho_1(z) &= \frac{1}{2} + {\sum_{n=-\infty}^{\infty}}' \frac{z^n}{1-q^{2n}}, \\ \rho_2(z) &= -\frac{1}{12} + {\sum_{n=-\infty}^{\infty}}' \frac{q^{2n}z^n}{(1-q^{2n})^2}, \end{aligned}$$

where $|q| < 1$ and the prime denotes that $n = 0$ is excluded in the summation. Then

a) The series ρ_1 converges for $|q^2| < |z| < 1$;

b)The series ρ_2 converges for $|q^2| < |z| < |q|^{-2}$;

c) Given any three constants α, β, γ satisfying $\alpha\beta\gamma$=1 and $|q| < |\alpha|, |\beta| < 1$, we have

$$\rho_1(\alpha)\rho_1(\beta) - \rho_1(\alpha\beta)(\rho_1(\alpha) + \rho_1(\beta)) = \rho_2(\alpha) + \rho_2(\beta) + \rho_2(\gamma). \qquad (1.3.2)$$

Venkatachaliengar given a proof of this theorem in ([36], p. 4) and bases Chapters 1 and 2 on this identity.

Proposition 1.3.2 **([36], p. 5)**

$$\rho_1(z) = \frac{1+z}{2(1-z)} + \sum_{n=1}^{\infty}\left(\frac{q^{2n}z}{1-q^{2n}z} - \frac{q^{2n}z^{-1}}{1-q^{2n}z^{-1}}\right), \tag{1.3.3}$$

$$\rho_2(z) = -\frac{1}{12} + \sum_{n=1}^{\infty}\left(\frac{2nq^{2n}z}{1-q^{2n}z} + \frac{2nq^{2n}z^{-1}}{1-q^{2n}z^{-1}}\right). \tag{1.3.4}$$

These formulas give the analytic continuations of ρ_1 and ρ_2, and they are valid for all z except where there are poles.

Corollary 1.3.3 **([36], p. 5)**

(a) ρ_1 *has simple poles at the points* $z = q^{2r}$, $r = 0, \pm 1, \pm 2, \ldots$, *and is otherwise analytic in* $0 < |z| < \infty$.

(b) $\rho_1(z) = -\rho_1(z^{-1})$. *(Symmetry)*

(c) $\rho_1(q^2 z) = \rho_1(z) + 1$. *(Quasi-periodicity)*

Corollary 1.3.4 **([36], p. 6)**

(a) ρ_2 *has simple poles at the points* $z = q^{2r}$, $r = \pm 1, \pm 2, \ldots$, *and is otherwise analytic in* $0 < |z| < \infty$.

(b) $\rho_2(z) = \rho_2(z^{-1})$. *(Symmetry)*

(c) $\rho_2(z) - \rho_2(q^2 z) = \frac{1}{2} + \rho_1(z)$. *(Quasi-periodicity)*

Notice that the generalised Ramanujan relation (1.3.2) can be extended to all values of α, β, and γ, using the analytic continuation, the properties of symmetry and quasi-periodicity of the functions ρ_1 and ρ_2. Therefore we may restate (1.3.2) as

Theorem 1.3.5 **([36], p. 7)** *For any three constants α, β, γ satisfying $\alpha\beta\gamma = 1$, we have*

$$\rho_1(\alpha)\rho_1(\beta) + \rho_1(\beta)\rho_1(\gamma) + \rho_1(\gamma)\rho_1(\alpha) = \rho_2(\alpha) + \rho_2(\beta) + \rho_2(\gamma). \tag{1.3.5}$$

Definition 1.3.6 **([36], p. 7)** *Put $z = e^{i\theta}$ and set $q = e^{i\pi\tau}$, where $\mathrm{Im}(\tau) > 0$. Then define*

$$\phi_1(\theta) = \rho_1(z)/2i, \tag{1.3.6}$$

$$\phi_2(\theta) = \rho_2(z)/2. \tag{1.3.7}$$

The following properties of ϕ_1 (1.3.6) and ϕ_2 (1.3.7) follow immediately from the corresponding properties of ρ_1 and ρ_2.

Proposition 1.3.7 **([36], p. 7)**

(i)

$$\phi_1(\theta) = \frac{1}{4}\cot\frac{\theta}{2} + \sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin 2n\theta, \tag{1.3.8}$$

(ii)

$$\phi_2(\theta) = -\frac{1}{24} + \sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}\cos 2n\theta, \tag{1.3.9}$$

(iii)

$$\phi_1(-\theta) = -\phi_1(\theta),$$

(iv)

$$\phi_2(-\theta) = \phi_2(\theta),$$

(v)

$$\phi_1(\theta + 2\pi) = \phi_1(\theta),$$

(vi)

$$\phi_2(\theta + 2\pi) = \phi_2(\theta),$$

(vii)

$$\phi_1(\theta + 2\pi\tau) = \phi_1(\theta) - i/2,$$

(viii)

$$\phi_2(\theta + 2\pi\tau) = \phi_2(\theta) - i\phi_1(\theta) - 1/4,$$

(ix)

ϕ_1*has simple poles at* $\theta = 2\pi m + 2\pi i\tau n, m, n \in Z$*, and is analytic elsewhere,*

(x)

ϕ_2*has simple poles at* $\theta = 2\pi m + 2\pi i\tau n, m, n \in Z, n \neq 0$*,and is analytic elsewhere.*

And, if $a + b + c = 0$, then we can rewrite $(1.3.5)$ as

$$\phi_1(a)\phi_1(b) + \phi_1(b)\phi_1(c) + \phi_1(c)\phi_1(a) = -\frac{1}{2}(\phi_2(a) + \phi_2(b) + \phi_2(c)). \qquad (1.3.10)$$

Venkatachaliengar has shown that the functions ϕ_1 and ϕ_2, together with the generalised Ramanujan identity (1.3.10), can be used as the foundation for the development of elliptic functions. Elliptic functions involve two real or complex numbers ω_1 and ω_2 whose the ratio $\frac{\omega_1}{\omega_2}$ is not a real number. A function satisfying the relation

$$f(z + 2\omega_1) = f(z), \qquad f(z + 2\omega_2) = f(z),$$

for all complex values of z at which $f(z)$ exists, is called a doubly periodic function of z with periods $2\omega_1$, $2\omega_2$. A doubly periodic function that is meromorphic in the finite part of the complex plane is called an elliptic function.

The identity (1.3.1) can be obtained from (1.3.10) in the following way. Set $c = -(a+b)$ in (1.3.10). Since $\phi_1(c)$ is an odd function, we have

$$\phi_1(a)\phi_1(b) - \phi_1(b)\phi_1(a+b) - \phi_1(a)\phi_1(a+b) = -\frac{1}{2}\left[\phi_2(a) + \phi_2(b) + \phi_2(a+b)\right]. \tag{1.3.11}$$

Observe that

$$\phi_1(b)\left(\phi_1(a+b) - \phi_1(a)\right) = \left(b\phi_1(b)\right)\left(\frac{\phi_1(a+b) - \phi_1(a)}{b}\right) \to \frac{1}{2}\phi_1'(a) \text{ as } b \to 0.$$

Now write (1.3.11) as

$$\phi_1(b)\left(\phi_1(a+b) - \phi_1(a)\right) + \phi_1(a)\phi_1(a+b) = \frac{1}{2}\left[\phi_2(a) + \phi_2(b) + \phi_2(a+b)\right],$$

and let $b \to 0$ to get

$$\frac{1}{2}\phi_1'(a) + \phi_1^2(a) = \phi_2(a) + \frac{1}{2}\phi_2(0). \tag{1.3.12}$$

By substituting the series (1.3.8) and (1.3.9) into (1.3.12), we see that this equivalent to (1.3.1). Hence equation (1.3.12) can be rewritten as

$$\phi_1^2(a) = \phi_2(a) + \frac{1}{2}\phi_2(0) - \frac{1}{2}\phi_1'(a). \tag{1.3.13}$$

Letting $q^2 \to q$ in the functions ϕ_1 and ϕ_2 enables us to write some results which are special cases of (1.3.13).

Putting $a = \pi$ into (1.3.13) simply gives a tautology.

Put $a = \frac{2\pi}{3}$ into (1.3.13), we have

$$\begin{aligned} &\left[\frac{1}{6} + \frac{q}{1-q} - \frac{q^2}{1-q^2} + \frac{q^4}{1-q^4} - \frac{q^5}{1-q^5} + \dots\right]^2 \\ = \ &\frac{1}{36} + \frac{1}{3}\left[\frac{q}{1-q} + \frac{2q^2}{1-q^2} + \frac{4q^4}{1-q^4} + \frac{5q^5}{1-q^5} + \dots\right], \end{aligned}$$

where 1, 2, 4, 5,... are the natural numbers without the multiples of 3. This result can be found in ([21], p. 139).

Put $a = \frac{\pi}{2}$ into (1.3.13), we have

$$\begin{aligned} & \left[\frac{1}{4} + \frac{q}{1-q} - \frac{q^3}{1-q^3} + \frac{q^5}{1-q^5} - \frac{q^7}{1-q^7} + ...\right]^2 \\ = \; & \frac{1}{16} + \frac{1}{2}\left[\frac{q}{1-q} + \frac{2q^2}{1-q^2} + \frac{3q^3}{1-q^3} + \frac{5q^5}{1-q^5} + \frac{6q^6}{1-q^6} + ...\right], \end{aligned} \tag{1.3.14}$$

where 1, 2, 3, 5, 6,... are the natural numbers without the multiples of 4. We will use (1.3.14) to obtain a formula for the number of representations of an integer as the sum of four squares in a later section. This result can also be found in ([21], p. 140).

Put $a = \frac{\pi}{3}$ into equation (1.3.13), we have

$$\begin{aligned} & \left[\frac{1}{2} + \frac{q}{1-q} + \frac{q^2}{1-q^2} - \frac{q^4}{1-q^4} - \frac{q^5}{1-q^5} + \frac{q^7}{1-q^7} + ...\right]^2 \\ = \; & \frac{1}{4} + \frac{q}{1-q} + \frac{2q^2}{1-q^2} + \frac{3q^3}{1-q^3} + \frac{4q^4}{1-q^4} + \frac{5q^5}{1-q^5} + \frac{7q^7}{1-q^7} + ..., \end{aligned}$$

where 1, 2, 3, 4, 5, 7,... are the natural numbers without the multiples of 6.

Put $a = \frac{\pi}{4}$ into (1.3.13), we have

$$\left(\sqrt{2}A + B\right)^2 = \sqrt{2}C + D + \frac{1}{2}E + \sqrt{2}F + G,$$

where

$$\begin{aligned} A &= \frac{1}{2}\left[\frac{q}{1-q} + \frac{q^3}{1-q^3} + \frac{q^5}{1-q^5} - \frac{q^7}{1-q^7} - \frac{q^9}{1-q^9} + ...\right], \\ B &= 1 + \frac{q^2}{1-q^2} - \frac{q^6}{1-q^6} + \frac{q^{10}}{1-q^{10}} - ..., \\ C &= \frac{1}{2}\left[\frac{q}{(1-q)^2} - \frac{q^3}{(1-q^3)^2} - \frac{q^5}{(1-q^5)^2} + \frac{q^7}{(1-q^7)^2} + \frac{q^9}{(1-q^9)^2} - ...\right], \\ D &= \frac{-1}{24} - \frac{q^4}{(1-q^4)^2} + \frac{q^8}{(1-q^8)^2} - \frac{q^{12}}{(1-q^{12})^2} + ..., \end{aligned}$$

$$\begin{aligned}
E &= \frac{-1}{24}+\frac{q}{1-q}+\frac{2q^2}{1-q^2}+\frac{3q^3}{1-q^3}+...,\\
F &= \frac{1}{4}\left[\frac{1}{2}-\frac{q}{1-q}+\frac{3q^3}{1-q^3}-\frac{5q^5}{1-q^5}+\frac{7q^7}{1-q^7}-\frac{9q^9}{1-q^9}+...\right],\\
G &= \frac{1}{4}+\frac{2q^4}{1-q^4}-\frac{4q^8}{1-q^8}+\frac{6q^{12}}{1-q^{12}}-....
\end{aligned}$$

Notice that similar results for $\frac{2\pi}{n}$, where $n = 5, 6, 7, 8,...$, can be given for other n in principle, but these are more complicated.

With $\alpha = e^{ia}$, equation (1.3.12) can be rewritten as

$$\alpha\rho_1'(\alpha)-\rho_1^2(\alpha)=2\rho_2(\alpha)+\rho_2(1).$$

1.4 Ramanujan's ${}_1\psi_1$ summation formula

Ramanujan gave another famous formula called the Ramanujan ${}_1\psi_1$ summation formula. G. H. Hardy ([20], p. 222) described it as "a remarkable formula with many parameters". The first published proofs appeared in 1949 and 1950, by W. Hahn [18] and M. Jackson [24] respectively. There are number of proofs of equation (1.4.1). For example Venkatachaliengar ([36], pp. 24-27, pp. 29-30) and R. Askey [4].

Ramanujan's summation formula is a very useful result. In Chapter 4 we will use it to investigate the sums of two and four squares and triangles formulas. From the top of the flow chart, we can see that Ramanujan's summation formula is related directly or indirectly to all the main results presented in this thesis. Ramanujan's summation formula is

Theorem 1.4.1 (Ramanujan's${}_1\psi_1$ summation formula)

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \frac{(ax;q)_\infty \left(\frac{q}{ax};q\right)_\infty (q;q)_\infty \left(\frac{b}{a};q\right)_\infty}{(x;q)_\infty \left(\frac{b}{ax};q\right)_\infty (b;q)_\infty \left(\frac{q}{a};q\right)_\infty}, \tag{1.4.1}$$

where $|q| < 1$ *and* $\left|\frac{b}{a}\right| < |x| < 1.$

Proof To obtain Ramanujan's summation formula, we let

$$f(x) = \frac{(ax;q)_\infty \left(\frac{q}{ax};q\right)_\infty}{(x;q)_\infty \left(\frac{b}{ax};q\right)_\infty}. \tag{1.4.2}$$

Then, by Laurent's theorem, we can expand (1.4.2) as a Laurent series

$$f(x) = \sum_{n=-\infty}^{\infty} C_n x^n, \tag{1.4.3}$$

valid in the annulus $\left|\frac{b}{a}\right| < |x| < 1$ where $f(x)$ is analytic. Next, consider the Laurent expansion of $f(qx)$, which exists for $\left|\frac{b}{aq}\right| < |x| < \frac{1}{|q|}$. So, we assume additionally that $\left|\frac{b}{aq}\right| < 1$. Now, both $f(x)$ and $f(qx)$ are valid in $\left|\frac{b}{aq}\right| < |x| < 1$ and $|q| < 1$. Then construct the functional equation

$$\begin{aligned}
\frac{f(x)}{f(qx)} &= \frac{(ax;q)_\infty \left(\frac{q}{ax};q\right)_\infty (qx;q)_\infty \left(\frac{b}{aqx};q\right)_\infty}{(x;q)_\infty \left(\frac{b}{ax};q\right)_\infty (aqx;q)_\infty \left(\frac{1}{ax};q\right)_\infty} \\
&= \frac{(1-ax)\left(1-\frac{b}{aqx}\right)}{\left(1-\frac{1}{ax}\right)(1-x)} \\
&= \frac{b-aqx}{q(1-x)}.
\end{aligned}$$

Hence

$$q(1-x) f(x) = (b-aqx) f(qx). \tag{1.4.4}$$

Then, substituting $(1.4.3)$ into $(1.4.4)$ gives

$$q\left(1-x\right)\sum_{n=-\infty}^{\infty}C_nx^n=\left(b-aqx\right)\sum_{n=-\infty}^{\infty}C_nq^nx^n. \tag{1.4.5}$$

Equating the coefficients of powers of x^n gives

$$qC_n-qC_{n-1}=bC_nq^n-aC_{n-1}q^n.$$

Simplifying gives the recurrence relation

$$C_n=\frac{\left(1-aq^{n-1}\right)}{\left(1-bq^{n-1}\right)}C_{n-1}, \tag{1.4.6}$$

and iterating , we get

$$\begin{aligned} C_n &= \frac{\left(1-aq^{n-1}\right)\left(1-aq^{n-2}\right)...\left(1-a\right)}{\left(1-bq^{n-1}\right)\left(1-bq^{n-2}\right)...\left(1-b\right)}C_0 \\ &= \frac{\left(a;q\right)_n}{\left(b;q\right)_n}C_0, \end{aligned} \tag{1.4.7}$$

for $n=1$, 2, 3,.... Next, replacing n by $1-n$ in $(1.4.6)$ gives

$$C_{-n}=\frac{\left(1-bq^{-n}\right)}{\left(1-aq^{-n}\right)}C_{-n+1}.$$

Iterate this to give

$$\begin{aligned} C_{-n} &= \frac{\left(1-bq^{-n}\right)}{\left(1-aq^{-n}\right)}\frac{\left(1-bq^{-n+1}\right)...\left(1-bq^{-1}\right)}{\left(1-aq^{-n+1}\right)...\left(1-aq^{-1}\right)}C_0 \\ &= \frac{\left(a;q\right)_{-n}}{\left(b;q\right)_{-n}}C_0, \qquad n=1,2,3,.... \end{aligned} \tag{1.4.8}$$

Combining the results of (1.4.7) and (1.4.8), we have

$$C_n=\frac{\left(a;q\right)_n}{\left(b;q\right)_n}C_0, \qquad n=0,\pm1,\pm2,.... \tag{1.4.9}$$

Now, put $(1.4.9)$ into $(1.4.3)$ to get

$$f\left(x\right)=C_0\sum_{n=-\infty}^{\infty}\frac{\left(a;q\right)_n}{\left(b;q\right)_n}x^n. \tag{1.4.10}$$

Finally, we need to find C_0. Abel's continuity theorem ([4], p. 504) states that:

$$\text{Suppose } f(x)=\sum_{n=0}^{\infty} a_n x^n \text{ and } \lim_{n\to\infty} a_n = a. \text{ Then } \lim_{x\to 1^-}(1-x)\sum_{n=-\infty}^{\infty} a_n x^n = a. \tag{1.4.11}$$

Apply (1.4.11) to (1.4.9) gives

$$\lim_{n\to\infty} C_n = C_0\frac{(a;q)_\infty}{(b;q)_\infty},$$

and thus we have verified that $\lim_{n\to\infty} a_n = a$. By Abel's continuity theorem

$$\lim_{x\to 1^-}(1-x)\,f(x) = C_0\frac{(a;q)_\infty}{(b;q)_\infty}.$$

Then we work out the limit

$$\lim_{x\to 1^-}(1-x)\frac{(ax;q)_\infty\left(\frac{q}{ax};q\right)_\infty}{(1-x)(1-qx)(1-q^2x)\ldots\left(\frac{b}{ax};q\right)_\infty} = C_0\frac{(a;q)_\infty}{(b;q)_\infty}.$$

We have

$$\frac{(a;q)_\infty\left(\frac{q}{a};q\right)_\infty}{(q;q)_\infty\left(\frac{b}{a};q\right)_\infty} = C_0\frac{(a;q)_\infty}{(b;q)_\infty}.$$

Therefore

$$C_0 = \frac{(b;q)_\infty\left(\frac{q}{a};q\right)_\infty}{(q;q)_\infty\left(\frac{b}{a};q\right)_\infty}. \tag{1.4.12}$$

Put the result of (1.4.12) into (1.4.10) to get

$$f(x) = \frac{(b;q)_\infty\left(\frac{q}{a};q\right)_\infty}{(q;q)_\infty\left(\frac{b}{a};q\right)_\infty}\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}x^n. \tag{1.4.13}$$

Put (1.4.13) into (1.4.2) to get

$$\frac{(ax;q)_\infty\left(\frac{q}{ax};q\right)_\infty}{(x;q)_\infty\left(\frac{b}{ax};q\right)_\infty} = \frac{(b;q)_\infty\left(\frac{q}{a};q\right)_\infty}{(q;q)_\infty\left(\frac{b}{a};q\right)_\infty}\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}x^n. \tag{1.4.14}$$

We can rewrite (1.4.14) as

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \frac{(ax;q)_\infty \left(\frac{q}{ax};q\right)_\infty (q;q)_\infty \left(\frac{b}{a};q\right)_\infty}{(x;q)_\infty \left(\frac{b}{ax};q\right)_\infty (b;q)_\infty \left(\frac{q}{a};q\right)_\infty},$$

which completes the proof. ■

By replacing q and x by q^2 and z respectively in (1.4.1), set $\frac{1}{a}$, $\frac{b}{q^2}$, and $\frac{-az}{q}$ by α, β, and z respectively. There is another way to write Ramanujan's summation formula (1.4.1)

$$\begin{aligned} &1 + \sum_{k=1}^{\infty} \frac{(1/\alpha;q^2)_k (-\alpha q)^k}{(\beta q^2;q^2)_k} z^k + \sum_{k=1}^{\infty} \frac{(1/\beta;q^2)_k (-\beta q)^k}{(\alpha q^2;q^2)_k} z^{-k} \\ = &\left\{ \frac{(-qz;q^2)_\infty (-q/z;q^2)_\infty}{(-\alpha qz;q^2)_\infty (-\beta q/z;q^2)_\infty} \right\} \left\{ \frac{(q^2;q^2)_\infty (\alpha\beta q^2;q^2)_\infty}{(\alpha q^2;q^2)_\infty (\beta q^2;q^2)_\infty} \right\}, \end{aligned} \tag{1.4.15}$$

where $|\beta q| < |z| < 1/|\alpha q|$. There is a connection between Ramanujan's summation formula and many different identities such as the following.

Corollary 1.4.2 (Jacobi triple product identity-Ramanujan's form)

$$(-a, -b, ab; ab)_\infty = 1 + \sum_{n=1}^{\infty} (ab)^{n(n-1)/2} (a^n + b^n). \tag{1.4.16}$$

Proof Set $qz = a$, $q/z = b$, (i.e., $q^2 = ab$, $z^2 = a/b$) and let $\alpha = \beta = 0$ in (1.4.15). ■

Thus the Jacobi triple product identity is a special case of Ramanujan's summation formula. There are two other common ways of writing the Jacobi triple product identity as

$$(-qz, -q/z, q^2; q^2)_\infty = \sum_{n=-\infty}^{\infty} q^{n^2} z^n, \tag{1.4.17}$$

and

$$(z, q/z, q; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} z^n. \tag{1.4.18}$$

Note that if we put $a = qz$ and $b = q/z$ in (1.4.16), then we get (1.4.17). Similarly putting $a = -z$ and $b = -q/z$ in (1.4.16) we have (1.4.18).

Corollary 1.4.3 (q-binomial theorem)

$$\frac{(ax;q)_\infty}{(x;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n. \tag{1.4.19}$$

Proof Put $b = q$ in (1.4.1). ■

Corollary 1.4.4 ([3], p. 500)

(i) Euler's formula

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}, \tag{1.4.20}$$

(ii) Jacobi's formula

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}, \tag{1.4.21}$$

(iii) Gauss's formula

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{1.4.22}$$

Equation (1.4.22) is the generating function for triangular numbers.

1.5 The Jordan-Kronecker function

The Jordan-Kronecker function is defined by

$$F(a,t) = \sum_{n=-\infty}^{\infty} \frac{t^n}{1 - aq^{2n}}, \tag{1.5.1}$$

where $|q^2| < |t| < 1$, $a \neq q^{2k}$, and $k = 0, \pm 1, \pm 2, \ldots$

Theorem 1.5.1 (Jordan-Kronecker function)

$$\sum_{n=-\infty}^{\infty} \frac{t^n}{1-aq^{2n}} = \frac{(at;q^2)_\infty \left(\frac{q^2}{at};q^2\right)_\infty (q^2;q^2)_\infty (q^2;q^2)_\infty}{(t;q^2)_\infty \left(\frac{q^2}{t};q^2\right)_\infty (a;q^2)_\infty \left(\frac{q^2}{a};q^2\right)_\infty}. \tag{1.5.2}$$

Proof To obtain (1.5.2), let $x = t$, $q \to q^2$, and $b = aq^2$ in (1.4.1) to give

$$\sum_{n=-\infty}^{\infty} \frac{(a;q^2)_n}{(aq^2;q^2)_n} t^n = \frac{(at;q^2)_\infty \left(\frac{q^2}{at};q^2\right)_\infty (q^2;q^2)_\infty (q^2;q^2)_\infty}{(t;q^2)_\infty \left(\frac{q^2}{t};q^2\right)_\infty (aq^2;q^2)_\infty \left(\frac{q^2}{a};q^2\right)_\infty}. \tag{1.5.3}$$

Now use the following identities

(i)

$$\frac{(a;q^2)_n}{(aq^2;q^2)_n} = \frac{1-a}{1-aq^{2n}},$$

(ii)

$$(1-a)\left(aq^2;q^2\right)_\infty = \left(a;q^2\right)_\infty.$$

We can rewrite (1.5.3) as

$$\sum_{n=-\infty}^{\infty} \frac{t^n}{1-aq^{2n}} = \frac{(at;q^2)_\infty \left(\frac{q^2}{at};q^2\right)_\infty (q^2;q^2)_\infty (q^2;q^2)_\infty}{(t;q^2)_\infty \left(\frac{q^2}{t};q^2\right)_\infty (a;q^2)_\infty \left(\frac{q^2}{a};q^2\right)_\infty},$$

which completes the proof. ■

Below are some elementary properties of function F.

Theorem 1.5.2 ([36], p. 38, [13], p. 66)

$$\begin{aligned} F(a,t) &= F(t,a), \\ F(a,t) &= -F(\frac{1}{a},\frac{1}{t}), \\ F(a,t) &= tF(aq^2,t) = aF(a,tq^2). \end{aligned}$$

Let $t = e^v$ in $(1.5.1)$ and recall that the Bernoulli numbers B_m are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_m t^m}{m!}, \qquad \text{for } |t| < 2\pi.$$

Then, write the Jordan-Kronecker function as

$$\begin{aligned} F(a, e^v) &= \sum_{n=-\infty}^{\infty} \frac{e^{vn}}{1 - aq^{2n}} \\ &= \frac{1}{1-a} + \frac{1}{1-e^v} - 1 + \sum_{n=1}^{\infty} \left(\frac{aq^{2n}e^{nv}}{1 - aq^{2n}} - \frac{a^{-1}q^{2n}e^{-nv}}{1 - a^{-1}q^{2n}} \right) \qquad (1.5.4) \\ &= \frac{1}{1-a} - 1 - \sum_{m=0}^{\infty} \frac{B_m v^{m-1}}{m!} \\ &\quad + \sum_{n=1}^{\infty} \left(\frac{aq^{2n}}{1 - aq^{2n}} \sum_{m=0}^{\infty} \frac{(nv)^m}{m!} - \frac{a^{-1}q^{2n}}{1 - a^{-1}q^{2n}} \sum_{m=0}^{\infty} \frac{(-nv)^m}{m!} \right) \\ &= -\frac{1}{v} + \left(\frac{1+a}{2(1-a)} + \sum_{n=1}^{\infty} \frac{aq^{2n}}{1 - aq^{2n}} - \frac{a^{-1}q^{2n}}{1 - a^{-1}q^{2n}} \right) \\ &\quad + \sum_{m=1}^{\infty} \left(-\frac{B_{m+1}}{m+1} + \sum_{n=1}^{\infty} \frac{n^m aq^{2n}}{1 - aq^{2n}} - \frac{(-1)^m n^m a^{-1} q^{2n}}{1 - a^{-1}q^{2n}} \right) \frac{v^m}{m!} \\ &= -\frac{1}{v} + \sum_{m=0}^{\infty} \rho_{m+1}(a) \frac{v^m}{m!}, \end{aligned}$$

where

$$\rho_1(a) = \frac{1+a}{2(1-a)} + \sum_{n=1}^{\infty} \left(\frac{aq^{2n}}{1 - aq^{2n}} - \frac{a^{-1}q^{2n}}{1 - a^{-1}q^{2n}} \right),$$

and

$$\rho_m(a) = -\frac{B_m}{m} + \sum_{n=1}^{\infty} \left(\frac{n^{m-1}aq^{2n}}{1 - aq^{2n}} + (-1)^m \frac{n^{m-1}a^{-1}q^{2n}}{1 - a^{-1}q^{2n}} \right), m = 2, 3, 4, ... \qquad (1.5.5)$$

The series (1.5.4) converges for $|\mathrm{Re}\, v| < |\mathrm{Im}\, 2\pi\tau|$. Observe that the functions ρ_1 and ρ_2 are exactly the same functions as these already encountered in (1.3.3) and (1.3.4). For $m = 3$, 4, 5,$\cdots$, we take (1.5.5) as the definition of ρ_m. We use these results later in Chapter 3 section 2. Venkatachaliengar derives a fundamental multiplicative identity for the function F which will be shown in the next section.

1.6 The fundamental multiplicative identity

In this section, we look at the fundamental multiplicative identity, then introduce the Weierstrass $\wp$ function and three special cases of the function $F(a,t)$ which we denote by f_1, f_2, and f_3.

Theorem 1.6.1 (Fundamental Multiplicative Identity) ([36], p. 40, [13], p. 66) *Let F be the Jordan-Kronecker function defined by* $(1.5.1)$ *and let ρ_1 be as in equation* $(1.3.4)$. *Then*

$$F(a,t)F(b,t) = t\frac{\partial}{\partial t}F(ab,t) + F(ab,t)(\rho_1(a)+\rho_1(b)). \tag{1.6.1}$$

Note that a simplified version of Venkatachaliengar's proof can be found in [13]. Now using the change of variable $t = e^v$, the fundamental multiplicative identity $(1.6.1)$ becomes

$$F(a,e^v)F(b,e^v) = \frac{\partial}{\partial v}F(ab,e^v) + F(ab,e^v)(\rho_1(a)+\rho_1(b)).$$

Expand both sides of this powers of v using $(1.5.4)$ to get

$$\begin{aligned}
&\left[-\frac{1}{v}+\sum_{m=0}^{\infty}\rho_{m+1}(a)\frac{v^m}{m!}\right]\left[-\frac{1}{v}+\sum_{m=0}^{\infty}\rho_{m+1}(b)\frac{v^m}{m!}\right]\\
=\;& \frac{1}{v^2}+\sum_{m=0}^{\infty}\rho_{m+2}(ab)\frac{v^m}{m!}+\left[-\frac{1}{v}+\sum_{m=0}^{\infty}\rho_{m+1}(ab)\frac{v^m}{m!}\right][\rho_1(a)+\rho_1(b)].
\end{aligned}$$

Equating coefficients of v^m, $m = 0, 1, 2, \ldots$, gives

$$\sum_{k=0}^{m} \binom{m}{k} \rho_{k+1}(a)\rho_{m+1-k}(b) = \frac{\rho_{m+2}(a) + \rho_{m+2}(b)}{m+1} + \rho_{m+2}(ab) + \rho_{m+1}(ab)(\rho_1(a) + \rho_1(b)). \tag{1.6.2}$$

We set $m = 0$, 1, 2, and 3 in (1.6.2) as follows

(i) $m = 0$

$$\rho_1(a)\rho_1(b) = \rho_2(a) + \rho_2(b) + \rho_2(ab) + \rho_1(ab)\left[\rho_1(a) + \rho_1(b)\right]. \tag{1.6.3}$$

(ii) $m = 1$

$$\rho_1(a)\rho_2(b) + \rho_2(a)\rho_1(b) = \frac{1}{2}\left[\rho_3(a) + \rho_3(b)\right] + \rho_3(ab) + \rho_2(ab)\left[\rho_1(a) + \rho_1(b)\right]. \tag{1.6.4}$$

(iii) $m = 2$

$$\begin{aligned} & \rho_1(a)\rho_3(b) + 2\rho_2(a)\rho_2(b) + \rho_3(a)\rho_1(b) \\ = \ & \frac{1}{3}\left[\rho_4(a) + \rho_4(b) + \rho_4(ab)\right] + \rho_3(ab)\left[\rho_1(a) + \rho_1(b)\right]. \end{aligned}$$

(iv) $m = 3$

$$\begin{aligned} & \rho_1(a)\rho_4(b) + 3\rho_2(a)\rho_3(b) + 3\rho_3(a)\rho_2(b) + \rho_4(a)\rho_1(b) \\ = \ & \frac{1}{4}\left[\rho_5(a) + \rho_5(b)\right] + \rho_5(ab) + \rho_4(ab)\left[\rho_1(a) + \rho_1(b)\right]. \end{aligned}$$

Note that formula (1.6.3) is the same as (1.3.5). We will use (1.6.3) to obtain a formula for the number of representations of an integer as the sum of two squares in Chapter 4 and use (1.6.4) to obtain Ramanujan's modulus 5 partition congruence in Chapter 3. Next, ([13], p.

68) we define the Weierstrass $\wp$ function with periods 2π and $2\pi\tau$ by

$$\wp(\theta)=\frac{1}{\theta^2}+\sum_{m,n}{}'\left[\frac{1}{(\theta-2\pi m-2\pi\tau n)^2}-\frac{1}{(2\pi m+2\pi\tau n)^2}\right].$$

It can be shown ([13], p.68) this is equivalent to

$$\wp(\theta)=-\frac{1}{12}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{\sin^2\pi\tau n}+\frac{1}{4}\sum_{n=-\infty}^{\infty}\frac{1}{\sin^2\left(\frac{\theta}{2}+\pi\tau n\right)}.$$

Recall that $q=e^{i\pi\tau}$. Then

$$\begin{aligned}\wp(\theta) &= \frac{-1}{12}+2\sum_{n=1}^{\infty}\frac{1}{(q^n-q^{-n})^2}-\sum_{n=1}^{\infty}\frac{1}{(e^{i\theta/2}q^n-e^{-i\theta/2}q^{-n})^2}\\ &= \frac{-1}{12}+2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}-\sum_{n=1}^{\infty}\frac{e^{i\theta}q^{2n}}{(1-e^{i\theta}q^{2n})^2} \qquad (1.6.5)\\ &= \frac{-1}{12}+2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}\\ &\quad -\frac{e^{i\theta}}{(1-e^{i\theta})^2}-\sum_{n=1}^{\infty}\left[\frac{e^{i\theta}q^{2n}}{(1-e^{i\theta}q^{2n})^2}+\frac{e^{-i\theta}q^{2n}}{(1-e^{-i\theta}q^{2n})^2}\right]\\ &= \frac{-1}{12}+2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}+i\frac{d}{d\theta}\rho_1\left(e^{i\theta}\right). \qquad (1.6.6)\end{aligned}$$

Thus if we let

$$P=1-24\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}, \qquad (1.6.7)$$

then substitute (1.6.7) in (1.6.6), we have

$$\wp(\theta)=i\frac{d}{d\theta}\rho_1\left(e^{i\theta}\right)-\frac{P}{12}. \qquad (1.6.8)$$

We now consider three special cases of the function $F(a,t)$, which we call f_1, f_2, and f_3. Their properties follow from those of $F(a,t)$ and Ramanujan's ${}_1\psi_1$ summation formula.

Definition 1.6.2 **([36], p. 111, [13], p. 69)** *Let*

$$f_1(\theta) = \frac{1}{i}F\left(e^{i\pi},e^{i\theta}\right), \qquad (1.6.9)$$

$$f_2(\theta) = \frac{e^{i\theta/2}}{i}F\left(e^{i\pi\tau}, e^{i\theta}\right), \tag{1.6.10}$$

$$f_3(\theta) = \frac{e^{i\theta/2}}{i}F\left(e^{i\pi+i\pi\tau}, e^{i\theta}\right). \tag{1.6.11}$$

The factors $1/i$ and $e^{i\theta/2}/i$ are included to make f_1, f_2, and f_3 real valued when θ is real. The Fourier expansions follow directly from $(1.5.1)$. Rewrite $(1.6.9)$ as

$$f_1(\theta) = \frac{1}{i}\sum_{n=-\infty}^{\infty}\frac{e^{in\theta}}{1+q^{2n}} \tag{1.6.12}$$

$$= \frac{1}{2}\cot\frac{\theta}{2} - 2\sum_{n=1}^{\infty}\frac{q^{2n}}{1+q^{2n}}\sin n\theta. \tag{1.6.13}$$

Similarly we can rewrite $(1.6.10)$ and $(1.6.11)$ as

$$f_2(\theta) = \frac{e^{i\theta/2}}{i}\sum_{n=-\infty}^{\infty}\frac{e^{in\theta}}{1-q^{2n+1}} \tag{1.6.14}$$

$$= \frac{1}{2}\csc\frac{\theta}{2} + 2\sum_{n=0}^{\infty}\frac{q^{2n+1}}{1-q^{2n+1}}\sin\left(n+\frac{1}{2}\right)\theta, \tag{1.6.15}$$

$$f_3(\theta) = \frac{e^{i\theta/2}}{i}\sum_{n=-\infty}^{\infty}\frac{e^{in\theta}}{1+q^{2n+1}} \tag{1.6.16}$$

$$= \frac{1}{2}\csc\frac{\theta}{2} - 2\sum_{n=0}^{\infty}\frac{q^{2n+1}}{1+q^{2n+1}}\sin\left(n+\frac{1}{2}\right)\theta. \tag{1.6.17}$$

The series (1.6.12), (1.6.14), and (1.6.16) converge for $0 < \operatorname{Im}\theta < 2\operatorname{Im}\tau$, and the series (1.6.13), (1.6.15), and (1.6.17) converge for $-2\operatorname{Im}\tau < \operatorname{Im}\theta < 2\operatorname{Im}\tau$. We use those series to obtain a formula for the number of representations of an integer as the sum of six squares and triangles formulas in Chapter 4. Using the infinite product formulas $(1.5.2)$ we can rewrite $(1.6.9)$, $(1.6.10)$, and $(1.6.11)$ as

$$f_1(\theta) = \frac{1\left(-e^{i\theta}, -q^2e^{-i\theta}, q^2, q^2; q^2\right)_\infty}{i\left(e^{i\theta}, q^2e^{-i\theta}, -1, -q^2; q^2\right)_\infty} \tag{1.6.18}$$

$$= \frac{1}{2}\frac{\left(q^2;q^2\right)_\infty^2}{\left(-q^2;q^2\right)_\infty^2}\cot\frac{\theta}{2}\prod_{n=1}^{\infty}\frac{\left(1+2q^{2n}\cos\theta+q^{4n}\right)}{\left(1-2q^{2n}\cos\theta+q^{4n}\right)}, \tag{1.6.19}$$

$$
\begin{aligned}
f_2(\theta) &= \frac{e^{i\theta/2}\left(qe^{i\theta}, qe^{-i\theta}, q^2, q^2; q^2\right)_\infty}{i\left(e^{i\theta}, q^2e^{-i\theta}, q, q; q^2\right)_\infty} && (1.6.20)\\
&= \frac{1}{2}\frac{\left(q^2;q^2\right)_\infty^2}{\left(q;q^2\right)_\infty^2}\csc\frac{\theta}{2}\prod_{n=1}^{\infty}\frac{\left(1-2q^{2n-1}\cos\theta+q^{4n-2}\right)}{\left(1-2q^{2n}\cos\theta+q^{4n}\right)}, && (1.6.21)\\
f_3(\theta) &= \frac{e^{i\theta/2}\left(-qe^{i\theta}, -qe^{-i\theta}, q^2, q^2; q^2\right)_\infty}{i\left(e^{i\theta}, q^2e^{-i\theta}, -q, -q; q^2\right)_\infty} && (1.6.22)\\
&= \frac{1}{2}\frac{\left(q^2;q^2\right)_\infty^2}{\left(-q;q^2\right)_\infty^2}\csc\frac{\theta}{2}\prod_{n=1}^{\infty}\frac{\left(1+2q^{2n-1}\cos\theta+q^{4n-2}\right)}{\left(1-2q^{2n}\cos\theta+q^{4n}\right)}. && (1.6.23)
\end{aligned}
$$

The functions f_1, f_2, and f_3 are connected with the Weierstrass $\wp$ function but we first define the Weierstrass invariants e_1, e_2, and e_3.

Definition 1.6.3 **([13], p. 71)** *Let*

$$
\begin{aligned}
e_1 &= \wp(\pi), && (1.6.24)\\
e_2 &= \wp(\pi\tau), && (1.6.25)\\
e_3 &= \wp(\pi+\pi\tau). && (1.6.26)
\end{aligned}
$$

Using (1.6.5), we can rewrite (1.6.24), (1.6.25), and (1.6.26) as

$$
\begin{aligned}
e_1 &= \frac{1}{6}+2\sum_{n=1}^{\infty}\frac{q^{2n}}{\left(1-q^{2n}\right)^2}+2\sum_{n=1}^{\infty}\frac{q^{2n}}{\left(1+q^{2n}\right)^2},\\
e_2 &= -\frac{1}{12}+2\sum_{n=1}^{\infty}\frac{q^{2n}}{\left(1-q^{2n}\right)^2}+2\sum_{n=1}^{\infty}\frac{q^{2n-1}}{\left(1-q^{2n-1}\right)^2},\\
e_3 &= -\frac{1}{12}+2\sum_{n=1}^{\infty}\frac{q^{2n}}{\left(1-q^{2n}\right)^2}+2\sum_{n=1}^{\infty}\frac{q^{2n-1}}{\left(1+q^{2n-1}\right)^2}. && (1.6.27)
\end{aligned}
$$

Equations (1.6.27) can be found in ([13], p. 71). Next we consider the relation of each of f_1, f_2, and f_3 to the Weierstrass $\wp$ function. First, let $b \to 1/a$ in the fundamental multiplicative identity (1.6.1) to get

$$
\lim_{b\to 1/a} F(a,t)\,F(b,t) = \lim_{b\to 1/a} t\frac{\partial}{\partial t}F(ab,t) + \lim_{b\to 1/a} F(ab,t)\left(\rho_1(a)+\rho_1(b)\right). \qquad (1.6.28)
$$

The left hand side of (1.6.28) is just $F(a,t)\,F(1/a,t)$. The first limit on the right hand side of (1.6.28) is

$$\begin{aligned}\lim_{b\to 1/a} t\frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty}\frac{t^n}{1-abq^{2n}} &= \lim_{b\to 1/a}\sum_{n=-\infty}^{\infty}{}'\frac{nt^n}{1-abq^{2n}}\\ &= \sum_{n=-\infty}^{\infty}{}'\frac{nt^n}{1-q^{2n}}\\ &= t\frac{d}{dt}\rho_1(t).\end{aligned}$$

Using (1.3.3) it follows that $\rho_1(b)=-\rho_1(1/b)$. Using this and the infinite product formula (1.5.2) for the function F, the remaining limit on the right hand side (1.6.28) becomes

$$\begin{aligned}&\lim_{b\to 1/a} F(ab,t)\left(\rho_1(a)+\rho_1(b)\right)\\ =\;& \lim_{b\to 1/a}(1-ab)\,F(ab,t)\lim_{b\to 1/a}\frac{\rho_1(a)+\rho_1(b)}{1-ab}\\ =\;& \lim_{b\to 1/a}(1-ab)\frac{(abt,q^2/abt,q^2,q^2;q^2)_\infty}{(t,q^2/t,ab,q^2/ab;q^2)_\infty}\lim_{b\to 1/a}\frac{\rho_1(a)-\rho_1(1/b)}{a-1/b}\left(\frac{-1}{b}\right)\\ =\;& (1)\,\rho_1'(a)\,(-a).\end{aligned}$$

Hence ([13], p. 72)

$$F(a,t)\,F(1/a,t)=t\frac{d}{dt}\rho_1(t)-a\frac{d}{da}\rho_1(a). \tag{1.6.29}$$

Then setting $a=e^{i\alpha}$, $t=e^{i\theta}$, and using (1.6.8), this becomes ([13], p. 72)

$$F\left(e^{i\alpha},e^{i\theta}\right)F\left(e^{-i\alpha},e^{i\theta}\right)=\wp(\alpha)-\wp(\theta). \tag{1.6.30}$$

Then letting $\alpha=\pi$, $\alpha=\pi\tau$, and $\alpha=\pi+\pi\tau$ respectively in (1.6.30), and simplifying, leads to ([13], p. 72)

$$f_1^2(\theta) = \wp(\theta)-e_1, \tag{1.6.31}$$

$$f_2^2(\theta) = \wp(\theta)-e_2, \tag{1.6.32}$$

$$f_3^2(\theta) \quad = \quad \wp(\theta) - e_3. \tag{1.6.33}$$

Letting $\theta = \pi\tau$ in $(1.6.31)$, $\theta = \pi + \pi\tau$ in $(1.6.32)$, $\theta = \pi$ in equation $(1.6.33)$, and using the infinite products for f_1, f_2, and f_3 gives ([13], p. 72)

$$e_1 - e_2 \quad = \quad \frac{1}{4}\frac{(-q;q^2)_\infty^4\,(q^2;q^2)_\infty^4}{(q;q^2)_\infty^4\,(-q^2;q^2)_\infty^4}, \tag{1.6.34}$$

$$e_3 - e_2 \quad = \quad 4q\frac{(-q^2;q^2)_\infty^4\,(q^2;q^2)_\infty^4}{(-q;q^2)_\infty^4\,(q;q^2)_\infty^4}, \tag{1.6.35}$$

$$e_1 - e_3 \quad = \quad \frac{1}{4}\frac{(q;q^2)_\infty^4\,(q^2;q^2)_\infty^4}{(-q;q^2)_\infty^4\,(-q;q^2)_\infty^4}. \tag{1.6.36}$$

Note that since $\operatorname{Im}\tau > 0$ this implies that $e_1 \neq e_2$, $e_1 \neq e_3$, and $e_2 \neq e_3$. If we combine the equations $(1.6.31)$, $(1.6.32)$, and $(1.6.33)$ two at a time to eliminate the $\wp(\theta)$ term, we obtain ([13], pp. 72-73)

$$f_2^2(\theta) - f_1^2(\theta) \quad = \quad e_1 - e_2, \tag{1.6.37}$$

$$f_2^2(\theta) - f_3^2(\theta) \quad = \quad e_3 - e_2, \tag{1.6.38}$$

$$f_3^2(\theta) - f_1^2(\theta) \quad = \quad e_1 - e_3. \tag{1.6.39}$$

Now we will look at the derivatives of $f_1(\theta)$, $f_2(\theta)$, and $f_3(\theta)$. By replacing t by $e^{i\theta}$ in the fundamental multiplicative identity $(1.6.1)$ it becomes

$$F(a,e^{i\theta})F(b,e^{i\theta}) = \frac{1}{i}\frac{\partial}{\partial\theta}F(ab,e^{i\theta}) + F(ab,e^{i\theta})(\rho_1(a) + \rho_1(b)). \tag{1.6.40}$$

Now let $a = e^{i\pi}$ and $b = e^{i\pi\tau}$. Since from $(1.3.3)$, we have $\rho_1(e^{i\pi}) = 0$ and $\rho_1(e^{i\pi\tau}) = \dfrac{1}{2}$, $(1.6.40)$ becomes

$$F\left(-1,e^{i\theta}\right)F\left(q,e^{i\theta}\right) = \frac{1}{i}\frac{\partial}{\partial\theta}F\left(-q,e^{i\theta}\right) + \frac{1}{2}F\left(-q,e^{i\theta}\right). \tag{1.6.41}$$

We can rewrite the left hand side of $(1.6.41)$ as

$$\begin{aligned} F\left(-1, e^{i\theta}\right) F\left(q, e^{i\theta}\right) &= i f_1(\theta)\, i e^{-i\theta/2} f_2(\theta) \\ &= -e^{i\theta/2} f_1(\theta) f_2(\theta), \end{aligned} \tag{1.6.42}$$

and the right hand side of $(1.6.41)$ as

$$\begin{aligned} & \frac{1}{i}\frac{\partial}{\partial\theta}\left(i e^{-i\theta/2} f_3(\theta)\right) + \frac{i}{2} e^{-i\theta/2} f_3(\theta) \\ = \;& e^{-i\theta/2} f_3'(\theta) - \frac{i}{2} e^{-i\theta/2} f_3(\theta) + \frac{i}{2} e^{-i\theta/2} f_3(\theta) \\ = \;& e^{-i\theta/2} f_3'(\theta). \end{aligned} \tag{1.6.43}$$

Combining the result of $(1.6.42)$ and $(1.6.43)$ gives ([13], p. 73)

$$f_3'(\theta) = -f_1(\theta) f_2(\theta). \tag{1.6.44}$$

Similarly by letting $a = e^{i\pi\tau}$, $b = e^{i\pi+i\pi\tau}$ and $a = e^{i\pi+i\pi\tau}$, $b = e^{i\pi}$ respectively in $(1.6.40)$ gives

$$f_1'(\theta) = -f_2(\theta) f_3(\theta), \tag{1.6.45}$$

$$f_2'(\theta) = -f_3(\theta) f_1(\theta). \tag{1.6.46}$$

We will use the functions of f_1, f_2, and f_3 to obtain formulas for the number of representations of a positive integer as sum of six squares and triangles in Chapter 4.

1.7 Series expansions

In this section, we give series expansions for $\phi_1(a)$, $\phi_2(a)$, and $\wp(a)$ and introduce the series P, Q, and R. Let us define

$$\begin{aligned} \phi_{r,s} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^r n^s q^{2mn}, && (1.7.1) \\ S_r &= \frac{-B_{r+1}}{2\,(r+1)} + \sum_{n=1}^{\infty} \frac{n^r q^{2n}}{1-q^{2n}} \\ &= \phi_{0,r} - \frac{B_{r+1}}{2(r+1)}. && (1.7.2) \end{aligned}$$

Theorem 1.7.1 **([36], p. 15)**

$$\begin{aligned} \phi_1(a) &= \frac{1}{2a} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} S_{2n-1} a^{2n-1}, && (1.7.3) \\ \phi_2(a) &= -\frac{1}{24} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \phi_{2n,1} a^{2n}, && (1.7.4) \\ \wp(a) &= \frac{1}{a^2} + 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!} S_{2n+1} a^{2n}. && (1.7.5) \end{aligned}$$

From equations (1.7.3), (1.7.4), and (1.7.5), the first few terms in these expansions are

$$\begin{aligned} \phi_1(a) &= \frac{1}{2a} - \frac{Pa}{24} - \frac{Q}{240}\frac{a^3}{3!} - \frac{R}{504}\frac{a^5}{5!} - \dots, \\ \phi_2(a) &= -\frac{P}{24} - \phi_{2,1}\frac{a^2}{2!} + \phi_{4,1}\frac{a^4}{4!} - \dots, \\ \wp(a) &= \frac{1}{a^2} + 2\left(\frac{Q}{240}\frac{a^2}{2!} + \frac{R}{504}\frac{a^4}{4!} + \dots\right), \end{aligned}$$

where

$$P = 1 - 24\sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} = -24S_1 = 1 - 24\phi_{0,1},$$

$$\begin{aligned} Q &= 1+240\sum_{n=1}^{\infty}\frac{n^3q^{2n}}{1-q^{2n}} = 240S_3 = 1+240\phi_{0,3}, \\ R &= 1-504\sum_{n=1}^{\infty}\frac{n^5q^{2n}}{1-q^{2n}} = -504S_5 = 1-504\phi_{0,5}. \end{aligned} \tag{1.7.6}$$

Ramanujan derived the results of equations $(1.7.1)$, $(1.7.2)$, and $(1.7.6)$ in ([21], pp. 137-140).

1.8 Summary

In this chapter, we have presented some of Ramanujan's identities and used some theorems and results due to Venkatachaliengar to prove them. Then we introduced Ramanujan's ${}_1\psi_1$summation formula, which is related to many different identities. The Jacobi triple product identity and the Jordan-Kronecker function are special cases of Ramanujan's summation formula. Then we introduced the fundamental multiplicative identity, the Weierstrass function and three special cases of $F(a,t)$. After that we wrote $\phi_1(a)$, $\phi_2(a)$, and $\wp(a)$ as a series and introduced the series P, Q, and R.

Chapter 2

The transformation of $S_{2n+1}(q)$, $\phi_{r,s}(q)$, $U_{2n}(q)$, and $V_{2n}(q)$

2.1 Introduction

In 1916, Ramanujan published a paper called "On certain arithmetical functions" ([21], pp. 136-143). In this paper, he gave formulae for S_n, $\phi_{r,s}$, and derived what are now called the Ramanujan differential equations. The aim of this chapter is to write the transformation of $S_{2n+1}(q)$, $\phi_{r,s}(q)$, $U_{2n}(q)$, and $V_{2n}(q)$ in terms of $P(p)$, $Q(p)$, and $R(p)$. Here $q = e^{i\pi\tau}$ and $p = e^{-i\pi/\tau}$.

2.2 The Ramanujan differential equations

Ramanujan has used his identity (1.3.1) to prove the following theorem.

Theorem 2.2.1 **([13], p. 13)**

(i) (Eisenstein and Weierstrass) $S_{2r-1}, r \geq 2$, *is a polynomial in* S_3 *and* S_5 *with rational coefficients.*

(ii) (Ramanujan) If $r+s$ *is odd and* $rs \neq 0$, *then* $\phi_{r,s}$ *is a polynomial in* P, Q, *and* R *with rational coefficients.*

In (i)

$$\begin{aligned} S_1 &= \frac{-P}{24}, \\ S_3 &= \frac{Q}{240}, \\ S_5 &= \frac{-R}{504}. \end{aligned} \tag{2.2.1}$$

Ramanujan gave the following recurrence relation for the S_{2n+3} in ([21], p. 140)

$$S_{2n+3} = \frac{12(n+1)(2n+1)}{(n-1)(2n+5)} \sum_{j=1}^{n-1} \binom{2n}{2j} S_{2j+1} S_{2n+1-2j}, \quad n = 2, 3, ..., \tag{2.2.2}$$

so S_{2n+3} is a polynomial in S_3, S_5, S_7,.... Since S_7, S_9,... are polynomials in S_3 and S_5 with rational coefficients, it follows that S_{2n+3} is a polynomial in S_3 and S_5 (or in terms of Q and R) with rational coefficients. The first few values are

$$\begin{aligned} S_7 &= \frac{1}{480} Q^2, \\ S_9 &= -\frac{1}{264} QR, \\ S_{11} &= \frac{7}{1040} Q^3 + \frac{25}{6552} R^2, \\ S_{13} &= -\frac{1}{24} Q^2 R, \\ S_{15} &= \frac{539}{5440} Q^4 + \frac{25}{204} QR^2, \\ S_{17} &= -\frac{203}{152} Q^3 R - \frac{1375}{7182} R^3, \\ S_{19} &= \frac{1617}{400} Q^5 + \frac{2425}{264} Q^2 R^2, \\ S_{21} &= -\frac{19061}{184} Q^4 R - \frac{5125}{138} QR^3, \\ S_{23} &= \frac{788557}{2080} Q^6 + \frac{17500}{13} Q^3 R^2 + \frac{257125}{3276} R^4. \end{aligned} \tag{2.2.3}$$

In (ii) Ramanujan gave the following recurrence relation of $\phi_{2n,1}$ in ([21], p. 141)

$$\phi_{2n,1} = \frac{2n+3}{2(2n+1)} S_{2n+1} - \sum_{j=1}^{n} \binom{2n}{2j-1} S_{2j-1} S_{2n+1-2j}, \qquad \text{where } n = 1, 2, 3, ..., \tag{2.2.4}$$

so $\phi_{1,2r}$ is a polynomial in S_1, S_3, S_5, S_7,.... Since S_7, S_9,... are polynomials in S_3 and S_5 with rational coefficients, $\phi_{1,2r}$ is a polynomial in S_1, S_3, and S_5 (or in terms of P, Q, and R) with rational coefficients.

Next we introduce Ramanujan differential equations.

$$\begin{aligned} q\frac{d}{dq}\phi_{r,s} &= \phi_{r+1,s+1}, &(2.2.5)\\ q\frac{dP}{dq} &= \frac{P^2-Q}{12}, &(2.2.6)\\ q\frac{dQ}{dq} &= \frac{PQ-R}{3}, &(2.2.7)\\ q\frac{dR}{dq} &= \frac{PR-Q^2}{2}. &(2.2.8) \end{aligned}$$

Equations (2.2.6), (2.2.7), and (2.2.8) are called the Ramanujan differential equations and proofs can be found in ([21], p. 142). By induction on the minimum of r and s using (2.2.5) and the Ramanujan Differential Equations, provided $r + s$ is odd, $\phi_{r,s}$ is a polynomial in P, Q, and R with rational coefficients. The first few are as follows:

$$\begin{aligned} \phi_{2,1} &= (Q - P^2)/288, \\ \phi_{4,1} &= (PQ - R)/720, \\ \phi_{6,1} &= (Q^2 - PR)/1008, \\ \phi_{8,1} &= (PQ^2 - QR)/720, \\ \phi_{10,1} &= (3Q^3 + 2R^2 - 5PQR)/1584, \end{aligned}$$

$$\begin{aligned}
\phi_{12,1} &= (441PQ^3 - 691Q^2R + 250PR^2)/65520, \\
\phi_{14,1} &= \left(3Q^4 + 4QR^2 - 7PQ^2R\right)/144, \\
\phi_{3,2} &= (3PQ - P^3 - 2R)/1728, \\
\phi_{5,2} &= (P^2Q + Q^2 - 2PR)/1728, \\
\phi_{7,2} &= (2PQ^2 - P^2R - QR)/1728, \\
\phi_{9,2} &= (9P^2Q^2 + 5Q^3 - 18PQR + 4R^2)/8640, \\
\phi_{11,2} &= (6PQ^3 + 4PR^2 - 5P^2QR - 5Q^2R)/1728, \\
\phi_{13,2} &= (-1382PQ^2R + 406QR^2 + 285Q^4 + 441P^2Q^3 \\
&\quad +250P^2R^2)/60480, \\
\phi_{4,3} &= (6P^2Q + 3Q^2 - 8PR - P^4)/6912, \\
\phi_{6,3} &= (P^3Q + 3PQ^2 - 3P^2R - QR)/3456, \\
\phi_{8,3} &= (6P^2Q^2 + Q^3 + R^2 - 2P^3R - 6PQR)/5184, \\
\phi_{10,3} &= (3P^3Q^2 + 5PQ^3 + 4PR^2 - 9P^2QR - 3Q^2R)/3456, \\
\phi_{12,3} &= (9P^2Q^3 + 2Q^4 - 3PQ^2R + 3QR^2 + 6P^2R^2 - 5P^3QR)/1728, \\
\phi_{5,4} &= (10P^3Q + 15PQ^2 - 20P^2R - 4QR - P^5)/20736, \\
\phi_{7,4} &= (7P^4Q + 42P^2Q^2 - 28P^3R + 3Q^3 - 28PQR + 4R^2)/41472, \\
\phi_{9,4} &= (2PR^2 + 2PQ^3 + 4P^3Q^2 - P^4R - 6P^2QR - Q^2R)/3456, \\
\phi_{11,4} &= (-132P^3QR + 88P^2R^2 + 110P^2Q^3 - 132PQ^2R + 20QR^2 \\
&\quad +13Q^4 + 33P^4Q^2)/41472, \\
\phi_{6,5} &= (75P^4Q + 225P^2Q^2 + 9Q^3 + 16R^2
\end{aligned}$$

$$\begin{aligned}
&\quad -5P^6 - 200P^3R - 120PQR)/248832, \\
\phi_{8,5} &= (7P^5Q + 70P^3Q^2 + 15PQ^3 + 20PR^2 \\
&\quad -35P^4R - 70P^2QR - 7Q^2R)/62208, \\
\phi_{10,5} &= (25P^2R^2 - 25PQ^2R - 50P^3QR + 25P^4Q^2 + 25P^2Q^3 \\
&\quad +2Q^4 - 5P^5R + 3QR^2)/20736, \\
\phi_{7,6} &= (105P^5Q + 525P^3Q^2 + 63PQ^3 + 112PR^2 \\
&\quad -5P^7 - 350P^4R - 420P^2QR - 30Q^2R)/497664, \\
\phi_{9,6} &= (7P^6Q + 60P^2R^2 + 45P^2Q^3 + 4QR^2 - 42P^5R + 105P^4Q^2 \\
&\quad -105P^3QR - 42PQ^2R + 3Q^4)/82944, \\
\phi_{8,7} &= 980P^6Q + 3136P^2R^2 + 1764P^2Q^3 + 128QR^2 - 3920P^5R \\
&\quad +7350P^4Q^2 - 7840P^3QR - 1680PQ^2R + 117Q^4 \\
&\quad -35P^8)/5971968. \qquad (2.2.9)
\end{aligned}$$

Ramanujan ([21], p. 143) has given a general form for (2.2.9) as follows

$$\phi_{r,s} = \sum K_{l,m,n} P^l Q^m R^n, \qquad (2.2.10)$$

where $l - 1$ does not exceed the smaller of r and s and

$$2l + 4m + 6n = r + s + 1,$$

and K is a rational number.

2.3 The transcendentals U_n and V_n

According to Venkatachaliengar ([36], p. 31), Ramanujan's notebooks contain two other transcendentals (2.3.1) and (2.3.2) which are also polynomials in P, Q, and R. These are

$$\begin{aligned} U_n &= \frac{\sum_{k=0}^{\infty}(-1)^k(2k+1)^{n+1}q^{k(k+1)/2}}{\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)/2}}, & (2.3.1) \\ V_n &= \frac{\sum_{k=-\infty}^{\infty}(-1)^k(6k+1)^n q^{k(3k+1)/2}}{\sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k+1)/2}}, & (2.3.2) \end{aligned}$$

where the denominators are the Jacobi and Euler products in (1.4.20) and (1.4.21) respectively.

Theorem 2.3.1 **([36], p. 31)**

$$\begin{aligned} U_0 = V_0 &= 1, \\ U_{n+2} &= PU_n + 8q\frac{dU_n}{dq}, & (2.3.3) \\ V_{n+2} &= PV_n + 24q\frac{dV_n}{dq}. & (2.3.4) \end{aligned}$$

Corollary 2.3.2 **([36], p. 31)** *U_{2n} and V_{2n} are polynomials in P, Q, and R. Furthermore the coefficients in the expression for U_{2n} are rational numbers whose denominators are powers of 3 only, and the coefficients in the expression for V_{2n} are integers.*

The first few polynomials are:

$$\begin{aligned} U_2 &= P, \\ U_4 &= (5P^2 - 2Q)/3, \\ U_6 &= (35P^3 - 42PQ + 16R)/9, \end{aligned}$$

$$\begin{aligned}
U_8 &= (35P^4 - 84P^2Q + 64PR - 12Q^2)/3, \\
U_{10} &= (385P^5 - 1540P^3Q + 1760P^2R - 660PQ^2 + 64QR)/9, \\
U_{12} &= (5005P^6 - 30030P^4Q + 45760P^3R - 25740P^2Q^2 \\
&\quad +4992PQR + 552Q^3 - 512R^2)/27, \\
U_{14} &= (-210210P^5Q - 3648Q^2R - 300300P^3Q^2 + 400400P^4R \\
&\quad -17920PR^2 + 19320PQ^3 + 87360P^2QR + 25025P^7)/27,
\end{aligned} \tag{2.3.5}$$

$$\begin{aligned}
V_2 &= P, \\
V_4 &= 3P^2 - 2Q, \\
V_6 &= 15P^3 - 30PQ + 16R, \\
V_8 &= 105P^4 - 420P^2Q + 448PR - 132Q^2, \\
V_{10} &= 945P^5 - 6300P^3Q + 10080P^2R - 5940PQ^2 + 1216QR, \\
V_{12} &= 10395P^6 - 103950P^4Q + 221760P^3R - 196020P^2Q^2, \\
&\quad +80256PQR - 2712Q^3 - 9728R^2, \\
V_{14} &= -1891890P^5Q + 138048Q^2R - 5945940P^3Q^2 + 5045040P^4R \\
&\quad -885248PR^2 - 246792PQ^3 + 3651648P^2QR + 135135P^7.
\end{aligned} \tag{2.3.6}$$

By using the results of Ramanujan differential equations and applying the induction in equation (2.3.3), we can rewrite the solutions of (2.3.5) as

$$U_{2n} = \sum G_{l,m,w} P^l Q^m R^w, \qquad \text{where } n = 1, 2, 3, ..., \tag{2.3.7}$$

where $2l+4m+6w=2n$ and G is a rational number. Similarly, the solutions of (2.3.6) can be represented as

$$V_{2n}=\sum H_{l,m,w}P^{l}Q^{m}R^{w}, \qquad \text{where } n=1,2,3,..., \tag{2.3.8}$$

where $2l+4m+6w=2n$ and H is a rational number.

2.4 The transformation of $S_{2n+1}(q)$, $\phi_{r,s}(q)$, $U_{2n}(q)$, and $V_{2n}(q)$

This section is the key section for this Chapter, in which we rewrite $S_{2n+1}(q)$, $\phi_{r,s}(q)$, $U_{2n}(q)$, and $V_{2n}(q)$ in terms of $P(p)$, $Q\left(p\right)$, and $R\left(p\right)$ where $q=e^{i\pi\tau}$ and $p=e^{-i\pi/\tau}$. According to Venkatachaliengar, the transformation formulae for various functions $S_{2n+1}(q)$, $\phi_{r,s}(q)$, $U_{2n}(q)$, and $V_{2n}(q)$ are able to be calculated. We have calculated these and written them in a simple form.

In ([36], p.32), the transformation of $P(p)$, $Q(p)$, and $R(p)$ are represented as

$$\begin{aligned} P(p) &= \tau^{2}P(q)+\frac{6\tau}{\pi i}, && (2.4.1)\\ Q(p) &= \tau^{4}Q(q), && (2.4.2)\\ R(p) &= \tau^{6}R(q). && (2.4.3) \end{aligned}$$

We can rewrite equations (2.4.1), (2.4.2), and (2.4.3) as

$$\begin{aligned} P\left(q\right) &= \frac{1}{\tau^{2}}P\left(p\right)+\frac{6i}{\pi\tau},\\ Q\left(q\right) &= \frac{1}{\tau^{4}}Q\left(p\right),\\ R\left(q\right) &= \frac{1}{\tau^{6}}R\left(p\right). \end{aligned} \tag{2.4.4}$$

Then putting these results in (2.2.3) we obtain the following results.

$$\begin{aligned}
S_1(q) &= -\frac{1}{24\tau^2}P(p)-\frac{i}{4\pi\tau},\\
S_3(q) &= \frac{1}{240\tau^4}Q(p),\\
S_5(q) &= -\frac{1}{504\tau^6}R(p),\\
S_7(q) &= \frac{1}{480\tau^8}Q^2(p),\\
S_9(q) &= -\frac{1}{264\tau^{10}}Q(p)R(p),\\
S_{11}(q) &= \frac{7}{1040\tau^{12}}Q^3(p)+\frac{25}{6552\tau^{12}}R^2(p),\\
S_{13}(q) &= -\frac{1}{24\tau^{14}}Q^2(p)R(p),\\
S_{15}(q) &= \frac{539}{5440\tau^{16}}Q^4(p)+\frac{25}{204\tau^{16}}Q(p)R^2(p),\\
S_{17}(q) &= -\frac{203}{152\tau^{18}}Q^3(p)R(p)-\frac{1375}{7182\tau^{18}}R^3(p),\\
S_{19}(q) &= \frac{1617}{400\tau^{20}}Q^5(p)+\frac{2425}{264\tau^{20}}Q^2(p)R^2(p).
\end{aligned} \tag{2.4.5}$$

By using the results of (2.4.4) and apply the induction in (2.2.1) and (2.2.2), the solution of (2.4.5) can be represented in the form of

$$S_{2n+1}(q)=\begin{cases} -\dfrac{1}{24\tau^2}P(p)-\dfrac{i}{4\pi\tau}, & \text{if } n=0,\\[2ex] \dfrac{1}{\tau^{2n+2}}\sum C_{m,w}Q^m(p)R^w(p) & \text{if } n=1,2,3,\dots, \end{cases} \tag{2.4.6}$$

where $4m+6w=2n+2$ and C is a rational number.

Next by putting the results of (2.4.4) in (2.2.9) we obtain the following results.

$$\begin{aligned}
\phi_{2,1}(q) &= \frac{Q(p)-P^2(p)}{288\tau^4}+\frac{36\tau-12iP(p)\pi}{\tau^3\pi^2},\\
\phi_{4,1}(q) &= \frac{P(p)Q(p)-R(p)}{720\tau^6}+\frac{iQ(p)}{120\tau^5\pi},\\
\phi_{6,1}(q) &= \frac{Q^2(p)-P(p)R(p)}{1008\tau^8}-\frac{6iR(p)}{\tau^7\pi},
\end{aligned}$$

$$\begin{aligned}
\phi_{3,2}(q) &= \frac{3P(p)\,Q(p) - P^3(p) - 2R(p)}{1728\tau^6} + \frac{iQ(p)\,\pi^2 - iP^2(p)\,\pi^2 + 6P(p)\,\tau\pi + 12i\tau^2}{96\tau^5\pi^3}, \\
\phi_{5,2}(q) &= \frac{P^2(p)\,Q(p) + Q^2(p) - 2P(p)\,R(p)}{1728\tau^8} + \frac{iP(p)Q(p)\,\pi - iR(p)\,\pi - 3Q(p)\,\tau}{144\tau^7\pi^2}, \\
\phi_{4,3}(q) &= \frac{6P^2(p)\,Q(p) + 3Q^2(p) - 8P(p)\,R(p) - P^4(p)}{6912\tau^8} \\
&\quad + \frac{3iP(p)\,Q(p)\,\pi^3 - 9Q(p)\,\tau\pi^2 - 2iR(p)\,\pi^3 - iP^3(p)\,\pi^3}{288\tau^7\pi^4} \\
&\quad + \frac{9P^2(p)\,\tau\pi^2 + 36iP(p)\,\tau^2\pi - 54\tau^3}{288\tau^7\pi^4}.
\end{aligned} \tag{2.4.7}$$

By substituting equations (2.2.10) into (2.4.4) we get

$$\phi_{r,s}(q) = \sum K_{l,m,n}\left(\frac{1}{\tau^2}P(p) + \frac{6i}{\pi\tau}\right)^l \left(\frac{1}{\tau^4}Q(p)\right)^m \left(\frac{1}{\tau^6}R(p)\right)^n, \tag{2.4.8}$$

where $2l + 4m + 6n = r + s + 1$. Observe that

$$\begin{aligned}
(a+b)^l &= \sum_{j=0}^{l} \begin{pmatrix} l \\ j \end{pmatrix} a^j b^{l-j} \\
&= \sum_{j+k=l} \begin{pmatrix} l \\ j \end{pmatrix} a^j b^k \\
&= a^l + \sum_{j=0}^{l-1} \begin{pmatrix} l \\ j \end{pmatrix} a^j b^{l-j}.
\end{aligned} \tag{2.4.9}$$

So we can rewrite (2.4.8) as

$$\begin{aligned}
\phi_{r,s}(q) &= \frac{\sum K_{l,m,n} P^l(p)\,Q^m(p)\,R^n(p)}{\tau^{2l+4m+6n}} \\
&\quad + \sum_{\substack{2l+4m+6n=r+s+1 \\ j+k=l}} K_{l,m,n} \begin{pmatrix} l \\ j \end{pmatrix} \left(\frac{1}{\tau^2}P(p)\right)^j \left(\frac{6i}{\pi\tau}\right)^k \left(\frac{1}{\tau^4}Q(p)\right)^m \left(\frac{1}{\tau^6}R(p)\right)^n,
\end{aligned} \tag{2.4.10}$$

where $2l + 4m + 6n = r + s + 1$, $j + k = l$, and K is a rational number. Equation (2.4.10) can be written in a simple form of

$$\phi_{r,s}(q) = \frac{\phi_{r,s}(p)}{\tau^{r+s+1}} + \sum \frac{E_{j,k,l,m,n} P^j(p)\,Q^m(p)\,R^w(p)}{\tau^{r+s+1-k}\pi^k}, \tag{2.4.11}$$

where $2l+4m+6n=r+s+1$, $j+k=l$, and E is a rational number.

Next by putting the results of (2.4.4) in (2.3.6) we obtain the following results of which the first few polynomials are

$$\begin{aligned}
U_2(q) &= \frac{P(p)}{\tau^2}+\frac{6i}{\tau\pi},\\
U_4(q) &= \frac{5P^2(p)-2Q(p)}{3\tau^4}+\frac{20iP(p)\pi-60\tau}{\tau^3\pi^2},\\
U_6(q) &= \frac{35P^3(p)-42P(p)Q(p)+16R(p)}{9\tau^6}\\
&\quad+\frac{70iP^2(p)\pi^2-420P(p)\tau\pi-840i\tau^2-28Q(p)\pi^2}{\tau^5\pi^3},\\
U_8(q) &= \frac{35P^4(p)-84P^2(p)Q(p)+64P(p)R(p)-12Q^2(p)}{3}\\
&\quad+\frac{2680iP^3(p)\pi^3-2520P^2(p)\tau\pi^2-10080iP(p)\tau^2\pi}{3\tau^7\pi^4}\\
&\quad+\frac{15120\tau^3-336iP(p)Q(p)\pi^3}{3\tau^7\pi^4},
\end{aligned}\tag{2.4.12}$$

$$\begin{aligned}
V_2(q) &= \frac{P(p)}{\tau^2}+\frac{6i}{\tau\pi},\\
V_4(q) &= \frac{3P^2(p)-2Q(p)}{\tau^4}+\frac{36iP(p)\pi-108\tau}{\tau^3\pi^2},\\
V_6(q) &= \frac{15P^3(p)-30P(p)Q(p)+16R(p)}{\tau^6}\\
&\quad+\frac{270iP(p)\pi^2-1620P(p)\tau\pi-3240i\tau^2-180iQ(p)\pi^2}{\tau^5\pi^3},\\
V_8(q) &= \frac{105P^4(p)-420P^2(p)Q(p)+448P(p)R(p)-132Q^2(p)}{\tau^8}\\
&\quad+\frac{2520iP^3(p)\pi^3-22680P^2(p)\tau\pi^2-90720iP\tau^2\pi+136080\tau^3}{\tau^7\pi^4}\\
&\quad+\frac{15120Q(p)\tau\pi^2-5040iP(p)Q(p)\pi^3+2688iR(p)\pi^3}{\tau^7\pi^4}.
\end{aligned}\tag{2.4.13}$$

By substituting equations (2.4.4) into (2.3.3) we get

$$U_{2n}(q)=\sum G_{l,m,w}\left(\frac{1}{\tau^2}P(p)+\frac{6i}{\pi\tau}\right)^l\left(\frac{1}{\tau^4}Q(p)\right)^m\left(\frac{1}{\tau^6}R(p)\right)^w, \tag{2.4.14}$$

where $2l + 4m + 6w = 2n$ and G is a rational number. Using the results of (2.4.9), we can rewrite (2.4.14) becomes

$$\begin{aligned} U_{2n}(q) &= \frac{\sum G_{l,m,w} P^{l}(p) Q^{m}(p) R^{w}(p)}{\tau^{2l+4m+6n}} \\ &+ \sum_{\substack{2l+4m+6w=2n \\ j+k=l}} G_{l,m,w} \begin{pmatrix} l \\ j \end{pmatrix} \left(\frac{1}{\tau^{2}} P(p)\right)^{j} \left(\frac{6i}{\pi\tau}\right)^{k} \left(\frac{1}{\tau^{4}} Q(p)\right)^{m} \left(\frac{1}{\tau^{6}} R(p)\right)^{n}, \end{aligned} \tag{2.4.15}$$

where $2l + 4m + 6w = 2n$, $j + k = l$, and G is a rational number. Equation (2.4.15) can be written in a simple form of

$$U_{2n}(q) = \frac{U_{2n}(p)}{\tau^{2n}} + \sum \frac{B_{j,k,l,m,w} P^{l}(p) Q^{m}(p) R^{w}(p)}{\tau^{2n-k} \pi^{k}}, \tag{2.4.16}$$

where $2l + 4m + 6w = 2n$, $j + k = l$, and B is a rational number. Similarly by substituting equations (2.4.4) into (2.3.4) to obtain

$$V_{2n}(q) = \frac{V_{2n}(p)}{\tau^{2n}} + \sum \frac{A_{j,k,l,m,w} P^{l}(p) Q^{m}(p) R^{w}(p)}{\tau^{2n-k} \pi^{k}}, \tag{2.4.17}$$

where $2l + 4m + 6w = 2n$, $j + k = l$, and A is a rational number.

2.5 Summary

In this chapter, we have written the transformation of $S_{2n+1}(q)$, $\phi_{r,s}(q)$, $U_{2n}(q)$, and $V_{2n}(q)$ in terms of $P(p)$, $Q(p)$, and $R(p)$ and got simple formulas for them. Next Chapter, we will introduce another important Ramanujan result called Ramanujan's congruence for partitions.

Chapter 3
Ramanujan's congruence for partitions

3.1 Introduction

A partition of a number n is a representation of n as the sum of any number of positive integral parts. Thus the integer 5 has 7 partitions:

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1.$$

We denote by $p\,(n)$ the number of partitions of n; thus $p\,(5) = 7$. Ramanujan was the first mathematician to discover the properties of $p\,(n)$ by studying the table of values of $p\,(n)$ constructed by MacMahon from $n = 1$ to 200. Congruences for the partition function have formed an important part of number theory.

According to Askey ([4], p. 569) Ramanujan discovered, and gave a simple proof of, the fact that

$$p\,(5n + 4) \equiv 0 \pmod 5, \qquad n = 0, 1, 2, \dots .$$

He also found expressions for the generating function of $p\,(5n + 4)$ as a product. Ramanujan ([21], pp. 210-213) sketched a proof of the result

$$\sum_{n=0}^{\infty} p\,(5n + 4)\, q^n = 5 \prod_{n=1}^{\infty} \frac{\left(1 - q^{5n}\right)^5}{\left(1 - q^n\right)^6} \qquad \text{for } |q| < 1, \tag{3.1.1}$$

and promised to give details, but he died a year later. However, Askey mentioned that ([4], p. 569) Ramanujan gave enough details in an unpublished manuscript for others to complete the proof.

Theorem 3.1.1 **([19], p. 286)** *The generating function for the partition is*

$$\frac{1}{(q;q)_\infty} = \sum_{n=1}^{\infty} p(n)\, q^n. \tag{3.1.2}$$

Theorem 3.1.2 **([19], p. 286)** *The recurrence relation for the partition is*

$$p(n) = \sum_{k=1}^{\frac{3k^2-k}{2} \le n} (-1)^{k+1} p\left(n - \frac{3k^2-k}{2}\right) + \sum_{k=1}^{\frac{3k^2+k}{2} \le n} (-1)^{k+1} p\left(n - \frac{3k^2+k}{2}\right). \tag{3.1.3}$$

Next we will give a proof for Ramanujan's modulus 5 partition congruence.

3.2 Ramanujan's modulus 5 partition congruence

Theorem 3.2.1

$$\sum_{n=0}^{\infty}\left[\frac{q^{5n+1}}{(1-q^{5n+1})^2} - \frac{q^{5n+2}}{(1-q^{5n+2})^2} - \frac{q^{5n+3}}{(1-q^{5n+3})^2} + \frac{q^{5n+4}}{(1-q^{5n+4})^2}\right] = q\prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)}, \tag{3.2.1}$$

for $|q| < 1$.

Proof Using the case $m = 1$ case of $(1.6.2)$ and replacing q^2 by q in the functions ρ_1, ρ_2, and ρ_3, we get

$$\rho_1(a)\,\rho_2(b) + \rho_2(a)\,\rho_1(b) = \frac{1}{2}\left(\rho_3(a) + \rho_3(b)\right) + \rho_3(ab) + \rho_2(ab)\left[\rho_1(a) + \rho_1(b)\right]. \tag{3.2.2}$$

Apply $a\dfrac{\partial}{\partial a} - b\dfrac{\partial}{\partial b}$ in (3.2.2) to get

$$a\rho_1'(a)\,\rho_2(b) + a\rho_2'(a)\,\rho_1(b) - b\rho_1(a)\,\rho_2'(b) - b\rho_2(a)\,\rho_1'(b)$$

$$= \frac{1}{2}\left(a\rho_3'(a) - b\rho_3'(b)\right) + \rho_2(ab)\left[a\rho_1'(a) - b\rho_1'(b)\right]. \tag{3.2.3}$$

We can write expansions of these functions using (1.5.5) and replacing q^2 by q in the functions ρ_1, ρ_2, and ρ_3 to get

$$\begin{aligned}
\rho_1(a) &= \frac{1+a}{2(1-a)} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(a^m q^{mn} - a^{-m}q^{mn}\right), \\
\rho_2(a) &= \frac{-1}{12} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(na^m q^{mn} + na^{-m}q^{mn}\right), \\
\rho_3(a) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(n^2a^m q^{mn} - n^2a^{-m}q^{mn}\right).
\end{aligned} \tag{3.2.4}$$

We now apply $a\dfrac{\partial}{\partial a}$ to (3.2.4) to obtain

$$\begin{aligned}
a\rho_1'(a) &= \frac{a}{(1-a)^2} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(ma^m q^{mn} + ma^{-m}q^{mn}\right), \\
a\rho_2'(a) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(mna^m q^{mn} - mna^{-m}q^{mn}\right), \\
a\rho_3'(a) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(mn^2a^m q^{mn} + mn^2a^{-m}q^{mn}\right) \\
&= q\frac{d}{dq}\left(\rho_2(a)\right).
\end{aligned} \tag{3.2.5}$$

Observe that, letting $\xi = e^{\frac{2\pi i}{5}}$ and substituting into (3.2.4) and (3.2.5), we have

$$\begin{aligned}
\rho_1\left(\xi^3\right) &= -\rho_1\left(\xi^2\right), \\
\xi^3\rho_1'\left(\xi^3\right) &= \xi^2\rho_1'\left(\xi^2\right), \\
\rho_2\left(\xi^3\right) &= \rho_2\left(\xi^2\right), \\
\rho_2\left(\xi^4\right) &= \rho_2(\xi), \\
\xi^3\rho_2'\left(\xi^3\right) &= -\xi^2\rho_3'\left(\xi^2\right), \\
\xi^3\rho_3'\left(\xi^3\right) &= \xi^2\rho_3'\left(\xi^2\right).
\end{aligned} \tag{3.2.6}$$

Next we put $(a,b)=\left(\xi,\xi^2\right)$ into (3.2.3), to get

$$\begin{aligned}
&\xi\rho_1'(\xi)\,\rho_2\left(\xi^2\right)+\xi\rho_2'(\xi)\,\rho_1\left(\xi^2\right)-\xi^2\rho_1(\xi)\,\rho_2'\left(\xi^2\right)-\xi^2\rho_2(\xi)\,\rho_1'\left(\xi^2\right)\\
=\;&\frac{1}{2}\left(\xi\rho_3'(\xi)-\xi^2\rho_3'\left(\xi^2\right)\right)+\rho_2\left(\xi^3\right)\left[\xi\rho_1'(\xi)-\xi^2\rho_1'\left(\xi^2\right)\right],
\end{aligned}$$

and use the result from (3.2.6) to get

$$\begin{aligned}
&\xi\rho_1'(\xi)\,\rho_2\left(\xi^2\right)+\xi\rho_2'(\xi)\,\rho_1\left(\xi^2\right)-\xi^2\rho_1(\xi)\,\rho_2'\left(\xi^2\right)-\xi^2\rho_2(\xi)\,\rho_1'\left(\xi^2\right)\\
=\;&\frac{1}{2}\left(\xi\rho_3'(\xi)-\xi^2\rho_3'\left(\xi^2\right)\right)+\rho_2\left(\xi^2\right)\left[\xi\rho_1'(\xi)-\xi^2\rho_1'\left(\xi^2\right)\right]. \qquad (3.2.7)
\end{aligned}$$

Similarly, we put $(a,b)=\left(\xi,\xi^3\right)$ in (3.2.3) and use (3.2.6) to get

$$\begin{aligned}
&\xi\rho_1'(\xi)\,\rho_2\left(\xi^3\right)-\xi\rho_2'(\xi)\,\rho_1\left(\xi^2\right)+\xi^2\rho_1(\xi)\,\rho_2'\left(\xi^2\right)-\xi^2\rho_2(\xi)\,\rho_1'\left(\xi^2\right)\\
=\;&\frac{1}{2}\left(\xi\rho_3'(\xi)-\xi^2\rho_3'\left(\xi^2\right)\right)+\rho_2(\xi)\left[\xi\rho_1'(\xi)-\xi^2\rho_1'\left(\xi^2\right)\right]. \qquad (3.2.8)
\end{aligned}$$

Then add (3.2.7) and (3.2.8) to get

$$\begin{aligned}
&2\xi\rho_1'(\xi)\,\rho_2\left(\xi^2\right)-2\xi^2\rho_2(\xi)\,\rho_1'\left(\xi^2\right)\\
=\;&\xi\rho_3'(\xi)-\xi^2\rho_3'\left(\xi^2\right)+\left[\rho_2(\xi)+\rho_2\left(\xi^2\right)\right]\left[\xi\rho_1'(\xi)-\xi^2\rho_1'\left(\xi^2\right)\right]. \qquad (3.2.9)
\end{aligned}$$

Observe that

$$\begin{aligned}
\rho_2(\xi) &= \frac{-1}{12}+2\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\cos\left(\frac{2\pi n}{5}\right)\\
&= \frac{-1}{12}+2\left(\begin{array}{c}
\frac{q}{(1-q)^2}\left(\frac{-1+\sqrt{5}}{4}\right)+\frac{q^2}{(1-q^2)^2}\left(\frac{-1-\sqrt{5}}{4}\right)\\
+\frac{q^3}{(1-q^3)^2}\left(\frac{-1-\sqrt{5}}{4}\right)+\frac{q^4}{(1-q^4)^2}\left(\frac{-1+\sqrt{5}}{4}\right)\\
+\frac{q^5}{(1-q^5)^2}+\ldots
\end{array}\right)\\
&= \frac{-1}{12}-\frac{1}{2}\left(\frac{q}{(1-q)^2}+\frac{q^2}{(1-q^2)^2}+\frac{q^3}{(1-q^3)^2}+\frac{q^4}{(1-q^4)^2}+\frac{q^6}{(1-q^6)^2}+\ldots\right)
\end{aligned}$$

$$\begin{aligned}
&+2\left(\frac{q^5}{\left(1-q^5\right)^2}+\frac{q^{10}}{\left(1-q^{10}\right)^2}+...\right)\\
&+\frac{\sqrt{5}}{2}\left(\frac{q^2}{\left(1-q^2\right)^2}-\frac{q^4}{\left(1-q^4\right)^2}-\frac{q^3}{\left(1-q^3\right)^2}+\frac{q^4}{\left(1-q^4\right)^2}+\frac{q^6}{\left(1-q^6\right)^2}-...\right)\\
=\;&\left(\frac{-1}{12}-\frac{1}{2}\sum_{5\nmid n}\frac{q^n}{\left(1-q^n\right)^2}+2\sum_{n=1}^{\infty}\frac{q^{5n}}{\left(1-q^{5n}\right)^2}\right)\\
&+\frac{2\sqrt{5}}{4}\sum_{n=0}^{\infty}\left(\frac{q^{5n+1}}{\left(1-q^{5n+1}\right)^2}-\frac{q^{5n+2}}{\left(1-q^{5n+2}\right)^2}-\frac{q^{5n+3}}{\left(1-q^{5n+3}\right)^2}+\frac{q^{5n+4}}{\left(1-q^{5n+4}\right)^2}\right)\\
=\;&2A+2B\sqrt{5},
\end{aligned}$$

$$\begin{aligned}
\text{where } A &= \frac{-1}{12}-\frac{1}{2}\sum_{5\nmid n}\frac{q^n}{\left(1-q^n\right)^2}+2\sum_{n=1}^{\infty}\frac{q^{5n}}{\left(1-q^{5n}\right)^2},\\
B &= \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{q^{5n+1}}{\left(1-q^{5n+1}\right)^2}-\frac{q^{5n+2}}{\left(1-q^{5n+2}\right)^2}-\frac{q^{5n+3}}{\left(1-q^{5n+3}\right)^2}+\frac{q^{5n+4}}{\left(1-q^{5n+4}\right)^2}\right).
\end{aligned}$$

Similarly, we can rewrite $\rho_2\left(\xi^2\right)$, $\xi\rho_1'\left(\xi\right)$, and $\xi^2\rho_1'\left(\xi^2\right)$ as

$$\begin{aligned}
\rho_2\left(\xi^2\right) &= 2A-2B\sqrt{5},\\
\xi\rho_1'\left(\xi\right) &= 2C+2D\sqrt{5},\\
\xi^2\rho_1'\left(\xi^2\right) &= 2C-2D\sqrt{5},
\end{aligned}$$

$$\begin{aligned}
\text{where } C &= \frac{-1}{4}-\frac{1}{4}\sum_{5\nmid n}\frac{nq^n}{1-q^n}+\sum_{n=1}^{\infty}\frac{5nq^{5n}}{1-q^{5n}},\\
D &= \frac{-1}{20}-\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{\left(5n+1\right)q^{5n+1}}{1-q^{5n+1}}-\frac{\left(5n+2\right)q^{5n+2}}{1-q^{5n+2}}\right.\\
&\qquad\left.-\frac{\left(5n+3\right)q^{5n+3}}{1-q^{5n+3}}+\frac{\left(5n+4\right)q^{5n+4}}{1-q^{5n+4}}\right).
\end{aligned}$$

Furthermore, we can rewrite $\xi\rho_3'\left(\xi\right)$ and $\xi^2\rho_3'\left(\xi^2\right)$ as

$$\begin{aligned}
\xi\rho_3'\left(\xi\right) &= q\frac{d}{dq}\left(\rho_2\left(\xi\right)\right)=2q\frac{d}{dq}\left(A+B\sqrt{5}\right),\\
\xi^2\rho_3'\left(\xi^2\right) &= q\frac{d}{dq}\left(\rho_2\left(\xi^2\right)\right)=2q\frac{d}{dq}\left(A-B\sqrt{5}\right).
\end{aligned}$$

Then, rewriting (3.2.9) in terms of A, B, C, and D gives

$$\begin{aligned} & 2\left(2C+2D\sqrt{5}\right)\left(2A-2B\sqrt{5}\right)-2\left(2A+2B\sqrt{5}\right)\left(2C-2D\sqrt{5}\right) \\ = \; & 2q\frac{d}{dq}\left(A+B\sqrt{5}\right)-2q\frac{d}{dq}\left(A-B\sqrt{5}\right) \\ & +\left[2A+2B\sqrt{5}+2A-2B\sqrt{5}\right]\left[2C+2D\sqrt{5}-2C+2D\sqrt{5}\right], \end{aligned}$$

which can be simplified to give

$$q\frac{dB}{dq}=-4BC.$$

Therefore

$$\begin{aligned} \frac{-4C}{q} &= \frac{1}{B}\frac{dB}{dq}, \\ \frac{1}{q}+\frac{1}{q}\sum_{5\nmid n}\frac{nq^n}{1-q^n}-\frac{4}{q}\sum_{n=1}^{\infty}\frac{5nq^{5n}}{1-q^{5n}} &= \frac{d\left(\ln B\right)}{dq}. \end{aligned} \tag{3.2.10}$$

Integrating both sides with respect to q in (3.2.10) gives

$$\ln q-\sum_{5\nmid n}\ln\left(1-q^n\right)+4\sum_{n=1}^{\infty}\ln\left(1-q^{5n}\right)=\ln B+\ln k \qquad \text{where } k \text{ is a constant,}$$

or

$$\ln\left(\frac{q\prod_{n=1}^{\infty}\left(1-q^{5n}\right)^5}{\prod_{n=1}^{\infty}\left(1-q^n\right)}\right)=\ln\left(kB\right).$$

Take the exponential of both sides and rewrite B as a series, which gives

$$\frac{q\left(q^5;q^5\right)_{\infty}^5}{\left(q;q\right)_{\infty}}=\frac{k}{4}\sum_{n=0}^{\infty}\left[\frac{q^{5n+1}}{\left(1-q^{5n+1}\right)^2}-\frac{q^{5n+2}}{\left(1-q^{5n+2}\right)^2}-\frac{q^{5n+3}}{\left(1-q^{5n+3}\right)^2}+\frac{q^{5n+4}}{\left(1-q^{5n+4}\right)^2}\right]. \tag{3.2.11}$$

Now equating the coefficient of q on each sides in (3.2.11), gives

$$k=4.$$

Put the value $k = 4$ in (3.2.11) gives

$$\frac{q\left(q^5;q^5\right)_\infty^5}{(q;q)_\infty} = \sum_{n=0}^{\infty}\left[\frac{q^{5n+1}}{\left(1-q^{5n+1}\right)^2} - \frac{q^{5n+2}}{\left(1-q^{5n+2}\right)^2} - \frac{q^{5n+3}}{\left(1-q^{5n+3}\right)^2} + \frac{q^{5n+4}}{\left(1-q^{5n+4}\right)^2}\right].$$

By using the notation in section (1.2), the above identity can be represented as

$$\sum_{n=0}^{\infty}\left[\frac{q^{5n+1}}{\left(1-q^{5n+1}\right)^2} - \frac{q^{5n+2}}{\left(1-q^{5n+2}\right)^2} - \frac{q^{5n+3}}{\left(1-q^{5n+3}\right)^2} + \frac{q^{5n+4}}{\left(1-q^{5n+4}\right)^2}\right] = q\prod_{n=1}^{\infty}\frac{\left(1-q^{5n}\right)^5}{\left(1-q^{n}\right)},$$

which completes the proof. ■

The above proof is the same as Venkatachaliengar's ([36], pp.49-52), except we use ρ_1 and ρ_2 instead of $\phi_1$and ϕ_2. The same technique can also be used to give the number of representations of an integer as a sum of two squares (in the next Chapter).

From identity (3.2.1) Ramanujan deduced another of his famous identities (3.1.1):

Theorem 3.2.2

$$\sum_{n=0}^{\infty} p\left(5n+4\right)q^n = 5\prod_{n=1}^{\infty}\frac{\left(1-q^{5n}\right)^5}{\left(1-q^{n}\right)^6} \qquad \textit{for } |q|<1, \tag{3.2.12}$$

where $p(n)$ is the number of partitions of n.

The following proof was given by H. H. Chan [10].

Proof First, rewrite the left hand side of (3.2.1) as

$$\begin{aligned} & \sum_{n=0}^{\infty}\left[\frac{q^{5n+1}}{\left(1-q^{5n+1}\right)^2} - \frac{q^{5n+2}}{\left(1-q^{5n+2}\right)^2} - \frac{q^{5n+3}}{\left(1-q^{5n+3}\right)^2} + \frac{q^{5n+4}}{\left(1-q^{5n+4}\right)^2}\right] \\ = & \sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{\left(1-q^n\right)^2} \\ = & \sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\sum_{k=1}^{\infty}kq^{nk}, \end{aligned} \tag{3.2.13}$$

where $\left(\frac{n}{5}\right)$ is the Legendre symbol defined by the following

$$\left(\frac{n}{5}\right)=\begin{cases} 0 & \text{if } 5 \mid n, \\ 1 & \text{if } n=5k\pm 1, \\ -1 & \text{if } n=5k\pm 2. \end{cases}$$

By (3.1.2), we rewrite the right hand side of (3.2.1) as

$$\begin{aligned} q\prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)} &= \frac{q\left(q^5;q^5\right)_\infty^5}{(q;q)_\infty} \\ &= q\left(q^5;q^5\right)_\infty^5\sum_{n=0}^{\infty}p(n)\,q^n. \end{aligned} \tag{3.2.14}$$

By combining the results of (3.2.13) and (3.2.14), the identity (3.2.1) can be represented as

$$q\left(q^5;q^5\right)_\infty^5\sum_{n=0}^{\infty}p(n)\,q^n=\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\sum_{k=1}^{\infty}kq^{nk}. \tag{3.2.15}$$

Next by equating the coefficients of powers of q which are multiples of 5 in (3.2.15) we have

$$\begin{aligned} \left(q^5;q^5\right)_\infty^5\sum_{n=0}^{\infty}p(5n+4)\,q^{5n+5} &= 5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\sum_{k=1}^{\infty}kq^{5nk} \\ &= 5q^5\frac{\left(q^{25};q^{25}\right)_\infty^5}{\left(q^5;q^5\right)_\infty}. \end{aligned} \tag{3.2.16}$$

Simplifying (3.2.16) gives

$$\sum_{n=0}^{\infty}p(5n+4)\,q^{5n}=5\frac{\left(q^{25};q^{25}\right)_\infty^5}{\left(q^5;q^5\right)_\infty^6}. \tag{3.2.17}$$

Then replacing q^5 with q in (3.2.17), gives Ramanujan's identity

$$\begin{aligned} \sum_{n=0}^{\infty}p(5n+4)\,q^n &= 5\frac{\left(q^5;q^5\right)_\infty^5}{(q;q)_\infty^6} \\ &= 5\prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)^6}. \end{aligned}$$

This completes the proof of (3.2.12). ■

Hardy ([4], p. 572) remarks that he would agree with Major MacMahon in selecting (3.1.1) if he had to select one beautiful formula from all of Ramanujan's work. Next, we give a proof of another of Ramanujan's formulas.

Theorem 3.2.3

$$\prod_{n=1}^{\infty}\frac{(1-q^n)^5}{1-q^{5n}}=1-5\left(\frac{q}{1-q}-\frac{2q^2}{1-q^2}-\frac{3q^3}{1-q^3}+\frac{4q^4}{1-q^4}+\frac{6q^6}{1-q^6}-\ldots\right). \tag{3.2.18}$$

Proof Use the result of fundamental multiplicative identity (1.6.30). Let $\alpha=\frac{2\pi}{5}$, $\theta=\frac{4\pi}{5}$, $w=e^{\frac{2\pi i}{5}}$, so that $w^5=1$, $\frac{1}{w}=w^4$, $\frac{1}{w^2}=w^3$, then substitute into (1.6.29) and replacing q^2 by q, the left hand side of (1.6.29) becomes

$$\begin{aligned}
&F\left(w,w^2\right)F\left(w^4,w^2\right)\\
=&\ \frac{\left(w^3,\frac{q}{w^3},w,\frac{q}{w},q,q,q,q;q\right)_\infty}{\left(w,\frac{1}{w},\frac{q}{w},wq,w^2,w^2,qw^3,qw^3;q\right)_\infty}\\
=&\ \frac{(w^3,qw^2,w,qw^4,q,q,q,q;q)_\infty}{(w,w^4,qw^4,wq,w^2,w^2,qw^3,qw^3;q)_\infty}\\
=&\ \frac{(1-w^3)(1-w)}{(1-w)(1-w^4)(1-w^2)(1-w^2)}\frac{(qw,qw^2,qw^3,qw^4,q,q,q,q;q)_\infty}{(qw,qw,qw^2,qw^2,qw^3,qw^3,qw^4,qw^4;q)_\infty}\\
=&\ \frac{\sqrt{5}}{5}\frac{(q;q)_\infty^5}{(q,qw,qw^2,qw^3,qw^4;q)_\infty}\\
=&\ \frac{\sqrt{5}}{5}\frac{(q;q)_\infty^5}{\prod_{n=1}^{\infty}(1-q^n)(1-wq^n)(1-w^2q^n)(1-w^3q^n)(1-w^4q^n)}\\
=&\ \frac{\sqrt{5}}{5}\frac{(q;q)_\infty^5}{\prod_{n=1}^{\infty}(1-q^{5n})}\\
=&\ \frac{\sqrt{5}}{5}\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty},
\end{aligned} \tag{3.2.19}$$

since $(1-x)(1-wx)(1-w^2x)(1-w^3x)(1-w^4x) = 1-x^5$. Then rewrite the right hand side of (1.6.30) and replacing q^2 by q as

$$\begin{aligned}
\wp(\alpha)-\wp(\theta) &= \left[\frac{-e^{i\alpha}}{(1-e^{i\alpha})^2} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(ne^{i\alpha n}q^{mn} + ne^{-i\alpha n}q^{mn}\right)\right] \\
&\quad - \left[\frac{-e^{i\theta}}{(1-e^{i\theta})^2} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(ne^{i\theta n}q^{mn} + ne^{-i\theta n}q^{mn}\right)\right] \\
&= \left[\frac{1}{4\sin^2\frac{\alpha}{2}} - \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\left(e^{i\alpha n}+e^{-i\alpha n}\right)\right] \\
&\quad - \left[\frac{1}{4\sin^2\frac{\theta}{2}} - \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\left(e^{i\theta n}+e^{-i\theta n}\right)\right]. \qquad (3.2.20)
\end{aligned}$$

Let $\theta = \dfrac{4\pi}{5}$, $\alpha = \dfrac{2\pi}{5}$, and substitute into (3.2.20)

$$\begin{aligned}
\wp(\alpha)-\wp(\theta) &= \left[\frac{1}{4\sin^2\frac{\pi}{5}} - 2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\cos\frac{2\pi n}{5}\right] \\
&\quad - \left[\frac{1}{4\sin^2\frac{2\pi}{5}} - 2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\cos\frac{4\pi n}{5}\right] \\
&= \left[\frac{2}{5-\sqrt{5}} - 2\left(\frac{q}{1-q}\cos\frac{2\pi}{5} + \frac{2q^2}{1-q^2}\cos\frac{4\pi}{5} + \frac{3q^3}{1-q^3}\cos\frac{6\pi}{5}\right.\right. \\
&\quad \left.\left. + \frac{4q^4}{1-q^4}\cos\frac{8\pi}{5} + \frac{5q^5}{1-q^5}\cos\frac{10\pi}{5} + \frac{6q^6}{1-q^6}\cos\frac{12\pi}{5} + \ldots\right)\right] - \\
&\quad \left[\frac{2}{5+\sqrt{5}} - 2\left(\frac{q}{1-q}\cos\frac{4\pi}{5} + \frac{2q^2}{1-q^2}\cos\frac{8\pi}{5} + \frac{3q^3}{1-q^3}\cos\frac{12\pi}{5}\right.\right. \\
&\quad \left.\left. + \frac{4q^4}{1-q^4}\cos\frac{16\pi}{5} + \frac{5q^5}{1-q^5}\cos\frac{20\pi}{5} + \frac{6q^6}{1-q^6}\cos\frac{24\pi}{5} + \ldots\right)\right] \\
&= \left[\frac{10+2\sqrt{5}}{20} - 2\left(\frac{q}{1-q}\left(\frac{\sqrt{5}-1}{4}\right) + \frac{2q^2}{1-q^2}\left(\frac{-\sqrt{5}-1}{4}\right)\right.\right. \\
&\quad + \frac{3q^3}{1-q^3}\left(\frac{-\sqrt{5}-1}{4}\right) + \frac{4q^4}{1-q^4}\left(\frac{\sqrt{5}-1}{4}\right) \\
&\quad \left.\left. + \frac{5q^5}{1-q^5} + \frac{6q^6}{1-q^6}\left(\frac{\sqrt{5}-1}{4}\right) + \ldots\right)\right] \\
&\quad - \left[\frac{10-2\sqrt{5}}{20} - 2\left(\frac{q}{1-q}\left(\frac{-\sqrt{5}-1}{4}\right) + \frac{2q^2}{1-q^2}\left(\frac{\sqrt{5}-1}{4}\right)\right.\right.
\end{aligned}$$

$$\begin{aligned}
& +\frac{3q^3}{1-q^3}\left(\frac{\sqrt{5}-1}{4}\right)+\frac{4q^4}{1-q^4}\left(\frac{-\sqrt{5}-1}{4}\right) \\
& +\frac{5q^5}{1-q^5}+\frac{6q^6}{1-q^6}\left(\frac{-\sqrt{5}-1}{4}\right)+...\Bigg)\Bigg] \\
= \;& \frac{\sqrt{5}}{5}-\frac{\sqrt{5}q}{1-q}+\frac{2\sqrt{5}q^2}{1-q^2}+\frac{3\sqrt{5}q^3}{1-q^3}-\frac{4\sqrt{5}q^4}{1-q^4}-\frac{6\sqrt{5}q^6}{1-q^6}+... \\
= \;& \frac{\sqrt{5}}{5}\left[1-5\left(\frac{q}{1-q}-\frac{2q^2}{1-q^2}-\frac{3q^3}{1-q^3}+\frac{4q^4}{1-q^4}+\frac{6q^6}{1-q^6}-...\right)\right].
\end{aligned} \tag{3.2.21}$$

Combining $(3.2.19)$ and $(3.2.21)$ we have

$$\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty}=1-5\left(\frac{q}{1-q}-\frac{2q^2}{1-q^2}-\frac{3q^3}{1-q^3}+\frac{4q^4}{1-q^4}+\frac{6q^6}{1-q^6}-...\right),$$

which completes the proof. ■

The above proof is the same as W. N. Bailey [6], except we use the fundamental multiplicative identity instead of the ${}_6\psi_6$ summation formula. Another way to represent the above formula is given below.

If

$$f(x)=x^{\frac{1}{5}}\frac{(1-x)(1-x^4)(1-x^6)(1-x^9)...}{(1-x^2)(1-x^3)(1-x^7)(1-x^8)...},$$

then

$$\frac{1}{f}\frac{df}{dx}=\frac{1}{5x}\prod_{n=1}^{\infty}\frac{(1-x^n)^5}{1-x^{5n}}.$$

A very complicated proof of this result was given by H. B. C. Darling [15] and Bailey [5] and this follows immediately from Theorem 3.2.3.

3.3 Summary

Ramanujan observed that the numbers of the partitions of numbers $5m+4$, $7m+5$, and $11m+6$ are divisible by 5, 7, and 11 respectively, that is

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, && (3.3.1) \\ p(7n+5) &\equiv 0 \pmod{7}, && (3.3.2) \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

$$\sum_{n=0}^{\infty} p(5n+4)\, q^n = 5 \prod_{n=1}^{\infty} \frac{\left(1-q^{5n}\right)^5}{\left(1-q^n\right)^6} \qquad \text{for } |q|<1, \tag{3.3.3}$$

$$\sum_{n=0}^{\infty} p(7n+5)\, q^n = 7q \prod_{n=1}^{\infty} \frac{\left(1-q^{7n}\right)^3}{\left(1-q^n\right)^4} + 49q^2 \prod_{n=1}^{\infty} \frac{\left(1-q^{7n}\right)^7}{\left(1-q^n\right)^8} \qquad \text{for } |q|<1, \tag{3.3.4}$$

$$\sum_{n=0}^{\infty} \left(\frac{n}{5}\right) \frac{q^n}{\left(1-q^n\right)^2} = q \prod_{n=1}^{\infty} \frac{\left(1-q^{5n}\right)^5}{1-q^n} \qquad \text{for } |q|<1, \tag{3.3.5}$$

$$\sum_{n=0}^{\infty} \left(\frac{n}{7}\right) q^n \frac{1+q^n}{\left(1-q^n\right)^3} = q \prod_{n=1}^{\infty} \left(1-q^n\right)^3 \left(1-q^{7n}\right)^3 + 8q^2 \prod_{n=1}^{\infty} \frac{\left(1-q^{7n}\right)^7}{1-q^n} \qquad \text{for } |q|<1. \tag{3.3.6}$$

We proved Ramanujan's modulus 5 partition congruence (3.2.1) by using the case $m=1$ case of (1.6.2) and letting $q^2 \to q$ in function ρ_1, ρ_2, and ρ_3. Ramanujan found a simple proof of (3.3.1) and (3.3.2) which used the properties of the elliptic functions, Euler's formula (1.4.20) and Jacobi's formula (1.4.21). Hardy ([21], p. 232) has given a new proof on the first two congruences and also proved the third congruence. Berndt ([21], p. 372) mentioned that Ramanujan stated without proof (3.3.4), most proofs of (3.3.3) and (3.3.4) are based, respectively, on Ramanujan's identities (3.3.5) and (3.3.6). Chan [10] mentioned that the first proof of (3.2.1) was given by Bailey [5] and the first proof of (3.3.6) was given by N. J. Fine [17]. Berndt ([21], p. 373) also mentioned that many people have given proofs of either (3.3.3) or (3.3.5), including H. B.C. Darling [15], L. J. Modell [30], H. Rademacher and H. S. Zuckerman ([31], [33]), S. Chowla [12], D. Kruswijk [28], Chan [10], and M. D. Hirschhorn ([23], [22]). Proofs of either equations (3.3.4) or (3.3.6) were given by Mordell [30], Rademacher and Zuckerman [31], Dobbie [16], Fine [17], O. Kolberg [27], Raghavan [34], and Chan [10].

We tried to prove (3.3.6) by considering the case $m = 2$ in equation (1.6.2) and putting $\xi = e^{\frac{2\pi i}{7}}$ instead of $e^{\frac{2\pi i}{5}}$, but we failed because we cannot write $e^{\frac{2\pi i}{7}}$ in rational form.

Chapter 4
The representations of sums of squares and triangles

4.1 Introduction

In this Chapter I prove formulae for the number of representations of integers as sums of two, four, six, and eight squares (and triangles) identities and give some relationships between squares and triangles. Let $r_k(n)$ be the number of ways of expressing n as a sum of k squares, and $t_k(n)$ be the number of ways of expressing n as a sum of k triangles. We define $r_k(0) = t_k(0) = 1$. Thus

$$\begin{aligned} \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^k &= \sum_{n=0}^{\infty} r_k(n)q^n, \\ \left(\sum_{n=0}^{\infty} q^{n(n+1)/2}\right)^k &= \sum_{n=0}^{\infty} t_k(n)q^n. \end{aligned}$$

4.2 Sums of Squares

The k squares problem is to count the number $r_k(n)$ of integral solutions $(x_1, x_2, \ldots, x_k)$ of the equation

$$x_1^2 + x_2^2 + \ldots + x_k^2 = n.$$

We need to be careful of the sign and order of the $x_1, x_2, \ldots, x_k$.

For example we can write the integer 25 as

$$\left.\begin{array}{l} 0^2+5^2 \\ 5^2+0^2 \\ 0^2+(-5)^2 \\ (-5)^2+0^2 \\ 3^2+4^2 \\ 4^2+3^2 \\ (-3)^2+4^2 \\ 4^2+(-3)^2 \\ 3^2+(-4)^2 \\ (-4)^2+3^2 \\ (-3)^2+(-4)^2 \\ (-4)^2+(-3)^2 \end{array}\right\},$$

so there are 12 ways to write 25 as the sum of two squares and therefore $r_2(25) = 12$.

In the next section we use Ramanujan's summation formula to obtain a formula for the number of representations of an integer as the sum of two and four squares For the sum of six squares we will use the three special cases of the fundamental multiplicative identity. We will also use the fundamental multiplicative identity to prove a formula for the number of representations of an integer as the sum of eight squares.

4.2.1 Sum of two squares

Theorem 4.2.1 *If* $|q| < 1$, *then*

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2 = 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{2n-1}}{1-q^{2n-1}}. \tag{4.2.1}$$

Proof Just as the $m = 1$ case of (1.6.2) can be used to obtain Ramanujan's modulus 5 partition congruence, the same procedure when applied to the $m = 0$ case of (1.6.2) leads to the number of representations of an integer as a sum of two squares. First considering the case $m = 0$ of $(1.6.2)$ and replacing q^2 with q in the functions ρ_1 and ρ_2, we obtain

$$\rho_1(a)\rho_1(b) = \rho_2(a) + \rho_2(b) + \rho_2(ab) + \rho_1(ab)\left(\rho_1(a) + \rho_1(b)\right). \tag{4.2.2}$$

Then applying $a\dfrac{\partial}{\partial a} - b\dfrac{\partial}{\partial b}$ in (4.2.2) we get

$$\begin{aligned} & a\rho_1'(a)\,\rho_1(b) - b\,\rho_1(a)\,\rho_1'(b) \\ = \; & a\rho_2'(a) - b\,\rho_2'(b) + \rho_1(ab)\left[a\rho_1'(a) - b\rho_1'(b)\right]. \end{aligned}$$

Put $a = e^{\frac{\pi i}{2}} = i$, $b = e^{\pi i} = -1$, and $ab = -i$ into the above equation and simplify using $\rho_1(-1) = 0$, $\rho_2'(-1) = 0$ to obtain

$$2\rho_1(i)\,\rho_1'(-1) = i\rho_2'(i) - \rho_1(i)\,i\rho_1'(i). \tag{4.2.3}$$

Let

$$\begin{aligned} \rho_1(i) &= A, \\ \rho_1'(-1) &= B, \\ i\rho_1'(i) &= C, \\ i\rho_2'(i) &= q\frac{d}{dq}\rho_1(i) = q\frac{dA}{dq}. \end{aligned} \tag{4.2.4}$$

Substitute (4.2.4) into (4.2.3) to get

$$2AB = q\frac{dA}{dq} - AC.$$

Then rearrange the above equation to obtain

$$\frac{1}{A}\frac{dA}{dq} = \frac{1}{q}\left(2B + C\right). \tag{4.2.5}$$

Integrate both sides with respect to q in $(4.2.5)$

$$\int \frac{d\ln A}{dq}dq = \int \frac{1}{q}\left(2B + C\right)\,dq,$$

implies to

$$\ln A + \ln k = \int \frac{1}{q}\left(2B + C\right)\, dq, \tag{4.2.6}$$

where k is a constant and

$$\begin{aligned} A &= \frac{i}{2} + 2i\left[\frac{q}{1-q} - \frac{q^3}{1-q^3} + \frac{q^5}{1-q^5} - \frac{q^7}{1-q^7} + ...\right], \\ B &= \frac{1}{4} + \frac{2q}{1-q} - \frac{4q^2}{1-q^2} + \frac{6q^3}{1-q^3} - \frac{8q^4}{1-q^4} + ..., \\ C &= -\frac{1}{2} - \frac{4q^2}{1-q^2} + \frac{8q^4}{1-q^4} - \frac{12q^6}{1-q^6} + ..., \end{aligned} \tag{4.2.7}$$

Put the results of (4.2.7) in $\frac{1}{q}\left(2B + C\right)$ we have

$$\begin{aligned} \frac{1}{q}\left(2B + C\right) &= \frac{1}{q}\left(\sum_{n=1}^{\infty} \frac{4\left(2n-1\right) q^{2n-1}}{1-q^{2n-1}} - \sum_{n=1}^{\infty} \frac{6\left(4n-2\right) q^{4n-2}}{1-q^{4n-2}} - \sum_{n=1}^{\infty} \frac{8nq^{4n}}{1-q^{4n}}\right) \\ &= \sum_{n=1}^{\infty} \frac{4\left(2n-1\right) q^{2n-2}}{1-q^{2n-1}} - \sum_{n=1}^{\infty} \frac{6\left(4n-2\right) q^{4n-3}}{1-q^{4n-2}} - \sum_{n=1}^{\infty} \frac{8nq^{4n-1}}{1-q^{4n}}. \end{aligned} \tag{4.2.8}$$

Integrating both sides with respect to q in (4.2.8) we get

$$\begin{aligned} \int \frac{1}{q}\left(2B + C\right) dq &= -\ln \prod_{n=1}^{\infty} \left(1 - q^{2n-1}\right)^4 + \ln \prod_{n=1}^{\infty} \left(1 - q^{4n-2}\right)^6 + \ln \prod_{n=1}^{\infty} \left(1 - q^{4n}\right)^2 \\ &= \ln \prod_{n=1}^{\infty} \frac{\left(1 - q^{4n-2}\right)^6 \left(1 - q^{4n}\right)^2}{\left(1 - q^{2n-1}\right)^4} \\ &= \ln \left[\frac{\left(1-q^2\right)^6 \left(1-q^4\right)^2 \left(1-q^6\right)^6 \left(1-q^8\right)^2 ...}{\left(1-q\right)^4 \left(1-q^3\right)^4 \left(1-q^5\right)^4 \left(1-q^7\right)^4 ...}\right]. \end{aligned} \tag{4.2.9}$$

Put the results of (4.2.7) in $\ln A + \ln k$ we have

$$\ln kA = \ln \left[\frac{\left(1-q^2\right)^6 \left(1-q^4\right)^2 \left(1-q^6\right)^6 \left(1-q^8\right)^2 ...}{\left(1-q\right)^4 \left(1-q^3\right)^4 \left(1-q^5\right)^4 \left(1-q^7\right)^4 ...}\right]. \tag{4.2.10}$$

Then, combining the results of (4.2.9) and (4.2.10) into (4.2.6) to get

$$\begin{aligned} & k\left[\frac{i}{2} + 2i\left[\frac{q}{1-q} - \frac{q^3}{1-q^3} + \frac{q^5}{1-q^5} - \frac{q^7}{1-q^7} + ...\right]\right] \\ = & \frac{\left(1-q^2\right)^6 \left(1-q^4\right)^2 \left(1-q^6\right)^6 \left(1-q^8\right)^2 ...}{\left(1-q\right)^4 \left(1-q^3\right)^4 \left(1-q^5\right)^4 \left(1-q^7\right)^4 ...}. \end{aligned} \tag{4.2.11}$$

Putting $q = 0$ in (4.2.11) gives

$$k = \frac{2}{i},$$

and thus

$$\begin{aligned} & 1 + 4\left[\frac{q}{1-q} - \frac{q^3}{1-q^3} + \frac{q^5}{1-q^5} - \frac{q^7}{1-q^7} + \ldots\right] \\ = \ & \frac{(1-q^2)^6(1-q^4)^2(1-q^6)^6(1-q^8)^2\ldots}{(1-q)^4(1-q^3)^4(1-q^5)^4(1-q^7)^4\ldots}. \end{aligned} \tag{4.2.12}$$

Rewrite the right hand side of (4.2.12) and using the notation in section (1.2) we have

$$\frac{(q^2;q^4)^6_\infty (q^4;q^4)^2_\infty}{(q;q^2)^4_\infty} = \frac{(q^2;q^2)^2_\infty (q^2;q^4)^4_\infty}{(q;q^2)^4_\infty}. \tag{4.2.13}$$

Using the fact that $(q^2;q^4)_\infty = (q;q^2)_\infty(-q;q^2)_\infty$, we can rewrite (4.2.13) as

$$\begin{aligned} \frac{(q^2;q^4)^6_\infty (q^4;q^4)^2_\infty}{(q;q^2)^4_\infty} &= \frac{(q^2;q^2)^2_\infty (q;q^2)^4_\infty (-q;q^2)^4_\infty}{(q;q^2)^4_\infty} \\ &= (-q;q^2)^4_\infty (q^2;q^2)^2_\infty \\ &= (-q,-q,q^2;q^2)^2_\infty. \end{aligned} \tag{4.2.14}$$

Put $z = 1$ in (1.4.17) to get

$$\sum_{n=-\infty}^{\infty} q^{n^2} = (-q,-q,q^2;q^2)_\infty. \tag{4.2.15}$$

Using the result of (4.2.15). We can rewrite (4.2.14) as

$$\frac{(q^2;q^4)^6_\infty (q^4;q^4)^2_\infty}{(q;q^2)^4_\infty} = \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2. \tag{4.2.16}$$

We can combine the results of (4.2.12) and (4.2.16) to obtain

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2 = 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{2n-1}}{1-q^{2n-1}},$$

which is the sum of two squares formula. ■

Notice that if we put $q \to q^2$, $x = q$, $a = -1$, and $b = -q^2$ into Ramanujan's summation formula $(1.4.1)$, then we get the same result of $(4.2.1)$.

Corollary 4.2.2 *Let $r_2(n)$ denote the number of the representations of n as a sum of two squares, let $d_{4,1}(n)$ denote the number of positive divisors of n that are congruent to $1 \pmod 4$ and let $d_{4,3}(n)$ denote the number of positive divisors of n that are congruent to $3 \pmod 4$. Then,*

$$r_2(n) = 4\left[d_{4,1}(n) - d_{4,3}(n)\right]. \tag{4.2.17}$$

Proof To prove (4.2.17) we expand the series $(4.2.1)$ to obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} r_2(n)\, q^n &= \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2 \\
&= 1 + 4\left(\frac{q}{1-q} - \frac{q^3}{1-q^3} + \frac{q^5}{1-q^5} - \frac{q^7}{1-q^7} + \dots\right) \\
&= 1 + 4\sum_{n=0}^{\infty} \frac{q^{4n+1}}{1-q^{4n+1}} - 4\sum_{n=0}^{\infty} \frac{q^{4n+3}}{1-q^{4n+3}} \\
&= 1 + 4\sum_{n=1}^{\infty} \frac{q^{4n-3}}{1-q^{4n-3}} - 4\sum_{n=1}^{\infty} \frac{q^{4n-1}}{1-q^{4n-1}} \\
&= 1 + 4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{m(4n-3)} - 4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{m(4n-1)} \qquad (4.2.18) \\
&= 1 + 4\sum_{n=1}^{\infty} d_{4,1}(n)\, q^n - 4\sum_{n=1}^{\infty} d_{4,3}(n)\, q^n.
\end{aligned}$$

By considering the coefficient of q^n in (4.2.18), it now follows that

$$r_2(n) = 4\left[d_{4,1}(n) - d_{4,3}(n)\right],$$

which completes the proof. ■

Venkatachaliengar ([36], p. 108) used the Jacobi triple product identity to prove the sum of two squares formula. Milne ([29], p. 3) mentioned that Diophantus (325-409 A.D.) knew that no integer of the form $4n-1$ is a sum of two squares, and that Girard conjectured in 1632 that n is a sum of two squares if and only if all prime divisors q of n with $q \equiv 3 \pmod 4$ occur in n to an even power.

4.2.2 Sum of four squares

Theorem 4.2.3 *If* $|q| < 1$, *then*

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2} \right)^4 = 1 + 8 \sum_{4 \nmid n} \frac{nq^n}{1-q^n}. \tag{4.2.19}$$

Proof To obtain the sum of four squares formula, we first square both sides of the sum of two squares formula and use the result of (1.3.14) to prove (4.2.19). This gives

$$\begin{aligned} \left(\sum_{n=-\infty}^{\infty} q^{n^2} \right)^4 &= \left(1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{2n-1}}{1-q^{2n-1}} \right)^2 \\ &= 1 + 8 \left(\frac{q}{1-q} + \frac{2q^2}{1-q^2} + \frac{3q^3}{1-q^3} + \frac{5q^5}{1-q^5} + \ldots \right) \\ &= 1 + 8 \sum_{4 \nmid n} \frac{nq^n}{1-q^n}, \end{aligned}$$

which completes the proof. ■

Corollary 4.2.4 *Let* $r_4(n)$ *denote the number of ways of writing* n *as sum of four squares, then*

$$r_4(n) = 8 \sum_{d|n, 4 \nmid d} d, \qquad n = 1, 2, 3, \ldots. \tag{4.2.20}$$

Proof Rewrite the right hand side of the series (4.2.19) to get

$$\begin{aligned}\sum_{n=0}^{\infty} r_4(n)\,q^n &= \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^4 \\ &= 1+8\left[\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-\sum_{n=1}^{\infty}\frac{4nq^{4n}}{1-q^{4n}}\right] \\ &= 1+8\left[\sum_{n=1}^{\infty}\sigma(n)\,q^n-4\sum_{n=1}^{\infty}\sigma\left(\frac{n}{4}\right)q^n\right] \qquad \text{as } \sigma(n)=\sum_{d|n} d \\ &= 1+8\sum_{n=1}^{\infty}\left[\sigma(n)-4\sigma\left(\frac{n}{4}\right)\right]q^n. \qquad (4.2.21)\end{aligned}$$

Equate the coefficients of q^n in (4.2.21) to obtain

$$\begin{aligned} r_4(n) &= 8\left(\sigma(n)-4\sigma\left(\frac{n}{4}\right)\right) \\ &= 8\sum_{d|n,\,4\nmid d} d, \qquad n=1,2,3,..., \end{aligned}$$

which completes the proof. ■

Venkatachaliengar ([36], pp. 108-109) has given a proof of the sum of four squares formula. Askey ([4], p. 508) used Ramanujan's summation formula to prove (4.2.19). About (4.2.20), Milne ([29], p.3) mentioned that Diophantus was aware that all positive integers are sums of four integral squares. Bachet conjectured this result in 1621, and Lagrange gave the first proof in 1770.

4.2.3 Sum of six squares

Lemma 4.2.5

$$e_1-e_2=\frac{1}{4}\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^4.$$

Proof By (1.6.34),

$$
\begin{aligned}
e_1 - e_2 &= \frac{\left(-q;q^2\right)_\infty^4 \left(q^2;q^2\right)_\infty^4}{4\left(q;q^2\right)_\infty^4 \left(-q^2;q^2\right)_\infty^4} \\
&= \frac{1}{4}\left(-q;q^2\right)_\infty^4 \left(-q;q^2\right)_\infty^4 \left(q^2;q^2\right)_\infty^4 \\
&= \frac{1}{4}\left(-q,-q,-q^2;q^2\right)_\infty^4 \\
&= \frac{1}{4}\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^4,
\end{aligned}
$$

which completes the proof. ■

Lemma 4.2.6

$$
\begin{aligned}
f_1''(\theta) &= f_1(\theta)\left[f_2^2(\theta) + f_3^2(\theta)\right], \\
f_2''(\theta) &= f_2(\theta)\left[f_1^2(\theta) + f_3^2(\theta)\right], \\
f_3''(\theta) &= f_3(\theta)\left[f_1^2(\theta) + f_2^2(\theta)\right].
\end{aligned}
$$

Proof Using (1.6.44) - (1.6.46) we have

$$f_1'(\theta) = -f_2(\theta) f_3(\theta).$$

Then differentiate to get

$$
\begin{aligned}
f_1''(\theta) &= -\left(f_2(\theta) f_3(\theta)\right)' \\
&= -f_2'(\theta) f_3(\theta) - f_2(\theta) f_3'(\theta) \\
&= f_1(\theta) f_3^2(\theta) + f_1(\theta) f_2^2(\theta) \\
&= f_1(\theta)\left[f_2^2(\theta) + f_3^2(\theta)\right].
\end{aligned}
$$

Similarly,

$$\begin{aligned} f_2''(\theta) &= f_2(\theta)\left[f_1^2(\theta)+f_3^2(\theta)\right], \\ f_3''(\theta) &= f_3(\theta)\left[f_1^2(\theta)+f_2^2(\theta)\right]. \end{aligned}$$

which completes the proof. ■

Higher derivatives can be calculated in the same way.

Lemma 4.2.7

$$f_2''(\pi) - i f_1''(\pi\tau) = (e_1-e_2)^{\frac{3}{2}}.$$

Proof By (1.6.18) - (1.6.23), (1.6.24) - (1.6.26), (1.6.31) - (1.6.33), the results of Lemma 4.2.6 imply

$$\begin{aligned} f_1''(\pi\tau) &= f_1(\pi\tau)\left[f_2^2(\pi\tau)+f_3^2(\pi\tau)\right] \\ &= \frac{(-q,-q,q^2,q^2;q^2)_\infty}{i\,(q,q,-1,-q^2;q^2)_\infty}\left[0+e_2-e_3\right] \\ &= \frac{(-q,-q,q^2,q^2;q^2)_\infty}{2i\,(q,q,-q^2,-q^2;q^2)_\infty}\left[e_2-e_3\right] \\ &= -i\,(e_1-e_2)^{\frac{1}{2}}(e_2-e_3), \end{aligned} \tag{4.2.22}$$

where (1.6.34) was used to obtain the last line. In the same way we also get

$$\begin{aligned} f_2''(\pi) &= f_2(\pi)\left[f_1^2(\pi)+f_3^2(\pi)\right] \\ &= \frac{(-q,-q,q^2,q^2;q^2)_\infty}{(-1,-q^2,q,q;q^2)_\infty}\left[e_1-e_1+e_1-e_3\right] \\ &= \frac{(-q,-q,q^2,q^2;q^2)_\infty}{2\,(-q^2,-q^2,q,q;q^2)_\infty}\left[e_1-e_3\right] \\ &= (e_1-e_2)^{\frac{1}{2}}(e_1-e_3). \end{aligned} \tag{4.2.23}$$

Combining (4.2.22) and (4.2.23) gives

$$\begin{aligned} f_2''(\pi) - if_1''(\pi\tau) &= (e_1 - e_2)^{\frac{1}{2}}(e_1 - e_3) - i\left(-i(e_1 - e_2)^{\frac{1}{2}}(e_2 - e_3)\right) \\ &= (e_1 - e_2)^{\frac{1}{2}}(e_1 - e_3 - e_2 - e_3) \\ &= (e_1 - e_2)^{\frac{3}{2}}, \end{aligned}$$

which completes the proof. ■

Theorem 4.2.8 *If* $|q| < 1$, *then*

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^6 = 1 + 4\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)^2 q^{2n-1}}{1-q^{2n-1}} + 16\sum_{n=1}^{\infty}\frac{n^2 q^n}{1+q^{2n}}. \qquad (4.2.24)$$

Proof The ideas of following proof can be traced back to V. Ramamani [35], although her methods are slightly different. Lemma 4.2.7 together with Lemma 4.2.5 gives

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^6 = 8f_2''(\pi) - 8if_3''(\pi). \qquad (4.2.25)$$

Lambert series for $f_2''(\pi)$ and $f_1''(\pi\tau)$ can easily be obtained using the series expansions (1.6.13) and (1.6.15).

Specifically,

$$\begin{aligned} 8f_2''(\pi) &= 1 + 4\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)^2 q^{2n-1}}{1-q^{2n-1}} \\ &= 1 - 4\left[\frac{1^2 q}{1-q} - \frac{3^2 q^3}{1-q^3} + \frac{5^2 q^5}{1-q^5} - \ldots\right], \end{aligned} \qquad (4.2.26)$$

and

$$\begin{aligned} -8if_1''(\pi\tau) &= 16\sum_{n=1}^{\infty}\frac{n^2 q^n}{1+q^{2n}} \\ &= 16\left[\frac{1^2 q}{1+q^2} + \frac{2^2 q^2}{1+q^4} + \frac{3^2 q^3}{1+q^6} + \ldots\right]. \end{aligned} \qquad (4.2.27)$$

Combining the results of (4.2.26) and (4.2.27) in (4.2.25) to obtain

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^6 = 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)^2 q^{2n-1}}{1-q^{2n-1}} + 16\sum_{n=1}^{\infty} \frac{n^2 q^n}{1+q^{2n}},$$

which completes the proof. ■

Corollary 4.2.9 *Let $r_6(n)$ denote the number of the representations of n as a sum of six squares*

$$r_6(n) = 4\sum_{\substack{d|n \\ d\equiv 1(4)}} \left(\frac{4n^2}{d^2} - d^2\right) - 4\sum_{\substack{d|n \\ d\equiv 3(4)}} \left(\frac{4n^2}{d^2} - d^2\right). \tag{4.2.28}$$

Proof Start with (4.2.24)

$$\begin{aligned}
\sum_{n=0}^{\infty} r_6(n) q^n &= \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^6 \\
&= 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)^2 q^{2n-1}}{1-q^{2n-1}} + 16\sum_{n=1}^{\infty} \frac{n^2 q^n}{1+q^{2n}} \\
&= 1 + 4\sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+1)^2 q^{2n+1}}{1-q^{2n+1}} + 16\sum_{n=0}^{\infty} \frac{(n+1)^2 q^{n+1}}{1+q^{2n+2}} \\
&= 1 + 4\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (-1)^{n+1} (2n+1)^2 q^{2n+1} q^{(2n+1)m} \\
&\quad +16\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (n+1)^2 q^{n+1} \left(-q^{2n+2}\right)^m \\
&= 1 + 4\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} (-1)^{k+1} (2k+1)^2 q^{(2k+1)(m+1)} \\
&\quad +4\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} (-1)^m (2k+2)^2 q^{(k+1)(2m+1)}.
\end{aligned} \tag{4.2.29}$$

Equate the coefficients of q^n in (4.2.29) to obtain

$$r_6(n) = 4\sum_{(2k+1)(m+1)=n} (-1)^{k+1} (2k+1)^2 + 16\sum_{(k+1)(2m+1)=n} (-1)^m (k+1)^2. \tag{4.2.30}$$

Let $d = 2k + 1$ in the first term and $d = 2m + 1$ in the second term of (4.2.30) to get

$$\begin{aligned} r_6(n) &= 4 \sum_{d(m+1)=n} (-1)^{k+1} d^2 + 16 \sum_{(k+1)d=n} (-1)^m \left(\frac{n}{d}\right)^2 \\ &= 4 \sum_{d(m+1)=n} (-1)^{\frac{d+1}{2}} d^2 + 16 \sum_{d(k+1)=n} (-1)^{\frac{d-1}{2}} \left(\frac{n}{d}\right)^2. \end{aligned} \tag{4.2.31}$$

Notice that $(-1)^{\frac{d+1}{2}} = -1$ when $d \equiv 1 (\bmod 4)$ and $(-1)^{\frac{d+1}{2}} = 1$ when $d \equiv 3 (\bmod 4)$. Similarly, $(-1)^{\frac{d-1}{2}} = -1$ when $d \equiv 3 (\bmod 4)$ and $(-1)^{\frac{d-1}{2}} = 1$ when $d \equiv 1 (\bmod 4)$. Using this fact and put them into (4.2.31) to get

$$\begin{aligned} r_6(n) &= -4 \sum_{\substack{d|n \\ d\equiv 1(4)}} d^2 + \sum_{\substack{d|n \\ d\equiv 3(4)}} d^2 - 16 \sum_{\substack{d|n \\ d\equiv 3(4)}} \frac{n^2}{d^2} + 16 \sum_{\substack{d|n \\ d\equiv 1(4)}} \frac{n^2}{d^2} \\ &= 4 \sum_{\substack{d|n \\ d\equiv 1(4)}} \left(\frac{4n^2}{d^2} - d^2\right) - 4 \sum_{\substack{d|n \\ d\equiv 3(4)}} \left(\frac{4n^2}{d^2} - d^2\right), \end{aligned}$$

which completes the proof. ■

Berndt ([8], p. 36) has given the formula (4.2.28).

4.2.4 Sum of eight squares

Theorem 4.2.10 *If* $|q| < 1$, *then*

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^8 = 1 + 16 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - (-q)^n}. \tag{4.2.32}$$

Proof The following proof is a simplified version of a proof by S. H. Chan [11]. He used fundamental multiplicative identity (1.6.29) to obtain the eight squares formula. First we divide both sides of (1.6.29) by $(1-at)\left(1-\frac{t}{a}\right)$, we have

$$\frac{F(a,t)\,F\left(\frac{1}{a},t\right)}{(1-at)\left(1-\frac{t}{a}\right)}=\frac{t\frac{d}{dt}\rho_1(t)-a\frac{d}{da}\rho_1(a)}{(1-at)\left(1-\frac{t}{a}\right)}. \tag{4.2.33}$$

Then we can rewrite (4.2.33) as

$$\begin{aligned}
&\frac{(1-at)\,(atq^2;q^2)_\infty\left(\frac{q^2}{at},q^2,q^2;q^2\right)_\infty}{(1-at)\left(t,\frac{q^2}{t},a,\frac{q^2}{a};q^2\right)_\infty}\frac{\left(1-\frac{t}{a}\right)\left(\frac{tq^2}{a};q^2\right)_\infty\left(\frac{aq^2}{t},q^2,q^2;q^2\right)_\infty}{\left(1-\frac{t}{a}\right)\left(t,\frac{q^2}{t},\frac{1}{a},aq^2;q^2\right)_\infty}\\
=\;&\sum_{n=1}^{\infty}\frac{n\,(t^n-a^n)}{(1-q^{2n})\,(1-at)\left(1-\frac{t}{a}\right)}+\sum_{n=1}^{\infty}\frac{-n\,(t^{-n}-a^{-n})}{(1-q^{-2n})\,(1-at)\left(1-\frac{t}{a}\right)}\\
&\frac{\left(atq^2,\frac{tq^2}{a},\frac{q^2}{at},\frac{aq^2}{t},q^2,q^2,q^2,q^2;q^2\right)_\infty}{\left(t,t,\frac{q^2}{t},\frac{q^2}{t},a,\frac{1}{a},\frac{q^2}{a},aq^2;q^2\right)_\infty}\\
=\;&-\sum_{n=1}^{\infty}\frac{an\,(t^n-a^n)}{(1-q^{2n})\,(1-at)\,(t-a)}+\sum_{n=1}^{\infty}\frac{an\,(t^{-n}-a^{-n})\,q^{2n}}{(1-q^{2n})\,(1-at)\,(a-t)}.
\end{aligned} \tag{4.2.34}$$

Next we let $a\to t$ in (4.2.34) to get

$$\begin{aligned}
&\frac{\left(t^2q^2,q^2,\frac{q^2}{t^2},q^2,q^2,q^2,q^2,q^2;q^2\right)_\infty}{\left(t,t,\frac{q^2}{t},\frac{q^2}{t},t,\frac{1}{t},\frac{q^2}{t},tq^2;q^2\right)_\infty}\\
=\;&\lim_{a\to t}\left(-\sum_{n=1}^{\infty}\frac{an\,(t^n-a^n)}{(1-q^{2n})\,(1-at)\,(t-a)}\right)+\lim_{a\to t}\left(\sum_{n=1}^{\infty}\frac{an\,(t^{-n}-a^{-n})\,q^{2n}}{(1-q^{2n})\,(1-at)\,(a-t)}\right).
\end{aligned} \tag{4.2.35}$$

By using L'Hôpital's rule on the right hand of (4.2.35) we get

$$\frac{(q^2t^2;q^2)_\infty\left(\frac{q^2}{t^2};q^2\right)_\infty(q^2;q^2)_\infty}{(t;q^2)_\infty^3\left(\frac{q^2}{t^2};q^2\right)_\infty^3(q^2t;q^2)_\infty\left(\frac{1}{t};q^2\right)_\infty}=-\sum_{n=1}^{\infty}\frac{n^2t^n}{(1-q^{2n})\,(1-t^2)}+\sum_{n=1}^{\infty}\frac{n^2t^{-n}q^{2n}}{(1-q^{2n})\,(1-t^2)}. \tag{4.2.36}$$

Rewrite the right hand side of (4.2.36) to get

$$\begin{aligned}
& -\sum_{n=1}^{\infty} \frac{n^2 t^n}{(1-q^{2n})(1-t^2)} + \sum_{n=1}^{\infty} \frac{n^2 t^{-n} q^{2n}}{(1-q^{2n})(1-t^2)} \\
= & -\sum_{n=1}^{\infty} \frac{n^2 t^n (1-q^{2n}+q^{2n})}{(1-q^{2n})(1-t^2)} + \sum_{n=1}^{\infty} \frac{n^2 t^{-n} q^{2n}}{(1-q^{2n})(1-t^2)} \\
= & -\sum_{n=1}^{\infty} \frac{n^2 t^n}{1-t^2} + \sum_{n=1}^{\infty} \frac{-n^2 t^n q^{2n} + n^2 t^{-n} q^{2n}}{(1-q^{2n})(1-t^2)} \\
= & \frac{-t}{(1-t)^4} + \sum_{n=1}^{\infty} \frac{(-t^n+t^n)\, n^2 q^{2n}}{(1-q^{2n})(1-t^2)}. \qquad (4.2.37)
\end{aligned}$$

Put the result of (4.2.37) into (4.2.36) to get

$$\frac{(q^2t^2;q^2)_\infty \left(\frac{q^2}{t^2};q^2\right)_\infty (q^2;q^2)_\infty}{(t;q^2)_\infty^3 \left(\frac{q^2}{t^2};q^2\right)_\infty^3 (q^2t;q^2)_\infty \left(\frac{1}{t};q^2\right)_\infty} = \frac{--t}{(1-t)^4} + \sum_{n=1}^{\infty} \frac{(-t^n+t^n)\, n^2 q^{2n}}{(1-q^{2n})(1-t^2)}. \qquad (4.2.38)$$

Put $t \to -1$ into (4.2.38) to get

$$\begin{aligned}
\frac{(q^2;q^2)_\infty^8}{(-1;q^2)_\infty^3 (-q^2;q^2)_\infty^3 (-q^2;q^2)_\infty (-1;q^2)_\infty} &= \frac{1}{(1-(-1))^4} + \lim_{t\to -1} \sum_{n=1}^{\infty} \frac{(t^{-n}-t^n)\, n^2 q^{2n}}{(1-q^{2n})(1-t^2)} \\
\frac{(q^2;q^2)_\infty^8}{(-1;q^2)_\infty^4 (-q^2;q^2)_\infty^4} &= \frac{1}{16} + \lim_{t\to -1} \sum_{n=1}^{\infty} \frac{(-nt^{-n-1} -- nt^{n-1})\, n^2 q^{2n}}{(1-q^{2n})(-2t)} \\
\frac{(q^2;q^2)_\infty^8}{16(-q^2;q^2)_\infty^8} &= \frac{1}{16} + \sum_{n=1}^{\infty} \frac{n^3 (-1)^n q^{2n}}{(1-q^{2n})}. \qquad (4.2.39)
\end{aligned}$$

From (4.2.39), replacing q^2 by $-q$ we get

$$\begin{aligned}
\frac{(-q;-q)_\infty^8}{(q;-q)_\infty^8} &= 1 + 16 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-(-q)^n} \\
\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^8 &= 1 + 16 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-(-q)^n},
\end{aligned}$$

which completes the proof. ■

Milne ([29], p.3) mentioned that Jacobi introduced elliptic and theta functions in 1829 and motivated by Euler's work on four squares, Jacobi then used his theory of elliptic and theta functions to derive remarkable identities for the two, four, six, and eight squares identities. The identities (4.2.17) and (4.2.20) and their interpretations were first discovered by Jacobi in 1829.

Corollary 4.2.11 *Let $r_8(n)$ denote the number of ways of writing n as sum of eight squares*

$$r_8(n) = 16\sum_{d|n}(-1)^{n+d}d^3. \tag{4.2.40}$$

Proof Start with (4.2.32)

$$\begin{aligned}\sum_{n=0}^{\infty} r_8(n)q^n &= \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^8 \\ &= 1+16\sum_{n=1}^{\infty}\frac{n^3q^n}{1-(-q)^n} \\ &= 1+16\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} n^3q^n\left(-q\right)^{nm} \\ &= 1+16\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(-1\right)^{nm+n}n^3q^{nm}. \end{aligned} \tag{4.2.41}$$

Equate the coefficients of q^n in (4.2.41) to obtain

$$r_8(n) = 16\sum_{d|n}(-1)^{n+d}d^3,$$

which is same as (4.2.40). ■

Andrews [2] has given a proof of the sum of two, four, and eight squares formulas. Jacobi has given the sum of two, four, six, and eight squares formulas.

4.3 Sums of Triangles

For sums of triangles, $t_k(n)$ is the number of solutions of

$$\frac{x_1(x_1+1)}{2}+\frac{x_2(x_2+1)}{2}+\frac{x_3(x_3+1)}{2}+...+\frac{x_k(x_k+1)}{2}=n, \tag{4.3.1}$$

for non negative integers x_1, x_2,..., x_k.

For example we can write the integer 16 as

$$\left.\begin{array}{l}1+15\\15+1\\6+10\\10+6\end{array}\right\},$$

so there are 4 ways to write 16 as the sum of two triangles and therefore $t_2(16)=4$

In the next section, we prove the sum of two and four triangles formulas by using Ramanujan's summation formula. To prove the formula of the sums of six triangles, we will use three special cases of the fundamental multiplicative identity. Then using the fundamental multiplicative identity to prove the sum of eight triangles.

4.3.1 Sums of two triangles

Theorem 4.3.1 *If* $|q|<1$, then

$$\left(\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}\right)^2=\sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{4n+1}}. \tag{4.3.2}$$

Proof By (1.4.22),

$$\begin{aligned}\left(\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}\right)^2 &= \left(\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\right)^2\\ &= \frac{(q^2,q^2,q^4,q^4;q^4)_\infty}{(q,q,q^3,q^3;q^4)_\infty}.\end{aligned} \tag{4.3.3}$$

Replacing q with q^2 into $(1.5.2)$ and then putting $t = q$, $a = q$ gives

$$\begin{aligned}\frac{(q^2,q^2,q^4,q^4;q^4)_\infty}{(q,q,q^3,q^3;q^4)_\infty} &= \sum_{n=-\infty}^{\infty}\frac{q^n}{1-q\left(q^{4n}\right)}\\ &= \sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{4n+1}}. \qquad (4.3.4)\end{aligned}$$

Combining (4.3.3) and (4.3.4) completes the proof. ■

Corollary 4.3.2 *Let $t_2(n)$ denote by the representation of n as a sum of two triangles*

$$t_2(n) = d_{4,1}(4n+1) - d_{4,3}(4n+1). \qquad (4.3.5)$$

Proof Rewrite the right hand side of the series $(4.3.2)$ to get

$$\begin{aligned}\sum_{n=0}^{\infty} t_2(n)\,q^n &= \left(\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}\right)^2\\ &= \sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{4n+1}}\\ &= \sum_{n=0}^{\infty}\frac{q^n}{1-q^{4n+1}} + \sum_{n=-1}^{-\infty}\frac{q^n}{1-q^{4n+1}}\\ &= \sum_{n=0}^{\infty}\frac{q^n}{1-q^{4n+1}} + \sum_{n=0}^{\infty}\frac{q^{-n-1}}{1-q^{-4n-3}}\left(\frac{q^{4n+3}}{q^{4n+3}}\right)\\ &= \sum_{n=0}^{\infty}\frac{q^n}{1-q^{4n+1}} - \sum_{n=0}^{\infty}\frac{q^{3n+2}}{1-q^{4n+3}}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q^n q^{(4n+1)m} - \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q^{3n+2}q^{(4n+3)m}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q^{[(4n+1)(4m+1)-1]/4} - \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q^{[(4n+3)(4m+3)-1]/4} \qquad (4.3.6)\\ &= \sum_{n=0}^{\infty} d_{4,1}(4n+1)\,q^n - \sum_{n=0}^{\infty} d_{4,3}(4n+1)\,q^n.\end{aligned}$$

Equate the coefficients of q^n in (4.3.6) to obtain

$$t_2\left(n\right) = d_{4,1}\left(4n+1\right) - d_{4,3}\left(4n+1\right),$$

which completes the proof. ■

Adiga [1] says that (4.3.5) can be obtained from Ramanujan's ${}_1\psi_1$ summation formula but does not give the explicit details.

4.3.2 Sums of four triangles

Theorem 4.3.3 *If* $|q| < 1$, *then*

$$\left(\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}\right)^4 = \sum_{n=0}^{\infty} \frac{(2n+1)\,q^n}{1-q^{2n+1}}. \tag{4.3.7}$$

Proof By using Gauss's formula (1.4.22), we can write the left hand side of (4.3.7) as

$$\left(\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}\right)^4 = \frac{(q^2,q^2,q^2,q^2;q^2)_\infty}{(q,q,q,q;q^2)_\infty}. \tag{4.3.8}$$

Now, put $b = aq^2$ into the Ramanujan's summation formula (1.4.1) to get

$$\begin{aligned}
(1-a)\sum_{n=-\infty}^{\infty} \frac{x^n}{1-aq^{2n}} &= \frac{\left(ax, \dfrac{q^2}{ax}, q^2, q^2; q^2\right)_\infty}{\left(x, \dfrac{q^2}{x}, aq^2, \dfrac{q^2}{a}; q^2\right)_\infty} \\
\sum_{n=-\infty}^{\infty} \frac{x^n}{1-aq^{2n}} &= \frac{\left(ax, \dfrac{q^2}{ax}, q^2, q^2; q^2\right)_\infty}{\left(x, \dfrac{q^2}{x}, a, \dfrac{q^2}{a}; q^2\right)_\infty},
\end{aligned}$$

which may be rewritten as

$$\frac{1}{\left(1-\dfrac{q^2}{ax}\right)} \sum_{n=-\infty}^{\infty} \frac{x^n}{1-aq^{2n}} = \frac{\left(ax, \dfrac{q^4}{ax}, q^2, q^2; q^2\right)_\infty}{\left(x, \dfrac{q^2}{x}, a, \dfrac{q^2}{a}; q^2\right)_\infty}. \tag{4.3.9}$$

Next, put $a = q$ in (4.3.9) to get

$$\frac{1}{\left(1-\frac{q}{x}\right)}\sum_{n=-\infty}^{\infty}\frac{x^n}{1-q^{2n+1}} = \frac{\left(qx, \frac{q^3}{x}, q^2, q^2; q^2\right)_\infty}{\left(x, \frac{q^2}{x}, q, q; q^2\right)_\infty}. \tag{4.3.10}$$

The left hand side of (4.3.10) may be broken up into sums

$$\begin{aligned}
\frac{1}{\left(1-\frac{q}{x}\right)}\sum_{n=-\infty}^{\infty}\frac{x^n}{1-q^{2n+1}} &= \frac{1}{\left(1-\frac{q}{x}\right)}\left[\sum_{n=0}^{\infty}\frac{x^n}{1-q^{2n+1}} + \sum_{n=-1}^{-\infty}\frac{x^n}{1-q^{2n+1}}\right] \\
&= \frac{1}{\left(1-\frac{q}{x}\right)}\left[\sum_{n=0}^{\infty}\frac{x^n}{1-q^{2n+1}} + \sum_{n=0}^{\infty}\frac{x^{-n}}{1-q^{-2n-1}}\left(\frac{q^{2n+1}}{q^{2n+1}}\right)\right] \\
&= \frac{x}{x\left(1-\frac{q}{x}\right)}\sum_{n=0}^{\infty}\frac{x^n-q^{2n+1}x^{-n-1}}{1-q^{2n+1}}\left(\frac{x^{n+1}}{x^{n+1}}\right) \\
&= \frac{x}{x-q}\sum_{n=0}^{\infty}\frac{x^{2n+1}-q^{2n+1}}{x^{n+1}(1-q^{2n+1})} \\
&= \sum_{n=0}^{\infty}\left(\frac{x^{2n+1}-q^{2n+1}}{x-q}\right)\left(\frac{1}{x^n\left(1-q^{2n+1}\right)}\right) \\
&= \sum_{n=0}^{\infty}\frac{x^{2n}+x^{2n-1}q+x^{2n+2}q^2+\ldots+q^{2n}}{x^n\left(1-q^{2n+1}\right)}.
\end{aligned} \tag{4.3.11}$$

Then, put the results of (4.3.11) in (4.3.10) to get

$$\sum_{n=0}^{\infty}\frac{x^{2n}+x^{2n-1}q+x^{2n+2}q^2+\ldots+q^{2n}}{x^n\left(1-q^{2n+1}\right)} = \frac{\left(qx, \frac{q^3}{x}, q^2, q^2; q^2\right)_\infty}{\left(x, \frac{q^2}{x}, q, q; q^2\right)_\infty}. \tag{4.3.12}$$

Now, take the limit of letting $x \to q$ into (4.3.12) we have

$$\sum_{n=0}^{\infty}\frac{(2n+1)\,q^n}{1-q^{2n+1}} = \frac{(q^2, q^2, q^2, q^2; q^2)_\infty}{(q, q, q, q; q^2)_\infty}. \tag{4.3.13}$$

Finally, combine the results of (4.3.8) and (4.3.13) to get

$$\left(\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}\right)^4 = \sum_{n=0}^{\infty}\frac{(2n+1)\,q^n}{1-q^{2n+1}},$$

which completes the proof. ■

Corollary 4.3.4 *Let $t_4(n)$ denote the representation of n as a sum of four triangles*

$$t_4(n) = \sum_{d|2n+1} d. \tag{4.3.14}$$

Proof Start with (4.3.7)

$$\begin{aligned}
\sum_{n=0}^{\infty} t_4(n) q^n &= \left(\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} \right)^4 \\
&= \sum_{n=0}^{\infty} \frac{(2n+1)\, q^n}{1 - q^{2n+1}} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (2n+1)\, q^n q^{(2n+1)m} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (2n+1)\, q^{2mn+m+n} \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (2k+1)\, q^{[(2k+1)(2m+1)-1]/2}. \qquad (4.3.15)
\end{aligned}$$

Equate the coefficients of q^n in (4.3.15) to obtain

$$t_4(n) = \sum_{2k+1|2n+1} (2k+1). \tag{4.3.16}$$

Put $d = 2k+1$ in (4.3.16) to get

$$t_4(n) = \sum_{d|2n+1} d,$$

which completes the proof. ■

C. Adiga [1] has also used Ramanujan's ${}_1\psi_1$ summation formula (1.4.15) to prove (4.3.14), but the proof we have given here is simpler.

4.3.3 Sums of six triangles

Lemma 4.3.5

$$e_3 - e_2 = 4q\left(\sum_{n=0}^{\infty} q^{n(n+1)}\right)^4.$$

Proof By $(1.6.35)$

$$\begin{aligned} e_3 - e_2 &= 4q\frac{\left(-q^2;q^2\right)_\infty^4\left(q^2;q^2\right)_\infty^4}{\left(-q;q^2\right)_\infty^4\left(q;q^2\right)_\infty^4} \\ &= 4q\left(-q^2,-q^2,q^2;q^2\right)^4 \\ &= 4q\left(\sum_{n=0}^{\infty} q^{n(n+1)}\right)^4, \end{aligned}$$

which completes the proof. ■

Lemma 4.3.6

$$f_2''\left(\pi+\pi\tau\right) - if_3''\left(\pi\tau\right) = \left(e_3-e_2\right)^{\frac{3}{2}}.$$

Proof By $(1.6.18)$ - $(1.6.23)$, $(1.6.24)$ - $(1.6.26)$, $(1.6.31)$ - $(1.6.33)$, the results of the Lemma 4.2.6 imply

$$\begin{aligned} f_3''\left(\pi\tau\right) &= f_3\left(\pi\tau\right)\left[f_1^2\left(\pi\tau\right)+f_2^2\left(\pi\tau\right)\right] \\ &= \frac{q\left(-q^2,-1,q^2,q^2;q^2\right)_\infty}{i\left(q,q,-q,-q;q^2\right)_\infty}\left[e_2-e_1+e_2-e_2\right] \\ &= -i\left(e_3-e_2\right)^{\frac{1}{2}}\left(e_2-e_1\right), \end{aligned} \tag{4.3.17}$$

where $(1.6.34)$ was used to obtain the last line. In the same way we also get

$$\begin{aligned} f_2''\left(\pi+\pi\tau\right) &= f_2\left(\pi+\pi\tau\right)\left[f_1^2\left(\pi+\pi\tau\right)+f_2^2\left(\pi+\pi\tau\right)\right] \\ &= q^{\frac{1}{2}}\frac{\left(-q^2,-1,q^2,q^2;q^2\right)_\infty}{\left(-q,-q,q,q;q^2\right)_\infty}\left[e_3-e_1+e_3-e_3\right] \\ &= 2q^{\frac{1}{2}}\frac{\left(-q^2,-q^2,q^2,q^2;q^2\right)_\infty}{\left(-q,-q,q,q;q^2\right)_\infty}\left[e_3-e_1\right] \\ &= \left(e_3-e_2\right)^{\frac{1}{2}}\left(e_3-e_1\right). \end{aligned} \tag{4.3.18}$$

Combining (4.3.17) and (4.3.18) gives

$$\begin{aligned} f_2''(\pi + \pi\tau) - i f_3''(\pi\tau) &= (e_3 - e_2)^{\frac{1}{2}}(e_3 - e_1) - -i(e_3 - e_2)^{\frac{1}{2}}(e_2 - e_1) \\ &= (e_3 - e_2)^{\frac{1}{2}}(e_3 - e_2) \\ &= (e_3 - e_2)^{\frac{3}{2}}, \end{aligned}$$

which completes the proof. ■

Theorem 4.3.7 *If* $|q| < 1$, *then*

$$\left(\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}\right)^6 = \frac{1}{16}\sum_{n=0}^{\infty}\frac{(2n+1)^2 q^{\frac{n-1}{2}}}{1+q^{\frac{2n+1}{2}}} - \frac{1}{16}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^2 q^{\frac{n-1}{2}}}{1-q^{\frac{2n+1}{2}}}. \tag{4.3.19}$$

Proof Lemma 4.3.6 together with Lemma 4.3.5 gives

$$\begin{aligned} (e_3 - e_2)^{\frac{3}{2}} &= \left[4q\left(\sum_{n=0}^{\infty} q^{n(n+1)}\right)^4\right]^{\frac{3}{2}} \\ &= 8q^{\frac{3}{2}}\left(\sum_{n=0}^{\infty} q^{n(n+1)}\right)^6. \end{aligned}$$

It can be written as

$$\left(\sum_{n=0}^{\infty} q^{n(n+1)}\right)^6 = \frac{1}{8q^{\frac{3}{2}}} f_2''(\pi + \pi\tau) - \frac{i}{8q^{\frac{3}{2}}} f_3''(\pi\tau). \tag{4.3.20}$$

Lambert series for $f_2''(\pi + \pi\tau)$ and $f_3''(\pi\tau)$ can easily be obtained using the series expansions (1.6.15) and (1.6.17), specifically,

$$\begin{aligned} \frac{1}{8q^{\frac{3}{2}}} f_2''(\pi + \pi\tau) &= \frac{-1}{16q^{\frac{3}{2}}}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^2 q^{\frac{2n+1}{2}}}{1-q^{2n+1}} \\ &= \frac{-1}{16q^{\frac{3}{2}}}\left[\frac{1^2 q^{\frac{1}{2}}}{1-q} - \frac{3^2 q^{\frac{3}{2}}}{1-q^3} + \frac{5^2 q^{\frac{5}{2}}}{1-q^5} - \ldots\right], \end{aligned} \tag{4.3.21}$$

where

$$
\begin{aligned}
f_2''(\pi+\pi\tau) &= \frac{-1}{4i}\sum_{n=-\infty}^{\infty}\frac{(2n+1)^2 e^{\left(\frac{2n+1}{2}\right)i(\pi+\pi\tau)}}{1-q^{2n+1}} \\
&= \frac{-1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^2 q^{\frac{2n+1}{2}}}{1-q^{2n+1}},
\end{aligned}
$$

and

$$
\begin{aligned}
f_3''(\pi\tau) &= \frac{i}{4}\sum_{n=-\infty}^{\infty}\frac{(2n+1)^2 e^{\left(\frac{2n+1}{2}\right)i\pi\tau}}{1+q^{2n+1}} \\
&= \frac{i}{2}\sum_{n=0}^{\infty}\frac{(2n+1)^2 q^{\frac{2n+1}{2}}}{1+q^{2n+1}},
\end{aligned}
\tag{4.3.22}
$$

where

$$
\begin{aligned}
\frac{-i}{8q^{\frac{3}{2}}}f_3''(\pi\tau) &= \frac{1}{16q^{\frac{3}{2}}}\sum_{n=0}^{\infty}\frac{(2n+1)^2 q^{\frac{2n+1}{2}}}{1+q^{2n+1}} - \\
&= \frac{1}{16q^{\frac{3}{2}}}\left[\frac{1^2q^{\frac{1}{2}}}{1+q}+\frac{3^2q^{\frac{3}{2}}}{1+q^3}+\frac{5^2q^{\frac{5}{2}}}{1+q^5}+\ldots\right].
\end{aligned}
$$

Combining (4.3.21) and (4.3.22) and put into (4.3.20) to get

$$
\left(\sum_{n=0}^{\infty}q^{n(n+1)}\right)^6 = \frac{1}{16q^{\frac{3}{2}}}\sum_{n=0}^{\infty}\frac{(2n+1)^2 q^{\frac{2n+1}{2}}}{1+q^{2n+1}} - \frac{1}{16q^{\frac{3}{2}}}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^2 q^{\frac{2n+1}{2}}}{1-q^{2n+1}}.
\tag{4.3.23}
$$

Then, let $q \to q^{\frac{1}{2}}$ in (4.3.23) to obtain

$$
\begin{aligned}
\left(\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}\right)^6 &= \frac{1}{16q^{\frac{3}{4}}}\sum_{n=0}^{\infty}\frac{(2n+1)^2 q^{\frac{2n+1}{4}}}{1+q^{\frac{2n+1}{2}}} - \frac{1}{16q^{\frac{3}{4}}}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^2 q^{\frac{2n+1}{4}}}{1-q^{\frac{2n+1}{2}}} \\
&= \frac{1}{16}\sum_{n=0}^{\infty}\frac{(2n+1)^2 q^{\frac{n-1}{2}}}{1+q^{\frac{2n+1}{2}}} - \frac{1}{16}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^2 q^{\frac{n-1}{2}}}{1-q^{\frac{2n+1}{2}}},
\end{aligned}
$$

which completes the proof. ■

Berndt ([8], pp. 38-39) has given a proof of (4.3.19).

Corollary 4.3.8 *Let $t_6\left(n\right)$ denote by the representation of n as a sum of six triangles*

$$t_6(n)=\frac{1}{8}\sum_{\substack{d|4n+3\\ d\equiv 3(4)}} d^2-\frac{1}{8}\sum_{\substack{d|4n+3\\ d\equiv 1(4)}} d^2. \tag{4.3.24}$$

Proof Start with (4.3.19)

$$\begin{aligned}\sum_{n=0}^{\infty} t_6(n)q^n &= \left(\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}\right)^6\\ &= \frac{1}{16}\sum_{n=0}^{\infty}\frac{\left(2n+1\right)^2 q^{\frac{n-1}{2}}}{1+q^{\frac{2n+1}{2}}}-\frac{1}{16}\sum_{n=0}^{\infty}\frac{\left(-1\right)^n\left(2n+1\right)^2 q^{\frac{n-1}{2}}}{1-q^{\frac{2n+1}{2}}}\\ &= \frac{1}{16}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(2n+1\right)^2 q^{\frac{n-1}{2}}\left(-q^{\frac{2n+1}{2}}\right)^m\\ &\quad -\frac{1}{16}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(-1\right)^n\left(2n+1\right)^2 q^{\frac{n-1}{2}}\left(q^{\frac{2n+1}{2}}\right)^m\\ &= \frac{1}{16}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\left(-1\right)^m-\left(-1\right)^n\right]\left(2n+1\right)^2 q^{[(2n+1)(2m+1)-3]/4}.\end{aligned}$$

$$\tag{4.3.25}$$

It is easy to see that when m and n are same parity, then the terms in the sum (4.3.25) become zero. Now rewrite (4.3.25) becomes

$$\begin{aligned}\sum_{n=0}^{\infty} t_6(n)q^n &= \frac{1}{16}\sum_{m \text{ even}}\sum_{n \text{ odd}} 2\left(2n+1\right)^2 q^{[(2n+1)(2m+1)-3]/4}\\ &\quad -\frac{1}{16}\sum_{m \text{ odd}}\sum_{n \text{ even}} 2\left(2n+1\right)^2 q^{[(2n+1)(2m+1)-3]/4}\\ &= \frac{1}{8}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(4n+3\right)^2 q^{[(4n+3)(4m+1)-3]/4}\\ &\quad -\frac{1}{8}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(4n+1\right)^2 q^{[(4n+1)(4m+3)-3]/4}\\ &= \frac{1}{8}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left(4k+3\right)^2 q^{[(4k+3)(4m+1)-3]/4}\\ &\quad -\frac{1}{8}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left(4k+1\right)^2 q^{[(4k+1)(4m+3)-3]/4}.\end{aligned} \tag{4.3.26}$$

Equate the coefficients of q^n in (4.3.26) to obtain

$$t_6(n) = \sum_{4k+3|4n+3} (4k+3)^2 - \sum_{4k+1|4n+3} (4k+1)^2. \tag{4.3.27}$$

From (4.3.27), put $d = 4k+3$ such that $d \equiv 3(4)$ into the first summation and $d = 4k+1$ such that $d \equiv 1(4)$ into the second summation to get

$$t_6(n) = \frac{1}{8} \sum_{\substack{d|4n+3 \\ d\equiv 3(4)}} d^2 - \frac{1}{8} \sum_{\substack{d|4n+3 \\ d\equiv 1(4)}} d^2,$$

which completes the proof. ■

Note that the sum of six triangles formula was not given by Ramanujan in his notebook. The above proof is the same as Berndt ([8], p. 40).

4.3.4 Sum of eight triangles

Theorem 4.3.9 *If* $|q| < 1$, *then*

$$\left(\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} \right)^8 = \sum_{n=1}^{\infty} \frac{n^3 q^{n-1}}{1-q^{2n}}. \tag{4.3.28}$$

Proof The following proof is a simplified version of a proof by Chan [11]. He used the result of fundamental multiplicative identity(1.6.29) to obtain the eight triangles formula. Divide both sides of (1.6.29) by $\left(1 - \frac{q^2}{at}\right)\left(1 - \frac{t}{a}\right)$ to get

$$\frac{F(a,t)\, F\left(\frac{1}{a}, t\right)}{\left(1 - \frac{q^2}{at}\right)\left(1 - \frac{t}{a}\right)} = \frac{t\frac{d}{dt}\rho_1(t) - a\frac{d}{da}\rho_1(a)}{\left(1 - \frac{q^2}{at}\right)\left(1 - \frac{t}{a}\right)}. \tag{4.3.29}$$

We consider the left hand side of (4.3.29) to get

$$\begin{aligned}
\frac{F(a,t)\,F\left(\frac{1}{a},t\right)}{\left(1-\frac{q^2}{at}\right)\left(1-\frac{t}{a}\right)} &= \frac{\left(at,\frac{q^4}{at},q^2,q^2;q^2\right)_\infty}{\left(t,\frac{q^2}{t},a,\frac{q^2}{a};q^2\right)_\infty}\frac{\left(\frac{tq^2}{a},\frac{q^2a}{t},q^2,q^2;q^2\right)_\infty}{\left(t,\frac{q^2}{t},\frac{1}{a},aq^2;q^2\right)_\infty} \\
&= \frac{\left(at,\frac{q^4}{at},\frac{q^2t}{a},\frac{q^2a}{t},q^2,q^2,q^2,q^2;q^2\right)_\infty}{\left(a,\frac{q^2}{a},\frac{1}{a},aq^2,t,t,\frac{q^2}{t},\frac{q^2}{t};q^2\right)_\infty}. \qquad (4.3.30)
\end{aligned}$$

Next consider right hand side of (4.3.29) to get

$$\begin{aligned}
&\frac{t\frac{d}{dt}\rho_1(t)-a\frac{d}{da}\rho_1(a)}{\left(1-\frac{q^2}{at}\right)\left(1-\frac{t}{a}\right)} \\
&= \sum_{n=1}^{\infty}\left[\frac{n(t^n-a^n)}{(1-q^{2n})\left(1-\frac{q^2}{at}\right)\left(1-\frac{t}{a}\right)}+\frac{n(t^{-n}-a^{-n})q^{2n}}{(1-q^{2n})\left(1-\frac{q^2}{at}\right)\left(1-\frac{t}{a}\right)}\right] \\
&= \sum_{n=1}^{\infty}\left[\frac{-an(t^n-a^n)}{(1-q^{2n})\left(1-\frac{q^2}{at}\right)(t-a)}+\frac{an(t^n-a^n)q^{2n}}{(at)^n(1-q^{2n})\left(1-\frac{q^2}{at}\right)(t-a)}\right] \\
&= \sum_{n=1}^{\infty}\frac{-an(t^n-a^n)}{(1-q^{2n})(t-a)}\left[\frac{1}{\left(1-\frac{q^2}{at}\right)}-\frac{q^{2n}}{(at)^n\left(1-\frac{q^2}{at}\right)}\right] \\
&= \sum_{n=1}^{\infty}\frac{-an(t^n-a^n)}{(1-q^{2n})(t-a)}\left[\frac{1-\frac{q^{2n}}{(at)^n}}{1-\frac{q^2}{at}}\right] \\
&= \sum_{n=1}^{\infty}\frac{-an(t^n-a^n)}{(1-q^{2n})(t-a)}\sum_{m=0}^{n-1}\frac{q^{2m}}{(at)^m}. \qquad (4.3.31)
\end{aligned}$$

Combining (4.3.30) and (4.3.31) we have

$$\frac{\left(at,\frac{q^4}{at},\frac{q^2t}{a},\frac{q^2a}{t},q^2,q^2,q^2,q^2;q^2\right)_\infty}{\left(a,\frac{q^2}{a},\frac{1}{a},aq^2,t,t,\frac{q^2}{t},\frac{q^2}{t};q^2\right)_\infty}=\sum_{n=1}^{\infty}\frac{-an\left(t^n-a^n\right)}{\left(1-q^{2n}\right)(t-a)}\sum_{m=0}^{n-1}\frac{q^{2m}}{(at)^m}. \tag{4.3.32}$$

Then putting $a\to q$ and $t\to q$ into (4.3.32) we have

$$\begin{aligned}
\frac{\left(q^2,q^2,q^2,q^2,q^2,q^2,q^2,q^2;q^2\right)_\infty}{\left(q,q,\frac{1}{q},q^3,q,q,q,q;q^2\right)_\infty} &= \sum_{n=1}^{\infty}\frac{-n^2q^n}{1-q^{2n}}\sum_{m=0}^{n-1}\frac{q^{2m}}{\left(q^2\right)^m}\\
\frac{(1-q)\left(q^2;q^2\right)_\infty^8}{\left(1-\frac{1}{q}\right)\left(q;q^2\right)_\infty^8} &= \sum_{n=1}^{\infty}\frac{-n^3q^n}{1-q^{2n}}\\
-q\frac{\left(q^2;q^2\right)_\infty^8}{\left(q;q^2\right)_\infty^8} &= \sum_{n=1}^{\infty}\frac{-n^3q^n}{1-q^{2n}}\\
\frac{\left(q^2;q^2\right)_\infty^8}{\left(q;q^2\right)_\infty^8} &= \sum_{n=1}^{\infty}\frac{n^3q^{n-1}}{1-q^{2n}}\\
\left(\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}\right)^8 &= \sum_{n=1}^{\infty}\frac{n^3q^{n-1}}{1-q^{2n}},
\end{aligned}$$

which is called the sum of eight triangles formula. ■

Corollary 4.3.10 *Let $t_8(n)$ denote by the representation of n as a sum of eight triangles*

$$t_8(n)=\sum_{\substack{d\mid n+1\\ d\ odd}}\left(\frac{n+1}{d}\right)^3. \tag{4.3.33}$$

Proof Start with (4.3.28)

$$\begin{aligned}
\sum_{n=0}^{\infty}t_8(n)q^n &= \left(\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}\right)^8\\
&= \sum_{n=1}^{\infty}\frac{n^3q^{n-1}}{1-q^{2n}}\\
&= \sum_{n=0}^{\infty}\frac{(n+1)^3q^n}{1-q^{2(n+1)}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(n+1)^3 q^n q^{2(n+1)m} \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(k+1)^3 q^{(k+1)(2m+1)-1}. \qquad (4.3.34)
\end{aligned}
$$

Equate the coefficients of q^n in (4.3.34) to obtain

$$t_8(n) = \sum_{(k+1)(2m+1)=(n+1)} (k+1)^3 . \qquad (4.3.35)$$

Put $d = 2m+1$ in (4.3.35) to get

$$t_8(n) = \sum_{\substack{d \mid n+1 \\ 2 \nmid d}} \left(\frac{n+1}{d}\right)^3 ,$$

which is the same as (4.3.33). ■

Ramanujan [21] has given an amazing sum of 24 squares formula as following

$$r_{24}(n) = \frac{16}{691}(-1)^n \sum_{d \mid n} (-1)^d d^{11} + \frac{128}{691}\left\{(-1)^{n-1}\tau(n) - 512\tau\left(\frac{n}{2}\right)\right\},$$

where $\tau(n)$ is defined by

$$q\left\{(1-q)\left(1-q^2\right)\ldots\right\}^{24} = \sum_{n=1}^{\infty}\tau(n) q^n.$$

Now we have got the sum of two, four, six, and eight squares and triangles formulas. The next question is what are they? We have written two programs can solve this problem. As an illustration we start with an example for $r_4(100)$. In Appendix A we put 100 and 4 into the command lines from the end of the program and indicated by (*). The result is

$$
\begin{bmatrix}
10 & 0 & 0 & 0 & 8 \\
9 & 3 & 3 & 1 & 192 \\
8 & 6 & 0 & 0 & 48 \\
8 & 8 & 4 & 2 & 192 \\
7 & 7 & 1 & 1 & 96 \\
7 & 5 & 5 & 1 & 192 \\
5 & 5 & 5 & 5 & 16
\end{bmatrix}
$$

$$7$$

$$total := 744$$

The program outputs 3 results: a matrix box, a value 7, and a total 744. The value 7 indicates that there are 7 groups in the matrix box. The first four values in each row mean a group, for example the first group containing four numbers, that is 10, 0, 0, 0 represents $10^2 + 0^2 + 0^2 + 0^2$ and this gives 8 representations. Similarly $9^2 + 3^2 + 3^2 + 1^2$ gives 192 representations and so on. When we adding all the representations then we have the 'total' equal 744 which is the number of ways of writing 1000 as a sum of four squares. Notice that this computer program can compute the number of ways and their nature of representing of an integer as a sum of k squares where k is any positive integer. Furthermore, the program in Appendix B is to compute the number of ways and their nature of representing of an integer as a sum of k triangles.

4.4 Relationships between squares and triangles

In this section, we will provide some relationships between squares and triangles. There are more interesting relationships between sums of squares and sums of triangles. Observe that multiplying equation (4.3.1) by 8 and adding k to both sides gives

$$(2x_1 + 1)^2 + (2x_2 + 1)^2 + ... + (2x_k + 1)^2 = 8n + k.$$

Thus $t_k(n)$ is 2^{-k} times the number of representations of $8n + k$ as a sum of k odd squares. Below are some other relationships between two, four, six and eight squares and triangles.

Theorem 4.4.1 [14]

$$\begin{aligned} r_2(4n+1) &= 4t_2(n), \\ r_4(2n+1) &= 8t_4(n), \\ r_6(4n+3) &= 160t_6(n), \\ r_8(2n+1) &= 16t_8(2n). \end{aligned}$$

4.5 Summary

In this Chapter, we have given a proof of the sum of two, four, six and eight squares and triangles formulae. We have used Ramanujan's summation formula to prove sum of two and four squares and triangles. For the sum of six squares and triangles, we used three special cases of the fundamental multiplicative identity to prove them. The method for the sum of eight squares and triangles was based on Chan [11] who used the fundamental multiplicative identity to prove them; Berndt [8] used Ramanujan's Lambert series to prove those results. Berndt ([8], p. 41) mentioned that Ramanujan has written "There are of course well known formulae for the number of representations of a number as the sum of two, four, six and eight squares or triangular numbers. There are also various other arithmetical problems in which the partition method gives the actual value. I shall quote a few examples and reserve the discussion of these to another paper." But the "another paper" was never written. Ramanujan gives a general approach for deriving formulas for $r_{2k}(n)$ and $t_{2k}(n)$ but he did not work out the details. So, it would be useful to fill out the omitted part by Ramanujan. I have written two programs as Appendix A and B, which compute the number of ways and their nature of representing of an integer as a sum of k squares and triangles.

Chapter 5
Conclusion

We recall that our goal for this thesis was to obtain a better understanding of Ramanujan's work. We have considered some important classical results on elliptic functions and have given proofs of these results using the methods which may have been used by Ramanujan.

In the first Chapter we introduced some basic definitions and used it to prove one of Ramanujan's most famous identities. Then we introduced Ramanujan's ${}_1\psi_1$ summation formula and all of the results presented here were directly or indirectly related to Ramanujan's summation formula, and can be referred to the flow chart. The Jacobi triple product identity and the Jordan-Kronecker function are special cases of Ramanujan's ${}_1\psi_1$ summation formula. Next we introduced the fundamental multiplicative identity, Weierstrass function, and three special cases of $F(a,t)$ namely f_1, f_2, and f_3. Then we wrote $\phi_1(a)$, $\phi_2(a)$, and $\wp(a)$ as series and introduced P, Q, and R.

The task for the second Chapter was to write the transformation of $S_{2n+1}(q)$, $\phi_{r,s}(q)$, $U_{2n}(q)$, and $V_{2n}(q)$ in terms of $P\left(p\right)$, $Q\left(p\right)$, and $R\left(p\right)$. According to Venkatachaliengar, we can calculate the transformation formulas and we got simple formulas for them.

Ramanujan was the first mathematician to discover the properties of $p\left(n\right)$ and observed that the numbers of the partitions of numbers $5m+4$, $7m+5$, and $11m+6$ are divisible by 5, 7, and 11 respectively. There are others who have given a proofs of those results. In the third Chapter, a further proof is presented for the Ramanujan's modulo 5 partition congruence.

The fourth Chapter provided a substantial amount of work by giving proofs of formulas for the sums of two, four, six, and eight squares and triangles, all based on Ramanujan's results. Ramanujan's summation formula was used to get the sum of two and four squares and triangles formulas. For the sum of six squares and triangles, three special cases of the fundamental multiplicative identity was used to prove them. To prove the sum of eight squares and triangles formulas, the fundamental multiplicative identity was used to prove them. Then we gave two programs ,which compute the number of ways of representing an integer as a sum of k squares and triangles and their nature. Lastly, we stated some relationships between squares and triangles.

For the future development, we can try to solve the problem of Ramanujan's modulus 7 or 11 partition congruence by using the same technique on Ramanujan's modulus 5 partition congruence. To do this will require the development of ideas related to such things as Ramanujan's summation formula and Ramanujan differential equations etc. This approach could be used to construct new proofs of some recent problems in elliptic function theory, and thus increase our understanding of the work of Ramanujan himself.

Appendix A
Program for sums of squares

The following is a Maple V release 5 program for finding the number of ways of writing an integer as a sums of squares and their nature.

```
> sumsqrt := proc(x,level)
>    local temp, temp2, i, j, k, l, m, n;
>    with(linalg):
>    if (x = 0) then
>       temp := matrix(1,level,0):
>    elif (level = 2) then
>       l := 1;
>       for i from trunc(sqrt(x)) by -1 to trunc(sqrt(x/2)) do
>          k := x - i^2;
>          j := sqrt(k);
>          if (j = trunc(j) and i >= j) then
>             temp2 := matrix(1,2,0);
>             temp2[1,1] := i;
>             temp2[1,2] := j;
>             if (l > 1) then
>                temp := stackmatrix(temp, temp2):
>             else
```

```
>         temp := copy(temp2):
>       fi;
>       l := l + 1:
>     fi;
>   od;
> else
>   l := 1;
>   for i from trunc(sqrt(x)) by -1 to trunc(sqrt(x/level)) do
>     k := x - i^2;
>     temp2 := sumsqrt(k,level-1):
>     if (type(temp2,matrix)) then
>       m := rowdim(temp2);
>       n := coldim(temp2);
>     else
>       m := 0;
>     fi;
>     if (m > 0) then
>       j := 1;
>       while (j <= m) do
>         if (temp2[j,1] > i) then
>           temp2 := submatrix(temp2,2..m,1..n);
>           m := m - 1;
```

```
>                   j := j - 1;
>                fi;
>                j := j + 1;
>             od;
>             if (l = 1) then
>                temp := augment(matrix(m,1,i),temp2);
>             else
>                temp2 := augment(matrix(m,1,i),temp2);
>                temp := stackmatrix(temp, temp2);
>             fi;
>             l := l + 1;
>          fi;
>       od;
>    fi;
>    if (type(temp,’matrix’)) then
>       RETURN(copy(temp));
>    else
>       RETURN(‘undefined‘);
>    fi;
> end:
> w_sumsqrt := proc(A)
>    local temp, st, temp2, st2, i, j, k, m, n, res, fin_res;
```

```
>  with(linalg):
>  m := rowdim(A):
>  n := coldim(A):
>  if (m = 0) then
>     return(0);
>  fi:
>  for i from 1 by 1 to m do
>     res := factorial(n)*2^n:
>     for j from 1 by 1 to n do
>        temp[j] := 1:
>     od:
>     j := 1:
>     st := n:
>     while (st > 0) do
>        if (temp[j] <> 0) then
>           st2 := 0:
>           for k from j by 1 to n do
>              if (A[i,k] = A[i,j]) then
>                 st := st - 1:
>                 st2 := st2 + 1:
>                 temp[k] := 0:
>              fi:
```

```
>             od:
>             res := res/factorial(st2):
>             if (A[i,j] = 0) then
>                res := res/(2^st2):
>             fi:
>          fi:
>          j := j + 1:
>       od:
>       if (i = 1) then
>          fin_res := matrix(1,1,res):
>       else
>          fin_res := stackmatrix(fin_res,matrix(1,1,res)):
>       fi:
>    od:
>    RETURN(copy(fin_res));
> end:
> SumCol := proc(B)
>    local m, i, res;
>    m := rowdim(B):
>    res := 0;
>    for i from 1 by 1 to m do
>       res := res + B[i,1]:
```

```
>    od:

>    RETURN(res):

> end:

> A := sumsqrt(100,4):        (*)

> B := w_sumsqrt(A):

> augment(A,B);

> rowdim(A);

> total := SumCol(B);
```

The following is the program output.

```
[ 10  0  0  0  8   ]
[ 9   3  3  1  192 ]
[ 8   6  0  0  48  ]
[ 8   8  4  2  192 ]
[ 7   7  1  1  96  ]
[ 7   5  5  1  192 ]
[ 5   5  5  5  16  ]

          7

    total := 744
```

Appendix B
Program for sums of triangles

The following is a Maple V release 5 program for finding the number of ways of writing an integer as a sums of triangles and their nature.

```
> sumtri := proc(x,level)
>    local temp, temp2, i, j, k, l, m, n;
>    with(linalg):
>    if (x = 0) then
>       temp := matrix(1,level,0):
>    elif (level = 2) then
>       l := 1;
>       for i from trunc((-1+sqrt(1+8*x))/2) by -1 to trunc((-1+sqrt(1+8*x/2))/2) do
>          k := x - i*(i+1)/2;
>          j := (-1+sqrt(1+8*k))/2;
>          if (j = trunc(j) and i >= j) then
>             temp2 := matrix(1,2,0);
>             temp2[1,1] := i*(i+1)/2;
>             temp2[1,2] := j*(j+1)/2;
>             if (l > 1) then
>                temp := stackmatrix(temp, temp2):
>             else
```

```
>             temp := copy(temp2):
>           fi;
>           l := l + 1:
>         fi;
>       od;
>     else
>       l := 1;
>       for i from trunc((-1+sqrt(1+8*x))/2) by -1 to trunc((-1+sqrt(1+8*x/level))/2)
do
>         k := x - i*(i+1)/2;
>         temp2 := sumtri(k,level-1):
>         if (type(temp2,matrix)) then
>           m := rowdim(temp2);
>           n := coldim(temp2);
>         else
>           m := 0;
>         fi;
>         if (m > 0) then
>           j := 1;
>           while (j <= m) do
>             if (temp2[j,1] > i*(i+1)/2) then
>               temp2 := submatrix(temp2,2..m,1..n);
```

```
>                 m := m - 1;
>                 j := j - 1;
>              fi;
>              j := j + 1;
>           od;
>           if (l = 1) then
>              temp := augment(matrix(m,1,(i*(i+1)/2)),temp2);
>           else
>              temp2 := augment(matrix(m,1,(i*(i+1)/2)),temp2);
>              temp := stackmatrix(temp, temp2);
>           fi;
>           l := l + 1;
>        fi;
>     od;
>  fi;
>  if (type(temp,'matrix')) then
>     RETURN(copy(temp));
>  else
>     RETURN('undefined');
>  fi;
> end:
> w_sumtri := proc(A)
```

```
>  local temp, st, temp2, st2, i, j, k, m, n, res, fin_res;
>  with(linalg):
>  m := rowdim(A):
>  n := coldim(A):
>  if (m = 0) then
>     return(0);
>  fi:
>  for i from 1 by 1 to m do
>     res := factorial(n):
>     for j from 1 by 1 to n do
>        temp[j] := 1:
>     od:
>     j := 1:
>     st := n:
>     while (st > 0) do
>        if (temp[j] <> 0) then
>           st2 := 0:
>           for k from j by 1 to n do
>              if (A[i,k] = A[i,j]) then
>                 st := st - 1:
>                 st2 := st2 + 1:
>                 temp[k] := 0:
```

```
>                fi:
>             od:
>             res := res/factorial(st2):
>          fi:
>          j := j + 1:
>       od:
>       if (i = 1) then
>          fin_res := matrix(1,1,res):
>       else
>          fin_res := stackmatrix(fin_res,matrix(1,1,res)):
>       fi:
>    od:
>    RETURN(copy(fin_res));
> end:
> SumCol := proc(B)
>    local m, i, res;
>    m := rowdim(B):
>    res := 0;
>    for i from 1 by 1 to m do
>       res := res + B[i,1]:
>    od:
>    RETURN(res):
```

```
> end:

> with(linalg):

> A := sumtri(18,4):

> B := w_sumtri(A):

> augment(A,B);

> rowdim(A);

> total := SumCol(B);
```

The following is the program output.

```
[ 15  3  0  0  12 ]
[ 15  1  1  1  4  ]
[ 10  6  1  1  12 ]
[ 6   6  6  0  4  ]
[ 6   6  3  3  6  ]

         5

    total := 38
```

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