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The Algebraic Structure of B-Series

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Abstract

Runge-Kutta methods are some of the most widely used numerical integrators for approximating the solution of an ordinary differential equation (ODE). These methods form a subset of a larger class of numerical integrators called B-series methods. B-Series methods are expressed in terms of rooted trees, a type of combinatorial graph, which are related to the vector field of the ODE that is to be solved. Therefore, the conditions for B-series methods to preserve important properties of the solution of an ODE, such as symplecticity and energy-preservation, may be expressed in terms of rooted trees. Certain linear combinations of rooted trees give conditions for a B-series to be Energy-preserving while other linear combinations give conditions for a B-series to be Hamiltonian. B-series methods may be conjugate (by another B-series) to an Energy-preserving or an Hamiltonian B-series. Such B-series methods are called conjugate-to-Energy preserving and conjugate-to-Hamiltonian, respectivley. The conditions for a B-series to be conjugate-to-Energy preserving or conjugate-to-Hamiltonian may also be expressed in terms of rooted trees.

The rooted trees form a vector space over the Real numbers. This thesis explores the algebraic structure of this vector space and its natural energy-preserving, Hamiltonian, conjugate-to-Energy preserving and conjugate-to-Hamitlonian subspaces and dual subspaces.

The first part of this thesis reviews important concepts of numerical integrators and introduces the general Runge-Kutta methods. B-series methods, along with rooted trees, are then introduced in the context of Runge-Kutta methods. The theory of rooted trees is developed and the conditions for a B-series to be Hamiltonian or have first integral are given and discussed. In the final chapter we interpret the conditions in the context of vector spaces and explore the algebraic structure of, and the relationships between, the natural vector subspaces and dual spaces.

"Do you like Phil Collins? I've been a big Genesis fan ever since the release of their 1980 album, Duke. Before that, I really didn't understand any of their work. Too artsy, too intellectual. It was on Duke where, uh, Phil Collins' presence became more apparent. I think Invisible Touch was the group's undisputed masterpiece. It's an epic meditation on intangibility. At the same time, it deepens and enriches the meaning of the preceding three albums. Christy, take off your robe."

-Brett Easton Ellis

"Wu-Tang Clan ain't nuttin to f' wit"

-The RZA

"Possibilities of sweetness on technicolor beaches had been trickling through my spine for some time..."

-Vladimir Nabokov

"I have to return some videotapes."

-Brett Easton Ellis

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